THORPES

COURSE IN MATHEMATICS 5

for the IIT-JEE & Other Engineering Entrance Examinations

Calculus I

K. R. Choubey
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ALWAYS LEARNING
PEARSON

COURSE IN MATHEMATICS

(FOR IIT JEE AND OTHER ENGINEERING ENTRANCE EXAMINATIONS)

CALCULUS

ONE

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PREFACE

When a new book is written on a well known subject like *Calculus* for class XI/XII Academics/AIEEE/IIT/State engineering entrance exams and NDA, several questions arise like—why, what, how and for whom? What is new in it? How is it different from the other books? For whom is it meant? The answers to these questions are often not mutually exclusive. Neither are they entirely satisfactory except perhaps to the authors. We are certainly not under the illusion that there are no good books. There are many good books available in the market.

However, none of them caters specifically to the needs of students. Students find it difficult to solve most of the problems of any of the books in the absence of proper planning. This inspired us to write this book *Calculus I*, to address the requirements of students of class XI/XII CBSE and State Board Academics. In this book, we have tried to give a connected and simple account of the subject. It gives a detailed, lecture wise description of basic concepts with many numerical problems and innovative tricks and tips. Theory and problems have been designed in such a way that the students can themselves pursue the subject. We have also tried to keep this book self contained. In each lecture all relevant concepts, prerequisites and definitions have been discussed in a lucid manner and also explained with suitable illustrated examples including tests.

Due care has been taken regarding the Board (CBSE/State) examination need of students and nearly 100 per cent articles and problems set in various examinations including the IIT-JEE have been included.

The presentation of the subject matter is lecturewise, intelligent and systematic, the style is lucid and rational, and the approach is comprehensible with emphasis on improving speed and accuracy. The basic motive is to attract students towards the study of mathematics by making it simple, easy and interesting and on a day-to-day basis. The instructions and method for grasping the lectures are clearly outlined topic wise. The presentation of each lecture is planned for better experiential learning of mathematics which is as follows:

1. Basic Concepts: Lecture Wise
2. Solved Subjective Problems (XII Board (C.B.S.E./State): For Better Understanding and Concept Building of the Topic.
3. Unsolved Subjective Problems (XII Board (C.B.S.E./State): To Grasp the Lecture Solve These Problems.
4. Solved Objective Problems: Helping Hand.
5. Objective Problem: Important Questions with Solutions.
6. Unsolved Objective Problems (Identical Problems for Practice)
   For Improving Speed with Accuracy.

7. Worksheet: To Check Preparation Level
8. Assertion-Reason Problems: Topic Wise Important Questions and Solutions with Reasoning
9. Mental Preparation Test: 01
10. Mental Preparation Test: 02
11. Topic Wise Warm Up Test: 01: Objective Test
12. Topic Wise Warm Up Test: 02: Objective Test
13. Objective Question Bank Topic Wise: Solve These to Master.

This book will serve the need of the students of class XI/XII board, NDA, AIEEE and SLEEE (state level engineering entrance exam) and IIT-JEE. We suggest each student to attempt as many exercises as possible without looking up the solutions. However, one should not feel discouraged if one needs frequent help of the solutions as there are many questions that are either tough or lengthy. Students should not get frustrated if they fail to understand some of the solutions in the first attempt. Instead they should go back to the beginning of the solution and try to figure out what is being done At the end of every topic, some harder problems with 100 per cent solutions and Question Bank are also given for better understanding of the subject.

There is no end and limit to the improvement of the book. So, suggestions for improving the book are always welcome.

We thank our publisher, Pearson Education for their support and guidance in completing the project in record time.

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PART A

Functions

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LECTURE

1

Definitions, Value of Functions

BASIC CONCEPTS

1. **Quantity** Any thing on which mathematical opration such as addition, subtraction, multiplication, division can be perfomed is called a Quantity.
2. **Constant** A Quantity which retains the same value through out a mathematical operation is called a constant.
3. **Absolute Constant** A constant which retains the same value in every mathematical operation is called an Absolute constant. Thus 1, 2, 3 .... $\sqrt{2}, \sqrt{3}, \pi, e$, etc are the absolute constant.
4. **Arbitrary Constant** A constant which retains the same value throughout in one problem but may have different values for different problems is called and arbitary constant. It is usually denoted by $a, b, c, \ldots\ldots l, m, n$ ...... etc
   For Example In the equation of a straight line $y = mx + c$; m and c are arbitrary constant as they have same values for one line and other values for the other lines.
5. **Variable** A quantity which can take a number of values is called a variable. They are generally denoted by
   x_1, y_1, z_1 .... u_1, v_1, w_1 .... etc.
   For Example In the equation: $y = mx + c$; $ax + by + c = 0$; $x^2 + y^2 = a^2$, x and y are variables.
6. **Independent Variable** A variable which can assume any value it likes is called an independent variable.
7. **Dependent Variable** A variable whose value depends on the value of another variable is called a dependent variable.
8. **Domain of a Variable** If the values of variable x lie between a and b, then the set of all values of x between a and b is called the Domain of the variable x.
9. **Open Interval** If the values of the variable x lie between a and b but cannot be equal to a or b, then the set of all values of x between a and b is called an open Domain and is denoted by (a, b) or $[a, b]$.
10. **Closed Domain** If the values of the variable x lies between a and b including a and b also, then the set of all values of x between a and b including a and b also is called closed domain and is denoted by $[a, b]$. The semi open domain are denoted $(a, b]$ or (a, b) internal $(a \,.\, b]$ is the domain open on the left and closed on the right i.e., It is the set of values of x between a and b including b also the $x \neq a$. $[a, b)$ is the domain closed on the left and open on the right i.e., it is the set of values of x between a and b including a also the $x \neq b$.
11. **A Continuous Variable** Is a variable that can take all the numerical values between two given numbers.
12. **Discrete Variable** Quantities which are incapable of taking all possible values between two given numbers are called discrete or discontinuous variables. A variable whose possilbe values form a discrete set (eg., Any randon variable that has only a finite numbers of value is necessarily discrete).

13. Discrete Set The set of integers is a discrete set. The set of rational numbers is not discrete, since any interval of non-zero length that contains a rational number contains other rational numbers.

14. Function Let X and Y be two non empty sets: If function 'f' defined from set X to Y is a rule or a collection of rules which associate to each element x in X a unique element y in Y. Symbolically we write it as $f: X \rightarrow Y$ and is read as function defined from the set X to Set Y. The word function is also replaced by "mapping" or "correspondance" or transformation.

NOTES

1. The unique y of Y is called the value of 'f' at x (the image of x under f): If is written as $f(x)$. Thus $y = f(x)$.
2. The element x of X is called preimage (or inverse image) of y.
3. The set X is called the domain of f. The domain of the function $y = f(x)$ is the set of all real x for which real function $f(x)$ is defined.
4. The set Y is called the co-domain of f.
5. The set consisting of all images of the elements of X under f is called the range of f. This is defined by $f(x)$.
   Thus range of $f : y = \{f(x) \mid \text{for all } x \in X\}$
   This is a subset of Y, which may or may not be equal to Y. In other words range of $y = f(x)$ is set of all values of the function $f(x)$ corresponding to each real number in the domain.
6. If X and Y have m and n distinct elements respectively, then the number of mappings from X to $Y = n^m$.

15. Some Important Fast Track Formulas

(i) If $y = \dfrac{ax+b}{cx-a}$ then $f(y) = x$

(ii) If $f(x) f(1/x) = f(x) + f(1/x)$, then $f(x) = \pm x^n + 1$

(iii) If $f(x + y) = f(x) f(y) \Rightarrow f(x) =$
$f(x) = a^{\lambda x}, \lambda$ is constant. or $f(x) = e^x$

(iv) If $f(x + y) = f(x) + f(y) \Rightarrow f(x) = \lambda x$ or $f(x)$ is odd function.

(v) If $f(xy) = f(x) + f(y) \Rightarrow f(x) = \lambda \log x$ or $f(x) = 0$, λ is constant

(vi) If $f(xy) = f(x) f(y) \Rightarrow f(x) = x^n$, $n \in R$

(vii) If $f\left(\dfrac{x}{y}\right) = f(x) - f(y)$, $f(e) = 1$, then $f(x) = \log_e x = \ln x$.

(viii) $f(x) f(y) - f(xy) = x + y$ for all $x, y \in R$ and $f(1) > 0$ then $f(x) = x + 1$.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)):
TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. If $f(x) = \begin{cases} 3x-1 & \text{when} \quad x > 3 \\ x^2 - 2 & \text{when} \quad -2 \le x \le 3 \\ 2x+3 & \text{when} \quad x < -2 \end{cases}$. Then find the values of $f(2)$, $f(4)$, $f(-1)$, $f(-3)$ and $f(0)$. ***[MP-2001]***

2. If $y = f(x) = \dfrac{ax-b}{bx-a}$, show that $x = f(y)$.

3. Given $f(x) = \begin{cases} 3x-8, & \text{for} \quad x \le 5 \\ 7, & \text{for} \quad x > 5 \end{cases}$. What is the value of the function.
   (i) at $x = 3$ and (ii) at $x = 7$
   [CBSE Sample Paper]

4. If $f(x) = \dfrac{x^2-1}{x^2+1}$, then prove that $f(x) + f(1/x) = 0$. ***[MP-99]***

5. (i) If $f(x) = \log x$, then find $f(1)$. (ii) If $f(x) = 1 + \sin x$, then find $f(\pi/3)$

6. If $f(x) = \dfrac{1-x^2}{1+x^2}$. Show that $f(\tan\theta) = \cos 2\theta$ and $f(\sqrt{\cos\theta}) = \tan^2(\theta/2)$, $0 < \theta < \pi/2$.

7. If $f(x) = \dfrac{x^2}{1+x^2}$, then find the value of $f(\tan\theta)$. ***[MP-2007; MP-98]***

8. If $f(x) = \log e^x$, when $x > 0$, then prove that $f(uvw) = f(u) + f(v) + f(w)$. ***[MP-2007]***

9. Let $f : R \to R$ be given by $f(x) = x^2 + 3$. Find
(a) $\{x : f(x) = 28\}$
(b) The pre images of 39 and 2 under f.

10. If a function $f : R \to R$ be defined by $f(x) = \begin{cases} 3x-2, & x<0 \\ 1, & x=0 \\ 4x+1, & x>0 \end{cases}$ Find $f(1)$, $f(-1)$, $f(0)$, $f(2)$

ANSWERS

1. 2; 11; –1; –3; –2.

3. (i) 1 (ii) 7

5. (i) 0 (ii) $1 + \dfrac{\sqrt{3}}{2}$

7. $\sin^2 Q$

9. (a) $\{-5, 5\}$
(b) 6 and 6, 2 does not have any pre-image.

10. 5, –5, 1, 9.

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $f(x+y, x-y) = xy$, then $\dfrac{f(x,y) + f(y,x)}{2} =$ ***[AMU-2002]***

(a) x (b) y
(c) 0 (d) None of these

Solution

(c) Let $x + y = u$ and $x - y = v$. Then, $x = \dfrac{u+v}{2}$

and $y = \dfrac{u-v}{2}$

$\therefore \quad f(x+y, x-y) = xy$

$\Rightarrow \quad f(u,v) = \left(\dfrac{u+v}{2}\right)\left(\dfrac{u-v}{2}\right)$

$\Rightarrow \quad f(u,v) = \dfrac{u^2 - v^2}{4}$

$\Rightarrow \quad f(v,u) = \dfrac{v^2 - u^2}{4}$ Thus,

$f(u,v) + f(v,u) = 0$

$\Rightarrow \quad \dfrac{f(u,v) + f(v,u)}{2} = 0$

$\Rightarrow \quad \dfrac{f(x,y) + f(y,x)}{2} = 0$

2. Let $f(1) = 1$ and $f(n) = 2\sum_{r=1}^{n-1} f(r)$. Then $\sum_{n=1}^{m} f(n)$ is equal to

(a) $3^m - 1$
(b) 3^m
(c) 3^{m-1}
(d) None of these

Solution

(c) $\quad f(n) = 2\sum_{r=1}^{n-1} f(r)$, $f(1) = 1$

$\therefore \quad f(2) = 2\sum_{r=1}^{1} f(r) =\uparrow 2f(1)$ and $f(1) = 1$ (given)

$f(3) = 2\sum_{r=1}^{2} f(r) = 2[f(1) + f(2)]$

$= 2[1 + 2] = 2.3 = 6$

$f(4) = 2[f(1) + f(2) + f(3)]$

$= 2[1 + 2 + 6] = 18$

$\therefore \quad \sum_{n=1}^{2} f(n) = 1 + 2 = 3 = 3^{2-1}$

$$\therefore \quad \sum_{n=1}^{3} f(n) = 1 + 2 + 6 = 9 = 3^2 = 3^{3-1}$$

$$\therefore \quad \sum_{n=1}^{m} f(n) = 3^{m-1}$$

3. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at (0, 0) and $(x, g(x))$ is $\sqrt{3}/4$, then the function $g(x)$ is

(a) $g(x) = \pm\sqrt{1-x^2}$ (b) $\sqrt{1+x^2}$

(c) $g(x) = -\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

Solution

(a) Since the area of the equilateral triangle is $\left(\frac{\sqrt{3}}{4}\right)a^2$, a being the side of the triangle, we must have $\frac{\sqrt{3}}{4}[(x-0)^2 + (g(x)-0)^2] = \frac{\sqrt{3}}{4}$

$\Rightarrow \quad x^2 + (g(x))^2 = 1$

$\Rightarrow \quad g(x) = \sqrt{1-x^2}$ or $g(x) = -\sqrt{1-x^2}$

Hence (a) is the correct answer.

4. If $f(x) = \dfrac{x-1}{x+1}$, then $f(2x)$ in terms of $f(x)$ is

(a) $\dfrac{f(x)+1}{f(x)+3}$ (b) $\dfrac{3f(x)+1}{f(x)+3}$

(c) $\dfrac{f(x)+3}{f(x)+1}$ (d) $\dfrac{f(x)+3}{3f(x)+1}$

Solution

(b) $\quad f(2x) = \dfrac{2x-1}{2x+1}$

$$\frac{3f(x)+1}{f(x)+3} = \frac{3\left(\frac{x-1}{x+1}\right)+1}{\frac{x-1}{x+1}+3}$$

$$= \frac{3x-3+x+1}{x-1+3x+3}$$

$$\Rightarrow \quad \frac{4x-2}{4x+2} = \frac{2x-1}{2x+1}$$

Hence $\quad f(2x) = \dfrac{3f(x)+1}{f(x)+3}$

5. If for a function $f(x)$, $f(x + y) = f(x) + f(y)$ for all reals x and y then $f(0)$ is equal to

(a) 1 (b) 0

(c) $f(x)\ \forall\, x \in R$ (d) None of these

Solution

(b) Given $f(x + y) = f(x) + f(y)$, therefore $\forall\, x \in R$,

We have $f(x) = f(x + 0) = f(x) + f(0)$

$\Rightarrow \quad f(0) = 0$

6. If $f(n + 1) + f(n - 1) = 2f(n)$ and $f(0) = 0$, then $f(n), n \in N$ is

(a) $nf(1)$ (b) $(f(1))^n$

(c) 0 (d) None of these

Solution

(a) Putting: $n = 1$, we get

$\Rightarrow \quad f(2) + f(0) = 2f(1)$

$\Rightarrow \quad f(2) = 2f(1) \qquad [\because f(0) = 0]$

Putting $n = 2$, we get

$\Rightarrow \quad f(3) + f(1) = 2f(2)$

$\Rightarrow \quad f(3) = 2f(2) - f(1)$

$\Rightarrow \quad = 4f(1) - f(1) = 3f(1)$ and so on

$\because \quad f(n) = nf(1)$

7. Let f be a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. If $f(1) = K$, then $f(n), n \in N$, is equal to

(a) K^n (b) nK

(c) K^{-n} (d) None of these

Solution

(b) Putting $x = 1, y = 1$, we get

$f(2) = f(1) + f(1) = 2f(1) = 2k$

Putting $x = 2, y = 1$, we get

$f(3) = f(2) + f(1) = 3f(1) = 3k$ and so on

$\therefore f(n) = nk$

8. If $f(x) = x^2 - 3x + 1$ and $f(2\alpha) = 2f(\alpha)$, then α is equal to

(a) $\dfrac{1}{\sqrt{2}}$ (b) $-\dfrac{1}{\sqrt{2}}$

(c) $\dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}}$ (d) None of these

Solution

(c) Given $f(2\alpha) = 2f(\alpha)$

$(2\alpha)^2 - 3(2\alpha) + 1 = 2(\alpha^2 - 3\alpha + 1)$

$4\alpha^2 - 6\alpha + 1 - 2\alpha^2 + 6\alpha = 2$

$$2\alpha^2 = 1 \Rightarrow \alpha^2 = \frac{1}{2} \Rightarrow \alpha = \pm\frac{1}{\sqrt{2}}$$

9. A function $f : R \to R$ is defined by $f(x) = \begin{cases} 1, & \text{if } x \in Q \\ -1, & \text{if } x \in (R-Q) \end{cases}$. The value of $f(\pi) - f\left(\frac{22}{7}\right)$ is

(a) 0 (b) 2
(c) –2 (d) None of these

Solution

(c) π is an irrational number and 22/7, a rational number i.e., $\frac{22}{7} \in \underset{\sim}{Q}$ and $\pi \in (R - \phi)$

10. If a function F is such that $F(0) = 2$, $F(1) = 3$, $F(n+2) = 2F(n) - F(n+1)$ for $n \geq 0$, then $F(5)$ is equal to

(a) –7 (b) –3
(c) 7 (d) 13

Solution

(d) $F(2) = 2F(0) - F(1) = 2 \times 2 - 3 = 1$
$F(3) = 2F(1) - F(2) = 2 \times 3 - 1 = 5$
$F(4) = 2F(2) - F(3) = 2 - 5 = -3$
and $F(5) = 2F(3) - F(4) = 10 - (-3) = 13$

11. If $f(x) = \frac{4^x}{4^x + 2}$, then $f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \ldots + \left(\frac{1996}{1997}\right)$ is equal to

(a) 998 (b) 1997
(c) 0 (d) None of these

Solution

(a) Since $f(x) = \frac{4^x}{4^x + 2}$

$$\therefore \quad f(1-x) = \frac{4^{1-x}}{4^{1-x}+2} = \frac{4.4^{-x}}{4.4^{-x}+2}$$

$$= \frac{4}{4+2.4^x} + \frac{2}{2+4^x}$$

$$\therefore \quad f(x) + f(1-x) = \frac{4^x+2}{4^x+2} = 1$$

$$\text{Now } f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + .. + f\left(\frac{1996}{1997}\right)$$

$$= \sum_{n=1}^{1996} f\left(\frac{n}{1997}\right) = \sum_{n=1}^{998} f\left(\frac{n}{1997}\right) + \sum_{n=999}^{1996} f\left(\frac{n}{1997}\right)$$

$$= \sum_{n=1}^{998} f\left(\frac{n}{1997}\right) + \sum_{n=1}^{998} f\left(\frac{1997-n}{1997}\right)$$

$$= \sum_{n=1}^{998} f\left(\frac{n}{1997}\right) + f\left(\frac{1997-n}{1997}\right)$$

$$= \sum_{n=1}^{998} \left[f\left(\frac{n}{1997}\right) + f\left(1 - \frac{n}{1997}\right) \right]$$

$$= \sum_{n=1}^{998} (1) = 998$$

12. If $f(\theta) = \tan\theta$, then $\frac{f(\theta) - f(\phi)}{1 + f(\theta)f(\phi)}$ is equal to

[PET (Raj.)-96]

(a) $f(\theta - \phi)$
(b) $f(\phi - \theta)$
(c) $f(\theta + \phi)$
(d) None of these

Solution

(a) $\frac{f(\theta) - f(\phi)}{1 + f(\theta)f(\phi)} = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} =$
$\tan(\theta - \phi) = f(\theta - \phi)$

13. If $e^{f(x)} = \frac{10+x}{10-x}$, $x \in (-10, 10)$ and $f(x) = kf\left(\frac{200x}{100+x^2}\right)$, then k is equal to

[EAMCET-2003]

(a) 0.8 (b) 0.7
(c) 0.6 (d) 0.5

Solution

(d) $f(x) = \log_e\left(\frac{10+x}{10-x}\right)$

$$\Rightarrow f\left(\frac{200x}{100+x^2}\right)$$

$$= \log\left[\frac{10(100+x^2) + 200x}{10(100+x^2) - 200x}\right]$$

$$= 2\log\left(\frac{10+x}{10-x}\right) = 2f(x)$$

$$\therefore \quad f(x) = \frac{1}{2} f\left(\frac{200x}{100+x^2}\right)$$

$\Rightarrow \quad k = 0.5$

14. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ when $-1 < x_1, x_2 < 1$, then $f(x_1) + f(x_2)$ equals **[DCE-97]**

(a) $f\left(\frac{x_1 + x_2}{1 + x_1x_2}\right)$ (b) $f\left(\frac{x_1 - x_2}{1 + x_1x_2}\right)$

(c) $f\left(\frac{x_1 + x_2}{1 - x_1x_2}\right)$ (d) $f\left(\frac{x_1 - x_2}{1 - x_1x_2}\right)$

Solution

(a) $f(x_1) + f(x_2) = \log\left(\frac{1+x_1}{1-x_1}\right) + \log\left(\frac{1+x_2}{1-x_2}\right)$

$= \log\frac{(1+x_1)(1+x_2)}{(1-x_1)(1-x_2)} = \log\frac{1+x_1+x_2+x_1x_2}{1-x_1-x_2+x_1x_2}$

$= \log\left[\frac{\left(1+\frac{x_1+x_2}{1+x_1x_2}\right)}{\left(1-\frac{x_1+x_2}{1+x_1x_2}\right)}\right] = f\left(\frac{x_1+x_2}{1+x_1x_2}\right)$

15. If $f: R \to R, f(x+y) = f(x) + f(y), \forall\, x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is equal to **[AIEEE-2003]**

(a) $\frac{7n(n+1)}{2}$ (b) $\frac{7n}{2}$

(c) $\frac{7(n+1)}{2}$ (d) $7n\,(n+1)$

Solution

(a) $f(2) = f(1+1) = f(1) + f(1) = 7 + 7 = 14$

$f(3) = f(2+1) = f(2) + f(1) = 14 + 7 = 21$

$f(4) = f(3+1) = f(3) + f(1) = 21 + 7 = 28$

$\therefore \sum_{r=1}^{n} f(r) = f(1) + f(2) + f(3) + \ldots\ldots\ldots\ldots + f(n)$

$= 7 + 14 + 21 + \ldots\ldots\ldots + n$ terms

$= \frac{n}{2}[14 + (n-1)7] = \frac{7n(n+1)}{2}$

16. The domain of the function $f(x) = {}^{18-x}C_{2x-5} + {}^{26-3x}P_{4x-13}$ where the symbols have their usual meanings is the set:

(a) $\{4, 5\}$ (b) $\{1, 2\}$

(c) $\{2, 2\}$ (d) $\{3, 3\}$

Solution

(a) For D_f, $18 - x > 0$, $2x - 5 > 0$, $26 - 3x > 0$ and $4x - 13 > 0$

Also $18 - x \geq 2x - 5$, $26 - 3x \geq 4x - 13$ and $x \in N$

$\Rightarrow 18 > x, x > \frac{5}{2}, \frac{26}{3} > x, x > \frac{13}{4}$

$18 + 5 \geq 3x, 39 \geq 7x$

$\frac{23}{3} \geq x, \frac{39}{7} \geq x$

$\Rightarrow x > 5/2$ and $x > 13/4$

Also $x \leq \frac{23}{3}, x \leq \frac{39}{7}, x < \frac{26}{3}, x < 18$

Hence $\max\left\{\frac{5}{2}, \frac{13}{4}\right\} < x \leq \min\left\{\frac{23}{3}, \frac{39}{7}, \frac{26}{3}, 18\right\}$

i.e., $\frac{13}{4} < x \leq \frac{39}{7}$

Hence D_f is $\{4, 5\}$. [$\because x \in N$]

Proved.

17. Let f be a function satisfying $2f(xy) = [f(x)]^y + [f(y)]^x$ and $f(1) = k \neq 1$, then $\sum_{x=1}^{n} f(x)$ is equal to:

(a) $\frac{k(k^n - 1)}{k-1}$ (b) $\frac{k(k^n + 1)}{k-1}$

(c) $\frac{k^n + 1}{k-1}$ (d) $\frac{k(k^n - 1)}{k+1}$

Solution

(a) $2f(xy) = [f(x)]^y + [f(y)]^x$

Put $y = 1$

$2f(x) = [f(x)]^1 + [f(1)]^x$

$\Rightarrow 2f(x) - f(x) = [f(1)]^x$

$\Rightarrow f(x) = k^x \quad (\because f(1) = k)$

Now $\sum_{x=1}^{n} f(x) = \sum_{x=1}^{n} k^x = k^1 + k^2 + \ldots\ldots + k^n$

$= \frac{k(k^n - 1)}{k-1}$

18. If $e^x = y + \sqrt{1+y^2}$, then y is equal to **[UPSEAT-2000]**

(a) $e^x + e^{-x}$ (b) $e^x - e^{-x}$

(c) $\frac{1}{2}(e^x - e^{-x})$ (d) $\frac{1}{2}(e^x + e^{-x})$

Solution

(c) $e^x = y + \sqrt{1+y^2}$

$\Rightarrow x = \log(y + \sqrt{1+y^2}) = \sin h^{-1} y$

$\Rightarrow y = \sin hx = \frac{1}{2}(e^x - e^{-x})$

19. If $f(x)$ is an even function and $g(x)$ is an odd function, and $x^2 f(x) - 2f(1/x) = g(x)$, then $f(5)$ is equal to

(a) 5 (b) 1/75
(c) 0 (d) $g(5)$

Solution

(c) $x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$ .... (1)

replace x by $\frac{1}{x}$

$\Rightarrow \frac{1}{x^2} f\left(\frac{1}{x}\right) - 2f(x) = g\left(\frac{1}{x}\right)$

$\Rightarrow 2f\left(\frac{1}{x}\right) - 4x^2 f(x) = 2x^2 g\left(\frac{1}{x}\right)$ ....(2)

$(1)+(2) \Rightarrow -3x^2 f(x) = g(x) + 2x^2 g\left(\frac{1}{x}\right)$

$\Rightarrow f(x) = -\frac{1}{3x^2}\left[g(x) + 2x^2 g\left(\frac{1}{x}\right)\right]$ .... (3)

$\Rightarrow f(-x) = -\frac{1}{3x^2}\left[g(-x) + 2x^2 g\left(-\frac{1}{x}\right)\right]$

$= \frac{1}{3x^2}\left[g(x) + 2x^2 g\left(\frac{1}{x}\right)\right]$

[$\because g(x)$ is odd] $= -f(x)$ [by (3)]

$\Rightarrow f(x)$ is odd function but $f(x)$ is given as an even function, so $f(x) = 0 \Rightarrow f(5) = 0$.

20. The function f satisfies the functional equation $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$ for all real $x \neq 1$. The value of $f(7)$ is **[kerala (Engg.)-2005]**

(a) 8 (b) 4
(c) –8 (d) 11

Solution

(b) $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$

For $x = 7$, $3f(7) + 2f(11) = 70 + 30 = 100$

For $x = 11$, $3f(11) + 2f(7) = 140$

$\frac{f(7)}{-20} = \frac{f(11)}{-220} = \frac{-1}{9-4} \Rightarrow f(7) = 4.$

21. If $f(x).f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(4) = 65$, then $f(6)$ is **[Orissa-JEE-2007]**

(a) 215 (b) 217
(c) 220 (d) None of these

Solution

(b) If $f(x) . f(1/x) = f(x) + f(1/x)$

Then $f(x) = 1 - x^n$ or $1 + x^n$

If $f(x) = 1 - x^n$ then $f(4) = -4^n + 1 \neq 65$

so $f(x) = 1 + x^n$

$\therefore$ $f(4) = 1 + 4^n = 65$

$\therefore$ $4^n = 4^3$

$n = 3$

$\therefore$ $f(6) = 1 + 6^3 = 217$

22. Let $f : R \to R$ be defined by $f(x) = 2x + |x|$, then $f(2x) + f(-x) - f(x)$ is equal to **[EAMCET-2000]**

(a) $2x$ (b) $2|x|$
(c) $-2x$ (d) $-2|x|$

Solution

(b) $f(2x) = 2|2x| + |2x| = 4x + 2|x|$

$f(-x) = -2x + |-x| = -2x + |x|$

$f(x) = 2x + |x|$

$\Rightarrow f(2x) + f(-x) - f(x) = 4x + 2|x| + |x| - 2x - 2x - |x| = 2|x|$

23. If x, y, z are distinct positive numbers different from 1 such that $(\log_y x . \log_z x - \log_x x) + (\log_x y . \log_z y - \log_y y) + (\log_x z . \log_y z - \log_z z) = 0$, what is the value of xyz **[NDA-2004]**

(a) 2 (b) 1
(c) –1 (d) 0

Solution

(b) $(\log_y x . \log_z x - \log_x x) + (\log_x y . \log_z y - \log_y y) + (\log_x z . \log_y z - \log_z z) = 0$

$\Rightarrow \log x [(\log x)^2 - \log y \log z] + \log y [(\log y)^2 - \log x . \log z] + \log z [(\log z)^2 - \log x \log y] = 0$

$\Rightarrow (\log x)^3 + (\log y)^3 + (\log z)^3 - 3 \log x \log y \log z = 0$

$\therefore$ $\log x + \log y + \log z = 0 = \log 1$

$\therefore$ $xyz = 1$

($\because$ If $a^3 + b^3 + c^3 = 3abc$ then $a + b + c = 0$)

24. Let $f(x) = |x - 1|$, then **[IIT-JEE-1983]**

(a) $f(x^2) = f(x))^2$
(b) $f(x + y) = f(x) + f(y)$
(c) $f(|x|) = |f(x)|$
(d) None of these

Solution

(d) $f(x) = |x-1| = \begin{cases} -x+1, & x<1 \\ x-1, & x\geq 1 \end{cases}$ Consider

$f(x)^2 = (f(x))^2$

If it is true it should be true $\forall\ x$ $\therefore$ Put $x = 2$

LHS $= f(2^2) = |4-1| = 3$ RHS $= (f(2))^2 = 1$

$\therefore$ (a) is not correct Consider $f(x+y) = f(x) + f(y)$

Put $x = 2, y = 5$ we get $f(7) = 6; f(2) + f(5) = 1 + 4 = 5$

$\therefore$ (b) is not correct

Consider $f(|x|) = |f(x)|$ Put $x = -5$ then $f(|-5|) = f(5) = 4$

$|(f-5)| = |-5-1| = 6$

$\therefore$ (c) is not correct.

Hence (d) is the correct alternative.

25. If x satisfies $|x-1| + |x-2| + |x-3| \geq 6$, then

[IIT-JEE-1983]

(a) $0 \leq x \leq 4$
(b) $x \leq -2$ or $x \geq 4$
(c) $x \leq 0$ or $x \geq 4$
(d) None of these

Solution

(c) $|x-1| + |x-2| + |x-3| \geq 6$

Consider $f(x) = |x-1| + |x-2| + |x-3|$

$$= \begin{cases} 6-3x, & x<1 \\ 4-x, & 1\leq x<2 \\ x & 2\leq x<3 \\ 3x-6, & x\geq 3 \end{cases}$$

Graph of $f(x)$ shows $f(x) \geq 6$ for $x \leq 0$ or $x \geq 4$.

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. Let $A = \{1, 2, 3\}, B = \{2, 3, 4\}$, then which of the following is a function from A to B?

(a) $\{(1, 2), (1, 3), (2, 3), (3, 3)\}$
(b) $\{(1, 3), (2, 4)\}$
(c) $\{(1, 3), (2, 2), (3, 3)\}$
(d) $\{(1, 2), (2, 3), (3, 2), (3, 4)\}$

2. If $y = f(x) = \dfrac{ax+b}{cx-a}$, then $f(y)$ ***[AMU-2001]***

(a) $1/f(x)$ (b) $1/f(y)$
(c) $y f(x)$ (d) x

3. If $f(x) = \dfrac{1-x}{1+x}$, then $f(\cos 2\theta) =$

[MPPET-1994, 2001; Pb.CET-02]

(a) $\sin^2\theta$ (b) $\tan^2\theta$
(c) $\tan^3\theta$ (d) None of these

4. If $f(x) = bx^2 + cx + d$, then find value of b and c for which the identity $f(x+1) - f(x) = 8x + 3$ is satisfied

(a) $b = 4, c = -1$ (b) $b = 3, c = 1$
(c) $b = 4, c = 1$ (d) None of these

5. $f(x) = \dfrac{a^x + a^{-x}}{2}$, then $f(x+y)\ f(x-y)$ is equal to

(a) $2f(x) \,.\, f(y)$ (b) $f(x) \,.\, f(y)$
(c) $f(x)/f(y)$ (d) None of these

6. If for non-zero x, $af(x) + bf\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 5$ where $a \neq b$, then if $f(x) = \lambda$ $\left[\left(\dfrac{a}{x} - bx - 5a + 5b\right)\right]$ then $\lambda = ?$

(a) $\dfrac{1}{a^2-b^2}$ (b) $\dfrac{1}{a^2+b^2}$
(c) $a^2 - b^2$ (d) $a^2 + b^2$

7. If $f(x) = \dfrac{x}{x+1}$, then $\dfrac{f(a/b)}{f(b/a)}$ is equal to

(a) a (b) a/b
(c) b/a (d) 1

8. If $f(x)$ is a polynomial stisfying $f(x) \,.\, f(1/x) = f(x) + f(1/x)$ and $f(3) = 28$, then $f(4) =$

(a) 63 (b) 65
(c) 17 (d) None of these

9. Let A be a set containing 10 distinct elements, then the total number of distinct functions from A to A is

(a) 10 !
(b) 10^{10}
(c) 2^{10}
(d) $2^{10} - 1$

10. $y = f(x) = \cos(\log x)$, then $f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right] =$ is equal to

[IIT-83; RPET-95; MPPET-95; Karnataka CET -99; UPSEAT-2001]

(a) 0
(b) $\cos(\log xy)$
(c) $\sin(\log(x+y))$
(d) None of these

11. If $f(x + 1/x) = x^2 + 1/x^2, x \neq 0$, then $f(x)$ is equal to

[Kerala (CEE)-2003; VIT-2004]

(a) $x^2 - 1$
(b) $x^2 + 2$
(c) $x^2 - 2$
(d) $x^2 + 1$

12. If $f(x) = \frac{x(x-1)}{2}$, then $f(x + 2)$ equals

[PET (Raj.)-1986]

(a) $(x+1)f(x)$
(b) $\left(\frac{x+1}{x}\right)f(x)$
(c) $\left(\frac{x+2}{x}\right)f(x+1)$
(d) $\left(\frac{x+2}{2}\right)f(x+1)$

13. If $f(x) = x^2 - x^{-2}$, then $f(1/x)$ equals

[SCRA-99]

(a) $1/f(x)$
(b) $-1/f(x)$
(c) $f(x)$
(d) $-f(x)$

14. If $f(x^2 + 1) = 3x - 1$, then $f(x)$ is equal to

[AITSE-99]

(a) $3\sqrt{x-1} + 1$
(b) $3\sqrt{x-1} - 1$
(c) $\sqrt{x-1} - 3$
(d) None of these

15. If $f(x) = x^3 - \frac{1}{x^3}$, then $f(x) + f\left(\frac{1}{x}\right) =$

(a) 0
(b) -1
(c) x^3
(d) None of these

16. If $f(x) = x^2 - 3x + 4$, then find the values of x satisfying the equation $f(x) = f(2x + 1)$

(a) 5/3
(b) 2/3
(c) 1
(d) None of these

17. If $g(x)$ is a polynomial function satisfying $g(x)g(y) = g(x) + g(y) + g(xy) - 2$ for all $x, y \in R$ and $g(2) = 5$, then find $g(3) =$

(a) 10
(b) 15
(c) 20
(d) 28

18. If $f(x) = \log\left(\frac{1+x}{1-x}\right), -1 < x < 1$, then $f\left(\frac{3x+x^3}{1+3x^2}\right) - f\left(\frac{2x}{1+x^2}\right)$ is

[Kerala PET-2008]

(a) $[f(x)]^3$
(b) $[f(x)]^2$
(c) $-f(x)$
(d) $f(x)$

SOLUTIONS

1. (c) By definition of function

2. (d) $y = f(x) = \frac{ax+b}{cx-a}, f(y) = \frac{ay+b}{cy-a}$

$$\therefore \quad f\left(\frac{ax+b}{cx-a}\right) = \frac{a\left(\frac{ax+b}{cx-a}\right)+b}{c\left(\frac{ax+b}{cx-a}\right)-a}$$

$$= \frac{a(ax+b)+bc(x-a)}{c(ax+b)-a(cx-a)}$$

$$\frac{(a^2+bc)x}{a^2+bc} = x$$

3. (b) $f(x) = \frac{1-x}{1+x}, f(\cos 2\theta)$

$$= \frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta$$

4. (a) $f(x) = bx^2 + cx + d$

$$f(x+1) - f(x) = 8x + 3$$

$$b(x+1)^2 + c(x+1) + d - [bx^2 + cx + d] = 8x + 3$$

$$2bx + b + c = 8x + 3$$

$$\therefore \quad 2b = 8; b + c = 3$$

$$b = 4; c = -1$$

5. (a) $\because\ f(x)=\dfrac{a^x+a^{-x}}{2}$

$f(x+y)+f(x-y)$

$=\dfrac{a^{x+y}+a^{-(x+y)}}{2}+\dfrac{a^{x-y}+a^{-(x-y)}}{2}$

$=a^x\left[\dfrac{a^y+a^{-y}}{2}\right]+a^{-x}\left[\dfrac{a^{-y}+a^y}{2}\right]$

$=2\left(\dfrac{a^x+a^{-x}}{2}\right)\left(\dfrac{a^y+a^{-y}}{2}\right)$

6. (a) $af(x)+bf\left(\dfrac{1}{x}\right)=\dfrac{1}{x}-5$ .... (1)

replace x by $\dfrac{1}{x}$ in the equation

$af\left(\dfrac{1}{x}\right)+bf(x)=x-5$ .... (2)

From (1) and (2) we get

$f(x)=\dfrac{1}{a^2-b^2}\left[\dfrac{a}{x}-bx-5a+5b\right]$

7. (b) $f(x)=\dfrac{x}{x+1},\dfrac{f(a/b)}{f(b/a)}=\dfrac{\dfrac{a/b}{\frac{a}{b}+1}}{\dfrac{b/a}{\frac{b}{a}+1}}=\dfrac{a}{b}$

8. (b) Given that $f(x)f\left(\dfrac{1}{x}\right)=f(x)+f\left(\dfrac{1}{x}\right)$

$\therefore f(x)=x^n+1, f(3)=28$

$3^n+1=28\Rightarrow n=3$

$\therefore\quad f(x)=x^3+1$

$f(4)=4^3+1=65$

9. (b) No. of functions = no. of ways in which each element can be assosciated = 10 ×10 × 10 .... 10 times = 10^{10}

10. (a) $f(x)=\cos(\log x)$

$f(xy)=\cos(\log xy)=\cos(\log x+\log y)$

$f\left(\dfrac{x}{y}\right)=\cos\left(\log\dfrac{x}{y}\right)=\cos(\log x-\log y)$

$\therefore f(x)f(y)-\dfrac{1}{2}\left[f\left(\dfrac{x}{y}\right)+f(xy)\right]$

$=\cos(\log x)\cos(\log y)-\dfrac{1}{2}$

$[\cos(\log x-\log y)+\cos(\log x+\log y)]$

$=\cos(\log x)\cos(\log y)-\dfrac{1}{2}$

$[2\cos(\log x)\cos(\log y)]=0$

11. (c) $f\left(x+\dfrac{1}{x}\right)=x^2+\dfrac{1}{x^2}=\left(x+\dfrac{1}{x}\right)^2-2$

$\therefore f(t)=t^2-2$

or $f(x)=x^2-2$

12. (c) ~~$f(x)=$~~ $\dfrac{x(x-1)}{2}$

$f(x+2)=\dfrac{(x+2)(x+2-1)}{2}$

$=\dfrac{(x+2)(x+1)}{2}$

$=\dfrac{(x+2)}{x}\dfrac{x(x+1)}{2}$

$=\dfrac{x+2}{x}\left[\dfrac{(x+1)(x+1-1)}{2}\right]=\dfrac{x+2}{x}f(x+1)$

13. (d) If $f(x)=x^2-\dfrac{1}{x^2}$

$f\left(\dfrac{1}{x}\right)=\dfrac{1}{x^2}-x^2=-\left(x^2-\dfrac{1}{x^2}\right)$

$=-f(x)$

14. (b) $f(x^2+1)=3x-1$

put $x^2+1=t\Rightarrow x=\sqrt{t-1}$

$f(t)=3\sqrt{t-1}-1$

$\therefore f(x)=3\sqrt{x-1}-1$

15. (a) Here $f(x)=x^3-\dfrac{1}{x^3}$, $f\left(\dfrac{1}{x}\right)=\dfrac{1}{x^3}-x^3$

$\Rightarrow f\left(\dfrac{1}{x}\right)=-\left(x^3-\dfrac{1}{x^3}\right)$

$f\left(\dfrac{1}{x}\right)=-f(x)=\Rightarrow f(x)+f\left(\dfrac{1}{x}\right)=0$

16. (b) $f(x) = x^2 - 3x + 4$ and $f(x) = f(2x + 1)$

$$x^2 - 3x + 4 = (2x + 1)^2 - 3(2x + 1) + 4$$

$$\Rightarrow \quad 3x^2 + x - 2 = 0$$

$$\Rightarrow \quad x = -1, 2/3$$

17. (a) Step 1:

Given $g(2) = 5$ and $g(x)\,g(y) = g(x) + g(y) + g(xy) - 2$ ....(1)

all x and $y \in R$

therefore on putting $x = 2$ and $y = 1$ in (1)

we get $g(2)g(1) = g(2) + g(1) + g(2) - 2$

$$5g(1) = 10 + g(1) - 2$$

$$4g(1) = 8 \Rightarrow g(1) = 2 \qquad \text{........ (2)}$$

Step 2: Now on putting $\frac{1}{x}$ for y in (1)

we get $g(x)g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right)$ using (2)

Step 3: Using formula

If $g(x)g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right)$ then

$$g(x) = x^n + 1 \qquad \text{....... (3)}$$

we get

$$g(2) = 2^n + 1 = 5$$

$$\therefore \quad 2^n = 4 = 2^2 \Rightarrow n = 2$$

Therefore: $g(3) = 3^n + 1$ from (3)

i.e., $g(3) = 3^2 + 1 = 10$

18. (d) $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then

$$f\left(\frac{3x+x^3}{1+3x^2}\right) = \log\left(\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}\right)$$

$$= \log\left(\frac{1+x}{1-x}\right)^3 = 3\log\left(\frac{1+x}{1-x}\right)$$

and $f\left(\frac{2x}{1+x^2}\right) = \log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right)$

$$= \log\left(\frac{1+x}{1-x}\right)^2 = 2\log\left(\frac{1+x}{1-x}\right)$$

Now

$$f\left(\frac{3x+x^3}{1+3x^2}\right) - f\left(\frac{2x}{1+x^2}\right)$$

$$= 3\log\left(\frac{1+x}{1-x}\right) - 2\log\frac{1+x}{1-x}$$

$$= \log\left(\frac{1+x}{1-x}\right) = f(x)$$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE) FOR IMPROVING SPEED WITH ACCURACY

1. If $f(x) = \cos(\log x)$, then $f(x^2)\,f(y^2)$

$-\frac{1}{2}\left[f\left(\frac{x^2}{y^2}\right) + f(x^2y^2)\right]$ has the value

[MNR-1992]

(a) –2 (b) –1

(c) 1/2 (d) None of these

2. If $f(x) = \frac{2^x + 2^{-x}}{2}$, then $f(x + y).\,f(x - y)$ is equal to

(a) $\frac{1}{2}[f(2x) + f(2y)]$

(b) $\frac{1}{4}[f(2x) + f(2y)]$

(c) $\frac{1}{2}[f(2x) - f(2y)]$

(d) $\frac{1}{4}[f(x) - f(2y)]$

3. If $f(x)=\begin{cases}2x-3, & x\ge 2\\ x, & x<2\end{cases}$, then $f(1)$ is equal to

(a) $2f(2)$ (b) $f(2)$
(c) $-f(2)$ (d) $\frac{1}{2}f(2)$

4. If $\log_{10}x = y$, then $\log_{10^3} x^2$ equals

(a) $\frac{1}{3}y$ (b) $\frac{2}{3}y$
(c) $\frac{3}{2}y$ (d) $3y$

5. If $f(x)=\frac{x}{x-1}$, then $\frac{f(a)}{f(a+1)}=$

[MPPET-1996]

(a) $f(-a)$ (b) $f\left(\frac{1}{a}\right)$
(c) $f(a^2)$ (d) $f\left(-\frac{a}{a-1}\right)$

6. If $\phi(x)=a^x$, then $[\phi(P)]^3$ is equal to

[MPPET-1999]

(a) $\phi(3P)$ (b) $3\phi(P)$
(c) $6\phi(P)$ (d) $2\phi(P)$

7. If $f(x)=\frac{1-x}{1+x}$, then $f\left(\frac{1-x}{1+x}\right)$ is equal to

(a) x (b) $\frac{1+x}{1-x}$
(c) $\frac{1-x}{1+x}$ (d) $\frac{1}{x}$

8. If $f(x)=\log\left(\frac{1+x}{1-x}\right)$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to

[MPPET-99; RPET-99; UPSEAT-2003]

(a) $f\left(\frac{1-x}{1+x}\right)$ (b) $f(x^2)$
(c) 1 (d) $2f(x)$

9. If $f(x)=\frac{x}{x-1}=\frac{1}{y}$, then $f(y)=$

[MPPET-1995, 97]

(a) x
(b) $x+1$
(c) $x-1$
(d) $1-x$

10. If $f(x)=\frac{\cos^2 x+\sin^4 x}{\sin^2 x+\cos^4 x}$ for $x\in R$, then $f(2002)$ is equal to **[EAMCET-02]**

(a) 1 (b) 2
(c) 3 (d) 4

11. If a polynomial function satisfies the condition $f(x)\,f(1/x) = f(x) + f(1/x)$. If $f(10)=1001$, then $f(20)=$

(a) 2001 (b) 3001
(c) 8001 (d) 5001

12. If $f(x) = 4x^3 + 3x^2 + 3x + 4$, then $x^3 f(1/x)$ is

[SCRA-1996]

(a) $f(-x)$ (b) $\frac{1}{f(x)}$
(c) $\left(f\left(\frac{1}{x}\right)\right)^2$ (d) $f(x)$

13. Let $f(x)$ be defined for all $x>0$ and be continuous. Let $f(x)$ satisfy $f\left(\frac{x}{y}\right)=f(x)-f(y)$ for all x, y and $f(e)=1$, then

(a) $f(x) = \ln x$
(b) $f(x)$ is bounded
(c) $f\left(\frac{1}{x}\right)\to 0$ as $x\to 0$
(d) $xf(x)\to 1$ as $x\to 0$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 9 minutes.
3. The worksheet consists of 9 questions. The maximum marks are 27.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. Which of the following is function
(a) $\{(2, 1), (2, 2), (2, 3), (2, 4)\}$
(b) $\{(1, 4), (2, 5), (1, 6), (3, 9)\}$
(c) $\{(1, 2), (3, 3), (2, 3), (1, 4)\}$
(d) $\{(1, 2), (2, 2), (3, 2), (4, 2)\}$

2. If $f(x) = \dfrac{4^x}{4^x + 2}$, then $f(x) + f(1 - x)$ equal to

[JEE (Orissa)-2000]

(a) 0 (b) -1
(c) 1 (d) 4

3. If $f(x) = \dfrac{x}{x-1} = \dfrac{1}{y}$, then $f(x) f(y)$ is equal to

[PET (Raj.)-88; MP-95, 97]

(a) $-x$ (b) x
(c) $-y$ (d) y

4. If $2f(x) - 3f(1/x) = x^2$, x is not equal to zero, then $-4f(2)$ is equal to

[IIT-1991]

(a) 25 (b) +7
(c) 5 (d) -7

5. If $f(x) + f(1/x) = f(x) f(1/x)$ and $f(5) = 626$, then $f(2) =$
(a) 17 (b) 65
(c) 576 (d) 257

6. If $y = f(x) = \dfrac{ax+b}{cx-a}$, then x is equal to

[AMU-2001]

(a) $1/f(x)$ (b) $1/f(y)$
(c) $y f(x)$ (d) $f(y)$

7. If $3f(x) + 5f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 3 \forall x\ (\neq 0) \in R$, then $f(x)$ is equal to
(a) $\dfrac{1}{14}\left(\dfrac{3}{x} + 5x - 6\right)$
(b) $\dfrac{1}{14}\left(-\dfrac{3}{x} + 5x - 6\right)$
(c) $\dfrac{1}{14}\left(-\dfrac{3}{x} + 5x + 6\right)$
(d) None of these

8. Let $f(x) = \sin(\log x)$, then the value of $f(xy) + f\left(\dfrac{x}{y}\right) - 2f(x) \cos(\log y)$ is

[Orissa JEE-2004]

(a) -1 (b) 0
(c) 2 (d) 1

9. If $2f(x+1) + f\left(\dfrac{1}{x+1}\right) = 2x$ and $x \neq -1$, then $f(2)$ is equal to

[MPPET-2004]

(a) -1 (b) 2
(c) 5/3 (d) 5/2

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. (c) $f(x)=\dfrac{4^x}{4^x+2},\ f(1-x)=\dfrac{4^{1-x}}{4^{1-x}+2}$

$$f(1-x)=\frac{4/4^x}{\frac{4}{4^x}+2}=\frac{4}{4+2.4^x}=\frac{2}{2+4^x}$$

$\therefore\quad f(x)+f(1-x)=\dfrac{4^x+2}{4^x+2}=1$

4. (b) Given $2f(x)-3f\left(\dfrac{1}{x}\right)=x^2$

replace x by $\dfrac{1}{x}$ in given equation

$$2f\left(\frac{1}{x}\right)-3f(x)=\frac{1}{x^2}$$

Eliminating $f\left(\dfrac{1}{x}\right)$, we get

$$f(x)=\frac{-1}{5}\left(2x^2+\frac{3}{x^2}\right)$$

$\therefore\quad -4f(2)=-4\left[\dfrac{-1}{5}\left(2.2^2+\dfrac{3}{2^2}\right)\right]$

$=7$

9. (c) Put $x+1=t, \therefore x=t-1$

$2f(t)+f\left(\dfrac{1}{t}\right)=2(t-1)$ .... (1)

replace t by $\dfrac{1}{t}$

$2f\left(\dfrac{1}{t}\right)+f(t)=2\left(\dfrac{1}{t}-1\right)$ .... (2)

From (1) and (2)

$$f(t)=\frac{4t}{3}-\frac{2}{3t}-\frac{2}{3}$$

put $t=2$ to get

$$f(2)=\frac{8}{3}-\frac{1}{3}-\frac{2}{3}=\frac{5}{3}$$

LECTURE

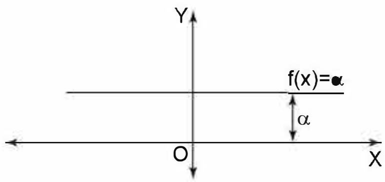
2

Domain and Range of Real Functions

BASIC CONCEPTS

Now we discuss some standard real functions which are frequently used in calculas.

1. **Constant Function** Let α be a fixed real number, then a real function f defined by $f(x) = \alpha$ for all $x \in R$ is called a constant function. The domain of the constant function is R,



$\because$ Constant function is defined for all real numbers. The range of f is set $\{\alpha\}$.

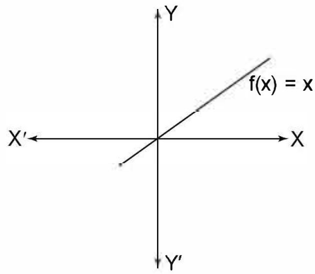
The graph of a constant function is a straight line parallel to x-axis at a distance $|\alpha|$ from it figure.

It is above or below x-axis according as α is positive or negative. If $\alpha = 0$, then its graph is x-axis itself.

2. **Identity Function** The real function f defined by $f(x) = x$ for all $x \in R$ is called identity function.

The domain of function is R and its range is also R.

The graph of identity function is a straight line passing through the origin making an angle of 45° with the x-axis.

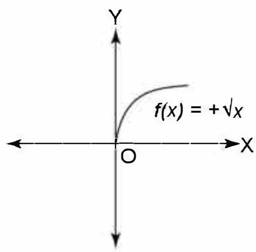


Remark: In general, the identity function is denoted by I, i.e., $I(x) = x$.

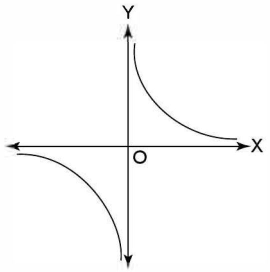
If identity function is defined from a set A to itself, then it is denoted by I_A, i.e., $I_A(x) = x$ for all $x \in A$.

3. **Square Root Function** The real function f defined by $f(x) = +\sqrt{x}$ is called the square root function.

The domain of square root function is $[0, \infty[$ as square root of negative numbers is not defined on real line. The range of square root function is also $[0, \infty[$.



4. **Reciprocal Function** The real function f defined by $f(x) = l/x$ is called the reciprocal function.
The domain of reciprocal function is $R - \{0\}$, as it is not defined at 0. The range of reciprocal function is also $R - \{0\}$



5. **Polynomial Function** A real function f defined by $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$, where $a_0, a_1, a_2, \ldots\ldots a_n$ are real numbers and n is a non-negative integer is called a polynomial function.
The domain and range of polynomial function are both R for example:
   (i) $f(x) = 2x + 3$
   (ii) $f(x) = 3x^2 + 2x + 3$
   (iii) $f(x) = (x + 1)(x + 3)$

6. **Rational Function** A real function f defined by $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function.

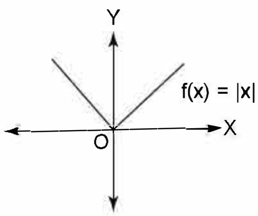
For example

(i) $f(x) = \dfrac{3x^2+1}{x+1}$

(ii) $f(x) = \dfrac{2x+3}{x^2+x+1}$

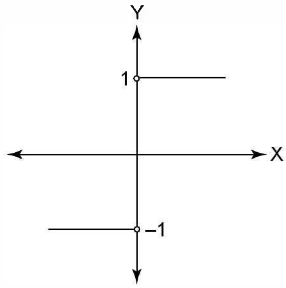
(iii) $f(x) = \dfrac{x-1}{x+2}$

7. **Modulus Function** The real function f defined by $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



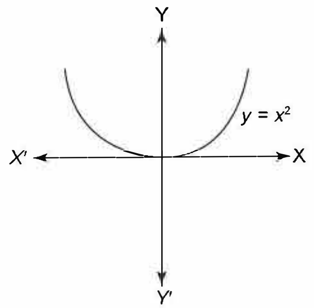
is called modulus function. The domain of modulus function is R. The range of modulus function is $[0, \infty[$.

8. **Signum Function** The real function f defined by $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$



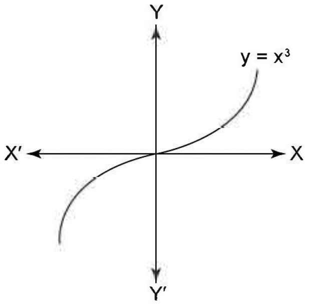
is called the signum function. The domain of signum function is R. The range of signum function is the set $(-1, 0, 1)$.

9. **Square Function** The function that associates a real number x to its square i.e., x^2 is called the square function. Since x^2 is defined for all $x \in R$. So, we define the square function as follows



The function $f : R \to R$ defined by $f(x) = x^2$ is called the square function.
Clearly, domain of the square function is R and its range is the set of all non-negative real numbers i.e., $[0, \infty)$. The graph of $f(x) = x^2$ is parabola as shown in figure.

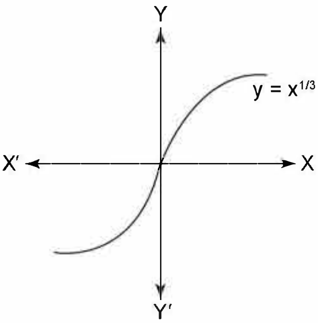
10. **Cube Function** The function that associate a real number x to its cube is called the cube function. We observe that x^3 is the meaningful for all $x \in R$.
So we define the cube function as follows
The function $f : R \to R$ defined by $f(x) = x^3$ is called the cube function.



We observe that the sign of x^3 is same as that of x and the values of x^3 increase with the increase in x. So, the graph of $f(x) = x^3$ is as shown in figure.

Clearly, the graph is symmetrical in opposite quadrants.

11. **Cube Root Function** The function that associates a real number x to its cube root $x^{1/3}$ is called the cube root function.
Clearly $x^{1/3}$ is defined for all $x \in R$.
So, we define the cube root function as follows



The function $f : R \to R$ defined by $f(x) = x^{1/3}$ is called the cube root function.
Clearly, domain and range of the cube root function are both equal to R.
Also, the sign of $x^{1/3}$ is same as that of x and $x^{1/3}$ increase with the increase in x.
So, the graph of $f(x) = x^{1/3}$ is as shown in figure.

12. **Polynomial Function**
$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_n x^n$, $a_n \neq 0$ where a, 's are real numbers and n is a non-negative integer is called a polynomial function of degree n. It is also called a algebraic function. A polynomial function of:
   (i) degree 0 is called a constant function i.e., $f(x) = c$
   (ii) degree 1 is called a linear function i.e., $f(x) = ax + b$.
   (iii) degree 2 is called a quadratic function and cubic function (degree 3) and Bi-quadratic (degree 4) function etc are also defined. i.e., $f(x) = ax^2 + bx + c$; $f(x) = ax^3 + bx^2 + cx + d$; $f(x) = ax^4 + bx^3 + cx^2 + dx + c$.

13. **Rational Function**
$$f(x) = \frac{x^3 - 5x + 3}{x^2 - 1}, \ f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$$
A function defined by the quotient of two polynomial function is called a rational function. Domain of the rational function is usually set of real numbers excluding those x for which denominator is zero.

14. **Irrational Function**
A function involving one or more radicals of polynomial or polynomials is called an irrational function.
Example
$$\sqrt{x^2 + 2x + 3};\ x^7 \frac{(x^3 + 2x^2 + 7)^{2/3}}{\sqrt{1 + x}}$$

NOTE

All polynomial, rational and irrational functions are **algebraic function**.

15. **Real Function $F : X \to Y$** If X and Y are subsets of real numbers then function is called real function $(f : R \to R)$
Examples
   (i) $y = \sqrt{9 - x^2}$ Domain $[-3, 3]$, Range $= [0, 3]$
   (ii) $y = \dfrac{1}{\sqrt{(x + 2)(5 - x)}} \Rightarrow$ Domain $(-2, 5)$

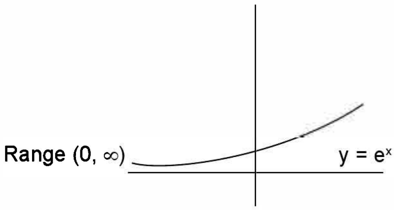
16. **Square Root Function**
$f(x) = \sqrt{x}$. (only + ve square root is associated)
Domain = set of all non negative real numbers $= [0, \infty)$ Range

17. **Transcendental Function** All functions which are not algebraic are called transcendental functions. A transcendental function is not expressed in a finite number of algebraic terms. Also expressed as the incommensurable powers of a variable as for example. x^e, $x^{\sqrt{5}}$ (general exponential): x^x, $x^{\log x}$; $x^{\cos x}$; ($\log_a^{\ x}$: $\log(1 + x)$; e^x; a^x)

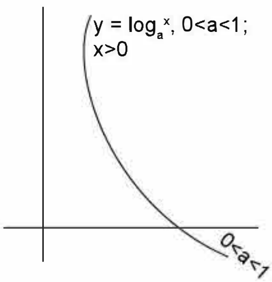
18. Indentity Function The function that associates each real no. x the same number x, is called the identity function.
Domain: set of real number; Range = set of real number.

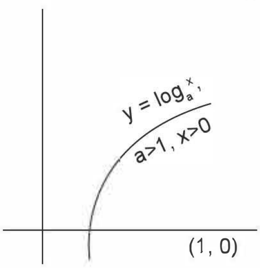
19. Reciprocal Function The function that associates to each non-zero real number to its reciprocal denoted by $1/x$ i.e., $y = 1/x$ is called reciprocal function. Domain of this function is denoted by $R - \{0\}$ Range is also $R - \{0\}$.

20. Exponential Function The function that associates number e^x to each real number x is called exponential function i.e., $y = e^x$. Domain: set of real numbers R; i.e., $(-\infty, +\infty)$



21. Logarithmic Function
The function that associates $\log x$ to x is called the logarithmic function. Domain $(0, \infty)$, Range $(-\infty, \infty)$





22. Modulus Function

$f(x) = |x| = \sqrt{x^2} = \max\{x, -x\}$

$= \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$; Domain R, Range $[0, \infty)$.

(i) $|x| < a \Rightarrow -a < x < a$

(ii) $|x - a| < l \Rightarrow a - l < x < a + l$

(iii) $|x + y| \leq |x| + |y|$

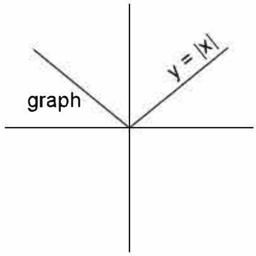
(iv) $|x - y| \geq |x| - |y|$

(v) $|xy| = |x||y|$

(vi) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}; y \neq 0$

(vii) If $|a - b| < l$; $|b - c| < m$, then $|a - c| < l + m$

(viii) $|x| > a \Rightarrow -a > x > a$



Example

(i) $y = |x| + |x - 1|$ for $x \leq 0$; is $y = 1 - 2x$

(ii) $|x + 1| \leq 2 \Rightarrow -3 \leq x \leq 1$

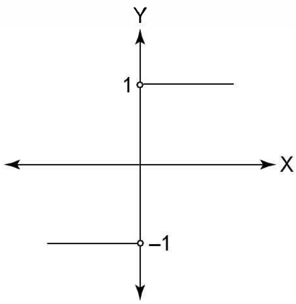
(iii) $0 < |x - 1| < 2 \Rightarrow -1 < x < 3$

(iv) $-3 \leq x \leq 7$ is equivalent to $|x - 2| \leq 5$
$|x - 3| \leq 4$ is equivalent to $-1 \leq x \leq 7$

(v) $|x + 2| < 10 \Rightarrow -12 < x < 8$

(vi) $|x| < 4 => -4 < x < 4$.

23. Signum Function The real function f defined by $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} 1 & \text{if} & x > 0 \\ 0 & \text{if} & x = 0 \\ -1 & \text{if} & x < 0 \end{cases}$



is called the signum function. The domain of signum function is R. The range of signum function is the set $(-1, 0, 1)$.

24. Greatest Integer Function Greatest integer function defined by $y = f(x) = [x]$ is the greatest integer less than or equal to x. In general $[x] = n$ for $n \le x < n + 1$ $n \in I$

A function 'f' defined by $f(x) = [x]$ is called greatest integer or integral part function. OR $[x] = x$ if x is an integer and an integer immediately on the left of x if x is not an integer.

Example

$[5]=5; [-3.9]=-4; [0]=0 \left[\frac{17}{3}\right] = 5; \left[\frac{-7}{2}\right] = -4$

$[x] \le x < [x]+1 [\pi] = 3, [\sqrt{5}] = 2$

$[n+h] = n \quad 0 \le x - [x] < 1 \Rightarrow x - [x]$ is called the fractional part of x and $[x-h] = n-1$ where $0 < h < 1$

Domain: set of real numbers; Range: set of integers.

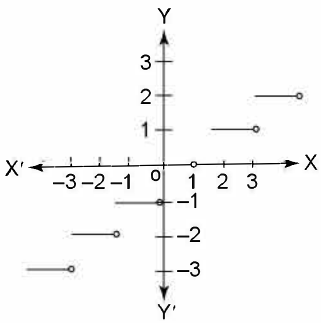
Graph: $0 \le x < 1 \Rightarrow [x] = 0,$

$1 \le x < 2 \Rightarrow [x] = 1,$

$2 \le x < 3 \Rightarrow [x] = 2,$

$-1 \le x < 0 \Rightarrow [x] -1,$

$-2 \le x < -1 \Rightarrow [x] = -2$



1. It consists of many broken pieces.
2. Each piece coincides with the graph of a constant function.
3. The graph lies within the first and third quadrant.
4. It is not continuous.
5. On each interval $[n, n+1]$ the function takes the constant value n.
6. x and $[x]$ have same sign for all x.
7. $f(x) = x - [x]$ is a periodic function with period 1 and is called the fractional part of x.
8. Let f be the greatest integer function and g be the modulus function, then $(gof)(-5/3) - fog\,(-5/3) = 1$

25. Explicit Function $y = f(x)$

26. Implicit Function $f(x, y) = 0$

27. Trigonometric Function (Domain and Range)

(i) Table of Trigonometric Functions

Function	Domain (x)	Range (y)
1. $y = \sin x$	$x \in R$	$y \in [-1, 1]$
2. $y = \cos x$	$x \in R$	$y \in [-1, 1]$
3. $y = \tan x$	$x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	$y \in R$
4. $y = \cot x$	$x \in R - \{n\pi, n \in I\}$	$y \in R$
5. $y = \sec x$	$x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	$y \in R - (-1, 1)$ $(-\infty, -1] \cup [1, \infty)$
6. $y = \text{cosec}\, x$	$x \in R - \{n\pi, n \in I\}$	$y \in R - (-1, 1)$

(ii) Table of Inverse Trigonometric Functions

Function	Domain (x)	Range (y)
1. $y = \sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
2. $y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3. $y = \tan^{-1} x$	R	$(-\pi/2, \pi/2)$
4. $y = \cot^{-1} x$	R	$(0, \pi)$
5. $y = \sec^{-1} x$	$R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$
6. $y = \text{cosec}^{-1} x$	$R - (-1, 1)(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$

(iii) Domain and Range of Some Standard Functions

Function	Domain	Range
Polynomial function	R	R
Identity function x	R	R
Constant function c	R	$\{c\}$
Reciprocal function $1/x$	R_0	R_0
x^2, $\|x\|$	R	$R^+ \cup \{0\}$
x^3, $x\|x\|$	R	R
Signum function	R	$\{-1, 0, 1\}$
$x + \|x\|$	R	$R^+ \cup \{0\}$
$x - \|x\|$	R	$R^- \cup \{0\}$
$[x]$	R	Z
$x - [x]$	R	$[0, 1)$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
a^x	R	R^+
$\log x$	R^+	R

WORKING RULE FOR FINDING DOMAIN AND RANGE OF A REAL FUNCTION

Step 1 : Solve for y in terms of x if not given in this form.

Step 2: Find those values of x for which y is well-defined and real.

Step 3: The set of values of x for which y is well defined and real is domain of f i.e., D_f.

Important: For range we repeat steps (1) and (2) with the roles of x and y interchanged.

i.e., Domain of f i.e., $D_f = \{x \in R : f(x) \in R\}$

Range of f i.e., $R_f = \{f(x) : x \in D_f\}$

TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Name the function $f(x) = \dfrac{x^3 - x^2 + 4x - 7}{3x + 11}$ and find its domain **[CBSE-1981]**

2. Find the domain and range of $f(x) = 1 + x - [x - 2]$.

3. Find the domain and the range of the function $f(x) = \dfrac{1}{5 - 2\cos 3x}$

4. Find the domain of $\dfrac{\sin^{-1} x}{x}$ ***[NCERT Book; HSB-96; J & K-1997]***

5. Find domain and range for the function $f(x) = \dfrac{x^2 + 3x + 10}{x^2 + 6x + 5}$ ***[PSB-1999]***

6. Find domain of $f(x) = \sqrt{\log_{0.5}^{(3x-8)} - \log_{0.5}^{(x^2+4)}}$ ***[AMU-1999]***

7. Find the domain and range of $f(x) = \dfrac{|x - 4|}{x - 4}$

8. Find the range of function $f(x) = 11 - 7 \sin x$ ***[MP-98]***

9. Find the domain and range of the function $f(x) = \dfrac{x}{x^2 + 1}$. Find the numbers in the domain which are associated with the number 3/10 in the range. Is this function one-one.

10. Find the domain and range of $f(x) = \sqrt{(x-1)(3-x)}$ ***[CBSE-78]***

11. Find the domain and range of $\dfrac{x}{|x|}$ ***[PSB-96]***

12. Find domain and range of $f(x) = \sqrt{4 - x^2}$ ***[PSB-97, 99; PET (Pb)-89]***

13. Find domain and range for the function $f(x) = \dfrac{4 - x}{x - 4}$ ***[NCERT Book]***

14. Find the domain of each of the following real valued function

(i) $f(x) = \dfrac{x - 1}{x - 3}$

(ii) $f(x) = \dfrac{2x - 3}{x^2 - 3x + 2}$

(iii) $f(x) = \sqrt{x - 2}$

(iv) $f(x) = \sqrt{4 - x^2}$

(v) $f(x) = \dfrac{1}{\sqrt{1 - x}}$

(vi) $f(x) = \dfrac{1}{\sqrt{x - |x|}}$

(vii) $f(x) = \sqrt{\dfrac{1 - |x|}{2 - |x|}}$

15. Find the range of each of the following functions

(i) $f(x) = \dfrac{1}{\sqrt{x-5}}$

(ii) $f(x) = \sqrt{16 - x^2}$

(iii) $f(x) = \dfrac{3}{2 - x^2}$

16. Find the domain and range of each of the following real valued functions of real variable

(i) $f(x) = \sqrt{9 - x^2}$

(ii) $f(x) = \sqrt{\dfrac{x-2}{3-x}}$

ANSWERS

1. $R - \left\{\dfrac{-11}{3}\right\}$

2. Set of all real numbers; $3 \le f(x) < 4$.

3. $D_f = R, R_f = \left[\dfrac{1}{7}, \dfrac{1}{3}\right]$

4. $D_f = [-1, 1]$ expect zero.

5. $D_f = R - \{-5, -1\}$

$R_f = \left(-\infty, \dfrac{-3}{4} - \sqrt{\dfrac{5}{2}}\right] \cup \left[\dfrac{-3}{4} + \sqrt{\dfrac{5}{2}}, \infty\right)$

6. $D_f = \left(\dfrac{8}{3}, \infty\right)$

7. Set of all real numbers other then 4; $\{-1, 1\}$

8. $R_f = [4, 18]$

9. $D_f = R, R_f = [-1/2, 1/2], f$ is not one-one.

10. $D_f = [1, 3], R_f = [0, 1]$

11. $D_f = R - \{0\}, R_f = \{-1, 1\}$

12. $D_f = [-2, 2], R_f = [0, 2]$

13. $D_f = R - \{4\}, R_f = \{-1\}$

14. (i) $R - \{3\}$

(ii) $R - \{1, 2\}$

(iii) $[2, \infty)$

(iv) $[-2, 2]$

(v) $(-\infty, 1)$

(vi) ϕ

(vii) $(-\infty, -2) \cup (2, \infty) \cup [-1, 1]$

15. (i) $(0, \infty)$

(ii) $[0, 4]$

(iii) $(-\infty, 0) \cup \left(\dfrac{3}{2}, \infty\right)$

16. Domain Range

(i) $[-3, 3]$ $[0, 3]$

(ii) $[2, 3)$ $[0, \infty)$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is ***[AIEEE-2004]***

(a) $\{1, 2, 3\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3, 4\}$ (d) $\{1, 2, 3, 4, 5\}$

Solution

(a) From the above solution, we find that the domain of $f(x)$ is, $\{3, 4, 5\}$. Now, $f(3) = {}^{7-3}P_0 = 1, f(4) = {}^3P_1 = 3$ and $f(5) = {}^2P_2 = 2$ Hence, range of $f = \{1, 2, 3\}$

2. Let $f(x) = (1 + b^2)\,x^2 + 2bx + 1$ and $m\,(b)$ the minimum value of $f(x)$ for a given b. As b varies, the range of $m(b)$ is

(a) $[0, 1]$ (b) $(0, 1/2]$
(c) $[1/2, 1]$ (d) $(0, 1]$

Solution

(d) $1 + b^2 > 0$

$\therefore$ $f(x)$ will have minimum value of

$$= \frac{4(1+b^2) - (2b)^2}{4(1+b^2)} = \frac{1}{1+b^2}$$

($ax^2 + bx + c$ have minimum value $\dfrac{4ac - b^2}{4a}$, when $a > 0$)

$\therefore$ $m(b) = \dfrac{1}{1+b^2}$, Range of $m(b)$ is $(0, 1]$.

3. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval

(a) $(2, \infty)$ (b) $(1, 2)$
(c) $(-2, -1)$ (d) None of these

Solution

(a) Since $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow \log_{0.3}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow \log_{0.3}(x-1) < \frac{1}{2}\log_{0.3}(x-1)$

$\Rightarrow \frac{1}{2}\log_{0.3}(x-1) < 0$

$\Rightarrow x-1 > (0.3)^0 = 1$ [$\because$ base < 1]

$\Rightarrow x > 2 \Rightarrow x \in (2, \infty)$

4. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where $[x]$ = greatest integer $\le x$, then ***[IIT-91]***

(a) $f(\pi/2) = -1$
(b) $f(\pi) = 1$
(c) $f(-\pi) = -1$
(d) $f(\pi/4) = 2$

Solution

(a) $\because [\pi^2] = 9$ and $[-\pi^2] = -10$, so

$f(x) = \cos 9x + \cos(-10)x$
$= \cos 9x + \cos 10x$

$\Rightarrow f(\pi/2) = \cos 9\pi/2 + \cos 5\pi = -1$

$f(\pi) = \cos 9\pi + \cos 10\pi = 0$

$f(-\pi) = \cos(-9\pi) + \cos(-10\pi) = 0$

$f(\pi/4) = \cos 9\pi/4 + \cos 5\pi/2 = 1/\sqrt{2}$

5. If x be real then the range of the function $f(x) = \dfrac{x}{1+x^2}$ is ***[Aligarh-98; NDA-2006]***

(a) $[-1/2, 1/2]$ (b) $(-2, 2)$
(c) $(-1, 1)$ (d) $(-1/2, 1/2)$

Solution

(a) Let $\dfrac{x}{1+x^2} = y \Rightarrow yx^2 - x + y = 0 \because x \in R$,

$\therefore B^2 - 4AC \ge 0$

$\Rightarrow 1 - 4y^2 \ge 0 \Rightarrow (1-2y)(1+2y) \ge 0$

$\Rightarrow (y - 1/2)(y + 1/2) \le 0 \Rightarrow -1/2 \le y \le 1/2$

$\therefore$ range $= [-1/2, 1/2]$

6. Domain of the function $f(x) = \log|\log x|$ is ***[CET (Pb.)-98]***

(a) $(0, \infty)$ (b) $(1, \infty)$
(c) $(0, 1)\cup(1, \infty)$ (d) $(-\infty, 1)$

Solution

(c) $f(x)$ is defined when $|\log x| > 0$

$\Rightarrow x \in (0, \infty), x \ne 1$

$\therefore D_f = (0, 1) \cup (1, \infty)$

7. Match list I with list II and select the correct answer using the codes given below the lists ***[SCRA-99]***

List I (function)	**List II (range)**
A. $y = \sin^{-1} x$	1. $-\pi/2 < y < \pi/2$, $y \ne 0$
B. $y = \sec^{-1} x$	2. $0 \le y \le \pi$
C. $y = \cot^{-1} x$	3. $0 \le y \le \pi, y \ne \pi/2$
D. $y = \mathrm{cosec}^{-1} x$	4. $0 < y \le \pi$
	5. $-\pi/2 \le y \le \pi/2$

Codes:

	A	B	C	D
(a)	2	1	4	5
(b)	1	3	2	5
(c)	5	4	3	1
(d)	5	3	4	1

Solution

(d) Refer to the table given in theory.

8. The range of the function $f(x) = \dfrac{e^x - e^{-|x|}}{e^x + e^{|x|}}$, then f is

(a) $[0, 1]$ (b) $[0, 1)$
(c) $[0, 1/2)$ (d) R

Solution

(c) $\because x \le 0 \Rightarrow f(x) = 0$

$x > 0 \Rightarrow f(x) = \dfrac{e^{2x}-1}{2e^{2x}} = \dfrac{1}{2}(1 - e^{-2x})$

$\Rightarrow f(x) \in (0, 1/2)$

[$\because x > 0 \Rightarrow e^{-2x} \in (0, 1)$] $\therefore R_f = [0, 1/2)$

9. Domain of the function $\sin^{-1}(\log_2 x/2)$ is ***[PET (Raj.)-2002; AIEEE-2002; Kerala (CEE)-2003, 2005]***

(a) $[0, 1]$ (b) $[1/4, 1/2]$
(c) $[1, 4]$ (d) $[-1, 1]$

Solution

(c) Given function is defined when $-1 \le \log_2 (x/2) \le 1$

$\Rightarrow 2^{-1} \le x/2 \le 2^1$ [$\because$ log function is increasing]

$\Rightarrow 1 \le x \le 4$

10. The range of the function $f(x) = \cos x - \sin x$ is ***[MP-95]***

(a) $(-1, 1)$ (b) $[-\sqrt{2}, \sqrt{2}]$
(c) $[-1, 1]$ (d) $(-\sqrt{2}, \sqrt{2})$

Solution

(b) $f(x) = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$ Also

$$-1 \le \cos\left(x + \frac{\pi}{4}\right) \le 1$$

$\therefore \quad -\sqrt{2} \le f(x) \le \sqrt{2}$

11. If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then the range of the function $f(x) = \cos[x]$ is ***[CET (Pb.)-94]***

(a) $\{1, \cos 1, \cos 2\}$ (b) $\{\cos 1, -\cos 1, 1\}$
(c) $\{-1, 0, 1\}$ (d) $\{-1, 1\}$

Solution

(a) $\because\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow [x] = -2, -1, 0, 1$

$\Rightarrow \quad f(x) = \cos(-2), \cos(-1), \cos 0, \cos 1 = \cos 2, \cos 1, 1$

12. The range of the function $f(x) = \sin^2 x + \cos^4 x$ is equal to ***[JEE (Orissa)-2000]***

(a) $[-1, 1]$ (b) $[3/4, 1]$
(c) $[3/4, \infty)$ (d) $(0, 2)$

Solution

(b) $f(x) = 1 - \cos^2 x + \cos^4 x = 1 + (\cos^4 x - \cos^2 x) = \frac{3}{4} + \left(\cos^2 x - \frac{1}{2}\right)^2$

This shows that max. $f(x) = \frac{3}{4} + \frac{1}{4} = 1$

$$f(x) = \frac{3}{4} + 0 = \frac{3}{4}$$

$\therefore \quad$ range = $[3/4, 1]$

13. The domain of the function

$f(x) = \sqrt{\log \frac{1}{|\sin x|}}$ is ***[PET (Raj.)-2001]***

(a) $R - (-\pi, \pi)$ (b) $(-\infty, \infty)$
(c) $R - \{2n\pi, n \in Z\}$ (d) $R - \{n\pi, n \in Z\}$

Solution

(d) $f(x)$ is defined when $\log \frac{1}{|\sin x|} \ge 0$

$\Rightarrow \quad -\log|\sin x| \ge 0$

$\Rightarrow \quad \log|\sin x| \le 0$

$\Rightarrow \quad |\sin x| \ne 0$

$\Rightarrow \quad x \ne n\pi, n \in Z$

$\therefore \quad$ Domain $= R - \{n\pi \mid n \in Z\}$

14. If $f : R \to R$ is defined by $f(x) = [2x] - 2[x]$ for $x \in R$, where $[x]$ is the greatest integer not exceeding x, then the range of f is ***[EAMCET-2006]***

(a) $\{x \in R : 0 \le x \le 1\}$ (b) $\{0, 1)$
(c) $\{x \in R : x > 0\}$ (d) $\{x \in R : x \le 0\}$

Solution

(b) Verification method:

For $\quad x = \frac{1}{2} \Rightarrow f(x) = 1, x = 1 \Rightarrow f(x) = 0$

$x = 10 \Rightarrow f(x) = 0$

$x = 9.9 \Rightarrow f(x) = 19 - 18 = 1$

$x = -1.5 \Rightarrow f(x) = -3 - 2(-2) = 1$

$x = -1 \Rightarrow f(x) = -2 + 2 = 0$ and so on.

15. If $f : R \to R$ is defined by $f(x) = x - [x] - \frac{1}{2}$ for $x \in R$, where $[x]$ is the greatest integer not exceeding x, then $\left\{x \in R : f(x) = \frac{1}{2}\right\}$ is equal to ***[EAMCET-2006]***

(a) Z, the set of all integers
(b) N, the set of all natural numbers
(c) ϕ, the empty set
(d) R

Solution

(c) We know that $[x] \le x < [x] + 1$

$0 \le x - [x] < 1$

$\therefore\ f(x) = x - [x] - \frac{1}{2}$ will never be equal to 1/2

$\therefore \quad$ solution set of x is the empty set.

16. The domain of the function

$$y = \frac{\log_{10}\log_{10}\log_{10}\ldots\ldots\log_{10} x}{n \text{ times}}$$

(a) $(10^n, +\infty)$ (b) $(10^{n-1}, \infty)$
(c) $(10^{n-2}, +\infty)$ (d) None of these

Solution

(d) $\log_{10} = \frac{\log_{10}\ldots\ldots\log_{10} x}{(n-1)\text{times}} > 0$

$\Rightarrow \quad \log_{10} \cdot \frac{\log_{10}\ldots\ldots\log_{10} x}{(n-2)\text{times}} > 10^0 = 1$

$\Rightarrow \quad \log_{10} \frac{\log_{10}\ldots\ldots\log_{10} x}{(n-3)\text{times}} > 10^1$

$\Rightarrow \quad \log_{10} \frac{\log_{10}\ldots\ldots\log_{10} x}{(n-4)\text{times}} > 10^{10}$ etc

$\therefore \quad$ (d) is true.

17. If $f : R \to R$ is defined by $f(x) = \dfrac{1}{2 - \cos 3x}$ for each $x \in R$, then the range of f is **[EAMCET-2007]**

(a) (1/3, 1) (b) [1/3, 1]
(c) (1, 2) (d) [1, 2]

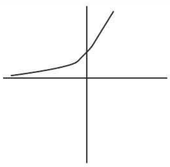
Solution

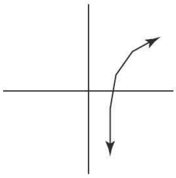
(b) We have $-1 \le -\cos 3x \le 1 \ \forall \ x \in R$

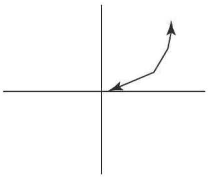
Hence $\dfrac{1}{3} \le f(x) \le 1 \ \forall \ x \in R$

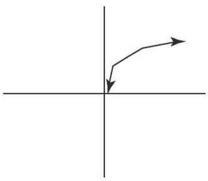
$\therefore$ Range of f = [1/3, 1]

18. Correct graph for $f(x) = e^x$ is **[MPPET-2007]**

(a) 

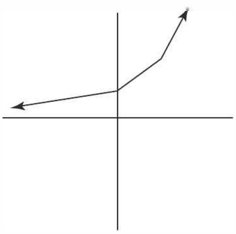
(b) 

(c) 

(d) 

Solution

(a) Value of $f(x) = e^x$, for positive or negative value, x is always positive and at $x = 0$, $y = e^0 = 1$ therefore the correct graph is

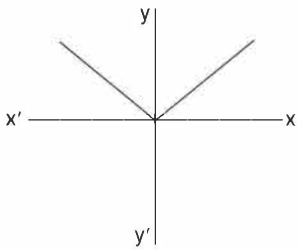


19. Graph of $y = |x|$ is **[MPPET-2007]**

(a) above x-axis
(b) below x-axis
(c) right side of y-axis
(d) left side of y-axis

Solution

(a) $y = |x|$ or $y = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases}$



Clearly this graph is above X-axis.

20. Domain of the function $f(x) = \sin^{-1}(1 + 3x + 2x^2)$ is

(a) $(-\infty, \infty)$
(b) $(-1, 1)$
(c) $\left[-\dfrac{3}{2}, 0\right]$
(d) $\left(-\infty, \dfrac{-1}{2}\right) \cup (2, \infty)$

Solution

(c) $-1 \le 1 + 3x + 2x^2 \le 1$

Case I: $2x^2 + 3x + 1 \ge -1$; $2x^2 + 3x + 2 \ge 0$

$x = \dfrac{-3 \pm \sqrt{9 - 16}}{6} = \dfrac{-3 \pm i\sqrt{7}}{6}$ (imaginary).

Case II: $2x^2 + 3x + 1 \le 1$

$\Rightarrow \ 2x^2 + 3x + \le 0 \quad \Rightarrow \ 2x\left(x + \dfrac{3}{2}\right) \le 0$

$\Rightarrow \ \dfrac{-3}{2} \le x \le 0 \Rightarrow x \in \left[-\dfrac{3}{2}, 0\right]$

In case I, we get imaginary value hence, rejected

$\therefore$ Domain of function = $\left[-\dfrac{3}{2}, 0\right]$,

21. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains

(a) $\left(-\infty, -\frac{3}{2}\right)$ (b) $\left(-\frac{3}{2}, -\frac{1}{4}\right)$

(c) $\left(-\frac{1}{4}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, 3\right)$

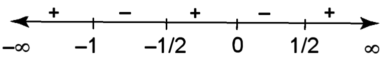
Solution

(a), (d) $\frac{2x-1}{2x^3+3x^2+x} > 0$

(a) option is not match to the question

$\Rightarrow \frac{(2x-1)}{x(2x^2+3x+1)} > 0$

or $\frac{(2x-1)}{x(2x+1)(x+1)} > 0$



Hence, the solution set is,

$x \in (-\infty, -1) \cup (-1/2, 0) \cup (1/2, \infty)$

22. Domain of the function $f(x) = \sqrt{\sin^{-1} 2x + \frac{\pi}{6}}$ is ***[IIT (Screening)-2003]***

(a) [–1/4, 1/2] (b) [–1/2, 1/2]

(c) [–1/2, 1/9] (d) [–1/4, 1/4]

Solution

(a) $f(x)$ is denied when

$0 \le \sin^{-1} 2x + \frac{\pi}{6} \le \frac{\pi}{2} + \frac{\pi}{6}$ $[\because \sin^{-1} 2x \le \frac{\pi}{2}]$

$\Rightarrow \quad -\frac{\pi}{6} \le \sin^{-1} 2x \le \frac{\pi}{2}$

$\Rightarrow \quad \sin\left(-\frac{\pi}{6}\right) \le 2x \le \sin\frac{\pi}{2}$

$\Rightarrow \quad -\frac{1}{4} \le x \le \frac{1}{2}$

23. What is the range of the function $f(x) = \log_2 \{(\sin x - \cos x + 3\sqrt{2})/\sqrt{2}\}$? ***[UPSC-2007]***

(a) [1, 2] (b) [0, 1]

(c) (1, 2) (d) (0, 1)

Solution

(a) The maximum and minimum value of the function sin x – cos x is $\sqrt{1^2+1^2}$ and $-\sqrt{1^2+1^2}$, i.e., $\sqrt{2}$ and $-\sqrt{2}$

$-\sqrt{2} \le \sin x - \cos x \le \sqrt{2}$

$\Rightarrow \quad -\sqrt{2} + 3\sqrt{2} \le \sin x$

$-\cos x + 3\sqrt{2} \le \sqrt{2} + 3\sqrt{2}$

$\Rightarrow \quad 2\sqrt{2} \le \sin x - \cos x + 3\sqrt{2} \le 4\sqrt{2}$

$\Rightarrow \quad 2 \le \frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}} \le 4$

$\Rightarrow \log_2 2 \le \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right) \le \log_2 4$

$\Rightarrow \quad 1 \le \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right) \le 2$ Thus the range is (1,2)

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The domain of definition of the function $y(x)$ given by the equation $2^x + 2^y = 2$

(a) $\infty < x < 1$ (b) $-\infty < x < 1$

(c) $1 < x < \infty$ (d) $1 < \infty < x$

2. Domain and range of $\sqrt{5-4x-x^2}$ respectively

(a) [–5, 1], [0, 3] (b) [5, 1], [3, 0]

(c) [5, –1], [3, 0] (d) None of these

3. Range of the function 1 – | x – 2 | is

(a) [1, ∞) (b) (–∞, 1)

(c) (–∞, 1] (d) (1, ∞)

4. If $f(x) = \cos[\pi^2]\,x + \cos[-\pi^2]x$, then ***[Orissa JEE-2002]***

(a) $f(\pi/4) = 2$ (b) $f(-\pi) = 2$
(c) $f(\pi) = 1$ (d) $f(\pi/2) = -1$

5. Domain of the function $\sqrt{\log_e(x^2 - 6x + 6)}$ is ***[Roorkee-1999; MPPET-2002]***

(a) $(-\infty, 3-\sqrt{3}] \cup [3+\sqrt{3}, \infty)$
(b) $(-\infty, 3-\sqrt{3}) \cup (3+\sqrt{3}, \infty)$
(c) $(-\infty, 1] \cup [5, \infty)$
(d) $(-\infty, 1) \cup (5, \infty)$

6. A real valued function $f(x)$ satisfies the functional equation $f(x - y) = f(x)f(y) - f(a - x)f(a + y)$ where a is a given constant and $f(0) = 1$, $f(2a - x)$ is equal to ***[AIEEE-2005]***

(a) $f(-x)$ (b) $f(a) + f(a - x)$
(c) $f(x)$ (d) $-f(x)$

7. Domain of $f(x) = 5\cot\dfrac{x}{3}$ is

(a) R
(b) $R - n\pi$
(c) $R - \{3n\pi; n \in I\}$
(d) None of these

8. The domain of $y = \dfrac{x}{\sqrt{x^2 - 3x + 2}}$ is

(a) $(\infty, 1)$
(b) $(2, \infty)$
(c) $(\infty, 1) \cup (2, \infty)$
(d) $(-\infty, 1) \cup (2, \infty)$

9. Domain of $\log_{10}[1 - \log_{10}(x^2 - 5x + 16)]$

(a) $(1, 3)$
(b) $(2, 3)$
(c) $(-1, 3)$
(d) None of these

10. Domain of $1 \le |x| \le 5$

(a) $[-5, -1] \cup [1, 5]$
(b) $[+5, 1] \cup [-1, -5]$
(c) $[-5, 1]$
(d) None of these

11. Domain and range of $y = \log(3x^2 - 4x + 5)$

(a) $(-\infty, \infty); \left(\log\dfrac{11}{3}, \infty\right)$
(b) $(3, -5); (5, 3)$
(c) $(3, -5)$
(d) None of these

12. Domain of the function $f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$

(a) $(2, 8)$ (b) $(6, 8)$
(c) $(8, 10)$ (d) None of these

13. Range of the function $f(x) = 3\sin\left[\sqrt{\dfrac{x^2}{16} + x^2}\right]$ is

(a) $[0, 3/\sqrt{2}]$
(b) $[-3/\sqrt{2}, 3/\sqrt{2}]$
(c) $[-3, 3]$
(d) None of these

14. The domain of the function $\sin^{-1}\left\{\log_2\left(\dfrac{x^2}{2}\right)\right\}$ is ***[MPPET-1998]***

(a) $(-2, -1) \cup (1, 2)$
(b) $(-1, -1) \cup (1, 2)$
(c) $(-2, -1) \cup (1, 2)$
(d) None of these

15. If $x^2 + y^2 + z^2 = 1$, then $xy + yz + zx$ lies in

(a) $[-1/2, 1]$
(b) $[1/2, 1]$
(c) $[-1/2, -1]$
(d) $[1/2, -1]$

16. Find domain and range of $\dfrac{x^2 - x + 4}{x^2 + 2x + 4}$?

(a) $[1, 5/4]$ (b) $[2, 5/4]$
(c) $[1, 4/4]$ (d) $[1, 4/5]$

17. The solution set of $\dfrac{x^2 - 3x + 4}{x + 1} > 1, x \in R$ is ***[UP SEE-2007]***

(a) $(3, +\infty)$
(b) $(-1, 1) \cup (3, +\infty)$
(c) $[-1, 1] \cup [3, +\infty]$
(d) None of these

SOLUTIONS

1. (b) Step 1: $2^y = 2 - 2^x > 0$ .... (1)
as $2^y > 0$
From (1) clearly $2^x < 2^1 \Rightarrow x < 1$ .... (2)

Step 2: Therefore from (2) domain of the definition of the function $y(x)$ is given by $(-\infty, 1)$

2. (a) Step 1: Given function is square root function. Therefore $5 - 4x - x^2 \geq 0$

i.e., $x^2 + 4x - 5 \leq 0$

or $(x+5)(x-1) \leq 0$

Step 2: Using the sign of quadratic interval of x
i.e., domain of the function is $[-5, 1]$

Step 3: Maximum value of the quadratic $ax^2 + bx + c$ is $\frac{4ac - b^2}{4a}$ when $a < 0$. Therefore maximum value of $5 - 4x - x^2$ is

$$\frac{4 \times (-1) \times 5 - (-4)^2}{4 \times (-1)} = \frac{-36}{-4} = 9$$

Now range of $\sqrt{5 - 4x - x^2} = [0, 3]$ as square root function can not take negative values.

3. (b) Step 1: $|x - 2|$ is always positive therefore $f(x) = 1 - |x - 2| < 1$
Hence its range is $(-\infty, 1)$

4. (d) Step 1: $[\pi^2] = [(3.14)^2] = [9.86] = 9$

$[-\pi^2] = [-9.86] = -10$

$\because \cos(-x) = \cos x$

$\therefore f(x) = \cos 9x + \cos 10x$

Step 2: $f\left(\frac{\pi}{4}\right) = \cos 9\frac{\pi}{4} + \cos 10\frac{\pi}{4}$

$$= \cos\left(2\pi + \frac{\pi}{4}\right) + \cos\left(2\pi + \frac{\pi}{2}\right)$$

$$= \cos\frac{\pi}{4} + \cos\frac{\pi}{2} = \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

$f(-\pi) = \cos 9\pi + \cos 10\pi$
$= (-1)^9 + (-1)^{10} = 0$

$f(\pi) = \cos 9\pi + \cos 10\pi = 0$

$$f\left(\frac{\pi}{2}\right) = \cos 9\frac{\pi}{2} + \cos\frac{10\pi}{2}$$
$$= 0 + (-1)^5 = -1$$

NOTE

$\cos n\pi = (-1)^n, n \in N$

$\cos(2n \pm 1)\frac{\pi}{2} = 0$

5. (c) Step 1: $\log_e (x^2 - 6x + 6) \geq 0$ as quantity within square root cannot be negative.
i.e., $x^2 - 6x + 6 \geq e^o = 1$

$x^2 - 6x + 5 \geq 0$

$(x-1)(x-5) \geq 0$

i.e., $x \leq 1$ and $x \geq 5$

or $x \in (-\infty, 1) \cup [5, \infty)$

NOTE

If $(x-a)(x-b) > 0$ and $a < b$ then $x \in (-\infty, a)$ (b, ∞)

6. (d) Step 1: can be verified for $f = \cos$ and $a = 90$ as follows
$f(x-y) = f(x)f(y) - f(a-x)f(a+y)$ is true for $f = \cos$ and $a = \frac{\pi}{2}$

e.g., $\cos(x-y) = \cos(x)\cos(y) - \sin x(-\sin y)$

Step 2: options can be verified as follows:
$\cos(2 \times 90° - x) = -\cos x = -f(x)$
(a) $\cos(-x) = \cos x = f(x)$
(b) $\cos 90° + \cos(90 - x)$
(c) $f(x) = \cos(x)$
(d) $-f(x) = -\cos x$

7. (c) Step 1: By definition, domain of $f(x) = \cot x$ is $R - \{n\pi\}$ because $\cot x$ is not defined when x is integer multiple of π i.e., $x \neq n\pi$ where n is any integer.

Step 2: $\frac{x}{3} \neq n\pi \Rightarrow x \neq 3n\pi$

Domain is: $R - \{3n\pi\}, n \in I$

8. (c) Step 1: $f(x)=\dfrac{x}{\sqrt{x^2-3x+2}}$ = A quotient function.

Step 2: Domain of

$\sqrt{x^2-3x+2}=\sqrt{(x-1)(x-2)}$

$\because$ $(x-1)(x-2)>0$ (as denominator cannot be zero and quantity within radical sign cannot be negative).

$\therefore$ Its domain is $(-\infty, 1)$ $(2, \infty)$.

9. (b) Step 1: As negative numbers donot have log therefore,

$\log_{10}(x^2-5x+16)<1$

or $x^2-5x+16<10$

or $x^2-5x+6<0$

or $(x-2)(x-3)<0$

or $2<x<3$ or $x\in(2, 3)$

NOTE

If $(x-a)(x-b)<0$ and $0<a<b$ then $a<x<b$ or $x\in(a, b)$

10. (a) Step 1: Given inequalities are equivalent to following two inequalities:

$|x|\geq 1 \Rightarrow x\leq -1, x\geq 1$ .... (1)

and $|x|\leq 5 \Rightarrow -5\leq x\leq 5$ .... (2)

Step 2: On taking common of both (1) and (2) we get domain as $[-5,-1]\cup[1,5]$

11. (a) Step 1: Range of quadratic

$f(x)=ax^2+bx+c$ is

$\left[\dfrac{4ac-b^2}{4a},\infty\right)$ if $a>0$

and $\left(-\infty,\dfrac{4ac-b^2}{4a}\right]$ if $a<0$

Step 2: Range of $f(x)=\log(3x^2-4x+5)$ is

$\left[\log\dfrac{4\times3\times5-16}{4\times3},\infty\right)$ or $\left[\log\dfrac{11}{3},\infty\right)$

Step 3: Sign of quadratic $f(x)=3x^2-4x+5$ is same as sign of coefficient of x^2

i.e., positive as $B^2-4AC=(-4)^2-4\times3\times5$

$=16-60=-44<0$

Therefore domain is set of real numbers i.e., $(-\infty,\infty)$

12. (c) For domain of

$f(x)=\log_4\left(\log_5\left(\log_3\left(18x-x^2-77\right)\right)\right)$

$\log_5\left(\log_3\left(18x-x^2-77\right)\right)>0$

$\log_3\left(18x-x^2-77\right)>5^0$

$18x-x^2-77>3^1$

$x^2-18x+80<0$

$(x-10)(x-8)<0$

$x\in(8,10)$

13. (a) Here consider $g(x)=\sqrt{\dfrac{\pi^2}{16}-x^2}$

whose range is $\left[0,\dfrac{\pi}{4}\right)$

$\left(\because g(0)=\dfrac{\pi}{4}; g\left(\dfrac{\pi}{4}\right)=0\right)$

$\therefore$ Range $=\left[3\sin 0, 3\sin\dfrac{\pi}{4}\right]=\left[0,\dfrac{3}{\sqrt{2}}\right]$

14. (a) $f(x)=\sin^{-1}\left(\log_2\left(\dfrac{x^2}{2}\right)\right)$ is defined when

$-1\leq\log_2\left(\dfrac{x^2}{2}\right)\leq 1 \Rightarrow 2^{-1}\leq\dfrac{x^2}{2}\leq 2$

$1\leq x^2\leq 4 \Rightarrow x\in(-2,-1)\cup(1,2)$

15. (a) If $x^2+y^2+z^2=1$; then $(x+y+z)^2\geq 0$

$x^2+y^2+z^2+2(xy+yz+zx)\geq 0$

$\Rightarrow$ $xy+yz+zx\geq\dfrac{-1}{2}$

Also $(x-y)^2+(y-z)^2+(z-x)^2\geq 0$

$x^2+y^2+z^2\geq xy+yz+zx \Rightarrow xy+yz+zx\leq 1$

$\therefore$ $xy+yz+zx\in\left[\dfrac{-1}{2},1\right]$

16. (a) Given $y=\dfrac{x^2-x+4}{x^2+2x+4}$;

Here y is defined when $x^2+2x+4\neq 0$ which is true for real x.

$\therefore$ Domain $\in R$

For range $y(x^2+2x+4)=x^2-x+4$

$$x^2(y-1)+x(2y+1)+4(y-1)=0$$

For real x,

$$D\ge 0 \Rightarrow (2y+1)^2-16(y-1)^2\ge 0$$

$$-12y^2+36y-15\ge 0$$

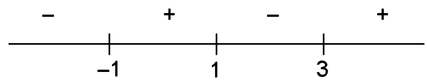
$$\Rightarrow 4y^2-9y+5\le 0$$

$$(4y-5)(y-1)\le 0 \Rightarrow y\in\left[1,\frac{5}{4}\right]$$

17. (b) $\dfrac{x^2-3x+4}{x+1}>1$ or $\dfrac{x^2-3x+4}{x+1}-1>0$

$$\frac{x^2-4x+3}{x+1}>0 \Rightarrow \frac{(x-1)(x-3)}{x+1}>0$$

number line will be



$$x\in(-1,1)\cup(3,\infty)$$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If x is real, then value of the expression $\dfrac{x^2+14x+9}{x^2+2x+3}$ lies between **[UPSEAT-2002]**

(a) 5 and 4
(b) 5 and –4
(c) –5 and 4
(d) None of these

2. The range of the function $f(x)=\dfrac{x+2}{|x+2|}$ is **[RPET-2002]**

(a) $\{0, 1\}$ (b) $\{-1, 1\}$
(c) R (d) $R-\{-2\}$

3. The domain of the function $y=\sqrt{\dfrac{1}{x}-1}$ is

(a) $x\le 1$ (b) $0\le x\le 1$
(c) $0\le x<1$ (d) $0<x\le 1$

4. If the domain of function $f(x)=x^2-6x+7$ is $(-\infty,\infty)$, then the range of function is **[MPPET-1996]**

(a) $(-\infty,\infty)$ (b) $(-2,\infty)$
(c) $(-2, 3)$ (d) $(-\infty,-2)$

5. Range of $f(x)=\dfrac{x^2+34x-71}{x^2+2x-7}$ is **[Roorkee-1983]**

(a) $[5, 9]$ (b) $[-\infty, 5]\cup[9,\infty)$
(c) $(5, 9)$ (d) None of these

6. The domain of the function $f(x)=\sqrt{2-2x-x^2}=\sqrt{3-(x+1)^2}$ **[Roorkee-92]**

(a) $[-1-\sqrt{3}, \sqrt{3}-1]$
(b) $[1-\sqrt{3}, \sqrt{3}-1]$
(c) $[-1+\sqrt{3}, \sqrt{3}+1]$
(d) None of these

7. Domain of the function $f(x)=\sqrt{4-x^2}$ is

(a) $(-\infty,-2)\cup(2,\infty)$
(b) $(-2, 2)$
(c) $[-2, 2]$
(d) None of these

8. The range of the function $f(x)=x^2$ is

(a) R (b) $(0,\infty)$
(c) $[0,\infty)$ (d) $[1,\infty)$

9. Range of the function $f(x)=\dfrac{x^2+x+2}{x^2+x+1}; x\in R$ is **[IIT (Screening)-2003]**

(a) $(1,\infty)$ (b) $(1, 11/7]$
(c) $(1, 7/3)$ (d) $(1, 7/5]$

10. Domain and range of $y=1+x-[x-3]$

(a) $(-\infty,\infty)$, $[4, 5)$
(b) (∞,∞), $[4, 5]$
(c) $(-\infty,\infty)$, $(4, 5)$
(d) (∞,∞), $[4, 5)$

11. The range of $f(x) = \sec\left(\frac{\pi}{4}\cos^2 x\right)$, x

$-\infty < x < \infty$ is

[Orissa JEE-2002]

(a) $[1, \sqrt{2}]$

(b) $[1, \infty]$

(c) $[-\sqrt{2}, -1] \cup [1, \sqrt{2}]$

(d) $(-\infty, -1] \cup [1, \infty)$

12. If $f(x) = \frac{1}{\sqrt{x + 2\sqrt{2x-4}}} + \frac{1}{\sqrt{x - 2\sqrt{2x-4}}}$ for $x > 2$, then $f(11)$ is equal to

(a) 7/6 (b) 5/6

(c) 6/7 (d) 5/7

13. $\log_{0.2}(x-2) < \log_{0.04}(x-2)$, then x lies in the integral

(a) $(3, \infty)$ (b) $(2, 3)$

(c) $(1, 2)$ (d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 11 minutes.
3. The worksheet consists of 11 questions. The maximum marks are 33.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If the domain of function $f(x) = x^2 - 8x + 11$ is $(-\infty, +\infty)$, then the range of function is

[KUKCEET-1994; MPCET-1996]

(a) $[-5, 5]$ (b) $[-5, \infty)$
(c) $(-\infty, -5]$ (d) $(-\infty, +\infty)$

2. The domain of the function

$f(x) = \dfrac{1}{\sqrt{x^2 - 3x + 2}}$ is

(a) $(-\infty, 1)$ (b) $(-\infty, 1) \cap (2, \infty)$
(c) $(-\infty, 1] \cup [2, \infty)$ (d) $(2, \infty)$

3. The range of the function is $y = \dfrac{x}{1 + x^2}$ is

(a) $\left[0, \dfrac{1}{2}\right]$ (b) $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
(c) $\left[-\dfrac{1}{2}, 0\right]$ (d) None of these

4. The range of the function $y = \dfrac{x^2}{1 + x^2}$ is

(a) $[0, 1[$ (b) $[0, 1]$
(c) $]0, 1[$ (d) None of these

5. The range of the function $f(x) = \sqrt{3x^2 - 4x + 5}$ is

(a) $\left[-\infty, \sqrt{\dfrac{11}{3}}\right]$ (b) $\left(-\infty, \sqrt{\dfrac{11}{3}}\right)$
(c) $\left[\sqrt{\dfrac{11}{3}}, \infty\right)$ (d) $\left(\sqrt{\dfrac{11}{3}}, \infty\right)$

6. The function $f(x) = \sqrt{\log_{10}\left(\dfrac{5x - x^2}{4}\right)}$ exists for

(a) $[1, 4]$ (b) $[1, 0]$
(c) $[0, 5]$ (d) $[5, 0]$

7. The range of the function $f(x) = \dfrac{5}{3 - x^2}$ is

(a) $(-\infty, 0) \cup \left[\dfrac{5}{3}, \infty\right)$
(b) $(-\infty, 0) \cup \left(\dfrac{5}{3}, \infty\right)$
(c) $(-\infty, 0] \cup \left[\dfrac{5}{3}, \infty\right)$
(d) None of these

8. The range of the function $\sin(\sin^{-1} x + \cos^{-1} x)$, $|x| \le 1$ is ***[Kerala CEE-2004]***

(a) $[-1, 1]$ (b) $[1, -1]$
(c) $\{-1\}$ (d) $\{1\}$

9. The domain of the function $f(x) = \exp\left(\sqrt{5x - 3 - 2x^2}\right)$ is ***[MP PET-2004]***

(a) $[3/2, \infty)$ (b) $[1, 3/2]$
(c) $(-\infty, 1]$ (d) $(1, 3/2)$

10. Domain and range of $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$ is

(a) $(-\infty, \infty)$; $[1, 3]$ (b) $[-\infty, \infty]$; $[1/3, 3]$
(c) $(-\infty, \infty]$ (d) None of these

11. Let n be a natural number. Then the range of the function $f(n) = {}^{8-n}P_{n-4}$, $4 \le n \le 6$, is ***[Kerala PET-2008]***

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3\}$ (d) $\{1, 2, 3, 4, 5\}$

ANSWER SHEET

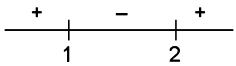
1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. (c) For domain of

$f(x) = \dfrac{1}{\sqrt{x^2 - 3x + 2}}, x^2 - 3x + 2 > 0$

$(x-1)(x-2) > 0$

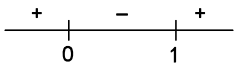


$\therefore x \in (-\infty, 1) \cup (2, \infty)$

4. (a) $y = \dfrac{x^2}{1+x^2}$, for range $y + yx^2 = x^2$

$x^2 = \dfrac{y}{y-1} \geq 0$

$\therefore$ Number line for $\dfrac{y}{y-1}$ is



$\therefore y \in (-\infty, 0] \cup (1, \infty)$

5. (c) Range of $f(x) = \sqrt{3x^2 - 4x + 5}$

We know that $3x^2 - 4x + 5 \geq \dfrac{-D}{4a}$

or $3x^2 - 4x + 5 \geq \dfrac{-(16 - 4.5.3)}{4.3}$

$3x^2 - 4x + 5 \geq \dfrac{+11}{3}$

$\therefore$ Range $\in \left[\sqrt{\dfrac{11}{3}}, \infty\right)$

6. (a) $f(x) = \sqrt{\log_{10}\left(\dfrac{5x - x^2}{4}\right)}$ will exist,

if $\log_{10}\left(\dfrac{5x - x^2}{4}\right) \geq 0$

$\dfrac{5x - x^2}{4} \geq 10^\circ \Rightarrow 5x - x^2 \geq 4$

$x^2 - 5x + 4 \leq 0$

$x \in [1, 4]$

8. (d) Range is $\left\{\sin\left(\dfrac{\pi}{2}\right)\right\}$

$y \in \{1\}$ $\left(\because \sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}\right)$

11. (c) For range n = 4, 5, 6

$\therefore$ Range $\in \{f(4), f(5), f(6)\}$

$\in \{{}^4P_0, {}^3P_1, {}^2P_2\}$

$\in \{1, 3, 2\}$

LECTURE

3

Algebra of Functions

BASIC CONCEPTS

(Sum, Difference, Product and Quotient of function) Let $f: A \to R$ and $g: B \to R$ be two real function with domains A and B respectively then

1. **Sum Function** $(f+g): A \cap B \to R$ is defined as $(f+g)(x) = f(x) + g(x)$: and domain of $f+g$ i.e., $D(f+g) = A \cap B$.
2. **Difference Function** $(f-g)$: $A \cap B \to R$ is defined as $(f-g)(x) = f(x) - g(x)$: and domain of $f-g$
   i.e., $D(f-g) = A \cap B$.
3. **Product Function** $fg: A \cap B \to R$ is defined as $fg(x) = f(x)\, g(x)$ and domain of fg is $A \cap B$.
4. **Quotient Function**
   $\frac{f}{g}: \{A \cap B - \{x \mid g(x) = 0\} \to$ R is defined as
   $$\frac{f}{g}(x) = \frac{f(x)}{g(x)}, g(x) \neq 0,$$
   $$D(f/g) = A \cap B - \{x \mid x \in A \cap B \text{ and } g(x) = 0\}$$

Example

f and g be two functions defined by
$f(x) = \sqrt{x-1}$; $g(x) = \sqrt{4-x^2}$ then

$$D(f) = [1, \infty)\ D(g) = [-2, 2]$$
$$D = D_f \cap D_g = [1, \infty) \cap [-2, 2] = [1, 2] \neq \phi$$

Then the following function with domain D are defined by $(f+g)(x) = f(x) + g(x) = \sqrt{x-1} + \sqrt{4-x^2}$

$$(f-g)(x) = f(x) - g(x) = \sqrt{x-1} - \sqrt{4-x^2}$$
$$(fg)(x) = f(x)\, g(x) = \sqrt{x-1}\sqrt{4-x^2}$$

NOTE

A great care is taken in finding domain of quotient function

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x-1}}{\sqrt{4-x^2}} \quad \text{Domain}(f/g) = [1, 2)$$
$$\frac{g}{f}(x) = \frac{\sqrt{4-x^2}}{\sqrt{x-1}}; \text{ i.e., } D\left(\frac{g}{f}\right) = (1, 2]$$

Example

Let f and g be two real functions defined by $f(x) = (x)$, $\forall\, x \in R$; $g(x) = |x|\ \forall\, x \in R$; find

1. $f + g: R \to R = f(x) + g(x)$
   $$= \begin{cases} 2x; & x \geq 0 \\ 0; & x < 0 \end{cases} \forall\, x \in R$$
2. $f - g: (x) = f(x) - g(x) = \begin{cases} 0 & \text{where } x \geq 0 \\ -2x & \text{where } x \leq 0 \end{cases}$
3. $fg: (x) = f(x)\, g(x) = \begin{cases} x^2 & \text{when } x \geq 0 \\ -x^2 & \text{when } x < 0 \end{cases}$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \begin{cases} \frac{x}{x} = 1, & \text{when } x \geq 0 \\ \frac{x}{(-x)} = -1, & \text{when } x < 0 \end{cases}$

EQUALITY OF TWO FUNCTIONS

Two functions $f(x)$ and $g(x)$ are said to be equal iff

(i) $D(f) = D(g)$ i.e., domain of f = domain of g.

(ii) $f(x) = g(x)$; $\forall x \in D$ domain f

Example

$f(x) = x\,|\,x\,|$ and $g(x) = x^2$ are equal if $D(f) = D(g) = [0, \infty)$

NOTE

If f and g are defined on different domains then they are not equal.

EVEN AND ODD FUNCTION

Even Function A function $y = f(x)$ is said to be an even function if $f(-x) = f(x)$, $\forall\ x \in D_f$.

Odd Function A function $y = f(x)$ is said to be an odd function if $f(-x) = -f(x)$, $\forall\ x \in D_f$.

NOTES

1. Inverse of an even function is not defined.
2. Every function can be expressed as the sum of an even and an odd function.
   i.e.,
   $$f(x) = \frac{1}{2}\{f(x) + f(-x)\} + \frac{1}{2}\{f(x) - f(-x)\}$$
   = {Even Function} + {odd function}
3. If $f(x) - f(-x) = 0$ then $f(x)$ is an even function and if $f(x) + f(-x) = 0$ then $f(x)$ is an odd function.
4. A function may neither be even nor odd.
5. $f(x) = 0$ is the only function which is defined on the entire number line is even and odd at the same time.
6. Every odd continuous function passes through origin.
7. Every even function $y = f(x)$ is not one-one $\forall\ x \in D_f$
8. The derivative of an odd function is an even function and derivative of an even function is an odd function.
9. If f and g both are even or any one of them is odd then fog will be even. If f and g both are odd then gof is odd.
10. The square of an even or an odd function is always an even function.
11. The graph of an even function is symmetrical about the Y-axis.
12. The graph of an odd function is symmetrical about the origin.
13. Table of two functions which are attached:

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$	$f(x)\ g(x)$	$\frac{f(x)}{g(x)}$	(fog) x
Even	Even	Even	Even	Even	Even	Even
Even	Odd	Neither even nor odd	Neither even nor odd	Odd	Odd	Even
Odd	Even	Neither even nor odd	Neither even nor odd	Odd	Odd	Even
Odd	Odd	Odd	Odd	Even	Even	Odd

Example

$$\frac{f(x) + f(-x)}{2}[g(x) - g(-x)] \text{ is odd function}$$

1. $x.\dfrac{e^x + 1}{e^x - 1}$ even.
2. $\log\left[x + \sqrt{1 + x^2}\right]$ odd
3. $\log\dfrac{(1 - x)}{1 + x}$ odd function
4. $\sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}$ odd function.
5. $x\dfrac{a^x + 1}{a^x - 1}$ even function
6. $\dfrac{e^x + 1}{e^x - 1}$ odd function
7. $-x\,|\,x\,|$ odd function.
8. $\dfrac{a^x - 1}{a^x + 1}$ is odd function
9. $|\,x\,|$ even
   Following are examples of neither even nor odd function: ($x^2 + x$: $\sin x + \cos x$: e^x . $[x]$, $|\,x - 2|$ etc.

PERIODIC FUNCTION

A function $f(x)$ is called a periodic function if there exists a positive number T such that $f(x + T) = f(x)$.

T is called the period of the function $f(x)$. If T is least then it is called fundamental period of the function.

Example: does there exist a function which is periodic but has no fundamental period:

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$$

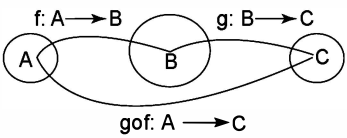
$$f(x+m) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$$

NOTES

1. The period of $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ is 2π.
2. The period of $\tan x$ and $\cot x$ is π.
3. If T is the fundamental period of the function $f(x)$ then the function $f(ax+b)$ where $a\ (>0)$ and b are any numbers is also a periodic function with fundamental period equal to $\frac{T}{a}$.
4. When a function consists of several terms, each of which is periodic, any common multiple of the periods of the terms is a period of the function. The least common multiple is the fundamental period of the function.
5. Period of $|\sin(ax+b)|$, $|\cos(ax+b)|$, $|\sec(ax+b)|$ or $|\operatorname{cosec}(ax+b)|$ is $\frac{\pi}{a}$
6. Period of $|\tan(ax+b)|$, $|\cot(ax+b)|$ is $\frac{\pi}{a}$
7. $\cos\sqrt{x}$ is not a periodic function.
8. $f(x) = x - [x]$ is a periodic function with period 1.

COMPOSITE FUNCTION

Let $f: A \to B$ and $g: B \to C$ be two functions then the composite function gof : $A \to C$ defined by:
gof$(x) = g[f(x)]$ is called the composite of two functions f and g.



$D(\text{gof}) \subseteq D(f)$; $R(\text{gof}) \subseteq R(g)$

NOTES

1. f and g are injective, gof is injective.
2. f and g are surjective => gof is surjective
3. f and g are bijective => gof is bijective
4. Composite of two functions is even if at least one of them is even function (composite of two odd function is odd)

KEY SKILLS

(i) The composite fof is defined when the domain of f is equal to the co-domain of f; f and g do not commute under the operation of composite function.

(ii) The product function fg and the composite function fog are different.

(iii) f and g commute under the product of two functions.

(vi) (fog) $oh = fo$ (goh); $(cf)\, og = fo\,(cg) = c$(fog)

(v) If $f: A \to B$; and $g: B \to C$ be one-one onto maps then (gof) is invertible and $(\text{gof})^{-1} = f^{-1} o g^{-1}$

Examples

1. If $f(x) = x^2 + 1$, $g(x) = 3x - 2$, the (gof) $(x) = g(f(x)) = g(x^2+1) = 3(x^2+1) - 2 = 3x^2 + 1$

2. If $A = \{2, 4, 6\}$ and function is defined $f: A \to A$, then $f = \{(2, 4), (4, 6); (6, 2)\}$, then f of is
(a) $\{(2, 6); (4, 2); (6, 4)\}$

3. If $f(x) = 2x$; $g(x) = 1 - x$; $h(x) = x + 1$; then [hogof] $(x) =$
(a) $1 - 2x$
(b) $1 - x$
(c) $2x + 1$
(d) $2(1 - x)$
Ans. (d)

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. $f : R \to R$ is an identity function. What is the fof and ff.

2. If $f(x) = \dfrac{3x}{x-1}$ and $\phi(x) = \dfrac{x}{x-3}$, prove that $f\{\phi(x)\} = \phi\{f(x)\}$. **[MP-2000]**

3. If $f(x) = x^3 + 3x - 4\sin^3 x$, then prove that $f(x)$ is an odd function of $f(x)$.

4. If $f(x) = e^{2x}$, $g(x) = \log\sqrt{x}$ where $x > 0$, then find the value of
   (i) $(f.g)\,x$ (ii) $(fog)\,x$ (iii) $(gof)\,x$
   Indicating the domains of the respective functions.

5. If $f(x) = (a - x^n)^{1/n}$. Prove that $f(f(x)) = x$. **[CET-1997]**

6. If $f(x) = e^x$ and $g(x) = \log_e x$ $(x > 0)$, find fog and gof. Is fog = gof. **[CBSE-2002]**

7. Prove that the fundamental period of $f(x) = 2\cos\frac{1}{3}(x - \pi)$ is 6π. **[DCEE-98]**

8. Find the value of $\log_2 \log_2(\sqrt{\sqrt{\ldots\sqrt{2}}})$. Square root being taken n times.

9. If the domains of f and g are R, then find the value of $f.g$, when $f(x) = x^5$ and $g(x) = x^2 + 1$.

10. If $f(x) = x^2 + 2x\sin x + 3$, then prove that $f(x)$ is an even function of x.

11. Find domain of $f(x) = \dfrac{\cos^{-1} x}{[x]}$.

12. If $f(x) = \sqrt{x}$ $(x \geq 0)$ and $g(x) = x^2 - 1$ are two real functions, find fog and gof. Is fog = gof. **[CBSE-2002]**

13. If $f(x) = \dfrac{2x-3}{x+1}$ and $g(x) = \dfrac{x+1}{x-3}$, then find the values of $f\{g(x)\}$ and $g\{f(x)\}$.

14. State, giving justification for your answer, which of the following pairs of functions are equal. **[PSB-1998]**
    (i) $f(x) = x$, $g(x) = \dfrac{x^2}{x}$
    (ii) $f(x) = \dfrac{(x^4 + x^2)(x-1)}{x^2(x^2+1)}$, $g(x) = \dfrac{x(x-1)}{x}$

15. If $f(x) = x^3 - 3x\cos x + 5x$, then prove that $f(x)$ is an odd function of x.

ANSWERS

1. x, x^2

4. (i) $e^{2x}.\log\sqrt{x}$
   (ii) x
   (iii) x, domain = $(-\infty, \infty)$

6. fog = x, gof = x, gof $\neq$ fog.

8. $-n$

9. $x^5(x^2 + 1)\ \forall\ x \in R$.

11. $D_f = [-1, 0) \cup \{1\}$.

12. $x - 1, \sqrt{x^2 - 1}$, fog $\neq$ gof

13. $\dfrac{11-x}{2x-2}; \dfrac{3x-2}{-x-6}$

14. (i) Not equal
    (ii) Equal

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0, \\ 1, & x > 0 \end{cases}$ then for all x, $f(g(x))$ is equal to (Here [.] represents greatest integer function)

[IIT (S)-2000, DEC-2005]

(a) 0 (b) 1
(c) –1 (d) 2

Solution

(b) $g(x) = 1 + \{x\}, g(x) \geq 1$

$\because$ $0 \leq x - [x] < 1$ Hence : $f(g(x)) = f(1) = 1$

2. If $f(x) = \dfrac{x-3}{x+1}$, then $f[f\{f(x)\}]$ is equal to

(a) x (b) $-x$
(c) $x/2$ (d) $-1/x$

Solution

(a) $f\{f(x)\} = \dfrac{f(x)-3}{f(x)+1} = \dfrac{\dfrac{x-3}{x+1}-3}{\dfrac{x-3}{x+1}+1}$

$= \dfrac{x-3-3x-3}{x-3+x+1} = \dfrac{-2x-6}{2x-2} = \dfrac{3+x}{1-x}$

Now $f[f\{f(x)\}] = f\left(\dfrac{3+x}{1-x}\right) = \dfrac{\left(\dfrac{3+x}{1-x}\right)-3}{\dfrac{3+x}{1-x}+1}$

$= \dfrac{3+x-3+3x}{3+x+1-x} = \dfrac{4x}{4} = x$

3. If $f(x) = -1 + |x - 1|, -1 \leq x \leq 3$; $g(x) = 2 - |x + 1|, -2 \leq x \leq 2$ then for $x \in (0, 1)$, (fog) (x) is equal to ***[Roorkee-90]***

(a) $x - 1$ (b) $1 - x$
(c) $x + 1$ (d) $-(x + 1)$

Solution

(a) $0 < x < 1 \Rightarrow f(x) = -1 - (x - 1) = -x$

$g(x) = 2 - (x + 1) = 1 - x$

$\therefore$ $x \in (0, 1) \Rightarrow (\text{fog})(x) = f[g(x)] = f(1 - x)$

$= -(1 - x) = x - 1.$

4. If $f(x) = \dfrac{\alpha x}{x+1}, x \neq -1$, then for what value of α, $f[f(x)] = x$

[IIT (Screening)-2001; Kerala (CEE)-2005]

(a) $\sqrt{2}$ (b) $-\sqrt{2}$
(c) 1 (d) –1

Solution

(d) $f[f(x)] = f\left(\dfrac{\alpha x}{x+1}\right) = \dfrac{\alpha\left(\dfrac{\alpha x}{x+1}\right)}{\dfrac{\alpha x}{x+1}+1}$

$= x$ (given)

$\Rightarrow$ $\dfrac{\alpha^2 x}{(\alpha+1)x+1} = x \Rightarrow \alpha^2 x = (\alpha + 1)x^2 + x$

$\Rightarrow$ $\alpha^2 = 1$ and $\alpha + 1 = 0 \Rightarrow \alpha = -1$

5. Function $f(x) = \log(x^3 + \sqrt{1 + x^6})$ is

[Roorkee (Screening)-2000]

(a) even function
(b) odd function
(c) algebraic function
(d) discontinuous function

Solution

(b) $\because$ $f(-x) = \log(-x^3 + \sqrt{1 + x^6})$

$\therefore$ $f(x) + f(-x) = \log(1 + x^6 - x^6) = 0$

$\Rightarrow$ $f(-x) = -f(x) \Rightarrow f$ is odd

6. For the graph of the function $y = \sin^{-1} x$, $-1 \leq x \leq 1$, correct statement is

[Kerala (CEE)-2003]

(a) graph is symmetrical about x-axis
(b) graph is symmetrical about y-axis
(c) line $y = 1$ is a tangent
(d) line $x = 1$ is a tangent

Solution

(d) By the graph of $\sin^{-1} x$, only (d) is correct.

7. If $f(x) = \dfrac{1}{1-x}$, $g(x) = f(f(x))$ and $h(x) = f(f(f(x)))$, then $f(x)\, g(x)\, h(x)$ is equal to

[VIT-2005]

(a) $\dfrac{1}{(1-x)^3}$ (b) $\dfrac{1}{1-x}$

(c) 1 (d) −1

Solution

(d) $g(x) = f(f(x)) = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{1-\dfrac{1}{1-x}}$

$= -\left(\dfrac{1-x}{x}\right)$; $h(x) = f(f(f(x))) = f\left(-\dfrac{1-x}{x}\right)$

$= \dfrac{1}{1+\dfrac{1-x}{x}} = x$

$\therefore \quad f(x)\,g(x)\,h(x) = \left(\dfrac{1}{1-x}\right)\left(-\dfrac{1-x}{x}\right)(x)$

$= -1$

8. Let $f(x) = ax + b$ and $g(x) = cx + d$. If (fog) (x) = (gof) (x), then

[Kerala PET-2008]
[NDA-2005]

(a) $f(a) = g(c)$ (b) $f(b) = g(a)$

(c) $f(c) = g(d)$ (d) $f(d) = g(b)$

Solution

(d) (fog) (x) = (gof) (x)

$\Rightarrow \quad a(cx + d) + b = c(ax + b) + d$

$\Rightarrow \quad ad + b = bc + d$

$\Rightarrow \quad f(d) = g(b)$

9. Given $f(x) = \log\dfrac{1+x}{1-x}$ and $g(x) = \dfrac{3x+x^3}{1+3x^2}$, then $f[g(x)]$ is equal to

[NDA-2006]

(a) $-f(x)$ (b) $3f(x)$

(c) $[f(x)]^3$ (d) $-3f(x)$

Solution

(a) $f[g(x)] = \log\left[\dfrac{1+g(x)}{1-g(x)}\right]$

$= \log\left(\dfrac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3}\right)$

$= \log\left(\dfrac{1+x}{1-x}\right)^3 = 3f(x)$

10. If $f: R \to R$ is defined by

$$f(x) = \begin{cases} x+4 & \text{for} \quad x < -4 \\ 3x+2 & \text{for} \quad -4 \le x < 4 \\ x-4 & \text{for} \quad x \ge 4 \end{cases}$$

then the correct matching of List I from List II is

[EAMCET-2006]

List-I	List-II
(A) $f(-5) + f(-4)$	(i) 14
(B) $f(\lvert f(-8)\rvert)$	(ii) 4
(C) $f(f(-7) + f(3))$	(iii) −11
(D) $f(f(f(f(0)))) + 1$	(iv) −1
	(v) 1
	(vi) 0

	(A)	(B)	(C)	(D)
(a)	(iii)	(vi)	(ii)	(v)
(b)	(iii)	(iv)	(ii)	(v)
(c)	(iv)	(iii)	(ii)	(i)
(d)	(iii)	(vi)	(v)	(ii)

Solution

(a) (A) : $f(-5) = -5 + 4 = -1$ and $f(-4) = 3 \times -4 + 2 = -10$

Then $f(-5) + f(-4) = -1 - 10 = -11$ Hence (A) → (iii)

(B) : $f(-8) = -8 + 4 = -4 \Rightarrow |f(-8)| = 4$

Now $f(|f(-8)|) = 4 - 4 = 0$ Hence $(B) \to$ (vi)

(C) : $f(-7) = -7 + 4 = -3$ and $f(3) = 3 \times 3 + 2 = 11$

$\Rightarrow \quad f(-7) + f(3) = 8$

Now $f(f(-7) + f(3)) = 8 - 4 = 4$

Hence (C) → (ii) Hence (a) is the correct answer.

11. If the function $f(x)$ is defined by $f(x) = a + bx$ and $f^r = fff$ ..... (repeated r times), then $f^r(x)$ is equal to ***[MPPET-2005]***

(a) $a + b^r x$

(b) $ar + b^r x$

(c) $ar + bx^r$

(d) $a\,(b^r - 1/b - 1) + b^r x$

Solution

(d) $f(x) = a + bx$

$f\{f(x)\} = a + b\,(a + bx)$

$= ab + a + b^2 x = a\,(1 + b) + b^2 x$

$f\{f\{f(x)\}\} = f\{a\,(1 + b) + b^2 x\}$

$= a - b\,(a\,(1 + b) + b^2 x)$

$= a\,(1 + b + b^2) + b^3 x$

$\therefore \quad f^r(x) = a\,(1 + b + b^2 + \ldots\ldots.. + b^{r-1}) + b^r x$

$$= a\left(\frac{b^r - 1}{b - 1}\right) + b^r x.$$

12. The domain of $f(x) = \dfrac{\log_2(x+3)}{x^2 + 3x + 2}$ is

[IIT (S)-2001]

(a) $R - \{-1, -2\}$ (b) $\{-2, \infty\}$
(c) $R - \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

Solution

(d) $f(x)$ is defined, if

$x + 3 > 0$ and $x^2 + 3x + 2 \neq 0$

$\Rightarrow \quad x > -3$ and $x \neq -1, -2$

$\Rightarrow \quad x \in (-3, \infty) - \{-1, -2\}$

Hence, the domain of $f(x)$ is $(-3, \infty) - \{-1, -2\}$

13. The domain of the function $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9 - x^2}}$ is

[AIEEE-2004]

(a) $[1, 2]$ (b) $[2, 3)$
(c) $(1, 2)$ (d) $[2, 3]$

Solution

(b) $\sin^{-1}(x - 3)$ is defined when $-1 \le x - 3 \le 1$

$\Rightarrow \quad 2 \le x \le 4$

$\therefore \quad$ domain of $\sin^{-1}(x - 3) = [2, 4]$

Also $\sqrt{9 - x^2}$ is defined when $9 - x^2 \ge 0$

$\Rightarrow \quad x^2 \le 9$

$\Rightarrow \quad |x| \le 3$

$\therefore \quad$ domain of $\dfrac{1}{\sqrt{9 - x^2}} = (-3, 3)$

Hence $D_f = [2, 4] \cap (-3, 3) = [2, 3)$

14. The domain of the real valued function $f(x) = 3e^{\sqrt{x^2-1}}\log(x - 1)$ is

[CET (Pb.)-91, 94]

(a) $R - \{1\}$
(b) $R - [-1, 1]$
(c) $[1, \infty)$
(d) $(1, \infty)$

Solution

(d) $f(x)$ will be real if $x^2 - 1 \ge 0$ and $x - 1 > 0$

$\Rightarrow \quad x^2 \ge 1$ and $x > 1$

$\Rightarrow \quad |x| \ge 1$ and $x > 1$

$\Rightarrow \quad x > 1 \;\therefore\; D_f = (1, \infty)$

15. Domain of the function $f(x) = \sin^{-1}\left(\dfrac{2-|x|}{4}\right) + \cos^{-1}\left(\dfrac{2-|x|}{4}\right) + \tan^{-1}\left(\dfrac{2-|x|}{4}\right)$ is

(a) R (b) $[0, 6]$
(c) $[-6, 6]$ (d) $[-3, 3]$

Solution

(c) If D_1, D_2, D_3 be domains of

$\sin^{-1}\left(\dfrac{2-|x|}{4}\right)$, $\cos^{-1}\left(\dfrac{2-|x|}{4}\right)$,

$\tan^{-1}\left(\dfrac{2-|x|}{4}\right)$ respectively, then

$D_1 = D_2 = [-6, 6], D_3 = R$

$\therefore D_f = D_1 \cap D_2 \cap D_3 = [-6, 6]$

16. The domain and range of the function

$$f(x) = \sin\left\{\log\left(\frac{\sqrt{4 - x^2}}{1 - x}\right)\right\}.$$

[IIT-85, DCE-98, PET (Raj.)-2003]

(a) $(-2, 1)$ and $[-1, 1]$
(b) $(2, 1)$ and $[-1, 1]$
(c) $(-2, -1)$ and $[-1, 1]$
(d) None of these

Solution

Since $\log x$ is defined for $x > 0$. Therefore, $f(x)$ is defined for $\dfrac{\sqrt{4 - x^2}}{1 - x} > 0 \Rightarrow 4 - x^2 > 0$ and $1 - x > 0 \Rightarrow x^2 - 4 < 0$ and $x < 1$

$\Rightarrow -2 < x < 2$ and $x < 1 \Rightarrow x \in (-2, 1)$

Hence, the domain of $f(x)$ is $(-2, 1)$

Since $f(x) = \sin\left\{\log\left(\dfrac{\sqrt{4 - x^2}}{1 - x}\right)\right\}$ and $\sin x$ assumes all values between -1 and 1. Therefore, range of $f(x) = [-1, 1]$.

17. If the function f satisfies the relation $f(x + y) + f(x - y) = 2 f(x) . f(y)$, $\forall\, x, y \in R$ and $f(0) \neq 0$, then prove that $f(x)$ is an even function.

Solution

Given $f(x + y) + f(x - y) = 2 f(x) f(y)$ .... (1)

Replacing x and y and y by x in (1) then $f(y + x) + f(y - x) = 2 f(y) f(x)$ .... (2)

$\therefore$ From (1) and (2) we get
$f(y - x) = f(x - y)$ Putting $y = 2x$ then
$f(x) = f(-x)$
Hence $f(x)$ is an even function.

18. If $f: R \to R$ and $g: R \to R$ are defined by $f(x) = x - [x]$ and $g(x) = [x]$ for $x \in R$, where $[x]$ is the greatest integer not exceeding x, then for every $x \in R, f(g(x)) =$ ***[EAMCET-2007]***

(a) x (b) 0
(c) $f(x)$ (d) $g(x)$

Solution

(b) $f(g(x)) = f([x]) = [x] - [x] = 0 \ \forall\ x \in R$
($\because$ $[(x)] = [x]$)

19. Domain of the function $f(x) = \dfrac{x-3}{(x-1)\sqrt{x^2-4}}$ is

(a) $(1, 2)$
(b) $(-\infty, -2) \cup (2, \infty)$
(c) $(-\infty, -2) \cup (1, \infty)$
(d) $(-\infty,\infty) - \{1, \pm 2\}$

Solution

(b) Obviously, here $|x| > 2$ and $x \neq 1$ i.e., $x \in (-\infty, -2) \cup (2, \infty)$

20. If a, b be two fixed positive integers such that $f(a + x) = b + [b^3 + 1 - 3b^2 f(x) + 3b f(x)^2 - (f(x))^3]^{1/3}$ for all real x, then $f(x)$ is a periodic function with period. ***[Orissa(JEE)-2003]***

(a) a (b) $2a$
(c) b (d) $2b$

Solution

(b) $f(a + x) = b + (1 + \{b - f(x)\}^3)^{1/3}$
$\Rightarrow$ $f(a + x) - b = (1 - \{f(x) - b\}^3)^{1/3}$
$\Rightarrow$ $\phi\,(a + x) = \{1 - \{\phi\,(x)\}^3\}^{1/3}$
$[\phi\,(x) = f(x) - b]$
$\Rightarrow$ $\phi\,(x + 2a) = 1\ \{1 - \{\phi\,(x + a\}^3\}^{1/3} = \phi\,(x)$
$\Rightarrow$ $f(x + 2a) - b = f(x) - b$
$\Rightarrow$ $f(x + 2a) = f(x)$
$\therefore$ $f(x)$ is periodic with period $2a$.

21. If $g : [-2, 2]\ \phi\ R$ where $g(x) = x^3 + \tan x + \left[\dfrac{x^2+1}{P}\right]$ is a odd function then the value of parametric P is ***[DCE-2005]***

(a) $-5 < P < 5$ (b) $P < 5$
(c) $P > 5$ (d) None of these

Solution

(c) $g(x) = x^3 + \tan x + \left[\dfrac{x^2+1}{P}\right]$

$$g(-x) = (-x^3) + \tan(-x) + \left[\frac{(-x)^2+1}{p}\right]$$

$$g(-x) = -x^3 - \tan x + \frac{x^2+1}{P}$$

$g(x) + g(-x) = 0$ because $g(x)$ is a odd function

$$\therefore \quad x^3 + \tan x + \left[\frac{x^2+1}{P}\right] - x^3$$
$$- \tan x + \left[\frac{x^2+1}{P}\right] = 0$$

$$\Rightarrow \quad 2\left[\frac{(x^2+1)}{P}\right] = 0$$

$\Rightarrow$ $0 \le \dfrac{x^2+1}{P} < 1 \because x \in [-2, 2] \Rightarrow 0 \le \dfrac{5}{P}$
$< 1 \Rightarrow P > 5$

22. If $f(x) = \cos^{-1}\left(\dfrac{2-|x|}{4}\right) + [\log(3 - x)]^{-1}$, then its domain is

(a) $[-2, 6]$ (b) $[-6, 2) \cup (2, 3)$
(c) $[-6, 2]$ (d) $[-2, 2) \cup (2, 3]$

Solution

(b) The domain of $\cos^{-1}\left(\dfrac{2-|x|}{4}\right)$ is given by $-1 \le \dfrac{2-|x|}{4} \le 1 \Leftrightarrow -6 \le -|x| \le 2 \Leftrightarrow -2 \le |x| \le 6 \Leftrightarrow |x| \le 6$

Thus, the domain of $\cos^{-1}\left(\dfrac{2-|x|}{4}\right)$ is $D_1 = [-6, 6]$.

The domain of $\dfrac{1}{\log(3-x)}$ is the set of all real numbers for which $3 - x > 0$ and $3 - x \neq 1$ i.e., $x < 3$ and $x \neq 2$

Hence, domain of the given function is
$\{x|\ -6 \le x \le 6\} \cap \{x|\ x \neq 2, x < 3\} = [-6, 2) \cup (2, 3)$

23. Let $f(x) = \sin x$ and $g(x) = \ln |x|$. If the ranges of the composition functions fog and gof are R_1 and R_2 respectively, then ***[IIT-JEE-1994]***

(a) $R_1 = \{u : -1 \le u < 1\}, R_2 = \{v : -\infty < v < 0\}$
(b) $R_1 = \{u : -\infty < u < 0\}, R_2 = \{v : -1 \le v \le 0\}$
(c) $R_1 = \{u : -1 < u < 1\}, R_2 = \{v : -\infty < v < 0\}$
(d) $R_1 = \{u : -1 \le u \le 1\}, R_2 = \{v : -\infty < v \le 0\}$

Solution

(d) We have fog(x) = $f(g(x))$ = sin (ln $|x|$)

$\therefore \quad R_1 = \{u : -1 \le u \le 1\}$

$(\because -1 \le \sin\theta \le 1, \forall\, \theta\}$

Also gof(x) = $g(f(x))$ = ln $|\sin x|$

$\because 0 \le |\sin x| \le 1$

$\Rightarrow \quad -\infty < \ln|\sin x| \le 0$

$\therefore R_2 = \{v : -\infty < v \le 0\}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The domain of the definition of the function $y = \dfrac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

[IIT-83; Roorkee-93]

(a) $[-2, 1) - [0]$ (b) $[2, -1]$
(c) $[-1, 2]$ (d) None of these

2. Domain of $y = \sqrt{1+x} + \sqrt{2-x}$

(a) $[1, 2], [0, \sqrt{6}\,]$ (b) $[-1, 2], [0, \sqrt{3}\,]$
(c) $[-1, 2]; [\sqrt{3}, \sqrt{6}]$ (d) None of these

3. Domain of $f(x) = \dfrac{1}{x} + 2^{\sin^{-1}x} + \dfrac{1}{\sqrt{x-2}}$

(a) $[1, 2]$ (b) $[2, 1]$
(c) $[0, 2]$ (d) ϕ

4. Domain of the function $f(x) = \dfrac{x-3}{(x-1)\sqrt{x^2-4}}$ is

(a) $(1, 2)$
(b) $(-\infty, -2) \cup (2, \infty)$
(c) $(-\infty, -2) \cup (1, \infty)$
(d) $(-\infty, \infty) - (1, \pm 2)$

5. For what value of x function be identical $f(x)$=log $(x - 1)$ –log$(x - 2)$ and $g(x) = \log\left(\dfrac{x-1}{x-2}\right)$

(a) $(1, \infty)$ (b) $(2, \infty)$
(c) $(3, \infty)$ (d) None of these

6. If $f(x) = \log\dfrac{1+x}{1-x}$, then $f(x)$ is

(a) Even Function
(b) $f(x_1)f(x_2) = f(x_1 + x_2)$
(c) $\dfrac{f(x_1)}{f(x_2)} = f(x_1 - x_2)$
(d) Odd Function

7. Which of the following is an odd function

(a) $f(x) = \cos x$ (b) $y = 2^{-x^2}$
(c) $y = 2^{x-x^2}$ (d) None of these

8. Which of the following function is even function

(a) $f(x) = \dfrac{a^x+1}{a^x-1}$
(b) $f(x) = x\left(\dfrac{a^x-1}{a^x+1}\right)$
(c) $f(x) = \dfrac{a^x - a^{-x}}{a^x + a^{-x}}$
(d) $f(x) = \sin x$

9. If the real valued function $f(x) = \dfrac{a^x-1}{x^n(a^x+1)}$, is even then n equals

(a) 2 (b) 2/3
(c) 1/4 (d) –1/3

10. If f is an even function defined on the interval $(-5, 5)$, then four real values of x satisfying the equation $f(x) = f\left(\dfrac{x+1}{x+2}\right)$ are

(a) $\dfrac{-3-\sqrt{5}}{2}, \dfrac{-3+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}$
(b) $\dfrac{-5+\sqrt{3}}{2}, \dfrac{-3+\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}, \dfrac{3-\sqrt{5}}{2}$
(c) $\dfrac{3-\sqrt{5}}{2}, \dfrac{3+\sqrt{5}}{2}, \dfrac{-3-\sqrt{5}}{2}, \dfrac{5+\sqrt{3}}{2}$
(d) $-3-\sqrt{5} - 3+\sqrt{5}, 3-\sqrt{5}, 3+\sqrt{5}$

11. Period of $\sin\left(2\pi x+\frac{\pi}{3}\right)+2\sin\left(3\pi x+\frac{\pi}{4}\right)+3\sin 5\pi x$ is

(a) 1 (b) 2/3
(c) 2/5 (d) 2

12. The period of $f(x)=\left|\sin^3\frac{x}{2}\right|$ is

(a) π (b) 2π
(c) 3π (d) None of these

13. The value of $n \in I$, for which the function $f(x)=\dfrac{\sin nx}{\sin\frac{x}{n}}$ has 4π as its period is n is equal to

(a) 2 (b) 3
(c) 5 (d) 4

14. If $f(x)$ = is an odd periodic function with period 2, then $f(4)$ equals

(a) 0 (b) 2
(c) 4 (d) –4

15. Period of $f(x)=\sin\frac{\pi}{2}x+2\cos\frac{\pi}{3}x-\tan\frac{\pi}{4}x$ is equal to

(a) 4 (b) 8
(c) 12 (d) 16

16. The function $f(x)=\frac{1}{2}\left\{\frac{|\sin x|}{\cos x}+\frac{\sin x}{|\cos x|}\right\}$ is periodic with period

(a) π (b) $\pi/2$
(c) 2π (d) 3π

17. Period of the function $y=\sin\dfrac{2t+3}{6\pi}$ is

(a) $3\pi^2$ (b) $5\pi^2$
(c) $7\pi^2$ (d) $6\pi^2$

18. If $g(x)=1+\sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$, then $f(x)$ is equal to

(a) $1 + 2x^2$ (b) $2 + x^2$
(c) $1 + x$ (d) $2 + x$

19. If $f(x)=\dfrac{x}{\sqrt{1+x^2}}$, then $fofof(x)$ is equal to **[RPET-2000]**

(a) $\dfrac{3x}{\sqrt{1+x^2}}$ (b) $\dfrac{x^3}{\sqrt{1+x^6}}$
(c) $\dfrac{x}{\sqrt{1+3x^2}}$ (d) None of these

20. If $g(x)=x^2+x-2$ and $\frac{1}{2}(gof)(x)=2x^2-5x+2$, then $f(x)$ is equal to

(a) $2x - 3$ (b) $2x + 3$
(c) $2x^2 + 3x + 1$ (d) $2x^2 - 3x - 1$

21. If $f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g(5/4) = 1$, then $gof(x)$ is equal to

(a) 1 (b) 2
(c) 5 (d) 4/5

22. Domain of definition of the function $f(x)=\dfrac{3}{4-x^2}+\log_{10}(x^3-x)$ is **[AIEEE-2003]**

(a) $(-1, 0) \cup (1, 2) \cup (2, \infty)$
(b) $(1, 2)$
(c) $(-1, 0) \cup (1, 2)$
(d) $(1, 2) \cup (2, \infty)$

23. If $f(x) = x$ and $g(x) = |x|$, then what is $(f + g)(x)$ equal to? **[NDA-2008]**

(a) 0 for all $x \in R$ (b) $2x$ for all $x \in R$
(c) $\begin{cases} 2x \text{ for } x \ge 0 \\ 0 \text{ for } x < 0 \end{cases}$ (d) $\begin{cases} 0 \text{ for } x \ge 0 \\ 2x \text{ for } x < 0 \end{cases}$

24. The period of the function $f(x) = a^{\{\tan(\pi x)+x-[x]\}}$, where $a > 0$, [.] denotes the greatest integer function and x is a real number, is **[Kerala PET-2007]**

(a) π (b) $\pi/2$
(c) $\pi/4$ (d) 1

25. If $f(x) = e^x$ and $g(x) = \log_e x$ then which of the following is true **[MPPET-2008]**

(a) $f\{g(x)\} \neq g\{f(x)\}$
(b) $f\{g(x)\} = g\{f(x)\}$
(c) $f\{g(x)\} + g\{f(x)\} = 0$
(d) $f\{g(x)\} - g\{f(x)\} = 1$

26. The domain of the real valued function $f(x)=\sqrt{1-2x}+2\sin^{-1}\left(\frac{3x-1}{2}\right)$ is **[Kerala PET-2007]**

(a) $\left[\frac{-1}{3},1\right]$ (b) $\left[\frac{1}{2},1\right]$
(c) $\left[\frac{-1}{2},\frac{1}{3}\right]$ (d) $\left[\frac{-1}{3},\frac{1}{2}\right]$

SOLUTIONS

1. (a) $y=\dfrac{1}{\log(1-x)}+\sqrt{x+2}$

For domain $\therefore$ $1-x>0;\ 1-x\neq 1$
$(\because \log 1=0)$

$x<1, x\neq 0$

Also $x+2\geq 0$, or $x\geq -2$
$\therefore$ Combining all intervals we get
$x\in(-2,1)-\{0\}$

2. (c) Domain $x+1\geq 0;\ 2-x\geq 0$

$x\geq -1$ or $x\leq 2$

$\therefore \quad D_f\in[-1,2]$

For range, $y=\sqrt{1+x}+\sqrt{2-x}$

for extreme values $\dfrac{dy}{dx}=0$

$$\frac{1}{2\sqrt{1+x}}-\frac{1}{2\sqrt{2-x}}=0$$

$$\sqrt{2-x}=\sqrt{1+x}\Rightarrow x=\frac{1}{2}$$

$\therefore \quad$ at $x=\dfrac{1}{2}\Rightarrow y=\sqrt{6}$

at $x=-1;\ y=\sqrt{3}$

at $x=2;\ y=\sqrt{3}$

$\therefore$ Range $\in\left[\sqrt{3},\sqrt{6}\right]$

3. (d) $f(x)=\dfrac{1}{x}+2^{\sin^{-1}x}+\dfrac{1}{\sqrt{x-2}}$

$\dfrac{1}{x}$ exists if $x\neq 0$ ....(1)

$\sin^{-1}x$ exist if $-1\leq x\leq 1$ ....(2)

$\dfrac{1}{\sqrt{x-2}}$ exist if $x-2>0\Rightarrow x>2$ .....(3)

$\therefore \quad$ Domain $\in$ (1)$\cap$(2)$\cap$(3) $\Rightarrow x\in\phi$

4. (b) $f(x)=\dfrac{x-3}{(x-1)\sqrt{x^2-4}}$

Here $x\neq 1$ ....(1), $x^2-4>0$

$(x-2)(x+2)>0$

$\Rightarrow x\in(-\infty,-2)\cup(2,\infty)$ ....(2)

From (1) and (2)

$x\in(-\infty,-2)\cup(2,\infty)$

5. (b) For identical function domain must be same domain of $f(x)$ be D_1

$x-1>0$ and $x-2>0$

$\therefore \quad D_1$ is $x\in(2,\infty)$

Domain of $g(x)$ be D_2

For D_2; $\dfrac{x-1}{x-2}>0$

$x\in(-\infty,1)\cup(2,\infty)$

$\therefore$ for equal function

$D_1\cap D_2\ x\in(2,\infty)$

6. $f(x)=\log\left(\dfrac{1+x}{1-x}\right)$

$f(-x)=\log\left(\dfrac{1-x}{1+x}\right)$

$=\log\left(\dfrac{1+x}{1-x}\right)^{-1}=-\log\left(\dfrac{1+x}{1-x}\right)$

$\therefore\ f(-x)=-f(x)$

$\therefore\ f(x)$ is odd function.

7. (d) For odd function

$f(-x)=-f(x)$

(a), (b) are even function (c) is neither even nor odd function.

$\therefore$ (d) is.

8. (b) For even function

$f(-x)=f(x)$

(a) $f(-x)=\dfrac{a^{-x}+1}{a^{-x}+1}=\dfrac{\dfrac{1}{a^x}+1}{\dfrac{1}{a^x}-1}$

$f(-x)=\dfrac{1+a^x}{1-a^x}=-f(x)$

$\therefore$ It is odd function.

(b) $f(x)=x\left(\dfrac{a^x-1}{a^x+1}\right)$

x is odd and $\dfrac{a^x-1}{a^x+1}$ is odd.

$\therefore\ f(x)=(\text{odd})(\text{odd})=\text{even}$

9. (d) $f(x) = \dfrac{a^x - 1}{x^n(a^x + 1)}$

For even function $f(-x) = f(x)$

$= \dfrac{a^{-x} - 1}{(-x)^n(a^{-x} + 1)} = \left(\dfrac{a^x - 1}{a^x + 1}\right)\dfrac{1}{x^n}$

$\left(\because a^{-x} = \dfrac{1}{a^x}\right)$

$= \dfrac{1 - a^x}{1 + a^x} \times \dfrac{1}{(-1)^n x^n} = \dfrac{a^x - 1}{a^x + 1} \cdot \dfrac{1}{x^n}$

$(-1)^n = -1$

$\therefore$ From option we get (d)

10. (a) Here $f(x) = f(-x) = f\left(\dfrac{x+1}{x+2}\right)$

$\therefore\ -x = \dfrac{x+1}{x+2} \Rightarrow -x^2 - 2x = x + 1$

$x^2 + 3x + 1 = 0$; $x = \dfrac{-3 \pm \sqrt{5}}{2}$

Also $x = \dfrac{x+1}{x+2} \Rightarrow x^2 + x - 1 = 0$

$x = \dfrac{-1 \pm \sqrt{5}}{2}$

11. (d) $f(x) = \sin\left(2\pi x + \dfrac{\pi}{3}\right) +$

$2\sin\left(3\pi x + \dfrac{\pi}{4}\right) + 3\sin(5\pi x)$

period = L.C.M. $\left(\dfrac{2\pi}{2\pi}, \dfrac{2\pi}{3\pi}, \dfrac{2\pi}{5\pi}\right)$

= L.C.M. $\left(1, \dfrac{2}{3}, \dfrac{2}{5}\right)$

= L.C.M. $\left(\dfrac{15}{15}, \dfrac{10}{15}, \dfrac{6}{15}\right)$

$= \dfrac{1}{15}$ L.C.M. (15, 10, 6) $= \dfrac{30}{15} = 2$

12. (b) We know that period of $\sin x$ is π

$\therefore$ period of $\sin^3\left(\dfrac{x}{2}\right) = \dfrac{\pi}{1/2} = 2\pi$

13. (a) Here period = 4π

L.C.M. $\left(\dfrac{2\pi}{n}, 2n\pi\right) = 4\pi$

$2n\pi = 4\pi \Rightarrow n = 2$

14. (a) For odd function $f(-x) = -f(x)$

put $x = 0$; $f(0) = -f(0)$

$\Rightarrow\ f(0) = 0$

$\because$ period = 2

$\therefore\ f(0) = f(2) = f(4) = 0$

15. (c) Period = L.C.M. $\left(\dfrac{2\pi}{\pi/2}, \dfrac{2\pi}{\pi/3}, \dfrac{\pi}{\pi/4}\right)$

L.C.M. (4, 6, 4) = 12

16. (c) Period of $\dfrac{|\sin x|}{\cos x}$ = L.C.M. $(\pi, 2\pi) = 2\pi$

Similarly, period of $\dfrac{\sin x}{|\cos x|} = 2\pi$

$\therefore$ Period = 2π

17. (d) $y = \sin\left(\dfrac{2t+3}{6\pi}\right)$

$= \sin\left(\dfrac{t}{3\pi} + \dfrac{1}{2\pi}\right)$

$\therefore$ Period $= \dfrac{2\pi}{1/3\pi} = 6\pi^2$

18. (b) $g(x) = 1 + \sqrt{x}$

$f(g(x)) = 3 + 2\sqrt{x} + x$

$f(1 + \sqrt{x}) = 2 + \left(\sqrt{x} + 1\right)^2$

$\therefore f(x) = 2 + x^2$

19. (c) $fofof(x) = f\left(f\left(\dfrac{x}{\sqrt{1+x^2}}\right)\right)$

$= f\left(\dfrac{\dfrac{x}{\sqrt{1+x^2}}}{\sqrt{\dfrac{1+x^2}{1+x^2}}}\right) = f\left(\dfrac{x}{\sqrt{1+2x^2}}\right)$

$= f\left(\dfrac{x}{\sqrt{1+2x^2}}\right) = \left(\dfrac{\dfrac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\dfrac{x^2}{1+2x^2}}}\right) = \left(\dfrac{x}{\sqrt{1+3x^3}}\right)$

20. (a) $g(x) = x^2 + x - 2; \dfrac{1}{2}g(f(x)) = 2x^2 - 5x + 2$

$g(f(x)) = 4x^2 - 10x + 4 = (2x-3)^2 + (2x-3) - 2$

$[f(x)]^2 + [f(x)] - 2 = (2x-3)^2 + (2x-3) - 2$

comparing we get $f(x) = 2x - 3$

21. (a)

$$f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$$

$$=1-\left[\cos^2 x-\sin^2\left(x+\frac{\pi}{3}\right)\right]+\cos x\cos\left(x+\frac{\pi}{3}\right)$$

$$=1-\cos\left(2x+\frac{\pi}{3}\right)\cos\frac{\pi}{3}+\frac{1}{2}\left[\cos\frac{\pi}{3}+\cos\left(2x+\frac{\pi}{3}\right)\right]$$

$$=1-\frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right)+\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}\cos\left(2x+\frac{\pi}{3}\right)=\frac{5}{4}$$

$g(f(x))=g\left(\frac{5}{4}\right)=1$ (Given in question)

22. (a) $f(x)=\dfrac{3}{4-x^2}+\log_{10}(x^3-x)$

For domain, $4-x^2\neq 0 \Rightarrow x=\pm 2$

and $x^3-x>0 \Rightarrow x(x-1)(x+1)>0$

$$\begin{array}{ccccccc} - & & + & & - & & + \\ \hline & -1 & & 0 & & 1 & \end{array}$$

$\therefore x\in(-1,0)\cup(1,\infty)$

But $x\neq 2,-2$

$\therefore x\in(-1,0)\cup(1,2)\cup(2,\infty)$

23. (c) $(f+g)(x)=f(x)+g(x)$

$=x+|x|$

$$=\begin{cases} x+x & \text{if } x\geq 0 \\ x-x & \text{if } x<0 \end{cases}$$

$$\therefore (f+g)(x)=\begin{matrix} 2x & \text{if } x\geq 0 \\ 0 & \text{if } x<0 \end{matrix}$$

24. (d) $f(x)=a^{\tan \pi x+x-[x]}$

$=a^{\tan \pi x+\{x\}}$

Period $=$ L.C.M. $\left(\frac{\pi}{\pi},1\right)=1$

25. (a) $f(x)=e^x$ and $g(x)=\log_e x$

Here $f(g(x))=f(\log_e x)=e^{\log_e x}$

$=x$, Here $x>0$

and $g(f(x))=g(e^x)$

$=\ln e^x=x$

Here $x\in R$

$\because$ Domain are different $f(g(x))\neq g(f(x))$

26. (d) $f(x)=\sqrt{1-2x}+2\sin^{-1}\left(\dfrac{3x-1}{2}\right)$

For domain

$1-2x\geq 0;\ -1\leq\dfrac{3x-1}{2}\leq 1$

$x\leq\dfrac{1}{2} \Rightarrow -2\leq 3x-1\leq 2\quad -\dfrac{1}{3}\leq x\leq 1$

taking common of both interval $x\in\left[\dfrac{-1}{3},\dfrac{1}{2}\right]$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $A=\{2, 4, 6\}$ and function is defined from $f: A\rightarrow A$, then for $f=\{(2, 4), (4, 6), (6, 2)\}$ fof is

(a) $\{(2, 6), (4, 2), (6, 4)\}$

(b) $\{(6, 2), (4, 2), (6, 4)\}$

(c) $\{(2, 6), (2, 4), (6, 4)\}$

(d) None of these

2. The equivalent function of $\log x^2$ is ***[MPPET-1997]***

(a) $2\log x$ (b) $2\log|x|$

(c) $|\log x^2|$ (d) $(\log x)^2$

3. If function $f(x)=\dfrac{1}{2}-\tan\left(\dfrac{\pi x}{2}\right)$, $-1<x<1$ and $g(x)=\sqrt{3+4x-4x^2}$, then the domain of gof is ***[IIT-1990]***

(a) $(-1, 1)$ (b) $[-1/2, 1/2]$

(c) $[-1, 1/2)$ (d) $[-1/2, -1)$

4. The fundamental period of the function $f(x)=2\cos\frac{1}{3}(x-\pi)$ is

(a) 6π (b) 4π

(c) 2π (d) π

5. If $f : R \to R$ satisfies the condition $f(x + y) = f(x) + f(y)$ for all x, y in R, then $f(x)$ is
(a) zero every where
(b) an even function
(c) an odd function
(d) defined at $x = 0$

6. The period of $f(x) = 3x-[3x]$, if it is periodic, is
(a) $f(x)$ is not periodic (b) 1/3
(c) 1 (d) 2

7. If f be the greatest integer function and g be the modulus function, then

$$\text{gof}\left(\frac{-5}{3}\right) - (\text{fog})\left(\frac{-5}{3}\right) =$$

(a) 1 (b) −1
(c) 2 (d) 4

8. If $f : R \to R$, $g : R \to R$ and $h : R \to R$ are such that $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$, then the value of $(\text{ho}(\text{gof}))(x)$ if $x = \sqrt{\pi}/2$ will be
(a) 0 (b) 1
(c) −1 (d) p

9. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then ***[IIT-1998]***
(a) $f(x) = \sin^2 x, g(x) = \sqrt{x}$
(b) $f(x) = \sin x$, $g(x) = |x|$
(c) $f(x) = x^2, g(x) = \sin\sqrt{x}$
(d) $f(x)$ and $g(x)$ cannot be determined

10. If $f(x) = 2x$, $g(x) = 1 - x$; $h(x) = x + 1$; then $[\text{hogof}]\ (x) = 1 - 2x;\ 1 - x;\ 2x + 1;\ 2\ (1 - x)$.

11. The graph of the function $y = f(x)$ is symmetrical about the line $x = 2$, then ***[AIEEE-2004]***
(a) $f(x) = -f(-x)$
(b) $f(2 + x) = f(2 - x)$
(c) $f(x) = f(-x)$
(d) $f(x + 2) = f(x - 2)$

12. The domain of the function
$f(x) = \log_{(3+x)}(x^2 - 1)$ is ***[Orissa JEE-2003]***
(a) $(-3, -1) \cup (1, \infty)$
(b) $[-3, -1) \cup [1, \infty)$
(c) $(-3, -2) \cup (-2, -1) \cup (1, \infty)$
(d) $[-3, -2) \cup (-2, -1) \cup [1, \infty)$

13. Domain of the function $y = \log_{1/2}(x - 1/2) + \log_2\sqrt{4x^2 - 4x + 5}$
(a) $(-\infty, +\infty)$ (b) $(-\infty, 1/2)$
(c) $(1/2, \infty)$ (d) $(-1/2, 1/2)$

14. The largest interval lying in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ for the function

$$\left[f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x) \right]$$

is defined, is ***[AIEEE-2007]***
(a) $[0, \pi]$ (b) $(-\pi/2, \pi/2)$
(c) $[-\pi/4, \pi/2)$ (d) $[0, \pi/2)$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If $f(x)$ is an odd periodic function with period 2, then $f(4)$ equals
(a) –4 (b) 4
(c) 2 (d) 0

2. If $f : R \to R$ and $g : R \to R$ defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, then the value of x for which $f(g(x)) = 25$ are **[EAMCET-2000]**
(a) ± 1 (b) ± 2
(c) ± 3 (d) ± 4

3. Let $f : R \to R : f(x) = x^2$ and $g : R \to R : g(x) = x + 5$, then gof is **[Kerala CEE-2004]**
(a) $(x^2 + 5)$ (b) $(x + 5^2)$
(c) $(x^2 + 5^2)$ (d) $(x + 5)^2$

4. If $f(x) = \dfrac{2x+1}{3x-2}$, then (fof) (2) is equal to **[Kerala CEE-2002]**
(a) 1 (b) 3
(c) 4 (d) 2

5. If $f : R \to R$ and $g : R \to R$ are defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, then the values of x such that $g(f(x)) = 8$ are **[EAMCET-2003]**
(a) 1, 2 (b) –1, 2
(c) –1, –2 (d) 1, –2

6. If $f(x) = \dfrac{x}{x-1}$, then $\underbrace{(fo\ \ fo....of)}_{19 \text{ times}}(x)$ is equal to **[MPPET-2004]**
(a) $\dfrac{x}{(x-1)}$ (b) $\left(\dfrac{x}{x-1}\right)^{19}$
(c) $\dfrac{19x}{(x-1)}$ (d) x

7. The domain of the function $f(x) = \sqrt{1-x} + \sqrt{6-x}$ is
(a) [1, 2) (b) (2, 1]
(c) [1, 2] (d) None of these

8. The domain of the function $f(x) = \dfrac{1}{1-x} + 2^{\sin^{-1}x} + \dfrac{1}{\sqrt{x-2}}$ is
(a) $(-\infty, \infty) - \{1\}$ (b) $(2, \infty)$
(c) $[-1, 1)$ (d) ϕ

9. The domain of the function $f(x) = \sqrt{x-1} + \sqrt{5-x}$ is
(a) $[1, \infty)$ (b) $(-\infty, 5)$
(c) (1, 5) (d) [1, 5]

10. The domain of the function $f(x) = \log_{10}(\sqrt{x-4} + \sqrt{6-x})$ is
(a) (4, 6) (b) [4, 6]
(c) [4, 6) (d) None of these

11. If $f(x) = x^3 - x$ and $\phi(x) = \sin 2x$, then
(a) $\phi(f(2)) = \sin 12$
(b) $\phi(f(1)) = 1$
(c) $f\left(\phi\left(\dfrac{\pi}{12}\right)\right) = \dfrac{-3}{8}$
(d) $f(f(1)) = 2$

12. If $f(x) = 3x + 2$, $g(x) = x^2 - 1$, then the value of (fog) $(x^2 - 1)$ is
(a) $3x^4 - 6x^2 + 2$
(b) $3x^4 + 3x^2 + 4$
(c) $6x^4 + 3x^2 + 2$
(d) $3x^2 + 6x + 2$

13. The natural domain of the real valued function defined by $f(x) = \sqrt{x^2-1} + \sqrt{x^2+1}$ is **[SCRA-1996]**
(a) $1 < x < \infty$
(b) $-\infty < x < \infty$
(c) $-\infty < x < -1$
(d) $(-\infty, \infty) - (-1, 1)$

14. The period of the function $\sin\left(\dfrac{2x}{3}\right) + \sin\left(\dfrac{3x}{2}\right)$ is **[Orissa-2004]**

(a) 2π (b) 10π
(c) 6π (d) 12π

15. If $g(x) = \sin x,\ x \in R$
and $f(x) = \dfrac{1}{\sin x},\ x \in \left(0, \dfrac{\pi}{2}\right)$ what is (gof) (x) equal to? ***[NDA-2008]***

(a) 1 (b) $\dfrac{1}{\sin(\sin x)}$

(c) $\dfrac{1}{\sin^2(x)}$ (d) $\sin\left(\dfrac{1}{\sin x}\right)$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	6. ⓐ ⓑ ⓒ ⓓ	11. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	7. ⓐ ⓑ ⓒ ⓓ	12. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	8. ⓐ ⓑ ⓒ ⓓ	13. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	9. ⓐ ⓑ ⓒ ⓓ	14. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	10. ⓐ ⓑ ⓒ ⓓ	15. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. (b) $f(x) = 2x + 3;\ g(x) = x^2 + 7$

Given $f(g(x)) = 25$

$f(x^2+7) = 2(x^2+7) + 3 = 25$

$2x^2 = 8 \Rightarrow x = \pm 2$

3. (a) $f(f(x)) = g(x^2) = x^2 + 5$

6. $f(x) = \dfrac{x}{x-1},\ f(f(x)) = \dfrac{\dfrac{x}{x-1}}{\dfrac{x}{x-1} - 1}$

$f(f(x)) = x,\ f(f(f(x))) = f(x)$

$\Rightarrow \underbrace{fof \ldots\ldots\ldots f(x)}_{19\ times} = f(x)$

$(\because f(f(f \ldots\ldots\ldots even\ times) = x$

$f(f(f \ldots\ldots\ldots odd\ times = \dfrac{x}{x-1}$

9. (d) for domain of the function $f(x)$
$x - 1 \geq 0$ and $5 - x \geq 0$

$x \geq 1$ and $x \leq 5\ \therefore\ x \in [1, 5]$

11. (c) Hence $f\left(\phi\left(\dfrac{\pi}{12}\right)\right) = f\left(\sin\left(2.\dfrac{\pi}{12}\right)\right)$

$= f\left(\sin\dfrac{\pi}{6}\right) = f\left(\dfrac{1}{2}\right) = \left(\dfrac{1}{2}\right)^3 - \dfrac{1}{2} = \dfrac{-3}{8}$

14. $f(x) = \sin\left(\dfrac{2x}{3}\right) + \sin\left(\dfrac{3x}{2}\right)$

Period = LCM $\left(\dfrac{2\pi}{\frac{2}{3}}, \dfrac{2\pi}{\frac{3}{2}}\right)$

$= \text{LCM}\left(3\pi, \dfrac{4\pi}{3}\right)$

$= \text{LCM}\left(\dfrac{12\pi}{3}, \dfrac{4\pi}{3}\right) = \dfrac{12\pi}{3} = 4\pi$

15. (b) $gof(x) = g[f(x)]$

$g\left(\dfrac{1}{\sin x}\right) = \sin\left(\dfrac{1}{\sin x}\right)$

LECTURE

4

Mapping of Functions

BASIC CONCEPTS

1. **Cartesian Product of Two Sets** X and Y symbolically written as $X \times Y = \{ (x, y) \mid x \in X; y \in Y\}$.
2. **Function as a Set of Ordered Pairs** A function is a set of ordered pairs if no two of the ordered pairs have the same first component:
   Domain $= \{ x \mid x \in (x,y) \}$ Set of 1st co-ordinates of ordered pair
   Range $= \{ y \mid y \in (x,y) \}$ set of 2nd co-ordinates of ordered pair
3. **Injective or One-One function**
   If different elements of the domain have different f image in Y
   i.e., $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Then the function is said to be one-one function. Also it is called Injective Function.
4. **Many One Function** If a function is not one-one, it is called a many one function. In this case at least one element of Y is the image of more than one element of X.
5. **Into Function** If there is at least one element of Y which is not the f image of any element in the domain X of the function so that Range of function is a proper subset of Y then the function is said to be an Into function: symbolically $f\colon X \xrightarrow{INTO} Y$.
6. **Onto Function (Surjective)** If the function $f\colon X \to Y$ is such that each element of Y is the f-image of at least one element in X: then we say that f is a function from X "ONTO" Y and this is symbolically expressed as: $f\colon X \xrightarrow{ONTO} Y$.
   In this case the range of f is the same as the co-domain of f.
7. **Bijective** If f is both injective and surjective then f is called Bijective. Thus f is bijective if every element of Y is the f image of exactly one element of A. Bijective function is also called One to One correspondence.
8. **(I) Methods to Test Many One**

NOTES

1. If X and Y are finite sets having n and m elements respectively then (i) Number of one-one functions from
   $$X \text{ to } Y = \begin{cases} {}^nP_m & \text{if } \quad n \geq m \\ = 0 & \text{if } \quad n < m \end{cases}$$
   Let X and Y be two sets having n elements each. Then the total number of bijective functions from X to Y is: $n\,!$.
2. The number of surjective from $A = \{1, 2, 3, ..., n)$, $n \geq 2$ to $B = \{a, b\}$ is $2^n - 2$.
3. If X and Y are two sets having m and n elements respectively such that $1 \leq n \leq m$, then number of ONTO functions from A to B is: $\sum_{r=1}^{n} (-1)^{n-r}\, {}^nC_r\, r^m$
4. $m, n = 2 \Rightarrow (-1)^{2-1}\, {}^2C_1\, (1)^m \to (-1)^{2-2}\, {}^2C_2\, (2)^m = 2^m - 2$
   (i) If any line parallel to x-axis cuts the graph of the function at atleast two points, then the function is many one.

(ii) Any function which is neither increasing nor decreasing in whole domain, then $f(x)$ is many one.
(iii) Any continuous function $f(x)$, which has at least one local maxima or local minima is called many ones.

(II) Methods to Test One-One

(i) If any line parallel to x-axis cuts the graph of the function at most at one point, then the function is 1-1 (one-one).
(ii) Any function which is entirely increasing or decreasing in whole domain, then $f(x)$ is one-one.

9. Inverse Function Let $f: X \to Y$ be a one-one onto function then the function $f^{-1}: Y \to X$ which associates to each element $b \in Y$ a unique element $a \in X$ such that $f(x) = y$ is called the inverse function of the function $f: X \to Y$.

OR

Invertible Function A mapping $f: A \to B$ is said to be invertible if there exists a mapping $g: B \to A$ such that fog $= I_B$; and gof $= I_A$ where I_A and I_B are the identity maps. In such a case the map g is called the inverse of f and is denoted by f^{-1}. If f is invertible if the inverse function f^{-1} is a mapping from B to A; f has an inverse iff if f is both one to one and onto.

9.1 Working Rule for Formula of inverse of Function $f: X \to Y$ or $y = f(x)$ is

Step 1: To start take $y = f(x)$

Step 2: Interchange x and y i.e., $x = f(y)$ and solve for y.

Step 3: $y = f^{-1}(x) =$ Inverse of desired function.

Q. If $f: R \to R$ be defined by $f(x) = 2x - 3$; find a formula for f^{-1}.

Solution

Step 1: Let y be the image of x under the mapping $f(x) = y = 2x - 3$

Step 2: Interchange x and y to obtain $x = 2y - 3$;

Step 3: Solve for y i.e., $y = \frac{x+3}{2}$. Thus the formula defining the inverse function is $f^{-1}(x) = \frac{x+3}{2}$

NOTE

y of step 3 is called the inverse function.

Q. Find a formula for the inverse of $g(x) = x^2 - 1$.

Solution

Let $y = x^2 - 1$ interchange x and y to get $x = y^2 - 1$

i.e., $y = \sqrt{x+1}$

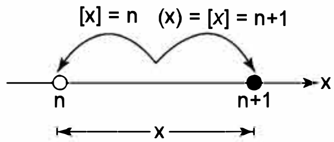
$\therefore \quad g^{-1}(x) = \sqrt{x+1}$

10. Least Integer Function

(x) or $\lceil x \rceil$ denotes the least integer function which is greater than or equal to x. It is also known as ceiling of x.

Thus, $(3.578) = 4$, $(0.87) = 1$, $(4) = 4$,

$\lceil -8.239 \rceil = -8$, $\lceil -0.7 \rceil = 0$



In general if n is an integer and x is any real number between n and $(n + 1)$

i.e., $n < x \le n + 1$ then $(x) = n + 1$

$\therefore \quad f(x) = (x) = [x]$

$D_f = R$, $R_f = [\,x\,] + 1$

NOTE

Domain and range of $\frac{1}{[x]}$ are $R - [0, 1)$ and $\frac{1}{n}$, $n \in I - \{0\}$ respectively.

For $x \in (-2, -1]$; $(x) = -1$.

11. Definition of the Extension of the Function

Extension of Function Let $f(x)$ be a given function defined in the interval $[0, a]$ and we are required to extend the function in the interval $[-a, a]$ to make it either even or odd in the interval $[-a, a]$. We define $f(x)$ in the interval $[-a, 0]$ such that for

(i) Odd Extension

$$g_0(x) = \begin{cases} f(x); & 0 \le x \le a \\ -f(-x); & -a \le x \le 0 \end{cases}$$

(ii) Even Extension

$$g_e(x) = \begin{cases} f(x); & 0 \le x \le a \\ f(-x); & -a \le x \le 0 \end{cases}$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. If $f(x) = \dfrac{x-1}{x+1}, x \neq 1, -1,$ show that fof^{-1} is the indentity function, *[CBSE-2002]*

2. Are the following invertible in their respective domain. If so, find the inverse in case $f(x) = \dfrac{1-x}{1+x}$.

3. If functions f and g are such that $f(x) = x^3$ and $g(x) = \sin x$, then find the value of(gof) x. *[MP-2000]*

4. If $f(x) = \sin x, x \in R$ and $g(x) = x^2, x \in R$, then prove that gof $\neq$ fog.

5. If $f(x) = x^2 - 1, g(x) = 3x + 1$, then describe the following functions (i) gog (ii) fof *[J & K Board-95]*

6. If $f(x) = x^2 + x + 1$ and $g(x) = \sin x$, then show that fog and gof are not equal. *[HSB-97 (S)]*

7. Find the value of $f + g$ for following function f and g.

(i) $f(x) = x^4, \forall x \in R$ and $g(x) = x - 1, \forall x \in R$.
(ii) $f(x) = x^5, \forall x \in A$ and $g(x) = x + 1, \forall x \in B$

8. Let f be the greatest integer function and g be the modulus function. Then find the value of following functions.

(1) $(\text{gof})\left\{\dfrac{-5}{3}\right\} - (\text{fog})\left\{\dfrac{-5}{3}\right\}$

(2) $(\text{gof})\left\{\dfrac{5}{3}\right\} - (\text{fog})\left\{\dfrac{5}{3}\right\}$

(3) $(f + 2g)(-1)$

9. Is $\cos\sqrt{t}$ periodic function if yes, find the period if not, give reason for your answer. *[Roorkee-82; I.S.M. Dhanbad-90]*

10. Let $f(x) = \dfrac{x^3-1}{x-1}$ and $g(x) = x^2 + x + 1$. Is $f = g$?
If not, modify f so that $f = g$.

11. Let f be the greatest integer function and g be the modulus function. Prove that $(\text{gof})\left\{\dfrac{-11}{7}\right\} - (\text{fog})\left\{\dfrac{-11}{7}\right\} = 1$

ANSWERS

2. $f^{-1} = R - \{-1\}$

3. $\sin x^3$

5. (i) $9x + 4$ (ii) $x^4 - 2x^2$

7. (i) $x^4 + x - 1 \ \forall x \in R$
(ii) $x^5 + x + 1 \ \forall x \in A \cap B$

8. (1) 1 (2) 0 (3) 1

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Period of the function
$f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+....+|\cos n\pi x|}$ is

(a) $2n\pi$ (b) 1
(c) Not periodic (d) $2\pi/n$

Solution

(b) Given $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+....+|\cos n\pi x|}$

Since period of $x - [x] = 1$

Period of $|\cos \pi x| = \dfrac{\pi}{\pi} = 1$

Period of $|\cos 2\pi x| = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$

Period of $|\cos n\pi x| = \dfrac{\pi}{n\pi} = \dfrac{1}{n}$.

So, period of $f(x)$ will be L.C.M. of all periods so period is 1.

Hence (b) is the correct answer.

2. Let $f : R \to R$ defined by $f(x) = \dfrac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$, then
(a) f is one-one but not onto
(b) f is neither one-one nor onto
(c) f is many one but onto
(d) f is one-one and onto

Solution

(b) $e^{x^2} - e^{-x^2} > 0 \, \forall x$ and

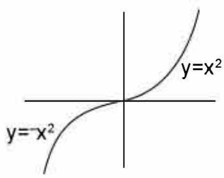
$$e^{x^2} - e^{-x^2} = 2\left[x^2 + \frac{x^6}{3!} + ...\right] > 0$$

$\therefore \quad f(x) > 0 \; \therefore f(x)$ is into $f'(x) = \dfrac{8x}{\left(e^{x^2} - e^{-x^2}\right)^2}$

$f'(x) = 0$, has real values of x. $\therefore f(x)$ is many one.

3. Function $f : R \to R, f(x) = x\,|\,x\,|$ is
[PET (Raj.)-91, 98; NDA-2004]
(a) one-one but not onto
(b) onto but not one-one
(c) one-one onto
(d) neither one-one nor onto

Solution



(c) Observing to the graph of this function we find that every line parallel to x-axis meets its graph only at a single point.

So it is one-one. Also range of $f = R$ = codomain, so it is also onto

4. Function $f : N \to N, f(x) = 2x + 3$ is
[IIT-73; MNR-83]
(a) one-one onto
(b) one-one into
(c) many one onto
(d) many one into

Solution

(b) f is one-one because $f(x_1) = f(x_2)$

$\Rightarrow \quad 2x_1 + 3 = 2x_2 + 3 \Rightarrow x_1 = x_2$

Further $f^{-1}(x) = \dfrac{x-3}{2} \notin N$ (domain) when $x = 1, 2, 3$ etc.

$\therefore \quad f$ is into which shows that f is one-one into.

5. Which of the following functions defined from R to R is onto
[PET (Raj.)-84, 85, 86]
(a) $f(x) = |\,x\,|$
(b) $f(x) = e^{-x}$
(c) $f(x) = x^3$
(d) $f(x) = \sin x$

Solution

(c) Range of $|\,x\,| \neq R$, range of $e^{-x} \neq R$ so that these two functions can not be onto. Also range of $\sin x = [-1, 1] \neq R$, so it is also not onto. The only alternative is that x^3 is onto.

6. Which of the following functions from $]-\pi/2, \pi/2[$ to R is a bijection ***[PET (Raj.)-86]***
(a) $\tan x$
(b) $\sin x$
(c) $\cos x$
(d) $e^x + e^{-x}$

Solution

(a) Values of $\sin x$ and $\cos x$ lie between -1 and 1, so these are not onto. Also $e^x + e^{-x} = 2 \cos hx$ assumes only positive values so it is not onto. Hence the remaining function $\tan x$ is a bijection.

7. Function $f : R \to R, f(x) = x^2 + x$ is
[PET (Raj.)-91, 99]
(a) one-one onto
(b) one-one into
(c) many one onto
(d) many one into

Solution

(d) $f(x_1) = f(x_2) \Rightarrow x_1^2 + x_1 = x_2^2 + x_2 \Rightarrow x_1^2 - x_2^2 + x_1 - x_2 = 0$

$\Rightarrow \quad (x_1 - x_2)(x_1 + x_2 + 1) = 0$

which shows that $f(x_1) = f(x_2)$ when $x_1 + x_2 + 1 = 0$. For example $-2 \neq 1$ but $f(-2) = 2 = f(1)$

$\therefore \quad f$ is many one.

Also $x^2 + x = y \Rightarrow x^2 + x - y = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1+4y}}{2}$$

$\Rightarrow \quad f^{-1}(y) = \dfrac{-1 \pm \sqrt{1+4y}}{2}$

which shows that $f^{-1}(-1) = \dfrac{-1 \pm \sqrt{-3}}{2} \notin$ domain $\therefore f$ is into

8. If $f(x) = (x+1)^2 - 1$ $(x \geq -1)$, then set $S = \{x \mid f(x) = f^{-1}(x)\}$ is equal to

[IIT Screening-95]

(a) ϕ
(b) $\{0, -1\}$
(c) $\{0, 1, -1\}$
(d) $\left\{0, -1, \dfrac{-3 \pm i\sqrt{3}}{2}\right\}$

Solution

(b) $\quad f^{-1}(x) = \sqrt{1+x} - 1 \qquad \therefore f(x) = f^{-1}(x)$

$\Rightarrow \quad (x+1)^2 - 1 = \sqrt{1+x} - 1 \Rightarrow (x+1)^4 = 1 + x$

$\Rightarrow \quad (x+1)\,[(x+1)^3 - 1] = 0 \Rightarrow x = -1$ or $x + 1 = 1, \omega, \omega^2$

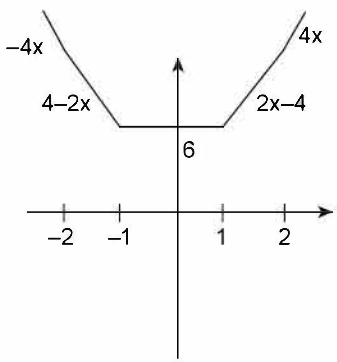
But $x \geq -1 \Rightarrow x \notin C \; \therefore S = \{-1, 0\}$

9. Value(s) of k for which ***[IIT-2009]*** $|x-1| + |x-2| + |x+1| + |x+2| = 4k$ has integer solution(s)

(a) 1 (b) 2
(c) 3 (d) 4

Solution

(b, c, d) Let
$f(x) = |x+2| + |x+1| + |x-1| + |x-2|$
Here for solution



$4k \geq 6 \Rightarrow k \geq \dfrac{3}{2}$

10. From the following, one-one function is

[PET (Raj.)-88]

(a) $f: R \to R, f(x) = x^2$
(b) $f: R \to R, f(x) = x + 1$
(c) $f: R \to R, f(x) = e^x + e^{-x}$
(d) None of these

Solution

(b) Range of $x^2 = [0, \infty] \neq R$
range of $(e^x + e^{-x})$ i.e., $\cos h\, x = [2, \infty) \neq R$
range of $(x + 1) = R$ (codomain). So it is onto.

11. If $f(x) = \dfrac{1-x}{1+x}(x \neq -1)$, then $f^{-1}(x)$ is equal to

(a) $-f(x)$ (b) $f(x)$
(c) $1/f(x)$ (d) $-1/f(x)$

Solution

(b) Let $f^{-1}(x) = y$, then $f(y) = x \Rightarrow \dfrac{1-y}{1+y} = x$

$\Rightarrow y = \dfrac{1-x}{1+x} = f(x)$

12. The inverse of the function $f(x) = |\sin x|$ will exist if its domain is ***[EAMCET-94]***

(a) $[0, \pi]$ (b) $[0, \pi/2]$
(c) $[-\pi/4, \pi/4]$ (d) None of these

Solution

(d) If $f: [0, \pi/2] \to [0, 1]$, then it will be a bijection. Only then f^{-1} will exist.

13. Function $y = \dfrac{x}{1+|x|}, x \in R, y \in R$ is

[IIIT (Hyderabad)-2001]

(a) one-one onto
(b) onto but not one-one
(c) one-one but not onto
(d) neither one-one nor onto

Solution

(c) Let $x_1, x_2 \in R$ and $x_1 \neq x_2$ then

$x_1 \neq x_2 \Rightarrow \dfrac{x_1}{1+|x_1|} \neq \dfrac{x_1}{1+|x_2|} \Rightarrow f(x_1) \neq f(x_2)$

$\Rightarrow \quad f$ is one-one Also $\dfrac{x}{1+|x|} < 1 \forall x \in R$

$\Rightarrow \quad$ range $\neq R \; \therefore$ It is not onto

14. If $f(x) = \sin x + \cos x$ and $g(x) = x^2 - 1$, then $g(f(x))$ is invertible in the domain

[IIT Screening-2004]

(a) $[0, \pi]$ (b) $[0, \pi/2]$
(c) $[-\pi/4, \pi/4]$ (d) $[-\pi/2, 0]$

Solution

(c) $g(f(x)) = g(\sin x + \cos x)$
$= (\sin x + \cos x)^2 - 1 = \sin 2x$

It is invertible in that domain where it is one-one. But $\sin 2x$ is one-one when $2x \in [-\pi/2, \pi/2]$ i.e., when $x \in [-\pi/4, \pi/4]$

15. The period of the function $f(x) = |\sin x + \cos x| + |\sin x - \cos x|$ is *[NDA-2005]*
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/2$ (d) π

Solution

(c) $f\left(\frac{\pi}{2}+x\right) = \left|\sin\left(\frac{\pi}{2}+x\right) + \cos\left(\frac{\pi}{2}+x\right)\right|$
$+\left|\sin\left(\frac{\pi}{2}+x\right) - \cos\left(\frac{\pi}{2}+x\right)\right|$
$= |\cos x - \sin x| + |\cos x + \sin x|$
$= |\sin x - \cos x| + |\sin x + \cos x| = f(x)$
$\therefore$ period of $f(x) = \pi/2$

16. If $f : R \to R, f(x) = (x - 1)(x - 2)$, then f is *[NDA-2005]*
(a) one-one but not onto
(b) onto but not one-one
(c) one-one onto
(d) neither one-one nor onto

Solution

(d) Obviously $f(x) = x^2 - 3x + 2$. It is a second degree polynomial function which is neither one-one nor onto R.

17. If $X = \{1, 2, 3, 4\}$, then total number of possible one-one onto functions from X to X for which $f(1) = 1, f(2) \neq 2, f(4) \neq 4$ will be *[Roorkee-2000]*
(a) 3 (b) 4
(c) 6 (d) 2

Solution

(a) $\because f(1) = 1, f(2) \neq 2, f(4) \neq 4$ and f is one-one onto, so $f(1) = 1$,
$f(2) = 3$ or $4, f(4) = 2$ or $3, f(3) =$ remaining fourth element
Hence f can be defined as follows: $f(1) = 1, f(2) = 3, f(4) = 2, f(3) = 4$
$f(1) = 1, f(2) = 4, f(4) = 2, f(3) = 3$
$f(1) = 1, f(2) = 4, f(4) = 3, f(3) = 2$

18. Function $f : R \to R, f(x) = 2^x + 2^{|x|}$ is
(a) one-one onto
(b) many one onto
(c) one-one into
(d) many one into

Solution

(c) $\because 2^x > 0, 2^{|x|} > 0 \; \forall \, x \in R$
$\therefore$ $R_f \neq R$ (codomain) $\Rightarrow f$ is into
Also $x_1 \neq x_2$
$\Rightarrow$ $2^{x_1} + 2^{|x_1|} \neq 2^{x_2} + 2^{|x_2|}$
$\Rightarrow$ $f(x_1) \neq f(x_2)$
$\Rightarrow$ f is one-one.

19. Let the function $f : R \to R$ be defined by $f(x) = 2x + \sin x, x \in R$. Then f is
(a) One-to-one and onto
(b) One-to-one but not onto
(c) Onto but not one-to-one
(d) Neither one-to-one nor onto

Solution

(a) $f'(x) = 2 + \cos x > 0$. So, $f(x)$ is strictly monotonic increasing. So $f(x)$ is one-to-one and onto.

20. $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$,

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$ Then $f - g$ is
(a) one-one and into
(b) neither one-one nor onto
(c) many one and onto
(d) one-one and onto

Solution

(d) $\phi(x) = f(x) - g(x) = \begin{cases} x, & x \in Q \\ -x, & x \notin Q \end{cases}$

Now to check one-one
Take any straight line parallel to x-axis which will intersect $\phi(x)$ only at one point $\Rightarrow$ one-one.
To check onto

As, $f(x) = \begin{cases} x, & x \in Q \\ -x, & x \notin Q \end{cases}$, which shows $y = x$ and $y = -x$ for rational and irrational values $\Rightarrow$ $y \in$ real numbers
$\therefore$ Range = Codomain $\Rightarrow$ onto Thus $f - g$ is one-one and onto.

21. Let $f : (2, 3) \to (0, 1)$ be defined by $f(x) = x - [x]$ then $f^{-1}(x)$ equals

[Orissa-JEE 2005]

(a) $x - 2$ (b) $x + 1$
(c) $x - 1$ (d) $x + 2$

Solution

(d) Given $f : (2, 3) \to (0, 1)$ and $f(x) = x - [x]$

$\therefore$ $f(x) = y = x - 2 \Rightarrow x = y + 2 = f^{-1}(y) \Rightarrow$ $f^{-1}(x) = x + 2$

22. Let $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined over the interval [0, 1]. The odd extension of $f(x)$ to interval [–1, 1] is

[UPSEAT-2000; MNR-94]

(a) $x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(b) $-x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(c) $-x^2 + x + \sin x - \cos x - \log(1 + |x|)$
(d) None of these

Solution

(b) Odd extension from [0, 1] to [–1, 1] means that function from given choices which satisfies the condition $f(-x) = -f(x)$

Now $|-x| = |x|$, $\cos(-x) = \cos x$

$\sin(-x) = -\sin x$, $(-x)^2 = x^2$.

So we observe that $f(-x) = x^2 - x - \sin x - \cos x + \log(1 + |x|)$

$= -$(function given in (b))

$\therefore$ (b) is the correct answer.

23. Let $f : R \to R$ be a function given by $f(x) = x^2 + 1$. Find $f^{-1}\{26\}$

(a) $\{-5, 5\}$ (b) $\{5, 5\}$
(c) $\{3, 2\}$ (d) $\{5, 6\}$

Solution

Recall that if $f : A \to B$ such that $y \in B$.
Then $f^{-1}\{y\} = \{x \in A : f(x) = y\}$
In other words, $f^{-1}\{y\}$ is the set of Pre-images of y.

Let $f^{-1}\{26\} = x$, Then $f(x) = 26$

$\Rightarrow$ $x^2 + 1 = 26$

$\Rightarrow$ $x = \pm 5$ $\therefore f^{-1}\{26\} = \{-5, 5\}$

24. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$ where

$Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$.

Show that f is invertible and its inverse is

[AIEEE-2008]

(a) $g(y) = \dfrac{y-3}{4}$ (b) $g(y) = \dfrac{3y+4}{3}$

(c) $g(y) = 4 + \dfrac{y+3}{4}$ (d) $g(y) = \dfrac{y+3}{4}$

Solution

(a) Let $f(x_1) = f(x_2)$, $x_1, x_2 \in N \Rightarrow 4x_1 + 3 = 4x_2 + 3 \Rightarrow x_1 = x_2$

Thus $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Hence the function is one-one. Let $y \in Y$ be a number of the form $y = 4k + 3$, for some $k \in N$, then $y = f(x) \Rightarrow 4k + 3 = 4x + 3 \Rightarrow x = k \in N$

Thus corresponding to any $y \in Y$ we have $x \in N$. The function then is onto.

The function, being both one-one and onto is invertible

$y = 4x + 3 \Rightarrow x = \dfrac{y-3}{4}$ $\therefore$ $f^{-1}(x) = \dfrac{x-3}{4}$

or $g(y) = \dfrac{y-3}{4}$ is the inverse of the function

25. If $f : R \to R$ is defined by $f(x) = |x|$, then

[Karnataka CET-2007]

(a) $f^{-1}(x) = -x$

(b) $f^{-1}(x) = \dfrac{1}{|x|}$

(c) the function $f^{-1}(x)$ does not exist

(d) $f^{-1}(x) = \dfrac{1}{|x|}$

Solution

(c) $f(x) = |x|$

$f(x) = x$ if $x \geq 0$

$= -x$ if $x < 0$

Therefore the function $f^{-1}(x)$ does not exist.

26. For real x, let $f(x) = x^3 + 5x + 1$, then

[AIEEE-2009]

(a) f is one-one but not onto R
(b) f is onto R but not one-one
(c) f is one-one and onto R
(d) f is neither one-one nor onto R

Solution

(c) Given, $f(x) = x^3 + 5x + 1$, Now

$f'(x) = 3x^2 + 5 > 0, \forall x \in R$

$\because$ $f(x)$ is strictly increasing function

$\therefore$ It is one-one.

Clearly, $f(x)$ is a continuous function and also increasing on R.

$\underset{x\to-\infty}{Lt} f(x) = -\infty$ and $\underset{x\to\infty}{Lt} f(x) = \infty$

$\therefore f(x)$ takes every value between $-\infty$ and ∞. Thus, $f(x)$ is onto function.

27. Let $f : R \to R$ be a function defined by $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$ Then

(a) f is both one-one and onto
(b) f is one-one but not onto
(c) f is onto but not one-one
(d) f is neither one-one nor onto

Solution

We have, if $x < 0$ $|x| = -x$

$\therefore \quad f(x) = \dfrac{e^{-x} - e^{-x}}{e^x + e^{-x}} = 0,$

$\therefore \quad f(x) = 0 \ \forall\, x < 0 \quad \therefore\ f(x)$ is not one-one.

Next if $x \ge 0$, $|x| = x$, $\therefore\ f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Let $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \Rightarrow y = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

$\therefore \quad e^{2x} = \dfrac{1+y}{1-y}$ for $x \ge 0$, $e^{2x} \ge 1$

$\therefore \quad \dfrac{1+y}{1-y} \ge 1 \Rightarrow \dfrac{2y}{1-y} \ge 0$

$\Rightarrow \quad y(y - 1) \le 0, y \ne 1 \Rightarrow 0 \le y < 1$

$\therefore \quad$ Range of $f(x) = [0, 1)$,

$\therefore \quad f(x)$ is not onto

28. The interval of values of α for which the function $f : R \to R$ defined by $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ is onto, is ***[IIT-1996]***

(a) [2, 14] (b) (2, 14)
(c) [2, 14) (d) None of these

Solution

(a) Let $y = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$\Rightarrow \quad (\alpha + 6x - 8x^2) = \alpha x^2 + 6x - 8$

$\Rightarrow \quad (\alpha + 8y)\, x^2 + 6(1 - y)x - (8 + \alpha y) = 0$

Since x is real

$\therefore \quad 36\,(1 - y^2) + 4\,(\alpha + 8y)\,(8 + 2y) \ge 0$

$\Rightarrow \quad 9(1 - 2y + y^2) + (8\alpha + \alpha^2 y + 64y + 8\,\alpha\, y^2 \ge 0$

$\Rightarrow \quad y^2(9 + 8\alpha) + y(\alpha^2 + 46) + (9 + 8\,\alpha) \ge 0$ ....(1)

(1) will hold for each $y \in R$ if $9 + 8\,\alpha > 0$ and $(46 + \alpha^2)^2 - 4(9 + 8\alpha)^2 \le 0$ (Disc. ≤ 0)

$\Rightarrow \quad \alpha > -\dfrac{9}{8}$ and $[46 + \alpha^2 - 2(9 + 8\alpha)]\,[46 + \alpha^2 + 2(9 + 8\alpha)] \le 0$

$\Rightarrow \quad \alpha > -\dfrac{9}{8}$ and $[\alpha^2 - 16\alpha + 28]\,[\alpha^2 + 16\alpha + 64] \le 0$

$\Rightarrow \quad \alpha > -\dfrac{9}{8}$ and $(\alpha - 2)\,(\alpha - 14)\,(\alpha + 8)^2 \le 0$

$\Rightarrow \quad \alpha > -\dfrac{9}{8}$ and $(\alpha - 2)\,(\alpha - 14) \le 0$

$(\because\ (\alpha + 8)^2 \ge 0)$

$\Rightarrow \quad \alpha > -\dfrac{9}{8}$ and $2 \le \alpha \le 14$

$\therefore \quad 2 \le \alpha \le 14$

Hence $f(x)$ will be onto if $2 \le \alpha \le 14$

$\therefore \quad$ required interval is [2,14]

29. If $f : [0, \infty) \to [0, \infty)$ and $f(x) = \dfrac{x}{1+x}$, then f is ***[IIT (Screening)-2003]***

(a) one-one and onto
(b) one-one but not onto
(c) onto but not one-one
(d) neither one-one nor onto

Solution

(b) $\quad f'(x) = \dfrac{1}{(1+x)^2} > 0 \ \forall\, x \in [0, \infty]$

$\Rightarrow \quad f(x)$ is increasing

$\Rightarrow \quad f(x)$ is one-one

Also $R_f = [0, 1] \ne [0, \infty) \quad \Rightarrow f$ is not onto

30. Let X, Y be two sets and $f : X \to Y$. If $\{f(c) = y, c \subset X. y \subset Y\}$ and $\{f^{-1}(d) = x; d \subset Y, x \subset X\}$, then correct statement is ***[IIT (Screening)-2005]***

(a) $f^{-1}(f(a)) = a$
(b) $f^{-1}(f(a)) = a, a \subset X$
(c) $f(f^{-1}(b)) = b$
(d) $f(f^{-1}(b)) = b, b \subset Y$

Solution

$\because$ (b) $f^{-1}(d) = x$

$\Rightarrow$ $f(x) = d$

so, if $a \subset x$, then $f(a) \subset f(x)$

$\Rightarrow$ $f(a) \subset d$

$\Rightarrow$ $f^{-1}(f(a)) = a$

[Note: If $a \not\subset x$ then this is not necessarily true]

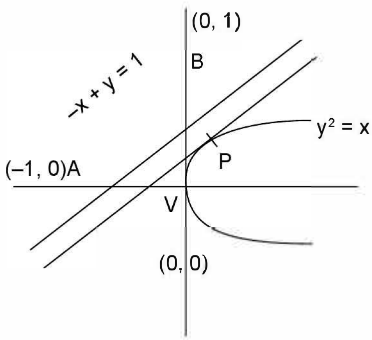
31. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is: ***[AIEEE-2009]***

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{2\sqrt{3}}{8}$

(c) $\frac{3\sqrt{2}}{5}$ (d) $\frac{\sqrt{3}}{4}$

Solution

(a) **Step-1:** For the shortest distance between the line and the curve. We have to find a point on the curve at which tangent drawn is parallel to the given line and then perpendicular distance of this point from the line will be required shortest distance.



Step-2: Given $x - y + 1 = 0$ .....(1)

$x = y^2$ ....(2)

Differentiate that,

$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y} =$ Slope of given line (1)

$\frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}$

$\Rightarrow y = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow (x, y) = \left(\frac{1}{4}, \frac{1}{2}\right)$. The shortest distance is

$\frac{\left(\frac{1}{4} - \frac{1}{2} + 1\right)}{\sqrt{1+1}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$

32. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is: ***[AIEEE-2010]***

(a) $y=1$ (b) $y = 2$

(c) $y = 3$ (d) $y = 0$

Solution

Step-1: If tangent is parallel to the x-axis, then

$\frac{dy}{dx} = 0$

$y = x + \frac{4}{x^2}$

$\frac{dy}{dx} = 1 - \frac{8}{x^3} = 0$

$\Rightarrow x^3 = 8$

$\Rightarrow x = 2$ and $y = 2 + \frac{4}{2^2} = 3$

(2, 3) is point of contact

Thus $y = 3$ is tangent.

Hence correct option is (c)

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. Set A has 3 elements and set B has 4 elements, the number of injections that can be defined from A to B is ***[UPSEAT-2001]***

(a) 144 (b) 12

(c) 24 (d) 64

2. The number of bijective functions from set A to itself when A contains 106 elements is

(a) 106 (b) $(106)^2$

(c) 106! (d) 2^{106}

3. A function f from the set of natural numbers to integer defined by
$$f(n) = \begin{cases} \dfrac{n-1}{2}, & \text{when } n \text{ is odd} \\ -n/2 & \text{when } n \text{ is even} \end{cases} \text{ is}$$
[ICS-2001]
[AIEEE-2003]
(a) neither one-one nor onto
(b) one-one but not onto
(c) onto but not one-one
(d) one-one and onto both

4. $f: R \to R$ given by $f(x) = 3 - 2 \sin x$ is
(a) 1-1 (b) onto
(c) bijective (d) None of these

5. $f: R \to R, f(x) = (x-1)(x-2)(x-3)$ is ***[Roorkee-1999]***
(a) one-one but not onto
(b) onto but not one-one
(c) both one-one and onto
(d) neither one-one nor onto

6. If $f: [0, \infty) \to [0, \infty)$ and $f(x) = \dfrac{x}{1+x}$, then f is ***[IIT Screening-2003]***
(a) one-one and onto
(b) one-one but not onto
(c) onto but not one-one
(d) neither one-one nor onto

7. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$, then the number of onto functions from E to F is ***[IIT Screening-2001]***
(a) 14 (b) 16
(c) 12 (d) 8

8. Let $f: R \to R$ be defined as $f(x) = x\,|\,x\,|$. Which one of the following is correct?
(a) f is only onto
(b) f is only one-one
(c) f is neither onto nor one-one
(d) f is one-one and onto

9. Let $f: N \to N$ defined by $f(x) = x^2 + x + 1, x \in N$, then f is
[AMU-2000]
(a) one-one onto
(b) Many one onto
(c) one-one but not onto
(d) None of these

10. If the functions $f(x)$ and $g(x)$ are defined on $R \to R$ such that
$$f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases},$$
$$g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases},$$
then $(f - g)(x)$ is
(a) one-one and onto
(b) neither one-one nor onto
(c) one-one but not onto
(d) onto but not one-one

11. Let $f: (-1, 1) \to B$, be a function defined by $f(x) = \tan^{-1}\dfrac{2x}{1-x^2}$,
Then f is both one-one and onto when B is the interval
(a) $[0, \pi/2)$ (b) $(0, \pi/2)$
(c) $(-\pi/2, \pi/2)$ (d) $[-\pi/2, \pi/2]$

12. If a function $f(x)$ is defined for $x \in [0, 1]$, then the function $f(2x + 3)$ is defined for
(a) $[3/2, 1]$ (b) $[-3/2, -1]$
(c) $[1, -3/2]$ (d) $[-1, 3/2]$

13. A condition for a function $y = f(x)$ to have inverse is that it should be
(a) defined for all x
(b) continuous everywhere
(c) an even function
(d) strictly monotonic and continuous in the domain

14. The inverse of function $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$
(a) $y = \dfrac{1}{2}\log_{10}\left(\dfrac{1+x}{1-x}\right)$ (b) $\log_{10}\dfrac{1-x}{1+x}$
(c) $2\log\dfrac{1+x}{1-x}$ (d) None of these

15. The inverse of the function $f(x) = \{1 - (x-3)^4\}^{1/7}$ is
(a) $(1 - x^4)^{1/7} + 3$ (b) $(1 - x^7)^{1/4} + 3$
(c) $(1 - x)^{4/7} - 3$ (d) None of these

16. Let $f(x) = (x + 1)^2 - 1, (x \geq -1)$, then set $s = \langle x : f(x) = f^{-1}(x) \rangle$ is
(a) Blank
(b) $< 0, -1 >$
(c) $< 0, 1, -1 >$
(d) $\left\langle 0, -1, \dfrac{-3 + i\sqrt{3}}{2}, \dfrac{-3 - i\sqrt{3}}{2} \right\rangle$

17. If $f(x) = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$, then $f^{-1}(x)$ is

(a) $\dfrac{1}{2}\log_2 \dfrac{x}{1-x}$ (b) $\dfrac{1}{2}\log_2 \dfrac{1+x}{1-x}$

c) $\dfrac{1}{2}\log_2 \dfrac{1+x}{x}$ (d) $\dfrac{1}{2}\log_2 \dfrac{2+x}{2-x}$

18. Which one of the following function is one-to-one? ***[Kerala PET-2008]***

(a) $f(x) = \sin x$, $x \in [-\pi, \pi)$

(b) $f(x) = \sin x$, $x \in \left[\dfrac{-3\pi}{2}, \dfrac{-\pi}{4}\right]$

(c) $f(x) = \cos x$, $x \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$

(d) $f(x) = \cos x$, $x \in [\pi, 2\pi)$

19. If $f(x) = 2x + 1$ and $g(x) = \dfrac{x-1}{2}$ for all real x, then $(\text{fog})^{-1}\left(\dfrac{1}{x}\right)$ is equal to

[Kerala PET-2008]

(a) x (b) $1/x$

(c) $-x$ (d) $-1/x$

20. If $f : R \to R$ is defined by $f(x) = x^3$ then $f^{-1}(8)$ is equal to

[Karnataka CET-2008]

(a) $\{2, -2\}$

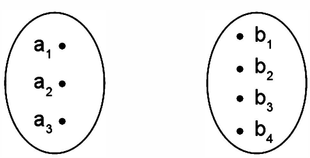
(b) $\{2, 2\}$

(c) $\{2\}$

(d) $\{2, 2\omega, 2\omega^2\}$

SOLUTIONS

1. (c) For injection, b_1, b_2, b_3, b_4 can have only one or no preimage.



No. of injections = 4 × 3 × 2 = 24 ways

2. (c) No. of bijection = 106 × 105 × 104 .... 3 × 2 × 1 = (106)!

3. (d) $f\,N \to I$

$f(n) = \dfrac{n-1}{2}$, when n is odd

Here $n \in \{1, 3, 5, ...\}$

$\therefore f(x)$ can have values

$\{0, 1, 2, 3, ...\}$

and $f(x) = \dfrac{-n}{2}$, when n is even

Here $n \in \{2, 4, 6, ...\}$

$f(x)$ can have values $\{-1, -2, -3, ...\}$

$\therefore$ Range will be $\{......, -3, -2, -1, 0, 1, 2, 3, ...\}$

$\therefore$ function is onto, and one-one.

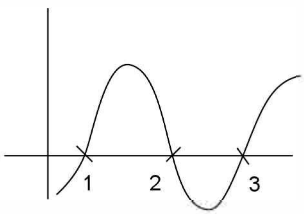
4. (d) $f(x) = 3 - 2\sin x$, Here $-1 \le \sin x \le 1$

Range $\in [1, 5] \ne$ codomain

Also $f'(x) = -2\cos x$ is positive as well as negative,

$f(x)$ is many one. Also function is into.

5. (b)



Graph of $f(x) = (x-1)(x-2)(x-3)$ by graph function is many are (a line | | to x-axis cuts graph more than once)

Also range = $(-\infty, \infty)$ = codomain

$\therefore$ it is onto.

6. (b) $f : [0, \infty) \to (0, \infty)$

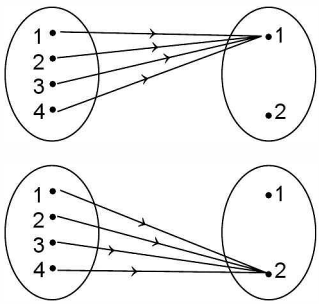
$$f(x) = \frac{x}{1+x}, f'(x) = \frac{1}{(1+x)^2} > 0$$

$\because$ $f(x)$ is monotonic, $f(x)$ is one-one function.

For range $f(0) = 0$;

$$f(\infty) = \lim_{x\to\infty}\frac{x}{1+x} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}$$

7. (a) Here all the function will be onto except two cases:



$\therefore$ Required case = $2^4 - 2 = 14$

8. (d) Here $f(x) = x|x|$

$= x^2; x \geq 0$

$\quad -x^2; x < 0$

$\therefore f(x)$ is monotonic, it is one-one function,

Also range = codomain, fv onto function.

9. (c) $f: N \to N; f(x) = x^2 + x + 1$

Here domain $\in \{1,2,3......\}$

Range $\in \{f(1), f(2), f(3), ...\}$

$\in \{3, 7, 13, ...\}$

$\neq$ codomain

$\therefore$ function is one one and into.

10. (a) Here $f - g(x) = \begin{cases} -x; & x \in Q \\ x; & x \notin Q \end{cases}$

Which is one-one and onto.

11. (d) $f: (-1,1) \to R$

If $x \in (-1,1)$ $f(x) = \tan^{-1}\frac{2x}{1-x^2}$

or $f(x) = 2\tan^{-1}x$

$\therefore f(x)$ is one one and onto if

$2\tan^{-1}x \in \left(2\tan^{-1}(-1), 2\tan^{-1}1\right)$

$\therefore f(x) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

12. (b) $\because f(2x+3)$ is defined if

$0 \leq 2x + 3 \leq 1$

$\frac{-3}{2} \leq x \leq -1$

13. (c) By definition.

14. (a) For inverse, replace x by y and y by x in

$$y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}} \quad \therefore \quad x = \frac{10^y - 10^{-y}}{10^y + 10^{-y}}$$

$$x = \frac{10^{2y} - 1}{10^{2y} + 1} \Rightarrow \frac{x+1}{x-1} = \frac{10^{2y}}{-1}$$

(By applying componendo and devidendo)

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

15. (b) Given function is $y = f(x)$

$y = (1-(x-3)^4)^{1/7}$

For inverse, replace y by x and x by y

$x = \left(1-(y-3)^4\right)^{1/7}$

$x^7 = 1-(y-3)^4$

$y - 3 = \pm\sqrt[4]{1-x^7}$

$y = 3 \pm \sqrt[4]{1-x^7}$

16. (b) $f(x) = f^{-1}(x)$ then $y = x$

Here $y = (x+1)^2 - 1$

$\therefore x = x^2 + 2x + 1 - 1$

$x = 0, -1$

$[\because f(x) = f^{-1}(x)$ is possible when $y = x]$

17. (b) $y = \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$ or $y = \frac{2^{2x} - 1}{2^{2x} + 1}$

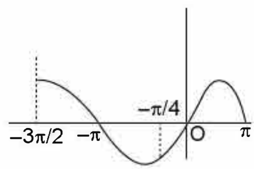
For inverse, $x = \frac{2^{2y} - 1}{2^{2y} + 1}$

$$\frac{x+1}{x-1} = \frac{2^{2y}}{-1}$$

$$y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

18. (c) For $f(x) = \sin x$

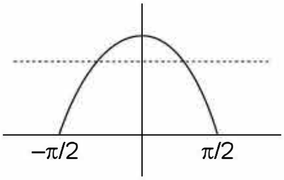
Graph is



In $[-\pi, \pi)$, function is one one and $\left[\frac{-3\pi}{2}, \frac{-\pi}{4}\right]$

where as for $f(x) = \cos x$ in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

it is many one



NOTE

If a line parallel to x axis cuts the graph more than once it is many one.

19. (b) Here

$f(x) = 2x + 1$; $g(x) = \frac{x-1}{2}$

then

$$(fog)x = f(g(x))$$

$$= f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right) + 1$$

$$= x$$

$$\therefore \ fog^{-1}x = x \ \text{ or } \ fog^{-1}\left(\frac{1}{x}\right) = \frac{1}{x}$$

NOTE

Inverse of $y = x$ is the function itself.

20. (c) $f(x) = x^3$; $f^{-1}(8)$, means value of x when $y = 8$ in $y = f(x)$

$8 = x^3 \Rightarrow x = 2$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. Let $f : R \to R$ be a function defined by $f(x) = \frac{x-m}{x-n}$ where $m \neq n$. Then **[UPSEAT-2001]**

(a) f is one-one onto (b) f is one-one into
(c) f is many one onto (d) f is many one into

2. The domain and range are same for
(a) a constant function
(b) Identity function
(c) an injective map
(d) A surjective map

3. The function $f : N \to N$, where N is the set of natural numbers, defined by $f(x) = 3x + 5$ is:
(a) Injective
(b) Surjective
(c) Injective but not surjective
(d) Many one

4. Function $f : R \to R, f(x) = x^2 + x$, is **[RPET-1999]**

(a) one-one onto (b) one-one into
(c) many one onto (d) many one into

5. On the set Z of all integers define $f : Z \to Z$ as follows

$$f(x) = \begin{cases} x/2, & \text{if } x \text{ is even} \\ 0, & \text{if } x \text{ is odd} \end{cases}, \text{ then } f \text{ is}$$

(a) onto but not one-one
(b) one-one and onto
(c) one-one but not onto
(d) into

6. Which one of the following is a bijective function on the set of real numbers **[Kerala (Engg.)-2002]**

(a) $2x - 5$ (b) $|x|$
(c) x^2 (d) $x^2 + 1$

7. If $f(x)$ is periodic function with period T then the function $f(ax + b)$ where $a > 0$, is periodic with period **[AMU-2000]**

(a) T/b (b) aT
(c) bT (d) T/a

8. If for two functions g and f, gof is both injective and surjective, then which of the following is true **[Kurukshetra CEE-1998]**

(a) g and f should be injective and surjective.
(b) g should be injective and surjective.
(c) f should be injective and surjective.
(d) None of them may be surjective and injective.

9. Let $f: R \to R$ be defined by $f(x) = 3x - 4$. Then $f^{-1}(x)$ is equal to ***[SCRA-1996]***

(a) $\frac{x+4}{3}$ (b) $\frac{x}{3} - 4$
(c) $3x + 4$ (d) None of these

10. If $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself, then the value of α is ***[Screening-1992]***

(a) -2 (b) -1
(c) 0 (d) 2

11. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is ***[IIT-1999]***

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$

(b) $\frac{1}{2}(1 + \sqrt{1 + 4\log_2 x})$

(c) $\frac{1}{2}(1 - \sqrt{1 + 4\log_2 x})$

(d) None of these

12. If $f(x) = \frac{3x+2}{5x-3}\left(x \neq \frac{3}{5}\right)$, then which one of the following is correct?

(a) $f^{-1}(x) = f(x)$
(b) $f^{-1}(x) = -f(x)$
(c) $(fof)(x) = -x$
(d) $f^{-1}(x) = -\frac{1}{19}f(x)$

13. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ is equal to ***[IIT Screening-2001]***

(a) $\frac{x + \sqrt{x^2 - 4}}{2}$ (b) $\frac{x + \sqrt{x^2 + 4}}{2}$

(c) $\frac{x - \sqrt{x^2 - 4}}{2}$ (d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 13 minutes.
3. The worksheet consists of 13 questions. The maximum marks are 39.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. Let $f: (-\infty, 2] \to (-\infty, 2]$ be a function defined by $f(x) = 4x - x^2$. Then $f^{-1}(x)$ is

(a) $2-\sqrt{4-x}$ (b) $2+\sqrt{4-x}$
(c) $\sqrt{4-x}$ (d) None of these

2. If $f(x) = x^2 + 1$, then $f^{-1}(17)$ and $f^{-1}(-3)$ will be **[UPSEAT-03]**

(a) 4, 1 (b) 4, 0
(c) 3, 2 (d) None of these

3. If $f(x) = 3x + 10$, $g(x) = x^2 - 1$, then $(fog)^{-1}$ is equal to **[UPSEAT-2001]**

(a) $\left(\frac{x-7}{3}\right)^{1/2}$ (b) $\left(\frac{x+7}{3}\right)^{1/2}$
(c) $\left(\frac{x-3}{7}\right)^{1/2}$ (d) $\left(\frac{x+3}{7}\right)^{1/2}$

4. Which of the following function is inverse function? **[AMU-2000]**

(a) $f(x) = \frac{1}{x-1}$
(b) $f(x) = x^2$, for all x
(c) $f(x) = x^2, x \geq 0$
(d) $f(x) = x^2, x \leq 0$

5. If $f: R \to R$ is defined by $f(x) = x^2 - 6x - 14$, then $f^{-1}(2)$ equal to **[Kerala CEE-2004]**

(a) $\{2, 8\}$ (b) $\{-2, 8\}$
(c) $\{-2, -8\}$ (d) $\{2, -8\}$

6. If $f(x) = \frac{1-x}{1+x}, (x \neq -1)$, then $f^{-1}(x)$ equals to **[Kerala CEE-2004]**

(a) $f(x)$ (b) $1/f(x)$
(c) $-f(x)$ (d) $-1/f(x)$

7. Let the function f be defined by $f(x) = \frac{2x+1}{1-3x}$, then $f^{-1}(x)$ is **[Kerala CEE-02]**

(a) $\frac{x-1}{3x+2}$ (b) $\frac{3x+2}{x-1}$
(c) $\frac{x+1}{3x-2}$ (d) $\frac{2x+1}{1-3x}$

8. Which one of the following is a bijective function on the set of real numbers? **[Kerala CEE-2002]**

(a) $2x - 5$ (b) $|x|$
(c) x^2 (d) $x^2 + 1$

9. If R denotes the set of all real numbers, then the function $f: R \to R$ defined $f(x) = |x|$ is **[Karnataka CET-2004]**

(a) one-one only
(b) onto only
(c) both one-one and onto
(d) neither one-one not onto

10. If $f(x) = \frac{1-x}{1+x} (x \neq -1)$, then $f^{-1}(x)$ equals to **[MPPET-2005]**

(a) $f(x)$ (b) $\frac{1}{f(x)}$
(c) $-f(x)$ (d) $-\frac{1}{f(x)}$

11. What is the inverse of the function $y = 5^{\log x}$? **[NDA-2008]**

(a) $x = 5^{1/\log y}$ (b) $x = y^{1/\log 5}$
(c) $x = 5^{\log y}$ (d) $x = y^{\log 5}$

12. Which one of the following is correct? The function $f: A \to R$ where $A = \left\{x \in R, -\frac{\pi}{2} < x < \frac{\pi}{2}\right\}$ defined by $f(x) = \tan x$ is **[NDA-2008]**

(a) Injective (b) Not injective
(c) Bijective (d) Not bijective

13. Which one of the following real valued functions is never zero? **[NDA-2008]**

(a) Polynomial function
(b) Trigonometric function
(c) Logarithmic function
(d) Exponential function

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (a) $f(x) = 4x - x^2$

Here $y = 4x - x^2$

For inverse $x = 4y - y^2$

$y^2 + 4y = -x$

$(y-2)^2 = 4 - x$

$y = 2 + \sqrt{4-x}$

or $y = 2 - \sqrt{4-x}$

$\because$ Range is $(-\infty, 2]$

$\therefore$ $y = 2 - \sqrt{4-x}$

2. (d) $f(x) = x^2 + 0$

For $f^{-1}(17); y = 17; f^{-1}(17) = x$

$17 = x^2 + 1 \Rightarrow x = 4, -4$

Similarly $-3 = x^2 + 1 \Rightarrow x^2 = -4$

No ϕ values of x is possible

$\therefore$ d is answer.

4. (a) Inverse of a function exist if it is one one onto

Here codomain is R.

$\because$ (b), (c), (d) has range as $[0, \infty)$,

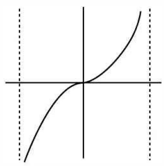
$\therefore$ it is an into function

NOTE

Whenever codomain is not given, it is considered as R

12. (c) Graph of $f(x) = \tan x$

Here $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$\therefore$ Function is one one onto i.e., bijection.

LECTURE

5

Test Your Skills

ASSERTION/REASONING

ASSERTION AND REASONING TYPE QUESTIONS

Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct.

(a) **Assertion** is True, **Reason** is True and **Reason** is a correct explanation for **Assertion.**
(b) **Assertion** is True, **Reason** is True and **Reason** is not a correct explanation for **Assertion.**
(c) **Assertion** is True and **Reason** is False.
(d) **Assertion** is False and **Reason** is True.

1. **Assertion (A):** If $f(x) = \ln x^2$ and $g(x) = 2\ln x$, then $f(x) = g(x)$.
   Reason (R): For $x < 0$, $g(x)$ is not defined,

2. **Assertion (A):** The function $f(x) = |x|$ is not one-one
   Reason (R): The negative real number are not the image of any real numbers

3. **Assertion (A):** If $f : R \to R$ and $g : R \to R$ be two mappings such that $f(x) = \sin x$ and $g(x) = x^2$, then $fog \neq gof$.
   Reason (R): $(fog)x = f(x)\, g(x) = (gof)x$.

4. **Assertion (A):** A function $y = f(x)$ is defined by $x^2 - \text{arc}\cos y = \pi$, then domain of $f(x)$ is R.
   Reason (R): $\cos^{-1} y \in [0, \pi]$

5. **Assertion (A):** If $f(x)$ is odd function and $g(x)$ is even function, then $f(x) + g(x)$ is neither even nor odd.
   Reason (R): Odd function is symmetrical in opposite quadrants and even function is symmetrical about the y-axis.

6. **Assertion (A):** Every even function $y = f(x)$ are not one-one, $\forall\, x \in D_f$
   Reason (R): Even function is symmetrical about they y-axis.

7. **Assertion (A):** A function $f : R \to R$ be defined by $f(x) = x - [x]$ (where $[x]$ is greatest integer $\leq x$) for all $x \in R$. f is not invertiable
   Reason (R): $f(x)$ is periodic function

8. **Assertion (A):** The function $f(x) = x^2 - x + 1$, $x \geq \frac{1}{2}$ and $g(x) = \frac{1}{2}+\left(\sqrt{x-\frac{3}{4}}\right)$, then the number of solutions of the equation $f(x) = g(x)$ is two.
   Reason (R): $f(x)$ and $g(x)$ are mutually inverstion.

9. **Assertion (A):** $f(x) = \sin x + \cos ax$ is a periodic function.
   Reason (R): a is rational number.

10. **Assertion (A):** The least period of the function, $f(x) = \cos(\cos x) + \cos(\sin x) + \sin 4x$ is π.
    Reason (R): $f(x + \pi) = f(x)$.

11. **Assertion (A):** If $f(x+y)+f(x-y)=2f(x)$. $f(y)\ \forall\ x, y \in R$ and $f(0) \neq 0$, then $f(x)$ is an even function.

Reason (R): If $f(-x)=f(x)$, then $f(x)$ is an even function.

12. **Assertion (A):** The equation $x^4 = (\lambda x - 1)^2$ has atmost two real solutions (is $\lambda > 0$)

Reason (R): Curves $f(x) = x^4$ and $g(x) = (\lambda x - 1)^2$ cut atmost two points.

13. **Assertion (A):** The domains of $f(x) = \sqrt{\cos(\sin x)}$ and $g(x) = \sqrt{\sin(\cos x)}$ are same.

Reason (R): $\because -1 \leq \cos(\sin x) \leq 1$ and $-1 \leq \sin(\cos x) \leq 1$

14. **Assertion (A):** If $f(x) = x^5 - 16x + 2$, then $f(x) = 0$ has only one root in the interval $[-1, 1]$.

Reason (R): $f(-1)$ and $f(1)$ are of opposite sign.

15. **Assertion (A):** The domain of the function $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$ is $[-1, 1]$

Reason (R): $\sin^{-1} x$ and $\cos^{-1} x$ is defined in $|x| \leq 1$ and $\tan^{-1} x$ is defined for all x.

16. **Assertion (A):** The period of $f(x) = \sin 3x \cos [3x] - \cos 3x \sin [3x]$ is $\frac{1}{3}$ where [.] denotes the greatest integer function $\leq x$.

Reason (R): The period of $\{x\}$ is 1, where $\{x\}$ denotes the fractional part function of x.

17. **Assertion (A):** The period of $f(x) = \sin 2x \cos [2x] - \cos 2x \sin [2x]$ is $\frac{1}{2}$

Reason (R): The period of $x - [x]$ is 1

18. **Assertion (A):** If $f(x) = |x-1| + |x-2| + |x-3|$ where $2 < x < 3$ is an identity function.

Reason (R): $f: A \to A$ defined by $f(x) = x$ is an identity function.

19. **Assertion (A):** $f: R \to R$ defined by $f(x) = \sin x$ is a bijection

Reason (R): If f is both one and onto it is bijection

20. **Assertion (A):** $f: R \to R$ is a function defined by $f(x) = \dfrac{2x+1}{3}$. Then $f^{-1}(x) = \dfrac{3x-1}{2}$

Reason (R): $f(x)$ is not a bijection

21. **Assertion (A):** If f is even function, g is odd function then $\frac{f}{g}$, $(g \neq 0)$ is an odd function.

Reason (R): If $f(-x) = -f(x)$ for every x of its domain, then $f(x)$ is called an odd function and if $f(-x) = f(x)$ for every x of its domain, then $f(x)$ is called an even function.

22. **Assertion (A):** Let $A = \{x/-1 \leq x \leq 1\}$, then $f: A \to A$ is defined by $f(x) = \sin \pi x,\ \forall\ x \in A$ is onto but not one-one.

Reason (R): If range set is equal to the codomain set then the function is onto.

23. Let $f(x) = (x+1)^2 - 1, x \geq -1$. ***[AIEEE-2009]***

Assertion: The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$

Reason: f is a bijection.

ASSERTION/REASONING: SOLUTIONS

1. (d) $\because$ $\ln x^2 = 2 \ln |x| = 2 \ln x$ and $g(x)$ is defined only when $x > 0$

2. (c) The function $f(x) = |x|$ is many one $\because$ $f(-x) = f(x)$ i.e., not one-one and the image of $-x$ is x.

3. (c) $\because$ (fog) $x = f(g(x)) = f(x^2) = \sin x^2$ and (gof) $x = g(f(x)) = g(\sin x) = \sin^2 x \Rightarrow \text{fog} \neq \text{gof}$

4. (d) $\because$ $x^2 - \text{arc} \cos y = \pi \Rightarrow \cos^{-1} y = (x^2 - \pi)$

$\because$ $0 \leq \cos^{-1} y \leq \pi$

$\Rightarrow$ $0 \leq x^2 - \pi \leq \pi$

$\Rightarrow$ $\pi \leq x^2 \leq 2\pi$

$\therefore$ $x \in \left[-\sqrt{2\pi}, -\sqrt{\pi}\right] \cup \left[\sqrt{\pi}, \sqrt{2\pi}\right]$

5. (b) $\because f(x)$ is odd $\Rightarrow f(-x) = -f(x)$ and $g(x)$ is even $\Rightarrow g(-x) = g(x)$

let $F(x) = f(x) + g(x)$

$\therefore \quad F(-x) = f(-x) + g(-x) = -f(x) + g(x)$

$\neq \pm F(x) \quad \therefore F(x)$ is neither even nor odd.

6. (a) Since every even function is symmetrical about the y-axis.
$\therefore$ any line parallel to x-axis cuts the graph move than one point, then $f(x)$ is not one-one $\forall\, x \in D_f$

7. (a) $\because f(x)$ is periodic function $\Rightarrow f(x)$ is many one
Hence, $f(x)$ is to invertible

8. (d) Let $y = x^2 - x + 1 \Rightarrow x^2 - x + 1 - y = 0$

$$\therefore \quad x = \frac{1 \pm \sqrt{1 - 4.1.(1-y)}}{2} = \frac{1}{2} \pm \sqrt{\left(y - \frac{3}{4}\right)}$$

$$= \frac{1}{2} + \sqrt{\left(y - \frac{3}{4}\right)}, \left(\because x \geq \frac{1}{2}\right) = g(y)$$

$\therefore \quad y = g^{-1}(x)$

$\Rightarrow \quad f(x) = g^{-1}(x)$

Hence, $f(x)$ and $g(x)$ are mutually inversion.

$\Rightarrow$ The graph of the original and inversion functions can intersect only on the straight line $y = x$

$\therefore \quad x = f(x) \Rightarrow x = x^2 - x + 1$

$\Rightarrow \quad x^2 - 2x + 1 = 0$

$\Rightarrow \quad (x - 1)^2 = 0$

$\therefore \quad x = 1$

9. (a) $\because$ Period of $\sin x$ is $\dfrac{2\pi}{1}$ and period of cos

$ax = \dfrac{2\pi}{a}$

Hence, period of $f(x)$= LCM of $\left\{\dfrac{2\pi}{1}, \dfrac{2\pi}{a}\right\}$

$$= \frac{\text{LCM of } \{2\pi, 2\pi\}}{\text{HCF of } \{1, a\}} = \frac{2\pi}{k}$$

when k is HCF of 1 and a

$\Rightarrow \quad \dfrac{1}{k}$ = integer = q (say) ($\neq 0$) and

$\dfrac{a}{k}$ = integer = p (say)

$$\because \quad \frac{a/k}{1/k} = \frac{p}{q} \Rightarrow a = \frac{p}{q}$$

$\Rightarrow \quad a$ is rational

10. (d) Let $f(x)$ be periodic with period λ, $\lambda \neq 0$, $\lambda > 0$

$\therefore f(x + \lambda) = f(x)$

$\Rightarrow \cos(\cos(x + \lambda)) + \cos(\sin(x + \lambda)) + \sin(4(x + \lambda))$

$= \cos(\cos x) + \cos(\sin x) + \sin 4x$

Put $x = 0$,

$\cos(\cos\lambda) + \cos(\sin\lambda) + \sin(4\lambda) = \cos(1) + \cos(0) + \sin 0$

$= \cos(\sin \pi/2) + \cos(\cos \pi/2) + \sin(2\pi)$

$\Rightarrow \quad \lambda = \pi/2$

11. (b) Given $f(x + y) + f(x - y) = 2 f(x) f(y)$ ....(i)

Replacing x by y by x in equation (i), then

$f(y + x) + f(y - x) = 2f(y) f(x)$ ....(ii)

$\therefore$ from equation (i) and (ii), we get $f(y - x) = f(x - y)$

Putting $y = 2x$, the $f(x) = f(-x)$

Hence, $f(x)$ is an even function.

12. (d) $\because x^4 = (\lambda x - 1)^2$

$x^2 = \pm(\lambda x - 1)$

$\Rightarrow \quad x^2 = \lambda x - 1 \therefore x^2 - \lambda x + 1 = 0$

$$\Rightarrow \quad x = \frac{\lambda \pm \sqrt{(\lambda^2 - 4)}}{2}$$

$\therefore \quad \lambda^2 - 4 \geq 0$

$\Rightarrow \quad \lambda \in (-\infty, -2) \cup [2, \infty)$

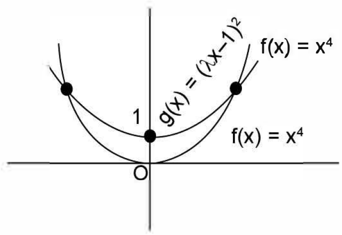
but $\lambda > 0 \therefore \lambda \in [2, \infty)$

and $x^2 = -(\lambda x - 1)$

$x^2 + \lambda x - 1 = 0$

$$x = \frac{-\lambda \pm \sqrt{(\lambda^2 + 4)}}{2}$$

$\lambda > 0 \Rightarrow$ infinite solutions cut at two points.



13. (d) $\cos(\sin x) \geq 0$

$$\Rightarrow \quad 2n\pi - \frac{\pi}{2} \leq \sin(x) \leq 2n\pi + \frac{\pi}{2}, n \in I$$

but $-1 \leq \sin x \leq 1 \Rightarrow -1 \leq \sin x \leq 1 \Rightarrow x \in R$

$\therefore \quad D_f = R$ Also, $\sin(\cos x) \geq 0$

$\Rightarrow \quad 2n\pi \leq \cos x \leq 2n\pi + \pi, n \in I$

but $-1 \leq \cos x \leq 1$

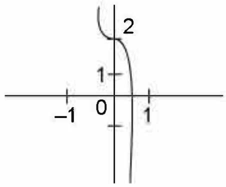
$\Rightarrow \quad 0 \leq \cos x \leq 1$

$\Rightarrow \quad x \in \left[2p\pi - \frac{\pi}{2}, 2p\pi + \frac{\pi}{2}\right], p \in I$

$\Rightarrow \quad D_g = \left[2p\pi - \frac{\pi}{2}, 2p\pi + \frac{\pi}{2}\right], p \in I$

$\Rightarrow \quad D_f \neq D_g$

14. (b) $\because f(-1) = -1 + 16 + 2 = 17$



and $f(1) = 1 - 16 + 2 = -13$

15. (a) $\because \sin^{-1} x$ is defined in $[-1, 1]$

$\cos^{-1} x$ is defined in $[-1, 1]$

and $\tan^{-1} x$ is defined in R

Hence, $f(x)$ is defined in $[-1, 1]$

16. (a) $\because f(x) = \sin 3x \cos [3x] - \cos 3x \sin [3x]$

$= \sin(3x - [3x]) = \sin(\{3x\})$

$\because \quad \{x\} = x - \{x\}$ is periodic with period 1

$\therefore \quad$ Period of $\{3x\}$ is $\frac{1}{3}$

17. (a) $\quad f(x) = x - [x]$

$f(x+1) = x + 1 - ([x] + 1) = x - [x]$

Period of $x - [x]$ is 1

$f(x) = \sin(2x - [2x])$

$f\left(x + \frac{1}{2}\right) = \sin\left(2\left(x + \frac{1}{2}\right) - \left[2\left(x + \frac{1}{2}\right)\right]\right)$

$= \sin(2x + 1 - [2x] - 1) = \sin(2x - [2x])$

period is $\frac{1}{2}$

18. (a) $2 < x < 3 \Rightarrow x - 1 > 0$

$x - 2 > 0$

$x - 3 < 0$

$\Rightarrow \quad f(x) = x - 1 + x - 2 + 3 - x = x$

$\Rightarrow \quad f$ is an identity function

19. (d) Range of $\sin x$ is $[-1, 1]$

$\Rightarrow \quad f : R \to R$ defined by $f(x) = \sin x$ in not onto

$\Rightarrow \quad$ it is not a bijection.

If f is both one and onto then f is bijection.
A is false R is true.

20. (c) $f : R \to R, f(x) = \frac{2x+1}{3}$ is a bijection.

$\Rightarrow f^{-1} = \frac{3x-1}{2}$

21. (a) Let $h(x) = \frac{f(x)}{g(x)}$ then

$h(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -h(x)$

$\therefore \quad h(x) = \frac{f}{g}$ is an odd function.

22. (a) A : $\sin x : -[-1\ 1] \to [-1\ 1]$

Also, $f\left(\frac{1}{2}\right) = f\left(\frac{5}{6}\right)$

$\Rightarrow f$ is not one-one

$\Rightarrow$ onto $R : def$

23. (c) There is no information about codomain, it will be consider as R

Also

$y = (x + 1)^2 - 1$

$\therefore \quad$ Range $\in [-1, \infty)$

Range $\neq$ codomain

$\therefore$ it is into.

MENTAL PREPARATION TEST

1. If $f(x) = \log_e \frac{1-x}{1+x}$, then prove that $f(a) + f(b) = f\left(\frac{a+b}{1+ab}\right)$ ***[MP-2000]***

2. Find the domain of the function $\frac{1}{\sqrt{1-x}\sqrt{x-2}}$ ***[MP-93]***

3. Find the domain and range of following function $y = \sin^{-1}(2x + 1)$ ***[MP-98]***

4. Determine the domain and range of the function $\frac{x^2-1}{x-1}$ ***[MP-2003]***

5. Find domain of six $x + \sin^{-1}x$. ***[HSB-95]***

6. Find the domain and range of the function $\sqrt{x-3}$

7. If $y = f(x) = \frac{x+2}{x-1}$, then prove that $x = f(y)$.

8. Draw the graph of the function $y = \frac{1}{x}$.

9. If $f(x) = \begin{cases} 2x+1, & \text{when } x \geq 2 \\ x, & \text{when } x < 2 \end{cases}$ Show that $f(x)$ does not exist when $x \to 2$, find $f(1)$ and $f(2)$. ***[CBSE-1986]***

10. If $f(x) = x^2 - 4x + 6$, then find the value of $f(2 + n)$. ***[MP-2001]***

11. Find the domain and range of the function $f(x) = \frac{x^2-1}{x-1}$. ***[MP-98]***

12. If $f(x) = \sin x$, then prove that $\frac{1}{f(x)} - f(x) = \cot x \cos x$. ***[MP-99]***

13. Find domain and range of $f(x) = 4 \sin x - 3 \cos x$. ***[PSB-2002 Similar]***

14. Find the domain of the function $\cos^{-1}(3x - 1)$.

15. Prove that $f(x) = 2^x + 2^{-x} - 2 > 0$ for $x > 0$ ***[W.B.J.E.E.-99]***

16. If $f(x) = x^4$ and $g(x) = \tan x$, then find the value of (gof) x.

17. If $f(x) = \log_e x$; $(x > 0)$. Then prove that $f(uvw) = f(u) + f(v) + f(w)$.

18. If $f(x) = x^2 + 2x - 3$, then find the value of $f(0)$, $f(-1)$, $f(1/3)$ and $f(\sin x)$. ***[MP-2001]***

19. Find the domain and range of $\frac{x+7}{x-5}$ ***[MP-98]***

20. If function f and g are such that $f(x) = 2x^2$ and $g(x) = \cos x$ then find the value of (gof) x. ***[MP-2000]***

21. Find k if $f(x) = x^3 - kx^2 + 2x$, $x \in R$ is odd function. ***[NCERT Book]***

22. If $\phi(x) = a^x$ prove that $[\phi(p)]^3 = \phi(3p)$ ***[M.M. CET-99]***

23. If $f(x) = 5x^6 - 4\tan^4 x + 3\cos^2 x$ then prove that $f(x)$ is an even function of x.

24. If $f(x) = x$ and $g(x) = \frac{1}{x}$, then prove that $f[g(x)] = g[f(x)]$

25. If $f(x) = \frac{x}{x-1}$, then prove that $f(\sec^2\theta) = \text{cosec}^2\theta$.

LECTUREWISE WARMUP TEST 1

1. If $[x]^2 - 5[x] + 6 = 0$, where [.] denotes the greatest integer function, then
(a) $x \in [3, 4]$ (b) $x \in [2, 3]$
(c) $x \in \{2, 3\}$ (d) $x \in [2, 4)$

2. The inverse of the function $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is equal to ***[NDA-2003]***
(a) $\log_{10}(2 - x)$ (b) $\dfrac{1}{2}\log_{10}(2x - 1)$
(c) $\dfrac{1}{4}\log_{10}\dfrac{2x}{2-x}$ (d) $\dfrac{1}{2}\log_{10}\left(\dfrac{1+x}{1-x}\right)$

3. If $3f(x) - 2f(1/x) = x$, then $5f(2) =$
(a) 2 (b) 3
(c) 8 (d) 7

4. If $f(x)$ is a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$. Then the value of n is
(a) 4 (b) 5
(c) 6 (d) None of these

5. If $f(x) = ax + b$ and $g(x) = cx + d$, then $f\{g(x)\} = g\{f(x)\}$ is equivalent to ***[NDA-2005]***
(a) $f(a) = g(c)$ (b) $f(b) = g(b)$
(c) $f(d) = g(b)$ (d) $f(c) = g(a)$

6. The function $f(x) = \sin(\log(x + \sqrt{x^2 + 1}))$ is ***[Orissa JEE-2002]***
(a) Even function
(b) Odd function
(c) Neither even nor odd
(d) Periodic function

7. If $f(x) = \log_a x$ and $F(x) = a^x$, then $F[f(x)]$ is ***[SCRA-1996]***
(a) $f[F(x)]$ (b) $f[F(2x)]$
(c) $F|f(2x)|$ (d) $F[(x)]$

8. If $f(x) = \dfrac{ax}{x+1}$, $x \neq -1$. Then, for what value of α is $f(f(x)) = x$ ***[IIT (Screening)-2001]***
(a) $\sqrt{2}$ (b) $-\sqrt{2}$
(c) 1 (d) -1

9. Let Function $f(x) = x^2 + x + \sin x - \cos x + \log(1 + |x|)$ be defined over the interval [0, 1]. The odd extentions of $f(x)$ to interval [–1, 1] is ***[UPSEAT-2000; MNR-94]***
(a) $x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(b) $-x^2 + x + \sin x + \cos x - \log(1 + |x|)$
(c) $-x^2 + x + \sin x - \cos x + \log(1 + |x|)$
(d) None of these

10. The inverse of the function $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 2$ is given by ***[Kurukshetra CEE-1996]***
(a) $\log_e\left(\dfrac{x-2}{x-1}\right)^{1/2}$ (b) $\log_e\left(\dfrac{x-1}{3-x}\right)^{1/2}$
(c) $\log_e\left(\dfrac{x}{2-x}\right)^{1/2}$ (d) $\log_e\left(\dfrac{x-1}{x+1}\right)^{-2}$

11. Given $f(x) = \log\left[\dfrac{(1+x)}{(1-x)}\right]$ and $g(x) = \dfrac{(3x + x^3)}{1 + 3x^2}$ then what is $f[g(x)]$ equal to ***[NDA-2006; DCE-1996]***
(a) $-f(x)$ (b) $3[f(x)]$
(c) $[f(x)]^3$ (d) $-3[f(x)]$

12. Let IR be the set of real numbers and let $f: IR \to IR$ be a function such that $f(x) = \dfrac{x^2}{1+x^2}$. What is the range of f? ***[NDA-2006]***
(a) IR (b) $IR - \{1\}$
(c) [0, 1] (d) [0, 1)

13. If $f: R \to R$ and $g: R \to R$ where $f(x) = |x|$ and $g(x) = [x]$, then $\{x \in R \mid g(f(x) \le f(g(x))\}$ is equal to ***[EAMCET-2003]***
(a) $Z \cup (-\infty, 0)$ (b) $(-\infty, 0)$
(c) Z (d) R

14. If $A = \{x : -1 \le x \le 1\}$ and $f: A \to A, f(x) = x|x|$, then $f =$
(a) one-one (b) onto
(c) one-one onto (d) many one-into

15. The number of all onto functions which can be defined from $A = \{1, 2, 3, \ldots\ldots\ldots, n\}, n \ge 2$ to $B = \{a, b\}$ is ***[EAMCET-1992]***
(a) $2^n - 2$ (b) $2^n - 1$
(c) 2^n (d) ${}^nP_2$

16. Which of the following is an even function *[PET (Raj.)-2000; MNR-1998]*

(a) $x\dfrac{a^x-1}{a^x+1}$ (b) $\tan x$

(c) $\dfrac{a^x-a^{-x}}{2}$ (d) $\dfrac{a^x+1}{a^x-1}$

17. Let $f(x) = \sin x$, $g(x) = \log |x|$. If ranges of function fog and gof are R_1 and R_2, respectively, then *[IIT Screening-1994]*

(a) $R_1 = (-1, 1), R_2 = (-\infty, 0)$
(b) $R_1 = (-\infty, 0), R_2 = [-1, 1]$
(c) $R_1 = [-1, 1), R_2 = (-\infty, 0)$
(d) $R_1 = [-1, 1], R_2 = (-\infty, 0]$

18. If $f(x_1) - f(x_2) = f\left(\dfrac{x_1-x_2}{1-x_1x_2}\right)$; $x_1, x_2 \in [-1, 1]$; then $f(x)$ is equal to *[Roorkee Screening-1998]*

(a) $\log\left(\dfrac{1-x}{1+x}\right)$ (b) $\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$

(c) $\log\left(\dfrac{2x}{1-x^2}\right)$ (d) $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

19. If for two functions f, g; gof is a bijection, then correct statement is *[Kurukshetra CEE-1998]*

(a) both g and f must be bijection
(b) g must be a bijection
(c) f must be a bijection
(d) neither of them may be a bijection

20. $f(x) = \cos\sqrt{x}$, correct statement is *[Kurukshetra CEE-1998]*

(a) $f(x)$ is periodic and its period $= \sqrt{2}\pi$
(b) $f(x)$ is periodic and its period $= 4\pi^2$
(c) $f(x)$ is periodic and its period $= \sqrt{\pi}$
(d) $f(x)$ is not periodic

21. Domain of the function $\sin\left[\log\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)\right]$ is *[IIT-85; Delhi (EEE)-98; PET (Raj.)-2003]*

(a) $[-2, 1]$ (b) $(-2, 1)$
(c) $[1, 2]$ (d) $(-1, 2)$

22. If $X = \{1, 2, 3, 4\}$, then total number of possible one-one onto functions from X to X for which $f(1) = 1, f(2) \neq 2, f(4) \neq 4$ will be

(a) 3 (b) 4
(c) 6 (d) 2

23. Range of $f(x) = 1 + x - [x]$ *[Roorkee-2000]*

(a) $[0, 1]$ (b) $[1, 2)$
(c) $(0, 2]$ (d) $[1, 2]$

24. If $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \dfrac{1}{x}$, then $f^{-1}(2)$ is equal to *[IIT Screening-2001]*

(a) -1 (b) 0
(c) 2 (d) 1

25. The domain of the function $\cos^{-1}\left(\log_2\dfrac{x^2}{2}\right)$ is *[MP-98]*

(a) $[-1/2] - \{0\}$ (b) $[1, 2]$
(c) $[-2, 2] - \{0\}$ (d) $[-2, 2] - (-1, 1)$

26. If $f(x) = \dfrac{1-x}{1+x}$, then $f[f(\cos 2\theta)] =$ *[MPPET-94, 01; Pb. CET-02]*

(a) $\tan^2\theta$ (b) $\sec 2\theta$
(c) $\cos 2\theta$ (d) $\cot 2\theta$

27. The domain of definition of the function $y(x)$ given by the equation $2^{2x} + 2^{2y} = 2$ is *[IIT Screening-2000]*

(a) $0 < x \leq 1$ (b) $-\infty < x < 0$
(c) $-\infty < x < \dfrac{1}{2}$ (d) $-\infty < x < 1$

28. Given the function $f(x) = \dfrac{a^x + a^{-x}}{2}$, $(a > 2)$. Then $f(x + y) + f(x - y) =$

(a) $2f(x) \cdot f(y)$ (b) $f(x) \cdot f(y)$
(c) $\dfrac{f(x)}{f(y)}$ (d) None of these

29. If $f(x + ay, x - ay) = axy$, then $f(x, y)$ is equal to *[MPPET-2009]*

(a) xy (b) $x^2 - a^2y^2$
(c) $\dfrac{x^2-y^2}{4}$ (d) $\dfrac{x^2-y^2}{a^2}$

LECTUREWISE WARMUP TEST 2

1. Let $f(x) = x^2$ and $g(x) = 2^x$, then the solution set of fog (x) = go$f(x)$ is
(a) R (b) {0}
(c) {0, 2} (d) None of these

2. If $f(x) = \begin{cases} -\dfrac{\sin(1+[x])}{[x]}, & \text{for } [x] \neq 0 \\ 0, & \text{for } [x] = 0 \end{cases}$ where $[x]$ denotes the greatest integer $\leq x$, then $\lim\limits_{x \to 0^-} f(x)$ equals
(a) 1 (b) 0
(c) –1 (d) None of these

3. The range of $3\sin x + 4\cos x + 5$ is
(a) [3, 4] (b) [–5, 10]
(c) [0, 10] (d) [3, 12]

4. The domain of the function $f(x) = \sqrt{\sin^{-1}(\log_2 x)} + \sin^{-1}\left(\dfrac{1}{1+x}\right)$ is
(a) [1, 3] (b) (0, 2]
(c) (0, 1] (d) [1, 2]

5. For $x \in [0, 2]$, let $f(x) = [x^2] - [x]^2$, then the range of f is
(a) {–1, 0} (b) {–1, 0, 1}
(c) {0, 1, 2} (d) {0}

6. If $f: R \to R$ and $g: R \to R$ are defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, then the value of x such that $g(f(x)) = 8$ are
(a) 1, 2 (b) –1, 2
(c) –1, –2 (d) 1, –2

7. Domain of $f(x) = \log|\log x|$ is
(a) $(0, \infty)$ (b) $(1, \infty)$
(c) $(0, 1) \cup (1, \infty)$ (d) $(\infty, 1)$

8. The domain of the function $f(x) = \dfrac{1}{\sqrt{(x-2)(1-x)}}$
(a) (1, 2] (b) [1, 2]
(c) (1, 2) (d) None of these

9. The set of all x for which $f(x) = \log_{\frac{x-2}{x+3}} 2$ and $g(x) = \dfrac{1}{\sqrt{x^2-9}}$ are both not defined as
(a) (–3, 2) (b) [–3, 2)
(c) (–3, 2] (d) [–3, 2]

10. Let $f(x) = \dfrac{x^2-4}{x^2+4}$, for $|x| > 2$ then the function $f: (-\infty, -2] \cup [2, \infty) \to (-1, 1)$ is ....
(a) one-one into
(b) one-one onto
(c) many one into
(d) many one onto

11. The domain of the function $f(x) = \dfrac{\sin^{-1}(3-x)}{\ln(|x|-2)}$ is
(a) [2, 4] (b) (3, 4]
(c) $[2, \infty)$ (d) $(-\infty, -3) \cup [2, \infty)$

12. Period of the function $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+\ldots\ldots+|\cos n\pi x|}$ where [.] denotes the greatest integer function is
(a) $2n\pi$ (b) 1
(c) not periodic (d) $2\pi/n$

13. The equivalent function of $\log_{y^3}^{x^4}$ is equal to
(a) $\dfrac{4}{3}\log_y^x$ (b) $\dfrac{4}{3}\log_{|y|}^{|x|}$
(c) $\dfrac{4}{3}\left|\log_y^x\right|$ (d) $\left|\log_y^x\right|^{4/3}$

14. The function $f: R \to R; f(x) = x; x \in R$ is **[MPCET-97]**
(a) Injection but not surjection
(b) Surjection but not injection
(c) Injection as well as surjection
(d) Neither injection nor surjection

15. $f(x) = \cos x + \cos 2x + \cos 4x$ is periodic with period
(a) π (b) 2π
(c) 3π (d) 4π

16. fog $(x) = F(x)$. If $F(x) = \sqrt{a^2 - x^2}$ and $g(x) = -x^2$, what is the value of $f(x)$?
(a) $\sqrt{x^2 - x}$ (b) $\sqrt{a^2 + x}$
(c) $\sqrt{a^2 + x^2}$ (d) $\sqrt{a^2 - x^2}$

17. A function $f: R \to R$ satisfies the equation $f(x) f(y) - f(xy) = x + y$ for all $x, y \in R$ and $f(1) > 0$, then

(a) $f(x) = x + \frac{1}{2}$

(b) $f(x) = x + 1$

(c) $f(x) = \frac{1}{2}x - 1$

(d) $f(x) = \frac{1}{2}x + 1$

18. If $f(x + 2) = \frac{1}{2}\left\{f(x+1) + \frac{4}{f(x)}\right\}$ and $f(x) > 0$ for all $x \in R$, then $f(x)$ is

(a) 1 (b) 2
(c) –2 (d) 0

19. If $\phi(x) = \frac{1}{1+e^{-x}}$, then the value of $\phi(5) + \phi(4) + \phi(3) + \ldots + \phi(-3) + \phi(-4) + \phi(-5)$ is

(a) 5 (b) 9/2
(c) 11/2 (d) None of these

20. If $f: (3, 4) \to (0,1)$ is defined by $f(x) = x - [x]$ where $[x]$ denotes the greatest integer function then $f^{-1}(x)$ is

(a) $\frac{1}{(x-[x])}$ (b) $[x] - x$
(c) $x - 3$ (d) $x + 3$

21. Mark one incorrect statement. If x and y are independent variable, then

(a) $f(x + y) = f(x)f(y) \Rightarrow f(x) = a^{\lambda x}$, λ is constant

(b) $f(x + y) = f(x) = f(y) \Rightarrow f(x) = \lambda x$, λ is constant

(c) $f(xy) = f(x) + f(y) \Rightarrow f(x) = \lambda$ in x or $f(x) = 0$, λ is constant

(d) $f(xy) = f(x) . f(y) \Rightarrow f(x) = e^{nx}$, $n \in R$.

22. If the function f satisfies the relation $f(x + y) + f(x - y) = 2 f(x) f(y)$ $\forall$ $x, y \in R$ and $f(0) \neq 0$, then $f(x)$ is

(a) an odd function
(b) an even function
(c) Periodic function
(d) a constant function

23. The fundamental period of the function $f(x) = 3x - [3x]$

(a) 1 (b) 1/3
(c) 3 (d) None of these

24. If $\log_2^7 = m$; then $\log_{49}^{28}$ is equal to

(a) $2(1 + 2m)$ (b) $\frac{1+2m}{2}$

(c) $\frac{2}{1+2m}$ (d) $1+m$

25. If f be a function defined as $f(x) = x^3 - 3x$, $-1 \le x \le 3$, then the range of f is

(a) [0, 24] (b) [–2, 2]
(c) [–2, 18] (d) None of these

26. The domain of the function $\frac{\sqrt{\sqrt{x+1}}}{x}$ is

[NDA-2003]

(a) $[-1, \infty) - \{0\}$ (b) $(-1, \infty)$
(c) R (d) None of these

27. The function $f(x) = \sqrt{x + 3\sqrt{x^3} + \sqrt{x}}$ is called

(a) Rational Function
(b) An Irrational Function
(c) Algebraic Function
(d) Transcendental Function

28. What is the equivalent definition of the function given by $f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$?

[NDA-2006]

(a) $f(x) = |x|$ (b) $f(x) = 2x$
(c) $f(x) = |x| + x$ (d) $f(x) = 2|x|$

29. If $f: R \to R^+$ such that $f(x) = (1/3)^x$, then what is $f^{-1}(x)$ equal to? ***[NDA-2006]***

(a) $(1/3)^x$ (b) 3^x
(c) $\log_{1/3}x$ (d) $\log_x(1/3)$

30. If $f(x) = (a - x^n)^{1/n}$, where $a > 0$ and n is a positive integer, then $f[f(x)]$ is equal to

[CET-97; Roorkee-91; Karnataka CET-98]

(a) x^3 (b) x^2
(c) x (d) None of these

31. The value of α for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is inverse of itself will be

[Screening Paper-1992]

(a) 1 (b) 2
(c) –1 (d) 0

32. If $f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then gof$(x)$ is equal to ***[IIT-1996]***

(a) 0 (b) 1
(c) sin1° (d) None of these

33. Let R be set of real numbers. If $f : R \to R$ defined by $f(x) = e^x$, then f is ***[Karnataka CET-02; UPSEAT-02]***

(a) surjective but not injective
(b) injective but not surgective
(c) bijective
(d) neither surjective nor injective.

34. If $f(x)=\frac{2^x+2^{-x}}{2}$, then $f(x+y)\,.\,f(x-y)=$

(a) $\frac{1}{2}[f(2x)+f(2y)]$ (b) $\frac{1}{4}[f(2x)+f(2y)]$
(c) $\frac{1}{2}[f(2x)-f(2y)]$ (d) $\frac{1}{4}[f(x)-f(2y)]$

35. For real values of x, range of the function $y=\frac{1}{2-\sin 3x}$ is ***[Delhi (EEE)-98]***

(a) $1/3 \le y \le 1$
(b) $-1/3 \le y < 1$
(c) $-1/3 > y > -1$
(d) $1/3 > y > -1$

LECTUREWISE WARMUP TEST 1: SOLUTIONS

1. (d) $[x]^2 - 5[x] + 6 = 0$
$\Rightarrow [x] = 2, 3 \Rightarrow x \in [2, 4)$

2. (d) $\because\ y=\frac{10^x-10^{-x}}{10^x+10^{-x}}$

$\Rightarrow \frac{1+y}{1-y}=\frac{2\times 10^x}{2\times 10^{-x}}=10^{2x}$

$\Rightarrow 2x=\log_{10}\left(\frac{1+y}{1-y}\right)$

$\Rightarrow x=\frac{1}{2}\log_{10}\left(\frac{1+y}{1-y}\right)$

$\therefore\ f^{-1}(y)=\frac{1}{2}\log_{10}\frac{1+y}{1-y}$

$\therefore\ f^{-1}(x)=\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$

3. (d) Given $3f(x) - 2f(1/x) = x$ ....(i)
Replace x by $1/x$, we get $3f(1/x) - 2f(x) = 1/x$ ....(ii)
Eliminate $f(1/x)$ between (1) and (2)

$5f(x)=\frac{3x^2+2}{x}$, So $5f(2)=\frac{3(2)^2+2}{2}$

$\Rightarrow 5f(2)=\frac{12+2}{2}=\frac{14}{2}=7$

4. (a) $f(1) = 3$ given and we have to make use of the relation $f(x + y) = f(x) f(y)$, $f(x)$
$= f(x - 1 + 1) = f(x - 1).\, f(1)$
$= f(x - 2)\,[f(1)]^2 = f(x - 3)\,[f(1)]^3$ ....
$f[x - (x - 1)][f(1)]^{x-1} = [f(1)f(1)]^{x-1} = [f(1)]^x$
or $f(x) = [f(1)]^x = 3^x$

Now $\sum_{x=1}^{n} f(x) = \sum_{x=1}^{n} 3^x = 3 + 3^2 + 3^3 + 3^4 + ... + 3^n$

or $\frac{3(3^n-1)}{3-1}=120$ (G.P.) (Given)

$\therefore\ 3^n-1=\frac{240}{3}=80$ or $3^n = 81 = 3^4$

$\Rightarrow n = 4$

5. (c) we have $f(x) = ax + b$, $g(x) = cx + d$ and $f\{g(x)\} = g\{f(x)\}$

$\Rightarrow f(cx + d) = g(ax + b) \Rightarrow a(cx + d) + b = c(ax + b) + d$

$\Rightarrow ad + b = cb + d \Rightarrow f(d) = g(b)$

6. (b) $f(x) = \sin\,[\log\,(x+\sqrt{1+x^2})]$

$\Rightarrow f(-x) = \sin\,[\log\,(-x+\sqrt{1+x^2}]$

$\Rightarrow \quad f(-x) = \sin\,\log\left(\dfrac{1}{x+\sqrt{1+x^2}}\right)$

$= \sin[-\log(x+\sqrt{1+x^2})]$

$= -\sin\log\,(x+\sqrt{1+x^2}) \Rightarrow f(-x) = -f(x)$

So $f(x)$ is odd function.

7. (a) $F[f(x)] = F(\log_a x) = a\,\log_a x = x$

$f\,[F(x)] = f(a^x) = \log_a a^x = x$

8. (d) $f\{f(x)\} = \dfrac{\alpha f(x)}{f(x)+1}$

$$= \frac{\alpha\left(\dfrac{\alpha x}{\alpha+1}\right)}{\left(\dfrac{\alpha x}{\alpha+1}+1\right)} = \frac{\alpha^2 x}{\alpha x + x + 1}$$

$\therefore \quad x = \dfrac{\alpha^2 x}{(\alpha+1)x+1}$ or $x\,\{(\alpha+1)\,x + 1 - \alpha^2\}$

$= 0$ or $(\alpha+1)x^2 + (1-\alpha^2)\,x = 0$. This should hold for all x

$\Rightarrow \quad \alpha + 1 = 0,\ 1-\alpha^2 = 0 \ \therefore\ \alpha = -1$

9. (b) Odd extension from [0, 1] to [–1, 1] means that function from given choices which satisfies the condition $f(-x) = -f(x)$

Now $|-x| = |x|$, $\cos(-x) = \cos x$,

$\sin(-x) = -\sin x$, $(-x)^2 = x^2$

so we observe that $f(-x) = x^2 - x - \sin x - \cos x + \log(1+|x|) = -$ (function given in (2))

$\therefore$ (b) is the correct answer.

10. (b) $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} + 2$

$\Rightarrow \quad y = \dfrac{e^{2x}-1}{e^{2x}+1} + 2$

$\Rightarrow \quad e^{2x} = \dfrac{1-y}{y-3} = \dfrac{y-1}{3-y}$

$\Rightarrow \quad x = \dfrac{1}{2}\log\left(\dfrac{y-1}{3-y}\right)$

$\therefore \quad f^{-1}(y) = \log\left(\dfrac{y-1}{3-y}\right)^{1/2}$ So $f^{-1}(x)$

$= \log_e\left(\dfrac{x-1}{3-x}\right)^{1/2}$

11. (b) $f\,[g(x)] = \log\left(\dfrac{1+g(x)}{1-g(x)}\right) = \log$

$$\left(\frac{1+\dfrac{3x+x^3}{1+3x^2}}{1-\dfrac{3x+x^3}{1+3x^2}}\right)$$

$$= \log\left(\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3}\right)$$

$$= \log\left(\frac{1+x}{1-x}\right)^3$$

$$= 3\log\left(\frac{1+x}{1-x}\right) = 3[f(x)]$$

12. (d) $f(x)$ is continuous function and takes non-negative values. Also we find that $f(0) = 0$ and $f(x) \to 1$ as $x \to \infty$.

Hence range of $f = [0, 1)$

13. (d) $g(f(x)) = g(|x|) = [|x|], f(g(x)) = f(|x|) = |[x]|$

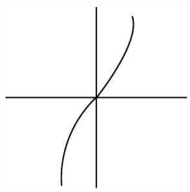
when $x \geq 0$, $[|x|] = [x] = |[x]|$ $\therefore$ $f(g(x)) = g(f(x))$

when $x < 0$, $[x] \leq x < 0 \Rightarrow |[x]| \geq |x|$

$\therefore \quad |[x]| \geq |x| \geq [|x|]$ ($\because$ $[x] \leq x$ for all x)

$\Rightarrow \quad f(g(x)) \geq g(f(x))$ Thus $g(f(x)) \leq f(g(x))$ for all $x \in R$.

14. (c) $f(x) = \begin{cases} x^2, & 0 < x < 1 \\ -x^2, & -1 < x < 0 \end{cases}$



1-1 onto.

15. (a) By formula

16. (a) odd function $(x) \times$ odd function

$\left(\dfrac{a^x-1}{a^x+1}\right) =$ even function

17. (d) Range of fog is range of $f[-1, 1]$ and range of gof is range of $g[-\infty, 0]$

18. (a) Verification method

$$f(x_1) - f(x_2) = \log\left\{\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1+x_2}{1-x_2}\right)\right\}$$

$$= \log\left\{\left(1-\frac{x_1-x_2}{1-x_1x_2}\right)\Big/\left(1+\frac{x_1-x_2}{1-x_1x_2}\right)\right\}$$

$$= f\left(\frac{x_1-x_2}{1-x_1x_2}\right)$$

19. (a) By definition.

20. (d) $f(x+T) = f(x) \Rightarrow \cos\sqrt{x+T} = \cos\sqrt{x}$

$\sqrt{x+T} = 2n\pi \pm \sqrt{x}\ ; n \in N$

$\Rightarrow$ $T = 4n^2\pi^2 \pm 4n\pi\sqrt{x}$ which is not independent of x

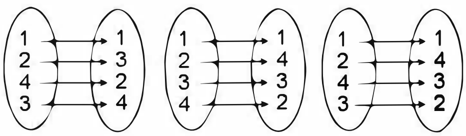
$\therefore$ $\cos\sqrt{x}$ is not periodic

21. (b) $\dfrac{\sqrt{4-x^2}}{1-x} > 0 \Rightarrow 4 - x^2 > 0$

$\Rightarrow |x| < 2 \Rightarrow 1 - x > 0$

$\Rightarrow x < 1 \Rightarrow -2 < x < 1$

22. (a)



23. (b) We know that $0 \le x - [x] < 1$ for all $x \in R$

$\Rightarrow$ $0 + 1 \le 1 + x - [x] < 1 + 1$ for all $x \in R$

$\Rightarrow$ $1 \le 1 + x - [x] < 2$ for all $x \in R$

$\Rightarrow$ $1 \le f(x) < 2$ for all $x \in R$

24. (d) Clearly, $f : [1, \infty) \to [2, \infty)$ is a bijection,

Let $f(x) = y$. Then, $f(x) = y \Rightarrow x + \dfrac{1}{x} = y$

$\Rightarrow x^2 - xy + 1 = 0$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2-4}}{2}$$

$$\Rightarrow \quad x = \frac{y + \sqrt{y^2-4}}{2}$$

$[\because x \ge 1]$

$$\Rightarrow \quad f^{-1}(y) = \frac{y + \sqrt{y^2-4}}{2}$$ Hence,

$$f^{-1}(x)\frac{x + \sqrt{x^2-4}}{2}$$ for all

$$x \in [1, \infty] \Rightarrow f^{-1}(2) = \frac{2 + \sqrt{2^2-4}}{2} = 1$$

25. (d) $\cos^{-1}\left(\log_2^{x^2/2}\right)$ is defined if $-1 \le \log_2^{x^2/2} \le 1$

if $2^{-1} \le x^2/2 \le 2^1$ if $1 \le x^2 \le 4$

if $1 \le x \le 2$ or $-2 \le x \le -1$

$\therefore$ domain is $[-2, -1] \cup [1, 2] = [-2, 2] - (-1, 1)$

26. (c) $f[f(\cos 2\theta)] = f\left[\dfrac{1-\cos 2\theta}{1+\cos 2\theta}\right]$

$$= f(\tan^2\theta) = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

27. (c) $2^{2y} = 2 - 2^{2x} > 0$ as exponential function 2^{2y} is always +ive

$\therefore$ $2^{2x} < 2^1 \Rightarrow x < \dfrac{1}{2} \Rightarrow x \in \left(-\infty, \dfrac{1}{2}\right)$ or $-\infty < x < \dfrac{1}{2}$

28. (a) We have $f(x+y) + f(x-y)$

$$= \frac{1}{2}[a^{x+y} + a^{-x-y} + a^{x-y} + a^{-x+y}]$$

$$= \frac{1}{2}[a^x(a^y + a^{-y}) + a^{-x}(a^y + a^{-y})]$$

$$= \frac{1}{2}(a^x + a^{-x})(a^y + a^{-y}) = 2f(x)f(y)$$

29. (c) Given $f(x + ay, x - ay) = axy$ ....(i)

Let $x + ay = u$ and $x - ay = v$

Then $x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2a}$ Substituting the value of x and y in equation (i), we obtain

$$f(u,v) = \frac{u^2-v^2}{4}$$

$$\Rightarrow \quad f(x,y) = \frac{x^2-y^2}{4}$$

LECTUREWISE WARMUP TEST 2: SOLUTIONS

1. (c) fog $(x) = f(g(x)) = f(2^x) = (2^x)^2 = 2^{2x}$ and

$\text{gof}(x) = g(f(x)) = g(x^2) = 2^{x^2}$

Thus the solution of $2^{x^2} = 2^{2x}$ is given by $x^2 = 2x$ which is $x = 0, 2$

2. (b) For $-1 < x < 0$, $[x] = -1$, So

$$\lim_{x\to 0^-} \frac{\sin(1+[x])}{[x]} = \frac{\sin 0}{-1} = 0$$

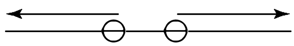
3. (c) Range of $f(x) = a \cos x + b \sin x + c$ is

$[c - \sqrt{a^2+b^2}, c + \sqrt{a^2+b^2}]$

$= [5 - \sqrt{4^2+3^2}, 5 + \sqrt{4^2+3^2}] = [5 - 5, 5 + 5] = [0, 10]$

4. (d) For the domain of the function $\sin^{-1}\left(\frac{1}{1+x}\right)$ i.e., $\text{cosec}^{-1}(1 + x)$, $1 + x \geq 1$ or $1 + x \leq -1$

$\Rightarrow \quad x \geq 0$ or $x \leq -2$



Also $\sqrt{\sin^{-1}(\log_2 x)}$ is defined when $\sin^{-1}(\log_2 x) \geq 0$ and $\log_2 x \leq 1 \Rightarrow x \in [1, 2]$ $\therefore D_f = [1, 2]$

5. (c) Obviously $x \in [0, 1) \Rightarrow f(x) = 0$

$x \in [1, \sqrt{2}) \Rightarrow f(x) = 0$ $x \in [\sqrt{2}, \sqrt{3})$

$\Rightarrow f(x) = 1$

$x \in [\sqrt{3}, 2) \Rightarrow f(x) = 2, f(2) = 0$ $\therefore R_f = \{0, 1, 2\}$

6. (c) $g(f(x)) = 8 \Rightarrow [f(x)]^2 + 7 = 8$

$\Rightarrow \quad (2x + 3)^2 = 1$

$\Rightarrow \quad 4x^2 + 9 + 12x = 1$

$\Rightarrow \quad 4x^2 + 12x + 8 = 0$

$\Rightarrow \quad x^2 + 3x + 2 = 0$

$\Rightarrow \quad (x + 1)(x + 2) = 0$

$\Rightarrow \quad x = -1, -2$

7. (c) $f(x)$ is defined when $|\log x| > 0$

$\Rightarrow \quad x \in (0, \infty), x \neq 1$

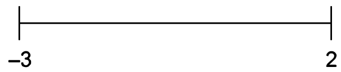
$\therefore \quad D_f = (0, 1) \cup (1, \infty)$

8. (c) The domain of $\frac{1}{\sqrt{(x-a)(b-x)}}$ when $a > b$ is (b, a) hence the domain of given function is $(1, 2)$.

9. (d) $f(x)$ is defined if $\frac{x-2}{x+3} > 0$ or

$$\frac{(x+3)(x-2)}{(x+3)^2} > 0$$

$\Rightarrow \quad x < -3$ or $x > 2 = D_1$ and also $x \neq 2, x \neq -3$ by definition of $\log_a b$



$g(x)$ is defined is $x^2 - 9 > 0 \Rightarrow (x + 3)(x - 3) > 0$

i.e., $x < -3$ or $x > 3 = D_2$

Hence both are defined for $D_1 \cap D_2$ i.e., for $x < -3$ and $x > 2$ ..(1)

Therefore both are not defined for $-3 \leq x \leq 2$ or $[-3, 2]$.

10. (c) Let $f(x) = f(y)$

$$\Rightarrow \quad \frac{x^2-4}{x^2+4} = \frac{y^2-4}{y^2+4}$$

$$\Rightarrow \quad \frac{x^2-4}{x^2+4} - 1 = \frac{y^2-4}{y^2+4} - 1$$

$\Rightarrow \quad x^2 + 4 = y^2 + 4$

$\Rightarrow \quad x = \pm y$

$\therefore \quad f(x)$ is many-one.

Now for each $y \in (-1, 1)$, there does not exist $x \in X$ such that $f(x) = y$.

Hence f is into.

11. (b) $f(x) = \frac{\sin^{-1}(3-x)}{\log[|x|-2]}$ Let $g(x) = \sin^{-1}(3 - x)$

$\Rightarrow \; -1 \leq 3 - x \leq 1$

Domain of $g(x)$ is $[2, 4]$ and let $h(x) = \log[|x| - 2] \Rightarrow |x| - 2 > 0$

$\Rightarrow \quad |x| > 2 \Rightarrow x < -2$ or $x > 2 \Rightarrow (-\infty, -2) \cup (2, \infty)$

We know that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $\forall\, x \in D_1 \cap D_2 - \{x \in R : g(x) = 0\}$

$\therefore$ Domain of $f(x) = (2, 4] - \{3\} = (2, 3) \cup (3, 4]$

12. (b) Since periods of $x - [x]$, $|\cos \pi x|$, $|\cos 2\pi x|$...$|\cos n\pi x|$

are 1, $\frac{1}{2}.\frac{2\pi}{\pi} = 1$, $\frac{1}{2}.\frac{2\pi}{2\pi} = \frac{1}{2}, \ldots \frac{1}{2}.\frac{2\pi}{n\pi} = \frac{1}{n}$

$\therefore$ Period of $f(x)$ is L.C.M. of 1, 1, $\frac{1}{2}, \frac{1}{2}, \ldots \frac{1}{4} = 1$ $\therefore$ (b) is answer.

13. (b) By formula $\log^{m}_{y^k x} = \frac{m}{k}\log_y{}^x$ Hence

$\log^{x^4}_{y^3} = \frac{4}{3}\log^{|x|}_{|y|}$

14. (c) Properties of identity function.

15. (b) Period of $\cos x = 2\pi$. Period of $\cos 2x = \pi$

Period of $\cos 4x = \pi/2$.

The required period is L.C.M. of 2π, π, $\pi/2$

16. (b) $f(g(x)) = f(-x^2) = \sqrt{a^2 - x^2}$

17. (b) By actual verification we find correct option.

$(x + 1)(y + 1) - (xy) - 1 = xy + x + y + 1 - xy - 1 = x + y$

18. (b) Clearly, $\lim\limits_{x\to\infty} f(x + 2) = \lim\limits_{x\to\infty} f(x + 1) = \lim\limits_{x\to\infty} f(x) = l$(say)

Then taking limit, $l = \frac{1}{2}\left(l + \frac{4}{l}\right)$ or $\frac{1}{2}l = \frac{2}{l}$

or $l^2 = 4$

$\therefore$ $l = 2$ [$\because f(x) > 0$ for all x]

19. (c) Here, $\phi(-x) = \frac{1}{1+e^x}$ So $\phi(x) + \phi(-x) = \frac{1}{1+e^{-x}} + \frac{1}{1+e^x}$

$\therefore$ $\phi(x) + \phi(-x) = \frac{e^x}{e^x+1} + \frac{1}{1+e^x} = \frac{e^x+1}{e^x+1} = 1$

$\therefore$ sum $= \{\phi(5) + \phi(-5)\} + \ldots + \{\phi(1) + \phi(-1)\} + \phi(0)$

$= 1 + 1 + 1 + 1 + 1 + \phi(0) = 5 + \frac{1}{1+e^0} = 5 + \frac{1}{2} = \frac{11}{2}$

20. (d) Let $x = 3 + k$, $0 < k < 1$

Then $y = f(x) = x - [x] = 3 + k - 3 = k = x - 3$

$\therefore$ $x = y + 3$ $\therefore f^{-1}(y) = y + 3$

$\therefore$ $f^{-1}(x) = x + 3$

21. (d) By the results of the function the incorrect option is (d). Because the correct result is $f(xy) = f(x).f(y)$

$\Rightarrow$ $f(x) = x^n$, $n \in R$.

22. (b) Given $f(x + y) + f(x - y) = 2f(x)f(y)$ ....(i)

Replacing x by y and y by x in (i) then

$f(y + x) + f(y - x) = 2f(y)f(x)$ ....(ii)

$\therefore$ From (i) and (ii) we get $f(y - x) = f(x - y)$

Putting $y = 2x$ then $f(x) = f(-x)$

Hence $f(x)$ is an even function.

23. (b) Period of function $f(x)$ is T. Then period of function $f(ax)$ is T/a.

Hence the period of $3x - [3x]$ is $1/3$.

24. (b) $\log^{28}_{7\times7} = \log^{4\times7}_{7^2} = \frac{1}{2}\log^{4\times7}_{7}$

$= \frac{1}{2}\left[\log^4_7 + \log^7_7\right]$

$= \frac{1}{2}\left[2\log^2_7 + 1\right] = \frac{1}{2}[2m + 1]$

25. (c) $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$\therefore$ $f'(x) \geq 0$ if $x \leq -1$ or $x \geq 1$

$f'(x) \leq 0$ if $-1 \leq x \leq 1$

$\therefore$ in $[-1, 1]$ it is monotonically decreasing and in $[1, 3]$ it is monotonically increasing.

$\therefore$ max $f(x)$ = greatest among $\{f(-1), f(3)\} = f(3) = 18$ min $f(x) = f(1) = -2$.

So, the range of $f = [-2, 18]$.

26. (a) Quantity within radical sign must be +ve or zero in numerater. Also denominater can not be zero.

$\sqrt{x+1} \geq 0 \Rightarrow x + 1 \geq 0 \Rightarrow x \geq -1$

27. (c) By definition of Algebraic function.

28. (c) x as well as $|x|$ has same sign.

Hence $f(x) = |x| + x$ is equivalent to the given function.

29. (c) $y = \left(\frac{1}{3}\right)^x \Rightarrow x = \left(\frac{1}{3}\right)^y$

$\Rightarrow y = \log_{1/3}^{x} = f^{-1}(x)$

30. (c) $f\{f(x)\} = [a - \{f(x)^n]^{1/n} = [a - (a - x)^n]^{1/n} = x$

31. (c) (1) $y = 1 + \alpha\, x$ (2) $x = 1 + \alpha\, y$

(3) $y = \frac{x-1}{\alpha} = f'(x) = 1 + \alpha\, x$

32. (b) $f(x) = \sin^2 x +$

$$\sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$$

$$= \sin^2 x + \left(\sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3}\right)^2$$

$$+ \cos x\left(\cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3}\right)$$

$$= \sin^2 x + \frac{\sin^2 x}{4} + \frac{3\cos^2 x}{4} + \frac{2\sqrt{3}}{2.2}$$

$$\sin x \cos x + \frac{\cos^2 x}{2} - \cos x \sin x \frac{\sqrt{3}}{2}$$

$$= \sin^2 x + \frac{\sin^2 x}{4} + \frac{3\cos^2 x}{4} + \frac{\cos^2 x}{2}$$

$$= \frac{5\sin^2 x}{4} + \frac{6\cos^2 x + 4\cos^2 x}{8}$$

$$= \frac{5}{4}(\sin^2 x + \cos^2 x) = \frac{5}{4}$$

$\therefore \quad gof(x) = g(f(x)) = g\left(\frac{5}{4}\right) = 1.$

33. (b) f is injective (i.e., one-one), since $x_1, x_2 \in R$ and $x_1 \neq x_2$.

$\Rightarrow \quad e^{x_1} \neq e^{x_2} \Rightarrow f(x_1) \neq f(x_2)$

f is not surjective, since $e^x > 0$ for all x and so negative real number can be the image of any real number.

For example, there is no real x such that $f(x) = -2$.

34. (a) $f(x + y) f(x - y)$

$$= \frac{1}{2}[2^{x+y} + 2^{-x-y}]\ \frac{1}{2}[2^{x-y} + 2^{-x+y}]$$

$$= \frac{1}{4}[2^{2x} + 2^{2y} + 2^{-2y} + 2^{-2x}]$$

$$= \frac{1}{4}[(2^{2x} + 2^{-2x}) + (2^{2y} + 2^{-2y})] = \frac{1}{2}[f(2x) + f(2y)]$$

35. (a) $\because\ y = \frac{1}{2 - \sin 3x}$

$\therefore \quad 2 - \sin 3x = 1/y$

$\Rightarrow \quad \sin 3x = 2 - 1/y$

Now since $-1 \leq \sin 3x \leq 1$

$\Rightarrow \quad -1 \leq 2 - 1/y \leq 1$

$\Rightarrow \quad -3 \leq -1/y \leq -1$

$\Rightarrow \quad 1 \leq 1/y \leq 3 \Rightarrow 1/3 \leq y \leq 1$

ANSWERS

LECTURE 1

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (d) **2.** (a) **3.** (b) **4.** (b) **5.** (c) **6.** (a) **7.** (a) **8.** (d) **9.** (d) **10.** (a) **11.** (c) **12.** (d) **13.** (a)

Worksheet: To Check the Preparation Level

1. (d) **2.** (c) **3.** (a) **4.** (b) **5.** (a) **6.** (d) **7.** (b) **8.** (b) **9.** (c)

LECTURE 2

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (c) **2.** (b) **3.** (d) **4.** (b) **5.** (b) **6.** (a) **7.** (c) **8.** (c) **9.** (c) **10.** (a) **11.** (a) **12.** (c) **13.** (a)

Worksheet: To Check the Preparation Level

1. (b) **2.** (b) **3.** (b) **4.** (a) **5.** (c) **6.** (a) **7.** (a) **8.** (d) **9.** (b) **10.** (b) **11.** (c)

LECTURE 3

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (a) **2.** (b) **3.** (a) **4.** (a) **5.** (c), (d) **6.** (b) **7.** (a) **8.** (a) **9.** (a) **10.** (d) **11.** (b) **12.** (c) **13.** (c) **14.** (d)

Worksheet: To Check the Preparation Level

1. (d) **2.** (b) **3.** (a) **4.** (d) **5.** (c) **6.** (a) **7.** [1, 6] **8.** (d) **9.** (d) **10.** (b) **11.** (c) **12.** (a) **13.** (d) **14.** (d) **15.** (d)

LECTURE 4

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (b) **2.** (b) **3.** (c) **4.** (d) **5.** (a) **6.** (a) **7.** (d) **8.** (a) **9.** (a) **10.** (a) **11.** (b) **12.** (a) **13.** (a)

Worksheet: To Check the Preparation Level

1. (a) **2.** (d) **3.** (a) **4.** (a) **5.** (b) **6.** (b) **7.** (a) **8.** (a) **9.** (a) **10.** (c) **11.** (a) **12.** (a) **13.** (d)

LECTURE 5

Mental Preparation Test

2. (1, 2)
3. $\{x : -1 \le x \le 0\}$; $(-\pi/2, \pi/2)$
4. $D_f = R - \{1\}, R_r = R - \{2\}$
5. $D_f = [-1, 1]$
6. $x \ge 3, y \ge 0$
9. 1 and 5
10. $n^2 + 2$
11. $D_f = R - \{1\}, R_f = R - \{2\}$
13. $D_f = R, R_f = [-5, 5]$
14. $-[0, 2/3]$
16. $\tan x^4$
18. $-3, -4, -20/9$; $\sin^2 x + 2 \sin x - 3$
19. $\cos (2x^2)$
20. $k = 0$

Part B

Limits

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LECTURE 1

Basic Definition, Evaluation of Limits (Basic)

BASIC CONCEPTS

INTRODUCTION

Left and Right Sides of a Point

1. If c is a point of an open interval $\{x \mid a < x < b\}$; then c divides the open interval (segment of the straight line) into three disjoint parts; viz
   (i) c
   (ii) $\{x \mid a < x < c\}$
   (iii) $\{x \mid c < x < b\}$
   The part $\{x \mid a < x < c\}$ is called the left side of the point $x = c$ and the part $\{x \mid c < x < b\}$ is called the right side of the point c.
2. **Neighbourhood of a point $x = c$**
   The open interval $\{x \mid c - h < x < c + h\}$ where h is a small +ve real number at our choice (i.e., any) is called the small i.e., h – nbd of the point $x = c$.
   The open interval = $\{x \mid c - h < x < c + h$ except $x = c\}$ is called the nbd of point c.
   Left nbd of $x = c$ is: $\{x \mid c - h < x < c\}$
   Right nbd of $x = c$ is: $\{x \mid c < x < c + h\}$
   Here 'nbd' means neighbourhood
3. **The meaning of $x \to c$ negatively i.e., $x \to c-$**
   This means x takes any value of the deleted left nbd of point $x = c$ i.e., $x \in \{c - h < x < c\}$ for all small +ve h : In other words x takes values nearer and nearer to $x = c$ from left.
4. The meaning of $x \to c$ positively i.e., $x \to c+$. It means x-takes any value of the right neighbourhood of c. i.e., $x \in \{c < x < c + h\}$ for small h. In other words, x takes values nearer and nearer to $x = c$ from right.

NOTE

In both the cases we never mean x must be equal to c for some x.

DEFINITION OF LIMIT

1. **Meaning of**
   $\lim\limits_{x \to c+} f(x) = l_1 = f(c+0) = \lim\limits_{h \to 0} f(c+h)$
   = Right hand limit of the function
   In words it means as x approaches c from right $f(x)$ approaches l which is called the R.H. limit of function.

2. **Meaning of**
   $\lim\limits_{x \to c-} f(x) = l = f(c-o) = \lim\limits_{h \to 0} f(c-h)$
   = Left hand limit of the function
   In other words it means as x approaches c from left then $f(x)$ approaches l_2. l_2 is called the left hand limit of function.

3. $\lim\limits_{x \to c} f(x)$ **exists if and only if $l_1 = l_2$ = Finite and unique** Common value l is called limit of the function and denoted by $\lim\limits_{x \to c} f(x) = l$.

NOTES

1. Here we never mean $f(x)$ must be equal to l for some x.
2. The concept of limit can some how extend the definition of a function to the point where it is not defined.
3. **Limit and value of a Function**
   The value and the limit of the functions are quite independent of each other.
4. The limit of $f(x)$ as x approaches c does not exist if at least one of the following is true:
   (i) Right hand limit does not exist.
   (ii) Left hand limit does not exist.
   (iii) Both right hand and left hand limits exist but they are not equal.
5. (i) $x \to \infty$ means that x grows without bounds in the positive direction.
   (ii) $x \to -\infty$ means that x is negative and $|x|$ increases without bounds.
   (iii) $\lim_{x\to a} f(x) = +\infty$ means that $f(x)$ grows without bounds in the positive direction as x - approaches a.
   (iv) $\lim_{x\to a} f(x) = -\infty$ means that $f(x)$ is negative and $|f(x)|$ grows without bounds as $x \to a$.
6. **Seven Indeterminate forms**
   $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, \infty^0, 0^0, 1^\infty$

EVALUATION OF LIMITS (WORKING RULE)

1. **Direct substitution Method** To evaluate $\lim_{x\to a} f(x)$, put $x = a$ and simplify. In this case limit as well as value of the function is same.
2. **Factor Method** To evaluate $\lim_{x\to a} \frac{f(x)}{g(x)}$ factorise both $f(x)$ and $g(x)$ and cancel the common factor $x - a$ and then put $x = a$ for finding the desired limit.
3. $\frac{\text{Something finite}}{0} \to \infty$ or $-\infty$
4. $\frac{\text{Something finite}}{0} \to \infty$
5. **Limits when $x \to \infty$** In case of limits when $x \to \infty$ i.e., $\frac{f(x)}{g(x)} \to \frac{\infty}{\infty}$ then divide $f(x)$ and $g(x)$ by maximum powers of x obtained from the numerator and denominator for finding desired limits.
6. **Substitution Method** To evaluate $\lim_{x\to a} f(x)$ put $x = a + h$ when $x \to a, h \to 0$ and simplify the numerator and denominator, and cancel common h and now obtain the limit by substituting $h = 0$.
7. **Rationalization Method** In case numerator or denominator are irrational functions (Factors having square root) rationalisation of numerator or denominator helps to obtain the limit.
8. **Problems Based on Expansion** The following expansion formula can also be used with advantage in evaluation of limits.

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\infty$

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\infty$

(iii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots\infty$

(iv) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\infty$

(v) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty$

(vi) $a^x = 1 + \frac{x\log a}{1!} + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + \ldots\ldots\infty$

(vii) $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\ldots\infty$

(viii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\infty$

(xi) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots\infty$, $-1 < x < 1, n \in Q$.

(x) $(1+x)^{\frac{1}{x}} = e\left\{1 - \frac{x}{2} + \frac{11}{24}x^2 + \ldots\ldots\right\}$

(xi) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61}{6!}x^6 + \ldots..$

(xii) $(\sin^{-1} x)^2 = \frac{2}{2!}x^2 + \frac{2.2^2}{4!}x^4 + \ldots..$

9. For the small value of x : $\sin x \to x$, $\cos x \to 1$, $\tan x \to x$, $\log(1 + x) \to x$, $\log(1 - x) \to -x$ and $(1 + x)^n = 1 + nx$, $e^x = 1 + x$, $a^x = 1 + x \log a$.

10. Some Important Results

(i) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$, x in radian

(ii) $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$, x in radian, $\lim\limits_{x\to 0}\dfrac{\sin ax}{x}=a$

(iii) $\lim\limits_{x\to 0}\cos x=1$, x in radian,

$\lim\limits_{x\to 0}\dfrac{\sin x^0}{x}=\dfrac{\pi}{180}$

(iv) $\lim\limits_{x\to \infty}\dfrac{\sin x}{x}=0$, x in radian.

(v) $\lim\limits_{x\to \infty}\dfrac{\cos x}{x}=0$, x in radian, $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}=\dfrac{a}{b}$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Evaluate $\lim\limits_{\alpha\to \frac{\pi}{4}}\dfrac{\sin\alpha-\cos\alpha}{\alpha-\frac{\pi}{4}}$

[CBSE-92C, 2002C, 2003, (Sample paper) 2003, Practice sample paper VIII]

Solution

Putting $\alpha=t+\dfrac{\pi}{4}$, when $\alpha\to\dfrac{\pi}{4}$, then $t\to 0$

$$\lim_{\alpha\to \frac{\pi}{4}}\frac{\sin\alpha-\cos\alpha}{\alpha-\frac{\pi}{4}}$$

$$=\lim_{t\to 0}\frac{\sin\left(t+\frac{\pi}{4}\right)-\cos\left(t+\frac{\pi}{4}\right)}{t+\frac{\pi}{4}-\frac{\pi}{4}}$$

$$=\lim_{t\to 0}\frac{\left(\sin t\cos\frac{\pi}{4}+\cos t\sin\frac{\pi}{4}\right)-\left(\cos t\cos\frac{\pi}{4}-\sin t\sin\frac{\pi}{4}\right)}{t}$$

$$=\lim_{t\to 0}\frac{\frac{\sin t}{\sqrt{2}}+\frac{\cos t}{\sqrt{2}}-\frac{\cos t}{\sqrt{2}}+\frac{\sin t}{\sqrt{2}}}{t}=\lim_{t\to 0}\frac{\frac{2\sin t}{\sqrt{2}}}{t}$$

$$=\lim_{t\to 0}\frac{\sqrt{2}\sin t}{t}=\sqrt{2}\lim_{t\to 0}\frac{\sin t}{t}$$

$$=\sqrt{2}\times 1=\sqrt{2}$$

2. $\lim\limits_{x\to 0}\dfrac{x\cos x-\sin x}{x^2\sin x}$ is equal to

[MP-1999]

Solution

We know that

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\text{........ and}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\text{........}$$

$$\text{Then }=\lim_{x\to 0}\frac{x\cos x-\sin x}{x^2\sin x}$$

$$=\lim_{x\to 0}\frac{x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}{x^2\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}$$

$$=\lim_{x\to 0}\frac{\left(\frac{1}{6}-\frac{1}{2}\right)+\left(\frac{1}{4!}-\frac{1}{5!}\right)x^2+\ldots\ldots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\ldots\ldots\ldots}$$

$$=\frac{\frac{1}{6}-\frac{1}{2}+0+0+\ldots\ldots\ldots}{1-0+0\ldots\ldots\ldots}$$

$$=\frac{1}{6}-\frac{1}{2}$$

$$=\frac{1-3}{6}=\frac{-2}{6}=\frac{-1}{3}$$

3. Evaluate the limit $\lim\limits_{x\to 0}\dfrac{x\tan 4x}{1-\cos 4x}$

[CBSE-2004]

Solution

$$\lim_{x\to0}\frac{x\tan 4x}{1-\cos 4x}$$

$$= \lim_{x\to0}\frac{x\frac{\sin 4x}{\cos 4x}}{2\sin^2 2x} = \lim_{x\to0}\frac{x\sin 4x}{2\sin^2 2x.\cos 4x}$$

$$= \lim_{x\to0}\frac{x.2\sin 2x\cos 2x}{\cos 4x.2\sin^2 2x} = \lim_{x\to0}\frac{\cos 2x}{\cos 4x}.\frac{2\sin 2x}{2\sin^2 2x}.x$$

$$= \lim_{x\to0}\frac{\cos 2x}{\cos 4x}.\frac{x}{\sin 2x}$$

$$= \frac{\lim\limits_{x\to0}\cos 2x}{\lim\limits_{x\to0}\cos 4x}.\lim_{x\to0}\frac{1}{2}\frac{2x}{\sin 2x}$$

$$= \frac{1}{1}.\frac{1}{2}\lim_{x\to0}\left(\frac{2x}{\sin 2x}\right) = 1.\frac{1}{2}.1 = \frac{1}{2}$$

4. Evaluate the limit $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{2-\operatorname{cosec}^2 x}{1-\cot x}$

[CBSE-2001]

Solution

$$\lim_{x\to\frac{\pi}{4}}\frac{2-\operatorname{cosec}^2 x}{1-\cot x} = \lim_{x\to\frac{\pi}{4}}\frac{2-(1+\cot^2 x)}{1-\cot x}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{1-\cot^2 x}{1-\cot x} = \lim_{x\to\frac{\pi}{4}}\frac{(1-\cot x)(1+\cot x)}{1-\cot x}$$

$$= \lim_{x\to\frac{\pi}{4}}(1+\cot x) = \left(1+\cot\frac{\pi}{4}\right)$$

$$= (1 + 1) = 2$$

5. Evaluate $\lim\limits_{x\to0}\dfrac{\tan 3x-2x}{3x-\sin^2 x}$ ***[AISSE-89]***

Solution

Dividing numerator and denominator by x

$$= \lim_{x\to0}\frac{\frac{\tan 3x}{x}-\frac{2x}{x}}{\frac{3x}{x}-\frac{\sin^2 x}{x}}$$

$$= \lim_{x\to0}\frac{\frac{\tan 3x}{x}-2}{3-\frac{\sin^2 x}{x}}$$

$$= \lim_{x\to0}\left(\frac{3.\frac{\tan 3x}{3x}-2}{3-\frac{\sin x}{x}.\sin x}\right) = \frac{3\lim\limits_{x\to0}\frac{\tan 3x}{3x}-2}{3-\lim\limits_{x\to0}\frac{\sin x}{x}.\lim\limits_{x\to0}\sin x}$$

$$= \frac{3\times1-2}{3-1\times\sin 0} = \frac{3-2}{3-1\times0} = \frac{1}{3}$$

6. Evaluate $\lim\limits_{x\to-1}\dfrac{x^3+1}{x+1}$

[HPB-2000; HPSB-2000]

Solution

When $x = -1$ the expression $\dfrac{x^3-1}{x-1}$ assumes the indeterminate form $\dfrac{0}{0}$. Therefore, $(x + 1)$ is a common factor in numerator and denominater. Factorizing the numerator and denominator, we have $\lim\limits_{x\to-1}\dfrac{x^3+1}{x+1}$ $\left(\text{form } \dfrac{0}{0}\right)$

$$= \lim_{x\to-1}\frac{(x+1)(x^2-x+1)}{(x+1)} = \lim_{x\to-1}x^2-x+1$$

$$= (-1)^2 - (-1) + 1 = 1 + 1 + 1 = 3$$

7. Evaluate the limit $\lim\limits_{x\to1}\dfrac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}}$

[West Bengal-79]

Solution

$$\lim_{x\to1}\frac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}}$$

$$= \lim_{x\to1}\frac{x^2-1}{\sqrt{3x+1}-\sqrt{5x-1}}\times\frac{\sqrt{3x+1}+\sqrt{5x-1}}{\sqrt{3x+1}+\sqrt{5x-1}}$$

$$= \lim_{x\to1}\frac{(x^2-1)(\sqrt{3x+1}+\sqrt{5x-1})}{(\sqrt{3x+1})^2-(\sqrt{5x-1})^2}$$

$$= \lim_{x\to1}\frac{(x^2-1)(\sqrt{3x+1}+\sqrt{5x-1})}{3x+1-5x+1}$$

$$= \lim_{x\to1}\frac{(x+1)(x-1)(\sqrt{3x+1}+\sqrt{5x-1})}{-2(x-1)}$$

$$= \lim_{x\to1}\frac{(x+1)(\sqrt{3x+1}+\sqrt{5x-1})}{-2}$$

$[\because x \neq 1 \Rightarrow x - 1 \neq 0]$

$$= -\frac{1}{2}\lim_{x\to 1}(x+1)(\sqrt{3x+1}+\sqrt{5x-1})$$

$$= -\frac{1}{2}(1+1)(\sqrt{3.1+1}+\sqrt{5.1-1})$$

$$= -\frac{1}{2}\ .\ 2\ .(2+2) = -1(4) = -4.$$

8. Evaluate $\lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}$ ***[MP-96, 97]***

Solution

$$\lim_{x\to\infty}\frac{(3x-1)(4x-2)}{(x+8)(x-1)}$$

$$= \lim_{x\to\infty}\frac{x\left(3-\frac{1}{x}\right).x.\left(4-\frac{2}{x}\right)}{x.\left(1+\frac{8}{x}\right).x.\left(1-\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty}\frac{x^2\left(3-\frac{1}{x}\right)\left(4-\frac{2}{x}\right)}{x^2\left(1+\frac{8}{x}\right)\left(1-\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty}\frac{\left(3-\frac{1}{x}\right)\left(4-\frac{2}{x}\right)}{\left(1+\frac{8}{x}\right)\left(1-\frac{1}{x}\right)} = \frac{(3-0)(4-0)}{(1+0)(1-0)}$$

$$= \frac{3\times 4}{1\times 1} = 12$$

9. Find the value of $\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x}$

[MP-2001]

Solution

$$\lim_{x\to 0}\frac{x(e^x-1)}{1-\cos x} = \lim_{x\to 0}\frac{e^x-1}{x}.\frac{x^2}{(1-\cos x)}$$

$$= \lim_{x\to 0}\frac{e^x-1}{x}\times\frac{x^2}{2\sin^2\frac{x}{2}}$$

$$= \lim_{x\to 0}\frac{e^x-1}{x}\times\frac{\frac{x}{2}\times\frac{x}{2}}{\sin\frac{x}{2}.\sin\frac{x}{2}}\times 2$$

$$= 2\lim_{x\to 0}\frac{e^x-1}{x}.\lim_{\frac{x}{2}\to 0}\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)^2$$

$$= 2\times 1\times 1 = 2$$

10. Evaluate the limit $\lim_{x\to 0}\frac{3^{2x}-2^{3x}}{x}$

[PSB-2002; PSB-2001(C)]

Solution

$$3^{2x} = 1 + (2x\log 3) + \frac{(2x\log 3)^2}{2!} + \ldots..$$

$$2^{3x} = 1 + (3x\log 2) + \frac{(3x\log 2)^2}{2!} + \ldots..$$

$$\therefore\quad \lim_{x\to 0}\frac{3^{2x}-2^{3x}}{x} = 2\log 3 - 3\log 2 = \log\frac{9}{8}$$

11. Show that $\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x} = 1$

[CBSE-2004]

Solution

We have $\lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x}$

$$= \lim_{x\to 0}\frac{\log(1+x^3)}{x^3}.\frac{x^3}{\sin^3 x}$$

$$= \lim_{x\to 0}\frac{\log(1+x^3)}{x^3}.\lim_{x\to 0}\left(\frac{x}{\sin^3 x}\right)^3$$

$$= 1\ .\ (1)^3 = 1$$

Proved.

12. Evaluate the limit $\lim_{x\to 1}\frac{1-\frac{1}{x}}{\sin\pi(x-1)}$

[PSB-99]

Solution

$$\lim_{x\to 1}\frac{1-\frac{1}{x}}{\sin\pi(x-1)}$$

$$= \lim_{x\to 1}\frac{x-1}{x\sin\pi(x-1)}$$

Put $x = 1 + h$, so that when $x \to 1, h \to 0$

$$= \lim_{h\to 0}\frac{(1+h)-1}{(1+h)\sin\pi(1+h-1)} = \lim_{h\to 0}\frac{h}{(1+h)\sin\pi h}$$

$$= \frac{1}{\lim\limits_{h\to 0}(1+h)}\times\left(\frac{1}{\lim\limits_{\pi h\to 0}\frac{\sin\pi h}{\pi h}}\times\frac{1}{\pi}\right)$$

$$= \frac{1}{(1+0)}\times\frac{1}{\pi}\times\frac{1}{\lim\limits_{\pi h\to 0}\frac{\sin\pi h}{\pi h}}$$

$$= \frac{1}{1}\times\frac{1}{\pi}\times\frac{1}{1} = \frac{1}{\pi}.\frac{1}{1} = \frac{1}{\pi}$$

13. Evaluate the limit $\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{1-\sin x}{(\pi-2x)^2}\right]$

[CBSE (foreign)-2000]

Solution

We have $\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{1-\sin x}{(\pi-2x)^2}\right]$

$$= \lim_{h\to 0}\frac{1-\sin\left(\frac{\pi}{2}+h\right)}{\left(\pi-\left(\frac{\pi}{2}+h\right).2\right)^2}\quad\left(\text{form }\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\frac{1-\cos h}{(\pi-\pi-2h)^2} = \lim_{h\to 0}\frac{1-\cos h}{4h^2}$$

$$= \lim_{h\to 0}\frac{2}{4}\frac{\sin^2\frac{h}{2}}{h^2} = \frac{1}{2}\lim_{h\to 0}.\frac{1}{4}\frac{\sin\frac{h}{2}}{h/2}.\frac{\sin\frac{h}{2}}{h/2}$$

$$= \frac{1}{2}.\frac{1}{4}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{h/2}\right).\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{h/2}\right)$$

$$= \frac{1}{8}\times 1\times 1 = \frac{1}{8}$$

14. Evaluate the limit $\lim\limits_{x\to 0}\frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{2x^2}$

[CBSE (O.D.)-93C]

Solution

When $x = 0$, the expression $\frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{2x^2}$ takes the form $\frac{0}{0}$. Rationalizing the numerator, we have $\lim\limits_{x\to 0}\frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{2x^2}$

$$= \lim_{x\to 0}\frac{\sqrt{1-x^2}-\sqrt{1+x^2}}{2x^2}\times\frac{\sqrt{1-x^2}+\sqrt{1+x^2}}{\sqrt{1-x^2}+\sqrt{1+x^2}}$$

$$= \lim_{x\to 0}\frac{1-x^2-1-x^2}{2x^2(\sqrt{1-x^2}+\sqrt{1+x^2})}$$

$$= \lim_{x\to 0}\frac{-2x^2}{2x^2(\sqrt{1-x^2}+\sqrt{1+x^2})}$$

$$= \lim_{x\to 0}\frac{-1}{\sqrt{1-x^2}+\sqrt{1+x^2}} = \frac{-1}{\sqrt{1}+\sqrt{1}} = \frac{-1}{2}$$

15. Evaluate the limit $\lim\limits_{x\to\frac{1}{\sqrt{2}}}\frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)}$

[CBSE (Foreign)-2005 (II); PSB-90]

Solution

Let $\sin^{-1}x = \theta$ Then $x = \sin\theta$

Now $x = \frac{1}{\sqrt{2}} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow\quad \theta = \pi/4$

$$\therefore\quad \lim_{x\to\frac{1}{\sqrt{2}}}\frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)}$$

$$= \lim_{\theta\to\frac{\pi}{4}}\frac{\sin\theta-\cos\theta}{1-\tan\theta} = \lim_{\theta\to\frac{\pi}{4}}\frac{\sin\theta-\cos\theta}{1-\frac{\sin\theta}{\cos\theta}}$$

$$\lim_{\cdot\to\frac{A}{4}}\frac{(\sin\theta-\cos\theta)\cos\theta}{\cos\theta-\sin\theta}$$

$$= \lim_{\theta\to\frac{\pi}{4}}\frac{-(\cos\theta-\sin\theta)\cos\theta}{(\cos\theta-\sin\theta)}$$

$$= \lim_{\theta\to\frac{\pi}{4}}-\cos\theta = -\cos\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

[redacted]
TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}$

[CBSE-88, 91]

2. Evaluate $\lim\limits_{x\to 0}\dfrac{x}{\sqrt{1+x}-1}$

[MP-97; Raj. 82, 86; AISSE-92]

3. Find the value of $\lim\limits_{x\to 1}\dfrac{x^2-4x+3}{x^2+2x-3}$.

4. Evaluate $\lim\limits_{n\to\infty}\dfrac{\sum n^3}{n^4}$

5. Evaluate $\lim\limits_{x\to 0}\left[\dfrac{3^x-1}{\sqrt{1+\sin x}-1}\right]$

[CBSE (Sample Paper) (III)]

6. If $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}=\lim\limits_{x\to k}\dfrac{x^3-k^3}{x^2-k^2}$. Then find the value of k.

7. Prove $\lim\limits_{x\to 0}\left(1+\dfrac{x}{a}\right)^{a/x}=e$

8. Evaluate $\lim\limits_{x\to\infty}2^x\sin\left(\dfrac{a}{2^x}\right)$

9. Evaluate $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$

10. If $f(x)=\begin{cases}\dfrac{x-|x|}{2}, & x\neq 0\\ 2, & x=0\end{cases}$ show that $\lim\limits_{x\to 0}f(x)$ does not exist.

11. Show that $\lim\limits_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

12. Let $f(x)$ be a function defined by

$$f(x)=\begin{cases}4x-5, & \text{if } x\le 2\\ x-\lambda, & \text{if } x>2\end{cases}.$$

Find λ, if $\lim\limits_{x\to 2}f(x)$ exists.

13. Evaluate the limit $\lim\limits_{x\to 0}\dfrac{\sin 2x+3x}{4x-\sin 5x}$.

[CBSE-99]

14. Evaluate the limit $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{1-\cos 2x}$.

[CBSE-2002(C)]

15. Evaluate the limit $\lim\limits_{x\to 0}\dfrac{\cos(2-x)-\cos(2+x)}{x}$

[CBSE-97 (C)]

16. Evaluate $\lim\limits_{x\to 0}\dfrac{(e^{2x}-1)(1-\cos 2x)}{x^3}$.

[Maharastha Board March-87]

17. Evaluate $\lim\limits_{x\to 0}\dfrac{a^{\sin x}-1}{b^{\sin x}-1}$

[Karnataka (CET)-2000]

18. Find k so that $\lim\limits_{x\to 2}f(x)$ may exists where $f(x)$

$$=\begin{cases}2x+3 & \text{if } x\le 2\\ x+3k & \text{if } x>2\end{cases}.$$

[CBSE-2001(C)]

19. Show that $\lim\limits_{x\to\infty}\left(\dfrac{x^2+3x+3}{x^2-x+2}\right)^x=e^4$

[CBSE (Practice Sample Paper) (II)]

20. If $\lim\limits_{x\to 3}\dfrac{x^n-3^n}{x-3}=405$, then find the value of n.

21. Prove $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a$

[MP-95, 98, 99, 2001]

ANSWERS

1. 4
2. 2
3. $-\frac{1}{2}$
4. 1/4
5. 2 log 3
6. 8/3
8. a
9. –1/2
12. $\lambda = -1$
13. –5
14. 4
15. 2 sin 2
16. 4
17. $\frac{\log a}{\log b}$
18. $k = 5/3$
20. 5

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. $\lim_{x\to 0}\frac{1+e^{-1/x}}{1-e^{-1/x}}$ is equal to

[PET (Raj.)-1994]

(a) 1 (b) –1
(c) 0 (d) does not exist

Solution

(d) $\text{LHL} = \lim_{h\to 0}\frac{1+e^{1/h}}{1-e^{1/h}} = \lim_{h\to 0}\frac{e^{-1/h}+1}{e^{-1/h}-1} = -1$

$\text{RHL} = \lim_{h\to 0}\frac{1+e^{-1/h}}{1-e^{-1/h}} = \frac{1+0}{1-0} = 1$

LHL $\neq$ RHL, so given limit does not exist.

2. If $f(x) = \begin{cases} x & \text{when } x\in Q \\ -x & \text{when } x\notin Q \end{cases}$, then $\lim_{x\to 0} f(x)$ equals

[Kurukshetra (CEE)-98]

(a) 0 (b) 1
(c) –1 (d) does not exist

Solution

(a) $\text{LHL} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\{-h \text{ or } h\} = 0$

$\text{RHL} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\{h \text{ or } -h\} = 0$

$\therefore$ Limit = 0

3. $\lim_{x\to -\infty}\frac{\sqrt{4x^2+5x+8}}{4x+5}$ is equal to

[Roorkee (Screening)-98]

(a) –1/2 (b) 0
(c) 1/2 (d) 1

Solution

(a) Putting $x = -t$

$$\text{limit} = \lim_{t\to\infty}\frac{\sqrt{4t^2-5t+8}}{-4t+5} = -\frac{1}{2}$$

4. If $\lim_{x\to 0}\frac{x(1-\cos x)-ax^2\sin x}{x^5}$. exists finitely, then a is equal to ***[ICS-2001]***

(a) 1 (b) 1/2
(c) 1/3 (d) 1/4

Solution

(b) $$\text{Limit} = \lim_{x\to 0}\frac{x\left(1-1+\frac{x^2}{2!}-\frac{x^4}{4!}+\ldots\ldots\right)-ax^2\left(x-\frac{x^3}{3!}+\ldots\ldots\right)}{x^5}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{2}-a\right)x^3+\left(-\frac{1}{24}+\frac{a}{6}\right)x^5+\ldots.}{x^5}.$$

It will exist finitely if $\frac{1}{2}-a=0$

i.e., when $a = 1/2$

Also then limit $=-\frac{1}{24}+\frac{1}{12}=\frac{1}{24}$

5. $\lim_{h\to 0}\left[\frac{1}{h\sqrt[3]{8+h}}-\frac{1}{2h}\right]$ is equal to

[NDA-2005]

(a) 1/12
(b) −1/12
(c) 1/48
(d) −1/48

Solution

(d) Limit $=\lim_{h\to 0}\frac{2-(8+h)^{1/3}}{2h(8+h)^{1/3}}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{h\to 0}\frac{-\frac{1}{3}(8+h)^{-2/3}}{2(8+h)^{1/3}-\frac{2}{3}h(8+h)^{-2/3}}=-\frac{1}{48}$$

(L-Hospitals')

6. $\lim_{x\to 0}\frac{\sin|x|}{x}$ is equal to

[NDA-2006]

(a) 1 (b) −1
(c) ∞ (d) does not exist

Solution

$$\text{LHL}=\lim_{h\to 0}\frac{\sin|-h|}{-h}=\lim_{h\to 0}\frac{\sin h}{-h}=-1$$

$$\text{RHL}=\lim_{h\to 0}\frac{\sin|h|}{h}=\lim_{h\to 0}\frac{\sin h}{h}=1$$

$\because$ LHL ≠ RHL, so limit does not exist.

7. $\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)$ is equal to

[UPSEAT-99]

(a) $\pi/4$ (b) $\pi^2/4$
(c) $\pi/2$ (d) $\pi^2/16$

Solution

(a) Let $T_r=\tan^{-1}\left(\frac{1}{2r^2}\right)$, then

$$T_r=\tan^{-1}\left(\frac{2}{4r^2}\right)$$

$$=\tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right)$$

$$=\tan^{-1}(2r+1)-\tan^{-1}(2r-1)$$

$\Rightarrow$ $T_1=\tan^{-1}3-\tan^{-1}1$, $T_2=\tan^{-1}5-\tan^{-1}3$, $T_3=\tan^{-1}7-\tan^{-1}5$, ...

$T_n=\tan^{-1}(2n+1)-\tan^{-1}(2n-1)$

$\therefore$ $\sum_{r=1}^{n}T_r=\tan^{-1}(2n+1)-\tan^{-1}1$

$$=\tan^{-1}(2n+1)-\frac{\pi}{4}$$

$\therefore$ Limit $=\tan^{-1}\infty-\frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

NOTE

Question on the same lines was repeated in JEE-2006 $\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{1}{2i^2}\right)=t$, then $\tan t = I$

[IIT-JEE-06]

8. $\lim_{\theta\to 0}\frac{4\theta(\tan\theta-\sin\theta)}{(1-\cos\theta)^2}$ is ***[Orissa JEE-2005]***

(a) $1/\sqrt{2}$ (b) 1/2
(c) 1 (d) 2

Solution

(b)

$$\lim_{\theta\to 0}\frac{4\theta(\tan\theta-\sin\theta)}{(1-\cos 2\theta)^2}=\lim_{\theta\to 0}\frac{4\theta\sin\theta(1-\cos\theta)}{4\sin^4\theta\cos\theta}$$

$$=\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right)\frac{2\sin^2\theta/2}{\sin^2\theta\cos\theta}$$

$$=\lim_{\theta\to 0}\frac{2\sin^2\theta/2}{(2\sin(\theta/2)\cos(\theta/2)^2)}\frac{1}{\cos\theta}$$

$$=\lim_{\theta\to 0}\frac{1}{2}\frac{1}{\cos^2(\theta/2).\cos\theta}=\frac{1}{2}$$

9. The value of the constant α and β such that $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-\alpha x-\beta\right)=0$ are respectively.

(a) (1, 1)
(b) (–1, 1)
(c) (1, –1)
(d) (0, 1)

Solution

(c) $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-\alpha x-\beta\right)=0$

$$\Rightarrow \lim_{x\to\infty}\frac{x^2(1-\alpha)-x(\alpha+\beta)+1-b}{x+1}=0$$

Since the limit of the given expression is zero, therefore degree of the polynomial in numerator must be less than denominator.

∴ $1-\alpha=0$ and
$\alpha+\beta=0$

⇒ $\alpha=1$ and
$\beta=-1$.

10. Find the values of a and b in order that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$$

Solution

Series expansion method:

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$$

$\left(\text{from } \frac{0}{0}\right)$

$$\Rightarrow \lim_{x\to 0}\frac{x\left\{1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\right\}-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}{x^3}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{x(1+a-b)+x^3\left(\frac{a}{2!}+\frac{b}{3!}\right)+x^5\left(\frac{a}{4!}-\frac{b}{5!}\right)-\ldots..}{x^3}=1$$

$$\Rightarrow \lim_{x\to 0}\frac{(1+a-b)}{x^2}+\lim_{x\to 0}\left(-\frac{a}{2!}+\frac{b}{3!}\right)+\lim_{x\to 0}\left(\frac{a}{4!}-\frac{b}{5!}\right)x^2-\ldots=1..(1)$$

Since R.H.S. is finite ∴ $1+a-b=0$

∴ $b=1+a$ ...... (2)

then (1) becomes $-\frac{a}{2!}+\frac{b}{3!}=1 \Rightarrow -3a+b=6$ .. (3)

From (2) and (3), we get $a=-5/2,\ b=-3/2$

11. $f(x)=\sin x,\ x\neq n\pi,\ n=0,\pm1,\pm2,\pm3,$ ....
= 2, otherwise
and $g(x)=x^2+1,\ x\neq 0, 2$
$=4,\ x=0$
$=5,\ x=2$, then $\lim_{x\to 0} g[f(x)]$ is

[IIT JEE-1986]

Solution

given that $f(x)=\sin x, x\neq n\pi, n=0,\pm1,\pm2,$.......
= 2, otherwise
and $g(x)=x^2+1,\ x\neq 0, 2$
$=4,\ x=0\quad =5,\ x=2,$

Then $\lim_{x\to 0} g[f(x)]=\lim_{x\to 0} g(\sin x)=\lim_{x\to 0}(\sin^2 x+1)=1$

12. $\lim_{x\to 0}\cos\frac{1}{x}$ is ***[UPSEAT-2002]***

(a) is continuous at $x=0$
(b) differentiable at $x=0$
(c) does not exist
(d) None of these

Solution

(c) The value of $\lim_{x\to 0}\cos\frac{1}{x}$ lies between –1 and 1 then limit does not exist.

13. If $f(x)=\begin{cases}\frac{\sin[x]}{[x]}, & [x]\neq 0\\ 0, & [x]=0\end{cases}$, then $\lim_{x\to 0} f(x)$ is equal to, where [.] denotes greatest integer function

(a) 1 (b) 0
(c) –1 (d) does not exist

[IIT-85; PET (Raj.)-95; Aligarh-98]

Solution

(d) When $-1 \le x < 0$, then $f(x) = \dfrac{\sin(-1)}{-1}$

$= \sin 1$ and when $0 \le x < 1$, then $f(x) = 0$

$[\because [x] = 0 \Rightarrow f(x) = 0]$

$\therefore \quad f(0-0) = \lim_{h\to 0} \sin 1 = \sin 1$

$f(0+0) = \lim_{h\to 0} (0) = 0$

$\because \quad f(0-0) \ne f(0+0)$

$\therefore \quad \lim_{x\to 0} f(x)$ does not exist.

14. The integral value of n for which $\lim_{x\to 0} \dfrac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a fixed non-zero number will be

(a) 1 (b) 2

(c) 3 (d) 4

[IIT (Screening)-2002]

Solution

(c)

$$\text{Limit} = \lim_{x\to 0} \frac{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-......\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-......-1-\frac{x}{1!}-\frac{x^2}{2!}-......\right)}{x^n}$$

$$= \lim_{x\to 0} \frac{x^3\left(-\frac{1}{2!}+\frac{x}{24}-......\right)(-1-x+......)}{x^n}$$

This clearly shows that its value will be 1/2 (a non zero number) if $n = 3$.

15. $\lim_{x\to 0} \dfrac{\left\{\sum_{i=2}^{2006} i^x\right\} - 2005}{x}$ is equal to

[Gujarat CET-2007]

(a) $\log_e 2006$ (b) $\log_e \left\{\sum_{i=1}^{2006} i\right\}$

(c) $\log_e \{2006!\}$ (d) None of these

Solution

(c) $\lim_{x\to 0} \dfrac{\left\{\sum_{i=2}^{2006} i^x\right\} - 2005}{x}$

$$= \lim_{x\to 0} \frac{2^x + 3^x + ...+(2006)^x - 2005}{x}$$

$$= \lim_{x\to 0} \frac{(2^x - 1) + (3^x - 1) + ...+(2006^x - 1)}{x}$$

$$= \left(\lim_{x\to 0} \frac{2^x - 1}{x}\right) + \left(\lim_{x\to 0} \frac{3^x - 1}{x}\right) + ...+ \lim_{x\to 0}\left(\frac{2006^x - 1}{x}\right)$$

$= \ln 2 + \ln 3 + ... + \ln 2006$

$= \ln (2 . 3 .... 2006) = \ln (2006)!$

$= \log_e (2006)!$

16. $\lim_{x\to 1} \dfrac{p}{1-x^p} - \dfrac{q}{1-x^q} =$

(a) $\dfrac{p-q}{2}$ (b) $\dfrac{q}{2}$

(c) $\dfrac{p}{2}$ (d) None of these

Solution

(a) $\lim_{x\to 1} \dfrac{p}{1-x^p} - \dfrac{q}{1-x^q} =$

$$\lim_{x\to 1} \frac{p - px^q - q + qx^p}{(1-x^p)(1-x^q)}$$

$$= \lim_{x\to 1} \frac{p(1-x^q) - q(1-x^p)}{(1-x^p)(1-x^q)}$$

using L-hospital rule

$$\lim_{x\to 1} \frac{pq(x^{p-1} - x^{q-1})}{-px^{p-1} - qx^{q-1} + (p+q)x^{p+q-1}}$$

$$\lim_{x\to 1} \frac{(pq)x^{q-1}(x^{p-q} - 1)}{px^{p-1}(x^q - 1) + qx^{q-1}(x^p - 1)}$$

$$\lim_{x\to 1} \frac{pqx^{q-1}(x^{p-q} - 1)}{px^{p-q}(x^q - 1) + q(x^p - 1)}$$

$$\frac{pq \lim_{x\to 1} x^{q-1} \lim_{x\to 1} \frac{(x^{p-q} - 1)}{x - 1}}{p \lim_{x\to 1} x^{p-q} \lim_{x\to 1} \frac{x^q - 1}{x - 1} + q \lim_{x\to 1} \frac{x^p - 1}{x - 1}}$$

$$= \frac{pq(p-q)}{p(q) + pq} = \left(\frac{p-q}{2}\right)$$

17. Let x be an irrational, then

$\underset{m\to\infty}{Lt}\ \underset{n\to\infty}{Lt}[(\cos n!\pi x)]^{2m}$ equals

(a) 0 (b) −1
(c) 1 (d) Indeterminate

Solution

(a) $\underset{m\to\infty}{Lt}\ \underset{n\to\infty}{Lt}[(\cos n!\pi x)]^{2m}$

$= \underset{m\to\infty}{Lt}[\cos(\underset{n\to\infty}{Lt}\ n!\pi x)]^{2m} = \underset{m\to\infty}{Lt}[\cos(\infty)]^{2m}$

$\because \underset{n\to\infty}{Lt}\ n!\pi = \infty$

$\because -1 \le \cos\infty \le 1;\ = (\cos\infty)^{2m} = 0$
(= irrational no.)

18. $\lim\limits_{x\to 0}\dfrac{e^x - e^{\sin x}}{2(x-\sin x)}$ ***[EAMCET-2007]***

(a) −1/2 (b) 1/2
(c) 1 (d) 3/2

Solution

(b) $\lim\limits_{x\to 0}\dfrac{e^x - e^{\sin x}}{2(x-\sin x)} = \lim\limits_{x\to 0}\left(\dfrac{e^{x-\sin x}-1}{x-\sin x}\right)\cdot\dfrac{e^{\sin x}}{2} = \dfrac{1}{2}$

19. If $f(x) = \begin{cases}\dfrac{\sin(1+[x])}{[x]} & \text{for } [x]\neq 0\\ 0 & \text{for } [x]=0\end{cases}$ where $[x]$ denotes the greatest integer not exceeding x, then $\lim\limits_{x\to 0^-} f(x)$ is equal to ***[EAMCET-2007]***

(a) −1 (b) 0
(c) 1 (d) 2

Solution

(b) $x \to 0^-$, $[x] = -1$

$\therefore \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0^-}\left(\dfrac{\sin(1+[x])}{[x]}\right) = \dfrac{0}{-1} = 0$

20. If $\lim\limits_{x\to 0}\dfrac{(x+3\sin x - x^3 - k\sin hx)}{1-\cos x + x^2 - 3x^3}$ exists, then what is the value of k? ***[UPSC-2007]***

(a) −1 (b) 2
(c) 3 (d) 4

Solution

(d) $\lim\limits_{x\to 0}\dfrac{x+3\sin x - x^3 - k\sinh x}{1-\cos x + x^2 - 3x^3}$

$$\lim_{x\to 0}\frac{x+3\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots\right) - x^3 - k\left[x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\ldots\right]}{1-\left\{1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}\ldots\right\}+x^2-3x^3}$$

$$\lim_{x\to 0}\frac{(4-k)x - x^3\left(\dfrac{1}{2}+1-\dfrac{k}{6}\right)+x^5+\ldots}{\dfrac{3x^2}{2}-3x^3-\dfrac{x^4}{4!}+\ldots}$$

For limit to exist $4-k=0 \Rightarrow k=4$

21. $\lim\limits_{x\to\infty}\dfrac{(2+x)^{40}.(4+x)^5}{(2-x)^{45}}$ is equal to ***[Gujarat CET-2007]***

(a) 32 (b) 16
(c) −1 (d) 1

Solution

(c) $\lim\limits_{x\to\infty}\dfrac{(2+x)^{40}(4+x)^5}{(2-x)^{45}}$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{x^{40}\left(\dfrac{2}{x}+1\right)^{40} x^5\left(\dfrac{4}{x}+1\right)^5}{x^{45}\left(\dfrac{2}{x}-1\right)^{45}}$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{\left(\dfrac{2}{x}+1\right)^{40}\left(\dfrac{4}{x}+1\right)^5}{\left(\dfrac{2}{x}-1\right)^{45}} = \dfrac{1.1}{-1} = -1$

22. $\lim\limits_{x\to 1}\dfrac{e^{-x}-e^{-1}}{x-1} =$ ***[Orissa-JEE-2007]***

(a) $1/e$ (b) e
(c) e^2 (d) $-1/e$

Solution

(d) $\lim\limits_{x\to 1}\dfrac{e^{-x}-e^{-1}}{x-1}$

(0/0 form)

Applying L-Hospital Rule

$= \lim\limits_{x\to 1}\dfrac{-e^{-x}-0}{1-0} = -e^{-1} = -\dfrac{1}{e}$

23. $\lim\limits_{x\to 1}\dfrac{\tan(x^2-1)}{x-1}$ is equal to

[Karnataka CET-2007]

(a) 2 (b) 1/2
(c) –2 (d) –1/2

Solution

(a) $\lim\limits_{x\to 1}\dfrac{\tan(x^2-1)}{x-1}\quad\left(\dfrac{0}{0}\right)$

$\lim\limits_{x\to 1}\dfrac{2x\sec^2(x^2-1)}{1}$

$\Rightarrow\quad = \lim\limits_{x\to 1} 2x\sec^2(x^2-1) = 2.1\sec^2(0)$

$= 2\,.\,1\,.\,1 = 2.$

24. If $\lim\limits_{x\to 0}\dfrac{\log(a+x)-\log a}{x}+k\lim\limits_{x\to e}\dfrac{\log x-1}{x-e}=1$, then ***[IIT Screening]***

(a) $k = e\,(1 - 1/a)$
(b) $k = e\,(1 + a)$
(c) $k = e\,(2 - a)$
(d) The equality is not possible

Solution

(a) Let $f(x) = \log x \Rightarrow f'(x) = \dfrac{1}{x}$

Therefore, given function $= f'(a) + kf'(e) = 1$

$\Rightarrow\quad \dfrac{1}{a}+\dfrac{k}{e}=1\Rightarrow k=e\left(\dfrac{a-1}{a}\right)$

Aliter: Applyg L-Hospital's rule to find both the limits

25. Value of $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x+3}$ is ***[MPPET-2007]***

(a) e (b) $1/e$
(c) e^3 (d) $e/3$

Solution

(a) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x+3} = \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}.\left(1+\dfrac{1}{x}\right)^{3}$

$= \lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}.\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{3} = e.1 = e$

26. Value of $\lim\limits_{x\to a}\dfrac{x^2-a^2}{x-a}$ ***[MPPET-2007]***

(a) a (b) 0
(c) $2a$ (d) $4a$

Solution

(c) $\lim\limits_{x\to a}\dfrac{x^2-a^2}{x-a} = \lim\limits_{x\to a}\dfrac{(x-a)(x+a)}{(x-a)}$

$= \lim\limits_{x\to a}(x+a) = 2a$

27. If $f(x)=1+\dfrac{\cos x}{\sin x}$ is continuous at $x=\dfrac{\pi}{2}$ then value of $f(\pi/2)$ is ***[MPPET-2007]***

(a) 0 (b) –1
(c) 1 (d) 2

Solution

(c) $f(x)=1+\dfrac{\cos x}{\sin x}$

$f(x) = \dfrac{1+\cos\pi/2}{\sin\pi/2} = \dfrac{1+0}{1} = 1$

28. $\lim\limits_{n\to\infty} n\cos\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{\pi}{4n}\right) = k$ then k is

[MPPET-2007]

(a) $\pi/4$ (b) $\pi/3$
(c) π (d) None of these

Solution

(a) $\lim\limits_{n\to\infty} n\cos\dfrac{\pi}{4n}\sin\dfrac{\pi}{4n}$

$\dfrac{1}{2}\lim\limits_{n\to\infty} n.2\sin\dfrac{\pi}{4n}\cos\dfrac{\pi}{4n} = \dfrac{1}{2}\lim\limits_{n\to\infty} n.\sin\dfrac{\pi}{4n}$

$= \dfrac{\pi}{4}\lim\limits_{n\to\infty}\dfrac{\sin\dfrac{\pi}{2n}}{\dfrac{\pi}{2n}} = \dfrac{\pi}{4}\quad\left\{\because n\to\infty\Rightarrow\dfrac{\pi}{2n}\to 0\right\}.$

29. The value of $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+9}-3}$ is ***[MPPET-2010]***

(a) 3 (b) 4
(c) 1 (d) 2

Solution

(a) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+9}-3} = \lim\limits_{x\to 0}\dfrac{x^2+1-1}{\sqrt{x^2+1}+1}$

$\times\dfrac{\sqrt{x^2+9}+3}{x^2+9-9}$ (upon rationalizing)

$= \lim\limits_{x\to 0}\dfrac{\sqrt{x^2+9}+3}{\sqrt{x^2+1}+1} = 3$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. $f(x) = [x - 1] + |x - 1|; x \neq 1$ where [.] denotes greatest integer function then $f(1 - 0)$ and $f(1 + 0)$ respectively will be

(a) $-1, 0$ (b) $0, -1$
(a) $-1, -1$ (b) $0, 0$

2. $\lim_{x\to 2^+}\left(\frac{[x]^3}{3} - \left[\frac{x}{3}\right]^3\right) =$

(a) 0 (b) $\frac{64}{27}$
(c) $\frac{8}{3}$ (d) None of these

3. If $f(x) = \begin{cases} x, & x < 0 \\ 1, & x = 0 \\ x^2, & x > 0 \end{cases}$, then $\lim_{x\to 0} f(x) =$ ***[DCE-2000]***

(a) Is 1
(b) Is zero
(c) Does not exist
(d) None of these

4. $\lim_{\alpha\to\beta}\left[\frac{\sin^2\alpha - \sin^2\beta}{\alpha^2 - \beta^2}\right] =$ ***[MPPET-2001]***

(a) 0 (b) 1
(c) $\frac{\sin\beta}{\beta}$ (d) $\frac{\sin 2\beta}{2\beta}$

5. $\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3} =$ ***[IIT-1977]***

(a) $-\frac{1}{10}$ (b) $\frac{1}{10}$
(c) $-\frac{1}{8}$ (d) None of these

6. If $f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $\lim_{x\to 0} f(x) =$

(a) 1 (b) 0
(c) -1 (d) None of these

7. Let the function f be defined by the equation

$f(x) = \begin{cases} 3x & \text{if } 0 \le x \le 1, \\ 5 - 3x & \text{if } 1 < x \le 2 \end{cases}$ then ***[SCRA-1996]***

(a) $\lim_{x\to 1} f(x) = f(1)$
(b) $\lim_{x\to 1} f(x) = 3$
(c) $\lim_{x\to 1} f(x) = 2$
(d) $\lim_{x\to 1} f(x)$ does not exist

8. $\lim_{n\to\infty}\left[\frac{1}{1-n^2} + \frac{2}{1-n^2} + ... + \frac{n}{1-n^2}\right]$ is equal to

(a) 0 (b) $-1/2$
(c) $1/2$ (d) None of these
[IIT-1984; DCE-2000; Pb. CET-2000]

9. $\lim_{n\to\infty}\left[\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... + \frac{1}{(2n+1)(2n+3)}\right]$ is equal to

(a) 0 (b) $1/2$
(c) $1/9$ (d) 2 ***[DCE-2005]***

10. $\lim_{x\to\infty}\frac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}} =$

(a) 16 (b) 24
(c) 32 (d) 8 ***[IIT-90]***

11. The value of $\lim_{x\to\infty}\left(\frac{x^2+bx+4}{x^2+ax+5}\right)$ is ***[MPPET-1993]***

(a) b/a (b) 1
(c) 0 (d) $4/5$

12. $\lim_{x\to\infty}\left(\sqrt{x^2+8x+3} - \sqrt{x^2+4x+3}\right)$ is equal to ***[MPPET-1997]***

(a) 0 (b) ∞
(c) 2 (d) $1/2$

13. If $x_n = \frac{1-2+3-4+5-6+.........2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$, then $\lim_{n\to\infty} x_n$ is equal to ***[AMU-2000]***

(a) $1/3$ (b) $-1/3$
(c) $2/3$ (d) $-2/3$

14. $\lim_{x\to\infty}\frac{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}-\sqrt{x^2+d^2}} =$

(a) $\frac{a^2-b^2}{c^2-d^2}$ (b) $\frac{a^2+b^2}{c^2-d^2}$

(c) $\frac{a^2+b^2}{c^2+d^2}$ (d) None of these

15. $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3} =$

(a) 1/2 (b) –1/2
(c) 2/3 (d) None of these

[IIT-1974; AICBSE-1986, 90; AISSE-1983, 86, 90; RPET-2000]

16. $\lim_{x\to 0}\frac{e^{\tan x}-e^x}{\tan x-x} =$

[EAMCET-1994; RPET-2001]

(a) 1 (b) e
(c) e^{-1} (d) 0

17. $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$ is equal to ***[RPET-2000]***

(a) 0 (b) 1
(c) –1 (d) 1/2

18. Values of constant a, b and c so that $\lim_{x\to 0}\frac{xae^x-b\log(1+x)+cxe^{-x}}{x^2\sin x}=2$, then $a =$, $b =$, $c =$

(a) 3, 12, 9 (b) 9, 6, 9
(c) 5, 10, 20 (d) None of these

19. $\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4} =$

(a) 1/8 (b) 1/2
(c) 1/4 (d) None of these

20. $\lim_{x\to 0}\frac{a^x-b^x}{e^x-1} =$ ***[Kerala (Engg.)-2002]***

(a) log (a/b)\
(b) log (b/a)
(c) log (ab)
(d) log ($a + b$)

21. The value of $\lim_{x\to 7}\left(\frac{2-\sqrt{x-3}}{x^2-49}\right)$ is

[MPPET-2003]

(a) 2/9 (b) –2/49
(c) –1/56 (d) –1/59

22. $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2} =$

(a) $-\pi$ (b) π
(c) $\pi/2$ (d) 1

[IIT Screening-2001; UPSEAT-2001; MPPET-2002]

23. $\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}, n\in N$, ([$x$] denotes greatest integer less than or equal to x)

[AIEEE-2002]

(a) Has value –1
(b) Has value 0
(c) Has value 1
(d) Does not exist

24. If $\lim_{x\to\infty}\left[\frac{x^3+1}{x^2+1}-(ax+b)\right]=2$, then

[Karnataka CET-2000]

(a) $a = 1$ and $b = 1$
(b) $a = 1$ and $b = -1$
(c) $a = 1$ and $b = -2$
(d) $a = 1$ and $b = 2$

25. $\lim_{x\to 0}\frac{a^{\sin x}-1}{b^{\sin x}-1} =$

[Karnataka CET-2000]

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$

(c) $\frac{\log a}{\log b}$ (d) $\frac{\log b}{\log a}$

26. $\underset{x\to 1}{Lt}\frac{x^n+x^{n-1}+x^{n-2}+.......+x^2+x-n}{x-1}$ is equal to

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n-1)}{2}$

(c) $\frac{n(n+2)}{2}$ (d) None of these

27. $\lim_{x\to\infty}x\sin\left(\frac{2}{x}\right)$ is equal to

[Karnataka CET-2008]

(a) 2 (b) 1/2
(c) ∞ (d) 0

28. What is the value of $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$?

[NDA-2008]

(a) 1 (b) 0
(c) ∞ (d) -1

29. If $\lim\limits_{x\to 1}\dfrac{ax^2+bx+c}{(x-1)^2}=2$, then (a, b, c) is

[Kerala PET-2007]

(a) $(2, -4, 2)$ (b) $(2, 4, 2)$
(c) $(2, 4, -2)$ (d) $(2, -4, -2)$
(e) $(-2, 4, 2)$

SOLUTIONS

1. $f(x)=[x-1]+|x-1|$

$f(1-0)=\lim\limits_{h\to 0}[1-h-1]+|1-h-1|=-1$

$f(1+0)=\lim\limits_{h\to 0}[1+h-1]+|1+h-1|=0$

2. (c) $\lim\limits_{x\to 2^+}\left(\dfrac{[x]^3}{3}-\left[\dfrac{x}{3}\right]^3\right)$, then putting $x = 2 + h$

$\dfrac{8}{3}-\left[\dfrac{2}{3}\right]^3=\dfrac{8}{3}-0=\dfrac{8}{3}$

3. (a) $f(x)=\begin{cases} x, & x<0 \\ 1, & x=0 \\ x^2, & x>0 \end{cases}$

LHL = RHL = 0

$\therefore\ \lim\limits_{x\to 0} f(x)=1$

4. (d) $\lim\limits_{\alpha\to\beta}\left[\dfrac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}\right]$; then

$\lim\limits_{\alpha\to\beta}\dfrac{\sin(\alpha-\beta)}{\alpha-\beta}\cdot\dfrac{\sin(\alpha+\beta)}{\alpha+\beta}=\dfrac{\sin 2\beta}{2\beta}$

5. (a) $\lim\limits_{x\to 1}\dfrac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$, then on rationalizing

$\lim\limits_{x\to 1}\dfrac{(2x-3)\left(\sqrt{x}-1\right)}{(x-1)(2x+3)}\times\dfrac{\sqrt{x}+1}{\sqrt{x}+1}=\dfrac{-1}{5.2}=\dfrac{-1}{10}$

6. (b) $\sin\dfrac{1}{x}$ lies between -1 and 1

so $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}=0\times$ finite $=0$

7. (d) $f(x)=\begin{cases} 3x & \text{if } 0\le x\le 1 \\ 5-3x & \text{if } 1<x\le 2 \end{cases}$; then

LHL $=\lim\limits_{x\to 1^-} f(x)=\lim\limits_{h\to 0} f(1-h)=\lim\limits_{h\to 0} 3(1-h)$

$=\lim\limits_{h\to 0}(3-3h)=3-3.0=3$

RHL $=\lim\limits_{x\to 1^+} f(x)=\lim\limits_{h\to 0} f(1+h)$

$=\lim\limits_{h\to 0}[5-3(1+h)]$

$=\lim\limits_{h\to 0}(2-3h)=2-3.0=2$

Hence $\lim\limits_{x\to 1} f(x)$ does not exist.

8. (b) $\lim\limits_{n\to\infty}\left[\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+\dots\dots+\dfrac{n}{1-n^2}\right]$

$=\lim\limits_{n\to\infty}\dfrac{\Sigma n}{1-n^2}=\dfrac{n(n+1)}{2(1-n^2)}=-\dfrac{1}{2}$

9. (b)

$\lim\limits_{n\to\infty}\left[\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dfrac{1}{5.7}+\dots\dots+\dfrac{1}{(2n+1)(2n+3)}\right]$;

then

$=\lim\limits_{n\to\infty}\dfrac{1}{2}\left[\left(1-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{5}\right)+\left(\dfrac{1}{5}-\dfrac{1}{7}\right)+\dots\dots+\left(\dfrac{1}{2n-1}-\dfrac{1}{2n+1}\right)\right]$

$=\lim\limits_{n\to\infty}\dfrac{1}{2}\left[1-\dfrac{1}{2n+1}\right]=\dfrac{1}{2}$

10. (c) $\lim\limits_{x\to\infty}\dfrac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}}$, then

$\lim\limits_{x\to\infty}\dfrac{\left(2+\dfrac{1}{x}\right)^{40}\left(4-\dfrac{1}{x}\right)^5}{\left(2+\dfrac{3}{x}\right)^{45}}=\dfrac{2^{40}4^5}{2^{45}}$

11. (b) $\lim_{x\to\infty}\left(\dfrac{x^2+bx+4}{x^2+ax+5}\right)$; then

$$\lim_{x\to\infty}\left(\frac{1+\frac{b}{x}+\frac{4}{x^2}}{1+\frac{a}{x}+\frac{5}{x^2}}\right)=1$$

12. (a) $\lim_{x\to\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right)$; on rationalizing, we get

$$=\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}}$$

$$=\lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{8}{x}+\frac{3}{x^2}}+\sqrt{1+\frac{4}{x}+\frac{3}{x^2}}}=2$$

13. (b) $x_n=\dfrac{1-2+3-4+5-6+........+2n}{\sqrt{n^2+1}+\sqrt{4n^2+1}}$; then

$$\lim_{n\to\infty}x_n=\frac{(1+3+5+.......+(2n-1))-(2+4+6+.....+2n)}{\sqrt{n^2+1}+\sqrt{4n^2+1}}$$

$$=\lim_{n\to\infty}\frac{n^2-n(n+1)}{\sqrt{n^2+1}+\sqrt{4n^2+1}}$$

$$=\lim_{n\to\infty}\frac{-n}{\sqrt{n^2+1}+\sqrt{4n^2+1}}=\lim_{n\to\infty}\frac{-1}{\sqrt{1+\frac{1}{n^2}}+\sqrt{4+\frac{1}{n^2}}}$$

$$=\frac{-1}{3}$$

14. (a) $\lim_{x\to\infty}\dfrac{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}-\sqrt{x^2+d^2}}$, on rationalising,

$$\lim_{x\to\infty}\frac{a^2-b^2}{c^2-d^2}\frac{\left[\sqrt{1+\frac{c^2}{x^2}}+\sqrt{1+\frac{d^2}{x^2}}\right]}{\left[\sqrt{1+\frac{a^2}{x^2}}+\sqrt{1+\frac{b^2}{x^2}}\right]}=\frac{a^2-b^2}{c^2-d^2}$$

15. (a) $\lim_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

$$=\lim_{x\to 0}\frac{\left(x+\frac{x^3}{3}+.......\right)-\left(x-\frac{x^3}{3!}+.......\right)}{x^3}$$

$$=\lim_{x\to 0}\frac{x^3\left(\frac{1}{3}+\frac{1}{3!}\right).........}{x^3}=\frac{1}{3}+\frac{1}{3!}=\frac{1}{2}$$

16. (a) $\lim_{x\to 0}\dfrac{e^{\tan x}-e^x}{\tan x-x}$, then

$$=\lim_{x\to 0}\frac{e^x\left[e^{\tan x-x}-1\right]}{\tan x-x}$$

$$=\lim_{x\to 0}e^x\lim\frac{e^{\tan x-x}-1}{\tan x-x}=e^{o}\times 1=1$$

$$\left(\because \lim_{t\to 0}\frac{e^t-1}{t}=1\right)$$

17. (d)

$$\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$$

$$=\lim_{x\to 0}\frac{\left(x+\frac{x^3}{6}+.......\right)-\left(x-\frac{x^3}{3}+.......\right)}{x^3}$$

$$=\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$$

18. (a) $\lim_{x\to 0}\dfrac{xae^x-b\log(1+x)+cxe^{-x}}{x^2\sin x}=2$

$$=\lim_{x\to 0}\frac{ax\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}.....\right)-b\left(x-\frac{x^2}{2!}+\frac{x^3}{3!}.....\right)+cx\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}.....\right)}{x^3}=2$$

($\because \sin x\to x$ as $x\to 0$)

$$=\lim_{x\to 0}\frac{x(a-b+c)+x^2\left(a+\frac{b}{2}-c\right)+x^3\left(\frac{a}{2!}-\frac{b}{3}+\frac{c}{2!}\right)..}{x^3}=2$$

$\therefore a-b+c=0;\ a+\dfrac{b}{2}-c=0$ and $\dfrac{a}{2}-\dfrac{b}{3}+\dfrac{c}{2}=2$

Solving we get $a=3$, $b=12$, $c=9$

19. (a)

$$\lim_{x\to 0}\frac{(1-\cos(1-\cos x))}{x^4}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{1-\cos x}{2}\right)}{x^4}$$

$$= \lim_{x\to 0}\frac{2\left(\dfrac{1-\cos x}{2}\right)^2}{x^4} = \lim_{x\to 0}\frac{\left(2\sin^2\dfrac{x}{2}\right)^2}{2x^4}$$

$$= \lim_{x\to 0} 2.\frac{\sin^4\dfrac{x}{2}}{x^4} = \lim_{x\to 0}\frac{2.\left(\dfrac{x}{2}\right)^4}{x^4} = \frac{1}{8}$$

NOTE

as $x\to 0$; $\sin x \to x$

20. (a) $\lim_{x\to 0}\frac{a^x-b^x}{e^x-1} = \lim_{x\to 0}\frac{a^x-b^x}{x}.\frac{x}{e^x-1}$

$$= \lim_{x\to 0}\left[\frac{a^x-1}{x}-\frac{b^x-1}{x}\right]\frac{x}{e^x-1}$$

$$= (\log_e a - \log_e b).\frac{1}{\log_e e}$$

$$= \log_e\left(\frac{a}{b}\right)$$

21. (c) $\lim_{x\to 7}\frac{2-\sqrt{x-3}}{x^2-49}$ $\left(\frac{0}{0}\text{form}\right)$

Now, Applying L-hospital's rule

$$\lim_{x\to 7}\frac{-1}{2\sqrt{x-3}(2x)}\;\frac{-1}{4\times 7\times 2}$$

$$= \frac{-1}{56}$$

22. (b) $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$; then

$$= \lim_{x\to 0}\frac{\sin(\pi(1-\sin^2 x))}{x^2}$$

$$= \lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{x^2}\times\frac{\pi\sin^2 x}{\pi\sin^2 x} = \pi$$

23. (a) $\lim_{x\to\infty}\frac{\log x^n - [x]}{[x]}$; then

$$= \lim_{x\to\infty}\frac{\log x^n}{[x]} - \lim_{x\to 0}\frac{[x]}{[x]} = 0-1 = -1$$

24. (c) $\lim_{x\to\infty}\left[\frac{x^3+1}{x^2+1}-(ax+b)\right] = 2$

$$= \lim_{x\to\infty}\left[\frac{x^3+1-(ax^3+bx^2+ax+b)}{x^2+1}\right] = 2$$

$$= \lim_{x\to\infty}\frac{x^3(1-a)-bx^2-ax-b}{x^2+1} = 2$$

$\therefore$ $1-a=0$; and $-b=2$

$a=1$, $b=-2$

25. (c) $\lim_{x\to 0}\frac{a^{\sin x}-1}{b^{\sin x}-1}$; then

$$= \lim_{x\to 0}\frac{a^{\sin x}-1}{\sin x}\times\frac{\sin x}{b^{\sin x}-1}$$

$$= \log_e a\times\frac{1}{\log_e b} = \frac{\log a}{\log b}$$

26. (a) $\lim_{x\to 1}\frac{x^n+x^{n-1}+x^{n-2}+......x^2+x-n}{x-1}$

$$= \lim_{x\to 1}\left[\frac{x^n-1}{x-1}+\frac{x^{n-1}-1}{x-1}+\frac{x^{n-2}-1}{x-1}.........+\frac{x-1}{x-1}\right]$$

$$= n+(n-1)+(n-2)+.....3+2+1$$

$$= \frac{n(n+1)}{2}$$

27. (d) $\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right)$, put $x=\frac{1}{t}$

$$= \lim_{t\to 0}\frac{\sin 2t}{t} = 2$$

28. (b) $\lim_{x\to\infty}\frac{\sin x}{x}$, Let $x=\frac{1}{y}$ or $y=\frac{1}{x}$, so that

$x\to\infty \Rightarrow y\to 0$

$$\therefore\ \lim_{x\to\infty}\left(\frac{\sin x}{x}\right) = \lim_{y\to 0}\left(y.\sin\frac{1}{y}\right)$$

$$= \lim_{y\to 0} y\times\lim_{y\to 0}\sin\frac{1}{y} = 0\times\text{finite} = 0$$

29. (a) $\lim_{x\to 1}\frac{ax^2+bx+c}{(x-1)^2} = 2$

put $x-1=t$ as $x\to 1, t\to 0$

$$\lim_{t\to 0}\frac{a(1+t)^2+b(1+t)+c}{t^2} = 2$$

$$\lim_{t\to 0}\frac{at^2+t(2a+b)+a+b+c}{t^2} = 2$$

$\therefore$ $2a+b=0$; $a+b+c=0$, $a=2$

$\therefore$ $b=-4$; $c=2$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. If $f(x) = \begin{cases} x, & \text{when } 0 \le x \le 1 \\ 2-x, & \text{when } 1 < x \le 2 \end{cases}$, then $\lim\limits_{x\to 1} f(x) =$

(a) 1 (b) 2
(c) 0 (d) Does not exist

2. $\lim\limits_{n\to\infty}\left[\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+.....+\frac{n}{n^2}\right]$ is

(a) 1/2 (b) 0
(c) 1 (d) ∞

3. $\lim\limits_{x\to 0}\left\{\dfrac{\sin x - x + \frac{x^3}{6}}{x^5}\right\} =$ **[MNR-1985]**

(a) $\frac{1}{120}$ (b) $-\frac{1}{120}$
(c) $\frac{1}{20}$ (d) None of these

4. $\lim\limits_{x\to 0}\frac{\tan^{-1}x}{x}$ is **[SCRA-1996]**

(a) 0 (b) ∞
(c) -1 (d) 1

5. $\lim\limits_{x\to 0}\frac{4^x-9^x}{x(4^x+9^x)} =$ **[EAMCET-2002]**

(a) log (2/3) (b) $\frac{1}{2}\log\left(\frac{3}{2}\right)$
(c) $\frac{1}{2}\log\left(\frac{2}{3}\right)$ (d) log (3/2)

6. If $f(x) = \sqrt{\frac{x-\sin x}{x+\cos^2 x}}$, then $\lim\limits_{x\to\infty} f(x)$ is **[DCE-2000]**

(a) 0 (b) ∞
(c) 1 (d) None of these

7. $\lim\limits_{x\to 0}\frac{e^{x^2}-\cos x}{x^2} =$ **[IIT Screening]**

(a) 3/2 (b) $-1/2$
(c) 1 (d) None of these

8. $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$ is **[Kurukshetra CEE-2002]**

(a) 0 (b) 1
(c) 2 (d) Not existent

9. The value of $\lim\limits_{x\to 0}\frac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}$ is

[MPPET-2000; UPSEAT-2000; Karnataka CET-2001]

(a) 10/3
(b) 3/10
(c) 6/5
(d) 5/6

10. The value of $\lim\limits_{x\to 2}\frac{3^{x/2}-3}{3^x-9}$ is **[MPPET-2000]**

(a) 0 (b) 1/3
(c) 1/6 (d) $ln3$

11. $\lim\limits_{x\to\infty}\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^n}$ equals **[RPET-1996]**

(a) 2 (b) -1
(c) 1 (d) 3

12. $\lim\limits_{x\to\infty}\frac{(2x-3)(3x-4)}{(4x-5)(5x-6)} =$ **[MPPET-1996]**

(a) 0 (b) 1/10
(c) 1/5 (d) 3/10

13. $\lim\limits_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x} = 2$, then a, b, c are respectively

(a) 1, 2, 3 (b) 1, 2, 1
(c) 2, 1, 2 (d) None of these

14. $\lim\limits_{x\to 0}\frac{e^{\alpha x}-e^{\beta x}}{x} =$

[MPPET-1994; DCE-2005]

(a) $\alpha+\beta$ (b) $\frac{1}{\alpha}+\beta$
(c) $\alpha^2-\beta^2$ (d) $\alpha-\beta$

15. $\lim\limits_{x\to 0} \dfrac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x} =$

(a) 1 (b) –1
(c) 0 (d) None of these

[IIT-1991; AIEEE-2002; RPET-2001, 2002]

16. $\lim\limits_{x\to 0} \dfrac{3\sin x - \sin 3x}{x^3} =$

(a) 4 (b) –4
(c) 1/4 (d) None of these

[AISSE-85]

17. $\lim\limits_{x\to 0} \dfrac{\sin mx}{\tan nx} =$ ***[DSSE-87]***

(a) n/m (b) m/n
(c) mn (d) None of these

18. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin^2 x} =$ ***[DSSE-87]***

(a) 1/2 (b) –1/2
(c) 2 (d) None of these

19. $\lim\limits_{x\to 2} \dfrac{|x-2|}{x-2} =$ ***[AICBSE-85]***

(a) 1 (b) –1
(c) does not exist (d) None of these

20. $\lim\limits_{x\to \pi/2} \dfrac{2x-\pi}{\cos x} =$ ***[IIT-1973]***

(a) 2 (b) 1
(c) –2 (d) None of these

21. $\lim\limits_{x\to 1} \dfrac{x-1}{2x^2-7x+5} =$ ***[IIT-76]***

(a) 1/3 (b) 1/11
(c) –1/3 (d) None of these

22. $\lim\limits_{x\to 1} \dfrac{\log x}{x-1} =$

(a) 1 (b) –1
(c) 0 (d) ∞

[RPET-1996; MPPET-1996, Pb. CET-2002]

23. If $\lim\limits_{x\to 0} \phi(x) = a^3, a \neq 0$, then $\lim\limits_{x\to 0} \phi\left(\dfrac{x}{a}\right)$ is

(a) a^2 (b) $1/a^3$
(c) $1/a^2$ (d) a^3

24. $\lim\limits_{x\to 0} (\cos x + a\sin bx)^{1/x}$ is equal to

(a) $e^{a/b}$ (b) $e^{b/a}$
(c) e^{ab} (d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 16 minutes.
3. The worksheet consists of 16 questions. The maximum marks are 48.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited

1. The value of $\lim_{n\to\infty}\frac{1-n^2}{\Sigma n}$ will be

(a) –2 (b) –1
(c) 2 (d) 1

[UPSEAT-1999]

2. $\lim_{x\to 4}\left[\frac{x^{3/2}-8}{x-4}\right]=$ ***[DCE-99]***

(a) 3/2 (b) 3
(c) 2/3 (d) 1/3

3. $\lim_{x\to\infty}\frac{2x^2+3x+4}{3x^2+3x+4}$ is equal to ***[SCRA-96]***

(a) 2/3 (b) 1
(c) 0 (d) ∞

4. $\lim_{x\to\pi/2}\frac{\tan 3x}{x}=$

(a) ∞ (b) 3
(c) 1/3 (d) 0

5. $\lim_{x\to 0}\frac{(1+x)^n-1}{x}=$

(a) n (b) 1
(c) –1 (d) None of these

[Kurukshetra CEE-2002]

6. $\lim_{x\to 0}\frac{\sin x - x}{x^3}=$

(a) 1/3 (b) –1/3
(c) 1/6 (d) –1/6

[MNR-1980, 1986]

7. The value of $\lim_{\theta\to 0}\left(\frac{\sin\frac{\theta}{4}}{\theta}\right)$ is ***[MPPET-1993]***

(a) 0 (b) 1/4
(c) 1 (c) not in existence

8. $\lim_{x\to 0}\frac{3x+4\tan x}{x}=$

(a) 0 (b) 1
(c) 7 (d) ∞

9. $\lim_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{2^2}+\dots\dots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+\dots\dots+\frac{1}{3^n}}$ is equal to

(a) 4/3 (b) 2/3
(c) 1/3 (d) None of these

10. If $f(x)$ is a polynomial satisfying $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ and $f(2)>1$, then $\lim_{x\to 1}f(x)$ is

(a) 2 (b) 1
(c) –1 (d) None of these

11. $\lim_{x\to\infty}\left[\sqrt{x^2+2x-1}-x\right]$ is equal to

[EAMCET-2006]

(a) ∞ (b) 1/2
(c) 4 (d) 1

12. $\lim_{x\to\frac{\pi}{6}}\frac{2\sin^2 x+\sin x-1}{2\sin^2 x-3\sin x+1}$ is equal to

[Kerala CEE-2003]

(a) 3 (b) –3
(c) 6 (d) 0

13. The value of $\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)$ is

[Karnataka CET-2001]

(a) 1/2 (b) ∞
(c) 1 (d) 0

14. $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+\dots\dots+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to

(a) 0 (b) 1
(c) 10 (d) 100

[AMU-2000]

15. True statement for $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{2+3x}-\sqrt{2-3x}}$ is

[BIT Ranchi-82]

(a) Lies between 0 and 1
(b) Lies between 0 and 1/2
(c) Lies between 1/2 and 1
(d) None of these

16. $\lim_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}=$

(a) 32
(b) $\sqrt{2}$
(c) $32\sqrt{2}$
(d) None of these

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

1. (a) $\lim_{n\to\infty}\frac{1-n^2}{\Sigma n}=\lim_{n\to\infty}\frac{1-n^2}{\frac{n(n+1)}{2}}$

$$=\lim_{n\to\infty}2\left[\frac{\frac{1}{n^2}-1}{1+\frac{1}{n}}\right]=-2$$

2. (b) $\lim_{x\to 4}\left[\frac{x^{3/2}-8}{x-4}\right]$

$$=\lim_{x\to 4}\frac{x^{3/2}-4^{3/2}}{x-4}=\frac{3}{2}.4^{\frac{3}{2}-1}=3$$

NOTE

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

8. (c) $\lim_{x\to 0}\frac{3x+4\tan x}{x}$

$$\lim_{x\to 0}\frac{3x+4\left(x+\frac{x^3}{3}+\ldots\ldots\right)}{x}$$

$$=\lim_{x\to 0}\left(7+\frac{4x^2}{3}+\ldots\ldots\right)$$

$$=7$$

9. (a) $\lim_{n\to\infty}\frac{1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\ldots\frac{1}{3^n}}=\frac{\frac{1}{1-\frac{1}{2}}}{\frac{1}{1-\frac{1}{3}}}=\frac{4}{3}$

NOTE

Sum of infinite GP.

$$=a+ar+ar^2+\ldots\ldots\ldots=\frac{a}{1-r}$$

10. (a) $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$

$\therefore\ f(x) = x^n + 1, (f(2) > 1)$

$$\lim_{x\to 1}\left(x^n + 1\right) = 1 + 1 = 2$$

11. (d)

$$\lim_{x\to\infty}(\sqrt{x^2+2x-1} - x) = \lim_{x\to\infty}\frac{2x-1}{\sqrt{x^2+2x-1}+x}$$

(on rationalising)

$$= \lim_{x\to\infty}\frac{2-\frac{1}{x}}{\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}+1}$$

(dividing numerator and denominator by x)

$$= \frac{2}{1+1} = 1$$

14. (d) $\lim\limits_{x\to\infty}\dfrac{(x+1)^{10}+(x+2)^{10}+........(x+100)^{100}}{x^{10}+10^{10}}$

$$= \lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+........}{1+\frac{10^{10}}{x^{10}}}$$

(Multiplying and dividing by x^{10})

$$= \frac{1+1+.........100 \text{ terms}}{1+0} = 100$$

16. (d)

$$\lim_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$$

$$= \lim_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)}{(4\theta-\pi)^4}$$

put $\theta - \dfrac{\pi}{4} = t$, as $\theta \to \dfrac{\pi}{4}, t \to 0$

$$\therefore \quad \lim_{t\to 0}\frac{\sqrt{2}(1-\cos t)}{4^2\left(\theta-\frac{\pi}{4}\right)^2}$$

$$= \lim_{t\to 0}\frac{\sqrt{2}}{16}\cdot\frac{2\sin^2\frac{t}{2}}{t^2}$$

$$= \lim_{t\to 0}\frac{\sqrt{2}}{8}\frac{\sin^2\frac{t}{2}}{(t/2)^2}\times\frac{1}{4}$$

$$= \frac{\sqrt{2}}{32}\lim_{t\to 0}\left(\frac{\sin\frac{t}{2}}{t/2}\right)^2$$

$$= \frac{1}{16\sqrt{2}}$$

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LECTURE

2

Evaluation of Limits

BASIC CONCEPTS

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. If l and m exist i.e., l and m are finite and unique, then:

1. $\lim_{x\to a}[f(x)+g(x)] = l+m$
2. $\lim_{x\to a}[f(x)-g(x)] = l-m$
3. $\lim_{x\to a}[f(x)g(x)] = l\times m$
4. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{l}{m}, m \neq 0$
5. $\lim_{x\to a} kf(x) = kl$, k is constant
6. If $f(x) \le g(x)$ then $l \le m$
7. $\lim_{x\to a}[f(x)]^{[g(x)]} = l^m$
8. $\lim_{x\to a} f(g(x)) = f\left[\lim_{x\to a} g(x)\right] = f(m)$, In particular $\lim_{x\to a}\log f(x) = \log\left(\lim_{x\to a} f(x)\right) = \log l$.
9. $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)}$

SOME IMPORTANT RESULTS

1. $\lim_{x\to 0}(1+ax)^{1/x} = e^{a}$
2. $\lim_{x\to 0}(1+ax)^{b/x} = e^{ab}$
3. $\lim_{x\to\infty}\left(1+\dfrac{a}{x}\right)^{bx} = e^{ab}$
4. $\lim_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$, $\lim_{x\to\infty}\left(\dfrac{x+a}{x+b}\right)^{x} = e^{a-b}$
5. $\lim_{x\to 0}\dfrac{a^x-1}{x}\log_e a$
6. $\lim_{x\to a}\{1+f(x)\}^{\frac{1}{f(x)}} = e$
7. $\lim_{x\to a}\{1+bf(x)\}^{\frac{c}{f(x)}} = e^{bc}$
8. $\lim_{x\to 0}\left(\dfrac{1+ax}{1+bx}\right)^{1/x} = e^{a-b}$
9. $\lim_{x\to a}\ [f(x)]\ [g(x)] = e^{\lim_{x\to a}[f(x)-1]g(x)}$

   If $f(a)^{g(a)}$ is of the type (1^∞).

LIMITS BASED ON 5-STEP RULE

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{d}{dx}f(x) = f'(x)$$

1. $\lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \dfrac{d}{dx}\left(\sqrt{x}\right) = \dfrac{1}{2\sqrt{x}}$
2. $\lim_{h\to 0}\dfrac{1}{h}\left(\dfrac{1}{x+h}-\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$

3. $\lim_{h\to 0}\dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}=\dfrac{1}{3\sqrt[3]{x^2}}$

4. $\lim_{x\to 0}\dfrac{\tan(x+a)-\tan a}{3x}=\dfrac{1}{3}\sec^2 a$

L-HOSPITAL'S RULE

1. Let $\phi(x)$ and $\psi(x)$ be two functions such that $\phi(a)=0$ and $\psi(a)=0$ then $\lim_{x\to a}\dfrac{\phi(x)}{\psi(x)}=\lim_{x\to a}\dfrac{\phi'(x)}{\psi'(x)}$ provided $\phi'(a)$ and $\psi'(a)$ are not both zero.
2. If $\phi'(a)$, $\phi''(a)$,......, $\phi^{n-1}(a)$ and $\psi'(a)$, $\psi''(a)$,......, $\psi^{n-1}(a)$ are all zero but $\phi^{n}(a)$ and $\psi^{n}(a)$ are not zero, then $\lim_{x\to a}\dfrac{\phi(x)}{\psi(x)}=\lim_{x\to a}\dfrac{\phi^{n}(x)}{\psi^{n}(x)}$

NOTES

1. While applying L-Hospital's Rule differentiate numerater and denominator separately.
2. If the form of function comes out to be any other indeterminate form, it should be converted to $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

TO FIND THE LIMIT BY FINDING L.H. LIMIT AND R.H. LIMIT

When the values of $f(x)$ are given by different functions of x for $x \le a$ and $x \ge a$, then limit of the function $f(x)$ at $x = a$ is obtained by definition i.e., by finding its right hand limit (R.H.L.) and left hand limit (L.H.L.).

SQUEEZING THEOREM OR SANDWICH THEOREM

Let $f(x)$, $g(x)$ and $h(x)$ be defined for the set of real numbers and satisfy $f(x) \le g(x) \le h(x)$; $\forall$ (for all) x and $\lim_{x\to a} f(x) = l = \lim_{x\to a} f(x)$

$\Rightarrow$ $\lim_{x\to a} g(x) = l$

$\Rightarrow$ Let $f: R \to R$ be a positive increasing function with $\lim_{x\to\infty}\dfrac{f(3x)}{f(x)}=1$,

Then $\lim_{x\to\infty}\dfrac{f(2x)}{f(x)}=$

(a) 1 (b) $\dfrac{2}{3}$

(c) $\dfrac{3}{2}$ (d) 3

Solution

Since given function is a positive in creasing function therefore $3x > 2x > x$ i.e.,

$f(3x) > f(2x) > f(x)$

or $\dfrac{f(3x)}{f(x)} > \dfrac{f(2x)}{f(x)} > \dfrac{f(x)}{f(x)}$

or $\lim_{x\to\infty}\dfrac{f(3x)}{f(x)} > \lim_{x\to\infty}\dfrac{f(2x)}{f(x)} > \lim_{x\to\infty} 1$

or $1 > \lim_{x\to\infty}\dfrac{f(2x)}{f(x)} > 1$

$\therefore$ By sandwich theorem $\lim_{x\to\infty}\dfrac{f(2x)}{f(x)}=1$

POINTS TO REMEMBER

1. If $x \to \infty$, then $[x] = x$.
2. (i) When x is irrational then $\cos^{2m}(x\, n!\, \pi)$ is 0 whatever m and n may be.

   $\therefore$ $\lim_{m\to\infty}\lim_{n\to\infty}\cos(n!\pi x)^{2m}=0$

   (ii) If x is a rational number and $n\to\infty$ then $n!\,\pi x$ = integral multiple of π.

   $\therefore$ $\cos(n!\,\pi x) = 1$ or -1

   $\therefore$ $\cos^{2m}(n!\,\pi x)=1$
3. $\lim_{x\to 0}\left\{\dfrac{1^x+2^x+3^x+......+n^x}{n}\right\}^{1/x}$

   $=\{n!\}^{1/n}$
4. $\lim_{n\to\infty}\left\{\dfrac{[1^2x]+[2^2x]+......+[n^2x]}{n^3}\right\}$

   $=\dfrac{x}{3}$
5. $\lim_{n\to\infty}\left\{\dfrac{[x]+[2x]+[3x]+......+[nx]}{n^3}\right\}$

   $=\dfrac{x}{2}$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Evaluate the limit $\lim_{x\to 0}\left(\dfrac{\cos Ax-\cos Bx}{x^2}\right)$

[MP-2002; CBSE-88C; PSB-2000]

Solution

We have $\lim_{x\to 0}\left(\dfrac{\cos Ax-\cos Bx}{x^2}\right)$ $\left(\text{Form } \dfrac{0}{0}\right)$

Using $\cos C-\cos D = 2\sin\dfrac{C+D}{2}\sin\dfrac{D-C}{2}$

$$=\lim_{x\to 0}\frac{2\sin\left(\frac{A+B}{2}\right)x\sin\left(\frac{B-A}{2}\right)x}{x^2}$$

$$=2\lim_{x\to 0}\left\{\frac{\sin\left(\frac{A+B}{2}\right)x}{\left(\frac{A+B}{2}\right)x}\cdot\left(\frac{A+B}{2}\right)\cdot\frac{\sin\left(\frac{B-A}{2}\right)x}{\left(\frac{B-A}{2}\right)x}\cdot\left(\frac{B-A}{2}\right)\right\}$$

$\left(\text{Form } \dfrac{0}{0}\right)$

$$=2\left(\frac{B+A}{2}\right)\left(\frac{B-A}{2}\right)\lim_{x\to 0}\left\{\frac{\sin\left(\frac{A+B}{2}\right)x}{\left(\frac{A+B}{2}\right)x}\right\}.\left\{\lim_{x\to 0}\frac{\sin\left(\frac{B-A}{2}\right)x}{\left(\frac{B-A}{2}\right)x}\right\}$$

$$=2\left(\frac{B^2-A^2}{4}\right).(1).(1)=\frac{B^2-A^2}{2}$$

2. Evaluate $\lim_{x\to 0}\dfrac{\tan 2x-\sin 2x}{x^3}$ ***[CBSE-96]***

Solution

We have $\lim_{x\to 0}\dfrac{\tan 2x-\sin 2x}{x^3}$ $\left(\text{Form } \dfrac{0}{0}\right)$

$$=\lim_{x\to 0}\frac{\frac{\sin 2x}{\cos 2x}-\sin 2x}{x^3}$$

$$=\lim_{x\to 0}\frac{\sin 2x-\sin 2x\cos 2x}{x^3\cos 2x}$$

$$=\lim_{x\to 0}\frac{\sin 2x(1-\cos 2x)}{x^3\cos 2x}=\lim_{x\to 0}\frac{\sin 2x.2\sin^2 x}{x^3\cos 2x}$$

$$=2\lim_{x\to 0}\frac{\sin 2x}{\cos 2x}\cdot\frac{\sin^2 x}{x^3}$$

$$=4\lim_{x\to 0}\frac{\tan 2x}{2x}\cdot\left(\frac{\sin x}{x}\right)^2$$

$$=4\lim_{x\to 0}\left(\frac{\tan 2x}{2x}\right).\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$=4\,.\,(1)\,.\,(1)=4$$

3. Evaluate the limit $\lim_{x\to 0}\dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$

[CBSE-2003]

Solution

$$\lim_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}\times\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}$$

$$=\lim_{x\to 0}\frac{(\sqrt{1+\sin x})^2-(\sqrt{1-\sin x})^2}{x(\sqrt{1+\sin x}+\sqrt{1-\sin x})}$$

$$=\lim_{x\to 0}\frac{1+\sin x-1+\sin x}{x(\sqrt{1+\sin x}+\sqrt{1-\sin x})}$$

$$= \lim_{x \to 0} \frac{2\sin x}{x(\sqrt{1+\sin x} + \sqrt{1-\sin x})}$$

$$= 2\lim_{x \to 0}\left(\frac{\sin x}{x}\right).\lim_{x \to 0} \frac{1}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$= 2.(1).\frac{1}{\sqrt{1+0} + \sqrt{1-0}} = 2.\frac{1}{1+1} = 2.\frac{1}{2} = 1$$

4. Evaluate the limit $\lim_{x \to \pi} \frac{\sin 3x - 3\sin x}{(\pi - x)^3}$

[CBSE-2004]

Solution

We have $\lim_{x \to \pi} \frac{\sin 3x - 3\sin x}{(\pi - x)^3}$

$$= \lim_{x \to \pi} \frac{3\sin x - 4\sin^3 x - 3\sin x}{(\pi - x)^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \pi} \frac{-4\sin^3 x}{(\pi - x)^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= -4\lim_{x \to \pi} \frac{\sin^3 x}{(\pi - x)^3} = -4\lim_{h \to 0} \frac{\sin^3(\pi + h)}{(\pi - (\pi + h))^3}$$

$$= -4\lim_{h \to 0} \frac{(-1)^3 \sin^3 h}{(-1)^3 h^3} = -4\lim_{h \to 0}\left(\frac{\sin h}{h}\right)^3$$

$$= -4.\ (1)^3 = -4.$$

5. Evaluate $\lim_{x \to 0} \frac{1 - \cos x\sqrt{\cos 2x}}{x^2}$

[CBSE (Foreign)-2005 (III)]

Solution

$\lim_{x \to 0} \frac{1 - \cos x\sqrt{\cos 2x}}{x^2}$ by Rationalize

$$\lim_{x \to 0} \frac{1 - \cos x\sqrt{\cos 2x}}{x^2} \times \frac{1 + \cos x\sqrt{\cos 2x}}{1 + \cos x\sqrt{\cos 2x}}$$

$$= \lim_{x \to 0} \frac{1^2 - (\cos\sqrt{\cos 2x})^2}{x^2(1 + \cos x\sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x(\cos 2x)}{x^2(1 + \cos x\sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x(1 - 2\sin^2 x)}{x^2(1 + \cos x\sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x + 2\sin^2 x\cos^2 x}{x^2(1 + \cos x\sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{\sin^2 x + 2\sin^2 x\cos^2 x}{x^2(1 + \cos x\sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{\sin^2 x(1 + 2\cos^2 x)}{x^2(1 + \cos x\sqrt{\cos 2x})}$$

$$= \lim_{x \to 0}\left(\frac{\sin x}{x}\right)^2.\lim_{x \to 0} \frac{1 + 2\cos^2 x}{1 + \cos x\sqrt{\cos 2x}}$$

$$= 1.\frac{1 + 2\times 1}{1 + 1.\sqrt{1}} = \frac{1+2}{1+1} = \frac{3}{2}$$

6. Evaluate the limit $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$

[CBSE-86; HPSB-88]

Solution

We have $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{h \to 0} \frac{1 + \cos 2\left(\frac{\pi}{2} + h\right)}{\left(\pi - 2\left(\frac{\pi}{2} + h\right)\right)^2}$

$$\left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{1 + \cos(\pi + 2h)}{(\pi - \pi - 2h)^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 2h}{4h^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{2\sin^2 h}{4h^2}$$

$$= \frac{2}{4}\lim_{h \to 0} \frac{\sin h}{h}.\frac{\sin h}{h}$$

$$= \frac{1}{2}\left(\lim_{h \to 0} \frac{\sin h}{h}\right).\left(\lim_{h \to 0} \frac{\sin h}{h}\right) = \frac{1}{2} \times 1 \times 1$$

$$= \frac{1}{2}$$

7. Evaluate the limit $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{\log(1+x)}$

[AISSE-92]

Solution

We have $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{\log(1+x)}$

$$= \lim_{x\to 0}\frac{e^{\sin x}-1}{\log(1+x)}.\frac{\sin x}{\sin x}.\frac{x}{x}$$

$$= \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}.\frac{\sin x}{x}.\frac{x}{\log(1+x)}$$

$$= \lim_{\sin x\to 0}\frac{e^{\sin x}-1}{\sin x}.\lim_{x\to 0}\left(\frac{\sin x}{x}\right).\lim_{x\to 0}\left(\frac{1}{\frac{\log(1+x)}{x}}\right)$$

8. Evaluate $\lim\limits_{x\to 1}\dfrac{1-x^{-1/3}}{1-x^{-2/3}}$ ***[CBSE-91; HSB-95C]***

Solution

$$\lim_{x\to 1}\frac{1-x^{-1/3}}{1-x^{-2/3}} = \lim_{x\to 1}\frac{\frac{x^{1/3}-1}{x^{1/3}}}{\frac{x^{2/3}-1}{x^{2/3}}}$$

$$= \lim_{x\to 1}\frac{x^{2/3}.x^{-1/3}(x^{1/3}-1)}{x^{2/3}-1} = \lim_{x\to 1}\frac{x^{1/3}(x^{1/3}-1)}{x^{2/3}-1}$$

$$= (1)^{1/3}\lim_{x\to 1}\left(\frac{x^{1/3}-1}{x-1}.\frac{x-1}{x^{2/3}-1}\right)$$

$$= (1)^{1/3}\lim_{x\to 1}\frac{x^{1/3}-1}{x-1}\div\frac{x^{2/3}-1}{x-1}$$

$\therefore$ using $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}$

$$= \frac{1}{3}(1)^{1/3-1}\div\frac{2}{3}(1)^{2/3-1}$$

$$= \frac{1}{3}\div\frac{2}{3}$$

$$= \frac{1}{2}$$

9. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}\right)$

[CBSE-92, 90, comptt.-92, Practice Paper sample (III); PSB-89, 93; HSB-99; J & K-95 C]

Solution

We have $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$

$$= \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \lim_{x\to 0}\frac{(\sqrt{1+x})^2-(\sqrt{1-x})^2}{\sin^{-1}x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0}\frac{1+x-1+x}{\sin^{-1}x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0}\frac{2x}{\sin^{-1}x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0}2.\left(\frac{x}{\sin^{-1}x}\right).\frac{1}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= 2\ .\ (1)\ .\ \frac{1}{\sqrt{1+0}+\sqrt{1-0}} = 2\ .\ \frac{1}{2} = 2$$

10. Evaluate the limit $\lim\limits_{x\to 0}\dfrac{x[1-\sqrt{1-x^2}]}{\sqrt{1-x^2}(\sin^{-1}x)^3}$

[CBSE (Delhi)-2005 (I), (II), (III); PSB-91C; Goa-97]

Solution

We have $\lim\limits_{x\to 0}\dfrac{x[1-\sqrt{1-x^2}]}{\sqrt{1-x^2}(\sin^{-1}x)^3}$

$$= \lim_{x\to 0}\frac{x(1-\sqrt{1-x^2})}{\sqrt{1-x^2}(\sin^{-1}x)^3}\times\frac{(1+\sqrt{1-x^2})}{(1+\sqrt{1-x^2})}$$

$$= \lim_{x\to 0}\frac{x[1^2-(\sqrt{1-x^2})^2]}{\sqrt{1-x^2}(\sin^{-1}x)^3(1+\sqrt{1-x^2})}$$

$$= \lim_{x\to 0}\frac{x[1-(1-x^2)]}{\sqrt{1-x^2}(\sin^{-1}x)^3(1+\sqrt{1-x^2})}$$

$$= \lim_{x\to 0} \frac{x^3}{\sqrt{1-x^2}(\sin^{-1}x)^3(1+\sqrt{1-x^2})}$$

$$= \lim_{x\to 0}\left(\frac{x}{\sin^{-1}x}\right)^3 \cdot \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{(1+\sqrt{1-x^2})}$$

$$= (1)^3 \cdot \frac{1}{\sqrt{1-0}} \cdot \frac{1}{(1+\sqrt{1-0})} = 1 \,.\, 1 \,.\, \frac{1}{2}$$

$$= \frac{1}{2}$$

11. Evaluate the limit $\lim_{x\to 0} \frac{e^x - 1}{\sqrt{1-\cos x}}$

[CBSE-2002, 2003]

Solution

We have $\lim_{x\to 0} \frac{e^x-1}{\sqrt{1-\cos x}} = \lim_{x\to 0} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|}$

$$\therefore \quad \lim_{x\to 0} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|} = -\frac{1}{\sqrt{2}} \lim_{x\to 0} \frac{e^x-1}{\sin\frac{x}{2}}$$

$$= \frac{-2}{\sqrt{2}} \lim_{x\to 0}\left(\frac{e^x-1}{x}\right) \times \left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right) = \frac{-2}{\sqrt{2}}.1.1 = -\sqrt{2}$$

and $\lim_{x\to 0^+} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|}$

$$= \frac{2}{\sqrt{2}} \lim_{x\to 0^+}\left(\frac{e^x-1}{x}\right)\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right)$$

$$= \frac{2}{\sqrt{2}}.1.1 = \sqrt{2}$$

$\therefore \quad \lim_{x\to 0^-} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|} \neq \lim_{x\to 0^+} \frac{e^x-1}{\sqrt{2}\left|\sin\frac{x}{2}\right|}$ Hence limit does not exist.

12. Evaluate $\lim_{x\to \pi} \frac{1+\cos x}{\tan^2 x}$ ***[MP-98]***

Solution

Putting $x = t + \pi$ when $x \to \pi$, then $t \to 0$, we get

$$\lim_{x\to\pi} \frac{1+\cos x}{\tan^2 x} = \lim_{t\to 0} \frac{1+\cos(\pi+t)}{\tan^2(\pi+t)}$$

$$= \lim_{t\to 0} \frac{1-\cos t}{\tan^2 t} = \lim_{t\to 0}\left(\frac{1-\cos t}{t^2}.\frac{t^2}{\tan^2 t}\right)$$

$$= \lim_{t\to 0}\left(\frac{1-\cos t}{t^2}\right).\lim_{t\to 0}\left(\frac{t}{\sin t}\right)^2$$

$$= \frac{1}{2} \,.\, 1 = \frac{1}{2}$$

13. Evaluate the limit $\lim_{x\to 0} \frac{x\cos x + \sin x}{x^2 + \tan x}$

[HSB-2002]

Solution

Dividing numerator and denominator by x

$$= \lim_{x\to 0} \frac{\frac{x\cos x}{x} + \frac{\sin x}{x}}{\frac{x^2}{x} + \frac{\tan x}{x}} = \lim_{x\to 0} \frac{\left(\cos x + \frac{\sin x}{x}\right)}{\left(x + \frac{\tan x}{x}\right)}$$

$$= \frac{\lim_{x\to 0}\left(\cos x + \frac{\sin x}{x}\right)}{\lim_{x\to 0}\left(x + \frac{\tan x}{x}\right)} = \frac{\lim_{x\to 0}\cos x + \lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0} x + \lim_{x\to 0}\frac{\tan x}{x}}$$

$$= \frac{\cos 0 + 1}{0+1} = \frac{1+1}{1} = \frac{2}{1} = 2$$

14. Evaluate the limit $\lim_{x\to 0} \frac{1-\cos 5x}{1-\cos 6x}$ ***[CBSE-97]***

Solution

$$\lim_{x\to 0} \frac{1-\cos 5x}{1-\cos 6x} = \lim_{x\to 0} \frac{2\sin^2\frac{5x}{2}}{2\sin^2 3x}$$

$(\because 1 - \cos 2\theta = 2\sin^2\theta)$

$$= \lim_{x\to 0} \frac{\sin^2\frac{5x}{2}}{\sin^2 3x}$$

$$= \lim_{x\to 0}\left\{\frac{\frac{\sin^2 \frac{5x}{2}}{\frac{25x^2}{4}}\times\frac{25x^2}{4}}{\frac{\sin^2 3x}{9x^2}\times 9x^2}\right\}$$

$$= \lim_{x\to 0}\frac{25x^2}{4\times 9x^2}\left\{\frac{\frac{\sin^2 \frac{5}{2}x}{\frac{25}{4}x^2}}{\frac{\sin^2 3x}{9x^2}}\right\}$$

$$= \frac{25}{36}\times\frac{\lim\limits_{x\to 0}\left(\frac{\sin\frac{5}{2}x}{\frac{5}{2}x}\right)^2}{\lim\limits_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^2}$$

$$= \frac{25}{36}\times\frac{(1)^2}{(1)^2} = \frac{25}{36}\times\frac{1}{1} = \frac{25}{36}$$

15. Prove that $\lim\limits_{x\to 0}\dfrac{6^x-1}{\sqrt{3-x}-\sqrt{3}} = -2\sqrt{3}\log_e 6$

[MP-2001]

Solution

$$\lim_{x\to 0}\frac{6^x-1}{\sqrt{3-x}-\sqrt{3}}$$

$$= \lim_{x\to 0}\frac{6^x-1}{\sqrt{3-x}-\sqrt{3}}\times\frac{\sqrt{3-x}+\sqrt{3}}{\sqrt{3-x}+\sqrt{3}}$$

$$= \lim_{x\to 0}\frac{(6^x-1)(\sqrt{3-x}+\sqrt{3})}{(\sqrt{3-x})^2-(\sqrt{3})^2}$$

$$= \lim_{x\to 0}\frac{(6^x-1)(\sqrt{3-x}+\sqrt{3})}{3-x-3}$$

$$= \lim_{x\to 0}\frac{(6^x-1)(\sqrt{3-x}+\sqrt{3})}{-x}$$

$$= -\lim_{x\to 0}\frac{6^x-1}{x}.\lim_{x\to 0}(\sqrt{3-x}+\sqrt{3})$$

$$= -\log_e 6(\sqrt{3-0}+\sqrt{3})$$

$$= -\log_e 6(\sqrt{3}+\sqrt{3})$$

$$= -2\sqrt{3}\log_e 6$$ **Proved.**

16. Evaluate $\lim\limits_{x\to a}\dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$

[CBSE-92C, 91; HPSB-98]

Solution

We have $\lim\limits_{x\to a}\dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$

$$\Rightarrow \lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a+2-2}$$

$$\Rightarrow \lim_{(x+2)\to(a+2)}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{(x+2)-(a+2)}$$

$$\Rightarrow \lim_{y\to b}\frac{y^{5/3}-b^{5/3}}{y-b} \text{ where } x+2=y,\ a+2=b$$

$$= \frac{5}{3}(b)^{5/3-1} \qquad \left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= \frac{5}{3}(a+2)^{2/3}$$

17. Evaluate $\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{(\cos^{-1}x)^2}$ ***[CBSE-92]***

Solution

Putting $\cos^{-1}x = \theta$ when $x\to 1$, then $\theta\to 0$, we get

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{(\cos^{-1}x)^2} = \lim_{\theta\to 0}\frac{1-\sqrt{\cos\theta}}{\theta^2}$$

$$= \lim_{\theta\to 0}\frac{1-\sqrt{\cos\theta}}{\theta^2}\times\frac{1+\sqrt{\cos\theta}}{1+\sqrt{\cos\theta}}$$

$$= \lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta^2}.\frac{1}{1+\sqrt{\cos\theta}}\right)$$

$$= \lim_{\theta\to 0}\left(\frac{2\sin^2\theta/2}{4(\theta/2)^2}.\frac{1}{1+\sqrt{\cos\theta}}\right)$$

$$= \lim_{\theta\to 0}\frac{1}{2}\left[\left(\frac{\sin^2\theta/2}{(\theta/2)^2}\right).\frac{1}{1+\sqrt{\cos\theta}}\right]$$

$$= \frac{1}{2}\lim_{\theta\to 0}\left(\frac{\sin\theta/2}{\theta/2}\right)^2 . \lim_{\theta\to 0}\frac{1}{1+\sqrt{\cos\theta}}$$

$$= \frac{1}{2}(1)^2 . \frac{1}{1+\sqrt{\cos\theta}} = \frac{1}{2}.1.\frac{1}{1+1} = \frac{1}{4}$$

18. Evaluate

(i) $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]$, where [.] denotes the greatest integer function.

Solution

Let $P = \lim_{x\to 0}\left[\frac{\sin x}{x}\right]$ for $x > 0$, $\sin x < x$

$\Rightarrow \quad \frac{\sin x}{x} < 1$

$\therefore \quad$ R.H.L. $= \lim_{x\to 0}\left[\frac{\sin x}{x}\right] = 0$

and for $x < 0$, $\sin x > x$

$\Rightarrow \quad \frac{\sin x}{x} < 1$

$\therefore \quad$ L.H.L. $= \lim_{x\to 0}\left[\frac{\sin x}{x}\right] = 0$

Hence $P = 0$

(ii) $\lim_{x\to 0}\left[\frac{\sin^{-1} x}{x}\right]$, where [.] denotes the greatest integer function

Solution

Let $P = \lim_{x\to 0}\left[\frac{\sin^{-1} x}{x}\right]$ for $x > 0$, $\sin^{-1} x > x$

$\Rightarrow \quad \frac{\sin^{-1} x}{x} > 1$

$\therefore \quad$ R.H.L. $= \lim_{x\to 0}\left[\frac{\sin^{-1} x}{x}\right] = 1$ for $x < 0$,

$\sin^{-1} x < x \Rightarrow \frac{\sin^{-1} x}{x} > 1$

$\therefore \quad$ L.H.L. $= \lim_{x\to 0}\left[\frac{\sin^{-1} x}{x}\right] = 1$,

Hence $P = 1$

(iii) $\lim_{x\to 0}$ [cot x], where [.] denotes the greatest integer function.

Solution

Let $P = \lim_{x\to 0}$ [cot x] for $x > 0$

R.H.L. $= \lim_{x\to 0+}$ [cot x] $= \infty$ and for $x < 0$

L.H.L. $= \lim_{x\to 0-}$ [cot x] $= -\infty$ Hence P does not exist.

(iv) $\lim_{x\to 5\pi/4}$ [sin x + cos x], where [.] denotes the greatest integer function.

Solution

$P = \lim_{x\to 5\pi/4} [\sin x + \cos x]$

$= \lim_{x\to 5\pi/4} [\sqrt{2}\sin(x+\pi/4)]$

Put $x + \pi/4 = t$

$\therefore\ P = \lim_{t\to 3\pi/2} [\sqrt{2}\sin t]$

According to figure, when $t = 3\pi/2$ then $\sqrt{2}\sin t \to -\sqrt{2}$

Hence $P = -2$.

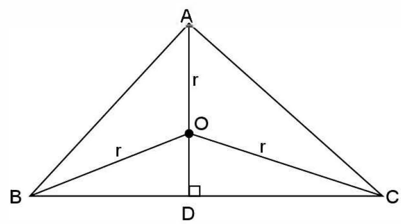
19. *ABC* is an isosceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC then the triangle ABC has perimeter $P = 2\left(\sqrt{2hr-h^2}+\sqrt{2hr}\right)$ and area $A =$ .... also $\lim_{h\to 0}\frac{A}{P^3} =$ ....

[IIT JEE-1989]

Solution

In ΔABC $AB = AC$

$AD \perp BC$ (D is mid *pt* of BC)



Let $AD = h$ $\quad r$ = radius of circumcircle

$\therefore \quad OA = OB = OC = r$

Now $BD = \sqrt{BO^2 - OD^2} = \sqrt{r^2 - (h-r)^2}$

$$= \sqrt{2rh - h^2}$$

$\therefore \quad BC = 2\sqrt{2rh - h^2}$

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2} \times BC \times AD$

$$= h\sqrt{2rh - h^2}$$

Also $\lim\limits_{h \to 0} \frac{A}{P^3} = \frac{h\sqrt{2rh - h^2}}{8\left(\sqrt{2rh - h^2} + \sqrt{2rh}\right)^3}$

$$= \lim_{h \to 0} \frac{h^{3/2}\sqrt{2r - h}}{8h^{3/2}\left(\sqrt{2r - h} + \sqrt{2r}\right)^3}$$

$$= \lim_{h \to 0} \frac{\sqrt{2r - h}}{8[\sqrt{2r - h} + \sqrt{2r}]^3}$$

$$= \frac{\sqrt{2r}}{8\left(\sqrt{2r} + \sqrt{2r}\right)^3}$$

$$= \frac{\sqrt{2r}}{8.\,8\,.\,2r.\,\sqrt{2r}} = \frac{1}{128r}$$

20. Use the formula $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \ln a$ to find $\frac{2^x - 1}{(1+x)^{1/2} - 1}$

[IIT JEE-1982]

Solution

$$\lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x} - 1} = \lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$$

$$= \lim_{x \to 0} \frac{(2^x - 1)\left(\sqrt{1+x} + 1\right)}{1 + x - 1}$$

$$= \lim_{x \to 0} \frac{2^x - 1}{x} . \lim_{x \to 0}(\sqrt{1+x} + 1)$$

$\ln 2 \,.\, (1 + 1) = 2 \ln 2$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Prove that $\lim\limits_{x \to 0} \frac{e^{\sin x} - 1}{x} = 1$

[MP-98; PSB-2000]

2. Evaluate $\lim\limits_{n \to \infty} \frac{(n-1)(2n+3)}{n^2}$ ***[MP-96]***

3. If f is an even function, then prove that $\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0+} f(x)$.

4. Evaluate $\lim\limits_{x \to 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x}$

[CBSE-2000 C]

5. Evaluate the limit $\lim\limits_{x \to 1} \frac{\cos^{-1} x}{\sqrt{1-x^2}}$. ***[CBSE-92]***

6. Evaluate the limit $\lim\limits_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$

[CBSE-2004]

7. Find the value of $\lim\limits_{x \to 0} \frac{3^{4x} - 3^{3x} - 3^x + 1}{x^2}$

[IMP]

8. Evaluate

$$\lim_{n \to \infty} \frac{2}{n}\left[7n + \frac{18}{n}\frac{n(n-1)}{2} + \frac{8}{n^2}.\frac{n(n-1)(2n-1)}{6}\right]$$

[CBSE (Sample Paper)-99]

9. Find the value of $\lim\limits_{x \to 0} \frac{x^3 \cot x}{1 - \cos x}$

[MP-2007]

10. Evaluate $\lim\limits_{\theta \to \frac{\pi}{2}} \frac{2\theta - \pi}{\cos\theta}$

11. Evaluate $\lim\limits_{x \to a} \frac{\sin x - \sin a}{x - a}$ ***[MP-98]***

12. Evaluate $\lim\limits_{x \to 0} \frac{\sin x \cos x}{3x}$

[MP-2001]

13. Evaluate the limit $\lim\limits_{x \to 0} \frac{\sin x^\circ}{x}$ ***[CBSE-93]***

14. Evaluate the limit $\lim\limits_{\theta \to 0} \frac{\sin 4\theta}{\tan 3\theta}$

[CBSE-95 (C)]

15. Evaluate the limit $\lim\limits_{x \to \frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1}$

[CBSE-2001; 2002 (C)]

16. Evaluate the limit $\lim\limits_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$

[CBSE-2001]

17. Evaluate $\lim\limits_{x \to 0} \frac{\cos 3x - \cos 2x}{x^2}$. ***[PSB-2000]***

18. Evaluate the limit $\lim\limits_{x \to 0} \frac{2^x - 1}{\sin x}$

[PSB-2001 (C)]

19. Evaluate the limit $\lim\limits_{x \to 0} \frac{\sin x^2}{x}$

[CBSE-81 (C)]

20. Evaluate the limit $\lim\limits_{x \to 0} \left(\frac{1 - \cos 2x}{\sin^2 2x} \right)$

[CBSE-2000, 2002]

21. Evaluate the limit $\lim\limits_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x \right) \sec x$

[CBSE (Foreign)-98]

22. Evaluate $\lim\limits_{\theta \to 0} \frac{\sin 3\theta - \sin\theta}{\sin\theta}$ ***[MP-2005 (B)]***

23. Find the value of $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

[MP-2007]

ANSWERS

2. 2

4. 3

5. 1

6. 1

7. 3 (log 3)^2

8. 112/3

10. –2

11. cos a

12. 1/3

13. $\frac{\pi}{180}$

14. 4/3

15. 2

16. 2

17. $\frac{-5}{2}$

18. log 2

19. 0

20. 1/2

21. 1

22. 2

23. 0

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. $\lim_{x\to 0}\dfrac{5^x-3^x}{\sqrt[3]{1+x}-1}$ is equal to

[JEE (WB)-2005]

(a) 3 log 5/3 (b) 3 log 3/5
(c) log 3/5 (d) 1

Solution

(a) $\dfrac{0}{0}$ form, so limit $=\lim_{x\to 0}\dfrac{5^x\log 5-3^x\log 3}{\frac{1}{3}(1+x)^{-2/3}}$

(Using L Hospital's) = 3 log 5/3

2. $\lim_{x\to 0}(\text{cosec } x)^{\frac{1}{\log x}}$ (cosec x)1/log x equals

[UPSEAT-2005]

(a) 1 (b) –1
(c) e (d) 1/e

Solution

(d) Let $A=\lim_{x\to 0}(\text{cosec } x)^{1/\log x}$

$\Rightarrow \log A=\lim_{x\to 0}\dfrac{1}{\log x}\log\text{cosec } x$

$=-\lim_{x\to 0}\dfrac{\log\sin x}{\log x}$

$=-\lim_{x\to 0}\dfrac{\cos x}{\sin x}x=-1 \quad \therefore A=1/e$

3. Let $f(x)=\dfrac{1}{\sqrt{18-x^2}}$, then $\lim_{x\to 3}\dfrac{f(x)-f(3)}{x-3}$ is equal to

(a) 0 (b) 1/3
(c) 1/9 (d) –1/9

[NDA-2006]

Solution

(c) By definition of derivative, limit = $f'(3)$ = 1/9

4. If $\lim_{x\to 0}(1+3x)^{(1+5x)/x}$ is equal to

[IIIT (Allahabad)-2001]

(a) $3e$ (b) e^5
(c) e^3 (d) e

Solution

(c) 1^∞ form, so limit = $e^{\lim_{x\to 0}(3x)\left(\frac{1+5x}{x}\right)}=e^3$

5. $\lim_{m\to\infty}\left(\cos\dfrac{x}{m}\right)^m$ is equal to

[AMU-2001]

(a) 0 (b) e
(c) $1/e$ (d) 1

Solution

(d) $\lim_{m\to\infty}\left(\cos\dfrac{x}{m}\right)^m=\lim_{m\to\infty}\left[1+\left(\cos\dfrac{x}{m}-1\right)\right]^m$

$=\lim_{m\to\infty}\left[1-\left(-\cos\dfrac{x}{m}+1\right)\right]^m$

$=\lim_{m\to\infty}\left[1-2\sin^2\dfrac{x}{2m}\right]^m$

$=e^{\lim_{m\to\infty}-\left(2\sin^2\frac{x}{2m}\right)m}=e^{\lim_{m\to\infty}-2\left(\frac{\sin\frac{x}{2m}}{\frac{x}{2m}}\right)^2\left(\frac{x^2}{4m^2}\right)m}$

$=e^{-2\lim_{m\to\infty}\frac{x^2}{4m}}=e^0=1$

6. $\lim_{n\to\infty}(3^n+4^n)^{1/n}$ is equal to

[Karnataka CET-2003]

(a) 3 (b) 4
(c) ∞ (d) e

Solution

(b) $\lim_{n\to\infty}(3^n+4^n)^{1/n}=\lim_{n\to\infty}(4^n)^{1/n}\left[\dfrac{3^n}{4^n}+1\right]^{1/n}$

$=\lim_{n\to\infty}4\left[1+\dfrac{1}{(4/3)^n}\right]^{1/n}$

$=4\lim_{n\to\infty}\left[1+\dfrac{1}{(4/3)^n}\right]^{1/n}$

$=4\left[1+\dfrac{1}{\infty}\right]^0=4\times(1)^0=4\times 1=4$

7. If $f(x)$ is differentiable and strictly increasing function, then $\lim_{x\to 0}\frac{f(x^2)-f(x)}{f(x)-f(0)}$ is equal to

(a) 1 (b) 0
(c) 2 (d) –1

[IIT (Screening)-2004]

Solution

(d) Limit is in $\frac{0}{0}$ form, so by Hospital rule

$$\text{Limit} = \lim_{x\to 0}\frac{2xf'(x^2)-f'(x)}{f'(x)}$$

$$= \lim_{x\to 0}\left[\frac{2xf'(x^2)}{f'(x)}-1\right] = 0-1$$

$[\because f'(x^2) > 0]$

$= -1$

8. If $G(x) = -\sqrt{25-x^2}$, then $\lim_{x\to 1}\frac{G(x)-G(1)}{x-1}$ is equal to ***[IIT-83]***

(a) 1/24 (b) 1/5
(c) $-\sqrt{24}$ (d) None of these

Solution

(d) $\because G(1) = -\sqrt{24}$ $\therefore$ limit

$$= \lim_{x\to 1}\frac{-\sqrt{25-x^2}+\sqrt{24}}{x-1}\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 1}\frac{x}{\sqrt{25-x^2}} = \frac{1}{\sqrt{24}}$$

(Using L Hospital's)

9. For $x > 0$, $\lim_{x\to 0}\{(\sin x)^{1/x} + (1/x)^{\sin x}\}$ is equal to

(a) 0 (b) 1
(c) –1 (d) 2

[IIT-JEE-2006]

Solution

(b) Limit $= \lim_{x\to 0}(\sin x)^{1/x} + \lim_{x\to 0}(1/x)^{\sin x}$ .... (1)

Since $x \to 0$, so $|\sin x| < 1$

$\Rightarrow \lim_{x\to 0}(\sin x)^{1/x} = 0$ $(0^\infty = 0)$

Now let $A = \lim_{x\to 0}(1/x)^{\sin x} \Rightarrow \log A$

$= \lim_{x\to 0}[-\sin x\log x]$

$$= \lim_{x\to 0}\frac{-\log x}{\operatorname{cosec} x}\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{1/x}{\operatorname{cosec} x.\cot x} \text{ [by Hospital rule]}$$

$$= \lim_{x\to 0}\frac{\sin^2 x}{x\cos x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)(\tan x)$$

$= 1\times 0 = 0$ $\therefore A = e^0 = 1$

Hence required limit $= 0 + 1 = 1$.

10. $\lim_{x\to\infty}\left(\frac{a_1^{1/x}+a_2^{1/x}+.......+a_n^{1/x}}{n}\right)^{nx}$ is equal to

(a) $a_1a_2a_3 .... a_n$ (b) $a_1+a_2+...+a_n$
(c) $\frac{1}{n}(a_1+a_2+...+a_n)$ (d) $[e^{a_1+a_2+......+a_n}]$

Solution

(a) Putting $x = 1/y$,

$$\text{Limit} = \lim_{y\to 0}\left(\frac{a_1^y+a_2^y+.......+a_n^y}{n}\right)^{n/y}$$

This is in 1^∞ form, so

$$\text{Limit} = e^{\lim_{y\to 0}\left(\frac{a_1^y+a_2^y+.........+a_n^y}{n}-1\right)\frac{n}{y}}$$

$$= e^{\lim_{y\to 0}\left\{\frac{a_1^y-1}{y}+\frac{a_2^y-1}{y}+.....+\frac{a_n^y-1}{y}\right\}}$$

$$= e^{\log a_1+\log a_2+\log a_n} = a_1a_2...a_n$$

11. Let $f:R\to R$ be a positive increasing function with $\lim_{x\to\infty}\frac{f(3x)}{f(x)} = 1$. Then $\lim_{x\to\infty}\frac{f(2x)}{f(x)} =$

[AIEEE-2010]

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$
(c) 3 (d) 1

Solution

(d) $\lim_{x\to\infty}\frac{f(3x)}{f(x)} = 1$

$f(x) < f(2x) < f(3x)$ Divide by $f(x)$

$$1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

using sandwich theorem $\Rightarrow \lim_{x\to\infty}\frac{f(2x)}{f(x)} = 1$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. For $x \in R$, $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$ is equal to

[IIT Screening-2000]

(a) e (b) e^{-1}
(c) e^{-5} (d) e^{5}

2. $\lim_{h\to 0}\frac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right]}{\sqrt{3}h(\sqrt{3}\cos h-\sin h)}$ is equal to ***[BIT Ranchi-1987]***

(a) –2/3 (b) –3/4
(c) $-2\sqrt{3}$ (d) 4/3

3. $\lim_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}$ is equal to

[IIT-1983; Karnataka CET-1999; Kerala CEE-2004]

(a) log 2 (b) log 4
(c) log $\sqrt{2}$ (d) None of these

4. $\lim_{x\to\frac{\pi}{2}}\frac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi-2x]^3}$ is ***[AIEEE-2003]***

(a) 1/8 (b) 0
(c) 1/32 (d) ∞

5. $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$ = (given that $f'(2) = 6$ and $f'(1) = 4$) ***[IIT Sc.-2004]***

(a) Does not exist (b) Is equal to –3/2
(c) Is equal to 3/2 (d) Is equal to 3

6. If $\lim_{x\to 0}\frac{[(a-n)nx-\tan x]\sin nx}{x^2}=0$, where n is non-zero real numbers, then a is equal to

[IIT Screening-2003]

(a) 0 (b) $\frac{n+1}{n}$
(c) n (d) $n+\frac{1}{n}$

7. Let $f(a) = g(a) = k$ and their n^{th} derivatives $f^n(1), g^n(1)$ exist and are not equal for some n. If $\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$, then the value of k is ***[AIEEE-2003]***

(a) 4 (b) 2
(c) 1 (d) 0

8. If $f(a) = 2, f'(a) = 1, g(a) = -3, g'(a) = -1$, then $\lim_{x\to a}\frac{f(a)g(x)-f(x)g(a)}{x-a}$ is equal to

[MPPET-1997; Karnataka CET-2003]

(a) 1 (b) 6
(c) –5 (d) –1

9. Let $f: R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$. Then $\lim_{x\to 0}\left\{\frac{f(1+x)}{f(1)}\right\}^{1/x}$ equals

[IIT Screening-2002]

(a) 1 (b) $e^{1/2}$
(c) e^2 (d) e^3

10. If $f(1) = 1, f'(1) = 2$, then $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ is

[AIEEE-2002]

(a) 2 (b) 4
(c) 1 (d) 1/2

11. Let $f(x) = 4$ and $f'(x) = 4$, then $\lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}$ equals ***[AIEEE-2002]***

(a) 2 (b) –2
(c) –4 (d) 3

12. $\lim_{x\to 0}\frac{\log_e(1+x)}{3^x-1}$ is equal to ***[MPPET-2002]***

(a) $\log_e 3$ (b) 0
(c) 1 (d) $\log_3 e$

13. $\lim_{x\to\frac{\pi}{2}}\frac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}$ is equal to

[Kerala (Engg.)-2001; J & K-2005]

(a) log a (b) log 2
(c) a (d) log x

14. Let α and β be the roots of $ax^2 + bx + c = 0$, then $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to **[AIEEE-05]**

(a) 0

(b) $\frac{1}{2}(\alpha-\beta)^2$

(c) $\frac{a^2}{2}(\alpha-\beta)^2$

(d) $-\frac{a^2}{2}(\alpha-\beta)^2$

15. $\lim_{x\to a}\frac{\cos x-\cos a}{\cot x-\cot a}$ is equal to **[BIT Ranchi-1987]**

(a) $\frac{1}{2}\sin^3 a$ (b) $\frac{1}{2}\operatorname{cosec}^2 a$

(c) $\sin^3 a$ (d) $\operatorname{cosec}^3 a$

16. $\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$ is equal to **[IIT-1989]**

(a) $a\cos a + a^2\sin a$

(b) $a\sin a + a^2\cos a$

(c) $2a\sin a + a^2\cos a$

(d) $2a\cos a + a^2\sin a$

17. $\underset{x\to 0}{Lt}(x+1)^{\cot x}$ is equal to

(a) e (b) 1

(c) 0 (d) $1/e$

18. $\underset{x\to 0}{Lt}\frac{x^2\sin(1/x)}{\sin x}$ is equal to

(a) 1 (b) 0

(c) −1 (d) None of these

19. If $f(x) = \left(\frac{x^2+5x+3}{x^2+x+3}\right)^x$, then $\lim_{x\to\infty} f(x)$ is equal to **[AIEEE-2002]**

(a) e^4 (b) e^3

(c) e^2 (d) 2^4

20. If $0 < x < y$, then $\lim_{n\to\infty}(y^n + x^n)^{1/n}$ is equal to **[EAMCET-06]**

(a) e (b) x

(c) y (d) None of these

21. $\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3}$ is **[MNR-1994; MPPET-2008]**

(a) 1 (b) e

(c) e^2 (d) e^3

SOLUTIONS

1. (c) $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$; then $=\lim_{x\to\infty}\left(\frac{x+2-5}{x+2}\right)^x$

$$=\lim_{x\to\infty}\left[\left(1-\frac{5}{x+2}\right)^{\frac{x+2}{-5}}\right]^{\frac{-5x}{x+2}} = e^{-5}$$

NOTE

$$\left\{\because \lim_{x\to\infty}\frac{-5x}{x+2} = \lim_{x\to\infty}\frac{-5}{1+\frac{2}{x}} = -5\right\}$$

2. (d) $\lim_{h\to 0}\frac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right]}{\sqrt{3}h\left(\sqrt{3}\cos h-\sin h\right)}$; then

$$=\lim_{h\to 0}\frac{\frac{4}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}\cos\left(\frac{\pi}{6}+h\right)\right]}{h\left(\sqrt{3}\cos h-\sin h\right)}$$

$$=\lim_{h\to 0}\frac{4}{\sqrt{3}}\cdot\frac{\sin h}{h}\cdot\frac{1}{\left(\sqrt{3}\cos h-\sin h\right)}=\frac{4}{3}$$

3. (b) $\lim_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}$, then

Applying L-hospital rule

$$=\lim_{x\to 0}\frac{2^x\log 2}{\frac{1}{2}(1+x)^{-1/2}}\quad\left\{\because \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\right\}$$

$$=2\log 2=\log 4$$

4. (c) $\lim_{x\to\frac{\pi}{2}}\frac{\left(1-\tan\left(\frac{x}{2}\right)\right)[1-\sin x]}{\left(1+\tan\left(\frac{x}{2}\right)\right)[x-2x]^3}$, then

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{(\pi-2x)^3}$$

Let $x=\frac{\pi}{2}+y,\ y\to 0$

$$\Rightarrow\lim_{y\to 0}\frac{\tan\left(\frac{-y}{2}\right)(1-\cos y)}{(-2y)^3}$$

$$=\lim_{y\to 0}\frac{-\tan\frac{y}{2}.2\sin^2\frac{y}{2}}{(-8)y^3}$$

$$=\lim_{y\to 0}\frac{1}{32}\frac{\tan\frac{y}{2}}{y/2}\left[\frac{\sin\frac{y}{2}}{y/2}\right]^2=\frac{1}{32}$$

5. (d) $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}\ \left(\frac{0}{0}\text{form}\right)$ and also $f'(2)=6$ and $f'(1)=4$, so then

Applying L-hospital rule

$$=\lim_{h\to 0}\frac{f'(2h+2+h^2)(2h+2)}{f'(h-h^2+1)(1-2h)}$$

$$=\lim_{h\to 0}\frac{f'(2)(2)}{f'(1)(1)}=\frac{6\times 2}{4}=3$$

6. (d) $\lim_{x\to 0}\frac{[(a-n)nx-\tan x]\sin nx}{x^2}=0$; then

$$=\lim_{x\to 0}n\frac{\sin nx}{nx}\lim_{x\to 0}\left((a-n)n-\frac{\tan x}{x}\right)=0$$

$$=n((a-n)n-1)=0$$

$$\Rightarrow(a-n)n=1\Rightarrow a=n+\frac{1}{n}$$

7. (a)

$$\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4,$$

and also $f(a)=g(a)=k$; then

Applying L-hospital's rule

$$\Rightarrow\lim_{x\to a}\frac{kg'(x)-kf'(x)}{g'(x)-f'(x)}=4$$

$$\Rightarrow\lim_{x\to a}k\left\{\frac{[g'(x)-f'(x)]}{g'(x)-f'(x)}\right\}=4$$

$$\therefore k=4$$

8. (a) $\lim_{x\to a}\frac{f(a)g(x)-f(x)g(a)}{x-a}$ and also

$f(a)=2, f'(a)=1$, $g(a)=-3, g'(a)=-1$;

then applying Hospital's rule

$$=\lim_{x\to a}\frac{f(a)g'(x)-f'(x)g(a)}{1}$$

$$=f(a)g'(a)-f'(a)g(a)$$

$$=2\times(-1)-1\times -3\ =-2+3=1$$

9. (c) Given that $f:R\to R$ such that $f(1)=3$ and $f'(1)=6$

Then $\lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{1/x}=e^{\lim_{x\to 0}\frac{1}{x}[\log f(1+x)-\log f(1)]}$

$e^{\lim_{x\to 0}\frac{\frac{1}{f(1+x)}f'(1+x)}{1}}$ [using L-hospital rule]

$$=e^{\frac{f'(1)}{f(1)}}=e^{6/3}=e^2$$

10. (a) $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ and also

$f(1)=1, f'(1)=2$ (given); then

Applying L-hospital rule

$$\Rightarrow\lim_{x\to 1}\frac{\frac{f'(x)}{2\sqrt{f(x)}}}{\frac{1}{2\sqrt{x}}}\Rightarrow\lim_{x\to 1}\frac{f'(x)\sqrt{x}}{\sqrt{f(x)}}\Rightarrow\frac{f'(1)}{\sqrt{f(1)}}=2$$

11. (c) $\lim\limits_{x\to 2}\dfrac{xf(2)-2f(x)}{x-2}$ and also

$f(x)=4, f'(x)=4$; then $f(2)=4, f'(2)=4$

Applying L-hospital rule

$$\Rightarrow \lim_{x\to 2}\frac{f(2)-2f'(x)}{1}$$

$$\Rightarrow f(2)-2f'(2)=4-2\times 4\ =-4$$

12. (d) $\lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{3^x-1}$; then

Applying L-Hospital's rules

$$\Rightarrow \lim_{x\to 0}\frac{\frac{1}{1+x}}{3^x\log 3}=\frac{1}{\log_e 3}$$

$$=\log_3 e$$

13. (a) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}$; then

$$=\lim_{x\to\frac{\pi}{2}}a^{\cos x}\left(\frac{a^{\cot x-\cos x}-1}{\cot x-\cos x}\right)$$

$$=a^{\cos(\pi/2)}\lim_{x\to\frac{\pi}{2}}\left(\frac{a^{\cot x-\cos x}-1}{\cot x-\cos x}\right)$$

$$=1.\log a=\log a$$

14. (c) $\lim\limits_{x\to\alpha}\dfrac{1-\cos(ax^2+bx+c)}{(x-a)^2}$; then

$$=2\lim_{x\to\alpha}\frac{\sin^2\left(\frac{ax^2+bx+c}{2}\right)}{(x-\alpha)^2}$$

$$=2\lim_{x\to\alpha}\frac{\sin^2\left\{\frac{a(x-\alpha)(x-\beta)}{2}\right\}}{(x-\alpha)^2}$$

[$\because$ α, β are roots of $ax^2+bx+c=0$

$\therefore$ $ax^2+bx+c=a(x-\alpha)(x-\beta)$]

$$=2\lim_{x\to\alpha}\left[\frac{\sin\left\{\frac{a(x-\alpha)(x-\beta)}{2}\right\}}{\frac{a(x-\alpha)(x-\beta)}{2}}\right].\frac{a^2}{4}(x-\beta)^2$$

$$=2(1)^2\frac{a^2}{4}(\alpha-\beta)^2$$

$$=\frac{a^2}{2}(\alpha-\beta)^2$$

15. (c) $\lim\limits_{x\to a}\dfrac{\cos x-\cos a}{\cot x-\cot a}$ $\left(\dfrac{0}{0}\text{form}\right)$; then

Applying L-Hospital rule;

$$\lim_{x\to a}\left(\frac{-\sin x}{-\text{cosec}^2 x}\right)=\lim_{x\to a}\sin^3 x$$

$$=\sin^3 a$$

16. (c) $\lim\limits_{h\to 0}\dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$; then

Applying L-hospital rule

$$=\lim_{h\to 0}\frac{2(a+h)\sin(a+h)+(a+h)^2\cos(a+h)}{1}$$

$$=2a\sin a+a^2\cos a$$

17. (a) $\lim\limits_{x\to 0}(x+1)^{\cot x}(1^\infty\text{form})$; then

$$=e^{\lim\limits_{x\to 0}[(x+1)-1]\cot x}=e^{\lim\limits_{x\to 0}x\cot x}=e^{\lim\limits_{x\to 0}\frac{x}{\tan x}}=e$$

18. (b) $\lim\limits_{x\to 0}\dfrac{x^2\sin\left(\frac{1}{x}\right)}{\sin x}$; then

$$=\lim_{x\to 0}\frac{\sin\left(\frac{1}{x}\right)}{1/x}\lim_{x\to 0}x\times\text{cosec }x$$

$= 0 \times$ finite $= 0$

19. (a) $f(x)=\left(\dfrac{x^2+5x+3}{x^2+x+3}\right)^x$, then

$$=1+\frac{4x}{x^2+x+3}=1+y\ \text{(say)}$$

and $\lim\limits_{x\to\infty}\left[1+\dfrac{4x}{x^2+x+3}\right]^x=e^{\lim\limits_{x\to\infty}\frac{4x}{x^2+x+3}.x}$

$$e^{\lim\limits_{x\to\infty}\frac{x^2}{x^2}\left[\frac{4}{1+\frac{x}{x^2}+\frac{3}{x^2}}\right]}=e^4$$

20. (c) If $0 < x < y$, then $\lim_{n\to\infty}\left(y^n + x^n\right)^{1/n}$

$$= \lim_{n\to\infty} y\left(1+\left(\frac{x}{y}\right)^n\right)^{1/n}$$

$$= y\lim_{n\to\infty}\left[1+\frac{1}{(y/x)^n}\right]^{1/n}$$

$$= y\left[1+\frac{1}{\infty}\right]^{\circ} = y\times(1)^{\circ} = y$$

21. (b) $\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x+3}$; then $= \lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{x+3}$

$$= \lim_{x\to\infty}\left[\left(1+\frac{1}{x+1}\right)^{x+1}\right]^{\frac{(x+3)}{(x+1)}} = e$$

$$\left\{\because \lim_{x\to\infty}\left(1+\frac{1}{x+1}\right)^{x+1} = e\right\}$$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. $\lim_{y\to 0}\frac{(x+y)\sec(x+y) - x\sec x}{y}$ is equal to **[AICBSE-1990]**

(a) $\sec x\,(x\tan x + 1)$
(b) $x\tan x + \sec x$
(c) $x\sec x + \tan x$
(d) None of these

2. $\lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ is equal to

(a) 1/8
(b) $\sqrt{2}/8$
(c) $\sqrt{2}$
(d) None of these

3. The value of $\underset{\theta\to\frac{\pi}{2}}{Lt}(\sec\theta - \tan\theta)$ is **[IIT-1976; AMU-1999]**

(a) 0 (b) 1
(c) $1/\sqrt{3}$ (d) ∞

4. $\lim_{x\to 0}\frac{(5^x - 1)^2}{x\log(1+x)}$ is equal to

(a) $\log_e 5$ (b) $(\log_e 5)^2$
(c) $3\log_e 5$ (d) None of these

5. $\lim_{x\to 0}(1+2x)^{(x+3)/x}$ is equal to

(a) e^3 (b) e^6
(c) e^9 (d) e^{12}

6. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}}$ is equal to

(a) 0 (b) 2
(c) 1 (d) 3

7. $\lim_{x\to 0}\frac{5\sin x - 7\sin 2x + 3\sin 3x}{x^2\sin x}$ is equal to

(a) 3 (b) –3
(c) 5 (d) –5

8. $\lim_{x\to\frac{\pi}{2}}\frac{1-\sin^3 x}{\cos^2 x}$ is equal to

(a) 1/2 (b) 1
(c) 3/2 (d) 12

9. $\lim_{x\to\infty}\frac{2x^2+2x-\sin^2 x}{3x^2-4x+\cos^2 x}$ is equal to

(a) 1/3 (b) 2/3
(c) 1 (d) 4/3

10. $\lim_{x\to 0}(\cos x)^{1/x}$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

11. If $f(a) = 2, f'(a) = 1, g(a) = 1, g'(a) = 2$, then $\lim_{x\to a}\frac{g(x)f(a) - g(a)f(x)}{x-a}$ is equal to **[DCE-1999; Karnataka CET-1999; MPPET-1995; Pb. CET-2004]**

(a) 1 (b) 2
(c) 3 (d) 4

12. If $\lim_{y\to 1}\frac{y^4-1}{y-1}=\lim_{y\to a}\frac{y^3-a^3}{y^2-a^2}$, then a is equal to

(a) 2/3 (b) 4/3
(c) 8/3 (d) 16

13. $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$ is equal to

(a) e^2 (b) e^3
(c) e^5 (d) e^7

14. $\lim_{x\to 2}\frac{x^6-24x-16}{x^3+2x-12}$ is equal to

(a) 6 (b) 12
(c) 18 (d) 24

15. $\lim_{x\to\tan^{-1}3}\frac{\tan^2 x-2\tan x-3}{\tan^2 x-4\tan x+3}$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

16. $\lim_{x\to\frac{\pi}{6}}\frac{\sin\left(x-\frac{\pi}{6}\right)}{\left(\frac{\sqrt{3}}{2}-\cos x\right)}$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

17. $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$ is equal to

[IIT-1993; RPET-2001]

(a) e (b) e^2
(c) e^0 (d) $1/e$

18. $\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$ is equal to

[Kerala (Engg.)-2005]

(a) 0 (b) 1
(c) 2 (d) 3

19. $\lim_{x\to 0^+}\frac{xe^{1/x}}{1+e^{1/x}}$ is equal to

(a) 0 (b) 1
(c) −1 (d) ∞

20. If a, b, c, d are positive, then $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$ is equal to ***[EAMCET-1992]***

(a) $e^{d/b}$ (b) $e^{c/a}$
(c) $e^{(c+d)/(a+b)}$ (d) e

21. $\lim_{x\to\infty}\left[1+\frac{1}{mx}\right]^x$ is equal to

[Kurukshetra CEE-1998]

(a) $e^{1/m}$ (b) $e^{-1/m}$
(c) e^m (d) m^e

22. $\lim_{x\to-\infty}\frac{2x-1}{\sqrt{x^2+2x+1}}$ is equal to

[Kerala PET-2007]

(a) 2 (b) −2
(c) 1 (d) −1

23. What is $\lim_{x\to 0}\frac{a^x-b^x}{x}$?

[NDA-2008]

(a) $\log\left(\frac{a}{b}\right)$ (b) $\log\left(\frac{b}{a}\right)$
(c) ab (d) $\log(ab)$

24. $\lim_{n\to\infty}\frac{4^{1/n}-1}{3^{1/n}-1}$ is equal to ***[UP-SEE-2007]***

(a) $\log_4 3$ (b) 1
(c) $\log_3 4$ (d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. $\lim\limits_{x\to 0}\dfrac{\cos(\sin x)-1}{x^2}$ is equal to

(a) 1 (b) −1
(c) 1/2 (d) −1/2

2. $\lim\limits_{x\to 0}\dfrac{\sec 4x-\sec 2x}{\sec 3x-\sec x}$ is equal to

(a) 1/2 (b) 1
(c) 3/2 (d) 2

3. $\lim\limits_{x\to a}\left(\dfrac{1}{x^2}-\cot^2 x\right)$ is equal to

(a) 1/3 (b) 2/3
(c) 1 (d) 4/3

4. $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}$: is equal to

(a) e (b) e^{-1}
(c) $\sqrt{e}$ (d) $e^{-1/2}$

5. If $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=80$, where n is a positive integer, then n =

(a) 2 (b) 5
(c) 3 (d) 9

6. $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/2}-(1-x)^{1/2}}{x}$ is equal to

[Roorkee-1979; RPET-1996]

(a) 0 (b) −1
(c) 1 (d) 1/2

7. The value of $\lim\limits_{x\to 0}\dfrac{e^x+\log(1+x)-(1-x)^{-2}}{x^2}$ is equal to ***[UPSEAT-2005]***

(a) 0 (b) −3
(c) −1 (d) infinitly

8. $\lim\limits_{x\to 0}\left[\dfrac{e^x-e^{\sin x}}{x-\sin x}\right]$ is equal to

[UPSEAT-2004]

(a) −1 (b) 0
(c) 1 (d) None of these

9. The value of $\lim\limits_{x\to\infty}\left(\dfrac{x+3}{x+1}\right)^{x+2}$ is

[UPSEAT-2003]

(a) e^4 (b) 0
(c) 1 (d) e^2

10. If $\lim\limits_{n\to\infty}\dfrac{1-(10)^n}{1+(10)^{n+1}}=\dfrac{-\alpha}{10}$, then the value of α is

[Orissa-2005]

(a) 0 (b) −1
(c) 1 (d) 2

11. $\lim\limits_{\theta\to 0}\dfrac{4\theta(\tan\theta-2\theta\tan\theta)}{(1-\cos 2\theta)}$ is ***[Orissa-2005]***

(a) $1/\sqrt{2}$ (b) 1/2
(c) 1 (d) 2

12. $\lim\limits_{x\to -2}\dfrac{\sin^{-1}(x+2)}{x^2+2x}$ is equal to

[Orissa 2002]

(a) 0 (b) ∞
(c) −1/2 (d) None of these

13. $\lim\limits_{x\to 0}\dfrac{\ln(\cos x)}{x^2}$ is equal to ***[AMU-2000]***

(a) 0 (b) 1
(c) 1/2 (d) −1/2

14. $\lim\limits_{x\to\pi/6}\left[\dfrac{3\sin x-\sqrt{3}\cos x}{6x-\pi}\right]$ is equal to

[EAMCET-2003]

(a) $\sqrt{3}$ (b) $1/\sqrt{3}$
(c) $-\sqrt{3}$ (d) $-1/\sqrt{3}$

15. The value of $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+\sin x}-\sqrt[3]{1-\sin x}}{x}$ is

(a) 2/3 (b) −2/3
(c) 3/2 (d) −3/2

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ
14. ⓐ ⓑ ⓒ ⓓ
15. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (d) $\lim\limits_{x\to 0}\dfrac{\cos(\sin x)-1}{x^2}$; then

$$\lim_{x\to 0}\frac{-2\sin^2\left(\dfrac{\sin x}{2}\right)}{x^2}=\frac{-2}{4}=\frac{-1}{2}$$

4. (b) $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x}(1^{\infty}\text{ form})$; then

$$=e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\tan 2x}$$

$$=e^{\lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\frac{2\tan x}{1-\tan^2 x}}$$

$$=e^{\lim\limits_{x\to\frac{\pi}{4}}\frac{-2\tan x}{1+\tan x}}=e^{-1}$$

5. (b) $\lim\limits_{x\to 2}\dfrac{x^n-2^n}{x-2}=80$; then

Applying L-hospital's rule

$$\lim_{x\to 2}nx^{n-1}=80\ =n2^{n-1}=80\ =n2^{n-1}=2^4\times 5$$

So $n=5$.

6. (c) $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/2}-(1-x)^{1/2}}{x}\ \left(\dfrac{0}{0}\text{form}\right)$

By L-hospital's rule

$$=\lim_{x\to 0}\frac{1}{2}(1+x)^{-1/2}+\frac{1}{2}(1-x)^{-1/2}$$

$$=\lim_{x\to 0}\frac{1}{2\sqrt{1+x}}+\frac{1}{2\sqrt{1-x}}=1$$

11. (c) $\lim\limits_{x\to a}\dfrac{1-(10)^n}{1+10^{n+1}}=\dfrac{-\alpha}{10}$; then

$$=\lim_{n\to\infty}\frac{10^n(10^{-n}-1)}{10^n(10^{-n}+10)}=\frac{-\alpha}{10}$$

$$=\frac{-1}{10}=\frac{-\alpha}{10}\Rightarrow\alpha=1$$

13. (c) $\lim\limits_{x\to -2}\dfrac{\sin^{-1}(x+2)}{x^2+2x}\ \left(\dfrac{0}{0}\text{form}\right)$; then

Applying L-Hospital's rule

$$\lim_{x\to -2}\frac{1}{\sqrt{1-(x+2)^2}\,(2x+2)}=-\frac{1}{2}$$

LECTURE

3

Test Your Skills

MENTAL PREPARATION TEST

1. Evaluate: $\lim_{x\to 1}\frac{x^2-1}{x-1}$ ***[CBSE-82, 84, 88]***

2. Evaluate: $\lim_{x\to 2}\frac{x^4-16}{x^3-8}$ ***[MP-2000]***

3. Evaluate: $\lim_{x\to 0}\frac{1-\cos 4x}{x^2}$ ***[MP-99, 2001; CBSE-91, 92]***

4. Evaluate: $\lim_{x\to\infty}\frac{(2x-3)(3x-4)}{(4x-5)(5x-6)}$ ***[MP-99]***

5. Evaluate the limit $\lim_{x\to 0}\left(\frac{\operatorname{cosec} x-\cot x}{x}\right)$ ***[CBSE-92, 96; HSB-93; HPSB-93]***

6. Evaluate: $\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2}$ ***[PSB-2000]***

7. Evaluate: $\lim_{x\to 2}\frac{x^3-8}{x^2-4}$ ***[MP-2000; PSB-2001; HPSEB-96, 2002C]***

8. Evaluate: $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$ ***[MP-95; 2004 (C)]***

9. Prove $\lim_{x\to 0}\frac{e^x-1}{x}=1$ ***[MP-95, 99, 2004 (B)]***

10. Evaluate: $\lim_{x\to\infty}\frac{2x^2-3x+1}{x^2-1}$ ***[MP-99]***

11. Evaluate: $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$ ***[AISSE-92C; CBSE-92 C]***

12. Evaluate the left hand and right hand limits of the function defined by $f(x)=\begin{cases}1+x^2, & \text{if } 0\le x\le 1\\ 2-x. & \text{if } x>1\end{cases}$ at $x = 1$. Also, show that $\lim_{x\to 1} f(x)$ does not exist. ***[CBSE-2001 (C)]***

13. Evaluate: $\lim_{x\to 0}\frac{1-\cos 2x}{x}$ ***[CBSE-87]***

14. Evaluate the limit $\lim_{x\to 0}\frac{1-\cos 2x}{3\tan^2 x}$ ***[AISSE-92; CBSE-92]***

15. Evaluate the limit $\lim_{x\to 0}\frac{5x+4\sin 3x}{4\sin 2x+7x}$ ***[HSB-2002]***

16. Evaluate the limit $\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$ ***[PSB-2001 (C); Karnataka (CEE)-99]***

17. Evaluate the limit $\lim\limits_{x\to 0}\dfrac{3^x-2^x}{x}$. ***[PSB-2001]***

18. Find the value of $\lim\limits_{x\to 2}\dfrac{x^3+8}{x+2}$ ***[MP-98]***

19. Evaluate: $\lim\limits_{x\to 0}\dfrac{1-\cos\theta}{\theta^2}$.

[MP-95, 98, 99, 2000, 2001, 2005(C); CBSE-92 C, 93, 2000; MP-96]

20. Prove that $\lim\limits_{x\to 0}\dfrac{e^{ax}-1}{x}=a$ ***[MP-99, 2000]***

21. Show that $\lim\limits_{x\to 2}\dfrac{|x-2|}{x-2}$ does not exist.

[CBSE-85]

22. Evaluate: $\lim\limits_{x\to 1}\dfrac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$ ***[PSB-2001]***

23. Find the left hand and right hand limits of the greatest integer function *f*(*x*) = [*x*] = greatest integer.
Less than or equal to x at $x=k$, where k is an integer. Also, show that $\lim\limits_{x\to k}f(x)$ does not exist.

24. Evaluate the limit $\lim\limits_{\theta\to 0}\dfrac{1-\cos 4\theta}{2\theta^2}$. ***[CBSE-93]***

25. Evaluate the limit $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos^2 x}{1-\sin x}$. ***[HSB-97]***

26. Evaluate the limit $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4}$. ***[MP-2003]***

27. Let $f(x)=\begin{cases} x & \text{if } 0\le x<\dfrac{1}{2} \\ 0 & \text{if } x=\dfrac{1}{2} \\ x-1 & \text{if } \dfrac{1}{2}<x\le 1\end{cases}$.

Then show that $\lim\limits_{x\to 1/2}f(x)$ does not exist.

[CBSE-85, PSB-95]

28. Evaluate the limit $\lim\limits_{x\to 2}\dfrac{e^x-e^2}{x-2}$.

[AISSE-2003]

LECTUREWISE WARMUP TEST

1. $\lim\limits_{x\to a}\dfrac{\log(x-a)}{\log(e^x-e^a)}$ is equal to

[MPPET-2005]

(a) 0 (b) 1
(c) a (d) does not exist

2. If $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x^x-a^a}=-1$, then a equal to

[EAMCET-2003; Karnataka CET-2000; MPPET-2005]

(a) 1 (b) 0
(c) e (d) $(1/e)$

3. $\lim\limits_{n\to\infty}\left(\dfrac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}$ equals

[AMU-2002]

(a) e (b) e^2
(c) e^{-1} (d) 1

4. $\lim\limits_{x\to\frac{\pi}{2}}\tan^2 x\left(\sqrt{2\sin^2 x+3\sin x+4}-\sqrt{\sin^2 x+6\sin x+2}\right)$ is

[AMU-2002]

(a) 1/10 (b) 1/11
(c) 1/12 (d) 1/8

5. $\lim_{x\to 0}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}=$ ***[IIT-1999]***

(a) 2 (b) –2
(c) 1/2 (d) –1/2

6. If $f(x)=\begin{cases}\frac{\sin[x]}{[x]}, & \text{when } [x]\neq 0\\ 0, & \text{when } [x]=0\end{cases}$ where $[x]$ is greatest integer function, then $\lim_{x\to 0^{\cdot}} f(x)=$

[RPET-95; IIT-1985]

(a) –1 (b) 1
(c) 0 (d) None of these

7. If $\lim_{n\to\infty}\frac{1-(10)^n}{1+(10)^{n+1}}=\frac{-\alpha}{10}$, then give the value of α is ***[Orissa JEE-2005]***

(a) 0 (b) –1
(c) 1 (d) 2

8. $\lim_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)=$

[IIT-78, 84; RPET-97, 2001; UPSEAT-2003; Pb.CET-2003]

(a) $\pi/2$ (b) π
(c) $2/\pi$ (d) 0

9. Let $f(x)=\frac{1}{\sqrt{18-x^2}}$. What is the value of; $\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}$ ***[MNR-1994]***

(a) 0 (b) –1/9
(c) 1/3 (d) 1/9

10. $\lim_{x\to 2}\frac{4^{x/2}-4}{4^x-16}=$ ***[MPCET-2000]***

(a) 1/8 (b) 0
(c) 1/4 (d) $ln\,4$

11. $\lim_{x\to 0}\frac{(1-\cos 2x)\sin 5x}{2x^2\sin 3x}=$

[KUKCEET-1998; MPCET-2000]

(a) 10/3 (b) 5/3
(c) 6/5 (d) 5/6

12. $\lim_{x\to\infty}\left(\sqrt{x^2+\lambda x+3}-\sqrt{x^2+4x+3}\right)=2$, then λ = ***[MPPET-1997]***

(a) 8 (b) 4
(c) 12 (d) 2

13. What is the value of $\lim_{x\to 0}\frac{10^x-2^x-5^x+1}{x\sin x}$?

[NDA-2004]

(a) $(ln\,2)\ (ln\,5)$ (b) $(ln\,3)\ (ln\,5)$
(c) $(ln\,10)\ (ln\,5)$ (d) 0

14. If $\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right)=A$ and $\lim_{x\to 0}x\sin\left(\frac{1}{x}\right)=B$, then which one of the following is correct?

[NDA-2004]

(a) A = 1 and B = 0
(b) A = 0 and B = 1
(c) A = 0 and B = 0
(d) A = 1 and B = 1

15. $\lim_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$ is given by

(a) 0 (b) 1/2
(c) log 2 (d) None of these

16. $\lim_{x\to 0}\left(\frac{e^x+e^{-x}-2}{x^2}\right)^{1/x^2}$ is

(a) $e^{1/2}$ (b) $e^{1/4}$
(c) $e^{1/3}$ (d) $e^{1/12}$

17. If $f(x)=\begin{cases}-\frac{\sin(1+[x])}{[x]}; & \text{for } [x]\neq 0\\ 0, & \text{for } [x]=0\end{cases}$ where $[x]$ denotes the greatest integer $\le x$, then $\lim_{x\to 0^-}f(x)$ equals

(a) 1 (b) 0
(c) –1 (d) None of these

18. $\lim_{x\to\infty}\left(1+\frac{-3}{x}\right)^x\left(1+\frac{2}{x}\right)^{-x}=$

(a) e^{-1} (b) e^{-5}
(c) e^5 (d) e^1

19. $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}=$

(a) e (b) e^2
(c) e^3 (d) $1/e$

20. $\lim_{n\to\infty}\frac{[1^2x]+[2^2x]+[3^2x]+....[n^2x]}{n^3}$ is where [.] denotes greatest integer function less then or equal to x

(a) $x/3$ (b) $x/6$
(c) $2x$ (d) $x/2$

21. $\lim\limits_{x\to\infty} \dfrac{x^3+px^2+qx+9}{x^3+ax^2+bx+13}$

(a) p/a (b) q/b
(c) 9/13 (d) 1

22. Which of the following is not true ***[REE qualifying exam.-1999]***

(a) $\lim\limits_{x\to\frac{\pi^+}{2}} \tan x = -\infty$

(b) $\lim\limits_{x\to\frac{\pi^-}{2}} \tan x = \infty$

(c) $\lim\limits_{x\to\frac{\pi}{2}} \tan x = \infty$

(d) $\lim\limits_{x\to\frac{\pi}{2}} \tan x$ does not exist

23. If $f(x+2) = \dfrac{1}{2}\left\{f(x+1)+\dfrac{4}{f(x)}\right\}$ and $f(x) > 0$ for all $x \in R$, then $\lim\limits_{x\to\infty} f(x)$ is

(a) 1 (b) 2
(c) –2 (d) 0

24. $\lim\limits_{x\to\frac{1}{2}} \dfrac{\sin 3[x]}{[x]}$, where $[x]$ denotes the greatest integer $\le x$, is equal to

(a) 3/2 (b) 3
(c) 1 (d) None of these

25. $\lim\limits_{x\to 0} \dfrac{\sin(\sin x)-\sin x}{ax^5+bx^3} =$

(a) 0 (b) –1/12
(c) 1/2 (d) 1/4

26. If $\lim\limits_{x\to 0}(1+ax+bx^2)^{2/x} = e^3$, then a and b in order are

(a) 3/2, 2/3 (b) 2/3, 3/2
(c) 6, 3/2 (d) 3, 2

24. $\lim\limits_{x\to 0} \dfrac{3\sin x^0 - \sin 3x^0}{x^3} =$

(a) $2\left(\dfrac{\pi}{180}\right)^3$ (b) $\left(\dfrac{\pi}{180}\right)^3$

(c) $3\left(\dfrac{\pi}{180}\right)^3$ (d) $4\left(\dfrac{\pi}{180}\right)^3$

28. What is the value of $\underset{x\to 0}{Lt} \dfrac{x\sin 5x}{\sin^2 4x}$? ***[NDA-2006]***

(a) 0 (b) 5/4
(c) 5/16 (d) 25/4

29. If $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right]$ exists, then which one of the following is correct? ***[NDA-2006]***

(a) Both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ must exist

(b) $\lim\limits_{x\to a} f(x)$ need not exist but $\lim\limits_{x\to a} g(x)$ must exist

(c) Both $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ need not exist

(d) $\lim\limits_{x\to a} f(x)$ must exist but $\lim\limits_{x\to a} g(x)$ need not exist.

30. The value of $\lim\limits_{x\to 0} \dfrac{(1+x)^{1/3}-(1-x)^{1/3}}{x}$ is

(a) 2/3 (b) 1/3
(c) 1 (d) None of these

31. $\lim\limits_{x\to 0} \dfrac{e^x+e^{-x}+2\cos x-4}{x^4}$ is equal to

(a) 0 (b) 1
(c) 1/6 (d) –1/6

32. $\lim\limits_{x\to 0}\left\{\dfrac{\sin x - x + \dfrac{x^3}{6}}{x^5}\right\} =$ ***[MNR-1985]***

(a) 1/120 (b) –1/120
(c) 1/20 (d) None of these

33. The value of $\lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e + \dfrac{1}{2}ex}{x^2}$ is ***[DCE-2001]***

(a) $11e/24$ (b) $-11e/24$
(c) $e/24$ (d) None of these

34. The value of $\lim\limits_{x\to 1} |\log_2 2x|^{\log_x 5}$ is

(a) 5 (b) $e^{\log_2 2}$
(c) $\log_2 5$ (d) $e^{\log_2 5}$

35. The value of $\lim\limits_{x\to 0} \dfrac{\tan[e^2]x^2 - \tan[-e^2]x^2}{\sin^2 x}$ is

(a) 7 (b) 8
(c) 15 (d) None of these

LECTUREWISE WARMUP TEST: SOLUTIONS

1. (b) $\lim_{x\to a}\dfrac{\log(x-a)}{\log(e^x-e^a)} = \lim_{x\to a}\dfrac{\frac{1}{x-a}}{\frac{1}{e^x-e^a}e^x}$

(By L'Hospital rule)

$=\lim_{x\to a}\dfrac{e^x-e^a}{e^x.(x-a)}$ (Again by L'Hospital rule)

$=\lim_{x\to a}\dfrac{e^x}{e^x(x-a)+e^x}$

$=\dfrac{e^a}{e^a}=1$

2. (a) $\lim_{x\to a}\dfrac{a^x-x^a}{x^x-a^a} = \lim_{x\to a}\dfrac{a^x\log_e a-ax^{a-1}}{x^x(1+\log x)}$

(By L'Hospital rule)

$=\dfrac{a^a\log_e a-a^a}{a^a(\log a+1)}=\dfrac{\log_e a-1}{\log_e a+1}$

$\because$ $\lim_{x\to a}\dfrac{a^x-x^a}{x^x-a^a}=-1$

$\therefore$ $\log_e a-1=-\log_e a-1$

$\Rightarrow$ $2\log_e a=0\Rightarrow a=1$

3. (b)

$\lim_{n\to\infty}\left(\dfrac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)}=\lim_{n\to\infty}\left(\dfrac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)}$

$=\lim_{n\to\infty}\dfrac{\left(1+\dfrac{1}{n(n-1)}\right)^{n(n-1)}}{\left(1-\dfrac{1}{n(n-1)}\right)^{n(n-1)}}=\dfrac{e}{e^{-1}}=e^2$

4. (c) Rationalising, we obtain the given limit as

$\lim_{x\to\frac{\pi}{2}}\dfrac{\tan^2 x(\sin^2 x-3\sin x+2)}{\sqrt{2\sin^2 x+3\sin x+4}+\sqrt{\sin^2 x+6\sin x+2}}$

$=\dfrac{\lim_{x\to\frac{\pi}{2}}\dfrac{\sin^2 x(\sin x-1)(\sin x-2)}{(1-\sin x)(1+\sin x)}}{\{\sqrt{2+3+4}+\sqrt{1+6+2}\}}$

$=\dfrac{-1(1-2)}{(1+1)(3+3)}=\dfrac{1}{12}$

5. (c) $\lim_{x\to 0}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$

$=\lim_{x\to 0}\dfrac{x\dfrac{2\tan x}{1-\tan^2 x}-2x.\tan x}{(2\sin^2 x)^2}$

$=\lim_{x\to 0}\dfrac{2x\tan x\left[\dfrac{1}{1-\tan^2 x}-1\right]}{4.\sin^4 x}$

$=\lim_{x\to 0}\dfrac{2x\tan x\left[\dfrac{1-1+\tan^2 x}{1-\tan^2 x}\right]}{4\sin^4 x}$

$=\lim_{x\to 0}\dfrac{1.x.\tan^3 x}{2\sin^4 x(1-\tan^2 x)}$

$=\lim_{x\to 0}\dfrac{1}{2}.\dfrac{x\left(\dfrac{\tan x}{x}\right)^3.x^3}{\sin^4 x(1-\tan^2 x)}$

$=\lim_{x\to 0}\dfrac{1\left(\dfrac{\tan x}{x}\right)^3}{2\left(\dfrac{\sin x}{x}\right)^4.(1-\tan^2 x)}$

$=\dfrac{1.(1)^3}{2.(1)^4.(1-0)}=\dfrac{1}{2}$

6. (d) As, $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & [x]\neq 0\\ 0, & [x]=0\end{cases}$,

$f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & x\in R-[0,1)\\ 0, & 0\le x<1\end{cases}$

$\therefore$ R.H.L. at $x=0\Rightarrow\lim_{x\to 0^+}0=0$

L.H.L. at $x=0\Rightarrow\lim_{x\to 0^-}\dfrac{\sin[x]}{[x]}$

$=\lim_{h\to 0}\dfrac{\sin[0-h]}{[0-h]}=\lim_{h\to 0}\dfrac{\sin(-1)}{-1}=\sin 1$

Since R.H.L. $\neq$ L.H.L. $\therefore$ limit does not exists.

7. (c) $\lim\limits_{n\to\infty}\dfrac{1-(10)^n}{1+(10)^{n+1}} =$

$$\lim_{n\to\infty}\frac{(10)^n\left[\left(\frac{1}{10}\right)^n-1\right]}{(10)^{n+1}\left(1+\frac{1}{10^{n+1}}\right)} = \frac{-1}{10}$$

$\therefore \quad \alpha = 1$

8. (a) $\lim\limits_{x\to1}(1-x) = \tan\dfrac{\pi x}{2}$

Here put $x - 1 = t$ or $x = 1 + t$

as $x \to 1$; $t \to 0$

$$\Rightarrow \quad \lim_{t\to0}(-t)\tan\frac{\pi}{2}(1+t)$$

$$= \lim_{t\to0} t\left(-\cot\frac{\pi t}{2}\right)$$

$$\Rightarrow \quad \lim_{t\to0}\frac{t}{\tan\left(\frac{\pi t}{2}\right)} = \frac{2}{\pi}$$

9. (d) $\lim\limits_{x\to3}\dfrac{f(x)-f(3)}{x-3} = \lim\limits_{x\to3}\dfrac{(18-x^2)^{-1/2}-(1/3)}{x-3}$

$$= \lim_{x\to3}\frac{3-\sqrt{18-x^2}}{3(x-3)\sqrt{18-x^2}}$$

$$= \lim_{x\to3}\left[\frac{9-(18-x^2)}{3(x-3)\sqrt{18-x^2}\,(3+\sqrt{18-x^2})}\right]$$

$$= \lim_{x\to3}\frac{x^2-9}{3(x-3)\sqrt{18-x^2}\,[3+\sqrt{18-x^2}]}$$

$$= \lim_{x\to3}\frac{x+3}{3\sqrt{18-x^2}\,[3+\sqrt{18-x^2}]}$$

$$= \frac{3+3}{3(3)(3+3)} = \frac{6}{9.6} = \frac{1}{9}$$

10. (a) $\lim\limits_{x\to2}\dfrac{4^{x/2}-4}{4^x-16} = \lim\limits_{x\to2}\dfrac{4^{x/2}-4}{(4^{x/2}-4)(4^{x/2}+4)}$

11. (b) $\lim\limits_{x\to0}\dfrac{(1-\cos 2x)\sin 5x}{2x^2\sin 3x}$

$$= \lim_{x\to0}.\frac{1}{2}.\frac{2\sin^2 x}{x^2}.\frac{\sin 5x}{5x}.5x.\frac{3x}{\sin 3x}.\frac{1}{3x}$$

$$= \lim_{x\to0}\left(\frac{\sin x}{x}\right)^2.\lim_{x\to0}\frac{\sin 5x}{5x}.\lim_{x\to0}\frac{3x}{\sin 3x}.\frac{5}{3}$$

$$= (1)\,.\,(1)\,.\,(1)\,.\,\frac{5}{3} = \frac{5}{3}$$

12. (a) On rationalization

$$\lim_{n\to\infty}\frac{(\lambda-4)x}{(\sqrt{x^2+\lambda x+3}+\sqrt{x^2+4x+3})} = 2$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{\lambda-4}{\left(\sqrt{1+\frac{\lambda}{x}+\frac{3}{x^2}}+\sqrt{1+\frac{4}{x}+\frac{3}{x^2}}\right)} = 2$$

$$\Rightarrow \quad \frac{\lambda-4}{2} = 2 \Rightarrow \lambda - 4 = 4 \quad \Rightarrow \lambda = 8$$

13. (a) $\lim\limits_{x\to\infty}\dfrac{10^x-2^x-5^x+1}{x\sin x}$

$$= \lim_{x\to0}\frac{2^x\times5^x-2^x-5^x+1}{x\sin x}$$

$$= \lim_{x\to0}\frac{(5^x-1)(2^x-1)}{x\sin x}$$

$$= \lim_{x\to0}\left(\frac{5^x-1}{x}\right)\left(\frac{2^x-1}{x}\right).\frac{x}{\sin x}$$

$= (ln\ 5)\ (ln\ 2)\ .\ 1 = (ln\ 5)\ (ln\ 2)$

14. (a) $\because\ \lim\limits_{x\to\infty} x\sin\left(\dfrac{1}{x}\right) = A \Rightarrow \lim\limits_{y\to0}\dfrac{1}{y}\sin y = A$

$\therefore \quad \lim\limits_{y\to0}\dfrac{\sin y}{y} = A \Rightarrow A = 1$ and

$\lim\limits_{x\to0} x\sin\left(\dfrac{1}{x}\right) = B \Rightarrow \lim\limits_{y\to\infty}\dfrac{1}{y}\sin y = B$

$\Rightarrow \quad \lim\limits_{y\to\infty}\dfrac{\sin y}{y} = B \Rightarrow B = 0$

$\therefore \quad A = 1$ and $B = 0$

15. (b) $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

$$= \lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1} = \frac{1}{2}$$

16. (d) For $x \neq 0$, let $u = \dfrac{e^x + e^{-x} - 2}{x^2}$

$$= \frac{1}{x^2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\right)\right]$$

$$+ \left[\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}....\right) - 2\right]$$

$$= \frac{2}{x^2}\left[\frac{x^2}{2!} + \frac{x^4}{4!} + ...\right] = \left[1 + \frac{x^2}{12} + \frac{2x^4}{6!} + ...\right]$$

$$\therefore \quad u^{1/x^2} = \left[1 + \frac{x^2}{12} + ...\right]^{1/x^2}$$

$$= \left[1 + \left(\frac{x^2}{12} + \frac{x^4}{360} - ......\right)\right]^{\frac{1}{(x^2/12 + x^4/360 + ...)}(1/12 + x^2/360 + ..)}$$

$$= \lim_{x \to 0} u^{1/x^2} = \lim_{y \to 0}(1 + y)^{1/y} \cdot \lim_{x \to 0}\left(\frac{1}{12} + \frac{x^2}{360} + ...\right)$$

$$= e^{1/12}\left(y = \frac{x^2}{12} + \frac{x^4}{360} + ...\right)$$

17. (b) For $-1 < x < 0$, $[x] = -1$,

So $\lim\limits_{x \to 0^-} \dfrac{\sin(1 + [x])}{[x]} = \dfrac{\sin 0}{-1} = 0$

18. (b) $\lim\limits_{x \to \infty}\left(1 + \dfrac{-3}{x}\right)^x \left(1 + \dfrac{2}{x}\right)^{-x}$

$$\Rightarrow \quad \lim_{x \to \infty}\left(\frac{x - 3}{x + 2}\right)^x = \lim_{x \to \infty}\left(\frac{x + 2 - 5}{x + 2}\right)^x$$

$$\Rightarrow \quad \lim_{x \to \infty}\left[\left(1 - \frac{5}{x + 2}\right)^{\frac{x+2}{-5}}\right]^{\frac{-5x}{x+2}}$$

$$= e^{-5}\left[\because \lim_{x \to \infty}\frac{-5x}{x + 2} = -5\lim_{x \to \infty}\frac{1}{1 + \frac{2}{x}} = -5.\frac{1}{1 + 0} = -5\right]$$

19. (b) $\lim\limits_{x \to 0}\left[(1 + 5x^2)(1 + 3x^2)^{-1}\right]^{1/x^2}$

$$= \lim_{x \to 0}\left[(1 + 5x^2)(1 - 3x^2 + ..)\right]^{1/x^2}$$

$$= \lim_{x \to 0}\left[(1 + 2x^2)^{1/2x^2}\right]^2 = e^2$$

20. (a) Since $1^2 x - 1 < [1^2 x] \leq 1^2 . x$

$2^2 x - 1 < [2^2 x] \leq 2^2 x$

$3^2 x - 1 < [3^2 x] \leq 3^2 x$

$n^2 x - 1 < [n^2 x] \leq n^2 x$

Adding all terms, we get

$$x\Sigma n^2 - n < [1^2 x] + [2^2 x] + [3^2 x] + .. + [n^2 x] \leq x\Sigma n^2$$

Dividing each term by n^3, we get

$$\frac{x(1 + 1/n)(2 + 1/n)}{6} - \frac{1}{n^2}$$

$$< \frac{[1^2 x] + [2^2 x] + .. + [n^2 x]}{n^3} \leq \frac{x(1 + 1/n)(2 + 1/n)}{6}$$

Let $n \to \infty$, we get

$$\frac{x}{6}(2) - 0 \leq \lim_{n \to \infty}\frac{[1^2 x] + [2^2 x] + .. + [n^2 x]}{n^3} \leq \frac{x(2)}{6}$$

$$\therefore \quad \lim_{n \to \infty}\frac{[1^2 x] + [2^2 x] + .. + [n^2 x]}{n^3} = \frac{x}{3}$$

21. (d) Devide the numerator and denominator by highest power of x.

$$\lim_{x \to \infty}\frac{1 + p/x + q/x^2 + 9/x^3}{1 + a/x + b/x^2 + 13/x^3} = \frac{1}{1} = 1$$

22. (c) L.H.L. (∞) and R.H.L. ($-\infty$) is not equal, hence (c) is not true.

23. (b) Clearly, $\lim\limits_{x \to \infty} f(x + 2) = \lim\limits_{x \to \infty} f(x + 1)$

$= \lim\limits_{x \to \infty} f(x) = l$(say)

Then taking limit,

$$l = \frac{1}{2}\left(l + \frac{4}{l}\right)$$

or $\dfrac{1}{2}l = \dfrac{2}{l}$

or $l^2 = 4$

$\therefore \quad l = 2 \quad [\because f(x) > 0$ for all $x]$

24. (b)

$$\text{R.H. limit} = \lim_{h \to 0}\frac{\sin 3\left[\frac{1}{2} + h\right]}{\left[\frac{1}{2} + h\right]} = \lim_{\theta \to 0}\frac{\sin 3\theta}{\theta} = 3$$

$$\text{L.H. limit} = \lim_{h \to 0}\frac{\sin 3\left[\frac{1}{2} - h\right]}{\left[\frac{1}{2} - h\right]} = \lim_{\theta \to 0}\frac{\sin 3\theta}{\theta} = 3$$

25. (b) $\lim\limits_{x\to 0}\dfrac{\sin(\sin x)-\sin x}{x^5+2x^3}$

$$= \lim_{x\to 0} 2\cos\frac{\left(\dfrac{\sin x + x}{2}\right)\sin\left(\dfrac{\sin x - x}{2}\right)}{x^5+2x^3}$$

$$= \lim_{x\to 0}\frac{2\left(\dfrac{\sin x - x}{2}\right)\times 1}{x^5+2x^3}$$

$$= \lim_{x\to 0}\frac{\sin x - x}{x^5+2x^3}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\cos x - 1}{5x^4+6x^2}\left(\frac{0}{0}\right) \quad \text{(By L-Hospital)}$$

$$= \lim_{x\to 0}\frac{-\sin x}{20x^3+12x}\left(\frac{0}{0}\right) \quad \text{(By L-Hospital)}$$

$$= \lim_{x\to 0}\frac{-\cos x}{60x^2+12} \quad \text{(By L-Hospital)}$$

$$= -\frac{1}{12}$$

26. (a) Let $P = \lim\limits_{x\to 0}\left(1+ax+bx^2\right)^{\frac{2}{x}}$ $\quad\left(\text{form } 1^\infty\right)$

$$= e^{\lim\limits_{x\to 0}\left(1+ax+bx^2-1\right)^{\frac{2}{x}}} = e^{\lim\limits_{x\to 0}(2a+2bx)}$$

$$= e^{2a} = e^3 \quad \text{(given)}$$

$\therefore \quad a = \dfrac{3}{2}$ and $b \in R.$

27. (d) $\lim\limits_{x\to 0}\dfrac{3\sin x^\circ - \sin 3x^\circ}{x^3}$

$$= \lim_{x\to 0}\frac{3\sin\left(\dfrac{\pi}{180}x\right)-\sin\left(\dfrac{3\pi}{180}x\right)}{x^3}$$

$$\lim_{x\to 0}\frac{3\left\{\dfrac{\pi x}{180}-\dfrac{\left(\dfrac{\pi x}{180}\right)^3}{3!}\right\}-\left\{\dfrac{3\pi}{180}x-\dfrac{\left(\dfrac{3\pi}{180}x\right)^3}{3!}\right\}}{x^3}$$

$$= \frac{1}{6}\left\{\frac{-3\pi^3}{(180)^3}+\frac{27\pi^3}{(180)^3}\right\}$$

$$= \frac{1}{6}\left\{\frac{24\pi^3}{(180)^3}\right\} = 4\left(\frac{\pi}{180}\right)^3$$

28. (c) $\lim\limits_{x\to 0}\dfrac{x\sin 5x}{\sin^2 4x} = \lim\limits_{x\to 0}\dfrac{x}{\sin^2 4x}.\sin 5x$

$$= \lim_{x\to 0}\frac{x^2}{\sin^2 4x}.\frac{\sin 5x}{x}$$

$$= \lim_{x\to 0}\frac{16x^2}{16\sin^2 4x}.\frac{5\sin 5x}{5x}$$

$$= \lim_{x\to 0}\frac{5}{16}\left(\frac{4x}{\sin 4x}\right)^2.\left(\frac{\sin 5x}{5x}\right) = \frac{5}{16}\times 1\times 1 = \frac{5}{16}$$

29. (a) It is the fudamental concept.

30. (a) $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/3}-(1-x)^{1/3}}{x}$

$$= \lim_{x\to 0}\frac{(1+x)-(1-x)}{(1+x)^{2/3}+(1+x)^{1/3}(1-x)^{1/3}+(1-x)^{2/3}}.\frac{1}{x}$$

$$\left[\text{using } a-b = \frac{a^3-b^3}{a^2+ab+b^2}\right]$$

$$= \lim_{x\to 0}\frac{2}{(1+x)^{2/3}+(1+x)^{1/3}(1-x)^{1/3}+(1-x)^{2/3}}$$

$$= \frac{2}{1+1+1} = \frac{2}{3}$$

31. (c) $\lim\limits_{x\to 0}\dfrac{e^x+e^{-x}+2\cos x-4}{x^4}\left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0}\frac{e^x-e^{-x}-2\sin x}{4x^3}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{12x^2}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{e^x-e^{-x}+2\sin x}{24x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x}{24} = \frac{4}{24} = \frac{1}{6}$$

32. (a) $\lim\limits_{x\to 0}\dfrac{\sin x - x + \dfrac{x^3}{6}}{x^5}$

$$= \lim_{x\to 0}\frac{\left(x-\dfrac{x^3}{6}+\dfrac{x^5}{120}-\ldots\right)-x+\dfrac{x^3}{6}}{x^5}$$

$$= \lim_{x\to 0}\frac{\dfrac{x^2}{120}-\ldots}{x^5} = \frac{1}{120}$$

33. (a) $(1+x)^{1/x} = e^{\frac{1}{x}\log(1+x)} = e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^5}{5}.....\right)}$

$= e^{1-\frac{x}{2}+\frac{x^2}{3}-...} = e.e^{-\frac{x}{2}+\frac{x^2}{3}-.....}$

$$= e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-.....\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-.....\right)^2+...\right]$$

$$= e\left[1-\frac{x}{2}+\frac{11}{24}x^2-....\right]$$

$$\therefore \quad \lim_{x\to 0}\frac{(1+x)^{1/x}-e+\dfrac{ex}{2}}{x^2}=\frac{11e}{24}$$

34. (d) $\lim_{x\to 1}(\log_2 2x)^{\log_x 5}$

$= \lim_{x\to 1}(\log_2 2+\log_2 x)^{\log_x 5}$

$$= \lim_{x\to 1}\left[(1+\log_2 x)^{\frac{1}{\log_2 x}}\right]^{\log_2 x\log_2 5}$$

$$= e^{\lim\limits_{x\to 1}\log_2 5} = e^{\log_2 5}$$

35. (c) $\lim_{x\to 0}\frac{\tan[e^2]x^2-\tan[-e^2]x^2}{\sin^2 x}\ \underset{x\to}{1}$

$$=\lim_{x\to 0}\frac{\tan 7x^2+\tan 8x^2}{\sin^2 x}(\because 7<e^2<8)$$

$$=\lim_{x\to 0}\frac{\dfrac{\tan 7x^2}{7x^2}7+\dfrac{\tan 8x^2}{8x^2}8}{\left(\dfrac{\sin x}{x}\right)^2}=\frac{7+8}{1}=15$$

ANSWERS

LECTURE 1

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (a)	**8.** (c)	**15.** (d)	**22.** (a)
2. (a)	**9.** (a)	**16.** (a)	**23.** (d)
3. (a)	**10.** (c)	**17.** (b)	**24.** (c)
4. (d)	**11.** (c)	**18.** (a)	
5. (a)	**12.** (d)	**19.** (c)	
6. (c)	**13.** (b)	**20.** (c)	
7. (a)	**14.** (d)	**21.** (c)	

Worksheet: To Check the Preparation Level

1. (a)	**5.** (a)	**9.** (a)	**13.** (c)
2. (b)	**6.** (d)	**10.** (a)	**14.** (d)
3. (a)	**7.** (b)	**11.** (d)	**15.** (b)
4. (a)	**8.** (c)	**12.** (b)	**16.** (d)

LECTURE 2

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (a)	**8.** (c)	**15.** (c)	**22.** (a)
2. (b)	**9.** (b)	**16.** (c)	**23.** (b)
3. (a)	**10.** (b)	**17.** (b)	**24.** (b)
4. (b)	**11.** (c)	**18.** (b)	
5. (b)	**12.** (c)	**19.** (a)	
6. (c)	**13.** (c)	**20.** (a)	
7. (d)	**14.** (b)	**21.** (d)	

Worksheet: To Check the Preparation Level

1. (d)	**5.** (b)	**9.** (d)	**13.** (d)
2. (c)	**6.** (c)	**10.** (c)	**14.** (b)
3. (b)	**7.** (b)	**11.** (d)	**15.** (a)
4. (b)	**8.** (c)	**12.** (c)	

LECTURE 3

Mental Preparation Test

1. 2	**15.** 17/15
2. 8/3	**16.** $2 \log_2$
3. 8	**17.** log (3/2)
4. 3/10	**18.** 4
5. 1/2	**19.** 1/2
6. $\dfrac{-(m^2 - n^2)}{2}$	**22.** 1/4
7. 3	**23.** $k - 1$; k
8. a/b	**24.** 1/4
10. 2	**25.** 2
11. 1/2	**26.** 8
12. 2; 1	**28.** e^2
13. 0	
14. 2/3	

Part C

Continuity

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LECTURE

1

Evaluation of Continuity

BASIC CONCEPTS

CONTINUITY

Continuity of a function at a point A function 'f' is said to be continuous at a point a in the domain of f if the following conditions are satisfied:

(i) $f(a)$ exists

(ii) $\lim_{x\to a} f(a)$ exists finitely

(iii) $\lim_{x\to a} f(x) = f(a)$

For the existence of $\lim_{x\to a} f(x)$ it is necessary that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist finitely and both are equal.

If any one or more of the above conditions fail to be satisfied, the function f is said to be discontinuous at the point $x = a$;

Geometrically speaking the graph of the function will exhibit a break at the point $x = a$.

Definition: Continuity of a function in an open interval (a, b) A function f is said to be continuous in (a, b). If f is continuous at each and every point of the open interval (a, b).

Definition: Continuity of a function in a closed interval $[a, b]$ A function f is said to be continuous in a closed interval $[a, b]$ if,

(i) f is continuous in the open interval (a, b) and

(ii) f is continuous at 'a' from the right i.e., $f(a)$ exists $\lim_{x\to a^+} f(x)$ exists finitely and $\lim_{x\to a+0} f(x) = f(a)$

(iii) f is continuous at 'b' from the left i.e., $f(b)$ exists; $\lim_{x\to b-0} f(x)$ exists finitely and $\lim_{x\to b^-} f(x) = f(b)$

Working Rule of Test the Continuity

To test the continuity of function $f(x)$ at $x = a$, find $f(a - 0)$, $f(a + 0)$ and $f(a)$. Then the function $f(x)$ is continuous at $x = a$; iff $f(a - 0) = f(a + 0) = f(a)$ which is the working rule to test the continuity at $x = a$.

Continuity from the left and continuity from the Right

(i) A function $f(x)$ is said to be continuous from the left at $x = a$; if $f(a - 0) = f(a)$

(ii) A function $f(x)$ is said to be continuous from the right at $x = a$ if $f(a + 0) = f(a)$.

Classification of Discontinuities

1. **Removable Discontinuity** The function $f(x)$ is said to have a removable discontinuity at a point $x = a$ if the limit of $f(x)$ at $x = a$ exists but not equal to $f(a)$.

i.e., $\lim_{x\to a} f(a) \neq f(a)$ or $f(a + 0) = f(a - 0) \neq f(a)$

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin^2 x}{(1+\cos x)x^2} = \frac{1}{2}$$

$\therefore\ f(0) = 1$

(i) Show that the function has a removable discontinuity at $x = 2$.

$$f = \begin{cases} 2x, & x < 2 \\ 2, & x = 2 \\ x^2, & x > 2 \end{cases}$$

$$\lim_{x\to 2} f(x) = 4$$

$$f(2) = 2$$

(ii) $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}; & x \neq 2 \\ 7; & x = 2 \end{cases}$

$f(2+0) = f(2-0) = 4$

$f(2) = 7$

(iii) $f(x) = \begin{cases} \dfrac{3x^3 - 2x^2 - 1}{x-1}; & \text{at } x \neq 1 \\ 2 & \text{at } x = 1 \end{cases}$

removal discontinualy.

2. **Discontinuity of the First Kind** (an ordinary discontinuity): In this case discontinuity is non removable. The point $x = a$ is called the point of discontinuity of the first kind if both $f(a-0)$ and $f(a+0)$ exists but are not equal and function $f(x)$ is said to have a discontinuity of the first kind at a point $x = a$.

(i) $f(x) = \begin{cases} 1 + x; & 0 \le x \le 2 \\ 3 - x; & 2 < x \le 3 \end{cases}$

Then discuss the continuity if $f(x)$ at $x = 2$

$\lim_{x\to 2^-} f(x) = 3; \lim_{x\to 2^+} f(x) = 1$

L.H.S. $\neq$ R.H.S.

$\Rightarrow$ Discontinuity of first kind at $x = 2$

(ii) $f(x) = \dfrac{1}{1 - e^{1/x}} : x \neq 0$

$f(0-0) = 1$

$f(0+0) = 0$

(iii) $f(x) = \lim_{x\to 1} x - [x]$

$\Rightarrow$ L.H.S. = 1; R.H.S. = 0

3. **Discontinuity of the Second Kind** A function $f(x)$ is said to have a discontinuity of the second kind at a point $x = a$. If the limits of the function on the left as well as on the right do not exist at $x = a$.

i.e., neither $\lim_{x\to a+0} f(x)$ nor $\lim_{x\to a-0} f(x)$ exists.

Example: $f(x) = \dfrac{1}{x}\sin\dfrac{1}{x}; f(x) = \dfrac{1}{x-a}$

$f(x) = \sin\left(\dfrac{1}{x}\right); f(x) = \cos\left(\dfrac{1}{x}\right)$

4. **Mixed Discontinuity**

A function $f(x)$ is said to have a mixed discontinuity at a point $x = a$ if one of the limits $\lim_{x\to a+0} f(x)$ and $\lim_{x\to a-0} f(x)$ exists and the other does not.

Example: $f(x) = \begin{cases} x^2; & x \le 0 \\ \cos\left(\dfrac{1}{x}\right); & x > 0 \end{cases}$

$f(0-h) = h^2 = 0$

$f(0+h) = \lim_{h\to 0} \cos\left(\dfrac{1}{h}\right)$ does not exists.

5. **Infinite Discontinuity** A function $f(x)$ is said to have an infinite discontinuity at a point $x = a$; If one or both of the limits is infinite.

$$f(x) = \begin{cases} \dfrac{1}{x-3}, & x \neq 3 \\ 1; & x = 3 \end{cases}$$

$f(3+h) = 0, f(3-h) = -\infty$

6. **Jump of a Function at a Point** If $f(a+0)$ and $f(a-0)$ both exists, then their non negative difference $|f(a+0) - f(a-0)|$ is called the jump in the function at $x = a$. Also a function $f(x)$ having a finite number of jumps in given interval is called sectionally or piecewise—continuous function.

[redacted]

TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Test the continuity of the function $f(x) = \dfrac{x(x-3)}{x-1}$ at $x = 3$.

2. If $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$ Is $f(x)$ continuous at $x = 0$. ***[MP-2001]***

3. If $f(x)=\begin{cases}\dfrac{x^2-1}{x+1}, & \text{for } x\neq -1\\ -2, & \text{for } x=-1\end{cases}$ Examine the continuity of the functions at $x=-1$.

[MP-2008]

4. A function $f(x)$ is defined as follows

$$f(x)=\begin{cases}\dfrac{5}{2}-x, & \text{for } x<2\\ 1, & \text{for } x=2\\ x-\dfrac{3}{2}, & \text{for } x>2\end{cases}$$

Prove that $f(x)$ is discontinuous at $x=2$.

5. Test the continuity/discontinuity of the function at the point $x=0$.

$$f(x)=\begin{cases}\dfrac{e^{1/x}}{1+e^{1/x}}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$$

6. If $f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & \text{when } x\neq 0\\ 2, & \text{when } x=0\end{cases}$

Then prove that $f(x)$ is continuous at $x=0$.

[MP-98; CBSE-92(C); PSB-96S, 2000S; AISSE-91]

7. Test the continuity of the function $f(x)$ at the origin $f(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 1, & x=0\end{cases}$

[CBSE-95 C; HSB-92]

8. Examine the function $f(t)$ given by

$$f(t)=\begin{cases}\dfrac{\cos t}{\frac{\pi}{2}-t}, & t\neq\dfrac{\pi}{2}\\ 1, & t=\dfrac{\pi}{2}\end{cases}$$ for continuity at $t=\dfrac{\pi}{2}$

[NCERT Book; PSB-90]

9. Find the value of the constant λ so that the function given below is continuous at $x=-1$

$$f(x)=\begin{cases}\dfrac{x^2-2x-3}{x+1}, & x\neq -1\\ \lambda, & x=-1\end{cases}$$

[CBSE-93(C); PSB-2000]

10. Discuss the continuity of the function at the indicated point(s).

$$f(x)=\begin{cases}\dfrac{|x^2-1|}{x-1}, & \text{for } x\neq 1\\ 2, & \text{for } x=1\end{cases} \text{ at } x=1.$$

[CBSE-1986]

11. Prove that $f(x)=\begin{cases}\dfrac{x-|x|}{x}, & x\neq 0\\ 2, & x=0\end{cases}$ is discontinuous at $x=0$.

[PSB-95 C, 2000 C, 2001]

12. Discuss the continuity of the function $f(x)$ at $x=2$ where $f(x)=\begin{cases}2-x, & x<0\\ 2+x, & x\geq 0\end{cases}$

[AISSE-1994]

13. Show that the function $\begin{cases}x^2\sin\dfrac{1}{x}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$

is continuous at $x=0$. ***[CBSE-1992]***

14. Test the continuity of the function $|x|$ at $x=0$.

[MP-99]

15. Prove that a function which is defined as follows, is continuous at $x=0$,

$$f(x)=\begin{cases}x\sin\dfrac{1}{x}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$$

[CBSE-92]

16. Test the continuity of the function

$$f(x)=\begin{cases}\dfrac{\sin ax}{\sin bx}, & \text{when } x\neq 0\\ a/b, & \text{when } x=0\end{cases} \text{ at } x=0.$$

[MP-2001]

17. Test the continuity of the function

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ 2, & x=1\end{cases} \text{ at } x=1.$$

[MP-98; CBSE-92; HSB-90; PSB-96; Kerala-93 (C)]

18. Test the continuity of the function

$$f(x)=\begin{cases}\dfrac{\sin^{-1}x}{x}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$$

[MP-1998]

19. Determine the value of the constant k so that the function $f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 1}, & \text{if } x \neq 1 \\ k, & \text{if } x = 1 \end{cases}$ is continuous at $x = 1$. **[CBSE-92 (C)]**

20. Find the value of a so that the function $f(x) = \begin{cases} ax + 5, & \text{if } x \leq 2 \\ x - 1, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$. **[CBSE-2002]**

21. If $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & \text{when } x \neq 5 \\ k, & \text{when } x = 5 \end{cases}$ is continuous at $x = 5$, find the value of k. **[CBSE-2007]**

22. If $f(x) = \dfrac{2x + 3\sin x}{3x + 2\sin x}, x \neq 0$ is continuous at $x = 0$ then find $f(0)$. **[GOA-1996]**

ANSWERS

1. continuous
2. not continuous
3. continuous
5. discontinuous
7. discontinuous
8. continuous
9. $\lambda = -4$.
10. discontinuous.
12. continuous at 2.
13. continuous
14. continuous
16. continuous
17. continuous.
18. continuous
19. $k = -1$
20. $a = -2$
22. 1

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $f(x) = \dfrac{1}{2}x - 1$, then, on the interval $[0, \pi]$

(a) $\tan[f(x)]$ and $1/f(x)$ are both continuous.
(b) $\tan[f(x)]$ and $1/f(x)$ are both discontinuous.
(c) $\tan[f(x)]$ and $f^{-1}(x)$ are both continuous
(d) $\tan[f(x)]$ is discontinuous but $1/f(x)$ is not.

Solution

(b) First note that $[x]$ means the greatest integer not exceeding x, keeping this in mind we find $[f(x)] = \left[\dfrac{1}{2}x - 1\right] = -1$, when $0 \leq x < 2$

$= 0$, when $2 \leq x \leq \pi$

$\therefore \quad \tan[f(x)] = \tan(-1) = -\tan 1, 0 \leq x < 2$

$= \tan 0 = 0, 2 \leq x \leq \pi$

The function $\tan[f(x)]$ is clearly discontinuous at $x = 2$.

Also the function $\dfrac{1}{f(x)} = \dfrac{1}{\left(\dfrac{1}{2}x - 1\right)}$ is discontinuous at $x = 2$.

These two functions are continuous at all other points in the interval $[0, \pi]$.

The function $f^{-1}(x)$ is defined by $f^{-1}(x) = y$

$\Rightarrow \quad f(y) = x$

$\Rightarrow \quad \dfrac{1}{2}y - 1 = x \Rightarrow y = 2x + 1$

Thus $f^{-1}(x) = 2x + 1$, which is continuous on $[0, \pi]$. Hence (b) is the only correct answer.

2. If $f(x) = \begin{cases} |x - 3|, & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$, then $f(x)$ is **[IIT-88]**

(a) continuous at $x = 1$ but not at $x = 3$
(b) continuous at $x = 3$ but not at $x = 1$

(c) continuous at $x = 1$ and $x = 3$
(d) discontinuous at $x = 1$ and $x = 3$

Solution

(c) $f(1) = 2, f(1-0)$

$$= \frac{1}{4} - \frac{3}{2} + \frac{13}{4} = 2, f(1+0) = 2$$

$\therefore f(x)$, is continuous at $x = 1$.

$f(3) = 0,\ f(3-0) = \lim_{h\to 0} |3 - h - 3| = 0$

$f(3+0) = \lim_{h\to 0} |3 + h - 3| = 0$

$\therefore$ $f(x)$, is also continuous at $x = 3$.

3. The function $f(x) = \dfrac{(27-2x)^{1/3} - 3}{9 - 3(243+5x)^{1/5}} (x \neq 0)$ is continuous function, then $f(0)$ is equal to **[DCE-98]**

(a) 2 (b) 4
(c) 6 (d) 2/3

Solution

(a) $f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{-\frac{2}{3}(27-2x)^{-2/3}}{-3(243+5x)^{-4/5}} = 2$

4. If $f(x) = [x],\ g(x) = \begin{cases} 0, & x \in Z \\ x^2, & x \in (R - Z) \end{cases}$, then **[Roorkee (Screening)-99]**

(a) $\lim_{x\to 1} g(x)$ exists but $g(x)$ is discontinuous at $x = 1$
(b) $\lim_{x\to 1} f(x)$ does not exist and $f(x)$ is not continuous at $x = 1$
(c) gof is continuous function
(d) all above

Solution

(d) $\because$ $g(1) = 0,\ g(1-0) = g(1+0) = 1^2 = 1 \neq g(1)$

$\Rightarrow$ $\lim_{x\to 1} g(x)$ exist but $g(x)$ is not continuous at $x = 1$

Also $f(1) = [1] = 1, f(1-0) = [1-h] = 0,$
$f(1+0) = [1+h] = 1 \lim_{x\to 1} f(1-0)$

$\Rightarrow$ $\lim_{x\to 1} f(x)$ does not exist and so $f(x)$ is discontinuous at $x = 1$

Further (gof) $(x) = g[f(x)] = g[x] = 0\ \forall\ x \in R$, which is continuous $\forall\ x \in R$.

5. If $f(x) = \begin{cases} \dfrac{(4^x - 1)^3}{\sin\left(\dfrac{x}{a}\right)\log\left(1 + \dfrac{x^2}{3}\right)}, & x \neq 0 \\ (\log 4)^3, & x = 0 \end{cases}$ is continuous at $x = 0$, then a is equal to **[JEE (Orissa)-2004]**

(a) 3 (b) 1/3
(c) 1 (d) 2

Solution

(b) $\because$ $f(x)$ is continuous at $x = 0$

$\because$ $\lim_{x\to 0} f(x) = f(0) = (\log 4)^3$

$$\Rightarrow \lim_{x\to 0} \frac{\left(x\log 4 + \dfrac{x^2(\log 4)^2}{2!} + \ldots\ldots\right)^3}{\left(\dfrac{x}{a} - \dfrac{x^3}{6a^3} + \ldots\right)\left(\dfrac{x^2}{3} - \dfrac{x^4}{18} + \ldots\right)}$$

$= (\log 4)^3$

$$\Rightarrow \frac{(\log 4)^3}{(1/3a)} = (\log 4)^3 \Rightarrow a = 1/3$$

6. If $f(x) = px^2 - q,\ x \in (0, 1) = x + 1,\ x \in (1, 2]$ and $f(1) = 2$, then the value of the pair (p, q) for which $f(x)$ cannot be continuous at $x = 1$ is

(a) (2, 0) (b) (1, – 1)
(c) (4, 2) (d) (1, 1)

Solution

(d) $f(x)$ is continuous at $x = 1$ if

$\lim_{h\to 0} f(1+h) = \lim_{h\to 0} f(1-h) = f(1) = 2$

$\lim_{h\to 0} f(1+h) = \lim_{h\to 0}\{1 + h + 1\} = 2$

$\lim_{h\to 0} f(1-h) = \lim_{h\to 0}\{p(1-h)^2 - q\}$
$= p - q$

$\therefore$ $f(x)$ is not continuous at $x = 1$ if $p - q \neq 2$.

7. Function $f(x) = \begin{cases} \dfrac{e^{1/(x-1)} - 2}{e^{1/(x-1)} + 2}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ **[Roorkee-2001]**

(a) is continuous at $x = 1$
(b) has a discontinuity of removable type
(c) has a discontiunity of the 1st kind
(d) has a discontinuity of the 2nd kind

Solution

(c) $\text{L.H.L} = \lim_{h\to 1}\dfrac{e^{1/(1-h-1)}-2}{e^{-1/(1-h-1)}+2}$

$= \lim_{h\to 1}\dfrac{e^{-1/h}-2}{e^{1/h}+2} = \dfrac{1-2e}{1+2e}$

$\text{R.H.L.} = \lim_{h\to 1}\dfrac{e^{1/(1+h-1)}-2}{e^{1/(1+h-1)}+2}$

$= \lim_{h\to 1}\dfrac{e^{1/h}-2}{e^{1/h}+2} = \dfrac{e-2}{e+2}$

L.H.L. $\neq$ R.H.L.

Hence limit exists but are not equal.
Thus function is not continuous at $x = 1$. i.e., function has a discontinuity of first kind.

8. If the function $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2} & \text{for } x \neq 0 \\ k & \text{for } x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is

[Karnataka CET-2007]

(a) 1 (b) 0
(c) 1/2 (d) −1

Solution

(c) Since $f(x)$ is continuous at $x = 0$. Therefore,

$\lim_{x\to 0} f(x) = f(0) \Rightarrow \lim_{x\to 0}\dfrac{1-\cos x}{x^2} = k$

$\lim_{x\to 0}\dfrac{-(-\sin x)}{2x} = k \Rightarrow \lim_{x\to 0}\dfrac{\sin x}{2x} = k$

$\dfrac{1}{2}\lim_{x\to 0}\dfrac{\sin x}{x} = k \Rightarrow \dfrac{1}{2}.1 = k \Rightarrow k = \dfrac{1}{2}$

9. Let $f: R \to R$ be a continuous function defined by

$f(x) = \dfrac{1}{e^x + 2e^{-x}}.$

Assertion: $f(c) = \dfrac{1}{3}$, for some $c \in R$

Reason: $0 < f(x) \leq \dfrac{1}{2\sqrt{2}}$, for all $x \in R$

[AIEEE-2010]

(a) Assertion is true, reason is true and reason is a correct explanation for assertion
(b) Assertion is true, reason is true and reason is NOT a correct explanation for assertion
(c) Assertion is true and reason is false
(d) Assertion is false and reason is true

Solution

(c) Step-1: Maximum value of

$e^x + 2e^{-x} \geq \sqrt[2]{e^x + 2e^{-x}} = 2\sqrt{2}, \forall x$

i.e. Minimum value of $e^x + 2e^{-x}$ is $2\sqrt{2}$.

Therefore, maximum value of $\dfrac{1}{e^x + 2e^{-x}}$ is $\dfrac{1}{2\sqrt{2}}$

Also e^x and e^{-x} are never negative

$\therefore \dfrac{1}{e^x + 2e^{-x}}$ is always greater than zero.

Step-2: Clearly, $f(0) = \dfrac{1}{e^0 + 2e^0} = \dfrac{1}{1+2} = \dfrac{1}{3}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If both the left as well as right hand limits exists as $x \to a$ and are not equal, then the function is said to have

(a) A discontinuity of the 1st kind at $x = a$
(b) A discontinuity of the 2nd kind at $x = a$
(c) Mixed continuity at $x = a$
(d) Removable discontinuity

2. Function $f(x) = \dfrac{1-\cos 4x}{8x^2}$, where $x \neq 0$ and $f(x) = k$ where $x = 0$ is a continuous function at $x = 0$ then the value of k will be

[AMU-2005]

(a) $k = 0$ (b) $k = 1$
(c) $k = -1$ (d) None of these

3. For the function $f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1} & , x\neq 0\\ 0 & , x=0\end{cases}$, which of the following is correct

[MPPET-2004]

(a) $\lim\limits_{x\to 0} f(x)$ does not exist

(b) $f(x)$ is continuous at $x=0$

(c) $\lim\limits_{x\to 0} f(x)=1$

(d) $\lim\limits_{x\to 0} f(x)$ exists but $f(x)$ is not continuous at $x=0$

4. The value of $f(0)$, so that the function $f(x)=\dfrac{1-\cos(1-\cos x)}{x^4}$ is continuous every where is

(a) 1/8 (b) 1/2

(c) 1/4 (d) None of these

5. The function $f(x)=\left(\dfrac{\pi}{2}-x\right)\tan x$ is not continuous at x is equal to

(a) π (b) 0

(c) $\pi/2$ (d) None of these

6. The function $f(x)=\dfrac{1+\sin x-\cos x}{1-\sin x-\cos x}$ is not defined at $x=0$. The value of $f(0)$ so that $f(x)$ is continuous at $x=0$ is

(a) 1 (b) −1

(c) 0 (d) None of these

7. In order that the function $f(x)=(x+1)^{\cot x}$ is continuous at $x=0$, then $f(0)$ must be defined as

[NDA-2007]

(a) $f(0)=0$ (b) $f(0)=e$

(c) $f(0)=e^{-1}$ (d) None of these

8. The number of points at which the function $f(x)=\dfrac{1}{\log|x|}$ is discontinuous is

(a) 1 (b) 2

(c) 3 (d) 4

9. If $f(x)=\dfrac{\tan\left(\dfrac{\pi}{4}-x\right)}{\cot 2x}$ at $x\neq\dfrac{\pi}{4}$. Then the value which should be assigned to it at $x=\dfrac{\pi}{4}$ so that the function is continuous at the point is

(a) 1/2 (b) 1

(c) 2 (d) None of these

10. The function defined by

$$f(x)=\frac{x-1}{1+e^{\frac{1}{x}-1}}\ x\neq 1;\ f(1)=0$$

(a) Is continuous for $x=1$

(b) Has a Discontinuity of the 1st kind at $x=1$

(c) Has mixed discontinuity at $x=1$

(d) Has removable discontinuity at $x=1$.

11. The function $g(x)=\begin{cases} x^2+5, & x<2\\ 10, & x=2,\\ (1+x^3)/(1-x), & x>2\end{cases}$

then mark one incorrect statement

(a) $\lim\limits_{x\to 2^+}(g(x))=-9$

(b) $\lim\limits_{x\to 2^-} g(x)=9$

(c) $\lim\limits_{x\to 2^+} g(x)=3$

(d) g is not continuous at $x=2$

12. If $f(x)=\dfrac{2-\sqrt{x+4}}{\sin 2x}(x\neq 0)$ is continuous function at $x=0$, then $f(0)$ equals

(a) 1/4 (b) − 1/4

(c) 1/8 (d) − 1/8

13. If $f(x)=[x]$ and $g[x]=\begin{cases}0, & x\in Z\\ x^2 & x\in R-Z\end{cases}$, then

(a) $\underset{x\to 1}{Lt}\, g(x)$ exist but $g(x)$ is not continuous at $x=1$

(b) $\underset{x\to 1}{Lt}\, f(x)$ does not exist and $f(x)$ is not continuous at $x=1$

(c) $g(x)$ is continuous for all x

(d) $f(x)$ is continuous for all x

14. The function $y=\dfrac{|3x-4|}{3x-4}$ is discontinuous when x is equal to

(a) 2/3 (b) 4/3

(c) −2/3 (d) −4/3

15. The function $f(x)=\dfrac{|x|}{x^2+2x}, x\neq 0$ and $f(0)$ is not continuous at $x=0$, because

(a) $\underset{x\to 0}{Lt}\, f(x) \neq f(0)$

(b) $\underset{x\to 0^+}{Lt}\, f(x)$ does not exist

(c) $\underset{x\to 0^-}{Lt}\, f(x)$ does not exist

(d) $\underset{x\to 0}{Lt}\, f(x)$ does not exist

16. The value of $f(0)$ so that $f(x) = \dfrac{(4^x - 1)^3}{\sin(x/4)\log(1+(x^2/3))}$ is continuous every where is
(a) $(3 \log 4)^3$
(b) $(4 \log 4)^3$
(c) $12 (\log 4)^3$
(d) $(15 \log 4)^3$

17. $f(x) = x^p \sin\dfrac{1}{x}$, when $x \neq 0$, $= 0$, when $x = 0$, is continuous at $x = 0$ if
(a) $p < -1$
(b) $p = 0$
(c) $p > 0$
(d) $-1 > p < 0$

18. The function $f : IR\backslash\{0\} \to IR$ given by $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining $f(0)$ as **[AIEEE-2007]**
(a) 2 (b) –1
(c) 0 (d) 1

19. For the function $f(x) = \begin{cases} \dfrac{x^3 - a^3}{x - a}, & x \neq a \\ b, & x = a \end{cases}$ if $f(x)$ is continuous at $x = a$, then b is equal to **[MP PET-2008]**
(a) a^2 (b) $2a^2$
(c) $3a^2$ (d) $4a^2$

20. If $f(x) = \dfrac{2x - 3\sin x}{3x + 4\tan x}$, $x \neq 0$, is continuous at $x = 0$, then $f(0)$ is equal to **[Kerala PET-2007]**
(a) 3 (b) 2/7
(c) –3/7 (d) –1/7

21. If $f(x) = \begin{cases} \dfrac{\sqrt{4+ax} - \sqrt{4-ax}}{x}, & -1 \le x < 0 \\ \dfrac{3x+2}{x-8}, & 0 \le x \le 1 \end{cases}$
is continuous in the interval [–1, 1], then the value of a is equal to **[Kerala PET-2007]**
(a) 1 (b) –1
(c) 1/2 (d) –1/2

22. Let $f(x) = \begin{cases} 3x - 4, & 0 \le x \le 2 \\ 2x + l, & 2 < x \le 9 \end{cases}$. If f is continuous at $x = 2$, then what is the value of l? **[NDA-2008]**
(a) 0 (b) 2
(c) –2 (d) –1

SOLUTIONS

1. (a) By definition

2. (b) For continuous function

$$k = \lim_{x\to 0}\frac{1-\cos 4x}{8x^2} = \lim_{x\to 0}\frac{2\sin^2 2x}{8x^2}$$

$$= \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1 \quad \therefore k = 1$$

3. (a) Here R.H.L. $= \lim\limits_{h\to 0}\dfrac{e^{1/h}-1}{e^{1/h}+1}$

$$= \lim_{h\to 0}\frac{1-e^{-1/h}}{1+e^{-1/h}} = \frac{1-0}{1+0} = 1$$

$$\text{L.H.L.} = \lim_{h\to 0}\frac{e^{-1/h}-1}{e^{-1/h}+1} = \frac{0-1}{0+1} = -1$$

$\therefore$ L.H.L. $\neq$ R.H.L., $\lim\limits_{x\to 0} f(x)$ doesn't exist.

4. (a) For continuous function

$$f(0) = \lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4}$$

$$= \lim_{x\to 0}\frac{1-\cos\left(2\sin^2\frac{x}{2}\right)}{x^4} = \lim_{x\to 0}\frac{2\sin^2\left(2\sin^2\frac{x}{2}\right)}{x^4}$$

$$= \lim_{x\to 0} \frac{2\left(\frac{2\sin^2\frac{x}{2}}{2}\right)^2}{x^4}$$

$$= \lim_{x\to 0} \frac{2\left(\sin\frac{x}{2}\right)^4}{16.(x/2)^4} = \frac{1}{8}$$

5. (c) $f(x) = \left(\frac{\pi}{2} - x\right)\tan x$ at $x = \frac{\pi}{2}$, $\tan x$ is not defined,

$\therefore f(x)$ is not defined at $x = \frac{\pi}{2}$

6. (b) $f(x) = \lim_{x\to 0} \frac{(1-\cos x)+\sin x}{1-\cos x - \sin x}$

$$= \lim_{x\to 0} \frac{2\sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \lim_{x\to 0} \frac{2\sin\frac{x}{2}\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)}{2\sin\frac{x}{2}\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)}$$

$$= \lim_{x\to 0} \frac{\sin\frac{x}{2} + \cos\frac{x}{2}}{\sin\frac{x}{2} - \cos\frac{x}{2}}$$

$$\therefore f(0) = \lim_{x\to 0} \frac{0+1}{0-1} = -1$$

7. (b) $f(0) = \lim_{x\to 0}(x+1)^{\cot x}$

$$= e^{\lim_{x\to 0} x\times\cot x} = e^{\lim_{x\to 0}\frac{x}{\tan x}} = e^1$$

8. (c) Point of discontinuity are those where $f(x)$ is not defined i.e., at $|x| = 0, 1$
$x = 0, 1, -1$
Number of points are 3.

9. (a) For contains function

$$f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}$$

put $= \frac{\pi}{4} - x = t$ or $x = \frac{\pi}{4} - t$

as $x \to \frac{\pi}{4}$; $t \to 0$

$$\lim_{t\to 0} \frac{\tan t}{\cot^2\left(\frac{\pi}{4} - t\right)} = \lim_{t\to 0} \frac{\tan t}{\tan 2t}$$

$$= \lim_{t\to 0} \frac{\tan t}{t} \times \frac{2t}{\tan 2t}$$

$$= \frac{1}{2}$$

10. (a) R.H.L. $= f(1+h) = \lim_{h\to 0} \frac{1+h-1}{1+e^{\frac{1}{1+h-1}}}$

$$= \lim_{h\to 0} \frac{h}{1+e^{\frac{1}{h}}} = 0$$

L.H.L. $= f(1-h) = \lim_{h\to 0} \frac{1-h-1}{1+e^{\frac{1}{1-h-1}}}$

$$= \lim_{h\to 0} \frac{-h}{1+e^{-1/h}} = 0$$

$\because f(1-h) = f(1+h) = f(1)$

$\therefore$ function is continuous.

11. (c) $g(x) = \begin{cases} x^2+5; & x<2 \\ 10; & x=2 \\ \frac{1+x^3}{1-x}; & x>2 \end{cases}$

$$\lim_{x\to 2^+} g(x) = \frac{1+2^3}{1-2} = -9$$

$$\lim_{x\to 2^-} g(x) = 2^2 + 5 = 9$$

Here $g(2+) \neq g(2-)$ function is discontinuous, (c) is incorrect.

12. (d) $f(0) = \lim_{x\to 0} \frac{2-\sqrt{x+4}}{\sin^2 x}$

$$= \lim_{x\to 0} \frac{2^2 - \left(\sqrt{x+4}\right)^2}{2+\sqrt{x+4}} \times \frac{1}{\sin 2x}$$

(On rationalising)

$$= \lim_{x\to 0} \frac{-x}{2+\sqrt{x+4}} \cdot \frac{1}{\sin 2x}$$

$$= \lim_{x\to 0} \frac{-1}{2+\sqrt{x+4}} \times \frac{2x}{\sin 2x} \cdot \frac{1}{2}$$

$$= \frac{-1}{8}$$

13. (a) $f(x)=[x]\quad g(x)=\begin{cases}0 & x\in Z\\ x^2 & x\in R-Z\end{cases}$

$f(x)$ is discontinuous at integer points lim of $f(x)$ doesn't exist of integral points

Now $g(x)=\begin{cases}0 & x\in Z\\ x^2 & x\notin Z\end{cases}$

$=\lim\limits_{x\to 1} g(x)=1^2=1$ ($\because$ $1+h$ or $1-h$ are non integer)

but $\lim\limits_{x\to 1} g(x)=1\neq g(1)\qquad (\because g(1)=0)$

14. (b) At $x=\dfrac{4}{3}$

$$f(1+)=\lim_{h\to 0}\frac{3h}{3h}=1$$

$$f(1-)=\lim_{h\to 0}\frac{3h}{-3h}=-1$$

$\therefore$ function is discontinuous at $x=\dfrac{4}{3}$

15. (d) $f(0+)=\lim\limits_{h\to 0}\dfrac{|h|}{h^2+2h}=\dfrac{1}{2}$

$$f(0-)=\lim_{h\to 0}\frac{|-h|}{h^2-2h}=\frac{-1}{2}$$

$\therefore$ limit doesn't exist.

16. (c) $f(0)=\lim\limits_{x\to 0}\dfrac{\left(\dfrac{4x-1}{x}\right)^3}{\dfrac{1}{4}\cdot\dfrac{\sin\frac{x}{4}}{x/4}\log\dfrac{\left(1+\frac{x^2}{3}\right)}{x^2/3}\cdot\dfrac{1}{3}}$

$=(\log 4)^3.4.3\ =12(\log 4)^3$

17. (c) $f(x)=x^p\sin\left(\dfrac{1}{x}\right)$ will be defined if $p>0$

18. (d) $f(0)=\lim\limits_{x\to 0}\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}$

$=\lim\limits_{x\to 0}\dfrac{e^{2x}-1-2x}{x(e^{2x}-1)}\left(\dfrac{0}{0}\right)$

Applying L-Hospital's rule

$$\lim_{x\to 0}\frac{2e^{2x}-2}{(e^{2x}-1)+2xe^{2x}}\left(\frac{0}{0}\right)$$

again using L-Hospital's rule

$$=\lim_{x\to 0}\frac{4e^{2x}}{4e^{2x}+4xe^{2x}}=1$$

$\therefore$ $f(x)$ is continuous at $x=0$, then

$\lim\limits_{x\to 0} f(x)=f(0)$

$\Rightarrow$ $f(0)=1$

19. (c) $f(x)=\dfrac{x^3-a^3}{x-a}, f(a)=b$

For continuity of $f(x)$

$\lim\limits_{x\to a} f(x)=b$

$$\lim_{x\to a}\frac{x^3-a^3}{x-a}=3a^2=b$$

20. (d) $f(0)=\lim\limits_{x\to 0}\dfrac{2x-3\sin x}{3x+4\tan x}$

$$=\lim_{x\to 0}\frac{2-3\dfrac{\sin x}{x}}{3+4\dfrac{\tan x}{x}}=\frac{2-3}{3+4}=\frac{-1}{7}$$

NOTE

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\lim_{x\to 0}\frac{\tan x}{x}=1$$

21. (c) For continuity

$f(0+)=f(0-)$

$$\lim_{h\to 0}\frac{\sqrt{4-ah}-\sqrt{4+ah}}{-h}=\lim_{h\to 0}\frac{3h+2}{h-8}$$

$$=\lim_{h\to 0}\frac{4-ah-(4+ah)}{\left(\sqrt{4-ah}+\sqrt{4+ah}\right)h}=\frac{2}{-8}$$

$$=\lim_{h\to 0}\frac{-2ah}{\left(\sqrt{4-ah}+\sqrt{4+ah}\right)h}=-\frac{1}{4}$$

$$=\frac{-2a}{4}=\frac{-1}{4}\Rightarrow a=\frac{1}{2}$$

22. (c) $f(x)=\begin{cases}3x-4; & 0\le x\le 2\\ 2x+l; & 2<x\le 9\end{cases}$

$f(2-)=f(2+)$

$3\times 2-4=2\times 2+l\Rightarrow l=-2$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The function f defined by $f(x) = (1 + 3x)^{1/x}$, $x \neq 0$ is continuous for $x = 0$ when $f(0)$ is defined as
(a) 1 (b) e
(c) e^2 (d) e^3

2. If $f(x) = \dfrac{\log(1+ax) - \log(1-bx)}{x}$ is continuous at $x = 0$, then $f(0)$ is equal to
(a) $a - b$ (b) $a + b$
(c) ab (d) $\log a - \log b$

3. If $f(x) = \dfrac{3x + \tan^2 x}{x}$ is continuous at $x = 0$, then $f(0)$ is equal to
(a) 1 (b) 2
(c) 4 (d) 3

4. Let f be defined as
$$f(x) = \begin{cases} \sin\left(x + \dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
. The function f is continuous except
(a) $x = 0$ (b) $x = 1/2$
(c) $x = 1$ (d) Always

5. The discontinuity of the function
$$f(x) = \begin{cases} \dfrac{3x + 4\tan x}{x}, & x \neq 0 \\ 3, & x = 0 \end{cases}$$
at $x = 0$ is of
(a) 1st kind (b) 2nd kind
(c) Removable type (d) Mixed kind

6. A function defined by
$$f(x) = \begin{cases} 1 + x & (x \leq 2) \\ 5 - x & (x \geq 2) \end{cases}$$
(a) Is continuous at $x = 2$
(b) has a discontinuity of the 1st kind at $x = 2$
(c) has a discontinuity of the 2nd kind at $x = 2$
(d) has a removable discontinuity at $x = 2$

7. The function defined by $f(x) = (1 + 3x)^{1/x}$, $x \neq 0$ $f(0) = e^3$
(a) has a discontinuity of the 1st kind at $x = 0$
(b) has a discontinuity of the 2nd at $x = 0$
(c) has removable discontinuity at $x = 0$
(d) Is continuous for $x = 0$

8. The function $f(x) = \begin{cases} \dfrac{\tan 2x}{3x}, & x \neq 0 \\ 2/3, & x = 0 \end{cases}$ is
(a) continuous at $x = 0$
(b) discontinuous at $x = 0$
(c) discontinuous at $x = -\pi$
(d) discontinuous at $x = \pi/2$

9. The no. of points of discontinuity of $f(x) = \dfrac{4 - x^2}{4x - x^3}$ is
(a) 2 (b) 6
(c) 0 (d) 3

10. The function $f(x) = \dfrac{\cos x - \sin x}{\cos 2x}$ is not defined at $x = \pi/4$; the value of $f(\pi/4)$ so that $f(x)$ is continuous every where is
(a) 1 (b) -1
(c) $\sqrt{2}$ (d) $1/\sqrt{2}$

11. The function $f(x)$ defined by
$$f(x) = x\left[1 + \frac{1}{3}\sin(\log x^2)\right], x \neq 0 \ = x, x = 0$$
(a) Is continuous at $x \equiv 0$
(b) has discontinuity of 1st kind at $x = 0$
(c) has discontinuity of 2nd kind at $x = 0$
(d) has removable discontinuity at $x = 0$

12. The function defined by $f(x) = \cos x$, $x \geq 0$; $= -\cos x$, $x < 0$
(a) has discontinuity of the 1st kind at $x = 0$
(b) has a discontinuity of the 2nd kind at $x = 0$
(c) has a removable discontinuity at $x = 0$
(d) Is continuous at $x = 0$

13. Which of the following functions is not continuous for $x = 0$?
(a) $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$
(b) $f(x) = \begin{cases} \dfrac{\sin^2 kx}{x^2}, & x \neq 0 \\ k^2, & x = 0 \end{cases}$
(c) $f(x) = \begin{cases} \dfrac{e^{x^2}}{e^{1/x^2} - 1}, & x \neq 0 \\ 1, & x = 0 \end{cases}$
(d) None of these

14. Which of the following functions is continuous at $x = 0$?

(a) $f(x) = \dfrac{\sin 2x}{x}, x \neq 1; f(0) = 1$

(b) $f(x) = (1 + x)^{1/x}, x \neq 0, f(0) = 1$

(c) $f(x) = \dfrac{\sin x}{x}, x \neq 0, f(0) = 1$

(d) $f(x) = e^{-1/x^2}, x \neq 0, f(0) = 1$

15. Which of the following functions is discontinuous at $x = 0$?

(a) $f(x) = \dfrac{e^{-1/x}}{1 + e^{1/x}}, x \neq 0, f(0) = 1$

(b) $f(x) = \dfrac{e^{1/x^2}}{e^{1/x^2} - 1}, x \neq 0, f(0) = 1$

(c) $f(x) = (1 + 2x)^{1/x}, x \neq 0, f(0) = e^2$

(d) $f(x) = \dfrac{\tan 2x}{3x}, x \neq 0, f(0) = \dfrac{2}{3}$

LECTURE

2

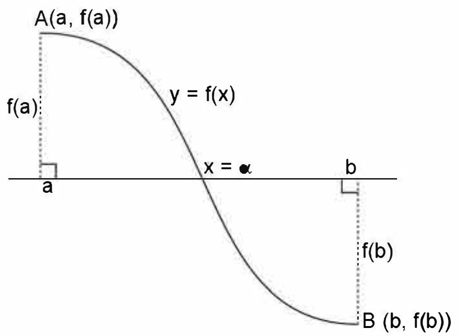
Continuous Functions

BASIC CONCEPTS

PROPERTIES OF CONTINUOUS FUNCTIONS

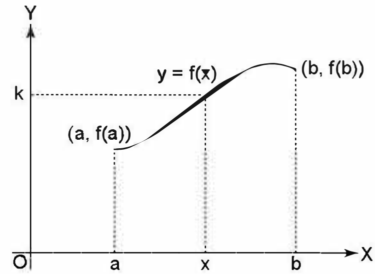
(i) In a closed interval $[a, b]$ a continuous function is bounded in that interval and attains its bounds at least once in the interval.

(ii) If a function $f(x)$ is continuous in the closed interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs then the function $f(x) = 0$; for at least one value of x such that $a < x < b$.



(iii) If a function $f(x)$ is continuous at $x = c$ and $f(c) \neq 0$, then a + ve number h can always be found such that the function $f(x)$ has the same sign as $f(c)$ $\forall$ x in $(c - h, c + h)$

(iv) If k is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation.
$f(x) = k$ in the open interval (a, b).
This is called intermediate value property and it states that a continuous function $f(x)$ attains every value lying between $f(a)$ and $f(b)$ at least once.



ALGEBRA OF CONTINUOUS FUNCTIONS

(i) The constant function $f(x) = c$ is continuous for all values of x.

(ii) The identity function $f(x) = x$ is continuous for all values of x.

(iii) Any polynomial $P_n(x)$ of degree n ($n = 0, 1, 2, ...n$); $P_n(x) = C_0 + C_1x + C_2x^2 + .. + C_nx^n$ is continuous for all values of x.

(iv) If $f(x)$ and $g(x)$ are two continuous function at $x = a$, then:

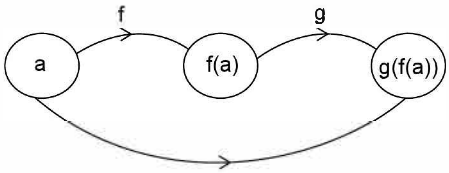
(a) $f(x) + g(x)$ is continuous at $x = a$

(b) $f(x) - g(x)$ is continuous at $x = a$

(c) $f(x)\,g(x)$ is continuous at $x = a$

(d) $\dfrac{f(x)}{g(x)}$ is continuous at $x = a$ if $g(a) \neq 0$

(e) **Continuity of Composite function:** If the function f is continuous at $x = a$ and g is continuous at $x = f(a)$ then composite function $f\{f(x)\}$ is continuous at $x = a$.



continuous (gof)

(f) If $f(x)$ is continuous then $|f(x)|$ is continuous but converse is not true.

POINTS TO REMEMBER

Remembering method

$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x) \cdot g(x)$	$f(x)/g(x)$ provided $g(x) \neq 0$
continuous	continuous	continuous	continuous	continuous
continuous	discontinuous	discontinuous	may be continuous or discontinuous	may be continuous or discontinuous
discontinuous	discontinuous	may be continuous or discontinuous	may be continuous or discontinuous	may be continuous or discontinuous

NOTE

Sum, difference, product, quotient and composite of two continuous functions is continuous

Example of discontinuous functions:

1. A function is said to be a discontinuous function if it is discontinuous at atleast one point in its domain. Following are examples of some discontinuous functions.

Functions	Point of discontinuous
(i) $[x]$	every integer
(ii) $x - [x]$	every integer
(iii) $1/x$	$x = 0$
(iv) $\tan x$, $\sec x$	$x = \pm \pi/2, \pm 3\pi/2, \ldots\ldots$
(v) $\cot x$, $\operatorname{cosec} x$	$x = 0, \pm \pi, \pm 2\pi, \ldots\ldots\ldots$
(vi) $\sin 1/x$, $\cos 1/x$	$x = 0$
(vii) $e^{1/x}$	$x = 0$
(viii) $\cot hx$, $\operatorname{cosec} hx$	$x = 0$

2. $e^{2x} + \sin x$ is a continuous function because it is the sum of two continuous function e^{2x} and $\sin x$.
3. $\cos(x^2+1)$ is a continuous function because it is the composite of two continuous function $\cos x$ and $x^2 + 1$.
4. The product of one continuous and one discontinuous function may or may not be continuous.

Example 1: $f(x) = x$ is continuous and $g(x) = \sin 1/x$ is discontinuous where as their product $x \sin 1/x$ is continuous.

Example 2: $f(x) = c$ is continuous and $g(x) = \cos 1/x$ is discontinuous where as their product $c \cos 1/x$ is discontinuous.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Let $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. If $f(x)$ is continuous at $x = 0$, show that $f(x)$ is continuous at all x. ***[PSB-91]***

Solution

Since $f(x)$ is continuous at $x = 0$. Therefore

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$$

$$\Rightarrow \quad \lim_{h\to 0} f(0+(-h)) = \lim_{h\to 0} f(0+h) = f(0)$$

$$\Rightarrow \quad \lim_{h\to 0}[f(0) + f(-h)]$$

$$= \lim_{h\to 0}[f(0) + f(h)] = f(0)$$

[using: $f(x + y) = f(x) + f(y)$]

$$\Rightarrow \quad f(0) + \lim_{h\to 0} f(-h)$$

$$= \lim_{h\to 0} f(h) + f(0) = f(0)$$

$$\Rightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = 0 \quad .... \quad (1)$$

Let a be any real number. Then, $\lim_{x\to a^-} f(x)$

$$= \lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+(-h))$$

$$= \lim_{h\to 0}[f(a)+f(-h)]$$

$[\because f(x+y) = f(x)+f(y)]$

$$= f(a) + \lim_{h\to 0} f(-h) = f(a) + 0$$

[using (1)]

$= f(a)$ and, $\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$

$= \lim_{h\to 0}[f(a)+f(h)]$ $[\because f(x+y) = f(x)+f(y)]$

$= f(a) + \lim_{h\to 0} f(h) = f(a) + 0$ [using (1)]

$= f(a)$

Thus, we have $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

$\Rightarrow$ $f(x)$ is continuous at $x = a$

Since a is an arbitrary real number.

So, $f(x)$ is continuous at all $x \in R$. **Proved.**

2. If $f(x) = \begin{cases} \dfrac{[x]-1}{x-1}, & x \neq 1 \\ -1, & x = 1 \end{cases}$ then show whether $f(x)$ is continuous at $x = 1$. ***[PB-95]***

Solution

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0} \frac{[1-h]-1}{(1-h)-1} = \lim_{h\to 0} \frac{0-1}{-h} = \lim_{h\to 0} \frac{1}{h}$$

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0^+} \frac{[1+h]-1}{(1+h)-1}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} 0 = 0$$

$\therefore \quad \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

$\therefore \quad \lim_{x\to 1} f(x)$ does not exist.

Hence $f(x)$ is not continuous at $x = 1$.

Proved.

3. $f(x) = \dfrac{\sqrt{1+px}-\sqrt{1-px}}{x}, -1 \le x < 0$

$= \dfrac{2x+1}{x-2}, 0 \le x \le 1$ is continuous in the interval $[-1, 1]$ then find the value of p.

[BITS (Ranchi)-1986]

Solution

$$f(0+0) = \lim_{h\to 0} \frac{2(0+h)+1}{(0+h)-2}$$

$$= \lim_{h\to 0} \frac{2h+1}{h-2} = -\frac{1}{2}$$

$$f(0-0) = \lim_{h\to 0} \frac{\sqrt{1-ph}-\sqrt{1+ph}}{0-h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-ph}-\sqrt{1+ph}}{-h} \times \frac{\sqrt{1-ph}\times\sqrt{1+ph}}{\sqrt{1-ph}\times\sqrt{1+ph}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{1-ph})^2-(\sqrt{1+ph})^2}{-h(\sqrt{1-ph}+\sqrt{1+ph})}$$

$$= \lim_{h\to 0} \frac{1-ph-1-ph}{-h(\sqrt{1-ph}+\sqrt{1+ph})}$$

$$= \lim_{h\to 0} \frac{-2ph}{-h(\sqrt{1-ph}+\sqrt{1+ph})} = \frac{-2p}{-(1+1)} = p$$

$f(x)$ is continuous at $x = 0$ in $(1, 1)$ if

$f(0+0) = f(0-0) = f(0)$

$\Rightarrow \quad p = -1/2$

4. The function

$$f(x) = \begin{cases} \dfrac{x^2}{a} & \text{if} \quad 0 \le x < 1 \\ a & \text{if} \quad 1 \le x < \sqrt{2} \\ \dfrac{2b^2-4b}{x^2} & \text{if} \quad \sqrt{2} \le x < \infty \end{cases}$$

is continuous for $0 \le x < \infty$, then find the most suitable values of a and b.

[BIT (Ranchi)-1984]

Solution

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} \frac{x^2}{a} = \frac{1}{a};$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} (a) = a$$

Since $f(x)$ is continuous for $0 \le x < \infty$

$\therefore \quad f(x)$ is continuous at $x = 1$

$\therefore \quad \lim_{x\to 1} f(x)$ exists

$\therefore \quad \dfrac{1}{a} = a \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

Also $\lim_{x\to\sqrt{2}-} f(x) = \lim_{x\to\sqrt{2}} (a) = a$

$$\lim_{x\to\sqrt{2}+} f(x) = \lim_{x\to\sqrt{2}} \frac{2b^2-4b}{x^2} = \frac{2b^2-4b}{2}$$

$$\therefore \quad \frac{2b^2 - 4b}{2} = a$$

$[\because f(x)$ is continuous at $x = \sqrt{2}\,]$

$$\Rightarrow \quad 2b^2 - 4b = 2a \Rightarrow b^2 - 2b = a$$

when $a = 1$

$$b^2 - 2b = 1 \quad \Rightarrow b^2 - 2b - 1 = 0$$

$$\Rightarrow \quad b = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2} \text{ when } a = -1$$

$b^2 - 2b + 1 = 0$ and $(b-1)^2 = 0 \Rightarrow b = 1$

Hence $a = -1$, $b = 1$ are most suitable values.

5. Find k for which the function

$$f(x) = \begin{cases} \dfrac{1-\sin x}{(\pi - 2x)^2}, & x \neq \dfrac{\pi}{2} \\ k, & x = \dfrac{\pi}{2} \end{cases}$$ is continuous at

$x = \pi/2$. ***[CBSE (Sample paper) (4)]***

Solution

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{(\pi - 2x)^2} = \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{4\left(\dfrac{\pi}{2} - x\right)^2}$$

$$= \lim_{\theta \to 0} \frac{1 - \sin\left(\dfrac{\pi}{2} + \theta\right)}{4(-\theta)^2}$$

$$\left(\begin{array}{l} \because x \to \dfrac{\pi}{2} \Rightarrow x - \dfrac{\pi}{2} \to 0 \Rightarrow \theta \to 0 \\ \text{as } \theta = x - \dfrac{\pi}{2} \end{array}\right)$$

$$= \lim_{\theta \to 0} \frac{1 - \cos\theta}{4\theta^2} = \lim_{\theta \to 0} \frac{\sin^2\theta}{4(1 + \cos\theta).\theta^2}$$

$$= \frac{1}{4}\left(\lim_{\theta \to 0} \frac{1}{1 + \cos\theta}\right)\left(\lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2}\right)$$

$$= \frac{1}{4}\left(\frac{1}{1+1}\right).1 = \frac{1}{8}$$

Also $f\left(\dfrac{\pi}{2}\right) = k$

Given that $f(x)$ is continuous at $x = \dfrac{\pi}{2}$

$$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \frac{1}{8} = k \Rightarrow k = \frac{1}{8}$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Discuss the continuity of the function $f(x)$ at the point $x = 0$.

$$f(x) = \begin{cases} x, & x > 0 \\ 1, & x = 0 \\ -x, & x < 0 \end{cases}$$

[CBSE-1994]

2. The function $f(x)$ is defined as follows

$$f(x) = \begin{cases} x^2 + ax + b, & 0 \le x < 2 \\ 3x + 2, & 2 \le x \le 4. \\ 2ax + 5b, & 4 < x \le 8 \end{cases}$$ If f is

continuous on $[0, 8]$, find the value of a and b. ***[GOA-1996]***

3. Determine the value of a, b and c for which the function

$$f(x) = \begin{cases} \sin(a+1)x + \sin x, & x < 0 \\ c, & x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{b\sqrt{x^3}}, & x > 0 \end{cases}$$ may be

continuous at $x = 0$. ***[PSB-99; CBSE-2008]***

4. For what value of k, the function

$$f(x) = \begin{cases} \dfrac{\sqrt{5x+2} - \sqrt{4x+4}}{x-2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$$ is

continuous at $x = 2$.

[CBSE (Sample Paper)-99; (Practice sample paper) (7)}

5. Test the continuity of the following function at $x = 0$

$$f(x)=\begin{cases} e^{1/x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

[MP-99; CBSE-91; HSB-2001]

6. Test the continuity of the function $f(x)$ at point $x = 0$ where

$$f(x)=\begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ 1/2, & x = 0 \end{cases}$$

[MP-2000, 2008; AICBSE-97; HSB-94]

7. Prove that the function f is discontinuous at $x = 5$ if

$$f(x)=\begin{cases} 5x, & \text{when } x < 5 \\ 5, & \text{when } x = 5 \\ x^2, & \text{when } x > 5 \end{cases}$$ ***[MP-2000]***

8. Prove that function f is discontinuous at $x = 3$. If

$$f(x)=\begin{cases} 3x, & x < 3 \\ 3, & x = 3 \\ x^2, & x > 3 \end{cases}$$ ***[MP-2000]***

9. Discuss the continuity of the function $f(x)$ at $x = 1/2$ where

$$f(x)=\begin{cases} \dfrac{1}{2-x}, & 0 \leq x < \dfrac{1}{2} \\ 1, & x = 1/2 \\ \dfrac{3}{2-x}, & \dfrac{1}{2} < x \leq 1 \end{cases}$$ ***[CBSE-1994]***

10. Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$.

[PSB-90; CBSE-2002]

11. Find the value of a if the function $f(x)$ defined by $f(x)=\begin{cases} 2x-1, & x < 2 \\ a, & x = 2 \\ x+1, & x > 2 \end{cases}$ is continuous at $x = 2$.

[HSB-86]

12. If the function $f(x)$ given by

$$f(x)=\begin{cases} 3ax+b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax-2b, & \text{if } x < 1 \end{cases}$$ is continuous at $x = 1$, find the value of a and b.

[CBSE-2002; PSB-2002]

13. Examine the function for continuity

$$f(x)=\begin{cases} \dfrac{|x-3|}{2(x-3)}, & \text{if } x \neq 3 \\ 0, & \text{if } x = 3 \end{cases}$$ at $x = 3$.

[PSB-2001]

14. Examine the function for continuity

$$f(x)=\begin{cases} e^{1/x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$ at $x = 0$

[AISSE-92]

15. For what value of k is the function $f(x)$ continuous at $x = 3$.

$$f(x)=\begin{cases} \dfrac{x^2-x-6}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

[CBSE (foreign)-94]

ANSWERS

1. discontinuous

2. $a = 3$, $b = -2$

3. $a = -3/2$; $c = 1/2$; b may have any real value.

4. $k = \dfrac{1}{4\sqrt{3}}$

5. discontinuous.

6. continuous

9. discontinuity of first kind.

11. $a = 3$

12. $a = 3$ and $b = 2$

13. Non-removable discontinuity at 3.

14. Non-removable discontinuity at $x = 0$.

15. 5

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $f(x)=\begin{cases} x^{\alpha}\cos 1/x, & x\neq 0 \\ 0, & x=0\end{cases}$ is continuous at $x = 0$, then **[ICS-2001]**

(a) $\alpha<0$ (b) $\alpha>0$
(c) $\alpha=0$ (d) $\alpha\geq 0$

Solution

(b) Since $f(x)$ is continuous at $x = 0$, so

$f(0-0)=f(0+0)=f(0)=0.$

But $f(0-0)=\lim_{h\to 0}(-h)^{\alpha}\cos(-1/h)=0,$

if $\alpha\geq 0$

$f(0+0)=\lim_{h\to 0}h^{\alpha}\cos 1/h=0,$ if $\alpha>0.$

Hence $f(x)$ is continuous at $x = 0$ when $\alpha > 0$

2. If $f(x)=\begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x} & , x<\pi/2 \\ a & , x=\pi/2 \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & , x>\pi/2\end{cases}$ is continuous at $x = \pi/2$, then value of a and b are **[Ranchi-87]**

(a) 1/2, 1/4 (b) 2, 4
(c) 1/2, 4 (d) 1/4, 2

Solution

(c) $$f\left(\frac{\pi}{2}-0\right)=\lim_{h\to 0}\frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}$$

$$=\lim_{h\to 0}\frac{1-\cos^3 h}{3\sin^2 h}$$

$$=\lim_{h\to 0}\frac{(1-\cos h)(1+\cos h+\cos^2 h)}{3(1-\cos h)(1+\cos h)}=\frac{1}{2}$$

$$f\left(\frac{\pi}{2}+0\right)=\lim_{h\to 0}\frac{b\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2}$$

$$=\lim_{h\to 0}\frac{b(1-\cos h)}{4h^2}=\lim_{h\to 0}\frac{2b\sin^2 h/2}{4h^2}=\frac{b}{8}$$

Now $f(x)$ is continuous at $x = \pi/2$

$$\Rightarrow\quad f\left(\frac{\pi}{2}-0\right)=f\left(\frac{\pi}{2}+0\right)=f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow\quad \frac{1}{2}=\frac{b}{8}=a$$

$\therefore\quad a = 1/2, b = 4$

3. If $f(x)=\begin{cases} \dfrac{\sin[x]}{[x]+1} & , x>0 \\ \dfrac{\cos\dfrac{x}{2[x]}}{[x]} & , x<0 \\ k & , x=0\end{cases}$ (where $[x]$ = greatest integer $\leq x$) is continuous at $x = 0$, then k is equal to **[Kurukshetra (CEE)-98]**

(a) 0 (b) 1
(c) −1 (d) indeterminate

Solution

(d) As given $f(0-0)=f(0+0)=k$

$$\text{Now } f(0-0)=\lim_{h\to 0}\frac{\cos\dfrac{(-h)}{2[-h]}}{[-h]}$$

$$=\lim_{h\to 0}\frac{\cos\left(\dfrac{-h}{2(-1)}\right)}{-1}=-1$$

$$f(0+0)=\lim_{h\to 0}\frac{\sin[h]}{[h]+1}=\lim_{h\to 0}\frac{\sin 0}{0+1}=0$$

$\because\quad f(0-0)\neq f(0+0)$, so k is indeterminate.

4. Function $f(x) = |\sin x| + |\cos x| + |x|$ is discontinuous at

(a) $x=0$ (b) $x=\pi/2$
(c) $x=\pi$ (d) no where

Solution

(d) $|\sin x|$, $|\cos x|$ and $|x|$ are continuous functions on R, hence their sum is also continuous on R.

5. The function defined by

$$f(x)=\begin{cases}\left(x^2+e^{\frac{1}{2-x}}\right)^{-1} & , x\neq 2\\ k & , x=2\end{cases}$$, is continuous from right at the point $x = 2$, then k is equal to *[Orissa JEE-2002]*

(a) 0 (b) 1/4

(c) –1/4 (d) None of these

Solution

(b) $f(x)=\left[x^2+e^{\frac{1}{2-x}}\right]^{-1}$ and $f(2) = k$

If $f(x)$ is continuous from right at $x = 2$ then

$$\lim_{x\to 2^+} f(x) = f(2) = k \Rightarrow \lim_{x\to 2^+}\left[x^2+e^{\frac{1}{2-x}}\right]^{-1} = k$$

$$\Rightarrow \quad k = \lim_{h\to 0} f(2+h)$$

$$\Rightarrow \quad k = \lim_{h\to 0}\left[(2+h)^2+e^{\frac{1}{2-(2+h)}}\right]^{-1}$$

$$\Rightarrow \quad k = \lim_{h\to 0}[4+h^2+4h+e^{-1/h}]^{-1}$$

$$\Rightarrow \quad k = [4+0+0+e^{-\infty}]^{-1} \Rightarrow k = \frac{1}{4}.$$

6. The function $f(x)=|x|+\dfrac{|x|}{x}$ is *[Karnataka CET-2003]*

(a) Continuous at the origin

(b) Discontinuous at the origin because $|x|$ is discontinuous there

(c) Discontinuous at the origin because $\dfrac{|x|}{x}$ is discontinuous there

(d) Discontinuous at the origin because both $|x|$ and $\dfrac{|x|}{x}$ are discontinuous there

Solution

(c) $|x|$ is continuous at $x = 0$ and $\dfrac{|x|}{x}$ is discontinuous at $x = 0$

$\therefore\ f(x)=|x|+\dfrac{|x|}{x}$ is discontinuous at $x = 0$.

7. If $f(x)=\begin{cases}x-1, & x<0\\ 1/4, & x=0\\ x^2, & x>0\end{cases}$, then *[Roorkee-1988]*

(a) $\lim\limits_{x\to 0+} f(x)=1$

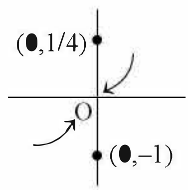
(b) $\lim\limits_{x\to 0-} f(x)=1$

(c) $f(x)$ is discontinuous at $x = 0$

(d) None of these

Solution

(c) Clearly from curve drawn of the given function $f(x)$ is discontinuous at $x = 0$.



8. If $f(x)=\begin{cases}\{\sin(a+2)x+\sin x\}/x, & x<0\\ b, & x=0\\ \{(x+3x^2)^{1/3}-x^{1/3}\}/x^{4/3}, & x>0\end{cases}$

is continuous at $x = 0$, then what are the values of a and b respectively? *[UPSC-2007]*

(a) –1, –1 (b) 1, –1

(c) 2, 1 (d) –2, 1

Solution

(d) $\lim\limits_{x\to 0}\dfrac{\sin(a+2)x+\sin x}{x}$

$$=\lim_{x\to 0}\frac{2\sin\left(\frac{a+3}{2}\right)x\cos\left(\frac{a-1}{2}\right)x}{x}$$

$$=\lim_{x\to 0}2\frac{\sin\frac{a+3}{2}x}{\frac{a+3}{2}x}.\left(\frac{a+3}{2}\right).\cos\left(\frac{a-1}{2}\right)x$$

$$=2.1.\left(\frac{a+3}{2}\right).1=a+3$$

Again $\lim\limits_{x\to 0}\dfrac{(a+3x^2)^{\frac{1}{3}}-x^{\frac{1}{3}}}{x^{\frac{4}{3}}}$

$$=\lim_{x\to 0}\frac{(x+3x^2)-x}{x^{\frac{4}{3}}\left[(x+3x^2)^{\frac{2}{3}}+(x+3x^2)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right]}$$

$$=\lim_{x\to 0}\frac{3x^2}{x^{\frac{4}{3}}.x^{\frac{2}{3}}\left[(1+3x)^{\frac{2}{3}}+\{(1+3x)^{\frac{1}{3}}+1\}\right]}$$

$$= \lim_{x\to 0}\frac{3x^2}{x^2\left[(1+3x)^{\frac{2}{3}}+\{(1+3x)^{1/3}+1\}\right]}$$

$$= \lim_{x\to 0}\frac{3}{[(1+3x)^{2/3}+\{(1+3x)^{1/3}+1\}]}$$

$$= \lim_{x\to 0}\frac{3}{1+1+1} = \frac{3}{3} = 1$$

As $f(x)$ is continuous at $x = 0$

$$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

$$\Rightarrow a+3 = b = 1$$

giving $b = 1, a = -2$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The number of points at which the function $f(x) = \dfrac{1}{x-[x]}$ is not continuous is

(a) 1 (b) 2
(c) 3 (d) ∞

2. If $f(x) = (1+\tan^2\sqrt{x})^{1/2x}$, the value of $f(0)$ that makes $f(x)$ continuous every where is

(a) e (b) 1/2
(c) $e^{1/2}$ (d) 0

3. Which of the following function is not continuous at $x = 0$?

(a) $f(x) = \begin{cases} x^2\sin\dfrac{1}{x}, & x \neq \\ 0, & x = 0 \end{cases}$

(b) $f(x) = x\sin\dfrac{1}{x};\ x \neq 0,\ f(0) = 0$

(c) $f(x) = \cos\dfrac{1}{x},\ x \neq 0$

(d) $f(x) = x\dfrac{1-e^{1/x}}{1+e^{1/x}},\ x \neq 0,\ f(0) = 0$

4. If $f: R \to R$ be defined be

$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$, then f is continuous at

(a) All rational points
(b) all irrational points
(c) all real points
(d) no real points

5. If the function $f(x) = \begin{cases} (\cos x)^{1/x}, & x \neq 0 \\ k, & x = 0 \end{cases}$, continuous at $x = 0$. Then the value of k is

(a) 1 (b) -1
(c) 0 (d) e

6. If the function $f(x) = \dfrac{x^2-(A+2)x+2A}{x-2}$ for $x \neq 2$, $= 2$ for $x = 2$, is continuous at $x = 2$, then

(a) $A = 0$ (b) $A = 1$
(c) $A = -1$ (d) None of these

7. The function defined by $f(x) = (x^2 + e^{1/(2-x)})^{-1}$, when $x \neq 2$ and $f(x) = k$ when $x = 2$, is continuous from right at the point $x = 2$. Then k is equal to

(a) 0 (b) 1/4
(c) $-1/4$ (d) None of these

8. If the function

$$f(x) = \begin{cases} 1+\sin\dfrac{\pi}{2}x & \text{for } -\infty < x \leq 1 \\ ax+b & \text{for } 1 < x < 3 \\ 6\tan\dfrac{x\pi}{12} & \text{for } 3 \leq x < 6 \end{cases}$$ is

continuous in the interval $(-\infty, 6)$, then the values of a and b are respectively

(a) (0, 2) (b) (1, 1)
(c) (2, 0) (d) (2, 1)

9. If function $f(x) = \begin{cases} x^2/a & 0 \leq x < 1 \\ a & 1 \leq x < \sqrt{2} \\ \dfrac{2b^2-4b}{x^2} & \sqrt{2} \leq x < \infty \end{cases}$ is

continuous for $0 \leq x < \infty$, then the most suitable value of a and b are respectively

(a) $1, -1$ (b) $-1, 1+\sqrt{2}$

(c) $-1, 1$ (d) $1, 1-\sqrt{2}$

10. The function $f(x) = [x] \cos [(2x - 1)/2]\pi$' [.] denotes the greatest integer function is discontinuous at

(a) All x

(b) All integer points

(c) no x

(d) x which is not integer

11. $f(x) = \begin{cases} \dfrac{\sqrt{1+px}-\sqrt{1-px}}{x}, & -1 \le x < 0 \\ \dfrac{2x+1}{x-2}, & 0 \le x \le 1 \end{cases}$ is continuous in the interval $[-1, 1]$, then $p =$

(a) -1 (b) $-1/2$

(c) $1/2$ (d) 1

12. Function $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{, for } x < 0 \\ c & \text{, for } x = 0 \\ \dfrac{\sqrt{(x+bx^2)}-\sqrt{x}}{bx\sqrt{x}} & \text{, for } x > 0 \end{cases}$

is continuous if

(a) $a = -3/2, c = 1/2, b = 0$

(b) $a = -3/2, c = -1/2$

(c) $a = -3/2, b =$ any value, $c = 1/2$

(d) None of these

13. The values of A and B such that the function

$f(x) = \begin{cases} -2\sin x & , \quad x \le -\dfrac{\pi}{2} \\ A\sin x + B & , \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \cos x & , \quad x \ge \dfrac{\pi}{2} \end{cases}$ is continuous every where are

(a) $A = 0, B = 1$ (b) $A = 1, B = 1$

(c) $A = -1, B = 1$ (d) $A = -1, B = 0$

14. The function $f(x) = [x]^2 - [x^2]$, (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at

(a) all integer

(b) all integers except 0 and 1

(c) all integers except 0

(d) all integers excpts 1

15. The function $f(x) = p\,[x+1] + q\,[x-1]$ where $[x]$ is the greatest integer function is continuous at $x = 1$, if

(a) $p - q = 0$ (b) $p + q = 0$

(c) $p = 0$ (d) $q = 0$

16. If $f(x) = 1/(1 - x)$, the point of discontinuity of the composite function $y = f(f(f(x)))$ are

(a) 0 (b) 1

(c) 2 (d) 4

17. Let $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{for } x < 0 \\ a, & \text{for } x = 0. \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{for } x > 0 \end{cases}$

The value of a for which $f(x)$ is continuous at $x = 0$ is

(a) 5 (b) 8

(c) 4 (d) 3

18. If $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$, then $f(x)$ is discontinuous at every real number except

[UPSEAT-2002]

(a) 0 (b) 1/2

(c) 1 (d) A rational point

SOLUTIONS

1. (d) $f(x) = \dfrac{1}{x-[x]}$ is not continuous at integer points,

$\therefore$ Number of points of discontinuity $= \infty$

2. (c) $f(0) = \lim\limits_{x \to 0}\left(1 + \tan^2\sqrt{x}\right)^{\frac{1}{2x}}$

$= e^{\lim_{x\to 0}\left(\tan^2\sqrt{x}\right)\frac{1}{2x}} = e^{1/2}$

3. (c) Is not continuous

$\because f(x)=\cos\left(\frac{1}{x}\right)$ is not defined at $x = 0$

4. (d) Here $f(x)=\begin{cases}1; & \text{if } x \text{ is rational}\\ 0; & \text{if } x \text{ is irrational}\end{cases}$

If x is rational number, its neighbourhood way contain rational or irrational points.

$\therefore$ it limit will be not defined similar for irrational number also, limit won't exist.

For defined limit, $f(x)$ should have same values for rational and irrational x.

5. (a) Here $k=\lim\limits_{x\to 0}(\cos x)^{1/x}$

$$= e^{\lim\limits_{x\to 0}(\cos x-1)^{1/x}}$$

$$= e^{\lim\limits_{x\to 0}\left(\frac{-2\sin^2\frac{x}{2}}{x}\right)}$$

$$= e^{\lim\limits_{x\to 0}\left(\frac{-\sin\frac{x}{2}}{x/2}\right)\left(-\sin\frac{x}{2}\right)}$$

$$= e^{(1)(0)} = e^0 = 1$$

6. (a) $\lim\limits_{x\to 2}\frac{x^2-(A+2)x+2A}{x-2}=2$

On factorising

$$\lim_{x\to 2}\frac{(x-A)(x-2)}{x-2}=2$$

$$\lim_{x\to 2}(x-A)=2$$

$$A=0$$

7. (b) $k=f(2+)$

$$k=\lim_{h\to 0}\frac{1}{(2+h)^2+e^{\frac{1}{2-(2+h)}}}$$

$$k=\lim_{h\to 0}\frac{1}{(2+h)^2+e^{-\frac{1}{h}}}=\frac{1}{4}$$

8. (c) $f(x)$ is contains at $x = 1$

$\therefore f(1-)=f(1+)$

$$1+\sin\frac{\pi}{2}=a+b$$

$a+b=2$ .... (1)

also at $x = 3$

$f(3-)=f(3+)$

$$3a+b=6\tan\left(\frac{3\pi}{12}\right)$$

$3a+b=6$ .... (2)

Solving from (1) and (2)

$a = 2, b = 0.$

9. (d) $f(x)$ is continuous at $x = 1$

$\therefore f(1-)=f(1+)$

$$\frac{1}{a}=a \Rightarrow a=\pm 1$$

at $x=\sqrt{2}$; $a=\frac{2b^2-4b}{(\sqrt{2})^2}=\frac{b^2-2b}{1}$

If $a=1$	If $a=-1$
$1=b^2-2b$	$-1=b^2-2b$
$b^2-2b-1=0$	$b^2-2b+1=0$
$b=1\pm\sqrt{2}$	$(b-1)^2=0$
	$b=1$

10. (b) $f(x)=[x]\cos\left((2x-1)\frac{\pi}{2}\right)$

$[x]$ is discontinuous at all integral points, but

$\cos(2x-1)\frac{\pi}{2}=0$ at all $x\in Z$

$\therefore$ $f(x)$ is continuous at all values including integers.

11. (b) $f(x)=\begin{cases}\frac{\sqrt{1+px}-\sqrt{1-px}}{x}; & -1\le x<0\\ \frac{2x-1}{x-2}; & 0\le x\le 1\end{cases}$

For continuity

$f(0-)=f(0+)$

$$\lim_{h\to 0}\frac{\sqrt{1-ph}-\sqrt{1+ph}}{-h}=\lim_{h\to 0}\frac{2h+1}{h-2}$$

On rationalising,

$$\lim_{h\to 0}\frac{(1-ph)-(1+ph)}{\sqrt{1-ph}+\sqrt{1+ph}}\cdot\frac{1}{-h}=\frac{-1}{2}$$

$$\frac{2p}{2}=\frac{-1}{2}\Rightarrow p=\frac{-1}{2}$$

12. (c) L.H.L. $=\lim_{x\to 0}\left(\frac{\sin x(a+1)x}{x}+\frac{\sin x}{x}\right)$

$=a+1+1=a+2$

R.H.L. $\lim_{x\to 0}\frac{\sqrt{x+bx^2}-\sqrt{x}}{bx\sqrt{x}}$

$$=\lim_{x\to 0}\frac{\sqrt{x}\left((1+bx)^{1/2}-1\right)}{bx\sqrt{x}}$$

$$=\lim_{x\to 0}\left(\frac{\left(1+\frac{bx}{2}+.....\right)-1}{bx}\right)=\frac{1}{2}$$

L.H.L. = R.H.L. = c

$$a+2=\frac{1}{2}=c$$

$$\therefore a=\frac{-3}{2}, c=\frac{1}{2}, b\in R$$

13. (c) At $x=\frac{-\pi}{2}$

$$f\left(\frac{-\pi}{2}-\right)=f\left(-\frac{\pi}{2}+\right)$$

$2=-A+B$ .... (i)

At $x=\frac{\pi}{2}$

$A+B=0$ .... (ii)

Solving from (i) and (ii)

$A=-1, B=1$

14. (d) At $x = 0$

$f(0+)=\lim_{h\to 0}[h]^2-[h^2]$

$=0-0=0$

$f(0-)=\lim_{h\to 0}[-h]^2-[(h)^2]=-1$

$\therefore$ discontinuous at $x = 0$

And; at $x = 1$

$f(1+)=\lim_{h\to 0}[1+h]^2-[(1+h)^2]=1-1=0$

$f(1-)=\lim_{h\to 0}[1-h]^2-[(1-h)^2]=0-0=0$

$\therefore\ f(1+)=f(1-)=f(1)$

Hence $f(x)$ is continuous at $x = 1$, ... from option (d).

15. (b) $f(x)=p[x+1]+q[x-1]$

$=p([x]+1)+q([x]-1)=(p+q)[x]+p-q$

$\because$ $[x]$ is discontinuous at integral points.

$\therefore$ For continuity of $f(x)$

$p+q=0$

16. (c) Here $f(x)=\frac{1}{1-x}$

$$f(f(x))=f\left(\frac{1}{1-x}\right)=\frac{1}{1-\frac{1}{1-x}}$$

$$=\frac{1-x}{-x}=1-\frac{1}{x}$$

$$f(f(f(x)))=f\left(1-\frac{1}{x}\right)=\frac{1}{1-\left(1-\frac{1}{x}\right)}=x$$

For discontinuity of $f(f(f(x)))$

$f(x)$ or $f(f(x))$ or $f(f(f(x)))$ may be discontinuous

$\therefore$ Point of discontinuity are $x=1,0$.

No. of points = 2

17. (b) $f(0-)=f(0)$

$$\lim_{h\to 0}\frac{1-\cos 4h}{h^2}=\lim_{h\to 0}\frac{2\sin^2 2h}{h^2}=9$$

$$\lim_{h\to 0}\frac{2(\sin 2h)^2}{\frac{1}{4}.(2h)^2}=9$$

$9=8$ $\left(\because \lim_{h\to 0}\frac{\sin x}{x}=1\right)$

18. (b) $f(x)=\begin{cases} x, & \text{if } x \text{ is rational}\\ 1-x, & \text{if } x \text{ is irrational}\end{cases}$

For continuity $f(x)$ must contain same value for rational and irrational values of x.

$\therefore\ x=1-x\ =\ x=\frac{1}{2}$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. Which of the following functions is continuous at $x = 0$?

(a) $f(x) = \dfrac{e^{1/x}}{1+e^{1/x}}, x \neq 0, f(0) = 0$

(b) $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}, x \neq 0, f(0) = 0$

(c) $f(x) = 1, x \leq 0;\ f(x) = x + \dfrac{1}{2}, x > 0$

(d) $f(x) = x.\dfrac{1-e^{1/x}}{1+e^{1/x}}, x \neq 0, f(0) = 0$

2. Which of the following is not a property of continuous functions?

(a) If $f(x)$ be continuous for $x = c$, $f(c) \neq 0$, then $\exists$ an interval $(c - \delta, c + \delta)$ around c such that $f(x)$ has the same sign as $f(c)$ for all values of x in this interval.

(b) If $f(x)$ be continuous in a closed interval $[a, b]$, then $\exists$ points c and d in $[a, b]$, where $f(x)$ assume its greatest value M and the least value m respectively.

(c) If $f(x)$ be continuous in a closed interval $[a, b]$ and let $f(a)$ & $f(b)$ have opposite signs. Then $f(x)$ is '+ve' for all values of x lying between a and b.

(d) In $c.f(x) = 0$ for at least one value of x_1 lying between a and b.

3. If a function $f(x)$ is continuous in $[a, b]$ and $f(a)$ & $f(b)$ are of the opposite signs, then $\exists c \in (a, b)$ such that

(a) $f(c) = f(a)$

(b) $f(c) > f(b)$

(c) $f(c) = 0$

(d) $f(c) > f(a)$

4. If a function $f(x)$ is continuous for at $x = c$ and $f(c) < 0$, then $\exists$ an interval $(c - \delta, c + \delta)$ such that for every x in the interval

(a) $f(x) = 0$

(b) $f(x) > 0$

(c) $f(x) = f(c)$

(d) $f(x) < 0$

5. Function $f(x) = \begin{cases} 2, & x \leq 0 \\ 3x+2, & 0 < x < 1 \\ \dfrac{x}{x-1}, & x > 1 \end{cases}$ is

(a) Continuous at $x = 0$ and $x = 1$

(b) continuous at $x = 0$ but discontinuous at $x = 1$

(c) discontinuous at $x = 0$ but continuous at $x = 1$

(d) discontinuous at both $x = 0$ and $x = 1$

6. The function $f(x) = \begin{cases} x, & 0 \leq x < \dfrac{1}{2} \\ 1, & x = \dfrac{1}{2} \\ 1-x, & \dfrac{1}{2} < x < 1 \end{cases}$ is

(a) continuous for all values of x in $(0, 1)$

(b) discontinuous at all values of x in $(0, 1)$

(c) discontinuous at $x = 0$

(d) discontinuous at $x = 0, 1/2, 1$

7. The function $f(x) = x^n$ is, where n being a '–ve' integer

(a) continuous for all x

(b) continuous for all values of $x \neq 0$

(c) continuous for all values of x except $x = 0$

(d) None of these

8. The function $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$

(a) continuous at every point of x

(b) discontinuous at every point of x

(c) continuous at every point x except $x = 0$

(d) discontinuous at every point of x except $x = 0$

9. If $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is

[MPPET-1999; AMU-1999; RPET-2003]

(a) 1 (b) –1

(c) 0 (d) 2

10. The function $f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$ is

(a) Continuous at $x = 1$

(b) discontinuous only at $x = 0$

(c) discontinuous only at 0 and 1

(d) discontinuous every where

11. If $f(x) = \begin{cases} \frac{3}{2}ax - (a+1); & x \neq 2 \\ 1; & x = 2 \end{cases}$ f is continuous at $x = 2$, then what is the value of a?

[Gujarat CET-2007]

(a) 3 (b) – 1

(c) 1 (d) 2

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 14 minutes.
3. The worksheet consists of 14 questions. The maximum marks are 42.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If $f(x) = \begin{cases} x+\lambda & x<4 \\ 7 & x=4 \\ 3x-5 & x>4 \end{cases}$ is continuous at $x = 4$, then λ is equal to

[PET (Raj.)-87; MP-2001]

(a) 4 (b) 3
(c) 2 (d) 1

2. Which one of the following statement is correct? $f(x) = \dfrac{1}{1+\tan x}$ *[NDA-2005]*

(a) is a continuous, real valued for all $x \in (-\infty, \infty)$.
(b) is discontinuous only at $x = 3\pi/4$
(c) has only finitely many discontinuties on $(-\infty, \infty)$
(d) has only infinitely many discontinuties on $(-\infty, \infty)$

3. The function $f(x) = \dfrac{2x^2+7}{x^3+3x^2-x-3}$ is discontinuous for *[J & K-2005]*

(a) $x = 1$ only
(b) $x = 1$ and $x = -1$ only
(c) $x = 1, x = -1, x = -3$ only
(d) $x = 1, x = -1, x = -3$ and other values of x

4. Let $f(x) = \dfrac{1-\tan x}{4x-\pi}, x \neq \dfrac{\pi}{4}, x \in \left[0, \dfrac{\pi}{2}\right]$. $f(x)$ is continuous in $[0, \pi/2]$, then $f(\pi/4)$ is

[AIEEE-2004]

(a) $-1/2$ (b) $1/2$
(c) 1 (d) -1

5. If $f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & -1 \le x < 0 \\ 2x^2+3x-2, & 0 \le x \le 1 \end{cases}$

is continuous at $x = 0$ then k is equal to

[EAMCET-2003]

(a) -4 (b) -3
(c) -2 (d) -1

6. The function $f(x) = \begin{cases} x+2, & 1 \le x \le 2 \\ 4, & x = 2 \\ 3x-2, & x>2 \end{cases}$ is continuous at *[DCE-1999]*

(a) $x = 2$ only (b) $x \le 2$
(c) $x \ge 2$ (d) None of these

7. If $f(x) = (1 + x)^{5/x}$ is continuous at $x = 0$, then what is the value of $f(0)$?

[NDA-2006]

(a) 0 (b) 1
(c) ∞ (d) e^5

8. Consider the following statements

1. The function $f(x)$ = greatest integer $\le x$, $x \in IR$ is a continuous function.
2. All trigonometric functions are continuous on IR

Which of the statements given above is/are correct? *[NDA-06]*

(a) 1 only
(b) 2 only
(c) Both 1 and 2
(d) Neither 1 nor 2

9. Let $f(x)\begin{cases} \sqrt{1+x^2}, & x<\sqrt{3} \\ \sqrt{3}x-1, & \sqrt{3} \le x < 4 \\ [x], & 4 \le x < 5 \\ |1-x|, & x \ge 5 \end{cases}$, where $[x]$ is the greatest integer $\le x$. The function is discontinuous at

(a) $\sqrt{3}$ (b) 4
(c) 5 (d) everywhere

10. If function $f(x) = \dfrac{(27-2x)^{1/3}-3}{9-3(243+5x)^{1/5}} (x \neq 0)$ is continuous function, then $f(0)$ is equal to ***[DCE-1998]***

(a) 2 (b) 4
(c) 6 (d) 2/3

11. If $f(x) = \begin{cases} 2^{\frac{1}{x-1}} & x < 1 \\ ax^2 + bx + 1, & x \geq 1 \end{cases}$ is continuous, then $a + b$

(a) 1 (b) –1
(c) 0 (d) 3/2

12. If $f(x) = \begin{cases} mx + 1 & x \leq \pi/2 \\ \sin x + n & x > \pi/2 \end{cases}$ is continuous at $x = \pi/2$, then which one of the following is correct? ***[NDA-06]***

(a) $m = 1, n = 0$ (b) $m = (n\pi/2) + 1$
(c) $n = m(\pi/2)$ (d) $m = n = \pi/2$

13. Let $f(x)$ be a continuous function given by $f(x) = \begin{cases} 2x, & |x| \leq 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$, then

(a) $a = +2, b = -1$
(b) $a = 1, b = -2$
(c) $a = 1, b = 2$
(d) $a = -1, b = -2$

14. If $f(x) = \dfrac{\log_e(1 + x^2 \tan x)}{\sin x^3}, x \neq 0,$ is to be continuous at $x = 0$, then $f(0)$ must be defined as ***[Kerala PET-2007]***

(a) 1 (b) 0
(c) 1/2 (d) –1
(e) 2

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**6.** ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

2. (d) $f(x) = \dfrac{1}{1 + \tan x}$ is discontinuous when

$\tan x = -1$, $x = 2n\pi - \dfrac{\pi}{4}, n \in Z$

$\therefore$ Infinitely many values of x.

3. (d) For discontinuity $x^3 + 3x^2 - x - 3 = 0$

$x^2(x+3) - (x+3) = 0$

$(x^2 - 1)(x+3) = 0$

$x = 1, -1, -3$

4. (a) $f\left(\dfrac{\pi}{4}\right) = \lim\limits_{x \to \frac{\pi}{4}} \dfrac{1 - \tan x}{4x - \pi}$

put $x - \dfrac{\pi}{4} = t$

$$f\left(\frac{\pi}{4}\right) = \lim_{t \to 0} \frac{1 - \tan\left(\frac{\pi}{4} + t\right)}{4t}$$

$$= \lim_{t \to 0} \frac{1 - \frac{1 + \tan t}{1 - \tan t}}{4t} = \lim_{t \to 0} \frac{-2\tan t}{1 - \tan t} \times \frac{1}{4t} = -\frac{1}{2}$$

5. (c) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}=2.0^2+3.0-2$

On rationalising L.H.S.

$$\lim_{x\to 0}\frac{1+kx-(1-kx)}{x\left(\sqrt{1+kx}+\sqrt{1-kx}\right)}=-2$$

$$\frac{2k}{2}=-2\Rightarrow k=-2$$

10. (c) $f(0)=\lim\limits_{x\to 0}\dfrac{3\left(\dfrac{2x}{27}\right)^{1/3}-3}{9-3.3\left(1+\dfrac{5x}{243}\right)^{1/5}}$

$$=\lim_{x\to 0}\frac{3}{9}\left[\frac{\left(1-\dfrac{2x}{27}\right)^{1/3}-1}{1-\left(1+\dfrac{5x}{243}\right)^{1/3}}\right]$$

$$=\frac{1}{3}\lim_{x\to 0}\left[\frac{1-\dfrac{2x}{81}-1}{1-\left(1+\dfrac{x}{729}\right)}\right]$$

$$=\frac{1}{3}\times\frac{2}{81}\times(729)=c\ =6=c$$

NOTE

For small x, $(1+x)^n=1=nx$.

LECTURE

3

Test Your Skills

IIT–BOOSTER

1. Let $f(x) = \begin{cases} (1+|\sin x|)^{a/|\sin x|} & -\pi/6 < x < 0 \\ b & x = 0 \\ e^{\tan 2x/\tan 3x} & 0 < x < \pi/6 \end{cases}$

Determine a and b such that f is continuous at $x = 0$ ***[IIT-1994]***

Solution

$L = R = V = b$ at $x = 0$ for continuity at $x = 0$

R.H.L. $= \underset{h\to 0}{Lt}\, f(0+h) = Lt\, e^{(\tan 2h/\tan 3h)}$

$= Lt\, e^{2h/3h} = e^{2/3}$

$\because \underset{\theta\to 0}{Lt} = \tan\theta = \theta$

L.H.L. $= \underset{h\to 0}{Lt} = f(0-h)$

$= Lt\,(1+|\sin(-h)|)^{a/|\sin(-h)|}$

$= Lt\,(1+\sin h)^{a/\sin h}$

$\because |-\sin h| = \sin h = y$, say

$= \underset{y\to 0}{Lt} = (1+y)^{a/y} = e^a$

as $h \to 0, y \to 0$

$\therefore e^a = e^{2/3} = b$

$\therefore a = 2/3, b = e^{2/3}$

2. (i) $f(x) = [\tan^2 x]$ is continuous at $x = 0$

(ii) The number of points where $f(x) = [\sin x + \cos x]$ (where [.] denotes the greatest integer function) $x \in (0, 2\pi)$ is discontinuous is ***[IIT-1993]***

(a) 3 (b) 4

(c) 5 (d) 6

Solution

(i) $\underset{h\to 0}{Lt}[\tan^2(0-h)]$

$= \underset{h\to 0}{Lt}[\tan^2(0+h)] = [\tan^2 0] = 0$

Since L.H.Lt. = R.H.Lt. = value.

$\therefore$ $f(x)$ is continuous at $x = 0$

(ii) (c)

$[\sin x]$ is non-differentiable at $x = \pi/2, \pi, 2\pi$ and $[\cos x]$ is non-differentiable at $x = 0, \pi/2, 3\pi/2, 2\pi$

Thus $f(x)$ is definitely non-differentiable at $x = \pi, 3\pi/2, 0$

Also $f\left(\frac{\pi}{2}\right) = 1, f\left(\frac{\pi}{2} - 0\right) = 0,$

$f(2\pi) = 1,$

$f(2\pi - 0) = -1$

Thus $f(x)$ is discontinuous and hence non-differentiable at $x = \frac{\pi}{2}$ and 2π.

3. Let $f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$. where [.] denotes the greatest integer function. The domain of f is .... and the points of discontinuity of f in the domain are .... ***[IIT-1996]***

Solution

The function is not defined for those values of x for which $[x + 1] = 0$. In other words it means that

$0 \le x+1 < 1$ or $-1 \le x < 0$ .... (1)

Hence the function is defined outside the region given by (1). In other words for $x \ge 0$ and $x < -1$ or $x \in]-\infty, -1[\cup [0, \infty[$

Now consider integral values of x say $x = n$.

$$\text{R.H.L.} = \lim_{h\to 0} [n+h] \sin\frac{\pi}{[n+1+h]}$$

$$= n\sin\frac{\pi}{n+1}$$

$$\text{L.H.L.} = \lim_{h\to 0} [n-h] \sin\frac{\pi}{[n+1-h]}$$

$$= (n-1)\sin\frac{\pi}{n}$$

$$V = n\sin\frac{\pi}{n+1}$$

Since $R \neq L = V$. Hence the given function is not continuous for integral values of $n (n \neq 0, -1)$.

At $x = 0$ $f(0) = 0$

$$\lim f(0+h) = \lim_{h\to 0} [h]\sin\frac{\pi}{[h+1]} = 0$$

The function is not defined for $x < 0$. Hence we cannot find $\lim f(0-h)$. Thus $f(x)$ is continuous at $x = 0$. Hence the point of discontinuity are given by $I - \{0\}$ where I is set of integers n except $n = -1$.

4. Let $f(x) = \begin{cases} \dfrac{(x^3+x^2-16x+20)}{(x-2)^2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$ If $f(x)$ is continuous for all x, then $k = \ldots\ldots.$

[IIT-1981]

Solution

We have $f(x) = \begin{cases} \dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$

Clearly $f(x)$ is continuous for all values of x except possibly at $x = 2$. It will be continuous at $x = 2$ if $\lim\limits_{x\to 2} f(x) = f(2)$

$$\Rightarrow \quad \lim_{x\to 2}\frac{x^3+x^2-16x+20}{(x-2)^2} = k$$

$$\Rightarrow \quad k = \lim_{x\to 2}\frac{(x-2)^2(x+5)}{(x-2)^2} = \lim_{x\to 2}(x+5)$$

$$= 7 \quad \therefore k = 7$$

5. Determine the values a, b, c for which the function $f(x) = \begin{cases} \dfrac{\sin(a+1)x+\sin x}{x} & , x < 0 \\ c & , x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & , x > 0 \end{cases}$

is continuous at $x = 0$

[IIT-1982]

Solution

Given function is,

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x+\sin x}{x} & , x < 0 \\ c & , x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & , x > 0 \end{cases}$$

Given that it is cont. at $x = 0$, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

Consider $\lim\limits_{x\to 0^-} f(x) = c$

$$\Rightarrow \lim_{h\to 0} f(0-h) = c$$

$$\Rightarrow \lim_{h\to 0}\frac{\sin(a+1)(-h)+\sin(-h)}{-h} = c$$

$$\Rightarrow \lim_{h\to 0}\frac{-[\sin(a+1)h+\sin h]}{-h} = c$$

$$\Rightarrow \lim_{h\to 0}\frac{2\sin\left(\dfrac{ah+h+h}{2}\right)\cos\left(\dfrac{ah+h-h}{2}\right)}{h} = c$$

6. Find the values of a and b so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & , 0 \le x < \pi/4 \\ 2x\cot x + b & , \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x & , \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$.

[IIT-89; Roorkee-98; AMU-2003]

Solution

Given that

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & , 0 \le x < \pi/4 \\ 2x\cot x + b & , \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x & , \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$.

$\therefore$ $f(x)$ must be continuous at $x = \pi/4$ and $x = \pi/2$

$\therefore \quad \lim_{x\to\pi/4^-} f(x) = f(\pi/4)$

$\Rightarrow \quad \lim_{h\to 0} f\left(\frac{\pi}{4} - h\right) = \frac{2\pi}{4}\cot\frac{\pi}{4} + b$

$\Rightarrow \quad \lim_{h\to 0}\left(\frac{\pi}{4} - h\right) + a\sqrt{2}\sin\left(\frac{\pi}{4} - h\right) = \frac{\pi}{2} + b$

$\Rightarrow \quad \frac{\pi}{4} + a = \frac{\pi}{2} + b \quad \Rightarrow \quad a - b = \frac{\pi}{4}$

Also, $\lim_{x\to\pi/2^+} f(x) = f(\pi/2)$

$\Rightarrow \quad \lim_{h\to 0} f\left(\frac{\pi}{2} + h\right)$

$= 2.\frac{\pi}{2}\cot\frac{\pi}{2} + b$

$\Rightarrow \quad \lim_{h\to 0} a\cos 2\left(\frac{\pi}{2} + h\right) - b\sin\left(\frac{\pi}{2} + h\right) = b$

$\Rightarrow \quad a\cos\pi - b\sin\pi/2 = b$

$\Rightarrow \quad -a - b = b \quad \Rightarrow \quad a + 2b = 0$

Solving (1) and (2),

we get, $a = \pi/6$

and $b = -\pi/12$.

7. The function $f(x) = \frac{\log(1+ax) - \log(1-bx)}{x}$ is undefined at point $x = 0$. If $f(x)$ is continuous at $x = 0$ then find the value of $f(0)$.

[IIT-1993]

Solution

Using the formula $\lim_{x\to 0}\frac{\log(1+ax)}{ax} = 1$

$\therefore$ Right hand limit

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{\log(1+ax) - \log(1-bx)}{x}$$

$$= \lim_{h\to 0}\frac{\log[1 + a(0+h) - \log[1 - b(0+h)]}{0+h}$$

[Put $x = 0 + h$]

$$= \lim_{h\to 0}\frac{\log(1+ah) - \log(1-bh)}{h}$$

$$= a\lim_{h\to 0}\frac{\log(1+ah)}{ah} + b\lim_{h\to 0}\frac{\log(1-bh)}{-bh}$$

$= a \times 1 + b \times 1 = a + b$ .... (1)

and left hand limit

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{\log(1+ax) - \log(1-bx)}{x}$$

$$= \lim_{h\to 0}\frac{\log[1 + a(0-h) - \log[1 - b(0-h)]}{0-h}$$

[Put $x = 0 - h$]

$$= a\lim_{h\to 0}\frac{\log(1-ah)}{-ah} + b\lim_{h\to 0}\frac{\log(1+bh)}{bh}$$

$= a \times 1 + b \times 1 = a + b$ .... (2)

If $f(x)$ is continuous, then

$f(0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$= a + b \Rightarrow f(0) = a + b$

8. Let $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & , \quad x < 0 \\ a & , \quad x = 0 \\ \frac{\sqrt{x}}{\sqrt{[16+\sqrt{x}]} - 4} & , \quad x > 0 \end{cases}$

If possible find the value of a so that the function may be continuous at $x = 0$.

[IIT-90]

Solution

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0}\frac{1 - \cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h\to 0}\frac{1-\cos 4h}{h^2} = \lim_{h\to 0}\frac{2\sin^2 2h}{h^2}$$

$$= \lim_{h\to 0}\frac{4.2\sin^2 2h}{4h^2} = \lim_{h\to 0} 8\left(\frac{\sin 2h}{2h}\right)^2 = 8.1 = 8$$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}\frac{\sqrt{0+h}}{\sqrt{[16+\sqrt{0+h}]} - 4}$$

$$\lim_{h\to 0}\sqrt{h}\frac{1}{\sqrt{[16+\sqrt{h}]} - 4} \times \frac{\sqrt{[16+\sqrt{h}]} + 4}{\sqrt{[16+\sqrt{h}]} + 4}$$

$$= \lim_{h\to 0}\frac{\sqrt{h}.(\sqrt{[16+\sqrt{h}]} + 4)}{(\sqrt{[16+\sqrt{h}]})^2 - 4^2}$$

$$= \lim_{h\to 0}\frac{\sqrt{h}.(\sqrt{[16+\sqrt{h}]} + 4)}{16 + \sqrt{h} - 16}$$

$$= \lim_{h\to 0}\sqrt{(16+\sqrt{h})} + 4$$

$= \sqrt{16+0} + 4 = 4 + 4 = 8$

The function $f(x)$ is continuous at $x = 0$ if

$\lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x) \Rightarrow a = 8$

Hence $a = 8$

MENTAL PREPARATION TEST

1. If $f(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ Test the continuity of $f(x)$ at $x = 0$.

2. Test of Continuity at $x = 0$ of the function
$$f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}.$$
[MP-99]

3. Determine the value of k for which the following function is continuous at $x = 3$,
$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$
[CBSE-94(C)]

4. Let $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$
Show that $f(x)$ is discontinuous at $x = 0$.
[PSB-2000 C; HSB-81C, 89C; HB-2003]

5. For what value of k is the function $f(x) = \begin{cases} \dfrac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.
[CBSE-92, 94; AISSE-93C; HSB-2001; PSB-94]

6. Discuss the continuity of the function $f(x)$ given by $f(x) = \begin{cases} 2x-1, & \text{if } x < 0 \\ 2x+1, & \text{if } x \geq 0 \end{cases}$
[CBSE-2002; HPSB-2000 C, 2002 C]

7. Test the continuity/discontinuity of the function at the point $x = 0$.
$$f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$
[PSB-94; BIT Mesra-98]

8. Prove that f is discontinuous at $x = 1$ if
$$f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1. \\ x^2+1 & x > 1 \end{cases}$$
[MP-2000]

9. Test the continuity of the function on $f(x)$ at the origin
$$f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$
[MP-2003; CBSE-92, 95(C)]

10. Discuss the continuity of the function $f(x)$ at the point $x = \dfrac{1}{2}$. $f(x) = \begin{cases} x, & 0 \leq x < \dfrac{1}{2} \\ \dfrac{1}{2}, & x = \dfrac{1}{2} \\ 1-x, & \dfrac{1}{2} < x \leq 1 \end{cases}$.
[CBSE-94]

11. For what value of k is the function $f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.
[AISSE-92C; CBSE-92, 94; PSB-94; HSB-2001]

12. If $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x \neq 0 \\ 4, & x = 0 \end{cases}$ find whether function $f(x)$ is continuous at $x = 0$
[MP-2003; CBSE-97]

13. Given $f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & \text{when } x \neq 0 \\ A, & \text{when } x = 0 \end{cases}$
Find the value of A if $f(x)$ is continuous at $x = 0$. ***[MP-99, PB-93C; HB-89 C]***

14. Discuss the continuity of the function $f(x)$ given by $f(x) = \begin{cases} 2-x, & x < 2 \\ 2+x, & x \geq 2 \end{cases}$ at $x = 2$.
[CBSE-94]

15. If $f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$ Find whether $f(x)$ is continuous at $x = 1$.

[CBSE-92C; PSB-99C]

16. For what value of k is the following function continuous at $x=1$. $f(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & x\neq 1\\ k, & x=1\end{cases}$

[CBSE-94(C)]

17. If $f(x)=\begin{cases}2x^2+k, & \text{if } x\geq 0\\ -2x^2+k, & \text{if } x<0\end{cases}$, then what should be the value of k so that $f(x)$ is continuous at $x = 0$.

[HSB-2001]

18. For what vlaue of k is the function $f(x)$, continuous at $x = 0$ $f(x)=\begin{cases}\dfrac{\sin x + x\cos x}{x}, & x\neq 0\\ k, & x=0\end{cases}$

when $x \neq 0$.

[CBSE (foreign)-94]

LECTUREWISE WARMUP TEST

1. If $f(x)=\begin{cases}\dfrac{\sin^2 ax}{x}, & x\neq 0\\ 1, & x=0\end{cases}$, then

[Bihar CEE-2003]

(a) $f(x)$ is discontinuous at $x = 0$
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is continuous at $x = 0$ if $f(0) = a^2$
(d) alternative (1) and (3)

2. If $f(x)=\begin{cases}\tan^{-1}x & \text{if } -1\leq x\leq 1\\ \frac{1}{2}(-x-1) & \text{if } x<-1\\ \frac{1}{2}(x-1) & \text{if } x>1\end{cases}$, then $f(x)$ is

[IIT (S)-2002]

(a) continuous at $x = -1$ and $x = 1$
(b) discontinuous at $x = -1$ and $x = 1$
(c) continuous at $x = -1$ and discontinuous at $x = 1$
(d) discontinuous at $x = -1$; continuous at $x = 1$

3. If $f(x)=\begin{cases}1+\sin\frac{\pi}{2}x & \text{for } -\infty<x\leq 1\\ ax+b & \text{for } 1<x<3\\ 6\tan\dfrac{x\pi}{12} & \text{for } 3\leq x<6\end{cases}$ is continuous at $x = 1$ and $x = 3$, then $a + b =$

[MP-98]

(a) 2 (b) 3
(c) 0 (d) 1

4. If $f(x)=\begin{cases}x+a\sqrt{2}\sin x, & 0<x<\pi/4\\ 2x\cot x+b, & \pi/4\leq x\leq\pi/2\\ a\cos 2x-b\sin x, & \pi/2<x\leq\pi\end{cases}$

is continuous at $x = \pi/4$, then $a - b$ is equal to

[IIT-89; Roorkee (Screening)-98; AMU-2003]

(a) $\pi/2$ (b) 0
(c) 1/4 (d) $\pi/4$

5. If $f(x)=\begin{cases}\dfrac{2^x-1}{\sqrt{1+x}-1}, & -1\leq x<\infty, x\neq 0\\ k, & x=0\end{cases}$ is continuous everywhere, then k is equal to

[Kerala (CEE)-2003]

(a) $\frac{1}{2}\log 2$ (b) log 4
(c) log 8 (d) log 2

6. If n is any integer then all the points of discontinuity of the function $f(x) = \sec 3x + \text{cosec } 3x$ are given completely by
(a) $x = n\pi$
(b) $x = n\pi/6$
(c) $x = (2n+1)\dfrac{\pi}{3}$
(d) $x = \dfrac{n\pi}{3}$

7. If $f(x) = \begin{cases} \dfrac{(4^x - 1)^3}{\sin\left(\dfrac{x}{a}\right)\log\left(1 + \dfrac{x^2}{3}\right)}, & x \neq 0 \\ 9(\log 4)^3, & x = 0 \end{cases}$ is a continuous function at $x = 0$, then the value of a is equal to ***[Orissa JEE-2004]***

(a) 3 (b) 1
(c) 2 (d) 0

8. The function $f(x) = |x| + \dfrac{|x|}{x}$ is ***[Karnataka CET-2003]***

(a) discontinuous at the origin because $|x|$ is discontinuous there
(b) continuous at the origin
(c) discontinuous at the origin because both $|x|$ and $\dfrac{|x|}{x}$ are discontinuous there
(d) discontinuous at the origin because $\dfrac{|x|}{x}$ is discontinuous there

9. The function $f(x) = p\,[x + 1] + q\,[x - 1]$, where $[x]$ is the greatest integer fucntion is continuous at $x = 1$ if ***[UPSEAT-2001; Orissa (JEE)-2002]***

(a) $p - q = 0$ (b) $p + q = 0$
(c) $p = 0$ (d) $q = 0$

10. Functions $f(x) = [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$ is discontinuous at ***[IIT (Screening)-95; AMU-2003]***

(a) every x
(b) no x
(c) every integral point
(d) every non-integral point

11. If the function

$$f(x) = \begin{cases} x + a^2\sqrt{2}\sin x &, \quad 0 \le x < \pi/4 \\ x\cot x + b &, \quad \pi/4 \le x < \pi/2 \\ b\sin 2x - a\cos 2x &, \quad \pi/2 \le x \le \pi \end{cases}$$

is continuous in the interval $[0, \pi]$, then the values of (a, b) are ***[Roorkee Qualifying-1998]***

(a) $(-1, -1)$ (b) $(0, 0)$
(c) $(-1, 1)$ (d) $(1, -1)$

12. If $f(x) = \dfrac{\log_e x}{x - 1}$, is continuous at $x = 1$, then $f(1) =$ ***[MP CET-1996; Delhi (CEE)-1997]***

(a) 1/2 (b) 0
(c) 1 (d) 2

13. Two function f and g are continuous at $x = a$; and $f + g, f - g, f/g, [g(a) \neq 0]$ and $f \times g$ are also continuous at $x = a$. If a function F is defined as $f(x) = \dfrac{e^x + e^{-x} - 2}{x\sin x}$ on $[-\pi/2, \pi/2]$, then which one of the following is correct? ***[NDA-2003]***

(a) $F(x)$ is continuous on $[-\pi/2, \pi/2]$
(b) $F(x)$ is not continuous on $[-\pi/2, \pi/2]$
(c) $F(x)$ is continuous on $[-\pi/2, \pi/2] - \{0\}$
(d) $F(x)$ is continuous on $(-\pi/2, \pi/2)$

14. Let f be a function defined on R by $f(x) = [x] + \sqrt{x - [x]}$, then

(a) f is not continuous at every $x \in 1$
(b) f is not continuous at every $x \in R \sim 1$
(c) f is a continuous function
(d) None of these

15. The value of k $(k > 0)$ for which the function $f(x) = \dfrac{(e^x - 1)^4}{\sin(x^2/k^2)\log\{1 + (x^2/2)\}}, x \neq 0;\ f(0) = 8$ may be continuous function is ***[IIT Hyderabad-2001]***

(a) 1 (b) 4
(c) 2 (d) 3

16. $f(x) = \begin{cases} (3/x^2)\sin 2x^2 & \text{if} \quad x < 0 \\ \dfrac{x^2 + 2x + c}{1 - 3x^2} & \text{if} \quad x \ge 0, x \neq \dfrac{1}{\sqrt{3}}, \\ 0 & \text{if} \quad x = 1/\sqrt{3} \end{cases}$

then in order that f be continuous at $x = 0$, the value of c is

(a) 2 (b) 4
(c) 6 (d) 8

17. Let $f(x) = [x]$, where $[x]$ denotes the greatest integer contained in x. Consider of the following statements ***[NDA-2004]***

1. $f(x)$ is not onto
2. $f(x)$ is continuous at $x = 0$
3. $f(x)$ is discontinuous for all positive integral values of x

Which of the statements given above are correct?

(a) 1 and 2 (b) 1 and 3
(c) 2 and 3 (d) 1, 2 and 3

18. $f(x) = \dfrac{x}{1+[x]}$, is discontinuous in set 'S' where 'S' is
(a) $\{[-1, 0) \cup \text{Integers}\}$
(b) {all real}
(c) {all positive rational}
(d) None of these

19. Let $f(x) = \sin^{\frac{1}{\pi - 2x}}, x \neq \dfrac{\pi}{2}$. If $f(x)$ is continuous at $x = \dfrac{\pi}{2}$, then $f\left(\dfrac{\pi}{2}\right)$ is
(a) e (b) 1
(c) 0 (d) None of these

LECTUREWISE WARMUP TEST: SOLUTIONS

1. (a) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin^2 ax}{x}$

$= \lim\limits_{x\to 0}\left(\dfrac{\sin ax}{ax}\right)^2 . a^2x$

$= 1\ (0) = 0 \neq f(0) = 1.$

Hence $f(x)$ is discontinuous at $x = 0$.

2. (b) $f(x)$ is discontinuous at $x = -1$ and $x = 1$.

3. (a) $f(x)$ is continuous at $x = 1$ and $x = 3$, Now $f(1) = 2 = f(1 - 0)$
$f(1 + 0) = a + b$, But $f(x)$ is continuous at $x = 1$
$\Rightarrow f(1 - 0) = f(1 + 0) = f(1) \Rightarrow a + b = 2$

4. (d) Continuous at $x = \pi/4$

$\Rightarrow f\left(\dfrac{\pi}{4} - 0\right) = f\left(\dfrac{\pi}{4} + 0\right) = f\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \dfrac{\pi}{4} + a = 2\left(\dfrac{\pi}{4}\right) + b = \dfrac{\pi}{2} + b$

$\Rightarrow a - b = \pi/4$

5. (b) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{(2^x \log 2) - 0}{\dfrac{1}{2\sqrt{1+x}} - 0}$

$= 2\log 2 = \log 4$

6. (b) $\dfrac{2(\sin 3x + \cos 3x)}{\sin 6x}$; $6x = n\pi$

$\Rightarrow x = \dfrac{n\pi}{6}$ $[\because \sin 6x \neq 0]$

7. (a) $\lim\limits_{x\to 0}\left(\dfrac{4^x - 1}{x}\right)^3\left(\dfrac{x}{\sin(x/a)}\right)\dfrac{x^2}{\log\left(1+\dfrac{x^2}{3}\right)}$

$= (\log 4)^3 \times a \times 3 = f(0)$. Hence $a = 3$

8. (d) Let $f(x) = |x| + \dfrac{|x|}{x} = f_1(x) + f_2(x)$ (say) At $x = 0, f_1(0) = 0$

L.H.L. $= \lim\limits_{x\to 0^-} |x| = 0$, R.H.L. $= \lim\limits_{x\to 0^+} |x| = 0$

$\therefore$ $f_1(x)$ is continuous at $x = 0$. Now consider $f_2(x)$ [At $x = 0$]

L.H.L. $= \lim\limits_{x\to 0^-} \dfrac{|x|}{x}$ [Put $x = 0 - h$]

$= \lim\limits_{h\to 0} \dfrac{|0-h|}{0-h} = \lim\limits_{h\to 0} -\dfrac{h}{h} = -1$ Similarly

RHL = 1

$\therefore$ $f_2(x)$ is discontinous at $x = 0$

$\therefore$ $f(x) = |x| + \dfrac{|x|}{x}$ is discontinuous at $x = 0$ because $\dfrac{|x|}{x}$ is discontinuous there.

9. (b) $f(x) = p\,[x + 1] + q\,[x - 1]$ and
$f(1) = p[1 + 1] + q[0] = 2p$
This function will be continuous at $x = 1$, then

$L\lim\limits_{x\to 1} f(x) = R\lim\limits_{x\to 1} f(x) = f(1)$

$\Rightarrow \lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} f(1+h) = f(1)$

$\Rightarrow \lim\limits_{h\to 0} f[1-h+1] + q[1-h-1]$

$= \lim\limits_{h\to 0} p[1+h+1] + q[1+h-1] = f(1)$

$\Rightarrow \lim\limits_{h\to 0} p[2-h] + q[-h]$

$= \lim\limits_{h\to 0} p[2+h] + q[h] = f(1)$

$\Rightarrow \lim\limits_{h\to 0} [p(1-h) + q(-h-1)]$

$= \lim\limits_{h\to 0} [p(1+h) + q(h-1)] = 2p$

$\Rightarrow p - q = 2p \Rightarrow p + q = 0$

10. (b) Obviously function is continuous between two successive integers. Further for any $k \in Z$, $f(k-0) = (k-1)\cos((2k-1)/2)\pi = 0$ $f(k+0) = k\cos((2k-1)/2)\pi = 0$, $f(k) = k\cos((2k-1)/2)\pi = 0$

$\therefore$ f is continuous at all integers.

11. (b) at $x = \frac{\pi}{4}$; L.H.L. $= \frac{\pi}{4} + a^2$

and R.H.L. $= \frac{\pi}{4} + b$

So $\frac{\pi}{4} + a^2 = \frac{\pi}{4} + b$

$\Rightarrow$ $a^2 = b$ ...... (i)

at $x = \frac{\pi}{2}$,

L.H.L. $= b$ and R.H.L. $= a$ So $a = b$ ..... (ii)

From (i) and (ii) $a^2 = a$

$\Rightarrow$ $a^2 - a = 0$

$\Rightarrow$ $a(a-1) = 0 \Rightarrow a = 0, 1$

Hence $b = 0, 1$ Thus $(a, b) = (0, 0)$

12. (c) $\lim\limits_{x\to 1}\frac{\log_e^x}{x-1} = \lim\limits_{x\to 1}\frac{1}{x} = 1$

(By L-hospital rule)

Therefore the value of function is same as the limit of a function.

13. (c) Since function is not defined at $x = 0$ therefore it is not continuous at $x = 0$.

14. (c) $f(x) = n + \sqrt{x-n}, n \le x < n+1$

If $x_0 = k \in I$ then

$\lim\limits_{x\to k^+} f(x) = \lim\limits_{x\to k} k + \sqrt{x-k)} = k$ and

$\lim\limits_{x\to k^-} f(x) = \lim\limits_{x\to k^-} (k-1) + \sqrt{x-(k-1)}$

$= \lim\limits_{x\to k} (k-1) + \sqrt{x-(k+1)} = k - 1 + 1 = k$

Hence f is continuous at every $x_0 = k \in I$. If $x_0 \in R \propto I$ then $[x]$ is continuous at x_0 which in turn gives that f is continuous at every $x_0 \in R \sim I$.

15. (c) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\left(\frac{e^x - 1}{x}\right)^4 \times$

$$\frac{x^2/k^2}{\sin(x^2/k^2)} \times \frac{x^2/2}{\log(1+x^2/2)} \times 2k^2$$

$= 2k^2$. So $2k^2 = 8 \Rightarrow k = \pm 2$

16. (c) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{3}{x^2}\sin 2x^2$

$= 6\lim\limits_{x\to 0}\frac{\sin 2x^2}{2x^2} = 6$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\frac{x^2 + 2x + c}{1 - 3x^2} = \frac{c}{1} = c$

Hence for f to be continuous $c = 6$.

17. (b) $\because R_f = Z \in R$, hence f is not onto. Also f is discontinuous at every integer, hence (a) and (c) are correct.

18. (a) $f(x) = \frac{x}{n+1}, n \le x < n+1$

(i) Clearly, if $n = -1$ then $f(x)$ is not defined i.e., $f(x)$ is not defined $\forall\, x \in [-1, 0]$.

(ii) If $n \ne -1$, $f(x) = \frac{-x}{2}$; $-3 \le x < -2$ $(n = -3)$

$= -x$; $-2 \le x < -1$ $(n = 2)$

$= x$; $0 \le x < 1$ $(n = 0)$

$= x/2$; $1 \le x < 2$ $(n = 1)$

= ....

Clearly, the function is discontinuous at all integral values of x

From (i) and (ii) $f(x)$ is not continuous in set 'S'. $S = \{[-1, 0) \cup$ Integers $\}$

19. (b) $f(\pi/2) = \lim\limits_{x\to\pi/2} f(x)$

$= \lim\limits_{h\to 0} f\left(\frac{\pi}{2} - h\right) = \lim\limits_{h\to 0} f\left(\frac{\pi}{2} + h\right)$

But $\lim\limits_{h\to 0} f\left(\frac{\pi}{2} - h\right) = \lim\limits_{h\to 0} (\cos h)^{1/2h}$ $[1^\infty$ form$]$

$e^{\lim\limits_{h\to 0}(\cos h)^{1/2h}} = e^0 = 1$

ANSWERS

LECTURE 1

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (b) **2.** (b) **3.** (d) **4.** (a) **5.** (c) **6.** (a) **7.** (d) **8.** (a) **9.** (d) **10.** (d) **11.** (a) **12.** (a) **13.** (c) **14.** (c) **15.** (a)

LECTURE 2

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (d) **2.** (c) **3.** (c) **4.** (d) **5.** (b) **6.** (d) **7.** (c) **8.** (d) **9.** (c) **10.** (d) **11.** (c)

Worksheet: To Check the Preparation Level

1. (b) **2.** (c) **3.** (c) **4.** (a) **5.** (c) **6.** (c) **7.** (d) **8.** (d) **9.** (b) **10.** (a) **11.** (b) **12.** (c) **13.** (a) **14.** (a)

LECTURE 3

Mental Preparation Test

1. Not continuous
2. Continuous
3. $k = 6$
5. $k = 5/3$
6. Continuous except at $x = 0$
7. Discontinuous
9. Discontinuous
10. Continuous
11. 2
12. Discontinuous
13. 1/2
14. Not continuous
15. Discontinuous
16. 2
17. k is any real number
18. k is any real number

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Part D

Differentiability

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LECTURE

1

Existence of Derivatives

BASIC CONCEPTS

1. **DERIVATIVE AT A POINT** If $f(x)$ is a function of x defined over the open interval (a, b) and if $x = c$ is any point of this interval; then $f(x)$ is said to be differentiable at the point $x = c$ iff $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ or $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists finitely;
The value of this limit is called the derivative (or differential coefficient) of $f(x)$ at the point $x = c$ and is denoted by: $f'(c); Df(c)$ or $\left[\frac{d}{dx}[f(x)]\right]_{x=c}$

2. **PROGRESSIVE AND REGRESSIVE DERIVATIVES** The progressive derivative or right hand derivative of $f(x)$ at $x = c$ is given by: $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}; h > 0$
If it exists finitely and is denoted by $Rf'(c)$ or by $f'(c + 0)$.
The regressive derivative or Left Hand Derivative of $f(x)$ at $x = c$ is given by:
$\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}; h > 0$
If it exists finitely and is denoted by $Lf'(c)$ or $f'(c - 0)$.

NOTES

The function $f(x)$ is said to be differentiable at $x = c$ if $Rf'(c)$ and $Lf'(c)$ both exists finitely and are equal; and their common value is called the derivative or differential coefficient at the point $x = c$. The function $f(x)$ is said to be non differentiable at $x = c$ if;

(i) Either or both $Rf'(c)$ and $Lf'(c)$ do not exist.
(ii) Both $Rf'(c)$ and $Lf'(c)$ exist but are not equal.
(iii) Either or both $Rf'(c)$ and $Lf'(c)$ are not finite.

3. **THEOREM Continuity** is a necessary but not a sufficient condition for the existence of a finite derivative (If a function f, has a finite derivative at a point it must be continuous at the point but continuity does not imply derivability)
But the converse of the theorem is not true:
Example 1: Prove that $f(x) = |x|$ is continuous at $x = 0$; but not differentiable at $x = 0$: we know that $\lim_{x\to 0}|x| = 0$ but
$f'(0 + h) = 1; f'(0 - h) = -1$
Left hand derivative $\neq$ Right hand derivative
Hence $f(x) = |x|$ is not differentiable at $x = 0$
Example 2: $f(x) = x\sin\frac{1}{x}$ is continuous at $x = 0$ but not differentiable at $x = 0$.
Example 3: $f(x) = |x| + |x - 1|$ is continuous at $x = 0$ and $x = 1$ but not differentiable at $x = 0$ and 1.

4. **EXISTENCE OF DERIVATIVE (differentiability on an interval)**:
The function $y = f(x)$ is said to be differentiable in the closed interval $[a, b]$, if:

(i) If f is differentiable at every point in the open interval (a, b) then $f(x)$ is said to be differentiable in the open interval (a, b).

(ii) For the points 'a' and 'b' if it is differentiable from the right at a and from the left at b or $f'(a+)$ and $f'(b-)$ both exist.

i.e., $\lim_{x\to a+} \dfrac{f(x)-f(a)}{x-a}$ and $\lim_{x\to b-} \dfrac{f(x)-f(b)}{x-b}$

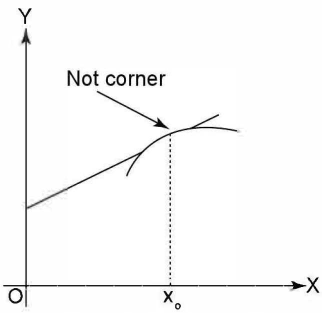
or $\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ and

$\lim_{h\to 0} \dfrac{f(b-h)-f(b)}{-h}$ both exist, then $f(x)$ is said to be differentiable in the closed interval $[a, b]$.

5. Graphical definition of Derivability

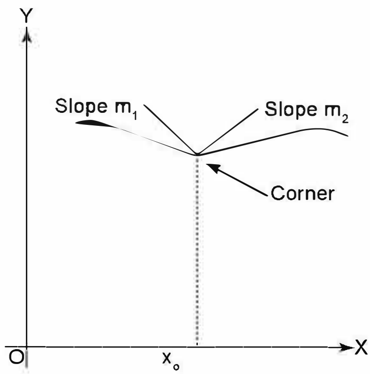
The function $y = f(x)$ is derivable if its graph is always smooth i.e., there should be no break or corner.

What is corner: If a curve has not unique tangent at a point then it is called its corner.



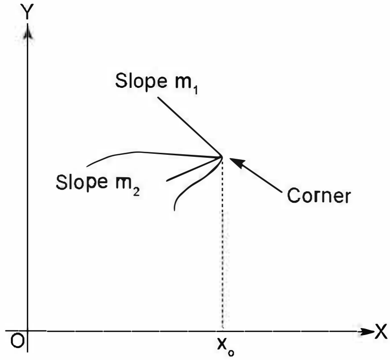
(Differentiable at $x = x_0$)

(a)



(Non differentiable at $x = x_0$) ($m_1 \neq m_2$)

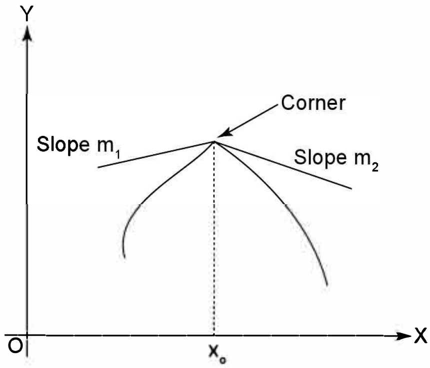
(b)



(Non differentiable at $x = x_0$)

($m_1 \neq m_2$)

(c)



(Non differentiable at $x = x_0$)

($m_1 \neq m_2$)

(d)

6. Derivability and Continuity

(i) If $Rf'(x_0)$ and $Lf'(x_0)$ exist finitely (both may or may not be equal) then $f(x)$ is continuous at $x = x_0$.

(ii) The converse of the above result (i) is not true

i.e., if $f(x)$ is continuous at $x = x_0$ then it may or may not be differentiable at $x = x_0$.

(iii) For a function $f(x)$

(a) Differentiable $\Rightarrow$ Continuous

(b) Not differentiable $\Rightarrow$ Not continuous

(c) Not continuous $\Rightarrow$ Not differentiable.

7. Solving the problems (Based on continuity and differentiability from following types:

Graphical Method

If graph of $f(x)$ has no break and no corner at $x = x_0$. Then $f(x)$ is differentiable at $x = x_0$ and if graph of $f(x)$ has no break or gap or hole at $x = x_0$ then $f(x)$ is continuous at $x = x_0$.

8. Important Results to Remember

1.

$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x).g(x)$
Differentiable	Differentiable	Differentiable	Differentiable
Differentiable	Non differentiable	Non differentiable	May be differentiable or non differentiable
Non differentiable	Non differentiable	May be differentiable or non differentiable	May be differentiable or non differentiable

2. If $f(x)$ and $g(x)$ are inverse functions of each other then $f'(x) = \dfrac{1}{f'(g(x))}$ and $f'(x) = \dfrac{1}{g'(f(x))}$

where $f'(g(x)) = \dfrac{df(g(x))}{d(g(x))}$

3. Derivative of an identity function is an identity function.
4. Every polynomial function is differentiable at each $x \in R$.
5. Every constant function is differentiable at each $x \in R$
6. The exponential function e^x, a^x ($a > 0$ and $a \neq 1$) are differentiable at each $x \in R$.
7. The logarithmic functions $\ln x$, $\log_a x$ ($a > 0$ and $a \neq 1$) are differentiable at each point in their domain.
8. The trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$ are differentiable in their domain.
9. The inverse trigonometric functions; $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$ are differentiable in their domain.
10. The composition of differentiable functions is also differentiable function.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. $f(x) = \begin{cases} ax^2 - b, & |x| < 1 \\ -\dfrac{1}{|x|}, & |x| \geq 1 \end{cases}$. The above function is continuous and differentiable, then prove that $a = 1/2$, $b = 3/2$.

Solution

$$f(x) = \begin{cases} -\dfrac{1}{|x|}, & |x| \geq 1 \text{ or } x^2 - 1 \geq 0 \\ & \text{or } x \leq -1, x \geq 1 \\ ax^2 - b, & |x| < 1 \text{ or } x^2 - 1 < 0 \end{cases} \qquad \ldots\ldots(1)$$

or $-1 < x < 1$ .... (2)

Also $|x| = x$ when $x = +\text{ve}$ .... (3)

$|x| = -x$ when $x = -\text{ve}$ .... (4)

Hence we redefine the function as under

$$f(x) = \left\{\begin{array}{ll} ax^2 - b, & x < 1 \text{ by } (2) \\ -\dfrac{1}{+x} = -\dfrac{1}{x}, & x \geq \text{ by } (1) \end{array}\right\} \qquad \ldots\ldots(A)$$

$$f(x) = \left\{\begin{array}{ll} -\dfrac{1}{-x} = \dfrac{1}{x}, & x \leq -1 \text{ by } (1) \\ ax^2 - b, & x > -1 \text{ by } (2) \end{array}\right\} \qquad \ldots\ldots(B)$$

At $x = 1$, for continuity $R = L = V$

$\therefore \quad a - b = 1$

For differentiability $R' = L'$

$\therefore \quad 2a = 1$ Solving $a = 1/2$, $b = 3/2$

At $x = -1$, for continuity $R = L = V$

$\therefore \quad a - b = -1$

For differentiability $R' = L' - 1 = -2a$

$\therefore \quad a = 1/2$, $b = 1/3$

2. Prove that $f(x) = |\log x|$ is continuous at $x = 1$ but is not differentiable at $x = 1$.

Solution

We know that $\log t = 0$, if $t = 1$; > 0 if $t > 1$, < 0, if $t < 1$, $|y| = y$, if $y > 0$, $|y| = -y$, if $y < 0$

R.H.L. $= \underset{h \to 0}{Lt} |\log(1 + h)|$

$= \underset{h \to 0}{Lt} \log(1 + h) = 0$

L.H.L $= \underset{h \to 0}{Lt} |\log(1 - h)|$

$= \underset{h \to 0}{Lt} -[\log(1 - h)] = 0$

$V = |\log 1| = 0.\quad \therefore R = L = V \quad \therefore$ continuous

$$R' = \underset{h\to 0}{Lt}\frac{|\log(1+h)| - |\log 1|}{h}$$

$$= \underset{h\to 0}{Lt}\frac{\log(1+h)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{h - \frac{h^2}{2} + \dots\dots\dots}{h} = h = 0$$

$L' = -1$ as above. Since $R' \neq L'$ therefore the function is not differentiable at $x = 1$.

3. Discuss the limit, continuity and differentiability of the function $f(x) = \begin{cases} \dfrac{x(3e^{1/x}+4)}{(2-e^{1/x})} & x \neq 0 \\ 0 & x = 0 \end{cases}$

at $x = 0$

[Roorkee-1995]

Solution

For continuity, $R = L = V$ at $x = 0$

$Lt\, e^{1/h} = \infty, Lt\ 1/e^{1/h} = 0$

$$R = \underset{h\to 0}{Lt}\frac{h(3e^{1/h}+4)}{2-e^{1/h}}$$

$$= Lt\frac{h\left(3+\frac{4}{e^{1/h}}\right)}{\left(\frac{2}{e^{1/h}}-1\right)} = Lth\left(\frac{3}{-1}\right) = 0$$

$$L = \underset{h\to 0}{Lt} - \frac{h(3e^{-1/h}+4)}{2-e^{-1/h}}$$

$$= \underset{h\to 0}{Lt} - h\left(\frac{4}{2}\right) = 0$$

Since R.H.L t = L. H. L t

$\therefore$ Lt exists at $x = 0$ and limit is equal to value and hence the function is continuous.

Differentiability:

$$R'\, \underset{h\to 0}{Lt}\frac{f(0+h)-f(0)}{h},$$

$$L' = \underset{h\to 0}{Lt}\frac{f(0-h)-f(0)}{-h}$$

$$R = \underset{h\to 0}{Lt}\frac{1}{h}\left[\frac{h(3e^{1/h}+4)}{(2-e^{1/h})} - 0\right] = \frac{3}{-1}$$

$$L = \underset{h\to 0}{Lt} - \frac{1}{h}\left[\frac{-h(3e^{-1/h}+4)}{(2-e^{-1/h})}\right] = \frac{4}{2} = 2$$

Since $R' \neq L'$ therefore the given function is not differentiable at $x = 0$.

4. Determine the values of x for which the following function fails to be continuous or differentiable:

$$f(x) = \begin{cases} 1-x, & x < 1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ 3-x, & x > 2 \end{cases}$$ Justify your answer.

[IIT Re-ex.-1997]

Solution

Consider $x = 1, 2$

$Lt\, f(1-h) = \underset{h\to 0}{Lt}\, 1-(1-h) = Lt\, h = 0$

$Lt\, f(1+h) = \underset{h\to 0}{Lt} - h(1-h) = 0, f(1) = 0$

Since $L = R = V = 0$

$\therefore$ f is continuous at $x = 1$

Consider differentiability

$$L'\ f'(1) = \underset{h\to 0}{Lt}\frac{f(1-h)-f(1)}{-h} = \frac{h-0}{-h} = -1$$

$$R'\ f'(1) = \underset{h\to 0}{Lt}\frac{f(1+h)-f(1)}{h}$$

$$= \frac{-h(1-h)-0}{h} = -1$$

$\therefore$ $f(x)$ is differentiable at $x = 1$

At $x = 2\quad L = 0\quad R = 1\quad V = 0$

Hence $f(x)$ is not continuous at $x = 2$ and as such it is not differentiable also at $x = 2$.

5. Show that the function f defined by

$f(x) = x\left\{1+\frac{1}{3}\sin(\log x^2)\right\}, x \neq 0,$ and $f(0) = 0$, is everywhere continuous but has no differential coefficient at the origin.

Solution

We consider $x = 0$, we have $f(0) = 0$.

$$f(0+0) = \lim_{h\to 0}\left\{(0+h)\left(1+\frac{1}{3}\sin\log(0+h)^2\right)\right\}$$

$$= \lim_{h\to 0}[h + (h/3)\sin\log h^2]$$

$= 0 + 0 \times a$ finite quantity $= 0$

[$\because$ sin log h^2 oscillates between -1 and 1 as $h \to 0$ hence finite]

and

$$f(0-0) = \lim_{h\to 0}(0-h).\left\{1+\frac{1}{3}\sin\log(0-h)^2\right\}$$

$$= \lim_{h\to 0}[-h-(h/3)\sin\log h^2] = 0 \text{ as before.}$$

Hence f is continuous at $x = 0$.

Now

$$Lf'(0) = \lim_{h\to 0} \frac{(0-h)\left\{1+\frac{1}{3}\sin\log(0-h)^2\right\}-0}{-h}$$

$$= \lim_{h\to 0}\left[1+\frac{1}{3}\sin\log h^2\right]$$

Now sin log h^2 oscillates between -1 and 1 as $h \to 0$ so that $\lim_{h\to 0}\sin(\log h^2)$ does not exist.

Hence $Lf'(0)$ does not exist.

Similarly

$$Rf'(0) = \lim_{h\to 0} \frac{(0+h)\left\{1+\frac{1}{3}\sin\log(0+h)^2\right\}-0}{h}$$

$= \lim_{h\to 0}\left\{1+\frac{1}{3}\sin\log h^2\right\}$, which does not exist as before

Hence f has no differential coefficient at $x = 0$.

6. $f(x+y+z) = f(x)f(y)f(z) \in x, y, z \in R$. If $f(2) = 4$, $f'(0) = 3$, then prove that $f(0) = 1$ and $f'(2) = 12$.

Solution

Put $y = z = -1$ $\therefore f(x-2) = f(x)f(-1)f(-1) = f(x)\,[f(-)]^2$

Now put $x = 2$ to find $f(0)$

$\therefore$ $f(0) = f(2)\,[f(-1)]^2 = 4\,[f(-1)]^2 = +$ ive .... (1)

Above shows that $f(0)$ is + ve

Now put $y = 0, z = 0$ $\therefore f(x) = f(x)\,\{f(0)\}^2$

$\therefore$ $[f(0)]^2 = 1$

$\therefore$ $f(0) = \pm 1$ but by (1) $f(0)$ is + ive

$\therefore$ we choose $f(0) = 1$ ...... (2)

Now we have to find $f'(2)$. Put $y = 2, z = 0$.

$\therefore$ $f(x+2) = f(x)f(2)\,.\,f(0) = 4f(x)$ by (2)

$\therefore$ $f'(x+2) = 4f'(x)$

Now put $x = 0$. $f'(2) = 4f'(0) = 4\,.\,3 = 12$

7. Let $f(x) = \begin{cases} (x-1)^2\sin\dfrac{1}{(x-1)} - |x| & \text{if } x \neq 1 \\ -1 & \text{if } x = 1 \end{cases}$

be a real-valued function. Then the set of points where $f(x)$ is not differentiable is .... ***[IIT-1981]***

Solution

Given

$$f(x) = \begin{cases} (x-1)^2\sin\dfrac{1}{(x-1)} - |x|, & x \neq 1 \\ -1, & x = 1 \end{cases}$$

We know that $|x|$ is not differentiable at $x = 0$

$\therefore$ $(x-1)^2\sin\dfrac{1}{x-1} - |x|$ is not differentiable at $x = 0$.

At all other values of x, $f(x)$ is differentiable.

$\therefore$ The req. set of points is $\{0\}$.

8. Let $f(x) = x\,|x|$. The set of points where $f(x)$ is twice differentiable is .... ***[IIT-1992]***

Solution

We have

$$f(x) = x\,|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

$$f'(x) = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases},$$

$$f''(x) = \begin{cases} -2, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

Clearly $f''(x)$ exists at every point except at $x = 0$

Thus $f(x)$ is twice differentiable on R $-$ $\{0\}$.

9. Let $f(x) = \dfrac{x^2}{2}$, $0 \leq x < 1$

$= 2x^2 - 3x + \dfrac{3}{2}$, $1 \leq x \leq 2$ Discuss the continuity of f, f' and f'' on $[0, 2]$. ***[IIT-1983]***

Solution

f and f' are continuous and f'' is discontinuous on $[0, 2]$

We have $f(x) = \dfrac{x^2}{2}, 0 \leq x < 1$

$$= 2x^2 - 3x + \frac{3}{2}, 1 \leq x \leq 2$$

Here $f(x)$ is continuous everywhere except possibly at $x = 1$

At $x = 1$ $Lf' = \dfrac{2}{2}\times 1 = 1$; $Rf' = 4\times 1 - 3 = 1$

$\Rightarrow$ f is differentiable and hence continuous at $x = 1$

$\therefore$ $f(x)$ is continuous on $[0, 2]$

$f'(x) = x, 0 \le x < 1$

$= 4x - 3, 1 \le x \le 2$

At $x = 1$,

$\lim_{x\to 1^-} f'(x) = \lim_{h\to 0} f'(1-h) = \lim_{h\to 0} (1-h) = 1$

$\lim_{x\to 1^+} f'(x) = \lim_{h\to 0} f'(1+h) = \lim_{h\to 0} 4(1+h) - 3 = 1$

$= f'(1) = 4 - 3 = 1$

$\therefore$ f' is continuos at $x = 1$,

$\therefore$ f' is continuos on $[0, 2]$

$$f''(x) = \begin{cases} 2, & 0 \le x < 1 \\ 4, & 1 \le x \le 2 \end{cases}$$

Clearly $f''(x)$ is discontinuous at $x = 1$.

$\therefore f''(x)$ is discontinuos on $[0, 2]$

10. Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, find its value. ***[IIT-1987]***

Solution

$f'(0) = 0$

Given that $f(x)$ is a function satifying $f(-x) = f(x)$, $\forall x \in R$ ....(1)

Also $f'(0)$ exists

$\Rightarrow$ $f'(0) = Rf'(0) = Lf'(0)$

Now, $Rf'(0) = f'(0)$

$$\Rightarrow \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f'(0)$$

Agian $Lf'(0) = f'(0)$ .....(2)

$$\Rightarrow \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(-h) - f(0)}{-h} = f'(0)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(h) - f(0)}{h} = -f'(0) \quad ...(3)$$

[Using eq. (1)]

From equations (2) and (3) we get, $f'(0) = -f'(0)$

$\Rightarrow$ $2f'(0) = 0$

$\Rightarrow$ $f'(0) = 0$

Alliter: A function satisfying $f(-x) = f(x)$ is $f(x) = \cos x$

Therefore, $f'(0) = -\sin 0 = 0$

11. Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α if and only if there is a function $g : R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$. ***[IIT-2001]***

Solution

Let us first prove that

(I) g is continuous at α and $f(x) - f(\alpha) = g(x)(x - \alpha)$, $\forall x \in R$

$\Rightarrow$ $f(x)$ is differentiable at α

Since g is continuous at $x = \alpha$ and

$$g(x) = \frac{f(x) - f(\alpha)}{x - \alpha}$$

We should have, $\lim_{x\to\alpha} g(x) = g(\alpha)$

$$\Rightarrow \lim_{x\to\alpha} \frac{f(x) - f(x)}{x - \alpha} = g(\alpha) \Rightarrow f'(x) = g(\alpha)$$

$\Rightarrow$ $f'(\alpha)$ exists and is equal to $g(\alpha)$.

Conversely now we prove.

(II) $f(x)$ is differentiable at $x = \alpha$

$\Rightarrow$ g is cont. at $x = \alpha$ and

$f(x) - f(\alpha) = g(x)(x - \alpha) \forall x \in R$

$\because$ $f(x)$ is differentiable at $x = \alpha$

$\therefore$ $\lim_{x\to\alpha} \dfrac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha)$ exists and is finite.

Let us define, $g(x) = \begin{cases} \dfrac{f(x) - f(\alpha)}{x - \alpha}, & x \ne \alpha \\ f'(\alpha), & x = \alpha \end{cases}$

Then, $f(x) - f(\alpha) = (x - \alpha) g(x)$, $\forall x \ne \alpha$

Now for continuity of $g(x)$ at $x = \alpha$

$$\lim_{x\to\alpha} g(x) = \lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha) = g(\alpha)$$

$\therefore$ g is continuous at $x = \alpha$.

12. If $f(x - y) = f(x) \,.\, g(y) - f(y) . g(x)$ and $g(x - y) = g(x). g(y) + f(x) \,.\, f(y)$ for all $x, y \in R$. If right hand derivative at $x = 0$ exists for $f(x)$. Find derivative of $g(x)$ at $x = 0$. ***[IIT-2005]***

Solution

Given that, $f(x - y) = f(x), g(y) - f(y) \,.\, g(x)$ ....(i)

$g(x - y) = g(x) \,.\, g(y) + f(x)\, f(y)$ ....(ii)

In eqn. (i) putting $x = y$ we get $f(0) = f(x) g(x) - f(x) g(x) \Rightarrow f(0) = 0$

Putting $y = 0$ in eqn. (i), we get

$f(x) = f(x)\, g(0) - f(0)\, g(x)$

$\Rightarrow \quad f(x) = f(x)\, g(0)$

$\Rightarrow \quad g(0) = 1$ [Using $f(0) = 0$]

putting $x = y$ in eqn. (ii), we get $g(0) = g(x)\, g(x) + f(x) f(x)$

$\Rightarrow \quad 1 = [g(x)]^2 + [f(x)]^2$ [Using $g(0) = 1$]

$\Rightarrow \quad [g(x)]^2 = 1 - [f(x)]^2$ ....(iii)

Clearly $g(x)$ will be differentiable only if $f(x)$ is differetiable.

$\therefore$ First we will check the differentiability of $f(x)$

Given that $Rf'(0)$ exists

i.e., $\lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ exists

i.e., $\lim\limits_{h\to 0} \dfrac{f(0)g(-h) - f(-h)g(0)}{h}$ exists

i.e., $\lim\limits_{h\to 0} \dfrac{-f(-h)}{h}$ exists (using $f(0) = 0$ and $g(0) = 1$)

which can be written as,

$$\lim_{h\to 0} \frac{f(0) - f(-h)}{-h} = Lf'(0)$$

$\Rightarrow \quad Lf'(0) = Rf'(0)$

$\therefore$ f is differentiable, at $x = 0$. Differentiating equation (iii) we get $2g(x) \,.\, g'(x) = -2f(x) \,.\, f'(x)$

For $x = 0$

$\Rightarrow \quad g(0).\, g'(0) = -f(0) f'(0)$

$\Rightarrow \quad g'(0) = 0$ [Using $f(0) = 0$ and $g(0) = 1$]

Alliter: Easily we can conclude that $f(x) = a \sin x$, $g(x) = b \cos x$

Therefore, $g'(x) = -\, b \sin x$

$\Rightarrow \quad g'(0) = 0$

[redacted]
TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Prove that the following function is continuous at $x = 0$, but not differentiable at $x = 0$.

$$f(x) = \begin{cases} x\cos\dfrac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

2. If $f(x)$ is differentiable at $x = a$, find

$$\lim_{x\to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$$

[PSB-96 (C)]; [Problem based on this appeared in IIT JEE-2007]

3. For what choice of a and b is the function $f(x) = \begin{cases} x^2, & x \le c \\ ax + b, & x > c \end{cases}$ is differentiable at $x = c$. ***[PSB-96]***

4. Show that the function f is continuous at $x = 1$ for all a, where $f(x) = \begin{cases} ax^2 + 1, & x \ge 1 \\ x + a, & x < 1 \end{cases}$. Find its left hand derivative at $x = 1$. Hence find the condition for the existence of derivative at $x = 1$.

[NCERT Book]

5. Show that $f(x) = \begin{cases} \dfrac{1}{1 + e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is not continuous at 0 and not differentiable at 0.

[BITS Ranchi-1999]

6. Show that $f(x) = x^{1/3}$ is not differentiable at $x = 0$.

7. Show that $f(x) = |\, x - 2|$ is continuous but not differentiable at $x = 2$.

8. Discuss the differentiability of $f(x) = |\log_e x|$ for $x > 0$.

9. If $f(x) = x^2 + 2x + 7$ find $f'(3)$.

10. For the function f given by $f(x) = x^2 - 6x + 8$. Prove that $f'(5) - 3f'(2) = f'(8)$.

11. Discuss the continuity and differentiability of $f(x) = e^{|x|}$.

12. If $f(x)=\begin{cases}2+x, & x\ge 0\\ 2-x, & x<0\end{cases}$. Then prove that $f(x)$ is not differentiable at $x=0$

13. If $f(2)=4$ and $f'(2)=1$, then find $\lim\limits_{x\to 2}\dfrac{xf(2)-2f(x)}{x-2}$.

14. Find the value of a and b so that the function $f(x)=\begin{cases}x^2+3x+a, & \text{if } x\le 1\\ bx+2, & \text{if } x>1\end{cases}$ is differentiable at each $x\in R$.

15. Show that the function $f(x)=\begin{cases}|2x-3|\,[x], & x\ge 1\\ \sin(\pi x/2), & x<1\end{cases}$ is continuous but not differentiable at $x=1$.

16. Examine the continuity and differentiability of the following function

$$f(x)=\begin{cases}(x-a)\sin\left(\dfrac{1}{x-a}\right), & x\ne a\\ 0, & x=a\end{cases}$$

at $x=a$.

17. If $f(x+y)=f(x)f(y)\ \forall\ x$ and y, then find $f'(5)$. Given $f(5)=2$ and $f'(0)=3$.

[IIT-1981]

18. $f(x)=\begin{cases}\dfrac{x}{1+e^{1/x}}, & x\ne 0\\ 0, & x=0\end{cases}$. Find the right hand derivative $f'(0^+)$

[IIT-1983]

ANSWERS

2. $2a\,f(a)-a^2 f'(a)$

3. $a=2c,\ b=-c^2$.

4. $Lf'(1)=1,\ Rf'(1)=2a$; $a=\dfrac{1}{2}$

9. 8

11. Not differentiable at $x=0$.

13. 2

14. $a=3,\ b=5$

16. Continuous but not differentiable at a.

17. 6

18. 0

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The function $f(x)$ is defined as

$$f(x)=\frac{1}{3}-x,\ x<\frac{1}{3}=\left(\frac{1}{3}-x\right)^2,\ x\ge\frac{1}{3}$$

then in the interval (0, 1), the mean value, theorem is not true because

[Screening-2003]

(a) $f(x)$ is not continuous

(b) $f(x)$ is not differentiable

(c) $f(0)\ne f(1)$

(d) None of these

Solution

(b), (c) The function given in (a) is not differentiable at $x=\frac{1}{3}$ in [0, 1] as $Lf'(1)=1$ and $Rf'(1)=-1$.

Hence it does not satisfy the mean value theorem $\Rightarrow$ (b)

Also $f(0)\ne f(1)\Rightarrow$ (c)

2. The function $f(x)=(x^2-1)\,|x^2-3x+2|+\cos(|x|)$ is NOT differentiable at ***[IIT-1999]***

(a) -1 (b) 0
(c) 1 (d) 2

Solution

We have $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

$\Rightarrow \quad |x^2 - 3x + 2| = |(x-1)(x-2)|$

$$= \begin{cases} (1-x)(x-2) & \text{if} \quad x \leq 1 \\ (x-1)(2-x) & \text{if} \quad 1 < x \leq 2 \\ (x-1)(x-2) & \text{if} \quad x \geq 2 \end{cases}$$

As $\cos(-\theta) = \cos\theta \Rightarrow \cos|x| = \cos x$

$\therefore$ Given function can be written as

$$\therefore \ f(x) = \begin{cases} (x^2-1)(x-1)(x-2) + \cos x & \text{if} \quad x \leq 1 \\ -(x^2-1)(x-1)(x-2) + \cos x & \text{if} \quad 1 < x \leq 2 \\ (x^2-1)(x-1)(x-2) + \cos x & \text{if} \quad x > 2 \end{cases}$$

This function is differentiable at all points except possibly at
$x = 1$ and $x = 2$.

$$Lf'(1) = \left\{\frac{d}{dx}[(x^2-1)(x-1)(x-2) + \cos x]\right\}_{x=1}$$

$= -\sin 1$

$$Rf'(1) = \left\{\frac{d}{dx}(-(x^2-1)(x-1)(x-2) + \cos x)\right\}_{x=1}$$

$= -\sin 1$

$\because \quad Lf'(1) = Rf'(1) \ \therefore f$ is differentiable at $x = 1$.

$$Lf'(2) = \left\{\frac{d}{dx}(-(x^2-1)(x-1)(x-2) + \cos x)\right\}_{x=2}$$

$= -3 - \sin 2$

$$Rf'(2) = \left\{\frac{d}{dx}((x^2-1)(x-1)(x-2) + \cos x)\right\}_{x=2}$$

$= 3 - \sin 2$

$\therefore \quad Lf'(2) \neq Rf'(2)$

$\therefore \quad f$ is not differentiable at $x = 2$.

3. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x. Then g is

[IIT Sc.-2000]

(a) onto if f is onto
(b) one-one if f is one-one
(c) continuous if f is continuous
(d) differentiable if f is differentiable

Solution

(c) f is continuous at $x = a$ if limit = value

i.e., $\lim\limits_{h \to 0} f(a-h) = \lim\limits_{h \to 0} f(a+h) = f(a)$.

$g(x) = |f(x)|$

$g(a-0) = \lim\limits_{h \to 0} g(a-h)$

$= \lim\limits_{h \to 0} |f(a-h)| = |f(a)|$

Similarly $g(a+0) = \lim\limits_{h \to 0} g(a+h)$

$= \lim\limits_{h \to 0} |f(a+h)| = |f(a)| = -g(a)$

$\therefore \quad g$ is continuous at $x = a$.
Hence g is continuous if f is continuous.

4. There exists a function $f(x)$ satisfying $f(0) = 1$, $f'(0) = -1$, $f(x) > 0$ for all x and

[IIT-1982, Kurukshetra CEE-1998]

(a) $f'(x) < 0$ for all x
(b) $-1 < f''(x) < 0$ for all x
(c) $-2 \leq f''(x) \leq -1$ for all x
s (d) $f''(x) < -2$ for all x.

Solution

(a) $f(x) = e^{-x}$ is one such function. It can be shown that there exists no function satisfying the conditions (b) or (c) or (d) and the given conditions $f(0) = 1$, $f'(0) = -1$, $f(x) > 0$ for all x.

5. If $f(x) = \begin{cases} e^x; & x \leq 0 \\ |1-x|; & x > 0 \end{cases}$, then

[Roorkee-1995]

(a) $f(x)$ is differentiable at $x = 0$
(b) $f(x)$ is dis-continuous at $x = 0$
(c) $f(x)$ is differentiable at $x = 1$
(d) $f(x)$ is continuous at $x = 1$

Solution

(d) $\quad f(x) = \begin{cases} e^x & ; \quad x \leq 0 \\ 1-x & ; \quad 0 < x \leq 1 \\ x-1 & ; \quad x > 1 \end{cases}$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{1-h-1}{h} = -1$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{e^{-h} - 1}{-h} = 1$$

So, it is not differentiable at $x = 0$.
Similarly, it is not differentiable at $x = 1$
But it is continuous at $x = 0, 1$

6. If $f(x) = \begin{cases} x\dfrac{e^{(1/x)} - e^{(-1/x)}}{e^{(1/x)} + e^{(-1/x)}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then which of the following is true

[Kurukshetra CEE-1998]

(a) f is continuous and differentiable at every point
(b) f is continuous at every point but is not differentiable
(c) f is differentiable at every point
(d) f is differentiable only at the origin.

Solution

(b) $f(0+0) = \lim_{h \to 0} f(x) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0} (0+h) \frac{e^{1/0+h} - e^{-1/0+h}}{e^{1/0+h} + e^{-1/0+h}}$$

$$= \lim_{h \to 0} h \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} = 0$$

and $f(0-0) = \lim_{h \to 0} f(0-h)$

$$= \lim_{h \to 0} - h \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} = 0$$

and $f(0) = 0$;

$\therefore \quad f(0+0) = f(0-0) = f(0)$

Hence f is continuous at $x = 0$.

At remaining points $f(x)$ is obviously continuous.

Thus it is everywhere continuous.

Again, $Lf'(0) = \lim_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{h.\dfrac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} - 0}{-h} = -1$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h\dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}}{h} = 1$$

$\because \quad Lf'(0) \neq Rf'(0)$

$\therefore \quad f$ is not differentiable at $x = 0$.

7. If $f(x) = x(\sqrt{x} - \sqrt{x+1})$, then

[IIT-1985]

(a) $f(x)$ is continuous but not differentiable at $x = 0$
(b) $f(x)$ is differentiable at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) None of these

Solution

(b) We have $f(x) = x(\sqrt{x} - \sqrt{x+1})$

Let us check differentiability of $f(x)$ at $x = 0$

$$Lf'(0) = \lim_{h \to 0} \frac{(0-h)[\sqrt{0-h} - \sqrt{0-h+1}] - 0}{-h}$$

$$= \lim_{h \to 0} [\sqrt{-h} - \sqrt{-h+1}] = 0 - \sqrt{+1} = -1$$

$$Rf'(0) = \lim_{h \to 0} \frac{(0+h)[\sqrt{0+h} - \sqrt{0+h+1}] - 0}{h}$$

$$= \lim_{h \to 0} \sqrt{h} - \sqrt{h+1} = -1$$

Since $Lf'(0) = Rf'(0)$

$\therefore f$ is differentiable at $x = 0$.

8. If $f(x) = sgn\ (x^3)$, then

[DCE-2001]

(a) f is continuous but not derivable at $x = 0$
(b) $f'(0^+) = 2$
(c) $f'(0^-) = 1$
(d) f is not derivable at $x = 0$

Solution

(d) Here,

$$f(x) = sgn\, x^3 = \begin{cases} \begin{cases} \dfrac{x^3}{|x^3|}, & \text{for } x^3 \neq 0 \\ 0, & \text{for } x^3 = 0 \end{cases} \\ \begin{cases} \dfrac{x}{|x|}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases} \\ \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases} \end{cases}$$

Thus, $f(x) = sgn\, x^3 = sgn\, x$, which is neither continuous nor derivable at 0. Note that

$$f'(0^+) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{1-0}{h} \to \infty$$

and $f'(0^-) = \lim_{h \to 0^-} \frac{f(0-h) - f(0)}{h}$

$$= \lim_{h \to 0^-} \frac{-1-0}{h} \to \infty$$

$\therefore$ $f'(0^+) \neq f'(0^-)$, $\therefore$ f is not derivable at $x = 0$.

9. Let [.] denote the greatest integer fuction and $f(x) = [\tan^2 x]$, then **[IIT-1993]**

(a) $\lim_{x \to 0} f(x)$ does not exist
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) $f'(0) = 1$

Solution

(b) We have $f(x) = [\tan^2 x]$
When $x = n$ (an integer)
$\lim_{x \to 0} f(x) = \lim_{x \to 0} [\tan^2 x] = 0$
and $f(0) = 0$ $\therefore f$ is continuous at $x = 0$

10. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases} \text{ is}$$

[IIT-2002 S]

(a) $R - \{0\}$ (b) $R - \{1\}$
(c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

Solution

(d) The given function is

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \frac{1}{2}(-x-1) & \text{if } \quad x < -1 \\ \tan^{-1} x & \text{if } \quad -1 \le x \le 1 \\ \frac{1}{2}(x-1) & \text{if } \quad x > 1 \end{cases}$$

Clearly L.H.L. at $(x = -1) = \lim_{h \to 0} f(-1-h)$

R.H.L. at $(x = -1) = \lim_{h \to 0} f(-1+h)$

$$= \lim_{h \to 0} \tan^{-1}(-1+h) = -3\pi/4$$

$\therefore$ L.H.L. $\neq$ R.H.L. at $x = -1$
$\therefore$ $f(x)$ is discontinuous at $x = -1$
Also we can prove in the same way, that $f(x)$ is discontinuous at $x = 1$
$\therefore$ $f'(x)$ can not be found for $x = \pm 1$ or domain of $f'(x) = R - \{-1, 1\}$

11. If $f(x)$ is differentiable and strictly increasing function, then the value of $\lim_{x \to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is **[IIT-2004 S]**

(a) 1 (b) 0
(c) – 1 (d) 2

Solution

(c) $\lim_{x \to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ [Using L.H. Rule]

$$\lim_{x \to 0} \frac{f'(x^2).2x - f'(x)}{f'(x)}$$

$$= \lim_{x \to 0} \frac{f'(x^2).2x}{f'(x)} - 1 = 0 - 1 = -1$$

12. The function $y = \begin{cases} 3^x, & -1 \le x \le 1 \\ 4 - x, & 1 < x < 4 \end{cases}$ is **[Roorkee-1994]**

(a) continuous and differentiable at $x = 1$
(b) continuous at $x = 1$
(c) differentiable at $x = 1$
(d) neither continuous nor differentiable at $x = 1$

Solution

(b) $f(1 + h) = f(1 - h) = f(1) = 3$

$\therefore$ continuous $Rf'(1) \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$$\lim \frac{4-(1+h)-3}{h} = -1$$

$$= Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim \frac{3^{1-h} - 3}{-h}$$

$$= \lim_{h \to 0} \frac{3(3^{-h} - 1)}{-h} \left(\frac{0}{0}\right) = \frac{3(-3^{-h} \log 3)}{-1} = 3 \log 3$$

$\therefore$ $Rf'(1) \neq Lf'(1)$ $\therefore$ not differentiable.

13. Let $f(x) = \begin{cases} (x-1)\sin\dfrac{1}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$.

[AIEEE-2008]

Then which one of the following is true?

(a) f is differentiable at $x = 1$ but not at $x = 0$

(b) f is neither differentiable at $x = 0$ nor at $x = 1$

(c) f is differentiable at $x = 0$ and at $x = 1$

(d) f is differentiable at $x = 0$ but not at $x = 1$

Solution

(b) By definition

$f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$, if the limit exists.

$\therefore \quad \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$.

$$= \lim_{h\to 0} \frac{(1+h-1)\sin\dfrac{1}{(1+h-1)} - 0}{h} = \lim_{h\to 0} \sin\frac{1}{h}$$

As the limit doesn't exist,

$\therefore$ it is not differentiable at $x = 1$

Again $f'(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$, if the limit exists

$$\lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$\therefore \quad = \lim_{h\to 0} \frac{(h-1)\sin\dfrac{1}{h-1} - \sin 1}{h}$$

But this limit doesn't exist. Hence it is not differentiable at $x = 0$

14. Let $f(x) = x^n \,|\, x \,|$ for real x. $f(x)$ is differentiable at the origin. If n is equal to which one of the following? ***[UPSC-2007]***

(a) -1

(b) 0

(c) any real number

(d) any positive integer

Solution

(d) For $n = -1$, we have $f(x) = \dfrac{|x|}{x}, x \neq 0$

This is not differentiable at origin

For $n = 0$, $f(x) = |\, x \,|$, not differentiable at origin. If n is a positive integer

$$\lim_{h\to 0} \frac{h^n \,|\, h \,| - 0}{h} = \lim_{h\to 0} |\, h \,| \,.\, h^{n-1}$$

$$= \lim_{n\to 0} h^{\alpha} \,|\, h \,|, \alpha \geq 1 = 0.$$

Then limit exist and so $f'(0)$ exists.

Hence $f(x)$ is differentiable at $x = 0$

Thus, $f(x) = x^n \,|\, x \,|$ is differentiable at the origin if n is any positive integer.

15. If $f(x) = \begin{cases} x-5 & \text{for } \quad x \leq 1 \\ 4x^2 - 9 & \text{for } \quad 1 < x < 2 \\ 3x+4 & \text{for } \quad x \geq 2 \end{cases}$ then

$f'(2+) =$ ***[EAMCET-2007]***

(a) 0 (b) 2

(c) 3 (d) 4

Solution

(c) $f'(2+) = \lim\limits_{x\to 2+} \left(\dfrac{f(x) - f(2)}{x-2} \right)$

$$\Rightarrow \quad \lim_{x\to 2+} \left(\frac{3x+4-10}{x-2} \right) = 3$$

16. If $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ exists finitely, then:

[MPPET-2010]

(a) $\lim\limits_{x\to c} f(x) = f(c)$

(b) $\lim\limits_{x\to c} f'(x) = f'(c)$

(c) $\lim\limits_{x\to c} f(x)$ does not exist

(d) $\lim\limits_{x\to c} f(x)$ may or not exist

Solution

(b) We know that $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$ if it exist finitely,

$\therefore$ we can say it is differentiable function

$\therefore \quad \lim\limits_{x\to c} f'(x) = f'(c)$

17. Let $f(x) = x|\, x \,|$ and $g(x) = \sin x$.

[AIEEE-2009]

Assertion: gof is differentiable at $x = 0$ and its derivative is continuous at that point

Reason: gof is twice differentiable at $x = 0$

Solution

(c) $f(x) = x|x|$ and $g(x) = \sin x$

$$gof(x) = \sin(x\,|\,x\,|) = \begin{cases} -\sin x^2, & x < 0 \\ \sin x^2, & x \geq 0 \end{cases}$$

$$\therefore (gof)'(x) = \begin{cases} -2x\cos x^2, & x < 0 \\ 2x\cos x^2, & x \geq 0 \end{cases}$$

Clearly, $L\,(gof)\,(0) = 0$ $\quad R(gof)'(0) = 0$

$\therefore$ gof is differentiable at $x = 0$ and also its derivative is continuous at x = 0

Now,

$$(gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2 \quad +\sin x^2, x < 0 \\ 2x\cos x^2 - 4x^2 \quad \sin x^2, x \geq 0 \end{cases}$$

$\therefore L(gof)''\,(0) = -2$ and $R\,(gof)''(0) = 2$

$\therefore L(gof)''\,(0)''\,R\,(gof)''(0)$

$\therefore gof(x)$ is not twice differentiable at $x = 0$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $f(x)$ is differentiable everywhere, then which one of the following is correct? ***[NDA-2007]***
(a) $|f|$ is differentiable everywhere
(b) $|f|^2$ is differentiable everywhere
(c) $f|f|$ is not differentiable at some points
(d) None of the above

2. If $f(x) = |\,x - 3|$, then f is ***[SCRA-96; RPET-97]***
(a) Discontinuous at $x = 2$
(b) Not differentiable $x = 2$
(c) Differentiable at $x = 3$
(d) Continuous but not differentiable at $x = 3$

3. If $f(x) = \begin{cases} x+1, \text{ when } x < 2 \\ 2x-1, \text{ when } x \geq 2 \end{cases}$, then $f'(2)$ equals ***[MPPET-97; Karnataka CET-2002]***
(a) 0 (b) 1
(c) 2 (d) Does not exist

4. At the point $x = 1$, the given function $f(x) = \begin{cases} x^3 - 1 & ; \quad 1 < x < \infty \\ x - 1 & ; \quad -\infty < x \leq 1 \end{cases}$ is ***[Roorkee-93]***
(a) Continuous and differentiable
(b) Continuous and not differentiable
(c) Discontinuous and differentiable
(d) Discontinuous and not differentiable

5. If $f(x) = \begin{cases} e^x + ax, & x < 0 \\ b(x-1)^2, & x \geq 0 \end{cases}$ is differentiable at $x = 0$, then (a, b) is ***[MPPET-2000]***
(a) $(-3, -1)$ (b) $(-3, 1)$
(c) $(3,1)$ (d) $(3, -1)$

6. The function $y = e^{-|x|}$ is ***[AMU-2000]***
(a) Continuous and differentiable at $x = 0$
(b) Neither continuous nor differentiable at $x = 0$
(c) Continuous but not differentiable at $x = 0$
(d) Not continuous but differentiable at $x = 0$

7. The function $f(x) = \begin{cases} e^{2x} - 1, & x \leq 0 \\ ax + \dfrac{bx^2}{2}, & x > 0 \end{cases}$ is continuous and differentiable for ***[AMU-2002]***
(a) $a = 1, b = 2$ (b) $a = 2, b = 4$
(c) $a = 2$, any b (d) Any a, $b = 4$

8. If $f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5} & \text{for } x \neq 1 \\ -\dfrac{1}{3} & \text{for } x = 1 \end{cases}$, then $f'(1) =$ ***[EAMET-2003]***
(a) $-1/9$ (b) $-2/9$
(c) $-1/3$ (d) $1/3$

9. If f is a real-valued differentiable function satisfying $|\,f(x) - f(y)\,| \leq (x - y)^2$, $x, y \in R$ and $f(0) = 0$, then $f(1)$ equal ***[AIEEE-2005]***
(a) 2 (b) 1
(c) -1 (d) 0

10. $f(x) = ||x|-1|$ is not differentiable at ***[IIT-Sc.-2005]***

(a) 0 (b) $\pm 1, 0$
(c) 1 (d) ± 1

11. The function which is continuous for all real values of x and differentiable at $x = 0$ is ***[MPPET-96]***

(a) $|x|$ (b) $\log x$
(c) $\sin x$ (d) $x^{1/2}$

12. If $f(x) = \begin{cases} x+2, & -1 < x < 3 \\ 5, & x = 3 \\ 8-x, & x > 3 \end{cases}$, then at $x = 3$,

$f'(x) =$ ***[MPPET-2001]***

(a) 1 (b) -1
(c) 0 (d) Does not exist

13. Let $g(x) = xf(x)$, where $f(x) = \begin{cases} x\sin\frac{1}{x}; & x \neq 0 \\ 0, & x = 0 \end{cases}$

at $x = 0$ ***[IIT-Sc.-94; UPSEAT-2004]***

(a) g is differentiable but g' is not continuous
(b) g is differentiable while f is not
(c) Both f and g are differentiable
(d) g is differentiable and g' is continuous

14. The function $f(x) = |x| + |x-1|$ is ***[RPET-96; CEE-02]***

(a) Continuous at $x = 1$, but not differentiable at $x = 1$
(b) Both continuous and differentiable at $x = 1$
(c) Not continuous at $x = 1$
(d) Not differentiable at $x = 1$

15. The set of all those points, where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable, is

(a) $(-\infty, \infty)$ (b) $[0, \infty]$
(c) $(-\infty, 0)\cup(0, \infty)$ (d) $(0, \infty)$

16. Let $f : IR \to IR$ be a function defined by $f(x) = \text{Min}\{x+1, |x|+1\}$. Then which of the following is true? ***[AIEEE-2007]***

(a) $f(x) \geq 1$ for all $x \in IR$
(b) $f(x)$ is not differentiable at $x = 1$.
(c) $f(x)$ is differentiable everywhere
(d) $f(x)$ is not differentiable at $x = 0$

17. The function $f(x) = x^n \sin(1/x)$; $x \neq 0$, $f(0) = 0$ is continuous and differentiable at $x = 0$. Which one of the following is correct? ***[NDA-2004]***

(a) $n \in [1, \infty)$ (b) $n \in (1, \infty)$
(c) $n \in (-\infty, 0)$ (d) $n \in (0, \infty)$

18. The function defined by

$f(x) = \begin{cases} |x-3|; & x \geq 1 \\ \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}; & x < 1 \end{cases}$ is ***[IIT-88]***

(a) Continuous at $x = 1$
(b) Continuous at $x = 3$
(c) Differentiable at $x = 1$
(d) all the above

19. The left-h and derivative of $f(x) = [x]\sin(\pi x)$ at $x = k$, k is an integer and $[x]$ = greatest integer $\leq x$, is ***[IIT-Sc.-2001]***

(a) $(-1)^k(k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$
(c) $(-1)^k k\pi$ (d) $(-1)^{k-1}k\pi$

20. The function $f(x) = \begin{cases} \frac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ is ***[MPPET-2008]***

(a) Continuous but not differentiable at $x = 0$
(b) discontinuous at $x = 0$
(c) continuous and differentiable at $x = 0$
(d) not defined at $x = 0$

21. Suppose $f(x)$ is differentiable at $x = 1$ and $\lim_{h\to 0}\frac{1}{h}f(1+h) = 5$, then $f'(1)$ equals ***[AIEEE-05]***

(a) 5 (b) 6
(c) 3 (d) 4

22. $f(x) = 2a - x$ in $-a < x < a$
$= 3x - 2a$ in $a \leq x$.
Then which of the following is true? ***[Karnataka CET-2008]***

(a) $f(x)$ is differentiable at all $x \geq a$
(b) $f(x)$ is continuous at all $x < a$
(c) $f(x)$ is discontinuous at $x = a$
(d) $f(x)$ is not differentiable at $x = a$

23. The function $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$, $x \neq 0$, is continuous at $x = 0$. Then

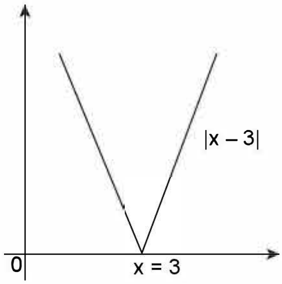
(a) $f(0) \neq 1$
(b) $f(x)$ is differentiable at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) $f'(0) = \dfrac{1}{3}$

SOLUTIONS

1. (c) If $f(x)$ is differential everywhere, then $f\,|f|$ is not differentiable at some points.

2. (d) $\lim\limits_{x\to3^-} f(x) = \lim\limits_{h\to0} f(3-h)$

$= \lim\limits_{h\to0} |3-h-3| = 0$



$\lim\limits_{x\to3^+} f(x) = \lim\limits_{h\to0} f(3+h)$

$= \lim\limits_{h\to0} |3+h-3| = 0$

$\therefore \quad \lim\limits_{x\to3^-} f(x) = \lim\limits_{x\to3^+} f(x) = f(3)$

Hence, f is continuous at $x = 3$.
Now,

$$\cancel{Lf'(3) = \lim_{h\to0}} \frac{f(3-h)-f(3)}{-h}$$

$$= \lim_{h\to0} \frac{|3-h-3|-0}{-h} = \lim_{h\to0} \frac{|h|}{-h} = -1$$

$$= \cancel{Rf'(3) = \lim_{h\to0}} \frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h\to0} \frac{|3+h-3|}{h} = 1$$

$\therefore \quad Lf'(3) \neq Rf'(3)$.

Hence f is not differentiable at $x = 3$.

Trick: Can be seen by graph, it is continuous but unique tangent is not defined at $x = 3$. Hence it is not differentiable.

3. (d) Step 1: The right hand derivative of the function at the point $x = 2$ is denoted by $Rf'(2)$ or $f'(2+0)$ and the function is defined when $x > 2$

$\Rightarrow \quad f(x) = 2x - 1;\ x > 2$

Hence

$$Rf'(x)\big|_{x=2} = 2 \Rightarrow f'(2+0) = 2 \qquad \text{..........(i)}$$

Step 2: The left hand derivative of the function at the point $x = 2$ is denoted by $Lf'(2)$ or $f'(2-0)$ and for this the function is defined when $x < 2$ i.e., $f(x) = x + 1;\ x < 2$

Hence $\left.\dfrac{d}{dx} f(x)\right|_{x=2} = 1 \Rightarrow f'(2-0) = 1 \qquad \text{....(ii)}$

Step 3: $f'(2+0)$ and $f'(2-0)$ both are exist but not equal. Hence function is not differentiable.

4. (b) Step 1: $f(1-0) = f(1+0) = f(1) = 0$
Hence, the function is continuous at $x = 1$

Step 2: $f'(1-0) = Lf'(1) = 1$

$f'(1+0) = Rf'(1) = 3$

Hence the function is not differentiable at $x = 1$

5. (b) Step 1: from left hand side, for continuity at $x = 0$

Function is defined when $x < 0$

Hence $f(0-0) =$ Left hand Limit

$$= e^0 + a \times 0 = 1 \qquad \text{..............(1)}$$

(In $e + ax$ by putting $x = 0$, we get. Left hand, Limit)

Step 2: From right side for the continuity at $x = 0$ function is defined when $x > 0$.

Hence $f(0+0) = b(0-1)^2 = b \qquad \text{............(2)}$

Put $x = 0$ in $(f(x)) = b(x-1)^2$, we get Right hand limit

Step 3: For the continuous function

Left hand limit = Right hand limit = b

Step 4: At the point $x = 0$ left hand derivative of function is denoted by $Lf'(0)$ or $f'(0-0)$ and for this function is defined when $x < 0$ i.e., $f(x) = e^x + ax$

Hence $f'(0-0) = e^x + a\big|_{x=0} = 1 + a \qquad \text{......(3)}$

Step 5: At the point $x = 0$, the right-hand derivative of the function is denoted by $Rf'(0)$

or $f'(0+0)$ and for this function is defined when $x>0$ i.e., $f(x)=b(x-1)^2$

Hence $f'(0+0)=2b(x-1)|_{x=0}=-2b$

Step 6: For the differentiability $f'(0-0)$ and $f'(0+0)$ both exist and are equal with eachother.

Hence $1+a=-2b$

$1+a=-2$ ($b=1$ from step 3)

$a=-3$

6. (c) We have, $f(x)=\begin{cases} e^{-x}, & x\geq 0 \\ e^{x}, & x<0 \end{cases}$

Clearly, $f(x)$ is continuous and differentiable for all non zero x.

Now,

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0} e^{x}=1, \lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} e^{-x}=1$

Also, $f(0)=e^0=1$. So, $f(x)$ is continuous for all x.

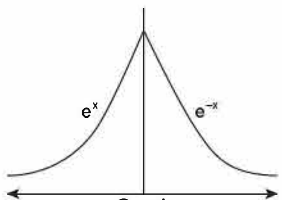
$$(\text{LHD at } x=0)=\left(\frac{d}{dx}(e^x)\right)_{x=0}=1$$

$$(\text{RHD at } x=0)=\left(\frac{d}{dx}(e^{-x})\right)_{x=0}=-1$$

So, $f(x)$ is not differentiable at $x=0$

Hence $f(x)=e^{-|x|}$ is everywhere continuous but not differentiable at $x=0$.

Aliter:



Graph

So, there is a sharp point at $x=0$ hence not differentiable.

7. (c) $\because f$ is continuous at $x=0$

$\therefore \quad f(0^-)=f(0^+)=f(0)=-1$

Also $Lf'(0)=Rf'(0)$

$$\Rightarrow \quad \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(0+h)-(0)}{h}$$

$$\Rightarrow \quad \lim_{h\to 0}\left(\frac{e^{-2h}-1+1}{-h}\right)=\lim_{h\to 0}\left(\frac{ah+\frac{bh^2}{2}-1+1}{-h}\right)$$

$$\Rightarrow \quad \lim_{h\to 0}\left(\frac{-2e^{-2h}}{-1}\right)=\lim_{h\to 0}\left(a+\frac{bh}{2}\right)$$

$\Rightarrow \quad 2=a+0 \Rightarrow a=2, b$ any number.

8. (b) Step 1: By definition

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

Step 2: Put the given data in the formula which is given in step 1, then we get:

$$\lim_{h\to 0}\frac{\frac{1}{2(1+h)-5}-\left(\frac{-1}{3}\right)}{h}=\lim_{h\to 0}\frac{\left(\frac{1}{2h-3}+\frac{1}{3}\right)}{h}$$

$$=\lim_{h\to 0}\left(\frac{3+2h-3}{3h(2h-3)}\right)=\lim_{h\to 0}\left(\frac{2h}{3h(2h-3)}\right)$$

$$=\lim_{h\to 0}\frac{2}{3(2h-3)}=\frac{2}{3(-3)}=\frac{-2}{9}$$

9. (b) Since $|f(x)-f(y)|\leq (x-y)^2$

$$\lim_{x\to y}\frac{|f(x)-f(y)|}{|x-y|}\leq \lim_{x\to y}|x-y|$$

$\Rightarrow \quad |f'(y)|\leq 0 \quad \Rightarrow \quad f'(y)=0$

$\Rightarrow \quad f(y)=$ constant.

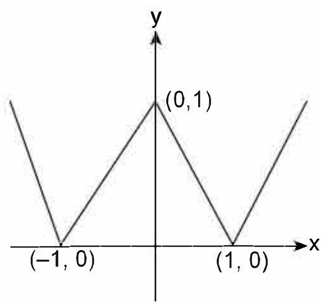
$\Rightarrow \quad f(y)=0 \quad [\because f(0)=0$ given]

$\Rightarrow \quad f(1)=0$

10. (b) $\begin{cases} |x|-1, & |x|-1\geq 0 \\ -|x|+1, & |x|-1<0 \end{cases}$

$$=\begin{cases} |x|-1, & x\leq -1 \text{ or } x\geq 1 \\ -|x|+1, & -1<x<1 \end{cases}$$

$$=\begin{cases} -x-1, & x\leq -1 \\ x+1, & -1<x<0 \\ -x+1, & 0\leq x<1 \\ x-1, & x\geq 1 \end{cases}$$



From the graph. It is clear that $f(x)$ is not differentiable at $x=-1,0$ and 1.

11. (c) Since $\dfrac{dy}{dx}=\cos x$ which is defined at $x=0$ and no other differential coefficient is defined at $x = 0$

12. (d) If $f(x)=\begin{cases} x+2, & -1<x<3 \\ 5, & x=3 \\ 8-x, & x>3 \end{cases}$

and $f(3)=5$

L.H.D.

$$= \lim_{x\to 3-}\frac{f(x)-f(3)}{x-3}=\lim_{h\to 0}\frac{f(3-h)-f(3)}{-h}$$

$$=\lim_{h\to 0}\frac{(3-h+2)-5}{-h}=\lim_{h\to 0}\frac{-h}{-h}=1$$

R.H.D.

$$=\lim_{x\to 3^+}\frac{f(x)-f(3)}{x-3}=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

$$=\lim_{h\to 0}\frac{8-(3+h)-5}{h}=\lim_{h\to 0}\frac{-h}{h}=-1$$

L.H.D. $\neq$ R.H.D.; $f(x)$ is not differentiable.

13. (b) Step 1: $g(x)=x^2\sin\left(\dfrac{1}{x}\right),\ x\neq 0$

$=0 \quad , x=0$

$$Lg'(0)=g'(0-0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{(0-h)^2\sin\left(\frac{1}{0-h}\right)-0}{-h}=0$$

$$Rg'(0)=g'(0+0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{(0+h)^2\sin\left(\frac{1}{0+h}\right)-0}{h}=0$$

Hence $Lg'(0)=Rg'(0)\Rightarrow g(x)$ which is a differentiable function.

Step 2: $f'(0-0)$ and $f'(0+0)$ both are not defined. Hence $f(x)$ is not differentiable.

Step 3: $g'(x)=2x\sin\left(\dfrac{1}{x}\right)-\cos\left(\dfrac{1}{x}\right)$

$Lg'(0)=g'(0-0)=$ Notdefined

$Rg'(0)=g'(0+0)=$ Notdefined

$g'(0)=$ Notdefined

Hence g' is not continuous

14. (a) We have $f(x)=|x|+|x-1|$

$$=\begin{cases} -2x+1 & x<0 \\ x-x+1 & 0\le x<1 \\ x+x-1 & x\ge 1\end{cases}=\begin{cases} -2x+1 & x<0 \\ 1 & 0\le x<1 \\ 2x-1 & x\ge 1\end{cases}$$

Clearly,

$\lim_{x\to 0^-}f(x)=1,\ \lim_{x\to 0^+}f(x)=1,\ \lim_{x\to 1^{-1}}f(x)=1$

and $\lim_{x\to 1^+}f(x)=1$. So, $f(x)$ is continuous at $x = 0,\ 1$

Now, $f'(x)=\begin{cases} -2, & x<0 \\ 0, & 0\le x<1 \\ 2, & x\ge 1\end{cases}$

Here $x=0, f'(0^+)=0$ while $f'(0^-)=-2$

and at $x=1, f'(1^+)=2$, while $f'(1^-)=0$

Thus, $f(x)$ is not differentiable at $x = 0$ and 1.

15. (b) Step 1: $f(x)=x^2,\ x\ge 0$ or $\dfrac{x}{1+x},\ x\ge 0$

$= -x^2, x<0 \quad \dfrac{x}{1-x}, x<0$

Clearly, for non-zero x, the $f(x)$ is differentiable

Step 2: Left-hand derivative

$=Lf'(0)=-2x\big|_{x=0}=0$ or $\left.\dfrac{1}{(1+x)^2}\right|_{x=0}=1$

Right-hand derivative = $Rf'(0)=2x\big|_{x=0}=0$ or

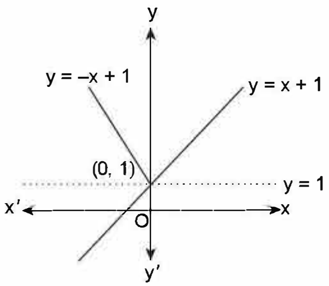
$\left.\dfrac{1}{(1-x)^2}\right|_{x=0}=1$

Hence: $Lf'(0)=Rf'(0)$

Hence function is differentiable in $(-\infty,+\infty)$

16. (a) $f(x)=\min\{x+1,|x|+1\}$

$f(x)=x+1, \forall x\in R$



It is clear from the figure that

$$f(x) \geq 1, \forall x \in R$$

17. (b) $\therefore$ $f(x)$ is continuous at $x = 0$

$\therefore$ $\lim_{x\to 0} f(x) = f(0) = 0$

$$\lim_{x\to 0} x^n \sin\left(\frac{1}{x}\right) = 0 \Rightarrow n > 0$$

and $f(x)$ is differentiable at $x = 0$, if

$$\lim_{x\to 0} \frac{x^n \sin\frac{1}{x} - 0}{x} \text{ exist finitely.}$$

$$= \lim_{x\to 0} x^{n-1} \sin\frac{1}{x} \text{ exist finitely}$$

$\Rightarrow$ $n - 1 > 0 \Rightarrow n > 1$ $\therefore$ $n \in (1, \infty)$

18. (d) Since $|x-3| = x - 3$ if $x \geq 3$

and $f(x) = -x + 3$, if $x < 3$

$\therefore$ The given function can be defined as

$$f(x) = \begin{cases} \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}, & x < 1 \\ 3 - x, & 1 \leq x < 3 \\ x - 3 & x \geq 3 \end{cases}$$

Now proceed to check the continuity and differentiability at $x = 1$.

19. (a) Step 1: $[x]$ means greatest integer function which is less than x and $[k - h] = -1$

Step 2: At $x = k$ Left-hand derivative

$$Lf'(k) = f'(k-0) = \lim_{h\to 0} \frac{f(k-h) - f(k)}{-h}$$

$$= \lim_{h\to 0} \frac{[k-h]\sin\pi(k-h) - [k]\sin\pi k}{-h}$$

$$= \lim_{h\to 0} \frac{(k-1)[\sin\pi k\cos\pi h - \cos\pi k\sin\pi h] - [k]\times\sin\pi k}{-h}$$

$$= \lim_{k\to 0} \frac{(k-1)[(0)\cos\pi h - (-1)^k \sin\pi h] - [k]\times 0}{-h}$$

$$= \lim_{h\to 0} \frac{(k-1)\{-(-1)^k \sin\pi h\}}{-h} = (-1)^k (k-1)\pi$$

NOTES

1. $\sin k\pi = 0, \cos k\pi = (-1)^k$

$$\lim_{h\to 0} \frac{\sin\pi h}{-h} = \pi$$

2. (Quicker Method)
For the left hand derivative at $x = 1$
$f(x) = (k-1)\sin x$ $\quad x < k$
Hence $Lf'(x)\big|_{x=k} = (k-1)\cos\pi x \times \pi\big|_{x=k}$
$= (k-1)\cos\pi k \times \pi = (-1)^k (k-1)\pi$

20. (a) $f(x) = \begin{cases} \frac{\tan x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

For $f(x)$ to be continuous at $x = 0$

$f(0) = \lim_{x\to 0} f(x)$ Here $f(0) = 1$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\tan x}{x} = 1$$

$\therefore$ $f(0) = \lim_{x\to 0} f(x)$

$\therefore$ $f(x)$ is continuous at $x = 0$, but not differentiable.

21. (a) $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h\to 0} \frac{f(1+h)}{h} - \lim_{h\to 0} \frac{f(1)}{h}$$

Since, $\lim_{h\to 0} \frac{f(1+h)}{h} = 5$

So, $\lim_{h\to 0} \frac{f(1)}{h}$ must be finite as $f'(1)$ exist and $\lim_{h\to 0} \frac{f(1)}{h}$ can be finite only. If $f(1) = 0$ and

$$\lim_{h\to 0} \frac{f(1)}{h} = 0$$

$\therefore$ $f'(1) = \lim_{h\to 0} \frac{f(1+h)}{h} = 5$

22. (d) $f(x)=\begin{cases} 2a-x & in\ -a<x<a \\ 3x-2a & in\ x\ge a \end{cases}$

Here $f(a+0)=a=f(a-0)$

Also $f'(a+0)=3, f'(a-0)=-1$

Here $f'(a+0)\ne f'(a-0)$,

$\therefore$ not differentiable.

23. (b) We have $\lim\limits_{x\to0} f(x)=\lim\limits_{x\to0}\left(\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}\right)$

$$\Rightarrow \quad \lim_{x\to0} f(x)=\lim_{x\to2}\frac{e^{2x}-1-2x}{x(e^{2x}-1)}$$

$$=\lim_{x\to0}\frac{\left(1+2x+\frac{(2x)^2}{2!}+\ldots..\right)-1-2x}{x\left(1+2x+\frac{(2x)^2}{2!}+\ldots\ldots..-1\right)}$$

$$=\lim_{x\to0}\frac{\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\ldots\ldots\ldots..}{x\left(2x+\frac{(2x)^2}{2!}+\ldots\ldots...\right)}$$

$$=\frac{\frac{2^2}{2!}}{2}=1=f(0)$$

So, $f(x)$ is continuous at $x=0$

Now, (LHD at $x=0$)

$$=\lim_{h\to0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to0}\frac{f(-h)-1}{-h}$$

$$=\lim_{h\to0}\frac{\left(-\frac{1}{h}-\frac{2}{e^{-2h}-1}-1\right)}{-h}$$

$$=\lim_{h\to0}\frac{e^{-2h}-1+2h+h(e^{-2h}-1)}{h^2(e^{-2h}-1)}$$

$$\left\{1-2h+\frac{(2h)^2}{2!}-\frac{(2h)^3}{3!}+\ldots..\right\}+h$$

$$\left\{-2h+\frac{(2h)^2}{2!}-\frac{(2h)^3}{3!}+\ldots..-1+2h\right\}$$

$$=\lim_{h\to0}\frac{-1+2h}{h^2\left(-2h+\frac{(2h)^2}{2!}-\frac{(2h)^3}{3!}+\ldots\ldots...\right)}$$

$$=\lim_{h\to0}\frac{\left(-\frac{8}{3!}+\frac{4}{2!}\right)h^3+\ldots..}{h^3\left(-2+\frac{4h}{2!}\ldots\ldots.\right)}=-\frac{1}{3}$$

Similarly, we have

(RHD at $x=0$) $=-\dfrac{1}{3}$

$\therefore$ (LHD at $x=0$) = (RHD at $x=0$)

Thus, $f(x)$ is differentiable at $x=0$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The function $f(x)=\begin{cases} x, \text{ if } 0\le x\le 1 \\ 1, \text{ if } 1<x\le 2\end{cases}$, is **[SCRA-96]**

(a) Continuous at all x, $0\le x\le 2$ and differentiable at all x except $x=1$ in the interval $[0, 2]$

(b) Continuous and differentiable at all x in $[0, 2]$

(c) Not continuous at any point in $[0, 2]$

(d) Not differentiable at any point $[0, 2]$

2. The function $f(x)=|x|$ at $x=0$ is **[MPPET-93]**

(a) Continuous but non-differentiable

(b) Discontinuous and Differentiable

(c) Discontinuous and non-differentiable

(d) Continuous and differentiable

3. Consider $f(x)=\begin{cases} \dfrac{x^2}{|x|}, & x\ne0 \\ 0, & x=0\end{cases}$ **[EAMCET-94]**

(a) $f(x)$ is discontinuous everywhere
(b) $f(x)$ is continuous everywhere
(c) $f'(x)$ exists in $(-1, 1)$
(d) $f'(x)$ exists in $(-2, 2)$

4. The function $y = |\sin x|$ is continuous for any x but it is not differentiable at

[AMU-2000]

(a) $x = 0$ only
(b) $x = \pi$ only
(c) $x = k\pi$ (k is an integer) only
(d) $x = 0$ and $x = k\pi$ (k is an integer)

5. A function $f(x) = \begin{cases} 1+x, & x \le 2 \\ 5-x, & x > 2 \end{cases}$ is

[AMU-2001]

(a) Not continuous at $x = 2$
(b) Differentiable at $x = 2$
(c) Continuous but not differentiable at $x = 2$
(d) None of these

6. The function $f(x) = x^2 \sin\frac{1}{x}$, $x \neq 0$, $f(0) = 0$ at $x = 0$

[MPPET-2003]

(a) Is continuous but not differentiable
(b) Is discontinuous
(c) Is having continuous derivative
(d) Is continuous and differentiable

7. If $f(x) = \dfrac{x}{1+|x|}$ for $x \in R$, then $f'(0) =$

[EAMCET-2003]

(a) 0 (b) 1
(c) 2 (d) 3

8. The value of m for which the function $f(x) = \begin{cases} mx^2, & x \le 1 \\ 2x, & x > 1 \end{cases}$ is differentiable at $x = 1$, is

[MPPET-98]

(a) 0 (b) 1
(c) 2 (d) Does not exist.

9. Let $f(x) = \begin{cases} 1, & \forall x < 0 \\ 1+\sin x, & \forall 0 \le x \le \pi/2 \end{cases}$, then what is the value of $f'(x)$ at $x = 0$

[Orissa JEE-2005]

(a) 1 (b) -1
(c) ∞ (d) does not exist

10. If $f(x) = \begin{cases} ax^2 + b; & x \le 0 \\ x^2; & x > 0 \end{cases}$ possesses derivative at $x = 0$, then
(a) $a = 0, b = 0$
(b) $a > 0, b = 0$
(c) $a \in R, b = 0$
(d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 14 minutes.
3. The worksheet consists of 14 questions. The maximum marks are 42.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The function which is continuous for all real values of x and differentiable at $x = 2$ is
(a) $\cos(x - 2)$ (b) $|x - 2|$
(c) $\log(x - 2)$ (d) $(x - 2)^{1/2}$

2. Let $f(x) = \begin{cases} \cos(\pi x/2) & \text{if } |x| \le 1 \\ |x-1| & \text{if } |x| > 1 \end{cases}$. Then f is
(a) a differentiable function
(b) a continuous function but not differentiable at $x = -1, 1$
(c) non-differentiable only at $x = 1$
(d) a discontinuous function.

3. A function $f(x)$ is defined as follows
$f(x) = -x^2, x \le 0$ $f(x) = 5x - 4, 0 < x \le 1$
$f(x) = 4x^2 - 3x, 1 < x \le 2$ $f(x) = 3x + 4, x > 2$
(a) $f(x)$ is not continuous at $x = 0$, but differentiable there
(b) $f(x)$ is continuous at $x = 1$, but not differentiable there
(c) $f(x)$ is continuous at $x = 2$, but not differentiable there
(d) None of the above

4. The function $f(x) = \sin|x|$ is **[DCE-2002]**
(a) continuous for all x
(b) continuous only at certain points
(c) differentiable at all points
(d) None of these

5. Given $f(0) = 0$ and $f(x) = \dfrac{1}{1 - e^{-1/x}}$ for $x \ne 0$.
Then only one of the following statement on $f(x)$ is true $f(x)$ is
(a) continuous at $x = 0$
(b) Not continuous at $x = 0$
(c) Both continuous and differentiable at $x = 0$
(d) Not defined at $x = 0$

6. If $f(x) = \begin{cases} 1/|x|, & |x| \ge 1 \\ ax^2 + b, & |x| < 1 \end{cases}$ is differentiable and continuous at $x = \pm 1$, then **[JEE (WB)-1999]**
(a) $a = 1/2, b = -3/2$
(b) $a = -1/2, b = 3/2$
(c) $a = -1/2, b = -3/2$
(d) $a = 1/2, b = 3/2$

7. Which of the following is not true? **[Kerala (Engg.)-2002]**
(a) A polynomial function is always continuous
(b) A continuous function is always differentiable
(c) A differentiable function is always continuous
(d) e^x is continuous for all x

8. If $f(x) = \begin{cases} \dfrac{|x+2|}{\tan^{-1}(x+2)}, & x \ne -2 \\ 2, & x = -2 \end{cases}$, then $f(x)$ is
(a) continuous at $x = -2$
(b) not continuous at $x = -2$
(c) differentiable at $x = -2$
(d) continuous but not derivable at $x = -2$

9. If $f : IR \to IR$ is an even function which is twice differentiable on IR and $f''(\pi) = 1$, then $f''(-\pi) =$
(a) -1 (b) 0
(c) 1 (d) 2

10. If $f : R \to R$, $f(x) = 15 - |x - 10|$, then the number of points where the function $g(x) = f\{f(x)\}$ is not differentiable, is **[ICS (Pre)-2004]**
(a) 0 (b) 1
(c) 2 (d) 3

11. If $f(x) = \dfrac{x}{\sqrt{x+1} - \sqrt{x}}$ be a real-valued function then
(a) $f(x)$ is continuous, but $f'(0)$ does not exist
(b) $f(x)$ is differentiable at $x = 0$
(c) $f(x)$ is not continuous at $x = 0$
(d) $f(x)$ is not differentiable at $x = 0$

12. Let $f(x) = x + |x|$. Then $f(x)$ is
(a) differentiable at all x
(b) continuous at all x except at $x = 0$
(c) differentiable everywhere except at $x = 0$
(d) discontinuous everywhere except at $x = 0$

13. If $f(x)=\begin{cases} e^x, & x<2 \\ a+bx, & x\ge 2\end{cases}$ is differentiable for all $x \in R$, then mark one incorrect statement
(a) $a + b = 0$
(b) $a + 2b = e^2$
(c) $b = e^2$
(d) None of these

14. If $f(x)=\begin{cases} x, & 0\le x\le 1 \\ 2x-1, & 1<x\end{cases}$, then **[Orissa JEE-2002]**
(a) f is discontinuous at $x = 1$
(b) f is differentiable at $x = 1$
(c) f is continuous but not differentiable at $x = 1$
(d) None of these

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**6.** ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

3. (c) $\lim\limits_{x\to 0^-} f(x)=0$

$f(0)=0,\ \lim\limits_{x\to 0^+} f(x)=4$

and $\lim\limits_{x\to 1^-} f(x)=1$ and $\lim\limits_{x\to 1^+} f(x)=1, f(1)=1$

Hence $f(x)$ is continuous at $x = 1$

Also, $\lim\limits_{x\to 2^-} f(x)=4(2)^2-3.2=10$

$f(2) = 10$ and $\lim\limits_{x\to 2^+} f(x)=3(2)+4=10$

Hence, $f(x)$ is continuous at $x = 2$

4. (a) If $g(x) = |x|$ and $h(x) = \sin x$, then $f(x) = (hog)(x)$ for all $x \in R$. As both g and h are continuous function, therefore, $f(x)$ is also continuous at all $x \in R$.

6. (b) We have,

$$f(x)=\begin{cases} -\dfrac{1}{x}, & x\le -1 \\ ax^2+b, & -1<x<1 \\ \dfrac{1}{x}, & x\ge 1\end{cases}$$

For $f(x)$ to be continuous at $x = 1$, we must have

$\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)$

$\lim\limits_{x\to 1^-} ax^2+b=\lim\limits_{x\to 1^+}\dfrac{1}{x}=1 \Rightarrow a+b=1$ .....(1)

$f(x)$ is differentiable at $x = 1$

$\therefore\ \lim\limits_{h\to 0} f'(1-h)=\lim\limits_{h\to 0} f'(1+h)$

$$\frac{d}{dx}(ax^2+b)_{at\ x=1} = \frac{d}{dx}\left(\frac{1}{x}\right)_{at\ x=1}$$

$$2a\times 1 = -\frac{1}{1^2}$$

$$a = -\frac{1}{2} \qquad \ldots\ldots\ldots\ldots\ (2)$$

From (1) and (2)

$$a = -\frac{1}{2},\ b = \frac{3}{2}$$

7. (b) A continuous function may or may not be differentiable. So, (b) is not true.

8. (b) $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-} \dfrac{|x+2|}{\tan^{-1}(x+2)}$

$$= \lim_{x\to -2^-} \frac{-(x+2)}{\tan^{-1}(x+2)} = -1$$

$$\left[\because \lim_{x\to 0^-} \frac{\tan^{-1}x}{x} = 1\right]$$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{|x+2|}{\tan^{-1}(x+2)}$$

$$= \lim_{x\to -2^+} \frac{x+2}{\tan^{-1}(x+2)} = 1$$

$\therefore$ $\lim\limits_{x\to -2} f(x)$ does not exist.

10. $f(x) = \begin{cases} 5+x\ ,\ x<10 \\ 25-x,\ x\ge 10 \end{cases}$

$$g(x) = \begin{cases} f(5+x) & f(x)<10 \\ f(25-x) & f(x)\ge 10 \end{cases}$$

$$= \begin{cases} 5+(5+x) & , \quad x<5 \\ 25-(5+x) & , \quad 5<x<15 \\ 5+(25-x) & , \quad x\ge 15 \end{cases}$$

$$= \begin{cases} 10+x & , \quad x<5 \\ 20-x & , \quad 5\le x<15 \\ 30-x & , \quad x\ge 15 \end{cases}$$

It is now clear to observe that $g(x)$ is not differentiable at $x = 5$ and $x = 15$.

13. (a) Clearly, $f(x)$ is everywhere continuous and differentiable except possible at $x = 2$ at $x = 2$, we have,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} e^x = e^2$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ax+b = 2a+b$$

$$f(2) = 2a+b$$

Also, (LHD at $x = 2$) $= \left(\dfrac{d}{dx}(e^x)\right)_{x=2} = e^2$

(RHD at $x = 2$) $= \left(\dfrac{d}{dx}(ax+b)\right)_{x=2} = a$

For $f(x)$ to be differentiable at $x = 2$. It should be both continuous and differentiable at $x = 2$.

$\therefore$ $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = f(2)$ and

(LHD at $x = 2$) = (RHD at $x = 2$)

$\Rightarrow$ $e^2 = 2a + b$ and $a = e^2$

$\Rightarrow$ $a = e^2$ and $b = -e^2$

$\therefore$ $a + b = e^2 - e^2 = 0$

14. Here, $\underset{x\to 1^-}{Lt}\ f(x) = \underset{x\to 1^-}{Lt}\ x = 1 = f(1)$

and also $\underset{x\to 1^+}{Lt}\ f(x) = \underset{x\to 1^+}{Lt}\ (2x-1) = 1 = f(1)$

But, $Lf'(1) = 1$ and $Rf'(1) = 2$

Thus, f is continuous but not derivable at 1.

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LECTURE

2

Test Your Skills

ASSERTION/REASONING

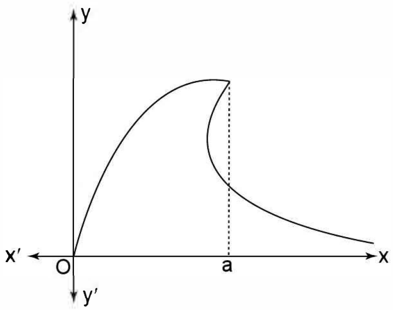
ASSERTION AND REASONING TYPE QUESTIONS

Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct.

(a) **Assertion** is True, **Reason** is True and **Reason** is a correct explanation for **Assertion.**
(b) **Assertion** is True, **Reason** is True and **Reason** is not a correct explanation for **Assertion.**
(c) **Assertion** is True and **Reason** is False.
(d) **Assertion** is False and **Reason** is True.

1. **Assertion (A):** The function $f(x) = |x|$ is discontinuous at $x = 0$.
   Reason (R): The function $f(x) = |x|$ is non-differentiable at $x = 0$.

2. **Assertion (A):** The function $f(x)$ in the figure is differentiable at $x = a$.



   Reason (R): The function $f(x)$ is continuous at $x = a$

3. **Assertion (A):** $f(x) = x \sin(1/x)$ is differentiable at $x = 0$.
   Reason (R): $f(x)$ is continuous at $x = 0$ ***[NDA-2008]***

4. **Assertion (A):** The function $f(x) = \dfrac{x}{1+|x|}$ is not differentiable at $x = 0$. ***[UPSC-2007]***
   Reason (R): $|x|$ and hene $(1 + |x|)$ is not differentiable at $x = 0$.

5. **Assertion (A):** For $x < 0, \dfrac{d}{dx}(\ln|x|) = -\dfrac{1}{x}$
   Reason (R): For $x < 0, |x| = -x$

6. **Assertion (A):** Derivative of $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ is 1 for $0 < x < 1$.
   Reason (R): $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ for $-1 \le x \le 1$

7. **Assertion (A):** $\dfrac{d}{dx}\left(x^{x^x}\right) = x^{x^x}.x(1+2\ln x)$
   Reason (R): $\because (x^x)^x = x^{x^2} = e^{x^2 \ln x}$

8. **Assertion (A):** Let $f : [0, \infty) \to [0, \infty]$, be a function defined by $y = f(x) = x^2$, then $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{d^2x}{dy^2}\right) = 1$
   Reason (R): $\left(\dfrac{dy}{dx}\right).\left(\dfrac{dx}{dy}\right) = 1$

9. Assertion (A): If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$, then $f'(x) = 0$.

Reason (R): Derivative of constant function is zero.

10. Assertion (A): $\frac{d}{dx}\{\tan^{-1}(\sec x + \tan x)\} = \frac{d}{dx}\{\cot^{-1}(\operatorname{cosec} x + \cot x)\}$

Reason (R): $\sec^2 x - \tan^2 x = 1 = \operatorname{cosec}^2 x - \cot^2 x$

11. Assertion (A): If $u = f(\tan x)$, $v = g(\sec x)$ and $f'(1) = 2,\ g'(\sqrt{2}) = 4$, then $\left.\frac{du}{dv}\right|_{x=\pi/4} = \frac{1}{\sqrt{2}}$

Reason (R): If $u = f(x)$, $v = g(x)$, then the derivative of f with respect to g is $\frac{du}{dv} = \frac{du/dx}{dv/dx}$

12. Assertion (A): If $e^{xy} + \ln(xy) + \cos(xy) + 5 = 0$, then $\frac{dy}{dx} = -\frac{y}{x}$.

Reason (R): $\frac{d}{dx}(xy) = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

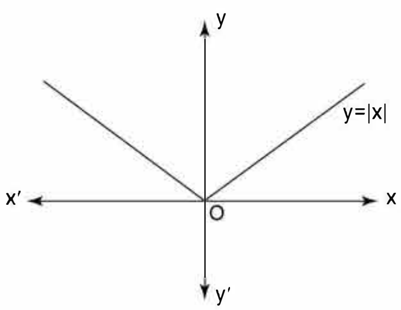
13. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x) \sin x$. ***[IIT JEE-2008]***

Assertion (A): $\lim\limits_{x \to 0} [g(x)\cot x - g(0) \operatorname{cosec} x] = f''(0)$

Reason (R): $f'(0) = g(0)$

ASSERTION/REASONING: SOLUTION

1. (d) It is clear from figure that $f(x)$ is continuous at $x = 0$ but non-differentiable at $x = 0$.



2. (d) At $x = a$, two tangents can be drawn. Hence, $f(x)$ is non-differentiable at $x = a$ but continuous at $x = a$.

3. (d)

4. (d)

5. (d) $\because \frac{d}{dx}(\ln|x|) = \frac{1}{x}$

Now, $\left(\because \frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{(-x)}(-1) = \frac{1}{x}\right)$

6. (c) $\because \sin^{-1}\left(\frac{2}{1+x^2}\right)$

$$= \begin{cases} \pi - 2\tan^{-1} x, & x > 1 \\ 2\tan^{-1} x, & -1 \le x \le 1 \\ -\pi - 2\tan^{-1} x, & x < -1 \end{cases}$$

and $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x, & x \ge 0 \\ -2\tan^{-1} x, & x < 0 \end{cases}$

For $0 < x < 1$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\therefore \quad \frac{du}{dv} = 1 \quad (\because u = v)$

7. (d) $\because \frac{d}{dx}\left(x^{x^x}\right) = \frac{d}{dx}(x)^{x^x} = \frac{d}{dx}e^{\ln(x)^{x^x}}$

$= \frac{d}{dx}e^{x^x . \ln x}$

$$= e^{x^x \cdot \ln x}\left\{x^2 \cdot \frac{1}{x} + \ln x . x^x .(1 + \ln x)\right\}$$

$$= x^{x^x} . x^{x-1}\{1 + x\ln x(1 + \ln x)\}$$

8. (d) $\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x \quad \Rightarrow \frac{d^2y}{dx^2} = 2$

Now, $y = x^2$

$$\Rightarrow \quad 1 = 2x\frac{dx}{dy} \Rightarrow \frac{dx}{dy} = \frac{1}{2x}$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = -\frac{1}{2x^2}.\frac{dx}{dy} = -\frac{1}{4x^3}$$

$$\therefore \quad \left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = (2)\left(-\frac{1}{4x^3}\right) = -\frac{1}{2x^3}$$

But $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \Rightarrow \left(\frac{dy}{dx}\right).\left(\frac{dx}{dy}\right) = 1$

9. (a) $\because f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right)$

$$+\cos x\cos\left(x + \frac{\pi}{3}\right)$$

$$= \frac{1}{2}\left\{2\sin^2 x + 2\sin^2\left(x + \frac{\pi}{3}\right)\right.$$

$$\left.+2\cos x\cos\left(x + \frac{\pi}{3}\right)\right\}$$

$$= \frac{1}{2}\left\{(1 - \cos 2x) + \left(1 - \cos\left(2x + \frac{2\pi}{3}\right)\right)\right.$$

$$\left.+\cos\left(2x + \frac{\pi}{3}\right) + \cos\frac{\pi}{3}\right\}$$

$$= \frac{1}{2}\left\{\frac{5}{2} + \cos\left(2x + \frac{\pi}{3}\right) - \cos 2x\right.$$

$$\left.-\cos\left(2x + \frac{2\pi}{3}\right)\right\} = \frac{1}{2}\left\{\frac{5}{2} + \cos\left(2x + \frac{\pi}{3}\right)\right.$$

$$\left.-2\cos\left(2x + \frac{\pi}{3}\right).\cos\frac{\pi}{3}\right\} = \frac{1}{2}\left\{\frac{5}{2} + 0\right\} = \frac{5}{4}$$

$$\therefore \quad f'(x) = 0$$

10. (b) $\because \frac{d}{dx}\{\tan^{-1}(\sec x + \tan x)\}$

$$= \frac{d}{dx}\left\{\tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right)\right\}$$

$$= \frac{d}{dx}\left\{\tan^{-1}\left(\frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right)\right\}$$

$$= \frac{d}{dx}\left\{\tan^{-1}\left(\left(\tan\frac{\pi}{4} + \frac{x}{2}\right)\right)\right\}$$

$$= \frac{d}{dx}\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1}{2}$$

and $\frac{d}{dx}\{\cot^{-1}(\text{cosec } x + \cot x)\}$

$$= \frac{d}{dx}\left\{\cot^{-1}\left(\frac{1 + \cos x}{\sin x}\right)\right\}$$

$$= \frac{d}{dx}\left\{\cot^{-1}\left(\cot\frac{x}{2}\right)\right\} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

11. (a) $\because u = f(\tan x) \Rightarrow \frac{du}{dx} = f'(\tan x)\sec^2 x$

$$v = g(\sec x) \Rightarrow \frac{dv}{dx} = g'(\sec x)\sec x\tan x$$

$$\therefore \quad \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{f'(\tan x)}{g'(\sec x)}.\frac{1}{\sin x}$$

$$\therefore \quad \left.\frac{du}{dv}\right|_{x=\frac{\pi}{4}} = \frac{f'(1)}{g'(\sqrt{2})}.\sqrt{2} = \frac{2}{4}.\sqrt{2} = \frac{1}{\sqrt{2}}$$

12. (a) $\because e^{xy} + ln(xy) + \cos(xy) + 5 = 0$

then $e^{xy}\frac{d}{dx}(xy) + \frac{1}{(xy)}\frac{d}{dx}$

$$(xy) - \sin(xy)\frac{d}{dx}(xy) = 0$$

$$\Rightarrow \quad \frac{d}{dx}(xy)\left\{e^{xy}\frac{1}{xy} - \sin(xy)\right\} = 0$$

$$\because \quad e^{xy} + \frac{1}{xy} - \sin(xy) \neq 0$$

$$\therefore \quad \frac{d}{dx}(xy) = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} + y.1 = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

13. (b) **1st Solution**

$f(x) = g(x)\sin x$ we have $f'(x) = g(x)\cos x + g'(x)\sin x$

$f'(0) = g(0)$

$f''(x) = 2g'(x)\cos x - g(x)\sin x + g''(x)\sin x$

$f''(0) = 2g'(0) = 0$

$$\lim_{x\to 0}[g(x)\cot x - g(0)\text{cosec } x]$$

$$= \lim_{x\to 0}\frac{g(x)\cos x - g(0)}{\sin x}$$

$$= \lim_{x\to 0} \frac{g'(x)\cos x - g(x)\sin x}{\cos x}$$

(using L 'Hospital's rule)

$g'(0) = 0 = f''(0)$

Statement-1 and statement 2 are both true but statement-2 doen't explain statement-1. Then the correct choice is (b).

2nd Solution:

(Appealing to definition of differentiable coefficient at a point)

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x}$$

$$= \lim_{x\to 0} \frac{g(x)\cos x + g'(x)\sin x - g(0)}{x}$$

$$= \lim_{x\to 0} \frac{g(x)\cos x - g(0)}{x} + \lim_{x\to 0} \frac{g'(x)\sin x}{x}$$

$$= \lim_{x\to 0} \frac{g(x)\cos x - g(0)}{\sin x}.\frac{\sin x}{x} + g'(0)$$

$$= \left(\lim_{x\to 0} \frac{g(x)\cos x - g(0)}{\sin x}\right).\left(\lim_{x\to 0} \frac{\sin x}{x}\right) + 0$$

$$= \lim_{x\to 0}(g(x)\cot x - g(0)\operatorname{cosec} x)$$

IIT-BOOSTER

1. Let $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Test whether

(i) $f(x)$ is continuous at $x = 0$

(ii) $f(x)$ is differentiable at $x = 0$

[IIT-1997]

Solution

(i) **Continuity:** $R = L = V = 0$ at $x = 0$

$\therefore$ continuous

$$R \lim_{h\to 0} he^{-\infty} = 0 \qquad L \lim_{h\to 0} -he^{-0} = 0 = V$$

(ii) **Differentiability:**

$$R' = \lim \frac{he^{-\infty} - 0}{h} = \lim e^{-\infty} = 0$$

$$L' = \lim \frac{-he^{-0} - 0}{-h} = 1$$

Since $R' \neq L'$ $\quad\therefore$ Not differentiable.

2. If f : $[-2a, 2a] \to R$ be an odd function such that left hand derivative at $x = a$ is zero and $f(x) = f(2a - x)$, $x \in (a, 2a)$ then find left hand derivative of f at $x = -a$.

[IIT-2003]

Solution

Given $f(-x) = -f(x)$ .... (1)

$$Lf'(a) = \underset{h\to 0}{Lt} \frac{f(a-h) - f(a)}{-h} = 0$$

.... (2)

since left hand derivative is zero.

$f(x) = f(2a - x)$, $x \in (a, 2a)$ .... (3)

We have to find $Lf'(-a)$.

$$Lf'(-a) = \underset{h\to 0}{Lt} \frac{f(-a-h) - f(-a)}{-h}$$

$$= \underset{h\to 0}{Lt} \frac{-f(a+h) + f(a)}{-h}, \text{ by (1)}$$

$$= \underset{h\to 0}{Lt} \frac{-f\{2a-(a-h)\} + f(a)}{-h}$$

$$= \underset{h\to 0}{Lt} \frac{-f(a-h) + f(a)}{-h} \text{ by (3)}$$

$$= \underset{h\to 0}{Lt} \frac{f(a-h) - f(a)}{-h} = 0, \text{ by (2)}$$

3. Discuss the differentiability of sin $\{\pi(x - [x])\}$ vwin $(-\pi/2, \pi/2)$ ***[IIT-1992]***

Solution

$f(x) = \sin[\pi(x - \{x\})]$

$$f(x) = \begin{cases} \sin\pi(x+2) = \sin(2\pi + \pi x) = \sin\pi x, I \\ \sin\pi(x+1) = \sin(\pi + \pi x) = -\sin\pi x, II \\ \sin\pi(x-0) = \sin(\pi x - 0) = \sin\pi x, III \\ \sin\pi(x-1) = -\sin(\pi - \pi x) = -\sin\pi x, IV \end{cases}$$

Consider $x = 1, x = -1$

$$L'(1) = \frac{\sin\pi(1-h) - 0}{-h} = \frac{\sin\pi h}{-h} = -\pi, III$$

$$R'(1) = \frac{-\sin\pi(1+h) - 0}{h} = \frac{\sin\pi h}{h} = \pi, IV$$

Since $L'(1) \neq R'(1)$ $\therefore$ function is not differentiable at $x = 1$.
Similarly, it can be shown that $L'(-1) = -\pi$, $R'(-1) = \pi$ hence not differentiable at $x = -1$ also.

4. Let $h(x) = \min\{x, x^2\}$, for every real number of x. Then ***[IIT-1998]***
   (a) h is continuous for all x
   (b) h is differentiable for all x
   (c) $h'(x) = 1$, for all $x > 1$
   (d) h is not differentiable at two values of x.

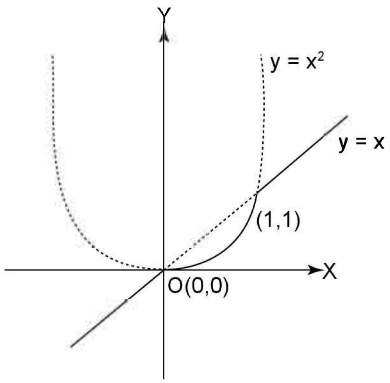
Solution

(a), (c), (d)

Let us consider continuity and differentiability for $x = 0, 1$.

Hence we define the function $h(x) = \min\{x, x^2\}$ as under:

$$h(x) = \begin{cases} x, & x < 0 \\ 0, & x = 0 \\ x^2, & 0 < x < 1 \\ x, & x > 1 \end{cases}$$



Clearly $h(x)$ being a polynomial in x is continuous for all $x \Rightarrow$ (a).
Also $h'(x) = 1\ \forall\, x > 1 \Rightarrow$ (c).
Again both at $x = 0$ and $x = 1$, $h(x)$ is not differentiable as
$L' \neq R' \Rightarrow$ (d)

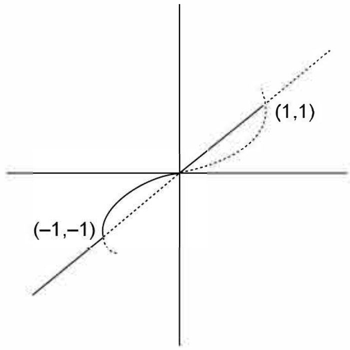
5. Let $f : R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is NOT differentiable is ***[IIT-2001S]***
   (a) $\{-1, 1\}$ (b) $\{-1, 0\}$
   (c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

Solution

(d) $f(x) = \max$,

$$\{x, x^3\} = \begin{cases} x & ; \ x < -1 \\ x^3 & ; \ -1 \le x \le 0 \\ x & ; \ 0 < x < 1 \\ x^3 & ; \ x \ge 1 \end{cases}$$

$$\therefore \quad f'(x) = \begin{cases} 1 & ; \ x < -1 \\ 3x^2 & ; \ -1 \le x \le 0 \\ 1 & ; \ 0 < x < 1 \\ 3x^2 & ; \ x \ge 1 \end{cases}$$



Clearly f is not differentiable at -1, 0 and 1

Alternative: Graph of $f(x) = \max\{x, x^3\}$ is as shown with solid lines. From graph at $x = -1$, 0, 1 we have sharp turns,

$\therefore$ $f(x)$ is not differentiable at $x = -1, 0, 1$.

6. Which of the following functions is differentiable at $x = 0$? ***[IIT-2001S]***
   (a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$
   (c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

Solution

(c), (a)

(A) $f(x) = \cos|x| + |x| = \begin{cases} \cos x - x, & x < 0 \\ \cos x + x, & x \ge 0 \end{cases}$

$$f'(x) = \begin{cases} -\sin x - 1, & x < 0 \\ -\sin x + 1, & x \ge 0 \end{cases} \qquad \begin{aligned} &At\ x = 0 \\ &LD = -1 \\ &RD = 1 \end{aligned}$$

$\therefore$ Not differentiable

(b) $f(x) = \cos|x| - |x| = \begin{cases} \cos x + x, & x < 0 \\ \cos x - x, & x \ge 0 \end{cases}$

Not differentiable at $x = 0$

(c) $f(x) \sin|x| - |x| = \begin{cases} -\sin x + x, & x < 0 \\ \sin x - x, & x \geq 0 \end{cases}$

$$f'(x) = \begin{cases} -\cos x + 1, & x < 0 \\ +\cos x - 1, & x \geq 0 \end{cases} \qquad \begin{matrix} At\ x = 0 \\ LD = 0 \\ RD = 0 \end{matrix}$$

$\therefore$ f is differentiable at $x = 0$.

(d) $f(x) = \sin|x| + |x| = \begin{cases} -\sin x - x, & x < 0 \\ +\sin x + x, & x \geq 0 \end{cases}$

Not differentiable at $x = 0$

7. $f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right), & -\dfrac{1}{2} < x < 0 \\ 1/2, & x = 0 \\ \dfrac{e^{\frac{ax}{2}} - 1}{x}, & 0 < x < \dfrac{1}{2} \end{cases}$

If $f(x)$ is differentiable at $x = 0$ and $|c| < 1/2$ then find the value of 'a' and prove that $64b^2 = (4 - c^2)$. ***[IIT-2004]***

Solution

For differentiability at $x = 0$, we must have $R' = L'$

or $\displaystyle \underset{h\to 0}{Lt} \frac{f(0+h) - f(0)}{h}$

$$= \underset{h\to 0}{Lt} \frac{f(0-h) - f(0)}{-h}$$

$$R' = \underset{h\to 0}{Lt} \frac{\dfrac{e^{\frac{a}{2}h} - 1}{h} - \dfrac{1}{2}}{h} = \underset{h\to 0}{Lt} \frac{\left(e^{\frac{a}{2}h} - 1\right) - \dfrac{h}{2}}{h^2}$$

$$= \underset{h\to 0}{Lt} \frac{\left[1 + \left(\dfrac{a}{2}h\right) + \dfrac{1}{2!}\left(\dfrac{a}{2}h\right)^2 + \ldots - 1\right] - \dfrac{h}{2}}{h^2}$$

$$= \underset{h\to 0}{Lt} \frac{(a-1)\dfrac{h}{2} + \dfrac{a^2}{8}h^2 + \ldots..}{h^2}$$

For the limit to exist we must have $a - 1 = 0$ i.e., $a = 1$ and in that case

$$R' = \underset{h\to 0}{Lt} \frac{\dfrac{1}{8}h^2 + \ldots..}{h^2} = \frac{1}{8},$$

$$L' = \underset{h\to 0}{Lt} \frac{b\sin^{-1}\dfrac{c-h}{2} - \dfrac{1}{2}}{-h} = R' = \frac{1}{8}$$

By expansion of $\sin^{-1} x$ the L.H.S. is

$$\underset{h\to 0}{Lt} \frac{b\left\{\dfrac{c-h}{2} + \dfrac{1^2}{2}\left(\dfrac{c-h}{2}\right)^3 + \dfrac{1^2.3^2}{5!}\left(\dfrac{c-h}{2}\right)^5 \ldots.\right\} - \dfrac{1}{2}}{-h}$$

The constant term in N^r will be zero and the coefficient of $-h$ in the N^r will be

$$\frac{b}{2}\left\{1 + \frac{1^2}{3!}3\left(\frac{c}{2}\right)^2 + \frac{1^2.3^2}{5!}.5\left(\frac{c}{2}\right)^4 + \ldots.\right\}$$

Above is clearly differentiation of $b\sin^{-1}\dfrac{x}{2}$ at $x = c$ which much be equal to $\dfrac{1}{8} = R'$

$$\therefore \quad \frac{b}{2}\frac{1}{\sqrt{1 - \dfrac{x^2}{4}}} \text{ at } x = c = \frac{1}{8}$$

$$\therefore \quad 64b^2 = 4\left(1 - \frac{c^2}{4}\right) = 4 - c^2.$$

8. For a real number y, let $[y]$ denote the greatest integer less than or equal to y. Then

$f(x) = \dfrac{\tan(\pi[x - \pi])}{1 + [x]^2}$ is

(a) discontinuous at some x

(b) continuous at all x, but the derivative $f'(x)$ does not exist for some x.

(c) $f'(x)$ exists for all x but second derivative $f''(x)$ does not exist.

(d) $f'(x)$ exists for all x.

Solution

(d) By definition $[x - \pi]$ is an integer whatever x may be and so $\pi[x - \pi]$ is an integral multiple of π.

Consequently $\tan(\pi[x - \pi]) = 0$ for all x.

And since $1 + [x]^2 \neq 0$ for any x, we conclude that $f(x) = 0$.

Thus $f(x)$ is constant function and so it is continuous and differentiable any number of times, that is $f'(x), f''(x)$,

$f'''(x) \ldots, f^n(x), \ldots..,$ all exist for every x, their value being 0 at every point x. Hence of all the given alternatives only (d) is correct.

9. Let $[x]$ denote the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is ***[IIT-1986]***

(a) continuous at $x = 0$
(b) continuous in $(-1, 0)$
(c) differentiable at $x = 1$
(d) differentiable in $(-1, 1)$

Solution

(a), (b) and (d) By def. of $[x]$, we easily see that $f(x) = [x \sin \pi x] = 0$ when $-1 \le x \le 1$

[Note that $-1 \le x \le 1 \Rightarrow 0 \le x \sin \pi x \le \frac{1}{2}$]

and $f(x) = [x \sin \pi x] = -1$ when $1 < x < 1 + h$, (h small)

[Note that: $x \sin \pi x$ becomes negative and numerically less than 1 when x is slightly greater than 1 and so by def.of $[x]$ $[x \sin \pi x] = -1$ when $1 < x < [1 + h]$]

Thus $f(x)$ is constant and equal to 0 in the closed interval $[-1, 1]$ and so $f(x)$ is continuous and differentiable in the open interval $(-1, 1)$. At $x = 1$,

$f(x)$ is clearly discontinuous since $f(1 - 0) = 0$ and $f(1 + 0) = -1$ and $f(x)$ is non-differentiable at $x = 1$. Hence (a), (b) and (d) are correct answers.

10. If $f(x)$ is twice differentiable polynomial function such that $f(1) = 1, f(2) = 4, f(3) = 9$, then ***[IIT (Screening)-2005]***

(a) $f''(x) = 2, \forall x \in R$
(b) There exist at least one $x \in (1, 3)$ such that $f''(x) = 2$
(c) There exist at least one $x \in (2, 3)$ such that $f'(x) = 5 = f''(x)$
(d) There exist at least one $x \in (1, 2)$ such that $f(x) = 3$

Solution

(b) Let a function be $g(x) = f(x) - x^2$

$\Rightarrow$ $g(x)$ has at least 3 real roots which are $x = 1, 2, 3$

$\Rightarrow$ $g'(x)$ has at least 2 real roots in $x \in (1,3)$

$\Rightarrow$ $g''(x)$ has at least 1 real roots in $x \in (1, 3)$

$\Rightarrow$ $f'(x) = 2$ for at least one $x \in (1, 3)$

11. Let $F(x) = f(x)\, g(x)\, h(x)$ for all real x, where $f(x)$, $g(x)$ and $h(x)$ are differentiable functions. At some point x_0, $F'(x_0) = 21F(x_0)$, $f'(x_0) = 4f(x_0)$, $g'(x_0) = -7g(x_0)$ and $h'(x_0) = kh(x_0)$. Then $k =$ ..... ***[IIT-1997C]***

Solution

$F(x) = f(x)\, g(x)\, h(x), \forall x \in R$

$f(x)$, $g(x)$, $h(x)$ are differentiable functions, therefore

$F'(x) = f'(x)\, g(x)\, h(x) + f(x)\, g'(x)\, h(x) + f(x)\, g(x)\, h'(x)$

At $x = x_0$

$F'(x_0) = f'(x_0)\, g(x_0)\, h(x_0) + f(x_0)\, g'(x_0)\, h(x_0) + f(x_0)\, g(x_0)\, h'(x_0)$ using the given values of $F'(x_0)$, $f'(x_0)$, g' and $h'(x_0)$ we get $21\, F(x_0) = 4f(x_0)\, g(x_0)\, h(x_0) - 7\, f(x_0)\, g(x_0)\, h(x_0) + kh(x_0)\, f(x_0)\, g(x_0)$

$\Rightarrow \quad 21 = 4 - 7 + k \Rightarrow k = 24$

$(\because F(x_0) = f(x_0)\, g(x_0)\, h(x_0))$

12. The function given by $y = |\,|x| - 1\,|$ is differentiable for all real numbers except the points ***[IIT-2005 S]***

(a) $\{0, 1, -1\}$ (b) ± 1
(c) 1 (d) -1

Solution

(a) Given function is $y = |\,|x| - 1\,|$

or $\quad y = \begin{cases} -|x| + 1 & \text{if } |x| < 1 \\ |x| - 1 & \text{if } |x| - 1 \end{cases}$

$= \begin{cases} -|x| + 1 & \text{if } -1 < x < 1 \\ |x| - 1 & \text{if } |x| \le -1 \text{ or } x \ge 1 \end{cases}$

$= \begin{cases} -x - 1 & \text{if } x \le -1 \\ x + 1 & \text{if } -1 < x < 0 \\ -x + 1 & \text{if } 0 \le x < 1 \\ x - 1 & \text{if } x \ge 1 \end{cases}$

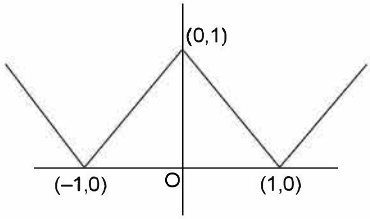
Here $Ly'(-1) = -1$ and $Ry'(-1) = 1$

$Ly'(0) = 1$ and $Ry'(0) = -1$ and $Ly'(1) = -1$ and $Ry'(1) = 1$

$\Rightarrow$ y is not differentiable at $x = -1, 0, 1$

$\therefore$ (a) is the correct option

Alternative:



Graph of $y = ||x|-1|$ is as follows:

Which has sharp turnings at $x = -1$, 0 and 1 and hence not differentiable at $x = -1, 0, 1$.

13. Let $f(x)$ be defined in the interval $[-2, 2]$ such that $f(x) = -1, -2 \le x \le 0 = x - 1, 0 < x \le 2$ and $g(x) = f(|x|) + |f(x)|$
Test the differentiablity of $g(x)$ in $(-2, 2)$.
[IIT-1986]

Solution

We have $f(x) = -1, -2 \le x \le 0$
$\qquad\qquad = x - 1, 0 < x \le 2$

and $\quad g(x) = f(|x|) + |f(x)|$

Hence $g(x)$ involves $|x|$ and $|x - 1|$ or $|-1| = 1$

Therefore we should divide the given interval $(-2, 2)$ into the following intervals.

I_1	I_2	I_3
$[-2, 2] = [-2, 0)$	$[0, 1)$	$[1, 2]$
$x = -$ve	+ve	+ve
$\|x\| = +$ve	+ve	+ve
$f(x) = -1$	-1	$x - 1 \quad x - 1$
$f(\|x\|) = \|x\| - 1$	$\|x\| - 1$	$\|x\| - 1$
$= -x - 1$	$= x - 1$	$= x - 1$
$\|f(x)\| = \|-1\|$	$\|x - 1\|$	$\|x - 1\|$
$= 1$	$= -(x - 1)$	$= x - 1$

$\therefore$ Using above we get $g(x) = f|x| + |f(x)|$
$\qquad = -x - 1 + 1 = -x$ in I_1

in $I_2 = x - 1 - (x - 1) = 0$

in $I_3 = (x - 1) + (x - 1) = 2(x - 1)$

Hence $g(x)$ is defined as follows:

$g(x) = -x, -2 \le x \le 0$
$\qquad = 0, \ 0 < x < 1$
$\qquad = 2(x - 1), 1 \le x \le 2$

Clearly $g'(x) = -1, 0, 2$ respectively as $g(x)$ is polynomial in x.

Also $Lg'(0) = -1$; $Rg'(0) = 0$ (not equal)
$Lg'(1) = 0$; $Rg'(1) = 2$ (not equal)
Hence $g(x)$ is not differentiable both at $x = 0$ and $x = 1$.

14. $f'(0) = \lim\limits_{n\to\infty} nf\left(\dfrac{1}{n}\right)$ and $f(0) = 0$. Using

this find $\lim\limits_{n\to\infty}\left((n+1)\dfrac{2}{\pi}\cos^{-1}\left(\dfrac{1}{n}\right) - n\right)$,

$\left|\cos^{-1}\dfrac{1}{n}\right| < \dfrac{\pi}{2}$ ***[IIT-2004]***

Solution

To find $\lim\limits_{n\to\infty}\left[(n+1)\dfrac{2}{\pi}\cos^{-1}\left(\dfrac{1}{n}\right) - n\right]$

$$= \lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)\frac{2}{\pi}\cos^{-1}\left(\frac{1}{n}\right) - 1\right]$$

$$= \lim_{n\to\infty} nf\left(\frac{1}{n}\right)$$

where $f(x) = \left[(1+x)\dfrac{2}{\pi}\cos^{-1}x - 1\right]$ s.t. $f(0) =$

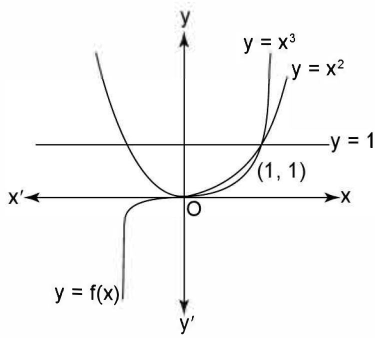
$$\left[(1+0)\frac{2}{\pi}\cos^{-1}0 - 1\right] = \frac{2}{\pi}\cdot\frac{\pi}{2} - 1 = 0$$

15. If $f(x) = \min\{1, x^2, x^3\}$, then
[IIT-2006]

(a) $f(x)$ is continuous $\forall\, x \in R$
(b) $f'(x) > 0, \ \forall x > 1$
(c) $f(x)$ is continuous but not differentiable $\forall$ $x \in R$
(d) $f(x)$ is not differentiable at two points

Solution

(a, c) from graph $f(x)$ is continuous every where but not differentiate at $x = 1$.



16. Let $g(x)=\dfrac{(x-1)^n}{\log\cos^m(x-1)}$; $0<x<2$, m and n are integers, $m\neq 0$, $n>0$, and let p be the left hand derivative of $|x-1|$ at $x=1$. If $\lim\limits_{x\to 1^+} g(x)=p$, then

[IIT JEE-2008]

(a) $n=1, m=1$
(b) $n=1, m=-1$
(c) $n=2, m=2$
(d) $n>2, m=n$

Solution

(c) First Solution: (Applying the definition of differential coefficient at a point)

$$g(x)=\frac{(x-1)^n}{\log\cos^m(x-1)};$$

$$p=\lim_{x\to 1^+} g(x)=\lim_{x\to 1^+}\frac{(x-1)^n}{\log\cos^m(x-1)}$$

Set $x=1+h$, then $h>0$. Also the limit reduces to

$$p=\lim_{h\to 0}\frac{h^n}{\log\cos^m h}=\lim_{h\to 0}\frac{h^n}{m\log(\cos h)}$$

$$=\frac{1}{m}\lim_{h\to 0}\frac{h^n}{\log[1-(1-\cos h)]}$$

$$=\frac{1}{m}.\lim_{h\to 0}\frac{h^n}{\dfrac{\log[1-(1-\cos h)]}{(1-\cos h)}.(1-\cos h)}$$

$$=\frac{1}{m}.\lim_{h\to 0} h^{n-2}.\frac{1}{\dfrac{\log[1-(1-\cos h)]}{1-\cos h}}.\frac{h^2}{1-\cos h}$$

$$=\frac{1}{m}\lim_{h\to 0} h^{n-2}.\frac{1}{\dfrac{\log[1-(1-\cos h)]}{1-\cos h}}.\frac{h^2}{2\sin^2\dfrac{h}{2}}$$

$$=\frac{1}{m}\left(\lim_{h\to 0} h^{n-2}\right)\left(\frac{1}{\lim\limits_{h\to 0}\dfrac{\log[1-(1-\cos h)]}{1-\cos h}}\right).$$

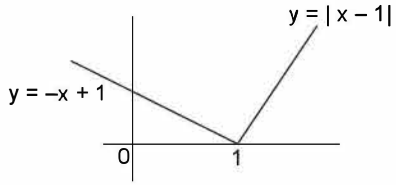
$$2\left\{\lim_{h\to 0}\left(\frac{h/2}{\sin\dfrac{h}{2}}\right)^2\right\}$$

$$=\frac{2}{m}.\left(\lim_{h\to 0} h^{n-2}\right)\left(\frac{1}{\lim\limits_{h\to 0}\dfrac{\log[1-(1-\cos h)]}{1-\cos h}}\right)$$

$$\left\{\lim_{h\to 0}\left(\frac{h/2}{\sin\dfrac{h}{2}}\right)^2\right\}$$

$$=\frac{2}{m}.\left(\lim_{h\to 0} h^{n-2}\right).(-1)(1)=-\frac{2}{m}\left(\lim_{h\to 0} h^{n-2}\right)$$

As p is the left hand derivative of $|x-1|$ at $x=1$ we have $p=-1$, as can be seen from the graph of $y=|x-1|$ using $p=-1$ in (A), we have $P=-\dfrac{2}{m}\left(\lim\limits_{h\to 0} h^{n-2}\right)$



$$\Rightarrow\quad -1=-\frac{2}{m}\left(\lim_{h\to 0} h^{n-2}\right)$$

$$\Rightarrow\quad 1=\frac{2}{m}\left(\lim_{h\to 0} h^{n-2}\right)$$

For the above to be satisfied, $n=2$ which gives, $1=\dfrac{2}{m}.1$

$\Rightarrow m=2$ Thus $m=2$ and $n=2$.

2nd Solution (Using L'Hospital's rule)

We have, as in the first solution

$$p=\lim_{h\to 0}\frac{h^n}{\log\cos^m h}=\lim_{h\to 0}\frac{h^n}{m\log(\cos h)}$$

$$=\frac{1}{m}\lim_{h\to 0}\frac{h^n}{\log(\cos h)}\left(\frac{0}{0}\text{form}\right)$$

$$=\frac{1}{m}\lim_{h\to 0}\frac{nh^{n-1}}{-\tan h}=-\frac{n}{m}\lim_{h\to 0}\frac{h^{n-1}}{\tan h}$$

$$=-\frac{n}{m}\lim_{h\to 0} h^{n-2}.\left(\frac{h}{\tan h}\right)$$

$$=-\frac{n}{m}\left(\lim_{h\to 0} h^{n-2}\right)\left(\lim_{h\to 0}\frac{h}{\tan h}\right)$$

$$=-\frac{n}{m}\left(\lim_{h\to 0} h^{n-2}\right).1$$

But $p = -1$, which gives

$$-1 - \frac{n}{m}\left(\lim_{h\to 0} h^{n-2}\right) \Rightarrow 1 = \frac{n}{m}\left(\lim_{h\to 0} h^{n-2}\right)$$

For the above to be satisfied $n = 2$, which then gives

$1 = \frac{n}{m}.\ 1 \Rightarrow m = n$ Thus, $m = n = 2$, as before.

17. If $f(x)$ is a differentiable function such that f : $R \to R$ and $f\left(\frac{1}{n}\right) = 0\ \forall\, n \geq 1, n \in I$ then

[IIT Screening-2005]

(a) $f(x) = 0\ \forall\ x \in (0, 1)$
(b) $f(0) = 0 = f'(0)$
(c) $f(0) = 0$ but $f'(0)$ may or may not be 0
(d) $|f(x)| \leq 1, \forall\ x \in (0,1)$

Solution

(b) $f(1) = f\left(\frac{1}{2}\right) = f\left(\frac{1}{3}\right)$

$= \ldots\ldots\ldots\ldots = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0$

Since there are infinitely many points in $x \in (0, 1)$

where $f(x) = 0$ and $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0 \Rightarrow f(0) = 0$

And since there are infinitely many points in the neighbourhood of $x = 0$ such that

$\Rightarrow$ $f(x)$ remains constant in the neighbourhood of $x = 0$
$\Rightarrow$ $f'(0) = 0$.

18. The function $f(x) = \max[(1 - x), (1 + x), 2]$, $x \in (-\infty, \infty)$, is

[IIT-1995]

(a) Continuous at all points
(b) Differentiable at all points
(c) Differentiable at all points except at $x = 1$ and $x = -1$
(d) Continuous at all points except at $x = 1$ and $x = -1$ where it is discontinuous.

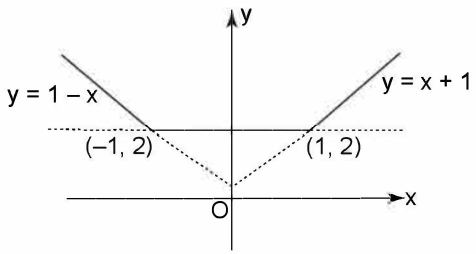
Solution

(a, c) $f(x) = \max\ \{(1 - x), (1 + x), 2\}$; $\forall\ x \in (-\infty, \infty)$

$$f(x) = \begin{cases} 1+x & ; \quad x > 1 \\ 2 & ; \quad -1 \leq x \leq 1 \\ 1-x & ; \quad x < -1 \end{cases}$$

Since $f(x) = 1 - x$ or $1 + x$ are polynomial functions and $f(x) = 2$ is a constant function.

$\therefore$ These are continuous at all points ... (i)



$\therefore$ $f(x)$ is differentiable at all the points, except at $x = 1$ and at $x = -1$. ....(ii)

MENTAL PREPARATION TEST

1. Prove that the greatest integer function $[x]$, is not differentiable at $x = 1$.

2. Prove that the function

$$f(x) = \begin{cases} x\tan^{-1}\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ is continuous at $x = 0$, but not differentiable at $x = 0$.

3. Show that $f(x) = \begin{cases} 12x - 13, & \text{if } x \leq 3 \\ 2x^2 + 5, & \text{if } x > 3 \end{cases}$ is differentiable at $x = 3$. Also, find $f'(3)$.

4. Prove that the function $f(x) = \frac{1}{x}$ is differentiable for all x, except $x = 0$.

5. Show that the function $f(x)=\begin{cases} x-1, & \text{if } x<2 \\ 2x-3, & \text{if } x\ge 2\end{cases}$ is not differentiable at $x = 2$.

6. Find the value of a and b is the function

$f(x)=\begin{cases} x^2, & \text{if } x\le 1 \\ 2ax+b, & \text{if } x>1\end{cases}$ derivable at $x = 1$.

7. Prove that there does not exist any differential coefficient at $x = 1$, of the function $f(x)=\begin{cases} x^2-1, & x\ge 1 \\ 1-x, & x<1\end{cases}$

8. Show that $f(x) = x^2$ is differentiable at $x = 1$ and find $f'(1)$.

9. Draw the graph of the function $y = [x] + |1 - x|$, $-1 \le x \le 3$. Determine the points, if any, where this function is not differentiable.

LECTUREWISE WARMUP TEST

1. If $f(x)$ is not differentiable finitely at $x = 2$ and $f(x) + g(x)$ is differentiable finitely at $x = 2$, then
(a) $g(x)$ must be differentiable finitely at $x = 2$
(b) $g(x)$ must be continuous at $x = 2$
(c) $g(x)$ may not be differentiable at $x = 2$
(d) $g(x)$ can not be differentiable at $x = 2$

2. Which of the following function is differentiable at $x = 0$? ***[IIT Screening-2001]***
(a) $\cos(|x|)+|x|$ (b) $\cos(|x|)-|x|$
(c) $\sin(|x|)+|x|$ (d) $\sin(|x|)-|x|$

3. Let $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & x\ne 0 \\ 0, & x=0\end{cases}$, then ***[AIEEE-2003]***
(a) $f(x)$ is continuous at $x = 0$
(b) $f(x)$ is differential at $x = 0$
(c) $f(x)$ is continuous as well as differentiable
(d) $f(x)$ is neither continuous nor differentiable

4. Let $f(x) = 1 - |\cos x|$ for all $x \in R$. Then mark one incorrect statement:
(a) $f'(\pi/2)$ does not exist
(b) $f(x)$ is continuous everywhere
(c) $f(x)$ is not differentiable anywhere
(d) $\lim\limits_{x\to\frac{\pi}{2}+0} f(x)=1$

5. If $f(1) = 8$, $f'(1) = 1/8$ and function f is differentiable also f is invertible and $g = f^{-1}$, then:
(a) $g'(1) = 8$ (b) $g'(1) = 1/8$
(c) $g'(8) = 8$ (d) $g'(8) = 1/8$

6. If $f: R\to R$ is continuous and differentiable function such that $f(1/n) = 0$ for $\forall n \in I$ and $n \ge 1$, then ***[IIT (Screening)-2005]***
(a) $f'(0) = 0 = f(0)$
(b) $f(0) = 0$ but $f'(0) \ne 0$
(c) $f(x) = 0, x \in (0, 1]$
(d) None of these

7. Consider the following statements with respect of the function, $f(x) = |ln|x||$
(i) It is symmetric about y-axis
(ii) It is continuous at every value of x
(iii) It is differentiable at $x = \pm 1$. The which of above is/are correct?
(a) 1 only (b) 2 only
(c) 1 and 3 (d) 1, 2 and 3

8. The function $f(x) = \sin^{-1}(\cos x)$ is
(a) continuous at $x = 0$
(b) discontinuous at $x = 0$
(c) differentiable at $x = 0$
(d) None of these

9. If $f(x)=\frac{1}{x}\log\left(\frac{1+px}{1-qx}\right)$ and $f(0) = p + q$, then
(a) Function has a discontinuity of the 1st kind
(b) Function has a removable discontinuity
(c) Function has a discontinuity of the 2nd kind
(d) Continuous function at $x = 0$

10. If $f(x)=\frac{\sin(2\pi[x-1])}{1+[x]+[x]^2}$ which one of the following is correct?
(a) $f(x)$ is a discontinuous function
(b) $f(x)$ is continuous, but $f'(x)$ is discontinuous
(c) $f'(x)$ is continous but $f''(x)$ is discontinuous
(d) All derivatives of $f(x)$ exists.

11. $f(x) = [\sin x] + [\cos x]$, $x \in [0, 2\pi]$, where [.] denotes the greatest integer function. Total number of points where $f(x)$ is non-differentiable is equal to

(a) 2 (b) 3
(c) 5 (d) 4

12. If $f(x) = |x - 3| + |x - 4|$, then in the interval [0, 5], the function $f(x)$ is

(a) differentiable at $x = 3$
(b) differentiable at $x = 4$
(c) not differentiable at $x = 3$ and $x = 4$
(d) not continuous in the interval [0, 5]

13. There exists a function $f(x)$ satisfying $f(0) = 1$, $f'(0) = -1$, $f(x) > 0$ for all x and

(a) $f'(x) < 0, \forall x$
(b) $-1 < f''(x) < 0, \forall x$
(c) $-2 < f''(x) \le -1, \forall x$
(d) $f''(x) < -2, \forall x$

14. Let $f(x) = x^p \cos(1/x)$, when $x \neq 0$ and $f(x) = 0$ when $x = 0$. Then $f(x)$ will be differentiable at $x = 0$ if ***[Orissa (JEE)-2002]***

(a) $p > 0$ (b) $p > 1$
(c) $0 < p < 1$ (d) $\frac{1}{2} < p < 1$

15. The function $f(x) = |x| + |x - 1|$ is

(a) Continuous at $x = 1$, but not differentiable
(b) both continuous and differentiable at $x = 1$
(c) not continuous at $x = 1$
(d) not differentiable at $x = 1$

16. f $f(x)$ is a differentiable function and $f''(0) = a$, then $\lim_{x \to 0} \frac{2f(x) - 3f(2x) + f(4x)}{x^2} =$

(a) $3a$ (b) $2a$
(c) $5a$ (d) $4a$

17. A function $f : R \to R$ satisfies $f(x + y) = f(x) f(y)$ for all $x, y \in R$ and $f(x) \neq 0$ for all $x \in R$. If $f(x)$ is differentiable at $x = 0$ and $f'(0) = 2$, then $f'(x)$ is equal to which one of the following ***[NDA-2004]***

(a) $f(x)$ (b) $-f(x)$
(c) $2f(x)$ (d) $f(x)/2$

18. Let $f(x)$ be a differentiable even function, consider the following statements

(i) $f'(x)$ is an even function
(ii) $f'(x)$ is an odd function
(iii) $f'(x)$ may be even or odd

Which of the above statements is/are correct? ***[NDA-2004]***

(a) (i) only (b) (ii) only
(c) (i) & (iii) (d) (ii) & (iii)

19. Let $g(x)$ be the inverse of an invertiable function $f(x)$ which is differentiable at $x = c$. Which one of the following is equal to $g'[f(c)]$? ***[NDA-2004]***

(a) $f'(c)$ (b) $1/f'(c)$
(c) $f(c)$ (d) $1/f(c)$

20. If $f(x)$ is differentiable function such that $f(0) = 0$, $f(1) = 1$ and $f'(x) > 0 \ \forall x \in [0, 1]$ and degree of $f(x) \le 2$, then

(a) $f(x) = \phi$
(b) $f(x) = ax + (1 - a)x^2$; $a \in R$
(c) $f(x) = ax + (1 - a)x^2$; $a \in (0, 2)$
(d) $f(x) = ax + (1 - a)x^2$; $a \in (0, \infty)$

21. If f is a real valued differentiable function satisfying $|f(x) - f(y)| \le (x - y)^2$, $x, y \in R$ and $f(0) = 0$, then $f(1)$ equals

(a) 2 (b) 1
(c) -1 (d) 0

22. The number of points in the interval (0, 2) where the derivative of the function $f(x) = |x - 1/2| + |x - 1| + \tan x$ does not exist is ***[MNR-1998]***

(a) 1 (b) 2
(c) 3 (d) 4

23. Function $f(x) = \begin{cases} |2x - 3|[x], & x \ge 1 \\ \sin(\pi x/2), & x < 1 \end{cases}$ is ***[DCE-1998]***

(a) continuous at $x = 1$ but not differentiable
(b) differentiable at $x = 1$
(c) continuous at $x = 2$
(d) None of these

24. If $f(x) = \begin{cases} x\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then correct statement is ***[Kurukshetra CEE-1998]***

(a) f is continuous at all points except $x = 0$
(b) f is continuous at every point but not differentiable
(c) f is differentiable at every point
(d) f is differentiable only at the origin

25. If $f(x) = \begin{cases} (\pi - x)\dfrac{\cos x}{|\sin x|}, & x \neq \pi \\ 1, & x = \pi \end{cases}$ then correct statement is

[Roorkee (Screening)-2000]

(a) $f(\pi - 0) = 1$
(b) $f(\pi + 0) = 1$
(c) $f(x)$ is continuous at $x = \pi$
(d) $f(x)$ is differentiable at $x = \pi$

26. Function $f(x) = \begin{cases} \dfrac{x}{1+|x|}, & |x| \geq 1 \\ \dfrac{x}{1-|x|}, & |x| < 1 \end{cases}$ is

[IIT(S)-2000; Roorkee-2000]

(a) differentiable at $x = 0$ and $x = -1$
(b) differentiable at $x = 0$ but not at $x = -1$
(c) not differentiable at $x = 0$ and $x = -1$
(d) None of these

27. If $f(x) = \begin{cases} x+2 & , \quad 1 < x < 3 \\ 5 & , \quad x = 3 \\ 8 - x & , \quad x > 3 \end{cases}$, then mark one incorrect statement

[MPPET-2001]

(a) Right hand derivative at $x = 3$ is -1
(b) Left hand derivatives at $x = 3$ is 1
(c) $f(x)$ is differentiable at $x = 3$
(d) $f'(x)$ at $x = 3$ does not exist

28. If $f(x) = \begin{cases} 210, & x < 0 \\ 210 + \sin x, & 0 \leq x < \dfrac{\pi}{2} \end{cases}$, then $f'(0) =$

(a) 210 (b) 0
(c) ∞ (d) None of these

29. The number of points at which the function $f(x) = |x| + |x - 1| + |x - 2|$ does not have a derivative in the interval [0, 2]

(a) 0 (b) 1
(c) 2 (d) 3

30. Let $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$, then

[MNR-98; MP-2003]

(a) $g(x) = \begin{cases} h(x)/x & x \neq 0 \\ 0, & x = 0 \end{cases}$ is a differentiable function
(b) h is a differentiable function
(c) h is not differentiable at $x = 0$
(d) None of these

31. Let [.] denotes the greatest integer function and $f(x) = [\tan^2 x]$, then

[IIT-1993; VIT-2008]

(a) $\lim_{x \to 0} f(x)$ does not exist
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) $f'(0) = 1$

32. Let $f(x) = \begin{cases} \sin x, & x \geq 0 \\ -\sin x, & x < 0 \end{cases}$. Then $f(x)$ is

(a) not continuous at $x = 0$
(b) differentiable at $x = 0$
(c) discontinuous at $x = 0$
(d) not differentiable at $x = 0$

33. If $f(x) = \cos^{-1}(\cos x)$ then $f(x)$ is

(a) not continuous at $x = \pi$
(b) discontinuous at $x = -\pi$
(c) differentiable at $x = 0$
(d) non differentiable at $x = \pi$

34. If $f(x) = \cot^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$ and

$g(x) = \cos^{-1}\left(\dfrac{1 - x^2}{1 + x^2}\right)$, then

$\lim_{x \to a} \dfrac{f(x) - f(a)}{g(x) - g(a)}, 0 < a < \dfrac{1}{2}$ is

[Orissa (JEE)-2002]

(a) $\dfrac{3}{2(1 + a^2)}$ (b) $\dfrac{3}{2(1 + x^2)}$
(c) 3/2 (d) $-3/2$

35. Let $f(x) = \begin{cases} \sin x, & \text{for } x \geq 0 \\ 1 - \cos x, & \text{for } x \leq 0 \end{cases}$ and $g(x) = e^x$. Then $(gof)'(0)$ is

[UPSEAT-2004]

(a) 1 (b) -1
(c) 0 (d) None of these

36. The derivative of function $f(x)$ is $\tan^4 x$. If $f(0) = 0$ then $\lim_{x\to 0}\frac{f(x)}{x} =$ ***[J & K-2005]***

(a) 1
(b) 0
(c) –1
(d) None of these

37. Let $f(x)=\begin{cases} x^p \sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$ then $f(x)$ is continuous but not differential at $x = 0$ if ***[DCE-2005]***

(a) $0 < p \le 1$
(b) $1 \le p < \infty$
(c) $-\infty < p < 0$
(d) $p = 0$

LECTUREWISE WARMUP TEST: SOLUTIONS

1. (d) As $f(x)$ is not differentiable at $x = 2$, $f(x) + g(x)$ can be differentiable at $x = 2$ only if $g(x)$ is also not differentiable at $x = 2$.

2. (d) R.H.D of

$$\sin|x| - |x| = \lim_{h\to 0}\frac{\sin h - h}{h} = 1 - 1 = 0 \text{ and}$$

$f(0) = 0$

$$\text{L.H.D. of } \sin|x| - |x| = \lim_{h\to 0}\frac{\sin|-h| - |-h|}{-h}$$

$$= \frac{\sin h - h}{-h} = 0$$

Therefore (d) is the answer.

3. (a) Here $f(x)=\begin{cases} xe^{-2/x}, & x>0 \\ x, & x<0 \\ 0, & x=0 \end{cases}$

$$\therefore \lim_{h\to 0} g(0+h) = \lim_{h\to 0} he^{-2/h}$$

$$= 0\times e^{-\infty} = 0\times 0 = 0$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0}(0-h) = 0,\ f(0) = 0$$

So, $f(x)$ is continuous at $x = 0$. Also $xe^{-2/x}$, x, 0 are continuous in their respective intervals.

$\therefore$ $f(x)$ is continuous everywhere

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{he^{-2/h}-0}{h} = 0$$

$$\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{-h-0}{-h} = 1$$

So, $f(x)$ is not differentiable at $x = 0$.

4. (c) $f(x) = 1 - \cos x$, $0 < x < \pi/2$
$= 1 + \cos x$, $\pi/2 < x < 0$
Then (c) is the answer.

5. (c) Since $f(1) = 8$ and $g = f^{-1}$, therefore, $g(8) = 1$.
Further, as $g = f^{-1}$, therefore, gof is the indentity function

$\Rightarrow$ $(gof)'(c) = 1\ \forall\ C \in D_{gof}$

$\Rightarrow$ $g'(f(c))f'(c) = 1$

$\Rightarrow$ $g'(f(1))f'(1) = 1 \Rightarrow g'(8).\ \frac{1}{8} = 1$

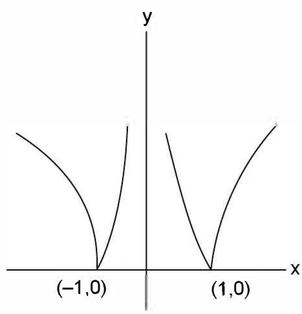
$\Rightarrow$ $g'(8) = 8$

6. (a) $f(1) = f(1/2) = f(1/3) = \ldots\ldots$
$= \lim_{n\to\infty} f(1/n) = 0$

$\therefore$ for infinite many points lying in (0, 1], $f(x) = 0$ and also $\lim_{n\to\infty} f(1/n) = 0$.

$\therefore$ $f(0) = 0$ also $f(x)$ is constant for all x lying in small neighbourhood of $x = 0$
$\therefore f'(x) = 0$.

7. (a) Clearly not continuous at $x = 0$ and also not differentiable at corner points (–1, 0) and (1, 0).



8. (a) $\because f(x) = \sin^{-1}(\cos x)$ $\therefore \lim_{x\to 0} \sin^{-1}(\cos x) =$ $\sin^{-1}(\cos 0) = \sin^{-1}(1) = \pi/2$ and $f(0) = \pi/2$

$\therefore$ Function is continuous at $x = 0$

But $f'(x) = \dfrac{-\sin x}{\sqrt{1-\cos^2 x}} = \dfrac{-\sin x}{|\sin x|}$

$$= \begin{cases} \dfrac{-\sin x}{-\sin x} = 1, & x < 0 \\ \dfrac{-\sin x}{\sin x} = -1, & x > 0 \end{cases}$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

9. (d) Let $f(x)$ is continuous at $x = 0$. So $f(0) = \lim_{x\to 0} f(x)$

$$\Rightarrow \quad f(0) = \lim_{x\to 0} \frac{\log(1+px) - \log(1-qx)}{x}$$

$$= \lim_{x\to 0}\left(\frac{p}{1+px} + \frac{q}{1-qx}\right) = f(0) = p + q$$

Hence (d) is the correct answer.

10. (d) $[x-1]$ is an integer $n \le x - 1$ $\therefore$ $\sin 2\pi x$ $= 0 \ \forall\ n$; $\dfrac{\sin(2\pi[x-1])}{1+[x]+[x]^2} = \dfrac{0}{1+[x]+[x^2]} = 0$

11. (c) $[\sin x]$ is non-differentiable at $x = \dfrac{\pi}{2}$, π, 2π and $[\cos x]$ is non-differentiable at $x = 0$, $\dfrac{\pi}{2}, \dfrac{3\pi}{2}, 2\pi$

Thus $f(x)$ is definitely non-differentiable at $x = \pi, 3\pi/2, 0$

Also $f\left(\dfrac{\pi}{2}\right) = 1, f\left(\dfrac{\pi}{2} - 0\right) = 0$, $f(2\pi) = 1$, $f(2\pi - 0) = -1$

Thus $f(x)$ is discontinuous and hence non-differentiable at $x = \pi/2$ and 2π.

12. (c) At $x = 3$; L.H.D. =

$$\lim_{x\to 3} \frac{|x-3| + |x-4| - |3-4|}{x-3}$$

$$= \lim_{h\to 0} \frac{|3-h-3| + |3-h-4| - 1}{3-h-3}$$

$$= \lim_{h\to 0} \frac{2h+1-1}{-h} = -2$$

and R.H.D. $= \lim_{x\to 3} \dfrac{|x-3| + |x-4| - |3-4|}{x-3}$

$$= \lim_{h\to 0} \frac{|3+h-3| + |3+h-4| - 1}{3+h-3}$$

$$= \lim_{h\to 0} \frac{2h-2}{h} = \infty$$

$\because$ L.H.D $\ne$ R.H.D.

$\therefore$ $f(x)$ is not differentiable at $x = 3$

At $x = 4$,

$$\text{L.H.D.} = \lim_{x\to 4} \frac{|x-3| + |x-4| - |3-4|}{x-4}$$

$$= \lim_{h\to 0} \frac{|4-h-3| + |4-h-4| - 1}{4-h-4}$$

$$= \lim_{h\to 0} \frac{1-h+h-1}{-h} = 0$$

$$\text{and R.H.D.} = \lim_{h\to 0} \frac{|4+h-3| + |4+h-4| - 1}{4+h-4}$$

$$= \lim_{h\to 0} \frac{1+h+h-1}{h} = 2$$

$\because$ L.H.D. $\ne$ R.H.D.

$\therefore$ $f(x)$ is also not differentiable at $x = 4$.

13. (a) $f'(x) < 0 \ \forall\ x$

14. (b) If $f(x) = x \cos(1/x)$ then $f'(0-0) \ne f'(0+0)$

15. (a) $f(x) = |x-a| + |x-b|$ is continuous at $x = a$ and $x = b$ but not differentiable.

16. (a) $\lim_{x\to 0} \dfrac{2f'(x) - 6f'(2x) + 4f'(4x)}{2x}$

$$= \lim_{x\to 0} \frac{2f''(x) - 12f''(2x) + 16f''(4x)}{2}$$

$$= \frac{2f''(0) - 12f''(0) + 16f''(0)}{2}$$

$$= \frac{2\times a + 4\times a}{2} = 3a$$

17. (c) $e^{x+y} = e^x \cdot e^y, f(x) = e^{2x}, f'(x) = 2e^{2x} = 2f(x)$

18. (b) Differentiation of even function is always odd.

19. (b) $g\{f(x)\} = x \Rightarrow g'(f(x))\, f'(x) = 1 \Rightarrow (g'(f(c)))\,(f'(c)) = 1$

20. (c) $f(x) = ax^2 + bx + c, f(0) = 0 \Rightarrow c = 0, f(1) = 1 \Rightarrow a + b = 1$

$f'(x) = 2(1-b)x + b > 0 \Rightarrow b \in (0, 2)$

$\Rightarrow \quad S = \{(1-a)x^2 + ax\}, a \in (0, 2)$

21. (d) $f'(y) = \dfrac{f(x) - f(y)}{x - y} \le (x - y)^2 = 0$

when $x \to y$ $\therefore$ f is constant $\Rightarrow f(0) = 0$ then $f(1) = 0$

22. (c) The function is not differentiable at change points 1/2, 1 and $\pi/2$.

23. (a) $f(1) = 1, f(1 + 0) = 1,$

$$f(1-0) = \sin\frac{\pi}{2} = 1$$

$$f'(1-0) = \frac{d}{dx}\left(\sin\frac{\pi x}{2}\right)$$

$$= \frac{1}{2}\cos\frac{\pi}{2}x\bigg|_{x=1} = \frac{1}{2}\times 0 = 0$$

$$f'(1-0) = \frac{d}{dx}(-2x+3) = -2, f'(1-0) \ne$$

$f'(1+0)$

24. (b) $f'(0-0) = -1, f'(0+0) = 1$

$\therefore$ $f(0-0) = f(0+0) = f(0) = 0$

25. (b) $f(\pi - 0) = f(\pi - h) = (\pi - (\pi - h))\,(-\cos h)/\sin h$

$= (h/\sin h)\,(-\cos h) = -1$

$f(\pi + h) = (-h)\,(-\cos h)/\sin h$

$$= \left(\frac{h}{\sin h}\right)(\cos h) = +1 = f(\pi + 0)$$

26. (b) $f(x) = \dfrac{x}{1-x}, x \le -1, \dfrac{x}{1+x};$

$$-1 < x \le 0, \frac{x}{1-x}, 0 < x < 1$$

$$= \frac{x}{1+x}, x \ge 1; f'(-1-0) = \frac{1}{(1-x)^2}$$

$$= \frac{1}{4}, f'(-1+0) = \frac{1}{(x+1)^2} \to \infty$$

$$f'(0-0) = \frac{1}{(x+1)^2} = 1, f'(0+0)$$

$$= \frac{1}{(1-x)^2} = 1$$

$\Rightarrow$ 1. $f'(-1-0) \ne f'(-1+0)$

2. $f'(0-0) = f'(0+0)$

27. (c) If $f(x) = \begin{cases} x+2, & 1 < x < 3 \\ 5, & x = 3 \\ 8 - x, & x > 3 \end{cases}$ and $f(3) = 5$

$$\text{L.H.D.} = \lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{h \to 0} \frac{f(3-h) - f(3)}{-h}$$

$$= \lim_{h \to 0} \frac{(3 - h + 2) - 5}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$$

$$\text{R.H.D.} = \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3}$$

$$= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{8 - (3 + h) - 5}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

L.H.D. $\ne$ R.H.D. Hence $f(x)$ is not differentiable at $x = 3$

28. (d) $Rf'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{210 + \sinh - 210}{h} = 1$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{210 - 210}{-h} = 0$$

Hence $f'(0)$ does not exist.

29. (d) Function $f(x) = |x| + |x - 1| + |x - 2|$ does not have a derivative at the points $x = 0, 1, 2 \in [0, 2]$

30. (b) The function $h(x) = x^2 \sin\,(1/x)$ is differentiable because

$$Rh'(x) = \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = 0,$$

$$Lh'(x) = \lim_{h \to 0} \frac{h^2 \sin\left(-\frac{1}{h}\right)}{-h} = 0$$

31. (b) $f(x) = [\tan^2 x]$ if $-45° < x < 45°$

$\Rightarrow$ $\tan(-45°) < \tan x < \tan(45°)$

$\Rightarrow$ $-\tan 45° < \tan x < \tan(45°)$

$\Rightarrow$ $-1 < \tan x < 1 \Rightarrow 0 < \tan^2 x < 1$

$\Rightarrow$ $[\tan^2 x] = 0$

i.e., $f(x)$ is zero for all values of x form $x = -45°$ to $45°$

Thus, $f(x)$ exists when $x \to 0$ and also it is continuous at $x = 0$, $f(x)$ is differentiable at $x = 0$ and has a value of zero. Therefore, (b) is the answer.

32. (d) $f(0 + 0) = \sin 0 = 0$, $f(0 - 0) = -\sin 0 = 0$ and $f(0) = 0$ So, $f(x)$ is continuous at $x = 0$

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sin(0+h) - 0}{h} = 1$$

$$\lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-\sin(0-h) - 0}{-h} = -1$$

So, $f(x)$ is not differentiable at $x = 0$

33. (d) Here, $f(x) = \cos^{-1}(\cos x) = 2\pi + x, -2\pi < x < -\pi$

$-x, -\pi \le x < 0$

$x, 0 \le x < \pi$

$2\pi - x, \pi < x < 2\pi$

Use these definitions for selecting the options.

34. (d) $f(x) = \cot^{-1}\left\{\dfrac{3x - x^3}{1 - 3x^2}\right\}$ and

$$g(x) = \cos^{-1}\left\{\frac{1 - x^2}{1 + x^2}\right\}$$

Put $x = \tan\theta$ in both equations

$$f(\theta) = \cot^{-1}\left\{\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right\}$$

$$= \cot^{-1}\cot\left(\frac{\pi}{2} - 3\theta\right) = \frac{\pi}{2} - 3\theta$$

$\Rightarrow\ f'(\theta) = 3$ .... (1)

and $g(\theta) = \cos^{-1}\left\{\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}\right\} = \cos^{-1}$

$(\cos 2\theta) = 2\theta$

$\Rightarrow\ g(\theta) = 2$ .... (2)

now $\lim_{x\to a}\left(\dfrac{f(x) - f(a)}{g(x) - g(a)}\right)$

$$= \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}\right)\frac{1}{\lim_{x\to a}\left(\dfrac{g(x) - g(a)}{x - a}\right)}$$

$$f'(x) = \frac{1}{g'(x)} = -3 \times \frac{1}{2} = \frac{-3}{2}$$

35. (c) $(gof)(x) = g[f(x)] = g[1 - \cos x] = e^{1-\cos x}$ for $x \le 0$ $(gof)'(x) = e^{1-\cos x}.\sin x$, for $x \le 0 \Rightarrow (gof)'(0) = 0$

36. (b) $\lim_{x\to 0} \dfrac{f'(x)}{1} = \lim_{x\to 0} \dfrac{\tan^4 x}{1} = \dfrac{0}{1} = 0$

37. (a) $f(x) = x^p \sin\dfrac{1}{x}, x \ne 0$ and $f(x) = 0, x = 0$

Since at $x = 0$, $f(x)$ is a continuous function

$\therefore\ \lim_{x\to 0} f(x) = f(0) = 0$

$\Rightarrow\ \lim_{x\to 0} x^p \sin\dfrac{1}{x} = 0 \Rightarrow p > 0$

$f(x)$ is differentiable at $x = 0$,

if $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ exists

$\Rightarrow\ \lim_{x\to 0} \dfrac{x^p \sin\frac{1}{x} - 0}{x - 0}$ exists

$\Rightarrow\ \lim_{x\to 0} x^{p-1}\sin\dfrac{1}{x}$ exists

$\Rightarrow\ p - 1 > 0$ or $p > 1$

If $p \le 1$, then $\lim_{x\to 0} x^{p-1}\sin\left(\dfrac{1}{x}\right)$ does not exist

and at $x = 0$ $f(x)$ is not differentiable.

$\therefore$ for $0 < p \le 1$ $f(x)$ is a continuous function at $x = 0$ but not differentiable.

ANSWERS

LECTURE 1

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (a)	**4.** (d)	**7.** (b)	**10.** (c)
2. (a)	**5.** (c)	**8.** (d)	
3. (b)	**6.** (d)	**9.** (d)	

Worksheet: To Check the Preparation Level

1. (a)	**5.** (b)	**9.** (c)	**13.** (d)
2. (b)	**6.** (b)	**10.** (d)	**14.** (c)
3. (c)	**7.** (b)	**11.** (b)	
4. (a)	**8.** (b)	**12.** (c)	

LECTURE 5

Mental Preparation Test

6. $a = 1, b = -1$

8. 2

PART E

Differentiation

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LECTURE

1 Simple Differentiation

BASIC CONCEPTS

DERIVATIVES OF SOME STANDRAD FUNCTIONS

Function		**Differential coefficient with respect to x**
1.1 (constant)	=	0
1.2 (x^n)	=	nx^{n-1}
1.3 $(\sqrt{x})$	=	$\dfrac{1}{2\sqrt{x}}$
1.4 $(1/x)$	=	$-1/x^2$
1.5 $\{1/x^n\}$	=	$-n/x^{n+1}$
1.6 (e^x)	=	e^x
1.7 (a^x)	=	$a^x \log_e a$
1.8 $(\log_e x)$	=	$1/x$
1.9 $(\log_a x)$	=	$\dfrac{\log_a e}{x} = \dfrac{1}{x \log_e a}$
1.10 $(\log_a (\log_a{}^x))$	=	$\dfrac{\log_a e}{(x)\log_e x} = \dfrac{1}{\log_e a (x \log_e x)}$
1.11 $(\sin x)$	=	$\cos x$
1.12 $(\cos x)$	=	$-\sin x$
1.13 $(\tan x)$	=	$\sec^2 x$
1.14 $(\cot x)$	=	$-\operatorname{cosec}^2 x$
1.15 $(\sec x)$	=	$\sec x \tan x$
1.16 $(\operatorname{cosec} x)$	=	$-\operatorname{cosec} x \cot x$
1.17 $(\sin^{-1} x)$	=	$\dfrac{1}{\sqrt{1-x^2}}, -1 < x < 1$
1.18 $(\cos^{-1} x)$	=	$-\dfrac{1}{\sqrt{1-x^2}}, -1 < x < 1$
1.19 $(\tan^{-1} x)$	=	$\dfrac{1}{1+x^2}, x \in R$
1.20 $(\cot^{-1} x)$	=	$-\dfrac{1}{1+x^2}, x \in R$
1.21 $(\sec^{-1} x)$	=	$\dfrac{1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$
1.22 $(\operatorname{cosec}^{-1} x)$	=	$-\dfrac{1}{\lvert x \rvert \sqrt{x^2-1}}, \lvert x \rvert > 1$
1.23 $\dfrac{d}{dx}\left[\dfrac{e^x + e^{-x}}{2}\right]$	=	or $\dfrac{d}{dx}(\cos hx) \sin hx$
1.24 $\dfrac{d}{dx}\left[\dfrac{e^x - e^{-x}}{2}\right]$	=	or $\dfrac{d}{dx}(\sin hx) \cos x$
1.25 $\dfrac{d}{dx}\left[\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right]$	=	$\sec h^2 x$
1.26 $\dfrac{d}{dx}\left[\log\left(x + \sqrt{x^2+a^2}\right)\right]$	=	$\dfrac{1}{\sqrt{x^2+a^2}}$
1.27 $\dfrac{d}{dx}\left[\log\left(x + \sqrt{x^2-a^2}\right)\right]$	=	$\dfrac{1}{\sqrt{x^2-a^2}}$
1.28 $\dfrac{d}{dx}\left[\dfrac{1}{2a}\log\dfrac{a+x}{a-x}\right], \lvert x \rvert < a$	=	$\dfrac{1}{a^2-x^2}$
1.29 $\dfrac{d}{dx}\left[\dfrac{1}{2a}\log\dfrac{a+x}{a-x}\right], \lvert x \rvert > a$	=	$\dfrac{1}{a^2-x^2}$
1.30 $(\operatorname{cosec} h^{-1} x)$	=	$-\dfrac{1}{\lvert x \rvert \sqrt{x^2+1}}$

1.31 $(e^{ax}\sin bx) = e^{ax}(a\sin bx + b\cos bx)$
$= \sqrt{a^2+b^2}\, e^{ax}\sin(bx + \tan^{-1} b/a)$

1.32 $(e^{ax}\cos bx) = e^{ax}(a\cos bx + b\sin bx)$
$= \sqrt{a^2+b^2}\, e^{ax}\cos(bx + \tan^{-1} b/a)$

1.33 $|x| \quad = \dfrac{x}{|x|}$ or $\dfrac{|x|}{x}, x \neq 0$

1.34 $\log|x| \quad = 1/x, x \neq 0$

1.35 $(x^x) \quad = x^x(1 + \log_e x)$

1.36 $(\log \sin x) \quad = \cot x$

1.37 $(\log \cos x) \quad = -\tan x$

1.38 $(\log \tan x) \quad = \sec x \operatorname{cosec} x$

1.39 $(\log \sec x) \quad = \tan x$

1.40 $(\log(\sec x + \tan x))$ or $\left[\log\left(\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right)\right]$
$= \sec x$

1.41 $[\log(\operatorname{cosec} x - \cot x)]$ or $\left[\log\left(\tan\dfrac{x}{2}\right)\right]$
$=\operatorname{cosec} x$

1.42 $\sin^{-1}(x/a) \quad = 1/\sqrt{a^2-x^2}$

1.43 $\sin^{-1}(ax) \quad = a/\sqrt{1-a^2x^2}$

1.44 $\left(\sin^{-1}\dfrac{bx}{a}\right) \quad = \dfrac{b}{\sqrt{a^2-b^2x^2}}$

1.45 $\left(\tan^{-1}\dfrac{x}{a}\right) \quad = \dfrac{a}{a^2-x^2}$

1.46 $(\tan^{-1} ax) \quad = a/1 + a^2x^2$

1.47 $\tan^{-1}(bx/a) \quad = ab/a^2 + b^2x^2$

1.48 $[\sec^{-1}(ax)] \quad = 1/x\sqrt{a^2x^2-1}$

1.49 $\left(\sec^{-1}\dfrac{x}{a}\right) \quad = a/x\sqrt{x^2-a^2}$

1.50 $\sec^{-1}\dfrac{bx}{a} \quad = a/x\sqrt{b^2x^2-a^2}$

1.51 $\{f(ax+b)\} \quad = af'(ax+b)$

FUNDAMENETAL THEOREMS

Let u, v, w ... be function of x whose derivatives exists (i) differential coefficient of constant is zero, i.e.,

(i) $\dfrac{d}{dx}(k) = 0$

(ii) $\dfrac{d}{dx}(ku) = k\dfrac{du}{dx}$

(iii) $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$ Sum or Difference Rule

(iv) $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ Product Rule

(v) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$ Quotient Rule

POINTS TO REMEMBER

1. $\dfrac{d}{dx}\sqrt{\dfrac{1+x}{1-x}} = \dfrac{1}{(1-x)\sqrt{1-x^2}}$

2. $\dfrac{d}{dx}\sqrt{\dfrac{1-x}{1+x}} = \dfrac{-1}{(1+x)\sqrt{1-x^2}}$

3. $\dfrac{d}{dx}\sqrt{\dfrac{x+1}{x-1}} = \dfrac{-1}{(x-1)\sqrt{x^2-1}}$

4. $\dfrac{d}{dx}\sqrt{\dfrac{x-1}{x+1}} = \dfrac{1}{(x+1)\sqrt{x^2-1}}$

5. $\dfrac{d}{dx}\left(\dfrac{ax+b}{cx+d}\right) = \dfrac{ab-bc}{(cx+d)^2}$

[redacted]

Directions: Q 1 to 23: Solve the questions using the first principle AB-INITIO Method/ Delta Method.

1. $\tan\sqrt{x}$

[CBSE-90, 95, 96C, 02, 04, 05; HB-94]

Solution

Let $f(x) = \tan\sqrt{x}$,

then $f(x+h) = \tan\sqrt{x+h}$

By definition of first principle

$$\frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(\tan\sqrt{x}) = \lim_{h\to 0}\frac{\tan\sqrt{x+h}-\tan\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\frac{\sin(\sqrt{x+h}-\sqrt{x})}{h\cos\sqrt{x+h}\,\cos\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{\sin(\sqrt{x+h}-\sqrt{x})}{(x+h-x)\cos\sqrt{x+h}\cos\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{\sin(\sqrt{x+h}-\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})(\sqrt{x+h}-\sqrt{x})\cos\sqrt{x+h}\cos\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{\sin(\sqrt{x+h}-\sqrt{x})}{(\sqrt{x+h}-\sqrt{x})}$$

$$\times\lim_{h\to 0}\frac{1}{(\sqrt{x+h}+\sqrt{x})\cos\sqrt{x+h}\cos\sqrt{x}}$$

$$= 1\times\frac{1}{2\sqrt{x}\cos\sqrt{x}\cos\sqrt{x}} = \frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$$

2. e^{x^2} ***[CBSE-1989]***

Solution

Let $f(x) = e^{x^2}$, then $f(x+h) = e^{(x+h)^2}$

By definition of first principle

$$\frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(e^{x^2}) = \lim_{h\to 0}\frac{e^{(x+h)^2}-e^{x^2}}{h}$$

$$= \lim_{h\to 0}\frac{e^{x^2}.e^{2hx+h^2}-e^{x^2}}{h}$$

$$= e^{x^2}\lim_{h\to 0}\frac{(e^{2hx+h^2}-1)(2hx+h^2)}{(2hx+h^2)h}$$

$$= e^{x^2}\lim_{h\to 0}\frac{(e^{2hx+h^2}-1)}{(2hx+h^2)}\times\lim_{h\to 0}\frac{h(2x+h)}{h}$$

$$= e^{x^2}\times 1\times 2x = 2x\,e^{x^2}$$

$$\therefore\ \frac{d}{dx}\{e^{x^2}\} = 2x\,e^{x^2}$$

3. $e^{\sqrt{2x}}$ ***[CBSE-93, 95]***

Solution

Let $f(x) = e^{\sqrt{2x}}$, then $f(x+h) = e^{\sqrt{2x+2h}}$

By definition of first principle

$$\frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(e^{\sqrt{2x}}) = \lim_{h\to 0}\frac{e^{\sqrt{2x+2h}}-e^{\sqrt{2x}}}{h}$$

$$= e^{\sqrt{2x}}\lim_{h\to 0}\frac{e^{\sqrt{2x+2h}-\sqrt{2x}}-1}{h}$$

$$= e^{\sqrt{2x}}\lim_{h\to 0}\left\{\frac{e^{\sqrt{2x+2h}-\sqrt{2x}}-1}{\sqrt{2x+2h}-\sqrt{2x}}\times\frac{\sqrt{2x+2h}-\sqrt{2x}}{h}\right\}$$

$$= e^{\sqrt{2x}}\times\lim_{h\to 0}\left(\frac{e^t-1}{t}\right)\times\lim_{h\to 0}\frac{\sqrt{2}\{\sqrt{x+h}-\sqrt{x}\}}{h}$$

where $t = \sqrt{2x+2h}-\sqrt{2x}$

$$= e^{\sqrt{2x}}\times 1\times\sqrt{2}\times\lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})(\sqrt{x+h}-\sqrt{x})}$$

$$= \frac{\sqrt{2}e^{\sqrt{2x}}}{2\sqrt{x}} = \frac{e^{\sqrt{2x}}}{\sqrt{2x}}$$

4. $\sin x^2$

[CBSE-90, 95, 01C, 02C; HB-94; MP-2000]

Solution

Let $f(x) = \sin x^2$,

then $f(x+h) = \sin(x+h)^2$

By definition of first principle

$$\frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}\{\sin x^2\} = \lim_{h\to 0}\frac{\sin(x+h)^2-\sin x^2}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\frac{(x+h)^2+x^2}{2}\sin\frac{(x+h)^2-x^2}{2}}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{x^2+h^2+2hx+x^2}{2}\right)\sin\left(\frac{x^2+h^2+2hx-x^2}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(x^2+\frac{2hx+h^2}{2}\right)\sin\frac{h(2x+h)}{2}}{h}$$

$$= \lim_{h\to 0} 2\cos\left(x^2+\frac{2hx+h^2}{2}\right)$$

$$\times \frac{\sin h\left(\frac{2x+h}{2}\right)}{h\left(\frac{2x+h}{2}\right)} \times \left(\frac{2x+h}{2}\right)$$

$$= \lim_{h\to 0} \cos\left(x^2+\frac{2hx+h^2}{2}\right)$$

$$\times \lim_{h\to 0} \frac{\sinh\left(\frac{2x+h}{2}\right)}{h\left(\frac{2x+h}{2}\right)} \times \lim_{h\to 0}(2x+h)$$

$= \cos x^2 \times 1 \times 2x = 2\,x \cos x^2$

5. $x^{-3/2}$ ***[CBSE-93]***

Solution

Let $f(x) = x^{-3/2}$, then $f(x+h) = (x+h)^{-3/2}$

By definition of first principle

$$\frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(x^{-3/2}) = \lim_{h\to 0}\frac{(x+h)^{-3/2}-x^{-3/2}}{h}$$

$$= \lim_{h\to 0}\frac{x^{-3/2}\left[\left(1+\frac{h}{x}\right)^{-3/2}-1\right]}{h}$$

$$= \lim_{h\to 0}\frac{x^{-3/2}}{h}\left[1-\frac{3}{2}\times\frac{h}{x}+\frac{\left(\frac{-3}{2}\right)\left(\frac{-3}{2}-1\right)}{2!}\cdot\frac{h^2}{x^2}+\ldots-1\right]$$

$$= \lim_{h\to 0}\frac{x^{-3/2}}{h}\times h\left[-\frac{3}{2x}+\frac{\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{2!}\frac{h}{x^2}+\ldots\right]$$

$$= (x^{-3/2})\left(-\frac{3}{2x}\right) = -\frac{3}{2}x^{-5/2}$$

$$\therefore \frac{d}{dx}(x^{-3/2}) = -\frac{3}{2}x^{-5/2}$$

6. $\dfrac{2x+3}{3x+2}$ ***[HB-83, 86]***

Solution

Let $f(x) = \dfrac{2x+3}{3x+2}$,

then $f(x+h) = \dfrac{2(x+h)+3}{3(x+h)+2}$.

By definition of first principle

$$\frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

$$\frac{d}{dx}\left(\frac{2x+3}{3x+2}\right) = \lim_{h\to 0}\frac{\frac{2(x+h)+3}{3(x+h)+2}-\frac{2x+3}{3x+2}}{h}.$$

$$= \lim_{h\to 0}\frac{\frac{(2x+3+2h)(3x+2)-(2x+3)}{(3x+2+3h)}}{h(3x+2+3h)(3x+2)}$$

$$= \lim_{h\to 0}\frac{(2x+3)(3x+2)+2h(3x+2)\; -(2x+3)(3x+2)-3h(2x+3)}{h(3x+2+3h)(3x+2)}$$

$$= \lim_{h\to 0}\frac{h\{6x+4-6x-9\}}{h(3x+2)(3x+2+3h)}$$

$$= \frac{-5}{(3x+2)(3x+2)}$$

$$= \frac{-5}{(3x+2)^2}$$

$$\therefore \frac{d}{dx}\left(\frac{2x+3}{3x+2}\right) = \frac{-5}{(3x+2)^2}$$

7. $x \sin x$ ***[CBSE-91C, 92; HB-94]***

Solution

Let $f(x) = x \sin x$, then $f(x+h) = (x+h)\sin(x+h)$

Now $\dfrac{d}{dx}\{f(x)\} = \lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{(x+h)\sin(x+h)-x\sin x}{h}$$

$$= \lim_{h\to 0} \frac{x[\sin(x+h) - \sin x] + h\sin(x+h)}{h}$$

$$= \lim_{h\to 0} \frac{x}{h}[\sin(x+h) - \sin x] + \lim_{h\to 0} \sin(x+h)$$

$$= \lim_{h\to 0} \frac{x}{h} \times 2\cos\left(\frac{x+h+x}{2}\right)$$

$$\sin\left(\frac{x+h-x}{2}\right) + \sin(x+0)$$

$$= \lim_{h\to 0} \frac{x \times 2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{2\times(h/2)} + \sin x$$

$$= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \times \lim_{h\to 0} x\cos\left(x+\frac{h}{2}\right) + \sin x$$

$$= 1 \times x\cos x + \sin x = x\cos x + \sin x$$

$$\therefore \quad \frac{d}{dx}(x\sin x) = x\cos x + \sin x$$

8. Using first principles, prove that $\frac{d}{dx}\left\{\frac{1}{f(x)}\right\} = -\frac{f'(x)}{[f(x)]^2}$ ***[PSB-1989]***

Solution

Let $\phi(x) = \frac{1}{f(x)}$, then $\phi(x+h) = \frac{1}{f(x+h)}$

$$\therefore \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h\to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$\frac{d}{dx}\left\{\frac{1}{f(x)}\right\} = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x) - f(x+h)}{hf(x+h)f(x)}$$

$$= \lim_{h\to 0} \frac{f(x) - f(x+h)}{h} \cdot \lim_{h\to 0} \frac{1}{f(x)f(x+h)}$$

$$= -\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \times \lim_{h\to 0} \frac{1}{f(x)f(x+h)}$$

$$= -f'(x) \times \frac{1}{f(x)f(x)} \quad \left\{\begin{array}{l} f(x) \text{ is differentiable} \\ \Rightarrow f(x) \text{ is continuous} \\ = \lim_{h\to 0} f(x+h) = f(x) \end{array}\right.$$

$$= \frac{-f'(x)}{[f(x)]^2}$$

$$\therefore \quad \frac{d}{dx}\left\{\frac{1}{f(x)}\right\} = \frac{-f'(x)}{\{f(x)\}^2}$$ **Proved.**

9. $\sin^{1/3} x$ or $\sqrt[3]{\sin x}$ ***[MNR-97]***

Solution

Let $f(x) = \sqrt[3]{\sin x}$

then $f(x+h) = \sqrt[3]{\sin(x+h)}$

$$\therefore \quad \frac{d}{dx}\{f(x)\} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}\{\sqrt[3]{\sin x}\} = \lim_{h\to 0} \frac{\sqrt[3]{\sin(x+h)} - \sqrt[3]{\sin x}}{h}$$

$$= \lim_{h\to 0} \frac{\{\sqrt[3]{\sin(x+h)}\}^3 - \{\sqrt[3]{\sin x}\}^3}{h(\sin^{2/3}(x+h) + \sin^{2/3} x + \sin^{1/3}(x+h)\sin^{1/3} x)}$$

$$\left[\because a - b = \frac{a^3 - b^3}{a^2 + b^2 + ab}\right]$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \times \frac{1}{\{\sin^{2/3}(x+h) + \sin^{2/3} x + \sin^{1/3}(x+h)\sin^{1/3} x\}}$$

$$= \lim_{h\to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{2(h/2)} \times \frac{1}{\{\sin^{2/3}(x+h) + \sin^{2/3} x + \sin^{1/3}(x+h)\sin^{1/3} x\}}$$

$$= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \times \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \times \lim_{h\to 0} \frac{1}{\sin^{2/3}(x+h) + \sin^{2/3} x + \sin^{1/3}(x+h)\sin^{1/3} x}$$

$$= 1 \times \cos x \times \frac{1}{\sin^{2/3} x + \sin^{2/3} x + \sin^{1/3} x \times \sin^{1/3} x}$$

$$= \frac{\cos x}{3\sin^{2/3} x}$$

$$\therefore \quad \frac{d}{dx}\{\sqrt[3]{\sin x}\} = \frac{\cos x}{3\sin^{2/3} x}$$

10. Differentiate $\sin^{-1}\sqrt{x}(0 < x < 1)$ from first principle.

[PSB-89, 90; HPSB-90; HSB-2001]

Solution

Let $f(x) = \sin^{-1}\sqrt{x}$,

then $f(x+h) = \sin^{-1}\sqrt{x+h}$

$$\therefore \quad \frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(\sin^{-1}\sqrt{x}) = \lim_{h\to 0}\frac{\sin^{-1}\sqrt{x+h}-\sin^{-1}\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}\{\sqrt{x+h}\sqrt{1-x}-\sqrt{x}\sqrt{1-x-h}\}}{h}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}z}{z}\times\frac{z}{h}$$

where $z = \sqrt{x+h}\sqrt{1-x}-\sqrt{x}\sqrt{1-x-h}$

$$= \lim_{h\to 0}\frac{z}{h}$$

$$\left[\because \lim_{h\to 0}\frac{\sin^{-1}z}{z} = \lim_{z\to 0}\frac{\sin^{-1}z}{z} = 1\right]$$

$$= \lim_{h\to 0}(\sqrt{x+h}\sqrt{1-x}-\sqrt{x}\sqrt{1-x-h})$$

$$(\sqrt{x+h}\sqrt{1-x}+\sqrt{x}\sqrt{1-x-h})/h(\sqrt{x+h}$$
$$\sqrt{1-x}+\sqrt{x}\sqrt{1-x-h})$$

$$= \lim_{h\to 0}\frac{(x+h)(1-x)-x(1-x-h)}{h(\sqrt{x+h}\sqrt{1-x}+\sqrt{x}\sqrt{1-x-h})}$$

$$= \lim_{h\to 0}\frac{h\{1-x+x\}}{h\{\sqrt{x+h}\sqrt{1-x}+\sqrt{x}\sqrt{1-x-h}\}}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h}\sqrt{1-x}+\sqrt{x}\sqrt{1-x-h}}$$

$$= \frac{1}{\sqrt{x}\sqrt{1-x}+\sqrt{x}\sqrt{1-x}}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}} \quad \therefore \quad \frac{d}{dx}\{\sin^{-1}x\} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

11. $\dfrac{\sin x}{x}$ ***[CBSE-2002 C]***

Solution

Let $f(x) = \dfrac{\sin x}{x}$,

then $f(x+h) = \dfrac{\sin(x+h)}{x+h}$

$$\therefore \quad \frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \lim_{h\to 0}\frac{\frac{\sin(x+h)}{x+h}-\frac{\sin x}{x}}{h}$$

$$= \lim_{h\to 0}\frac{x\sin(x+h)-(x+h)\sin x}{hx(x+h)}$$

$$= \lim_{h\to 0}\frac{x[\sin(x+h)-\sin x]-h\sin x}{hx(x+h)}$$

$$= \lim_{h\to 0}\frac{x2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)-h\sin x}{hx(x+h)}$$

$$= \lim_{h\to 0}\frac{2x\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{x(x+h)2(h/2)} - \lim_{h\to 0}\frac{h\sin x}{hx(x+h)}$$

$$= \lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)}{x+h}\times\lim_{h\to 0}\frac{\sin\frac{h}{2}}{h/2} - \lim_{h\to 0}\frac{\sin x}{x(x+h)}$$

$$= \frac{\cos x\times 1}{x}-\frac{\sin x}{x\times x} = \frac{\cos x}{x}-\frac{\sin x}{x^2}$$

$$= \frac{x\cos x-\sin x}{x^2}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x-\sin x}{x^2}$$

12. $x\tan^{-1}x$

Solution

Let $f(x) = x\tan^{-1}x$. Then $f(x+h) = (x+h)\tan^{-1}(x+h)$

$$\therefore \quad \frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(x\tan^{-1}x)$$

$$= \lim_{h\to 0}\frac{(x+h)\tan^{-1}(x+h)-x\tan^{-1}x}{h}$$

$$= \lim_{h\to 0}\frac{x\{\tan^{-1}(x+h)-\tan^{-1}x\}+h\tan^{-1}(x+h)}{h}$$

$$= \lim_{h\to 0}\frac{x}{h}\{\tan^{-1}(x+h)-\tan^{-1}x\}$$
$$+\lim_{h\to 0}\frac{h\tan^{-1}(x+h)}{h}$$

$$= \lim_{h\to 0}\frac{x}{h}\left\{\tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)\right\}+$$
$$\lim_{h\to 0}\tan^{-1}(x+h)$$

$$= x\lim_{h\to 0}\frac{\tan^{-1}\left(\frac{h}{1+x(x+h)}\right)}{h}+\tan^{-1}x$$

$$= x \lim_{h \to 0} \frac{\tan^{-1}\left\{\frac{h}{1+x(x+h)}\right\}}{\frac{h}{(1+x(x+h)} \times \{1+x(x+h)\}} + \tan^{-1} x$$

$$= x \times \lim_{h \to 0} \frac{1}{1+x(x+h)}$$

$$\times \lim_{h \to 0} \frac{\tan^{-1}\left\{\frac{h}{1+x(x+h)}\right\}}{\frac{h}{1+x(x+h)}} + \tan^{-1} x$$

$$= \frac{x \times 1}{(1+x^2)} \times 1 + \tan^{-1} x = \frac{x}{1+x^2} + \tan^{-1} x$$

$$\therefore \frac{d}{dx}(x \tan^{-1} x) = \frac{x}{1+x^2} + \tan^{-1} x$$

13. $\cos^{-1}(2x+3)$

Solution

Let $f(x) = \cos^{-1}(2x+3)$, then $f(x+h) = \cos^{-1}(2x+3+2h)$

$$\therefore \frac{d}{dx}\{f(x)\} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}\cos^{-1}(2x+3)$$

$$= \lim_{h \to 0} \frac{\cos^{-1}(2x+3+2h) - \cos^{-1}(2x+3)}{h}$$

$$= \lim_{h \to 0} \frac{\left\{\frac{\pi}{2} - \sin^{-1}(2x+3+2h)\right\} - \left\{\frac{\pi}{2} - \sin^{-1}(2x+3)\right\}}{h}$$

$$= \lim_{h \to 0} \frac{\sin^{-1}(2x+3) - \sin^{-1}(2x+3+2h)}{h}$$

$$= \lim_{h \to 0} \frac{\sin^{-1}\left\{\begin{matrix}(2x+3)\sqrt{1-(2x+3+2h)^2} \\ -(2x+3+2h)\sqrt{1-(2x+3)^2}\end{matrix}\right\}}{h}$$

$$= \lim_{h \to 0} \frac{\sin^{-1} z}{z} \times \frac{z}{h}$$

where $z = (2x+3)\sqrt{1-(2x+3+2h)^2}$

$-(2x+3+2h)\sqrt{1-(2x+3)^2}$

$$= \lim_{h \to 0} \frac{z}{h} \quad \left\{\because \lim_{h \to 0} \frac{\sin^{-1} z}{z} = 1\right\}$$

$$= \lim_{h \to 0} \frac{(2x+3)\sqrt{1-(2x+3+2h)^2} - (2x+3+2h)\sqrt{1-(2x+3)^2}}{h}$$

$$= \lim_{h \to 0} \frac{(2x+3)^2\{1-(2x+3+2h)^2\} - (2x+3+2h)^2\{1-(2x+3)^2\}}{h\{(2x+3)\sqrt{1-(2x+3+2h)^2} + (2x+3+2h)\sqrt{1-(2x+3)^2}\}}$$

$$= \lim_{h \to 0} \frac{(2x+3)^2 - (2x+3+2h)^2}{h\{(2x+3)\sqrt{1-(2x+3+2h)^2} + (2x+3+2h)\sqrt{1-(2x+3)^2}\}}$$

$$= \lim_{h \to 0} \frac{-4h(2x+3) - 4h^2}{h\{(2x+3)\sqrt{1-(2x+3+2h)^2} + (2x+3+2h)\sqrt{1-(2x+3)^2}\}}$$

$$= \lim_{h \to 0} \frac{4h(2x+3+h)}{h\{(2x+3)\sqrt{1-(2x+3+2h)^2} + (2x+3+2h)\sqrt{1-(2x+3)^2}\}}$$

$$= \frac{-4(2x+3)}{2(2x+3)\sqrt{1-(2x+3)^2}} = \frac{-2}{\sqrt{1-(2x+3)^2}}$$

$$\therefore \frac{d}{dx}\cos^{-1}(2x+3) = \frac{-2}{\sqrt{1-(2x+3)^2}}$$

14. $e^{\sqrt{\tan x}}$

Solution

Let $f(x) = e^{\sqrt{\tan x}}$, then $f(x+h) = e^{\sqrt{\tan(x+h)}}$

$$\therefore \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}(e^{\sqrt{\tan x}}) = \lim_{h \to 0} \frac{e^{\sqrt{\tan(x+h)}} - e^{\sqrt{\tan x}}}{h}$$

$$= \lim_{h \to 0} \frac{e^{\sqrt{\tan x}}\{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1\}}{h}$$

$$= e^{\sqrt{\tan x}} \lim_{h \to 0} \frac{\{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1\}}{(\sqrt{\tan(x+h)} - \sqrt{\tan x})}$$

$$\frac{(\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}$$

$$= e^{\sqrt{\tan x}} \times 1 \times \lim_{h \to 0} \frac{(\sqrt{\tan(x+h)} - \sqrt{\tan x})(\sqrt{\tan(x+h)} + \sqrt{\tan x})}{h(\sqrt{\tan(x+h)} + \sqrt{\tan x})}$$

$$e^{\sqrt{\tan x}} \times \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h(\sqrt{\tan(x+h)} + \sqrt{\tan x})}$$

$$= e^{\sqrt{\tan x}} \times \lim_{h \to 0} \frac{\sin(x+h-x)}{h} \times \lim_{h \to 0} \frac{1}{\cos(x+h)\cos x}$$

$$\times \lim_{h \to 0} \frac{1}{(\sqrt{\tan(x+h)} + \sqrt{\tan x})}$$

$$= e^{\sqrt{\tan x}} \times \frac{1}{\cos x . \cos x} \times \frac{1}{(\sqrt{\tan x} + \sqrt{\tan x})}$$

$$= \frac{e^{\sqrt{\tan x}}}{2\sqrt{\tan x}} \sec^2 x$$

$$\therefore \frac{d}{dx}\{e^{\sqrt{\tan x}}\} = \frac{e^{\sqrt{\tan x}}}{2\sqrt{\tan x}} \sec^2 x$$

15. $\sin(x^2 + 1)$ ***[CBSE-95, 2001]***

Solution

Let $f(x) = \sin(x^2 + 1)$, then $f(x + h) = \sin\{(x+h)^2 + 1\}$

$$\therefore \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}\sin(x^2+1) = \lim_{h \to 0} \frac{\sin\{(x+h)^2+1\} - \sin(x^2+1)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left\{2\cos\left\{\frac{x^2+h^2+2hx+1+x^2+1}{2}\right\} \sin\left\{\frac{x^2+h^2+2hx+1-x^2-1}{2}\right\}\right\}$$

$$= \lim_{h \to 0} \frac{2\cos\left\{x^2+1+\dfrac{h^2+2hx}{2}\right\}\sin h\left\{\dfrac{h+2x}{2}\right\}}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left\{x^2+1+\dfrac{h^2+2hx}{h}\right\}\sin h\left\{\dfrac{h+2x}{2}\right\}}{h\left(\dfrac{h+2x}{2}\right)} \times \left(\frac{h+2x}{2}\right)$$

$$= \lim_{h \to 0} \cos\left(x^2+1+\frac{h^2+2hx}{2}\right) \times \lim_{h \to 0} \frac{\sin\left\{\left(\dfrac{h(h+2x)}{2}\right)\right\}}{\dfrac{h(h+2x)}{2}} \times \lim_{h \to 0}(h+2x)$$

$$= \cos(x^2 + 1) \times 1 \times 2x = 2x\cos(x^2 + 1)$$

$$\therefore \frac{d}{dx}\sin(x^2+1) = 2x\cos(x^2+1)$$

16. $\dfrac{ax+b}{cx+d}$

Solution

Let $f(x) = \dfrac{ax+b}{cx+d}$

then $f(x+h) = \dfrac{a(x+h)+b}{c(x+h)+d}$

$$\therefore \frac{d}{dx}\{f(x)\} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}\left(\frac{ax+b}{cx+d}\right) = \lim_{h \to 0} \frac{\dfrac{a(x+h)+b}{c(x+h)+d} - \dfrac{ax+b}{cx+d}}{h}$$

$$= \lim_{h \to 0} \frac{(ax+b+ah)(cx+d) - (ax+b)(cx+d+ch)}{h\{cx+d+ch\}\{cx+d\}}$$

$$= \lim_{h \to 0} \frac{(ax+b)(cx+d) + ah(cx+d) - (ax+b)(cx+d) - (ax+b)ch}{h\{cx+d+ch\}\{cx+d\}}$$

$$= \lim_{h \to 0} \frac{h\{acx+ad-acx-bc\}}{h(cx+d+ch)(cx+d)}$$

$$= \frac{ad-bc}{(cx+d)(cx+d)}$$

$$= \frac{da-bc}{(cx+d)^2} \quad \therefore \frac{d}{dx}\left(\frac{ax+b}{cx+d}\right) = \frac{ad-bc}{(cx+d)^2}$$

17. $\tan^{-1}(x^2)$

Solution

Let $f(x) = \tan^{-1} x^2$, then $f(x + h) = \tan^{-1}(x+h)^2$

$$\therefore \frac{d}{dx}\{f(x)\} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}\tan^{-1}x^2 = \lim_{h\to 0}\frac{\tan^{-1}(x+h)^2 - \tan^{-1}x^2}{h}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}\left(\dfrac{x^2+h^2+2hx-x^2}{1+x^2(x+h)^2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}\left\{\dfrac{h(h+2x)}{1+x^2(x+h)^2}\right\}}{\dfrac{h(h+2x)}{1+x^2(x+h)^2}}\times\left(\frac{h+2x}{1+x^2(x+h)^2}\right)$$

$$= \lim_{h\to 0}\frac{\tan^{-1}\left\{\dfrac{h(h+2x)}{1+x^2(x+h)^2}\right\}}{\dfrac{h(h+2x)}{1+x^2(x+h)^2}}\times\lim_{h\to 0}\left(\frac{h+2x}{1+x^2(x+h)^2}\right)$$

$$= 1\times\frac{2x}{1+x^2\times x^2}$$

$$= \frac{2x}{1+x^4}$$

$$\therefore\ \frac{d}{dx}(\tan^{-1}x^2) = \frac{2x}{1+x^4}$$

18. sec $(2x - 1)$

[CBSE (Sample paper)-2006]

Solution

Let $f(x) = \sec(2x-1)$ then $f(x+h) = \sec(2x+2h-1)$

$$\therefore\ \frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}\{\sec(2x-1)\}$$

$$= \lim_{h\to 0}\frac{\sec(2x+2h-1)-\sec(2x-1)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{1}{\cos(2x+2h-1)}-\dfrac{1}{\cos(2x-1)}}{h}$$

$$= \lim_{h\to 0}\frac{\cos(2x-1)-\cos(2x+2h-1)}{h\cos(2x-1)\cos(2x+2h-1)}$$

$$= \lim_{h\to 0}\frac{2\sin\left(\dfrac{2x-1+2x+2h-1}{2}\right)\sin\left(\dfrac{2x+2h-1-2x+1}{2}\right)}{h\cos(2x-1)\cos(2x+2h-1)}$$

$$= \lim_{h\to 0}\frac{\sin h}{h}\times\lim_{h\to 0}\frac{2\sin(2x+h-1)}{\cos(2x-1)\cos(2x+2h-1)}$$

$$= \frac{2\sin(2x-1)}{\cos^2(2x-1)} = 2\tan(2x-1)\sec(2x-1)$$

$$\therefore\ \frac{d}{dx}\{\sec(2x-1)\} = 2\sec(2x-1)\tan(2x-1)$$

19. $\sqrt{\sin x}$

[CBSE-93, 96, 2003, 2006; MP-2001; PB-93, 94]

Solution

Let $f(x) = \sqrt{\sin x}$

then $f(x+h) = \sqrt{\sin(x+h)}$

$$\therefore\ \frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(\sqrt{\sin x}) = \lim_{h\to 0}\frac{\sqrt{\sin(x+h)}-\sqrt{\sin x}}{h}$$

$$= \lim_{h\to 0}\frac{\{\sqrt{\sin(x+h)}-\sqrt{\sin x}\}\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}{h\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}$$

$$= \lim_{h\to 0}\frac{2\cos\left(x+\dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)}{\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}h}$$

$$= \lim_{h\to 0}\frac{\cos\left(x+\dfrac{h}{2}\right)}{\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}\times\lim_{h\to 0}\frac{\sin\left(\dfrac{h}{2}\right)}{(h/2)}$$

$$= \frac{\cos x}{2\sqrt{\sin x}}\times 1$$

$$\therefore\ \frac{d}{dx}(\sqrt{\sin x}) = \frac{\cos x}{2\sqrt{\sin x}}$$

20. $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ ***[Sample paper-2006]***

Solution

Let $f(x) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

then $f(x+h) = \tan^{-1}\left(\dfrac{2(x+h)}{1-(x+h)^2}\right)$

$$\therefore\ \frac{d}{dx}\{f(x)\} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}\left\{\tan^{-1}\frac{2x}{1-x^2}\right\}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}\left(\frac{2x+2h}{1-(x+h)^2}\right)-\tan^{-1}\left(\frac{2x}{1-x^2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\tan^{-1}\left\{\frac{\frac{2x+2h}{\{1-(x+h)^2\}}-\frac{2x}{(1-x^2)}}{1+\frac{2(x+h)2x}{\{1-(x+h)^2\}(1-x^2)}}\right\}$$

$$= \lim_{h\to 0}\frac{1}{h}\tan^{-1}\left\{\frac{\begin{matrix}2(x+h)(1-x^2)-2x\\ \{1-(x+h)^2\}\end{matrix}}{\begin{matrix}\{1-(x+h)^2\}(1-x^2)\\ +2(x+h)2x\end{matrix}}\right\}$$

$$= \lim_{h\to 0}\frac{1}{h}\tan^{-1}\left\{\frac{\begin{matrix}2x\{1-x^2-1+x^2+h^2\\ +2hx\}+2h(1-x^2)\end{matrix}}{\begin{matrix}\{1-(x+h)^2\}(1-x^2)\\ +2(x+h)2x\end{matrix}}\right\}$$

$$= \lim_{h\to \bullet}\frac{\tan^{-1}\left\{\frac{2h\{hx+2x^2+1-x^2\}}{\{1-(x+h)^2\}(1-x^2)+2(x+h)2x}\right\}}{h}$$

$$= \lim_{h\to 0}\frac{\tan^{-1}\left\{\frac{2h\{x^2+hx+1\}}{\{1-(x+h)^2\}(1-x^2)+2(x+h)2x}\right\}}{\frac{2h(x^2+hx+1)}{\{1-(x+h)^2\}(1-x^2)+2(x+h)2x}}$$

$$\times 2\left\{\frac{x^2+hx+1}{\{1-(x+h)^2\}(1-x^2)+2(x+h)2x}\right\}$$

$$= 1\times\lim_{h\to 0}\frac{2(x^2+hx+1)}{\{1-(x+h)^2\}(1-x^2)+2(x+h)2x}$$

$$= \frac{2(x^2+1)}{(1-x^2)^2+2x+2x} = \frac{2(x^2+1)}{1+x^4-2x^2+4x^2}$$

$$= \frac{2(x^2+1)}{x^4+2x^2+1} = \frac{2(x^2+1)}{(x^2+1)^2} = \frac{2}{1+x^2}$$

$$\therefore\ \frac{d}{dx}\left\{\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\} = \frac{2}{1+x^2}$$

21. $\cos^{-1}(4x^3-3x)$

[CBSE (Sample Paper)-2006]

Solution

Let $y = \cos^{-1}(4x^3-3x)$

$4x^3-3x = \cos y$ (i)

If Δy be an increment in y corresponding to an increment Δx in x.

$\therefore$ $4(x+\Delta x)^3 - 3(x+\Delta x) = \cos(y+\Delta y)$ .... (ii)

Subtracting equation (i) from equation (ii) we get

$4x^3 + 4\Delta x^3 + 12x^2\Delta x + 12x\Delta x^2 - 3x - 3\Delta x - 4x^3 + 3x$

$= \cos(y+\Delta y) - \cos y$

$\Rightarrow (4\Delta x^2 + 12x\Delta x + 12x^2 - 3)\Delta x$

$$= 2\sin\left(y+\frac{\Delta y}{2}\right)\sin\left(\frac{-\Delta y}{2}\right)$$

$$(4\Delta x^2 + 12x\Delta x + 12x^2 - 3)\frac{\Delta x}{\Delta y}$$

$$= \frac{-2\sin\left(y+\frac{\Delta y}{2}\right)\sin\left(\frac{\Delta y}{2}\right)}{\Delta y}$$

Obviously as $\Delta x \to O$ then $\Delta y \to 0$

$$\Rightarrow \text{Now } \lim_{\Delta x\to 0}\frac{(4\Delta x^2+12x\Delta x+12x^2-3)}{(\Delta y/\Delta x)}$$

$$= -\lim_{\Delta y\to 0}\frac{\sin\left(y+\frac{\Delta y}{2}\right)\sin\left(\frac{\Delta y}{2}\right)}{(\Delta y/2)}$$

$$\Rightarrow \frac{\lim_{\Delta x\to 0}(4\Delta x^2+12x\Delta x+12x^2-3)}{\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)}$$

$$= -\lim_{\Delta y\to 0}\sin\left(y+\frac{\Delta y}{2}\right)\times\lim_{\Delta y\to 0}\frac{\sin\left(\frac{\Delta y}{2}\right)}{\left(\frac{\Delta y}{2}\right)}$$

$$\Rightarrow \frac{12x^2-3}{\frac{dy}{dx}} = -\sin y;\ \frac{dy}{dx} = \frac{-(12x^2-3)}{\sin y}$$

$$\frac{d}{dx}\{(\cos^{-1}(4x^3-3x)\}$$

$$= \frac{3-12x^2}{\sqrt{1-\cos^2 y}} = \frac{3-12x^2}{\sqrt{1-(4x^3-3x)^2}}$$

22. Find from first principles, the derivative of $\sqrt{\sec x}$ with respect to x. ***[CBSE-1998]***

Solution

Let $y = \sqrt{\sec x} \Rightarrow y + \delta y = \sqrt{\sec(x + \delta x)}$

$\Rightarrow \delta y = \sqrt{\sec(x + \delta x)} - \sqrt{\sec x}$

$$= \frac{1}{\sqrt{\cos(x + \delta x)}} - \frac{1}{\sqrt{\cos x}}$$

$$= \frac{\sqrt{\cos x} - \sqrt{\cos(x + \delta x)}}{\sqrt{\cos(x + \delta x)}.\sqrt{\cos x}} \Rightarrow \frac{\delta y}{\delta x}$$

$$= \frac{\cos x - \cos(x + \delta x)}{\delta x.\sqrt{\cos(x + \delta x)}.\sqrt{\cos x}[\sqrt{\cos x} + \sqrt{\cos(x + \delta x)}]}$$

$$= \frac{2\sin\frac{2x + \delta x}{2}\sin\frac{\delta x}{2}}{\delta x\sqrt{\cos(x + \delta x)}.\sqrt{\cos x}[\sqrt{\cos x} + \sqrt{\cos(x + \delta x)}]}$$

$$= \frac{\sin\frac{2x + \delta x}{2}\left(\sin\frac{\delta x}{2}\right)\Big/\frac{\delta x}{2}}{\sqrt{\cos(x + \delta x)}.\sqrt{\cos x}[\sqrt{\cos x} + \sqrt{\cos(x + \delta x)}]}$$

Proceeding to limit as $\delta x \to 0$, we get

$$\frac{dy}{dx} = \frac{\sin x.1}{\sqrt{\cos x}\sqrt{\cos x}[\sqrt{\cos x} + \sqrt{\cos x}]}$$

$$= \frac{\sin x}{\cos x.2\sqrt{\cos x}}$$

$$= \frac{1}{2}\sqrt{\sec x}\tan x$$

[redacted]
TO GRASP THE TOPIC SOLVE THESE PROBLEMS

Directions from 1 to 4: Differentiate the following work x

1. $\dfrac{(x^2 + 1)(x + 3)}{x}$ ***[MP-98,93; NCERT Book; HPSB-99S; 2001S]***

2. $\dfrac{7x^2 - 3x + 2}{\sqrt{x}}$

3. $\dfrac{(2x^2 - 7x)^2}{x^4}$ ***[MP-2000]***

4. $\dfrac{(x + a)^3}{\sqrt{x}}$ ***[MP-2000]***

5. Find $\dfrac{dy}{dx}$, if $y = \dfrac{x^4}{4} - \dfrac{x^3}{3} + \dfrac{x^2}{2} - x + 3$ ***[MP-2003]***

6. If $y = \dfrac{\cos x \sin x + 3\cos x + 1}{\sin x}$ find $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}$.

7. Differentiate $\sqrt{\sec^2 x + \text{cosec}^2 x}$ with respect to x ***[PSB-92]***

8. If $y = x^6$, then find the successive differential coefficients.

9. $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ ***[MP-2000, 2001, 2004]***

10. $\dfrac{\log_e x}{\cos x}$ ***[MP-95, 2000]***

11. $e^x \sin x + x^p \cos x$ ***[MP-99]***

12. $\dfrac{e^x}{\sin x}$ ***[MP-2000, 2005]***

13. $\dfrac{x\cos x}{\log x}$ ***[MP-97, 98, 99, 2002]***

14. $\dfrac{x^n}{\log_e x}$ ***[MP-2001]***

15. $\dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}$

16. $\dfrac{x\tan x}{\sec x + \tan x}$

17. $\dfrac{\log x}{1 + x\log x}$

18. If $y = \frac{x}{x+4}$ then prove that $x\frac{dy}{dx} = y(1-y)$. **[MP-97, 99]**

19. $\frac{(x^3-2x)^2}{x^5}$ **[MP-2000]**

20. $e^x + 3\cos x + \log x^2$ **[MP-98, 2000]**

21. $x^7 \log_a x$ **[MP-99]**

22. $\frac{\sin x}{1+\cos x}$ **[MP-98, 99]**

23. $\frac{\log_e x}{\sin x}$ **[MP-2001]**

24. $\frac{e^x + \log x}{\sin 3x}$ **[MP-97; CBSE-93]**

25. $\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$ **[AICBSE-88]**

ANSWERS

1. $2x + 3 - \frac{3}{x^2}$

2. $\frac{21}{2}x^{1/2}\frac{3}{2}x^{-1/2} - x^{-3/2}$

3. $-\frac{98}{x^3} + \frac{28}{x^2}$

4. $\frac{5}{2}x^{3/2} - \frac{a^3x^{-3/2}}{2} + \frac{9a}{2}x^{1/2} + \frac{3a^2}{2}x^{-1/2}$

5. $x^3 - x^2 + x - 1$

6. $\frac{-1}{\sqrt{2}} - 6 - \sqrt{2}$

7. $-4 \operatorname{cosec} 2x \cot 2x$

8. $y_1 = 6x^5$, $y_2 = 30x^4$, $y_3 = 120x^3$, $y_4 = 360x^2$, $y_5 = 720x$, $y_6 = 720$, $y_7 = 0$

9. $\frac{-4}{(e^x - e^{-x})^2}$

10. $\frac{\cos x + x\log_e x \sin x}{x\cos^2 x}$

11. $e^x(\sin x + \cos x) + x^{p-1}(p\cos x - x\sin x)$

12. $\frac{e^x(\sin x - \cos x)}{\sin^2 x}$

13. $\frac{(-x\sin x + \cos x)\log x - \cos x}{(\log x)^2}$

14. $\frac{x^{n-1}[n\log x - 1]}{(\log_e x)^2}$

15. $\frac{\sqrt{a}}{\sqrt{x}(\sqrt{a}-\sqrt{x})^2}$

16. $\frac{x\sec x(\sec x - \tan x) + \tan x}{\sec x + \tan x}$

17. $\frac{1 - x(\log x)^2}{x(1 + x\log x)^2}$

19. $1 - \frac{12}{x^4} + \frac{4}{x^2}$

20. $e^x - 3\sin x + \frac{2}{x}$

21. $7x^6 \log_a x + x^6 \log_a e$

22. $\frac{1}{1+\cos x}$

23. $\frac{\frac{\sin x}{x} - \cos x \log_e x}{\sin^2 x}$

24. $e^x \operatorname{cosec} 3x\,(1 - 3\cot 3x) - 3\log x \operatorname{cosec} 3x \cot 3x + \frac{\operatorname{cosec} 3x}{x}$

25. $\frac{-8}{(e^{2x}+e^{-2x})^2}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The derivative of function $f(x)$ is $\tan^4 x$, If $f(0) = 0$ then $\lim_{x\to 0}\frac{f(x)}{x}$ is equal to **[J&K-2005]**

(a) 1
(b) 0
(c) –1
(d) None of these

Solution

(b) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{f(x) - 0}{x - 0}$

$= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$

$= f'(x)|_{x=0} = \tan^4 x|_{x=0} = 0$

2. Let $f(x + y) = f(x) f(y)$ and $f(1) = 2 \forall x, y \in R$, where $f(x)$ is continuous function. Then $f'(1)$ is equal to ***[NDA-2006]***

(a) $2 ln2$ (b) $ln2$
(c) 1 (d) 0

Solution

(a) Obviously f is an exponential function, so let $f(x) = a^x$.

$\because$ $f(1) = 2$, so $a^1 = 2$

$\Rightarrow$ $a = 2 \Rightarrow f(x) = 2^x \Rightarrow f'(x) = 2x\, ln2$

$\Rightarrow$ $f'(1) = 2\, ln2$

3. If $y = |\cos x| + |\sin x|$ then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is

(a) $\frac{1-\sqrt{3}}{2}$ (b) 0

(c) $\frac{1}{2}(\sqrt{3} - 1)$ (d) None of these

Solution

(c) $x = \frac{2\pi}{3}$, $|\cos x| = -\cos x$ and $|\sin x|$ $= \sin x$

$\therefore$ $y = -\cos x + \sin x$

$\therefore$ $\frac{dy}{dx} = \sin x + \cos x.$

At $x = \frac{2\pi}{3}, \frac{dy}{dx} = \sin\frac{2\pi}{3} + \cos\frac{2\pi}{3}.$

4. If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$ then $\frac{dy}{dx}$ at $x = 1$ is

(a) 2 (b) 1
(c) –2 (d) None of these

Solution

(a) $y = f(x^2)$

$\Rightarrow$ $\frac{dy}{dx} = f'(x^2).\,2x = 2x.\sqrt{2(x^2)^2 - 1}$

At $x = 1, \frac{dy}{dx} = 2.1.\sqrt{2-1}$

5. If $P(x)$ is a polynomial such that $P(x^2 + 1) = \{P(x)\}^2 + 1$ and $P(0) = 0$ then $P'(0)$ is equal to

(a) 1 (b) 0
(c) -1 (d) None of these

Solution

(a) $\because$ $P(x^2 + 1) = \{P(x)\}^2 + 1$ and $P(x)$ is a polynomial, we have $P(x) = x$ which also satisfies $P(0) = 0$. Therefore, $P'(x) = 1$. So $P'(0) = 1$

6. If $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, then $\frac{dy}{dx}$ is equal to ***[Karnataka CET-2005]***

(a) $\sec h^2 x$ (b) $\operatorname{cosec} h^2 x$
(c) $-\sec h^2 x$ (d) $-\operatorname{cosec} h^2 x$

Solution

(d) $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$= \frac{\frac{e^x + e^{-x}}{2}}{\frac{e^x - e^{-x}}{2}} = \frac{\cos hx}{\sin hx} = \cot hx$

$\frac{dy}{dx} = -\operatorname{cosec} h^2 x$

7. If $f(x) = (x - x_0)\phi(x)$ and $\phi(x)$ is continuous at $x = x_0$. then $f'(x_0)$ is equal to ***[NDA-2006]***

(a) $\phi'(x_0)$ (b) $\phi(x_0)$
(c) $x_0\phi(x_0)$ (d) $2\phi(x_0)$

Solution

(b) $f(x) = (x - x_0)\phi'(x) + \phi(x) \Rightarrow f'(x_0) = 0 + \phi(x_0) = \phi(x_0)$

8. If $y = \frac{\sqrt{x}(2x+3)^2}{\sqrt{x+1}}$, then $\frac{dy}{dx} =$ ***[MPPET-2007]***

(a) $y\left[\frac{1}{2x} + \frac{4}{2x+3} - \frac{1}{2(x+1)}\right]$

(b) $y\left[\frac{1}{3x} + \frac{4}{2x+3} - \frac{1}{2(x-1)}\right]$

(c) $y\left[\frac{1}{3x} + \frac{4}{2x+3} + \frac{1}{x+1}\right]$

(d) None of these

Solution

(a) $y = \dfrac{\sqrt{x}(2x+3)^2}{\sqrt{x+1}}$

$\Rightarrow \quad \log y = \dfrac{1}{2}\log x + 2\log(2x+3) - \dfrac{1}{2}\log(x+1)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2x} + \dfrac{2}{(2x+3)}\cdot 2 - \dfrac{1}{2(x+1)}$

$\dfrac{dy}{dx} = y\left[\dfrac{1}{2x} + \dfrac{4}{2x+3} - \dfrac{1}{2(x+1)}\right]$

9. Let $y(x) = ax^n$ and δy denote small change in y. What is limit of $\dfrac{\delta y}{\delta x}$ as $\delta x \to 0$?

[AIEEE-2009]

(a) 0
(b) 1
(c) anx^{n-1}
(d) $ax^n \log(ax)$

Solution

(a) $\lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \left(\dfrac{dy}{dx}\right)_{at\ x=0}$

$= \left(\dfrac{d}{dx}ax^n\right)_{at\ x=0} = (anx^{n-1})_{at\ x=0} = 0$

10. If $f(x) = \tan x + e^{-2x} - 7x^3$, then what is the value of $f'(0)$? *[NDA-2009]*

(a) –2
(b) –1
(c) 0
(d) 3

Solution

(b) $\because f(x) = \tan x + e^{-2x} - 7x^3$

On differentiating w.r.t. x, we get

$f'(x) = \sec^2 x - 2e^{-2x} - 21x^2$

$\Rightarrow f'(0) = \sec^2 0 - 2e^0 - 21 \times 0 = 1 - 2 = -1$

11. If $3^x + 3^y = e^{x+y}$, then what is $\dfrac{dy}{dx}$ equal to?

[NDA-2009]

(a) $\dfrac{3^{x+y} - 3^x}{3^y}$
(b) $\dfrac{3^{x-y}(3^y - 1)}{1 - 3^x}$
(c) $\dfrac{3^x + 3^y}{3^x - 3^y}$
(d) $\dfrac{3^x + 3^y}{1 + 3^{x+y}}$

Solution

(b) $3^x + 3^y = 3^{x+y}$

On differentiating w.r.t. x, we get

$3^x \log 3 + 3^y \log 3 \dfrac{dy}{dx} = 3^{(x+y)} \log 3\left(1 + \dfrac{dy}{dx}\right)$

$\Rightarrow \quad 3^x + 3^y \dfrac{dy}{dx} = 3^{x+y} + 3^{(x+y)}\dfrac{dy}{dx}$

$\Rightarrow \quad \dfrac{dy}{dx}(-3^{x+y} + 3^y) = 3^{x+y} + 3^{(x+y)}\dfrac{dy}{dx}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{3^x(3^y - 1)}{3^y(1 - 3^x)} = \dfrac{3^{x-y}(3^y - 1)}{1 - 3^x}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. Derivative of $x^6 + 6^x$ with respect to x is

(a) $12x$
(b) $x + 4$
(c) $6x^5 + 6^x \log 6$
(d) $6x^5 + x6^{x-1}$

2. If $y = \log_{10} x^2$, then $\dfrac{dy}{dx}$ is equal to

(a) $2/x$
(b) $\dfrac{2}{x\log_e 10}$
(c) $\dfrac{1}{x\log_e 10}$
(d) $\dfrac{1}{10x}$

3. If $f(x) = \dfrac{1}{1-x}$ then derivative of composite function $f[f\{f(x)\}]$ is *[Orissa JEE-2003]*

(a) 0
(b) 1/2
(c) 1
(d) 2

4. Derivative of $f(x) = |x^2 - x|$ at $x = 2$ is

(a) –3
(b) 0
(c) 3
(d) Undefined

5. If $y = e^{(1+\log_e x)}$, then $\dfrac{dy}{dx} =$

[MPPET-1996; Pb.CET-2001]

(a) e
(b) 1
(c) 0
(d) $\log_e xe^{\log ex}$

6. If $y = e^x \log x$, then $\frac{dy}{dx}$ is **[SCRA-1996]**

(a) $\frac{e^x}{x}$ (b) $e^x\left(\frac{1}{x} + x\log x\right)$

(c) $e^x\left(\frac{1}{x} + \log x\right)$ (d) $\frac{e^x}{\log x}$

7. If $f(x) = e^x g(x)$, $g(0) = 2$, $g'(0)=1$, then $f'(0)$ is

(a) 1 (b) 3

(c) 2 (d) 0

8. If $y = \log_{10}x + \log_x 10 + \log_x x + \log_{10}10$, then $\frac{dy}{dx} =$

(a) $\frac{1}{x\log_e 10} - \frac{1}{(x\log_e x)^2}$

(b) $\frac{1}{x\log_e 10} - \frac{1}{x\log_{10} e}$

(c) $\frac{1}{x\log_e 10} - \frac{\log_e^{10}}{x(\log_e x)^2}$

(d) None of these

9. If $f(x) = x + 2$, then at $x = 4$, $f'[f(x)]$ is equal to

(a) 1 (b) 4

(c) 8 (d) 2

10. If $y = f(x)$ is an even function such that $f'(0)$ exists, then $f'(0) =$ **[IIT-1987]**

(a) 0 (b) –1

(c) 1 (d) None of these

11. If $f(x)$ an odd differentiable function defined on $(-\infty, +\infty)$ such that $f'(3) = 2$, then $f'(-3)$ is **[IIT-JEE 1992]**

(a) 0 (b) 1

(c) 2 (d) 4

12. The function f is differentiable with $f(1) = 8$ and $f'(1) = \frac{1}{8}$. If f is invertible and $g = f^{-1}$. Then $g'(8) =$

(a) 8 (b) 1/8

(c) 0 (d) 1

13. Given $f(x + y + z) = f(x) f(y) f(z)$ for all x, y, z. If $f(2)=4$ and $f'(0) = 3$, then $f'(2) =$

(a) 12 (b) –12

(c) + 12 or – 12 (d) 1

14. If $f(x) = \sqrt{ax} + \frac{a^2}{\sqrt{ax}}$, then $f'(a) =$

(a) –1 (b) 1

(c) 0 (d) a

15. The differential coefficient of the function $|x - 1| + |x - 3|$ at the point $x = 2$ is **[RPET-2002; PbCET-2000, 04]**

(a) –2 (b) 0

(c) 2 (d) undefined

16. If $f(x^5) = 5x^3$, then $f'(x) =$ **[Karnataka CET-2008]**

(a) $\frac{3}{x}$ (b) $\sqrt[5]{x}$

(c) $\frac{3}{\sqrt[5]{x^2}}$ (d) $\frac{3}{\sqrt[5]{x}}$

17. If $f(x) = mx^2 + nx + p$, Then $f'(1) + f'(4) - f'(5)$ is equal to **[MPPET-2008]**

(a) m (b) $-m$

(c) n (d) $-n$

SOLUTIONS

1. (c) Step 1: Differentiation $x^6 + 6^x$ with respect to $x = 6x^5 + 6^x \log 6$

NOTE

By using formula $\frac{d}{dx}x^n = nx^{n-1}$

and $\frac{d}{dx}(a^x) = a^x \log a$

2. (b) Step 1: $y = \log_{10} x^2$

$$\frac{dy}{dx} = \frac{2}{x\log_e 10}$$

NOTE

By using formula $\frac{d}{dx}(\log_a x) = \frac{\log_a e}{x} = \frac{1}{x\log_e a}$

3. (c) Step 1: $f(x)=\dfrac{1}{1-x}$

$$f(f(x))=\frac{1}{1-\dfrac{1}{1+x}}=\frac{1-x}{-x}=\frac{x-1}{x}$$

$$f\left[f\{f(x)\}\right]=\frac{1}{1-\dfrac{x-1}{x}}=x$$

then derivative of composite function $f(f(f(x)))$ is $f'\left[f\{f(x)\}\right]=1$

4. (c) Step 1: $f(x) = |x^2 - x| = x^2 - x$

Since $f(x) = x^2 - x > 0$ in the neighbourhood of $x = 2$

Step 2:

$$\frac{d}{dx}f(x)\bigg|_{x=2}=\frac{d}{dx}(x^2-x)\bigg|_{x=2}$$

$$=2x-1\big|_{x=2}=2\times2-1=3$$

5. (a) Step 1: $y=e.e^{\log_e x}=ex$

$$\frac{dy}{dx}=e$$

6. (c) Step 1: $y = e^x \log x$

$$\frac{dy}{dx}=e^x\log x+e^x\frac{1}{x}=e^x\left[\frac{1}{x}+\log x\right]$$

NOTE

By using formula $\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$ (Product Rule)

7. (b) Step 1: $f(x) = e^x g(x)$

$f'(x)= e^x g(x) + e^x g'(x)$

$\therefore\ f'(0) = e^0 g(0) + e^0 g'(0)$

$= g(0) + g'(0) = 2 + 1 = 3$

8. (c) Step 1: $y=\dfrac{\log_e x}{\log_e 10}+\dfrac{\log_e 10}{\log_e x}+1+1$

$$\therefore \frac{dy}{dx}=\frac{1}{x\log_e 10}-\frac{\log_e 10}{(\log_e x)^2}\times\frac{1}{x}$$

10. (a) since $f(x)$ is even

$\therefore\ f(-x) = f(x)$

$\therefore\ f'(-x)(-1) = f'(x)$

$\Rightarrow\ f'(-x) = -f'(x)$

$\therefore\ f'(-0) = -f'(0) \Rightarrow f'(0) = -f'(0)$

$\Rightarrow\ 2f'(0) = 0 \Rightarrow f'(0) = 0$

11. (c) Step 1: Differentiation of odd function is an even function

Hence $f'(x)$ is even function.

Step 2: Here $f'(x) = f'(-x)$

$f'(3) = f'(-3) \Rightarrow f'(-3) = 2$

12. (a) Step 1: given $g = f^{-1}$ i.e., $f(g(x)) = g(f(x)) = x$

Step 2: Differentiating $g(f(x)) = x$ with respect to x we get

$g'(f(x))f'(x) = 1$

for $x = 1$ we have

$g'(f(1))f'(1) = 1$

$$g'(8)\frac{1}{8}=1\Rightarrow g'(8)=8$$

13. (c) Step 1: Given $f(x + y + z) = f(x) f(y) f(z)$ .... (1)

for all x, y, z

i.e., $f(x + y + z) = f(x) f(y) f(z)$

or $f(x) = f(x)\,(f(0))^2 \Rightarrow (f(0))^2 = 1$

$\Rightarrow\ f(0) = \pm 1$ .... (2)

Step 2: $f(x + 2 + 0) = f(x) f(2) f(0)$

On differentiation $f'(x + 2) = f'(x) f(2) f(0)$

for $x = 0$ we have

$f'(0 + 2) = f'(0) f(2) f(0)$

i.e., $f'(2) = 3 \times 4 \times (\pm 1) = \pm 12$

14. (b) Step 1: Given $f(x)=\sqrt{a}\sqrt{x}+a^{3/2}x^{-1/2}$

$$f'(x)=\frac{\sqrt{a}}{2\sqrt{x}}+a^{3/2}\left(\frac{-1}{2x^{3/2}}\right)$$

$$\therefore f'(a)=\frac{\sqrt{a}}{2\sqrt{a}}+\frac{a^{3/2}}{2a^{3/2}}=\frac{1}{2}+\frac{1}{2}=1$$

15. (b) Step 1: $f(x) = |x - 1| + |x - 3|$

$$f(x)=\begin{cases}-(x-1)-(x-3), & x<1\\ (x-1)-(x-3), & x>1\\ (x-1)-(x-3), & x<3\\ (x-1)+(x-3), & x>3\end{cases}$$

$$=\begin{cases}4-2x & x<1\\ 2 & 1<x<3\\ 2x-4 & x>3\end{cases}$$

At $x = 2$, $f(x) = 2$. Hence $f'(x) = 0$

16. (c) $f(x^5) = 5x^3$

$f(t) = 5t^{3/5}$

$(\because t = x^5, x^3 = t^{3/5})$

$$f'(t) = 5 \times \frac{3}{5} t^{-2/5}$$

$$= \frac{3}{\sqrt[5]{t^2}}$$

$$\therefore f'(x) = \frac{3}{\sqrt[5]{x^2}}$$

17. (c) Step 1: Given:

$f(x) = mx^2 + nx + p$

$f'(x) = 2mx + n$

$f'(1) = 2m + n$

$f'(4) = 8m + n$

$f'(5) = 10m + n$

Step 2: $\therefore f'(1) + f'(4) - f'(5)$

$= 2m + n + 8m + n - 10m - n$

$= n$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 13 minutes.
3. The worksheet consists of 13 questions. The maximum marks are 39.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The derivative of $f(x) = |x|^3$ at $x = 0$ is ***[RPET-2001; Kurukshetra CEE-2002]***

(a) 0 (b) 1
(c) −1 (d) not defined

2. Let $3f(x) - 2f(1/x) = x$, then $f'(2)$ is equal to

(a) 2/7 (b) 1/2
(c) 2 (d) 7/2

3. The value of x, at which the first derivative of the function $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ with respect to x is 3/4, are ***[MPPET-98]***

(a) ± 2 (b) $\pm 1/2$
(c) $\pm\sqrt{3}/2$ (d) $\pm 2/\sqrt{3}$

4. If $y = \frac{(1-x)^2}{x^2}$, then $\frac{dy}{dx}$ is ***[MPPET-1999]***

(a) $\frac{2}{x^2} + \frac{2}{x^3}$

(b) $-\frac{2}{x^2} + \frac{2}{x^3}$

(c) $-\frac{2}{x^2} - \frac{2}{x^3}$

(d) $-\frac{2}{x^3} + \frac{2}{x^2}$

5. If $f(x) = 4x^8$, then =

(a) $f'(1/2) = f'(-1/2)$
(b) $f'(-1/2) = f'(1/2)$
(c) $f(-1/2) = f(1/2)$
(d) $f(1/2) = f'(-1/2)$

6. The derivative of $\tan x - x$ with respect to x is ***[SCRA-1996]***

(a) $1 - \tan^2 x$ (b) $\tan x$
(c) $-\tan^2 x$ (d) $\tan^2 x$

7. If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\infty$, then $\frac{dy}{dx} =$ ***[Karnataka CET-1999]***

(a) y (b) $y - 1$
(c) $y + 1$ (d) None of these

8. If $5f(x) + 3f(1/x) = x + 2$ and $y = xf(x)$, then $(dy/dx)_{x=1}$ is equal to

(a) 14 (b) 7/8
(c) 1 (d) None of these

9. Let $f(x + y) = f(x)f(y)$ for all x and y. Suppose that $f(3) = 3$ and $f'(0) = 11$, then $f'(3)$ is given by ***[Kerala PET-2001]***

(a) 22 (b) 44
(c) 28 (d) 33

10. If $pv = 81$, then $\frac{dp}{dv}$ is at $v = 9$ equal to ***[MPPET-1999]***

(a) 1 (b) −1
(c) 2 (d) None of these

11. If $y = (1 + x^{1/4})(1 + x^{1/2})(1 - x^{1/4})$, then $\frac{dy}{dx} =$ ***[MPPET-1994]***

(a) 1 (b) −1
(c) x (d) $\sqrt{x}$

12. The differential coefficient of x^6 with respect to x^3 is ***[EAMCET-88; UPSEAT-2000]***

(a) $5x^2$ (b) $3x^3$
(c) $5x^5$ (d) $2x^3$

13. If $x = t^2 + \frac{1}{t^2}$, $y = t^4 + \frac{1}{t^4}$, then $\frac{dy}{dx} =$

(a) $2x$ (b) x
(c) x^2 (d) None of these

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. Step 1: $3f(x)-2f\left(\frac{1}{x}\right)=x$ .... (1)

replace x by $\frac{1}{x}$ in equation (1), we get

$$3f\left(\frac{1}{x}\right)-2f(x)=\frac{1}{x} \quad \text{........... (2)}$$

Now eliminating $f\left(\frac{1}{x}\right)$ by using both equation (1) and (2)

$$f(x)=\frac{1}{5}\left[3x+\frac{2}{x}\right]$$

$$f'(x)\Big|_{x=2}=\frac{1}{5}\left[3-\frac{2}{x^2}\right]\Bigg|_{x=2}=\frac{1}{2}$$

NOTE

If $af(x)+bf\left(\frac{1}{x}\right)=x$ then

$$f(x)=\frac{1}{a^2-b^2}\left(ax-\frac{b}{x}\right)$$

3. Step 1: Differential coefficient of $y = f(x)$ at $x = 0$ is represented by $\left.\frac{dy}{dx}\right|_{x=a}$ or $f'(a)$ (It is obtained by differentiating function and then substituting $x = a$)

Step 2: $y=f(x)=\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=x+\frac{1}{x}+2$

$$\frac{dy}{dx}=f'(x)=1-\frac{1}{x^2}$$

Step 3: $1-\frac{1}{x^2}=\frac{3}{4}\Rightarrow x^2=4\Rightarrow x=\pm 2$

4. Hint: $y=\frac{1}{x^2}-\frac{2}{x}+1$

Formula: $\frac{d}{dx}\left(\frac{1}{x^n}\right)=\frac{-n}{x^{n+1}}$

5. Step 1: $f'(x) = 32x^7$: Power of x is odd

Hence, $f\left(-\frac{1}{2}\right)=f\left(\frac{1}{2}\right)$

7. Step 1: $y = e^x$

$$\frac{dy}{dx}=e^x=y$$

10. Hint: $p=\frac{81}{v},\ \frac{dp}{dv}=\left.\frac{-81}{v^2}\right|_{v=9}=-1$

11. Step 1: $y=\left(1-x^{1/4}\right)\left(1+x^{1/4}\right)\left(1+x^{1/2}\right)$

Step 2: $y=\left(1-x^{1/2}\right)\left(1+x^{1/2}\right)=1-x$

Step 3: $\frac{dy}{dx}=-1$

(differentiating with respect to x)

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LECTURE

2

Fundamental Theorems

BASIC CONCEPTS

1. If $y = f(t)$ and $t = \phi(x)$ Function of a function (chain Rule) then $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

2. If $u = f(y)$, then $\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = f'(y)\frac{dy}{dx}$

3. $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{dx/dy}$

4. **Differentiation of one function with respect to another function.**
   Let $y = f(x)$, $z = \phi(x)$ and we have to differentiate $f(x)$ with respect to $\phi(x)$ i.e., y with respect to z so that we have to find the value of $\frac{dy}{dz} = \frac{dy}{dx} \div \frac{dz}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dz}$.

5. **Logarithmic Differentiation:**
   If $y = [f_1(x)]^{f_2(x)}$ or $y = f_1(x)f_2(x)f_3(x)...$
   or $y = \frac{f_1(x) f_2(x)..}{\phi_1(x)\, \phi_2(x)..}$ then it will be convenient to take log of both sides before performing differentiation.
   Note: Quicker Method:
   $$\frac{d}{dx}[f(x)]^{f(x)} = [f(x)^{g(x)}]\left[\frac{d}{dx}\{g(x)\log f(x)\}\right]$$
   $$= [f(x)]^{g(x)}\left[\frac{d}{dx}(\text{exponent times log of base})\right]$$

6. **Parametric Equations:**
   If $x = f(t)$, $y = \phi(t)$, then $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$.

7. **Differentiation of Implicit Function $f(x, y) = c$:** Differentiate each term with respect to x and note that $\frac{d}{dx}(\phi(y)) = \frac{d}{dy}(\phi(y))\frac{dy}{dx}$.

   For example $x^3 + y^3 - 3axy = 0$
   Differentiating with respect to x
   $$3x^2 + 3y^2\frac{dy}{dx} - 3a\left(1.y + x\frac{dy}{dx}\right) = 0,$$
   $$(x^2 - ay) + (y^2 - ax)\frac{dy}{dx} = 0$$
   $$\therefore \quad \frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax}$$

8. **By the help of partial differentiation: (Quicker method)**
   If $f(x, y) = c$, then we can find $\frac{dy}{dx}$ by the help of partial differentiation as under $\frac{dy}{dx} = -\frac{f_x}{f_y}$
   where f_x is differential coefficient of $f(x, y)$ with respect to x treating y as constant. Similarly f_y is differentiation of $f(x, y)$ with respect to y treating x as constant. For example if $f(x, y) = x^3 + y^3 - 3axy = 0$, then $f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$
   $$\therefore \quad \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x^2 - ay}{y^2 - ax}$$

[redacted]

1. If $y = a^{x^{a^{x^{\cdots\infty}}}}$, then $\dfrac{dy}{dx} = \dfrac{y^2 \log y}{x(1 - y\log x \log y)}$

Solution

$y = a^{x^y}$ $\therefore$ $\log y = x^y \log a$

$\therefore$ $\log \log y = y \log x + \log (\log a)$

$$\frac{1}{\log y}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}.\log x + y.\frac{1}{x}$$

$$\therefore \left(\frac{1}{y\log y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y^2\log y}{x(1 - y\log x\log y)}$$

2. If $\sin y = x \sin (a + y)$, then

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a} \text{ or } \frac{\sin a}{1 + x^2 - 2x\cos a}$$

Solution

$\sin y = x \sin (a + y)$, $\therefore$ $x = \dfrac{\sin y}{\sin(a+y)}$

Differentiate with respect to x

$$\therefore 1 = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}\cdot\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y-y)} = \frac{\sin^2(a+y)}{\sin a}$$

2nd form:

In this form we want the value of $\dfrac{dy}{dx}$ in terms of x where as the value found earlier was in terms of y only. Hence from given relation we have

$$\tan y = \frac{x\sin a}{1 - x\cos a} \quad \therefore \quad y = \tan^{-1}\frac{x\sin a}{1 - x\cos a}$$

$\therefore$ $\dfrac{dy}{dx}$ as given.

3. If $f(x) = \sin (\log x)$ and $y = f\left[\left(\dfrac{2x+3}{3-2x}\right)\right]$ find $\dfrac{dy}{dx}$.

Solution

Since $f(x) = \sin \log x$, we have

$$y = f\left(\frac{2x+3}{3-2x}\right) = \sin\log\left(\frac{2x+3}{3-2x}\right)$$

$$= \sin [\log (2x + 3) - \log (3 - 2x)]$$

$$\therefore \frac{dy}{dx} = \cos[\log(2x+3) - \log(3-2x)].$$

$$\left(\frac{2}{2x+3} + \frac{2}{3-2x}\right)$$

$$= \left(\frac{12}{9-4x^2}\right)\cos\log\left(\frac{2x+3}{3-2x}\right)$$

[redacted]

TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. $e^{ax} \cos (bx + c)$ **[CBSE-89]**

2. $\sqrt{\cos\sqrt{x}}$ **[MP-98, 99]**

3. $\log\sqrt{\dfrac{1+\sin x}{1-\sin x}}$ **[MP-99, 2005]**

4. If $y = a \cos (\log x) + b \sin (\log x)$ then find $\dfrac{dy}{dx}$ **[MP-98]**

5. If $y = \cos x°$ then find $\dfrac{dy}{dx}$

6. If $y = 5^{\log (\sin x)}$, then find $\dfrac{dy}{dx}$ **[CBSE-90]**

7. If $y = \dfrac{e^{\sin x}}{\sin x^n}$ then find the value of $\dfrac{dy}{dx}$ **[MP-2004]**

8. If $y = \log_e \log_e (\log_e x)$ then find $\frac{dy}{dx}$ ***[MP-98, 99, 2001]***

9. If $y = \sqrt[3]{(x^2+2)^2}$, find $\frac{dy}{dx}$

10. Differentiate $\log_{\sin x} \cos x$ with respect to x

11. Find the derivative of $\cos(\sin x^2)$ at $x = \sqrt{\pi/2}$ ***[DSB-97]***

12. If $y = \frac{1-\cos x}{1+\cos x}$, then find $\frac{dy}{dx}$. ***[MP-2004]***

13. If $y = \sin^2 x^2$ then find $\frac{dy}{dx}$ ***[MP-2001]***

14. Find the differential coefficient of $e^{\sin x}$ with respect to x. ***[MP-2007]***

15. Find $\frac{dy}{dx}$, when $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ***[AICBSE-90; J & K-95]***

16. If $(\cos x)^y = (\sin y)^x$ find $\frac{dy}{dx}$. ***[CBSE-90]***

17. If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$ ***[CBSE-90C, 99, 2000C; SP-2006; MLNR-84; UP-81, 83; HPSB-98; HB-2002C; MP-2007]***

18. Find $\frac{dy}{dx}$, when $x^{2/3} + y^{2/3} = a^{2/3}$.

19. If $x = a\cos^3 t$ and $y = a\sin^3 t$ then find $\frac{dy}{dx}$. ***[MP-2000; CBSE-90, 91; HSB-93]***

20. If $\sin y = x\cos(a+y)$, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a}$. ***[MP-2007]***

21. Find the differential coefficient of $(\tan x)^{\cot x} + (\cot x)^{\tan x}$ ***[MP-2000, 2001, 2006]***

22. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\ldots\ldots + \infty}}}$ then prove that $\frac{dy}{dx} = \frac{\sin x}{1-2y}$. ***[MP-98, 2000, 2006; PB-2001C]***

ANSWERS

1. $e^{ax}[a\cos(bx+c) - b\sin(bx+c)]$

2. $\frac{-1 \times \sin\sqrt{x}}{4\sqrt{x}\sqrt{\cos\sqrt{x}}}$

3. $\sec x$

4. $\frac{-a\sin(\log x) + b\cos(\log x)}{x}$

5. $\frac{-\pi}{180}\sin x°$

6. $5^{\log(\sin x)}(\cot x)(\log 5)$

7. $\frac{dy}{dx} = \frac{\sin x^n e^{\sin x}\cos x - e^{\sin x}\cos x^n \,.\, nx^{n-1}}{\sin^2 x^n}$

8. $\frac{dy}{dx} = \frac{1}{x\log x\log(\log x)}$

9. $\frac{4x}{3\sqrt[3]{(x^2+2)}}$

10. $\frac{-\{\tan x\log\sin x + \cot x\log\cos x\}}{(\log\sin x)^2}$

11. 0

12. $\tan x/2 \sec^2 x/2$

13. $4x\sin x^2\cos x^2$

14. $\cos x\, e^{\sin x}$

15. $\frac{dy}{dx} = \frac{-(ax+hy+g)}{hx+by+f}$

16. $\frac{\log\sin y + y\tan x}{\log\cos x - x\cot y}$

18. $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$

19. $-\tan t$

21. $(\tan x)^{\cot x}\operatorname{cosec}^2 x\{1 - \log(\tan x)\} + (\cot x)^{\tan x}\sec^2 x\{\log(\cot x) - 1\}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $f(x) = \cos\left\{\frac{\pi}{2}[x] - x^3\right\}, 1 < x < 2,$ and $[x]$ = the greatest integer $\leq x$, then $f'\left(\sqrt[3]{\frac{\pi}{2}}\right)$ is equal to

(a) 0 (b) $3\left(\frac{\pi}{2}\right)^{2/3}$

(c) $-3\left(\frac{\pi}{2}\right)^{3/2}$ (d) None of these

Solution

(a) Around $x = \sqrt[3]{\frac{\pi}{2}}, [x] = 1.$

So, $f(x) = \cos\left\{\frac{\pi}{2} - x^3\right\} = \sin x^3$

around $x = \sqrt[3]{\frac{\pi}{2}}.$ $\therefore f'(x) = 3x^2 \cos x^3$

$$\therefore \quad f'\left(\sqrt[3]{\frac{\pi}{2}}\right) = 3.\left(\frac{\pi}{2}\right)^{2/3}.\cos\frac{\pi}{2} = 0$$

2. If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to

[IIT JEE-1994]

(a) $(\sin x)^{\tan x}(1 + \sec^2 x \log \sin x)$
(b) $\tan x(\sin x)^{\tan x - 1}. \cos x$
(c) $(\sin x)^{\tan x} \sec^2 x \log \sin x$
(d) $\tan x (\sin x)^{\tan x - 1}$

Solution

(a) Differentiating with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \log\sin x + \tan x.\frac{1}{\sin x}.\cos x$$

$$\frac{dy}{dx} = (\sin x)^{\tan x}[1 + \sec^2 x \log\sin x]$$

3. If $r = [2\phi + \cos^2(2\phi + \pi/4)]^{1/2}$ then what is the value of the derivative of $dr/d\phi$ at $\phi = \pi/4$.

[Orissa JEE 2005]

(a) $2\left(\frac{1}{\pi+1}\right)^{1/2}$ (b) $2\left(\frac{2}{\pi+1}\right)^{-1/2}$

(c) $2\left(\frac{1}{\pi+1}\right)^{-1/2}$ (d) $2\left(\frac{2}{\pi+1}\right)^{1/2}$

Solution

(d) $$\frac{dr}{d\phi} = \left[2\phi + \cos^2\left(2\phi + \frac{\pi}{4}\right)\right]^{-1/2}$$

$$\left[2 - 2\times 2\sin\left(2\phi + \frac{\pi}{4}\right)\times\cos\left(2\phi + \frac{\pi}{4}\right)\right]$$

$$\left(\frac{dr}{d\phi}\right)_{\phi=\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi}{4} + \cos^2\frac{3\pi}{4}\right]^{-1/2}$$

$$\times 2\left[\left(1 - \sin\left(\pi + \frac{\pi}{2}\right)\right)\right]$$

$$\left(\frac{dr}{d\phi}\right)_{\phi=\frac{\pi}{4}} = \frac{1}{2}\left(\frac{\pi}{2} + \frac{1}{2}\right)^{-1/2}$$

$$\times 2(1+1) = 2\times\left(\frac{2}{\pi+1}\right)^{1/2}$$

4. If $\ln(x + y) = 2xy$, then $y'(0) =$

[IIT Screening-2004]

(a) 1 (b) −1
(c) 2 (d) 0

Solution

(a) $\ln(x + y) = 2xy$ Differentiate both sides with respect to x,

$$\left(\frac{1}{x+y}\right)\left(1 + \frac{dy}{dx}\right) = 2\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 - 2xy - 2y^2}{2x^2 + 2xy - 1}$$

As at $x = 0, y = 1$, (From $\ln(x + y) = 2xy$)

Hence $y'(0) = \frac{1-2}{-1} = 1$

5. A curve is given by the equations $x = a\cos\theta + \frac{1}{2}b\cos 2\theta,\ y = a\sin\theta + \frac{1}{2}b\sin 2\theta,$ then the points for which $\frac{d^2y}{dx^2} = 0$, is given by

[Kurukshetra CEE-2002]

(a) $\sin\theta = \frac{2a^2 + b^2}{5ab}$

(b) $\tan\theta = \frac{3a^2 + 2b^2}{4ab}$

(c) $\cos\theta = \dfrac{-(a^2+2b^2)}{3ab}$

(d) $\cos\theta = \dfrac{(a^2-2b^2)}{3ab}$

Solution

(c) $x = a\cos\theta + \frac{1}{2}b\cos 2\theta,$

$y = a\sin\theta + \frac{1}{2}b\sin 2\theta$

$\dfrac{dy}{d\theta} = a\cos\theta + b\cos 2\theta,$

$\dfrac{dx}{d\theta} = -a\sin\theta - b\sin 2\theta$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\cos\theta + b\cos 2\theta}{-a\sin\theta - b\sin 2\theta}$

$\dfrac{d}{dx} = \left(\dfrac{dy}{dx}\right) = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{d\theta}{dx}$

$$= \left[\frac{(a\sin\theta + b\sin 2\theta)(a\sin\theta + 2b\sin 2\theta)}{(a\sin\theta + b\sin 2\theta)^2} + \frac{(a\cos\theta + b\cos 2\theta)(a\cos\theta + 2b\cos 2\theta)}{(a\sin\theta + b\sin 2\theta)^2}\right]\cdot\frac{d}{d\theta}$$

but $\dfrac{d^2y}{dx^2} = 0$

$\Rightarrow a^2 + 2b^2 + 3ab\,[\sin 2\theta\,\sin\theta + \cos 2\theta.\,\cos\theta] = 0$

$\Rightarrow a^2 + 2b^2 = -\,3ab\cos(2\theta - \theta)$

$\therefore \cos\theta = -\left(\dfrac{a^2+2b^2}{3ab}\right)$

6. If $x^2 + y^2 = t - \frac{1}{t}$, $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $\dfrac{dy}{dx}$ equals **[PET (Raj.)-90, 99; NDA-2005]**

(a) $\dfrac{1}{x^2y}$ (b) $\dfrac{1}{xy^3}$

(c) $-\dfrac{1}{x^3y}$ (d) $-\dfrac{1}{xy^3}$

Solution

(a) Squaring the first equation, we have

$x^4 + y^4 + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$

$\Rightarrow \quad t^2 + \frac{1}{t^2} + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$

(from second equation)

$\Rightarrow \quad x^2y^2 = -1 \qquad \Rightarrow y^2 = -\dfrac{1}{x^2}$

$\therefore \quad 2y\dfrac{dy}{dx} = \dfrac{2}{x^3} \qquad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x^3y}$

7. If $y = f\left(\dfrac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin^2 x$, then $\dfrac{dy}{dx}$ is equal to **[PET (Raj.) 90; IIT 82]**

(a) $\dfrac{2(1+x-x^2)}{(1+x^2)^2}\sin\left(\dfrac{2x-1}{x^2+1}\right)^2$

(b) $\dfrac{2(1+x-x^2)}{(1+x^2)^2}\sin^2\left(\dfrac{2x-1}{x^2+1}\right)$

(c) $\sin^2\left(\dfrac{2x-1}{x^2+1}\right)$

(d) $\sin\left(\dfrac{2x-1}{x^2+1}\right)^2$

Solution

$\dfrac{d}{dx}\left(\dfrac{2x-1}{x^2+1}\right) = \dfrac{(x^2+1)2-(2x-1)2x}{(x^2+1)^2}$

$= \dfrac{2(1+x-x^2)}{(1+x^2)^2}$

$\therefore \quad \dfrac{dy}{dx} = f'\left(\dfrac{2x-1}{x^2+1}\right)\cdot\dfrac{d}{dx}\left(\dfrac{2x-1}{x^2+1}\right)$

$= \sin^2\left(\dfrac{2x-1}{x^2+1}\right)\cdot\dfrac{2(1+x-x^2)}{(x^2+1)^2}$

8. If f and g are two differentiable functions such that g' (a) = 2, g (a) = b and $fog = I$ (identity function) then $f'(b)$ is equal to **[DCE-94]**

(a) 1/2 (b) 2

(c) 2/3 (d) 1

Solution

(a) $\because (fog)(x) = x \;\forall\; x$

$\Rightarrow \quad f[g(x)] = x$

$\Rightarrow \quad f'[g(x)]\,g'(x) = 1$

[on differentiating both sides]

$\Rightarrow \quad f'[g(a)]\,g'(a) = 1$

$\Rightarrow \quad f'(g(a)) = \dfrac{1}{g'(a)} = \dfrac{1}{2}$

$\Rightarrow \quad f'(b) = 1/2\;[\because g(a) = b]$

9. If $xe^{xy} = y + \sin^2 x$, then at $x = 0, \dfrac{dy}{dx}$ is equal to ***[IIT-96]***

(a) 1 (b) −1
(c) 2 (d) −2

Solution

(a) On differentiating $e^{xy} + xe^{xy}\left(y + x\dfrac{dy}{dx}\right) = \dfrac{dy}{dx} + \sin 2x$

$$e^{xy}\left[1 + xy + x^2\frac{dy}{dx}\right] = \frac{dy}{dx} + \sin 2x$$

Putting $x = 0, 1 + 0 = \left(\dfrac{dy}{dx}\right)_0 + 0$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_0 = 1$

10. If $y = \sin x^\circ$ and $z = \log_{10} x$, then $\dfrac{dy}{dz}$ is equal to ***[NDA-2005]***

(a) $\dfrac{x^0 \cos x^0}{\log_{10} e}$ (b) $\dfrac{x \cos x^0}{\log_e 10}$

(c) $\dfrac{x \cos x^0}{\log_{10} e}$ (d) $\dfrac{x^0 \cos x^0}{\log_e 10}$

Solution

(a) $\dfrac{dy}{dz} = \dfrac{\left(\dfrac{dy}{dx}\right)}{\left(\dfrac{dz}{dx}\right)} = \dfrac{\dfrac{\pi}{180}\cos x^0}{\left(\dfrac{1}{x\log_e 10}\right)}$

$$= \frac{\left(\dfrac{\pi}{180}x\right)\cos x^0}{\log_{10} e} = \frac{x^o \cos x^o}{\log_{10} e}$$

11. $g(x)$ is the inverse function of an invertible function $f(x)$ which is differentiable at $x = c$, then $g'[f(c)]$ is equal to

(a) $f'(c)$ (b) $1/f'(c)$
(c) $f(c)$ (d) $1/f(c)$

Solution

(b) Since $g(x)$ is the inverse of function $f(x)$, therefore $gof(x) = I_x$ for all x. Now $gof(x) = I(x), \forall x$

$\Rightarrow gof(x) = x, \forall x$

$\Rightarrow (gof)'(x) = 1, \forall x$

$\Rightarrow g'(f(x))f'(x) = 1, \forall x$ (using chain rule)

$\Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}, \forall x$

$\Rightarrow g'(f(c)) = \dfrac{1}{f'(c)}$ (putting x = c)

12. Derivative of function $f(x) = \log_5 (\log_7 x)$, $x > 7$ is

(a) $\dfrac{1}{x(In5)(In7)(\log_7 x)}$ (b) $\dfrac{1}{x(In5)(In7)}$

(c) $\dfrac{1}{x(Inx)}$ (d) None of these

Solution

(a) $f(x) = \log_5 (\log_7 x)$

$\Rightarrow f(x) = \log_5\left(\dfrac{\log_e x}{\log_e 7}\right)$

$\Rightarrow f(x) = \log_5 \log_e x - \log_5 \log_e 7$

$\Rightarrow f(x) = \dfrac{\log_e \log_e x}{\log_e 5} - \log_5 \log_e 7$

$\Rightarrow f'(x) = \dfrac{1}{x\log_e x \log_e 5} - 0$

$\Rightarrow f'(x) = \dfrac{1}{x\log_e x \dfrac{\log_e 5}{\log_e 7}\log_e 7}$

$= \dfrac{1}{x(In5)(In7)(\log_7 x)}$

13. If $\sin y + e^{-x\cos y} = e$ then $\dfrac{dy}{dx}(1, \pi)$ is

(a) $\sin y$ (b) $-x \cos y$
(c) e (d) $\sin y - x \cos y$

Solution

(c) $\sin y + e^{-x\cos y} = e$,

$\Rightarrow \cos y\dfrac{dy}{dx} + e^{-x\cos y}\left\langle(-x)\left(-\sin y\dfrac{dy}{dx}\right) + \cos y(-1)\right\rangle = 0$

$\Rightarrow \cos y\dfrac{dy}{dx} + x\sin y e^{-x\cos y}\left(\dfrac{dy}{dx} - \cos y e^{-x\cos y}\right) = 0$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(1,\pi)} = \dfrac{\cos \pi e^{-\cos\pi}}{\cos \pi e + \sin \pi e^{-\cos\pi}}$

$= \dfrac{(-1)e}{-1+0} = e$

14. If $y=\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$, then $\dfrac{dy}{dx}$ equal to

(a) $\dfrac{ay}{x\sqrt{a^2-x^2}}$ (b) $\dfrac{ay}{\sqrt{a^2-x^2}}$

(c) $\dfrac{ay}{x\sqrt{x^2-a^2}}$ (d) None of these

Solution

(c) $y=\dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$

$$\Rightarrow \quad y=\frac{\left(\sqrt{a+x}-\sqrt{a-x}\right)^2}{(a+x)-(a-x)}.$$

$$=\frac{(a+x)+(a-x)-2\left(\sqrt{a^2-x^2}\right)}{2x}$$

$$=\frac{(2x-2\sqrt{a^2-x^2}}{2x}$$

or $\quad y=\dfrac{a-\sqrt{a^2-x^2}}{x}$ ....(i)

Differentiation with respect to x of y, we get

$$\Rightarrow \frac{dy}{dx}$$

$$=\frac{x\left[-\frac{1}{2\sqrt{a^2-x^2}}(-2x)\right]-\left(a-\sqrt{a^2-x^2}\right)}{x^2}.$$

$$=\frac{x^2-a\sqrt{a^2-x^2}+a^2-x^2}{x^2\sqrt{a^2-x^2}}.$$

$$=\frac{a\left(a-\sqrt{a^2-x^2}\right)}{x^2\sqrt{a^2-x^2}}.$$

$$=\frac{a}{x\sqrt{a^2-x^2}}\left[\frac{a-\sqrt{a^2-x^2}}{x}\right]$$

$$=\frac{ay}{x\sqrt{a^2-x^2}}\text{ [By (1)]}$$

15. If $f(x)=5^{\log_x 5}$ then $f'(5)$ =?
($x \in R^+ - \{1\}$)

(a) –1 (b) 5
(c) 1/5 (d) None of these

Solution

(a) $\quad f(x)=5^{\log_x 5},\ x\in R^+-\{1\}$

$$=5^{\log_x e \log_e 5}=\left(5^{\log_e 5}\right)^{\log_x e}$$

(Recall that $\log_a b=\log_a c\,\log_c b$)

Differentiating with respect to x using chain rule

$$f'(x)=\frac{d}{dx}\left[\left(5^{\log_e 5}\right)^{\log_x e}\right]$$

$$=\frac{d}{dx}\left\{\left(5^{\log_e 5}\right)^{\frac{1}{\ln x}}\right\}$$

$$\left[\log_a b=\frac{1}{\log_b a}\right]$$

$$=\left(5^{\log_e 5}\right)^{\frac{1}{\ln x}}.\ln\left(5^{\log_e 5}\right).\left\{\frac{-1}{x(\ln x)^2}\right\}$$

Putting $x=5$

$$f'(5)=\left(5^{\log_e 5}\right)^{\frac{1}{\ln 5}}.\ln\left(5^{\log_e 5}\right).\left\{-\frac{1}{5(\ln 5)^2}\right\}$$

$$=5.(\ln 5)(\ln 5).\left\{-\frac{1}{5(\ln 5)^2}\right\}=-1$$

16. If $y=\log x^x$. then the value of $\dfrac{dy}{dx}$ is:

[MPPET-2009]

(a) $x^x(1+\log x)$ (b) $\log(ex)$

(c) $\log\left(\dfrac{e}{x}\right)$ (d) $\log\left(\dfrac{x}{e}\right)$

Solution

Given, $y = x\log x$
On differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{x}{x}+\log x\Rightarrow\frac{dy}{dx}$$

$$=\log e+\log x$$

$$\Rightarrow\quad\frac{dy}{dx}=\log(ex)$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ equals to

[RPET-2001]

(a) $\frac{\sin x}{2y-1}$ (b) $\frac{\cos x}{2y-1}$

(c) $\frac{\sin x}{2y+1}$ (d) $\frac{\cos x}{2y+1}$

2. If $y = \sqrt{\frac{1+\tan x}{1-\tan x}}$, then $\frac{dy}{dx} =$

[AISSE-1981, 83, 84, 85; DSSE-1985; AICBSE-1981, 83]

(a) $\frac{1}{2}\sqrt{\frac{1-\tan x}{1+\tan x}}.\sec^2\left(\frac{\pi}{4}+x\right)$

(b) $\sqrt{\frac{1-\tan x}{1+\tan x}}.\sec^2\left(\frac{\pi}{4}+x\right)$

(c) $\frac{1}{2}\sqrt{\frac{1-\tan x}{1+\tan x}}.\sec\left(\frac{\pi}{4}+x\right)$

(d) None of these

3. If $f(x) = \log\frac{u(x)}{v(x)}$, $u'(2) = 4$, $v'(2) = 2$, $u(2) = 2$, $v(2) = 1$. Then $f'(2) =$

(a) 0 (b) 1

(c) –1 (d) None of these

4. If $x = y\sqrt{1-y^2}$, then $\frac{dy}{dx} =$

(a) 0 (b) x

(c) $\frac{\sqrt{1-y^2}}{1-2y^2}$ (d) $\frac{\sqrt{1-y^2}}{1+2y^2}$

5. $\frac{d}{dx}\log_{\sqrt{x}}(1/x)$ is equal to

(a) $-\frac{1}{2\sqrt{x}}$ (b) –2

(c) $-\frac{1}{x^2\sqrt{x}}$ (d) 0

6. The differential coefficient of $f[\log(x)]$ when $f(x) = \log x$ is

[Kurukshetra CEE-1988; DCE-2000]

(a) $x \log x$ (b) $x/\log x$

(c) $1/x\log x$ (d) $\log x/x$

7. For the curve $\sqrt{x}+\sqrt{y} = 1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4},\frac{1}{4}\right)$ is

[Karnataka CET-1993]

(a) 1/2 (b) 1

(c) –1 (d) 2

8. If $f(x) = 3e^{x^2}$, then $f'(x) - 2xf(x) + \frac{1}{3}f(0) - f'(0) =$

(a) 0 (b) 1

(c) $\frac{7}{3}e^{x^2}$ (d) None of these

9. If $y = \log_{\sin x}(\tan x)$, then $\left(\frac{dy}{dx}\right)_{\pi/4} =$

(a) $\frac{4}{\log 2}$ (b) $-4\log 2$

(c) $\frac{-4}{\log 2}$ (d) None of these

10. $\frac{d}{dx}\{2^{\log x}\}$

(a) $\frac{2^{\log x}.\log 2}{x}$ (b) $2\log x \,.\, \log 2$

(c) $\frac{-2^{\log x}.\log 2}{x}$ (d) $\frac{2\log x.\log 2}{x}$

11. $\frac{d}{dx}\{e^x\log(1+x^2)\} =$ ***[AICBSE-1987]***

(a) $e^x\left[\log(1+x^2)+\frac{2x}{1+x^2}\right]$

(b) $e^x\left[\log(1+x^2)-\frac{2x}{1+x^2}\right]$

(c) $e^x\left[\log(1+x^2)+\frac{x}{1+x^2}\right]$

(d) $e^x\left[\log(1+x^2)-\frac{x}{1+x^2}\right]$

12. $\frac{d}{dx}e^{x\sin x} =$ ***[DSSE-1979]***

(a) $e^{x\sin x}(x\cos x + \sin x)$

(b) $e^{x\sin x}(\cos x + x\sin x)$

(c) $e^{x\sin x}(\cos x + \sin x)$

(d) None of these

13. If $y = f\left(\dfrac{5x+1}{10x^2-3}\right)$ and $f'(x)= \cos x$, then $\dfrac{dy}{dx}$ is equal to

(a) $\cos\left(\dfrac{5x+1}{10x^2-3}\right)$

(b) $\cos\left(\dfrac{5x+1}{10x^2-3}\right)\dfrac{d}{dx}\left(\dfrac{5x+1}{10x^2-3}\right)$

(c) $\dfrac{5x+1}{10x^2-3}\cos\left(\dfrac{5x+1}{10x^2-3}\right)$

(d) None of these

14. If $f(x) = \log_{x^3}{(\log x)^{27}}$, then $f'(e) =$

(a) $1/3e$ (b) $81e$
(c) $9/e$ (d) 1

15. If $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\text{to } \infty}}}$, then $\dfrac{dy}{dx} =$ ***[RPET-02; NDA-2007]***

(a) $\dfrac{x}{2y-1}$ (b) $\dfrac{2}{2y-1}$

(c) $\dfrac{-1}{2y-1}$ (d) $\dfrac{1}{2y-1}$

16. The rate of change of $\sqrt{(x^2+16)}$ with respect to $x/(x-1)$ at $x = 3$ is

(a) 2 (b) 11/5
(c) –12/5 (d) –3

17. If $2^x + 2^y = 2^{x+y}$, then the value of $\dfrac{dy}{dx}$ at $x = y = 1$ is

(a) 0 (b) –1
(c) 1 (d) 2

18. If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ $\dfrac{dy}{dx} =$ ***[DCE-99]***

(a) $\cos\theta$ (b) $\tan\theta$
(c) $\sec\theta$ (d) $\operatorname{cosec}\theta$

19. The derivative of $\sqrt{\sqrt{x}+1}$ is ***[SCRA-1996]***

(a) $\dfrac{1}{\sqrt{x}(\sqrt{x}+1)}$

(b) $\dfrac{1}{\sqrt{x}\sqrt{x}+1}$

(c) $\dfrac{4}{\sqrt{x(\sqrt{x}+1)}}$

(d) $\dfrac{1}{\sqrt[4]{x(\sqrt{x}+1)}}$

20. If $x^3 + y^3 - 3axy = 0$, then $\dfrac{dy}{dx}$ equals ***[RPET-1996]***

(a) $\dfrac{ay - x^2}{y^2 - ax}$ (b) $\dfrac{ay - x^2}{ay - y^2}$

(c) $\dfrac{x^2 + ay}{a^2 + ax}$ (d) $\dfrac{x^2 + ay}{ax - y}$

21. If $\sin(x+y) = \log(x+y)$, then $\dfrac{dy}{dx} =$

(a) 2 (b) –2
(c) 1 (d) –1

22. If $f(x) = (\log_{\cot x}{\tan x})(\log_{\tan x}{\cot x})^{-1}$, then $f'(2) =$

(a) 2 (b) 0
(c) 1/2 (d) –2

23. If $x^m y^n = 2(x+y)^{m+n}$ then $\dfrac{dy}{dx}$ is ***[MPPET-03; JEEWB-92]***

(a) $x + y$ (b) x/y
(c) y/x (d) $x - y$

24. Derivative of $F[f\{\phi(x)\}]$ ***[AMU-02]***

(a) $F'[f\{\phi(x)\}]$
(b) $F'[f\{\phi(x)\}]f'\{\phi(x)\}$
(c) $F'[f\{\phi(x)\}]f'\{\phi(x)\}$
(d) $F'[f\{\phi(x)\}]f'\{\phi(x)\}\phi'(x)$

25. $\dfrac{d}{dx}\left[\log\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right] =$

(a) $-\dfrac{\sqrt{x}}{1-x}$ (b) $\dfrac{1}{\sqrt{x}(1-x)}$

(c) $\dfrac{\sqrt{x}}{1+x}$ (d) $\dfrac{1}{\sqrt{x}(1+x)}$

26. If $y = (\tan x)^{\cot x}$, then $\dfrac{dy}{dx} =$ ***[AISSE-1985]***

(a) $y\operatorname{cosec}^2 x(1 - \log\tan x)$
(b) $y\operatorname{cosec}^2 x(1 + \log\tan x)$
(c) $y\operatorname{cosec}^2 x(\log\tan x)$
(d) None of these

27. $x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$, then $\dfrac{dy}{dx}$ at $(2, -1)$

(a) 22/7 (b) 6/7
(c) 6 (d) None of these

28. If $x = e^{y + e^{y + ... \text{to} \infty}}$, then $\dfrac{dy}{dx}$ is ***[AIEEE-2004]***

(a) $\dfrac{x}{1+x}$ (b) $\dfrac{1}{x}$

(c) $\dfrac{1-x}{x}$ (d) None of these

29. If $x^y = e^{x-y}$, then $\frac{dy}{dx} =$

[MPPET-1987, 2004; MNR-1984; Roorkee 1954; BIT Ranchi-1991; RPET-2000]

(a) $\log x \,.\, [\log (ex)]^{-2}$ (b) $\log x\, [\log (ex)]^2$
(c) $\log x \,.\, (\log x)^2$ (d) None of these

30. If $\varphi(x) = \log_5 \log_3 x$, then $\varphi'(e) =$

[Kerala PET-2008]

(a) $\frac{1}{e}\log_5 e$ (b) 1
(c) $e \log_e 5$ (d) $\log_5 e$

31. If $f(x+y) = 2f(x)f(y), f'(5) = 1024 (\log 2)$ and $f(2) = 8$, then the value of $f'(3)$ is

[Kerala PET-2008]

(a) 64 (log 2) (b) 128 (log 2)
(c) 256 (d) 256 (log 2)
(e) 1024 (log 2)

32. If $x = \sin t - t\cos t$ and $y = t\sin t + \cos t$, then what is $\frac{dy}{dx}$ at point $t = \frac{\pi}{2}$?

[NDA-2008]

(a) 0 (b) $\pi/2$
(c) $-\pi/2$ (d) 1

SOLUTIONS

1. (b) Step 1: $y = \sqrt{\sin x + y}$

Square on both side, we get

$$y^2 = \sin x + y$$

Step 2: differentiate with respect to x

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

2. (a) Step 1: $y = \sqrt{\frac{1+\tan x}{1-\tan x}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{1+\tan x}{1-\tan x}}}.\sec^2(\pi/4 + x)$$

$$\left(\because \frac{1+\tan x}{1-\tan x} = \tan(\pi/4 + x)\right)$$

NOTE

Using formula $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

3. (a) Step 1:

$$f(x) = \log\frac{u(x)}{v(x)} = \log[u(x)] - \log[v(x)]$$

$$f'(x) = \frac{1}{u(x)}\frac{d}{dx}[u(x)] - \frac{1}{v(x)}\frac{d}{dx}[v(x)]$$

$$f'(x) = \frac{u'(x)}{v(x)} - \frac{v'(x)}{v(x)}$$

Now, put the value of $x = 2$

$$f'(2) = \frac{u'(2)}{u(2)} - \frac{v'(2)}{v(2)} = \frac{4}{2} - \frac{2}{1} = 0$$

NOTE

by using formula $\frac{d}{dx}(\log x) = \frac{1}{x}$

4. (c) Step 1: differentiate with respect to y

$$\frac{dx}{dy} = 1\sqrt{1-y^2} + y \times \frac{1}{2\sqrt{1-y^2}}(-2y)$$

$$\frac{dx}{dy} = \sqrt{1-y^2} - \frac{y^2}{\sqrt{1-y^2}} = \frac{1-2y^2}{\sqrt{1-y^2}}.$$

Hence, $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{1-2y^2}$

NOTE

$$\frac{d}{dx}(u,v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

5. (d) Step 1: $\frac{d}{dx}\log_{\sqrt{x}}\left(\frac{1}{x}\right)$

$$= \frac{d}{dx}2\log_x(x^{-1}) = \frac{d}{dx}(-2) = 0$$

(differentiation of constant term is zero)

6. (c) Step 1: $f(\log x) = \log(\log x)$,

$$\frac{d}{dx}(\log(\log x)) = \frac{1}{x\log x}$$

Now comparing with the given data in the question.

7. (c) Step 1: $\sqrt{x} + \sqrt{y} = 1$

Differentiating both side w.r.t. x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}} \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\left(\frac{1}{4},\frac{1}{4}\right)} = -1$$

NOTE

By using formula $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

8. (b) Step 1: $f(x) = 3e^{x^2}$

$$f'(x) = 3e^{x^2}\frac{d}{dx}(x^2) = 3e^{x^2}2x = 6xe^{x^2}$$

$$f(0) = 3e^0 = 3$$

$$f'(0) = 0$$

from equation (1), put the values

$$6xe^{x^2} - 6xe^{x^2} + \frac{1}{3}[3-0] = 1$$

NOTE

By using formula $\frac{d}{dx}(e^x) = e^x$

9. (c) Step 1: $y = \log_{\sin x}(\tan x)$

$$y = \log_{\sin x}\left(\frac{\sin x}{\cos x}\right)$$

$$y = \log_{\sin x}\sin x - \log_{\sin x}\cos x$$

$$y = 1 - \log_{\sin x}\cos x = 1 - \frac{\log\cos x}{\log\sin x}$$

$$\left(\because \log_b a = \frac{\log_e a}{\log_e b}\right)$$

Now, differentiating both side w.r.t. x

$$\frac{dy}{dx} = -\left[\frac{\log\sin x\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\log\cos x}{(\log\sin x)^2}\right]$$

$$\left.\frac{dy}{dx}\right|_{x=\pi/4} = -\frac{\left[\frac{1}{2}\log 2 + \frac{1}{2}\log 2\right]}{\left(-\frac{1}{2}\log 2\right)^2}$$

$$= -\left[\frac{4\log 2}{(\log 2)^2}\right] = -\frac{4}{\log 2}$$

NOTE

By using formula $\frac{d}{dx}\left(\frac{U}{V}\right) = \frac{V\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

10. (a) Step 1: $\frac{d}{dx}\{2^{\log x}\}$

$$\frac{d}{dx}\{2^{\log x}\} = 2^{\log x}\log 2\frac{d}{dx}(\log x) = \frac{2^{\log x}\log 2}{x}$$

NOTE

By using formula $\frac{d}{dx}(a^x) = a^x\log a$ and

$\frac{d}{dx}\log x = \frac{1}{x}$

11. (a) Step 1: $\frac{d}{dx}\{e^x\log(1+x^2)\}$

$$\frac{d}{dx}\{e^x\log(1+x^2)\} = e^x\log(1+x^2) + \frac{2x}{1+x^2}e^x$$

$$= e^x\left[\log(1+x^2) + \frac{2x}{1+x^2}\right]$$

NOTE

By using product rule formula

$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$

12. (a) Step 1: $\frac{d}{dx}e^{x\sin x} = e^{x\sin x}\frac{d}{dx}[x\sin x]$

$= e^{x\sin x}[x\cos x + \sin x]$

NOTE

By using formula $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$ (Product rule)

13. (b) Step 1: $y = f\left(\frac{5x+1}{10x^2-3}\right)$

and $f'(t) = \cos t$

$$t = \frac{5x+1}{10x^2-3}(given)$$

Step 2: $f'(x) = \cos\left(\frac{5x+1}{10x^2-3}\right)\frac{d}{dx}\left(\frac{5x+1}{10x^2-3}\right)$

14. (c) Step 1: $f(x) = \log_{x^3}(\log x)^{27}$

$$f(x) = \frac{27}{3}\log_x(\log x)$$

$$f(x) = 9\frac{\log_e(\log x)}{\log_e x}$$

$$f'(x)\Big|_{x=e} = \frac{9\left[\log_e x\left(\frac{1}{\log x}\cdot\frac{1}{x}\right) - \log_e(\log_e x)\frac{1}{x}\right]}{(\log_e^x)^2}$$

$$= 9\left[\frac{\frac{1}{e}-0}{1}\right] = 9/e$$

NOTE

By using formula $\log_{a^k} b = \frac{l}{k}\log_a b$

Quotient rule and $\frac{d}{dx}\left(\frac{u}{v}\right) = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right)/v^2$

15. (d) Step 1: $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$

$$y = \sqrt{x+y}$$

Squaring both side, we get,

$$y^2 = x + y \qquad ....(1)$$

Step 2: Differentiating the equation (1) w.r.t. x

$$2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y-1) = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

16. Step-1 :

Let $u = \sqrt{x^2+16}$, $v = \frac{x}{x-1} = 1 + \frac{1}{x-1}$

Then a function u with respect to another function v from the formula of differential co-efficient :

$$\frac{du}{dv}\Big|_{x=3} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}\Bigg|_{x=3} = \frac{\frac{2x}{2\sqrt{x^2+16}}}{-\frac{1}{(x-1)^2}}\Bigg|_{x=3} = \frac{\frac{3}{5}}{-\frac{1}{4}} = \frac{-12}{5}$$

17. (d) Step 1: Differentiate by formula

$$\frac{d}{dx}(a^x) = a^x \log a$$

$$2^x\log 2 + 2^y\log 2\frac{dy}{dx} = 2^{x+y}\log 2\left(1 + \frac{dy}{dx}\right)$$

By cancelling log2 and adjusting the terms, we get

$$\frac{dy}{dx} = \frac{2^{x-y}(2^y-1)}{1-2^x}$$

18. Step 1:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(\cos\theta - \cos\theta + \theta\sin\theta)}{a(-\sin\theta + \sin\theta + \theta\cos\theta)} = \tan\theta$$

19. (d) Step 1: $y = \sqrt{\sqrt{x}+1}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\sqrt{x}+1}}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\sqrt{x}+1}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{\sqrt{x}+1}}$$

$$= \frac{1}{4\sqrt{x(\sqrt{x}+1)}}$$

NOTE

By using formula $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

20. (a) Step 1: Given $f(x,y) = x^3 + y^3 - 3axy$

Also $\frac{dy}{dx} = -\frac{\left.\frac{df(x,y)}{dx}\right|_{y=\text{constant}}}{\left.\frac{df(x,y)}{dy}\right|_{x=\text{constant}}} = -\frac{f_x}{f_y}$

Here f_x = partial differential coefficient of $f(x, y)$ with respect to x.

and f_y = partial differential coefficient of $f(x, y)$ with respect to y.

Step 2: $\frac{dy}{dx} = -\left(\frac{3x^2 + 0 - 3ay}{0 + 3y^2 - 3ax}\right) = \frac{ay - x^2}{y^2 - ax}$

21. (d) Step 1: $f(x,y) = \sin(x+y) - \log(x+y)$

$$f_x = \left.\frac{df(x,y)}{dx}\right|_{y=\text{constant}} = \cos(x+y) - \frac{1}{x+y}$$

$$f_y = \left.\frac{df(x,y)}{dy}\right|_{x=\text{constant}} = \cos(x+y) - \frac{1}{x+y}$$

Step 2:

$$\frac{dy}{dx} = -\frac{fx}{fy} = -\left\{\frac{\cos(x+y) - \frac{1}{x+y}}{\cos(x+y) - \frac{1}{x+y}}\right\} = -1$$

22. (b) Step 1:

$$\log_{\cot x} \tan x = \log_{(\tan x)^{-1}} \tan x$$

$$= \frac{1}{-1}\log_{\tan x} \tan x = -1$$

Therefore $f(x) = (1)(1)^{-1} = 1$

Step 2: $\frac{df(x)}{dx} = \frac{d}{dx}(1) = 0$

clearly $f'(x)$ is zero for all x

$\therefore \quad f'(2) = 0$

23. (c) Step 1: Given $x^m y^n = 2(x+y)^{m+n}$

Taking log of both sides, we get $m \log x + n \log y = \log 2 + (m+n) \log (x+y)$

Step 2: Differentiating both sides we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = 0 + \frac{m+n}{x+y}\left\{1 + \frac{dy}{dx}\right\}$$

This is true for all m and n therefore

Let us take $m = 1, n = 0$ to make easy calculation as follows:

$$\frac{1}{x} = \frac{1}{x+y}\left\{1 + \frac{dy}{dx}\right\}$$

$$\frac{x+y}{x} = 1 + \frac{dy}{dx}$$

or $\frac{dy}{dx} = \frac{x+y}{x} - 1 = \frac{y}{x}$

24. (d) Step 1: $y = F[f\{\phi(x)\}]$ can be differentiated by chain rule as follows:

$y = F(t),\ t = f(u),\ u = \phi(x)$

Step 2: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$

$= F'(t) \times f'(u) \times \phi'(x)$

$= F'(f(\phi(x)))f'(\phi(x)) \times \phi'(x)$

25. (b) Step 1: $y = \log\left(1+\sqrt{x}\right) - \log\left(1-\sqrt{x}\right)$

$$\frac{dy}{dx} = \frac{1}{1+\sqrt{x}} \times \frac{1}{2\sqrt{x}} - \frac{1}{1-\sqrt{x}} \times \left(\frac{-1}{2\sqrt{x}}\right)$$

$$= \frac{\left(1-\sqrt{x}\right) + \left(1+\sqrt{x}\right)}{2\left(1+\sqrt{x}\right)\left(1-\sqrt{x}\right)\sqrt{x}}$$

$$= \frac{1}{\left(1+\sqrt{x}\right)\left(1-\sqrt{x}\right)\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}(1-x)}$$

26. (a) Step 1: If $y = u^v$ where u and v both are functions of x then $\frac{dy}{dx} = u^v\left\{\frac{d}{dx}(v \log u)\right\}$

Also $\frac{d}{dx}\{\log \tan x\} = \sec x \operatorname{cosec} x$

Step 2: $\frac{d}{dx}\{(\tan x)^{\cot x}\}$

$= (\tan x)^{(\cot x)}\left\{\frac{d}{dx}(\cot x \log \tan x)\right\}$

$= (\tan x)^{(\cot x)}\{-\text{cosec}^2 x \log \tan x + \cot x \times \sec x \text{ cosec } x\} = y \text{ cosec}^2 x\{-\log \tan x + 1\}$

27. (b) Step 1: On solving given equations at $x = 2$ and $y = -1$ we get $t = 2$ (common value)

Also $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ at $t = 2$

Step 2: $\frac{dy}{dx} = \left.\frac{4t-2}{2t+3}\right|_{t=2} = \frac{6}{7}$

28. (c) Step 1: Given equation may be written as $x = e^{y+x}$

or $\quad y + x = \log_e x$

Step 2: Differentiating with respect to x we get

$$\frac{dy}{dx} + 1 = \frac{1}{x} \text{ or } \frac{dy}{dx} = \frac{1}{x} - 1 = \frac{1-x}{x}$$

29. (a) Step 1: According to Question $e^{x-y} = x^y$ log on both side, we get

$x - y = y \log x$

or $\quad y + y\log x = x$ or $y = \frac{x}{1+\log x}$

Step 2: From Quotient Rule

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

30. (a) Step 1: By base changing theorem $\log_b a = \frac{\log_e a}{\log_e b}$

Given $\phi(x) = \log_5\left(\frac{\log_e x}{\log_e 3}\right)$

$\phi(x) = \log_5(\log_e x) - \log_5(\log_e 3)$

or $\quad \phi(x) = \frac{\log_e(\log_e x)}{\log_e 5} - \text{constant}$

Step 2: $\phi'(x) = \frac{1}{\log_e x \times x \log_e 5} - 0$

$\therefore \phi'(e) = \frac{1}{1 \times e \times \log_e 5} = \frac{\log_5 e}{e}$

31. (a) Step 1: $f(x+y) = 2f(x)f(y)$ for all x and y

$f'(5) = 1024 \log 2$ and $f(2) = 8$

Step 2: $f(x+2) = 2f(x)f(2)$

On differentiating we get

$f'(x+2) = 2f'(x)f(2)$

put $x = 3$ in the above we get

$f'(5) = 2f'(3)f(2)$

$1024 \log 2 = 2f'(3) \times 8$

or $\quad 2^4 f'(3) = 2^{10} \log 2$

or $\quad f'(3) = 2^6 \log 2 = 64 \log 2$

32. (a) Step 1: $\frac{dy}{dx} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=\frac{\pi}{2}}$

$\frac{dy}{dt} = \sin t + t\cos t - \sin t = t\cos t$

$\frac{dx}{dt} = \cos t - \cos t + t\sin t = t\sin t$

Step 2: $\frac{dx}{dt} = \frac{t\cos t}{t\sin t} = \cot t$

$\therefore \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \cot 90° = 0$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. $\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$ equals to

[RPET-2001]

(a) 1

(b) $\frac{x^2+1}{x^2-4}$

(c) $\frac{x^2-1}{x^2-4}$

(d) $e^x\frac{x^2-1}{x^2-4}$

2. $\frac{d}{dx}[e^{ax}\cos(bx+c)] =$ **[AISSE-1989]**

(a) e^{ax} [acos $(bx + c) - b \sin (bx + c)$]
(b) e^{ax} [a sin $(bx + c) - b \cos (bx + c)$]
(c) e^{ax} [cos $(bx + c) - \sin(bx + c)$]
(d) None of these

3. If $y = \log.e^{(\tan x + x^2)}$, then $\frac{dy}{dx} =$ **[AICBSE-1985]**

(a) $e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x + x)\log x\right]$

(b) $e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x - x)\log x\right]$

(c) $e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x + 2x)\log x\right]$

(d) $e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x - 2x)\log x\right]$

4. $\frac{d}{dx}(x^2 + \cos x)^4 =$ **[DSSE-79]**

(a) $4(x^2 + \cos x)(2x - \sin x)$
(b) $4(x^2 - \cos x)^3(2x - \sin x)$
(c) $4(x^2 + \cos x)^3(2x - \sin x)$
(d) $4(x^2 + \cos x)^3(2x + \sin x)$

5. If $y = \sin\left(\frac{1+x^2}{1-x^2}\right)$, then $\frac{dy}{dx} =$

(a) $\frac{4x}{1-x^2}.\cos\left(\frac{1+x^2}{1-x^2}\right)$

(b) $\frac{x}{(1-x^2)^2}.\cos\left(\frac{1+x^2}{1-x^2}\right)$

(c) $\frac{x}{(1-x^2)}.\cos\left(\frac{1+x^2}{1-x^2}\right)$

(d) $\frac{4x}{(1-x^2)^2}.\cos\left(\frac{1+x^2}{1-x^2}\right)$

6. If $x = a\cos^4\theta, y = a\sin^4\theta$, then $\frac{dy}{dx}$ at $\theta = \frac{3\pi}{4}$ **[Kerala (Engg.)-2002]**

(a) -1 (b) 1
(c) $-a^2$ (d) a^2

7. If $f(x) = \frac{1}{\sqrt{x^2+a^2} + \sqrt{x^2+b^2}}$, then $f'(x)$ is equal to **[Kurukshetra CEE-1998]**

(a) $\frac{x}{(a^2-b^2)}\left[\frac{1}{\sqrt{x^2+a^2}} - \frac{1}{\sqrt{x^2+b^2}}\right]$

(b) $\frac{x}{(a^2+b^2)}\left[\frac{1}{\sqrt{x^2+a^2}} - \frac{2}{\sqrt{x^2+b^2}}\right]$

(c) $\frac{x}{(a^2-b^2)}\left[\frac{1}{\sqrt{x^2+a^2}} + \frac{1}{\sqrt{x^2+b^2}}\right]$

(d) $(a^2-b^2)\left[\frac{1}{\sqrt{x^2+a^2}} - \frac{2}{\sqrt{x^2+b^2}}\right]$

8. If $f(1) = 3, f'(1) = 2$, then $\frac{d}{dx}\{\log f(e^x + 2x)\}$ at $x = 0$ is **[AMU-1999]**

(a) 2/3 (b) 3/2
(c) 2 (d) 0

9. If $x^3 + 8xy + y^3 = 64$ then $\frac{dy}{dx} =$ **[AICBSE-1979]**

(a) $-\frac{3x^2+8y}{8x+4y^2}$ (b) $\frac{3x^2+8y}{8x+4y^2}$

(c) $\frac{3x+8y^2}{8x^2+3y}$ (d) None of these

10. $\frac{d}{dx}\{(\sin x)^{\log x}\} =$ **[DSSE-84]**

(a) $(\sin x)^{\log x}\left[\frac{1}{x}\log\sin x + \cot x\right]$

(b) $(\sin x)^{\log x}\left[\frac{1}{x}\log\sin x + \cot x\log x\right]$

(c) $(\sin x)^{\log x}\left[\frac{1}{x}\log\sin x + \log x\right]$

(d) None of these

11. If $y = x^{(x^x)}$, then $\frac{dy}{dx} =$ **[AISSE-1989]**

(a) $y[x^x(\log ex).\log x + x^x]$
(b) $y[x^x(\log ex).\log x + x]$
(c) $y[x^x(\log ex).\log x + x^{x-1}]$
(d) $y[x^x(\log_e x).\log x + x^{x-1}]$

12. If $x^y = y^x$, then $\frac{dy}{dx} =$ **[DSSE-1996; MPPET-1997]**

(a) $\frac{y(x\log_e y + y)}{x(y\log_e x + x)}$ (b) $\frac{y(x\log_e y - y)}{x(y\log_e x - x)}$

(c) $\frac{x(x\log_e y - y)}{y(y\log_e x - x)}$ (d) $\frac{x(x\log_e y + y)}{y(y\log_e x + x)}$

13. Function $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \infty}}}$ satisfies any one of the following which is ***[MPPET-1998; Pb. CET-2001]***

(a) $(2y-1)\frac{dy}{dx} - \sin x = 0$

(b) $(2y-1)\cos x + \frac{dy}{dx} = 0$

(c) $(2y-1)\cos x - \frac{dy}{dx} = 0$

(d) $(2y-1)\frac{dy}{dx} - \cos x = 0$

14. If $y = (\sin x)^{(\sin x)^{(\sin x)\ldots\infty}}$

(a) $\frac{y^2 \cot x}{1 - y \log \sin x}$ (b) $\frac{y^2 \cot x}{1 + y \log \sin x}$

(c) $\frac{y \cot x}{1 - y \log \sin x}$ (d) $\frac{y \cot x}{1 + y \log \sin x}$

15. Differential coefficient of $y = \sin\sqrt{\cos x}$ ***[MP PET-2007]***

(a) $\frac{-\sin x \cos\sqrt{\cos x}}{2\sqrt{\cos x}}$ (b) $\frac{\sin x \cos x \sqrt{\cos x}}{2\sqrt{\cos x}}$

(c) $-\sin x \cos x$ (d) $\sin x \cos x$

16. If $2x^2 - 3xy + y^2 + x + 2y - 8 = 0$ then $\frac{dy}{dx} =$ ***[EAMCET-2007]***

(a) $\frac{3y - 4x - 1}{2y - 3x + 2}$ (b) $\frac{3y + 4x + 1}{2y + 3x + 2}$

(c) $\frac{3y - 4x + 1}{2y - 3x - 2}$ (d) $\frac{3y - 4x + 1}{2y + 3x + 2}$

17. $\frac{d}{dx}[\sin^{-1}(3x - 4x^3)] = ?\ \left(|x| < \frac{1}{2}\right)$ ***[Gujarat CET-2007]***

(a) $\frac{4}{\sqrt{1-x^2}}$ (b) $\frac{3}{\sqrt{x^2-1}}$

(c) $\frac{-3}{\sqrt{1-x^2}}$ (d) $\frac{3}{\sqrt{1-x^2}}$

18. If $y = \log_e\left(\sec(e^{x^2})\right)$ then $\frac{dy}{dx}$ is equal to

(a) $x^2.e^{x^2}.\tan\left(e^{x^2}\right)$

(b) $e^{x^2}.\tan e^{x^2}$

(c) $2x.\left(\tan e^{x^2}\right).e^{x^2}$

(d) $2x.\left(e^{x^2}.\sec e^{x^2}\right)\tan e^{x^2}$

19. If $x^x = y^y$, then $\frac{dy}{dx}$ is ***[Karnataka CET-2007]***

(a) $-\frac{y}{x}$ (b) $-\frac{x}{y}$

(c) $1 + \log\left(\frac{x}{y}\right)$ (d) $\frac{1 + \log x}{1 + \log y}$

20. If $f(x) = \sin x$, the derivative of $f(\log x)$ with respect to x is ***[Kerala PET-2008]***

(a) $\cos x$ (b) $f'(\log x)$

(c) $\cos(\log x)$ (d) $\frac{\cos(\log x)}{x}$

(e) $\frac{1}{x}$

21. If $y = \frac{1}{\log_{10} x}$, then what is $\frac{dy}{dx}$ equal to? ***[NDA-2008]***

(a) x

(b) $x \log_e 10$

(c) $-\frac{(\log_x 10)^2(\log_{10} e)}{x}$

(d) $x \log_{10} e$

22. If $f(x) = e^{\sin(\log \cos x)}$ and $g(x) = \log(\cos x)$, then what is the derivative of $f(x)$ with respect to $g(x)$? ***[NDA-2008]***

(a) $f(x)\cos(g(x))$

(b) $f(x)\sin(g(x))$

(c) $g(x)\cos(f(x))$

(d) $g(x)\sin(f(x))$

23. If $y = \sqrt{\frac{1-x}{1+x}}$ then $(1-x^2)\frac{dy}{dx} + y$ is equal to ***[MP PET-2008]***

(a) 1 (b) –1

(c) 2 (d) 0

24. If $x = a(t\cos t - \sin t)$; $y = a(t\sin t + \cos t)$

Then $\frac{dy}{dx}$ is equal to ***[MP PET-2008]***

(a) $-\tan t$ (b) $-\cot t$

(c) $\tan t$ (d) $\cot t$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 16 minutes.
3. The worksheet consists of 16 questions. The maximum marks are 48.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If $y = 3^{x^2}$, then $\frac{dy}{dx}$ is equal to

(a) $(x^2)3^{x^2-1}$ (b) $3^{x^2}.2x$

(c) $3^{x^2}.2x.\log 3$ (d) $(x^2 - 1).3$

2. Differential coefficient of $\sqrt{\sec\sqrt{x}}$ is *[MPPET-1996]*

(a) $\frac{1}{4\sqrt{x}}(\sec\sqrt{x})^{3/2}\sin\sqrt{x}$

(b) $\frac{1}{4\sqrt{x}}\sec\sqrt{x}\sin\sqrt{x}$

(c) $\frac{1}{2}\sqrt{x}(\sec\sqrt{x})^{3/2}\sin\sqrt{x}$

(d) $\frac{1}{2}\sqrt{x}\sec\sqrt{x}\sin\sqrt{x}$

3. $\frac{d}{dx}\left[\log\left(x+\frac{1}{x}\right)\right] =$ *[MPPET-1995]*

(a) $\left(x+\frac{1}{x}\right)$ (b) $\frac{\left(1+\frac{1}{x^2}\right)}{\left(1+\frac{1}{x}\right)}$

(c) $\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)}$ (d) $\left(1+\frac{1}{x}\right)$

4. $\frac{d}{dx}\left(e^{x^3}\right)$ is equal to *[MPPET-1995]*

(a) $3xe^{x^3}$ (b) $3x^2e^{x^3}$

(c) $3x^2\left(e^{x^3}\right)^2$ (d) $2x^2e^{x^3}$

5. At $x = \sqrt{\frac{\pi}{2}}, \frac{d}{dx}\cos(\sin x^2) =$

(a) –1 (b) 1

(c) 0 (d) None of these

6. If $y = \sec x^0$, then $\frac{dy}{dx} =$ *[MPPET-1997]*

(a) $\sec x \tan x$ (b) $\sec x^0 \tan x^0$

(c) $\frac{\pi}{180}\sec x^0 \tan x^0$ (d) $\frac{180}{\pi}\sec x^0 \tan x^0$

7. If $y = \sqrt{\frac{1+e^x}{1-e^x}}$, then $\frac{dy}{dx} =$

(a) $\frac{e^x}{(1-e^x)\sqrt{1+e^{2x}}}$ (b) $\frac{e^x}{(1-e^x)\sqrt{1-e^x}}$

(c) $\frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$ (d) $\frac{e^x}{(1-e^x)\sqrt{1+e^x}}$

8. If $x = a\cos^3\theta, y = a\sin^3\theta$, then $\sqrt{1+\left(\frac{dy}{dx}\right)^2} =$ *[EAMCET-1992]*

(a) $\tan^2\theta$ (b) $\sec^2\theta$

(c) $\sec\theta$ (d) $|\sec\theta|$

9. If $x = a\left(\cot + \log\tan\frac{t}{2}\right)$ and $y = a\sin t$, then $\frac{dy}{dx} =$ *[RPET-1997; MPPET-2001]*

(a) $\tan t$ (b) $-\tan t$

(c) $\cot t$ (d) $-\cot t$

10. The derivative of $\sin^2 x$ with respect to $\cos^2 x$ is *[DCE-2002]*

(a) $\tan^2 x$ (b) $\tan x$

(c) $-\tan x$ (d) None of these

11. If $y = \log_e\left(x+\sqrt{x^2-a^2}\right)$, then $\frac{dy}{dx} =$

(a) $\frac{1}{x+\sqrt{x^2-a^2}}$ (b) $\frac{1}{\sqrt{x^2-a^2}}$

(c) $x-\sqrt{x^2-a^2}$ (d) Undefined

12. If $t = e^x$ and $y = t^2 - 1$, then $\frac{dx}{dy}$ at $t = 1$

(a) $1/2e^2$ (b) $1/2$
(c) 2 (d) $2e^2$

13. If $x = a\,(t - \sin t)$ and $y = a(1 - \cos t)$, then $\frac{dy}{dx} =$

[AISSE-84; Roorkee-74; SCRA-96; Karnataka CET-03]

(a) $\tan(t/2)$
(b) $-\tan(t/2)$
(c) $\cot(t/2)$
(d) $-\cot(t/2)$

14. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots\infty}}}$, then $\frac{dy}{dx} =$

(a) $\frac{x}{2y-1}$ (b) $\frac{x}{2y+1}$
(c) $\frac{1}{x(2y-1)}$ (d) $\frac{1}{x(1-2y)}$

15. If $y = \log\log x$, then $e^y\frac{dy}{dx} =$

[MPPET-1994, 95]

(a) $\frac{1}{x\log x}$ (b) $\frac{1}{x}$
(c) $\frac{1}{\log x}$ (d) e^y

16. Differential coefficient $\log_e\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$ of with respect to x is ***[MPPET-1993]***

(a) $\operatorname{cosec} x$ (b) $\tan x$
(c) $\cos x$ (d) $\sec x$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

2. Step 1: From Chain rule

$$\frac{d}{dx}\left(\sqrt{\sec\sqrt{x}}\right) = \frac{1}{2\sqrt{\sec\sqrt{x}}} \times \sec\sqrt{x}\tan\sqrt{x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{(\sec\sqrt{x})^{3/2}\sin\sqrt{x}}{4\sqrt{x}}$$

3. Step 1: From formula $\frac{d}{dx}\{\log(y)\} = \frac{1}{y}\frac{dy}{dx}$

$$= \frac{1}{x + \frac{1}{x}}\left\{1 - \frac{1}{x^2}\right\}$$

6. Step 1: Before differentiation, the degree is to be converted into radian.

Hence $$y = \sec\left(\frac{\pi}{180}x\right)$$

and $$\frac{dy}{dx} = \left(\frac{\pi}{180}\right)\sec\left(\frac{\pi}{180}x\right)\tan\left(\frac{\pi}{180}x\right)$$

$$= \frac{\pi}{180}\sec x^\circ \tan x^\circ$$

8. Step 1:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta(-\sin\theta)} = -\tan\theta$$

Step 2:

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta|$$

15. Step 1: From formula $e^{\log_e N} = N$

Step 2: $e^y = e^{\log(\log x)} = \log x$

Step 3: Differentiate with respect x,

$$e^y\frac{dy}{dx} = \frac{1}{x}$$

NOTE

Remember forever

$$e^{\log_e N} = N \Rightarrow e^{2\log_e N} = N^2, e^{-\log_e N} = \frac{1}{N}$$

16. Step 1:

$$\sqrt{\frac{1+\sin x}{1-\sin x}} = \frac{1+\sin x}{\sqrt{1-\sin^2 x}} = \sec x + \tan x$$

Step 2: $\frac{d}{dx}[\log(\sec x + \tan x)] = \sec x$

By given formula

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LECTURE

3

Differentiation of Inverse Function

BASIC CONCEPTS

1. $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}$

2. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \dfrac{x+y}{1-xy}$

3. $2\tan^{-1} x = \tan^{-1} \dfrac{2x}{1-x^2} = \cos^{-1} \dfrac{1-x^2}{1+x^2}$
$= \sin^{-1} \dfrac{2x}{1+x^2}$

4. $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

5. $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$

6. $\sin^{-1} x + \cos^{-1} x = \tan^{-1} x + \cot^{-1} x = \sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$,

7. $\sin^{-1} \dfrac{1-x^2}{1+x^2} = \dfrac{\pi}{2} - \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$,
$\tan^{-1} \dfrac{1-x^2}{2x} = \dfrac{\pi}{2} - 2\tan^{-1}(x)$

8. $\sin^{-1} \cos x = \sin^{-1} \sin\left(\dfrac{1}{2}\pi - x\right) = \dfrac{1}{2}\pi - x$

9. $\tan^{-1}(\tan\theta) = \theta$, $\sin^{-1}(\sin\theta) = \theta$, $\cos^{-1}(\cos\theta) = \theta$

10. $\sin^{-1} \dfrac{x}{1} = \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \dfrac{x}{\sqrt{1-x^2}}$
$= \cot^{-1} \dfrac{\sqrt{1-x^2}}{x} = \sec^{-1} \dfrac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1} \dfrac{1}{x}$

11. $\tan^{-1} \dfrac{x}{1} = \sin^{-1} \dfrac{x}{\sqrt{1+x^2}} = \cos^{-1} \dfrac{1}{\sqrt{1+x^2}}$
$= \cot^{-1} \dfrac{1}{x} = \sec^{-1} \dfrac{\sqrt{1+x^2}}{1} = \operatorname{cosec}^{-1} \dfrac{\sqrt{1+x^2}}{x}$

12. $\dfrac{1}{2}\tan^{-1} x = \tan^{-1}\left\{\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)\right\}$;
$\tan^{-1} \dfrac{x-y}{1+xy} = \tan^{-1} \dfrac{\frac{1}{y} - \frac{1}{x}}{1+(1/y)(1/x)}$

13. $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$, $2\sin^{-1} x = \cos^{-1}(1 - 2x^2) = \sin^{-1} 2x\sqrt{1-x^2}$

14. $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $\sin^{-1}$
$\dfrac{1-x}{1+x} = \dfrac{\pi}{2} - 2\tan^{-1}\sqrt{x}$
$3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$, $3\tan^{-1} x = \tan^{-1} \dfrac{3x - x^3}{1 - 3x^2}$

15. $4\cos^{-1} x = \cos^{-1}(8x^4 - 8x^2 + 1)$

16. $5\cos^{-1} x = \cos^{-1}(16x^5 - 20x^3 + 5x)$

17. $5\sin^{-1} x = \sin^{-1}(16x^5 - 20x^3 + 5x)$

18. $\sin 2\tan^{-1} x = \dfrac{2x}{1+x^2}, \cos 2\tan^{-1} x$

$= \dfrac{1-x^2}{1+x^2}, \tan 2\tan^{-1} x = \dfrac{2x}{1-x^2}$

19. $\cos 2\cos^{-1} x = 2x^2 - 1$, $\cos 2\sin^{-1} x = 1 - 2x^2$,
$\cos 3\cos^{-1} x = 4x^3 - 3x$

20. $\sin^{-1}(-x) = -\sin^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$,
$\tan^{-1}(-x) = -\tan^{-1} x$

21. $\tan^{-1}\left\{\dfrac{a\cos x + b\sin x}{b\cos x - a\sin x}\right\}$

$= \tan^{-1}\left\{\dfrac{\frac{a}{b} + \tan x}{1 - (a/b)(\tan x)}\right\} = \tan^{-1}\dfrac{a}{b} + x$

22. $\tan^{-1}\left\{\dfrac{4x}{1+5x^2}\right\} = \tan^{-1} 5x - \tan^{-1} x$

23. $\tan^{-1}\left\{\dfrac{1}{1-x+x^2}\right\}$

$= \tan^{-1}\left\{\dfrac{x-(x-1)}{1+x(x-1)}\right\} = \tan^{-1} x - \tan^{-1}(x-1)$

24. $\tan^{-1}\left[\dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right] = \dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}(x^2)$,

$\tan^{-1}\dfrac{a+bx}{b-ax} = \tan^{-1}\dfrac{a}{b} + \tan^{-1} x$

25. $\tan^{-1}\left\{\dfrac{\log ex^2}{\log(e/x^2)}\right\} = \tan^{-1}\left\{\dfrac{1+2\log x}{1-2\log x}\right\}$

$= \tan^{-1} 1 + \tan^{-1}(2\log x)$

$\tan^{-1}\sqrt{\dfrac{1-x}{1+x}} = \dfrac{1}{2}\cos^{-1} x$

26. $\tan^{-1}\left\{\dfrac{\frac{3x}{a} + \frac{2x}{a}}{1 - \left(\frac{3x}{a}\right)\frac{2x}{1}}\right\} = \tan^{-1}\left\{\dfrac{5ax}{a^2 - 6x^2}\right\}$;

$\tan^{-1}\dfrac{x}{1+\sqrt{1-x^2}} = \dfrac{1}{2}\sin^{-1} x$

27. $\sin^{-1} x + \sin^{-1}\sqrt{x} = \sin^{-1}\left\{x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right\}$

$= \sin^{-1}\left\{x\sqrt{1-x} + \sqrt{x-x^3}\right\}$

SOME IMPORTANT DIFFERENTIATIONS

1. $\log\left[\sqrt{x^2+1} - x\right] = \dfrac{-1}{\sqrt{x^2+1}}$

2. $f'(x) = kf(x)$, then $f(x) = e^{kx}$

3. $\dfrac{d}{dx}\left(vers^{-1}\left(\dfrac{x}{a}\right)\right) = \dfrac{1}{\sqrt{2ax - x^2}}$

$(1 - \cos\theta = vers\theta$ and $1 - \sin\theta = \text{covers}\ \theta)$

4. $\dfrac{d}{dx}\left(\dfrac{1}{2a}\log\dfrac{a+x}{a-x}\right) = \dfrac{1}{a^2 - x^2}$

5. $\dfrac{d}{dx}\left(\dfrac{1}{2a}\log\dfrac{x-a}{x+a}\right) = \dfrac{1}{x^2 - a^2}$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Differentiate $\sin^{-1}\dfrac{1-x}{1+x}$ with respect to $\sqrt{x}$

Solution

$\sqrt{x} = \tan\theta$

$\therefore x = \tan^2\theta$

$y = \sin^{-1}\cos 2\theta$

$= \dfrac{\pi}{2} - 2\theta = \dfrac{\pi}{2} - 2\tan^{-1}\sqrt{x}$

$y = \dfrac{\pi}{2} - 2\tan^{-1} z$, where $z = \sqrt{x}$;

$\dfrac{dy}{dz} = -\dfrac{2}{1+z^2} = -\dfrac{2}{1+x}$

2. Differentiate $x^{\sin^{-1}x}$ with respect to $\sin^{-1}x$.

Solution

Put $\sin^{-1}x = t \therefore x = \sin t, \sqrt{(1-x^2)} = \cos t$ .... (i)

$\therefore \quad y = (\sin t)^t$ and $z = t$,

we have to find $\dfrac{dy}{dz}$ or $\dfrac{dy}{dt}$

$\log y = t \log \sin t$. Differentiate with respect to t

$$\frac{1}{y}\frac{dy}{dt} = \log\sin t + t.\frac{\cos t}{\sin t}$$

$$\therefore \quad \frac{dy}{dt} = x^{\sin^{-1}x}\left[\log x + \sin^{-1}x\frac{\sqrt{(1-x^2)}}{x}\right]$$

3. If $y = \cos^{-1}\left(\dfrac{2\cos x + 3\sin x}{\sqrt{(13)}}\right)$, then $\dfrac{dy}{dx} = 1$

Solution

Put $2 = r\cos\alpha$, $3 = r\sin\alpha$

$\therefore \quad r = \sqrt{13}, \tan\alpha = 3/2$

$$\therefore \quad y = \cos^{-1}\frac{(\cos x\cos\alpha + \sin x\sin\alpha)}{\sqrt{(13)}}$$

$= \cos^{-1}\cos(x-\alpha) = x - \alpha$

or $\quad y = x - \tan^{-1}\dfrac{3}{2}, \therefore \dfrac{dy}{dx} = 1.$

4. If $y = \dfrac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left[\left\{\sqrt{\left(\dfrac{a-b}{a+b}\right)}\right\}\tan\dfrac{x}{2}\right]$

then show that $\dfrac{dy}{dx} = \dfrac{1}{a+b\cos x}$ and $\dfrac{d^2y}{dx^2} = \dfrac{b\sin x}{(a+b\cos x)^2}$

Solution

$$y = \frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\frac{x}{2}\right\}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{\sqrt{(a^2-b^2)}}.\frac{1}{\left(1+\dfrac{a-b}{a+b}.\dfrac{\sin^2(x/2)}{\cos^2(x/2)}\right)}\sqrt{\left(\frac{a-b}{a+b}\right)}.\sec^2\frac{x}{2}.\frac{1}{2}$$

$$= \left(\frac{1}{a+b}\right)\left(\frac{(a+b)\cos^2(x/2)\sec^2(x/2)}{a\{\cos^2(x/2)+\sin^2(x/2)\} + b\{\cos^2(x/2)-\sin^2(x/2)\}}\right)$$

$$= \frac{1}{a+b\cos x}$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = -\frac{1}{(a+b\cos x)^2}.(-b\sin x)$$

$$= \frac{b\sin x}{(a+b\cos x)^2}$$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. $(3-2x)\sin^{-1}2x$ ***[AICBSE-88]***

2. If $y = \sin^{-1}(x/a)$ then find $\dfrac{dy}{dx}$.

3. If $y = \cos^{-1}\left\{\dfrac{1-x^{2n}}{1+x^{2n}}\right\}$ then find $\dfrac{dy}{dx}$

4. If $y = \tan^{-1}\left(\dfrac{5x}{1-6x^2}\right)$ then find $\dfrac{dy}{dx}$.

5. Differentiate with respect to x

$$\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$$

[MNR-1984; PSB-93, 95C, 2002]

6. If $y = \tan^{-1}\left(\dfrac{x}{\sqrt{a^2 - x^2}}\right)$ then find $\dfrac{dy}{dx}$

[HSB-93]

7. If $y = \cos^{-1}\sqrt{\dfrac{1+x^2}{2}}$, then find $\dfrac{dy}{dx}$.

[CBSE-88]

8. Differentiate $\cos^{-1}\left(\dfrac{5\cos x - 12\cos x}{13}\right)$ with respect to x.

[CBSE (SP)-2006]

9. Differentiate with respect to x . $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

[MP-2003, 2004]

10. If $y = \tan^{-1}\sqrt{\dfrac{1+\cos x}{1-\cos x}}$, find $\dfrac{dy}{dx}$. ***[CBSE-92]***

11. If $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ then find dy/dx

[MP-95, 2002]

12. If $y = \tan^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$ then find $\dfrac{dy}{dx}$.

[MP-2000]

13. If $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ then find $\dfrac{dy}{dx}$.

14. If $y = \tan^{-1}\left(\dfrac{1+\sin x}{\cos x}\right)$ then find $\dfrac{dy}{dx}$.

15. If $y = \sin^{-1}\left(\dfrac{\sqrt{1+x} - \sqrt{1-x}}{2}\right)$ then find $\dfrac{dy}{dx}$.

[MP-2007]

16. If $y = \cos^{-1}\left(\dfrac{x - x^{-1}}{x + x^{-1}}\right)$ then find $\dfrac{dy}{dx}$.

17. Find the differential coefficient of $\sin^{-1} x$ with respect to $\cos^{-1}\sqrt{1-x^2}$

[MP-2001]

18. Find the differential coefficient of $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$ at $x = 1/2$.

[MP-2001]

19. Differentiate $y = \tan^{-1}\dfrac{x}{1+\sqrt{1-x^2}}$ with respect to x.

[MP-2007]

20. Find the differential coefficient of $y = \sin^{-1}\dfrac{x}{9}$ with respect to x.

[MP-2007]

ANSWERS

1. $\dfrac{2(3-2x)}{\sqrt{1-4x^2}} - 2\sin^{-1}2x$

2. $\dfrac{1}{\sqrt{a^2-x^2}}$

3. $\dfrac{2nx^{n-1}}{1+x^{2n}}$

4. $\dfrac{3}{1+9x^2} + \dfrac{2}{1+4x^2}$

5. 0

6. $\dfrac{1}{\sqrt{a^2-x^2}}$

7. $\dfrac{-x}{\sqrt{1-x^4}}$

8. 1

9. $\dfrac{2}{1+x^2}$

10. $-1/2$

11. $\dfrac{2}{\sqrt{1-x^2}}$

12. $\dfrac{-1}{x\sqrt{x^2-1}}$

13. $\dfrac{1}{1+x^2}$

14. $\dfrac{1}{2}$

15. $\dfrac{1}{2\sqrt{1-x^2}}$

16. $\dfrac{-2}{1+x^2}$

17. 1

18. $\dfrac{2}{x}$ and 4

19. $\dfrac{1}{2\sqrt{1-x^2}}$

20. $\dfrac{1}{\sqrt{81-x^2}}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. If $y = \cos^{-1}\cos(|x| - f(x))$, where

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}, \text{ then } \left.\frac{dy}{dx}\right|_{x=\frac{5\pi}{4}} \text{ is}$$

[J&K-2005]

(a) -1 (b) 1
(c) 0 (d) Indeterminate

Solution

(b) $y = \cos^{-1}\cos(x-1),\ x > 0$

$\Rightarrow\ y = x - 1,\ x > 0$ and $0 \le x - 1 \le \pi$

$\therefore\ y = x - 1,\ 1 \le x \le \pi + 1$

we have, $1 < \frac{5\pi}{4} < \pi + 1$

$\therefore y = x - 1,\ 1 \le x \le \pi + 1$ and $\frac{5\pi}{4} \in [1, \pi + 1]$

$$\left.\frac{dy}{dx}\right|_{x=\frac{5\pi}{4}} = 1\ \Big|_{x=\frac{5\pi}{4}} = 1$$

2. $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\sqrt{x}(3-x)}{1-3x}\right)\right] =$

[Kerala (Engg.) 2005]

(a) $\frac{1}{2(1+x)\sqrt{x}}$ (b) $\frac{3}{(1+x)\sqrt{x}}$

(c) $\frac{2}{(1+x)\sqrt{x}}$ (d) $\frac{3}{2(1+x)\sqrt{x}}$

Solution

(d) $\frac{d}{dx}\left(\tan^{-1}\frac{(\sqrt{x}(3-x)}{1-3x}\right)$

Put $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$

$$\frac{d}{dx}\left(\tan^{-1}\frac{(\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta}\right)$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{(3\tan\theta-\tan^3\theta)}{1-3\tan^2\theta}\right)$$

$$\frac{d}{dx}\left(\tan^{-1}(\tan 3\theta) = \frac{d}{dx}(3\theta)\right)$$

$$\frac{d}{dx}\left(3.\tan^{-1}\sqrt{x}\right) = \frac{3}{2\sqrt{x}(1+x)}$$

3. $y = \log\left\{\left(\frac{1+x}{1-x}\right)^{1/4}\right\} - \frac{1}{2}\tan^{-1}(x)$, then $\frac{dy}{dx} =$

[EAMCET-2007]

(a) $\frac{x}{1-x^2}$ (b) $\frac{x^2}{1-x^4}$

(c) $\frac{x}{1+x^4}$ (d) $\frac{x}{1-x^4}$

Solution

(b) $y = \log\left(\frac{1+x}{1-x}\right)^{1/4} - \frac{1}{2}\tan^{-1}x$

$= \frac{1}{2}\tan h^{-1}x - \frac{1}{2}\tan^{-1}x$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1-x^2} - \frac{1}{1+x^2}\right) = \frac{x^2}{1-x^4}$$

4. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is: ***[IIT-2009]***

(a) 0 (b) 1
(c) 2 (d) 3

Solution

(c) $f(0) = 1, f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$

$\because f^{-1}(x) = g(x) \Rightarrow f(g(x)) = x$

$\Rightarrow f'(g(x))\, g'(x) = 1$

Put $x = 0 \Rightarrow g'(1) = \frac{1}{f'(0)} = 2.$

5. If $x^m y^n = (x+y)^{m+n}$, then $\frac{dy}{dx} =$

[MIPPET-2009]

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$

(c) $\frac{x+y}{xy}$ (d) $\frac{xy}{x+y}$

Solution

(b) Given, $x^m y^m = (x+y)^{m+n}$

Taking log on both sides, we get

$m\log x + n\log y = (m+n)\log(x+y)$

$$\Rightarrow\ \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = (m+n).\frac{1}{x+y}.\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow\ \frac{dy}{dx}\left(\frac{n}{y} - \frac{(m+n)}{x+y}\right) = \frac{m+n}{(x+y)} - \frac{m}{x}.$$

$$\Rightarrow\ \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right) = \frac{nx-my}{x(x+y)} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

6. If $f(x) = \sin^2 x^2$, then what is $f'(x)$ equal to?

[NDA-2009]

(a) $4x\sin(x^2)\cos(x^2)$ (b) $2\sin(x^2)\cos(x^2)$
(c) $4\sin(x^2)\sin^2 x$ (d) $2x\cos^2(x^2)$

Solution

() $\because f(x) = \sin^2 x^2$

$\therefore f'(x) = 2\sin x^2 \cos x^2 . 2x = 4x \sin x^2 \cos x^2$

7. If $f(x) = \cos x$, $g(x) = \ln x$ and $y = (gof)(x)$, then what is the value of $\frac{dy}{dx}$ at $x = 0$?

(a) 0 (b) 1
(c) –1 (d) 2

Solution

(a) $\because f(x) = \cos x, g(x) = \log x$

$\therefore \quad y = gof(x) = g\{f(x)\} = \log(\cos x)$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) = -\tan x$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{at\ x=0} = -\tan 0 = 0$

8. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals: ***[AIEEE-2009]***

(a) –1 (b) 1
(c) log 2 (d) –log 2

Solution

(a) $x^{2x} - 2x^y \cot y - 1 = 0$ ....(1)

Now $x = 1$, $1 - 2\cot y - 1 = 0 \Rightarrow \cot y - 0$

$\Rightarrow y = \frac{\pi}{2}$

Now differentiating eq. (1) w.r.t. 'x'

$2x^{2x}(1 + \log x)$

$$-2\left[x^y(-\cos ec^1 y)\frac{dy}{dx} + \cot x^y(1+\log x)\right] = 0.$$

Now at $\left(1, \frac{\pi}{2}\right)$

$$2(1+\log 1) - 2\left(1(-1)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} + 0\right) = 0$$

$$\Rightarrow 2 + 2\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = -1$$

9. If $x^y = y^x$, then $\frac{dy}{dx} =$ ***[MPPET-2009]***

(a) $\frac{y(x\log y - y)}{x(y\log x - x)}$ (b) $\frac{y(x\log y + y)}{x(y\log x + x)}$

(c) $\frac{y(y\log x - x)}{x(x\log y - y)}$ (d) $\frac{y(y\log x + x)}{x(x\log y + y)}$

Solution

(a) Given $x^y = y^x \Rightarrow y\log x = x\log y$

On differentiating w.r.t. x we get

$$\frac{y}{x} = \log x\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\Rightarrow \frac{dy}{dx}\left(\log x - \frac{x}{y}\right) = \log y - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x\log x - y)}{x(y\log x - x)}$$

10. If $y = \left[x + \sqrt{1+x^2}\right]^n$, then the value of $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is: ***[MPPET-2009]***

(a) n^2y (b) $-n^2y$
(c) ny (d) $-ny$

Solution

() Given, $y = [x + \sqrt{1+x^2}]^n$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = n[x+\sqrt{1+x^2}]^{n-1}\left[1 + \frac{2x}{2\sqrt{1+x^2}}\right]$$

$$= \frac{n[x+\sqrt{1+x^2}]^n}{\sqrt{1+x^2}}$$

$$= \frac{ny}{\sqrt{1+x^2}} \Rightarrow (1+x^2)\left(\frac{dy}{dx}\right)^2 = n^2y^2$$

Agaubm differentiaing w.r.t.x, we get

$$\Rightarrow \quad 2(1+x^2)\frac{dy}{dx}\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 2n^2y\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y$$

11. Let $f:(-1, 1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0) =$ ***[AIEEE-2010]***

(a) –4 (b) 0
(c) –2 (d) 4

Solution

Step-1: $g(x) = [f(2f(x)) + 2]^2$ with $f(0) = -1$, $f^2(0) = 1$ Differentiating both sides with respect to x by chain rule, we get as follows.

$t = f(2f(x))$, $t = f(u) + 2$, $u = 2f(x)$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dt} = 2t \times f'(u) \times 2f'(x)$$

$g'(x) = 2[f(2f(x) + 2)]f'(2f(x) + 2)2f'(x)$

$g'(0) = 2[f(2f(0) + 2)]f'(2f(0) + 2)2f'(0)$

$=2[f(-2 + 2]f'(2(-1) + 2))2f'(0) = 2 = 2f(0)$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. If $y = \sin^{-1}\sqrt{x}$, then $\frac{dy}{dx} =$ ***[MPPET-1995]***

(a) $\frac{2}{\sqrt{x}\sqrt{1-x}}$ (b) $\frac{-2}{\sqrt{x}\sqrt{1-x}}$

(c) $\frac{1}{2\sqrt{x}\sqrt{1-x}}$ (d) $\frac{1}{\sqrt{1-x}}$

2. The differential coefficient of $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$ is ***[MPPET-03]***

(a) $\sqrt{1-x^2}$ (b) $1/\sqrt{1-x^2}$

(c) $1/2\sqrt{1-x^2}$ (d) x

3. Differential coefficient of $\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1} x$, is ***[Kurukshetra CEE-2002]***

(a) 1/2 (b) 1

(c) 2 (d) 3/2

4. The derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is ***[Karnataka CET-2000; Pb. CET-2004]***

(a) –1 (b) 1

(c) 2 (d) 4

5. If $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, then $\frac{dy}{dx} =$ ***[MPPET-1999]***

(a) $-\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{x}{\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{1-x^2}}$ (d) $\frac{\sqrt{1-x^2}}{x}$

6. If $y = \cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$, then $\frac{dy}{dx} =$

(a) 1/2 (b) 2/3

(c) 3 (d) 1

7. Derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1+3x}$ at $x = -\frac{1}{3}$ ***[EAMCET-1991]***

(a) 0 (b) 1/2

(c) 1/3 (d) None of these

8. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, then $(1-x^2)\frac{dy}{dx}$ is ***[RPET-1995]***

(a) $x+y$ (b) $1+xy$

(c) $1-xy$ (d) $xy-2$

9. If $y = \frac{a^{\cos^{-1} x}}{1+a^{\cos^{-1} x}}, z = a^{\cos^{-1} x}$ then $\frac{dy}{dz} =$ ***[MPPET-1994]***

(a) $\frac{1}{1+a^{\cos^{-1} x}}$ (b) $-\frac{1}{1+a^{\cos^{-1} x}}$

(c) $\frac{1}{\left(1+a^{\cos^{-1} x}\right)^2}$ (d) None of these

10. If $y = \sin(2\sin^{-1}x)$ then $\frac{dy}{dx} =$ ***[AICBSE-1983]***

(a) $\frac{2-4x^2}{\sqrt{1-x^2}}$ (b) $\frac{2+4x^2}{\sqrt{1-x^2}}$

(c) $\frac{2-4x^2}{\sqrt{1+x^2}}$ (d) $\frac{2+4x^2}{\sqrt{1+x^2}}$

11. If $f(x) = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$, then $f'(\pi/3) =$

(a) $\frac{1}{2(1+\cos x)}$ (b) 1/2

(c) 1/4 (d) None of these

12. Differential coiffiecient $\cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}$ of with respect to x is ***[MPPET-93]***

(a) $-\frac{1}{2\sqrt{1-x^2}}$ (b) $\frac{1}{2\sqrt{1-x^2}}$

(c) $\frac{1}{\sqrt{1-x}}$ (d) $\sin^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}$

13. If $y = \cot^{-1}\left(\frac{1+x}{1-x}\right)$, then $\frac{dy}{dx} =$

(a) $\frac{1}{1+x^2}$ (b) $-\frac{1}{1+x^2}$

(c) $\frac{2}{1+x^2}$ (d) $-\frac{2}{1+x^2}$

14. If $y = \tan^{-1}\left[\frac{\log(e/x^2)}{\log(ex^2)}\right] + \tan^{-1}\left[\frac{3+2\log x}{1-6\log x}\right]$, then $\frac{dy}{dx} =$

(a) 0 (b) –2

(c) 2 (d) 1

15. Differential coefficient of $\sin^{-1}\frac{1-x}{1+x}$ with respect $\sqrt{x} =$ ***[Roorkee-84]***

(a) $\frac{1}{2\sqrt{x}}$ (b) $\frac{\sqrt{x}}{\sqrt{1-x}}$

(c) 1 (d) $\frac{-2}{1+x}$

16. If $y = \sin^{-1}\left(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right)$, then $\frac{dy}{dx} =$

(a) $\frac{-2x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x-x^2}}$

(b) $\frac{-1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$

(c) $\frac{+1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x-x^2}}$

(d) None of these

17. If $y = \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$, then $\frac{dy}{dx} =$

(a) $\frac{1}{2\sqrt{1-x^2}}$ (b) $\frac{-2x}{\sqrt{1-x^2}}$

(c) $\frac{-x}{\sqrt{1-x^2}}$ (d) $\frac{x}{\sqrt{1-x^2}}$

18. If $\sin^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \log a$, then $\frac{dy}{dx}$ is equal to

(a) x/y

(b) y/x

(c) y/x^2

(d) $x^2 - y^2/x^2 + y^2$

19. $\frac{d}{dx}\left[\sin^{-1}\left(\frac{3x}{2} - \frac{x^3}{2}\right)\right]$ is equal to

(a) $\frac{3}{\sqrt{4-x^2}}$ (b) $\frac{-3}{\sqrt{4-x^2}}$

(c) $\frac{1}{\sqrt{4-x^2}}$ (d) $\frac{-1}{\sqrt{4-x^2}}$

20. $\frac{d}{dx}\left[\sin^2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right]$ is equal to

(a) 0 (b) 1/2

(c) –1/2 (d) –1

21. The derivatives of $\tan^{-1}\sqrt{\frac{1-x^2}{1+x^2}}$ with respect to $\cos^{-1}x^2$ is

(a) 1/2 (b) –1/2

(c) 1 (d) 0

22. $\frac{d}{dx}\left[\cos^{-1}\left(\frac{4x^3}{27} - x\right)\right] =$

(a) $\frac{3}{\sqrt{9-x^2}}$ (b) $\frac{1}{\sqrt{9-x^2}}$

(c) $\frac{-3}{\sqrt{9-x^2}}$ (d) $\frac{-1}{\sqrt{9-x^2}}$

23. If $f(x) = (x+1)\tan^{-1}(e^{-2x})$, then what is the value of $f'(0)$? ***[NDA-2007]***

(a) $(\pi/4) + 1$

(b) $(\pi/4) - 1$

(c) $(\pi/2) + 1$

(d) $\pi/4$

SOLUTIONS

1. Step 1: Formula $\frac{d}{dx}\left(\sin^{-1} y\right)=\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$

2. (c) Let $y=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

put $x=\cos\theta, 1+x=2\cos^2\frac{\theta}{2}$

$$1-x=2\sin^2\frac{\theta}{2}$$

$$\therefore y=\tan^{-1}\left(\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right)$$

$$y=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)=\frac{\pi}{4}-\frac{\theta}{2}$$

$$y=\frac{\pi}{4}-\frac{\cos^{-1}x}{2}\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$$

3. (a) $y=\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$

put $x=\sin\theta$, $y=\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$

$$=\tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$y=\frac{\theta}{2}=\frac{1}{2}\sin^{-1}x$$

$$\therefore\frac{d}{d(\sin^{-1}x)}\left(\frac{1}{2}\sin^{-1}x\right)=\frac{1}{2}$$

4. (b) Step 1: we know that

$$2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Step 2: Given equation is equivalent to differentiate $2\tan^{-1}x$ with respect to $2\tan^{-1}x$

i.e., $\frac{d(2\tan^{-1}x)}{d(2\tan^{-1}x)}=1$

5. (c) Step 1: Formula

$$\sin^{-1}\left(\frac{\text{perpendicular}}{\text{hypotenuese}}\right)=\cos^{-1}\left(\frac{\text{Base}}{\text{hypotenuese}}\right)$$

$$=\tan^{-1}\left(\frac{\text{perpendicular}}{\text{Base}}\right)$$

$$y=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\sin^{-1}\frac{x}{1};\ \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}$$

Here perpendicular = x, and base = $\sqrt{1-x^2}$
Hence hypotenuese = 1

6. (a) Here

$$1+\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2}$$

$$=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

Similarly $1-\sin x=\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2$

$$\therefore y=\cot^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right)$$

$$=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}$$

$$\frac{dy}{dx}=\frac{1}{2}$$

7. (a) put $x=\cos\theta$ in

$$\sec^{-1}\left(\frac{1}{2x^2-1}\right)=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)$$

$= \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1} x$

$$= \frac{\frac{d}{dx}(2\cos^{-1} x)}{\frac{d}{dx}\left(\sqrt{1+3x}\right)} = \frac{-2}{\sqrt{1-x^2}} \times \frac{2\sqrt{1+3x}}{3} = 0$$

at $x = \frac{-1}{3}$

8. (b) $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\, y = \sin^{-1} x$

differentiating we get

$$\frac{-x}{\sqrt{1-x^2}} y + \sqrt{1-x^2}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$-xy + \left(\sqrt{1-x^2}\right)^2 \frac{dy}{dx} = 1$$

$$(1-x^2)\frac{dy}{dx} = 1 + xy$$

9. (c) Step 1: $y = \frac{z}{1+z} = \frac{1+z-1}{1+z} = 1 - \frac{1}{1+z}$

Step 2: $\frac{dy}{dz} = 0 + \frac{1}{(1+z)^2} = \frac{1}{(1+a^{\cos^{-1} x})^2}$

10. (a) $y = \sin(2\sin^{-1} x) = 2x\sqrt{1-x^2}$

$$\frac{dy}{dx} = 2\sqrt{1-x^2} + 2x\left(\frac{-x}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = \frac{2(1-x^2) - 2x^2}{\sqrt{1-x^2}} = \frac{2-4x^2}{\sqrt{1-x^2}}$$

11. (d) Step 1: If

$$f(x) = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right) = \tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$f(x) = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

Step 2: $\frac{dy}{dx} = f'(x) = \frac{1}{2}$

12. (a) Step 1: Formula

$$\frac{d}{dx}(\cos^{-1} y) = \frac{-1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

and $\frac{d}{dx}\sqrt{ax+b} = \frac{1}{2\sqrt{ax+b}}.a$

Step 2: $\frac{-1}{\sqrt{1-\left(\sqrt{\frac{1+x}{2}}\right)^2}} \times \frac{1}{2\sqrt{\frac{1+x}{2}}} \times \frac{1}{2}$

$$= -\frac{1}{\frac{\sqrt{1-x}}{\sqrt{2}} \times \frac{2\sqrt{1+x}}{\sqrt{2}}} \times \frac{1}{2} = \frac{-1}{2\sqrt{1-x^2}}$$

13. (b) Step 1: $y = \cot^{-1}\frac{1+x}{1-x} = \tan^{-1}\frac{1-x}{1+x}$

$y = \tan^{-1} 1 - \tan^{-1} x$

Step 2: $\frac{dy}{dx} = 0$

$= \frac{-1}{1+x^2}$

14. (a) Step 1: $\log e = 1, \log(x^2) = 2\log x$

$$\tan^{-1}\frac{x+y}{1-xy} = \tan^{-1} x + \tan^{-1} y$$

$$\log_a mn = \log_a m + \log_a n;\ \log_a\left(\frac{m}{n}\right)$$

$$= \log_a m - \log_a n$$

Step 2:

$$y = \tan^{-1}\left\{\frac{1-2\log x}{1+2\log x}\right\} + \tan^{-1}\left\{\frac{3+2\log x}{1-6\log x}\right\}$$

$y = \tan^{-1} 1 - \tan^{-1}(2\log x) + \tan^{-1}(3) + \tan^{-1}(2\log x)$

$y = \tan^{-1} 1 + \tan^{-1} 3$ = a constant

Hence, $\frac{dy}{dx} = 0$

15. (d) Let $\sqrt{x} = t \Rightarrow x = t^2$

$$\sin^{-1}\left(\frac{1-t^2}{1+t^2}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{\pi}{2} - 2\tan^{-1} t$$

$$\therefore \frac{d}{dt}\left(\sin^{-1}\left(\frac{1-t^2}{1+t^2}\right)\right) = \frac{d}{dt}\left(\frac{\pi}{2} - 2\tan^{-1} t\right)$$

$$= \frac{-2}{1+t^2} = \frac{-2}{1+(\sqrt{x})^2} = \frac{-2}{1+x}$$

16. (c) $y=\sin^{-1}\left(x\sqrt{1-\left(\sqrt{x}\right)^2}+\sqrt{x}\sqrt{1-x^2}\right)$

$$=\sin^{-1}x+\sin^{-1}\sqrt{x}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

17. (c) $y=\sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$

Let $x=\cos\theta, \sqrt{\frac{1-x}{1+x}}=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}$

$$=\tan\frac{\theta}{2}$$

$$\therefore y=\sin\left[2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right]$$

$$y=\sin\left[2.\frac{\theta}{2}\right)=\sin\theta$$

$$y=\sqrt{1-\cos^2\theta}=\sqrt{1-x^2}$$

$$\frac{dy}{dx}=\frac{-x}{\sqrt{1-x^2}}$$

18. (b) $\sin^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\log a$

$$\frac{x^2-y^2}{x^2+y^2}=\sin(\log a)$$

Using componendo and devidendo,

$$\frac{x}{y}=\sqrt{\frac{1+\sin(\log a)}{1-\sin(\log a)}}$$

Differentiating w.r.t. x, $\frac{y-x\frac{dy}{dx}}{y^2}=0$

$$\frac{dy}{dx}=\frac{y}{x}$$

19. (a) $\frac{d}{dx}\left(\sin^{-1}\left(3\left(\frac{x}{2}\right)-4\left(\frac{x}{2}\right)^3\right)\right)$

$$=\frac{d}{dx}\left(3\sin^{-1}\frac{x}{2}\right)=\frac{3}{\sqrt{1-\frac{x^2}{4}}}.\frac{1}{2}=\frac{3}{\sqrt{4-x^2}}$$

20. (a) Step 1: Given expression

$$y=\frac{1}{\text{cosec}^2\cot^{-1}\left[\sqrt{\frac{1-x}{1+x}}\right]}$$

$$=\frac{1}{\text{cosec}^2\theta}\text{ when }\cot\theta=\sqrt{\frac{1-x}{1+x}}$$

$$=\frac{1}{1+\frac{1-x}{1+x}}=\frac{1+x}{2}\Rightarrow\frac{dy}{dx}=\frac{1}{2}$$

21. (a)

$$y=\tan^{-1}\sqrt{\frac{1-x^2}{1+x^2}}=\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, x^2=\cos\theta$$

$$y=\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}=\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$y=\frac{\theta}{2}=\frac{1}{2}\cos^{-1}x^2\ \therefore\frac{dy}{d(\cos^{-1}x^2)}=\frac{1}{2}$$

22. Step 1: From Formula

$$3\cos^{-1}x=\cos^{-1}(4x^3-3x)$$

$$\cos^{-1}\left(\frac{4x^3}{27}-x\right)=3\cos^{-1}\left(\frac{x}{3}\right)$$

Step 2: $\frac{d}{dx}\left(3\cos^{-1}\frac{x}{3}\right)=3\left(\frac{-1}{\sqrt{9-x^2}}\right)$

23. (b) $f(x)=(x+1)\tan^{-1}(e^{-2x})$

$$f'(x)=\tan^{-1}(e^{-2x})+(x+1).\frac{\left(-2e^{-2x}\right)}{1+(e^{-2x})^2}$$

put $x=0; f'(0)=\tan^{-1}(1)+\frac{(-2)}{1+1}$

$$=\frac{\pi}{4}-1$$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The derivatie of $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\cot^{-1}\left(\frac{1-3x^2}{3x-x^3}\right)$ is ***[Karnataka CET-2003]***

(a) 1 (b) 3/2
(c) 2/3 (d) 1/2

2. If $x = \sin^{-1}(3t - 4t^3)$ and $y = \cos^{-1}\left(\sqrt{1-t^2}\right)$, then $\frac{dy}{dx}$ is equal to ***[Kerala (Engg.)-2002]***

(a) 1/2 (b) 2/5
(c) 3/2 (d) 1/3

3. If $y = (1 + x^2)\tan^{-1} x - x$, then $\frac{dy}{dx} =$ ***[Karnataka CET-2001]***

(a) $\tan^{-1}x$ (b) $2x\tan^{-1}x$
(c) $2x\tan^{-1}x - 1$ (d) $\frac{2x}{\tan^{-1}x}$

4. Differential coefficient of $\cos^{-1}(\sqrt{x})$ with respect to $\sqrt{(1-x)}$ is ***[MPPET-1997]***

(a) $\sqrt{x}$ (b) $-\sqrt{x}$
(c) $1/\sqrt{x}$ (d) $-1/\sqrt{x}$

5. If $y = \sin^{-1}\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{2}$, then $\frac{dy}{dx} =$

(a) $\frac{1}{\sqrt{(1-x^2)}}$

(b) $-\frac{1}{\sqrt{(1-x^2)}}$

(c) $-\frac{1}{2\sqrt{(1-x^2)}}$

(d) None of these

6. If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$ then $\frac{dy}{dx} =$

(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $\frac{1-2x}{\sqrt{1-x^2}}$
(c) $\frac{1-2x}{2\sqrt{1-x^2}}$ (d) $\frac{1}{1+x^2}$

7. Darivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}x$ is ***[Kurukshetra CEE-1998; RPET-1999]***

(a) 1/2
(b) −1/2
(c) 1
(d) None of these

8. $\frac{d}{dx}\left(\cos^{-1}\sqrt{\frac{1+\cos x}{2}}\right) =$ ***[AICBSE-1982]***

(a) 1 (b) 1/2
(c) 1/3 (d) None of these

9. If $y = \tan^{-1}\left(\frac{x^{1/3}+a^{1/3}}{1-x^{1/3}.a^{1/3}}\right)$, then $\frac{dy}{dx} =$

(a) $\frac{1}{3x^{2/3}(1+x^{2/3})}$ (b) $\frac{a}{3x^{2/3}(1+x^{2/3})}$
(c) $-\frac{1}{3x^{2/3}(1+x^{2/3})}$ (d) $-\frac{a}{3x^{2/3}(1+x^{2/3})}$

10. Differential coefficient of $\frac{\tan^{-1}x}{1+\tan^{-1}x}$ with respect to $\tan^{-1}x$ is

(a) $\frac{1}{1+\tan^{-1}x}$ (b) $\frac{-1}{1+\tan^{-1}x}$
(c) $\frac{1}{(1+\tan^{-1}x^2)}$ (d) $\frac{-1}{2(1+\tan^{-1}x)^2}$

11. If $y = \tan^{-1}\dfrac{4x}{1+5x^2} + \tan^{-1}\dfrac{2+3x}{3-2x}$, then $\dfrac{dy}{dx} =$

[Kerala PET-08]

(a) $\dfrac{1}{1+25x^2} + \dfrac{2}{1+x^2}$ (b) $\dfrac{5}{1+25x^2} + \dfrac{2}{1+x^2}$

(c) $\dfrac{5}{1+25x^2}$ (d) $\dfrac{1}{1+25x^2}$

12. If $y = \sin^{-1}\dfrac{2x}{1+x^2} + \sec^{-1}\dfrac{1+x^2}{1-x^2}$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{4}{1-x^2}$

(b) $\dfrac{1}{1+x^2}$

(c) $\dfrac{4}{1+x^2}$

(d) $\dfrac{-4}{1+x^2}$

13. $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{\cos x}{1+\sin x}\right) =$

(a) $-1/2$ (b) $1/2$

(c) -1 (d) 1

14. The differential coefficient of $\operatorname{cosec}^{-1}\dfrac{1}{2x^2-1}$ with respect to $\sqrt{1-x^2}$ at $x = \dfrac{1}{2}$ is

(a) -4 (b) 4

(c) -1 (d) None of these

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 17 minutes.
3. The worksheet consists of 17 questions. The maximum marks are 51.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. If $f(x) = x\tan^{-1}x$, then $f'(1) =$

(a) $1+\frac{\pi}{4}$ (b) $\frac{1}{2}+\frac{\pi}{4}$
(c) $\frac{1}{2}-\frac{\pi}{4}$ (d) 2

2. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ is

[DCE-2002; Kurukshetra CEE-2001]

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) $\frac{-x}{\sqrt{1+x^2}}$
(c) $\frac{x}{\sqrt{1-x^2}}$ (d) None of these

3. $\frac{d}{dx}\left[\tan^{-1}\left(\frac{a-x}{1+ax}\right)\right] =$

[Karnataka CET-2001; Pb. CET-2001]

(a) $-\frac{1}{1+x^2}$ (b) $\frac{1}{1+a^2}-\frac{1}{1+x^2}$
(c) $\frac{1}{1+\left(\frac{a-x}{1+ax}\right)^2}$ (d) $\frac{-1}{\sqrt{1-\left(\frac{a-x}{1+ax}\right)^2}}$

4. If $y = \tan^{-1}\left(\frac{\sqrt{x}-x}{1+x^{3/2}}\right)$, then $y'(1)$ is

[AMU-2000; NDA-2007]

(a) 0 (b) 1/2
(c) –1 (d) –1/4

5. If $f(x) = \cos^{-1}\left[\frac{1-(\log x)^2}{1+(\log x)^2}\right]$, then the value of $f'(e) =$

[Karnataka CET-99; Pb. CET-2000]

(a) 1 (b) $1/e$
(c) $2/e$ (d) $2/e^2$

6. If $u = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$ and $v = 2\tan^{-1}x$, then $\frac{du}{dv}$ is equal to ***[RPET-1997]***

(a) 4 (b) 1
(c) 1/4 (d) –1/4

7. If $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

[EAMCET-1991; RPET-1996]

(a) $\frac{2}{1-x^2}$ (b) $\frac{1}{1+x^2}$
(c) $\pm\frac{2}{1+x^2}$ (d) $-\frac{2}{1+x^2}$

8. If $y = \tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}$, then $\frac{dy}{dx}$ is

[Roorkee-1995]

(a) 0 (b) –1/2
(c) 1/2 (d) 1

9. If $y = \cos^{-1}\left(\frac{3\cos x - 4\sin x}{5}\right)$, then $\frac{dy}{dx} =$

(a) 0 (b) 1
(c) –1 (d) 1/2

10. $\frac{d}{dx}\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} =$

(a) $\frac{a}{a^2+x^2}$ (b) $\frac{-a}{a^2+x^2}$
(c) $\frac{1}{a\sqrt{a^2-x^2}}$ (d) $\frac{1}{\sqrt{a^2-x^2}}$

11. If $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, then $\frac{dy}{dx}$ is equal to

(a) 1 (b) 0
(c) –1 (d) –2

12. $\frac{d}{dx}\left[\cot^{-1}\left(\frac{3-2\tan x}{2+3\tan x}\right)\right] =$

(a) –1 (b) 0
(c) 2 (d) 1

13. If $\sec^{-1}\left(\frac{1+x}{1-y}\right)=a$, then $\frac{dy}{dx}$ is **[Karnataka CET-2007]**

(a) $\frac{y-1}{x+1}$ (b) $\frac{y+1}{x-1}$

(c) $\frac{x-1}{y-1}$ (d) $\frac{x-1}{y+1}$

14. Value of $\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right)$ is **[MPPET-2007]**

(a) $\frac{1}{1+x}$ (b) $\frac{1}{1-x^2}$

(c) $\frac{1}{1+x^2}$ (d) $\frac{1}{1+x^2}$

15. $\frac{d}{dx}\left[\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}\right]=$ **[MPPET-2007]**

(a) $-1/2$ (b) 0

(c) $1/2$ (d) 1

16. If $y=\sin^{-1}(3x-4x^3)+\cos^{-1}(4x^3-3x)+\tan^{-1}(e)$, then $\frac{dy}{dx}=$ **[Kerala PET-2008]**

(a) $\frac{3}{\sqrt{1-x^2}}-\frac{3}{\sqrt{x^2-1}}$ (b) 0

(c) $\frac{\pi}{2}$ (d) $\frac{3\sqrt{x^2-1}}{\sqrt{1-x^2}\sqrt{x^2+1}}$

(e) $\frac{2}{\sqrt{1-x^2}}$

17. If $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$, what is $\frac{dy}{dx}$ equal to? **[NDA-2008]**

(a) $\cos^{-1}x+\cos^{-1}\sqrt{1-x^2}$

(b) $\frac{1}{\cos x}+\frac{1}{\cos\sqrt{1-x^2}}$

(c) $\frac{\pi}{2}$

(d) 0

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	**17.** ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

2. (a) Step 1: $f(x)=\sqrt{1+x^2}$

From formula

$$\sec\left(\tan^{-1}\frac{\text{perpendicular}}{\text{Base}}\right)=\frac{\text{hypotenuese}}{\text{Base}}$$

(Here perpendicular = x and base = 1)

Step 2:

$$\frac{d}{dx}(f(x))=f'(x)=\frac{1}{2\sqrt{1+x^2}}.2x=\frac{x}{\sqrt{1+x^2}}$$

(From formula $\frac{d}{dx}\sqrt{y}=\frac{1}{2\sqrt{y}}\frac{dy}{dx}$)

5. (b) Step 1: From formula

$$2\tan^{-1} y = \cos^{-1}\frac{1-y^2}{1+y^2}$$

Given question can be written as:

$f(x) = 2\tan^{-1}(\log x)$

Step 2: $f'(x)\Big|_{x=e} = \dfrac{2}{1+(\log x)^2}\times\dfrac{1}{x}\Big|_{x=e} = \dfrac{1}{e}$

17. (d) Step 1: Perpendicular in $\sin^{-1}\sqrt{1-x^2}$ $= \sqrt{1-x^2}$ and hypotenuese = 1. Hence base $= x$

Step 2: From formula

$$\sin^{-1}\left(\frac{\text{perpendicular}}{\text{hypotenuese}}\right)$$

$$= \cos^{-1}\left(\frac{\text{Base}}{\text{hypotenuese}}\right)$$

Given question:

$$y = \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} = \text{constant}$$

Hence $\dfrac{dy}{dx} = 0$

ANSWERS

LECTURE 1

Worksheet: To Check the Preparation Level

1. (a)	**5.** (c)	**9.** (d)	**13.** (a)
2. (b)	**6.** (d)	**10.** (b)	
3. (a)	**7.** (a)	**11.** (b)	
4. (d)	**8.** (b)	**12.** (d)	

LECTURE 2

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (c)	**8.** (c)	**15.** (a)	**22.** (a)
2. (a)	**9.** (d)	**16.** (a)	**23.** (d)
3. (a)	**10.** (b)	**17.** (d)	**24.** (b)
4. (c)	**11.** (c)	**18.** (c)	
5. (d)	**12.** (b)	**19.** (d)	
6. (a)	**13.** (d)	**20.** (d)	
7. (a)	**14.** (a)	**21.** (c)	

Worksheet: To Check the Preparation Level

1. (c)	**6.** (c)	**11.** (b)	**16.** (d)
2. (a)	**7.** (c)	**12.** (b)	
3. (c)	**8.** (d)	**13.** (c)	
4. (b)	**9.** (a)	**14.** (c)	
5. (c)	**10.** (d)	**15.** (b)	

LECTURE 3

Unsolved Objective Problems (Identical Problems for Practice): for Improving Speed with Accuracy

1. (c)	**5.** (c)	**9.** (a)	**13.** (a)
2. (d)	**6.** (c)	**10.** (c)	**14.** (a)
3. (b)	**7.** (a)	**11.** (c)	
4. (c)	**8.** (b)	**12.** (c)	

Worksheet: To Check the Preparation Level

1. (b)	**6.** (c)	**11.** (d)	**16.** (b)
2. (a)	**7.** (d)	**12.** (d)	**17.** (d)
3. (a)	**8.** (b)	**13.** (a)	
4. (d)	**9.** (b)	**14.** (c)	
5. (b)	**10.** (d)	**15.** (c)	

Part F

Applications of Derivatives

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LECTURE

1

Monotonic (Increasing and Decreasing) Functions, Mean Value Theorem and Approximations

BASIC CONCEPTS

MONOTONIC FUNCTION

1. **Monotonocity:** A function $f(x)$ is said to be monotonic on an interval (a, b) if it is either increasing or decreasing function.
2. **Strictly Increasing Function:** A function $f(x)$ is said to be a strictly increasing function on (a, b) if

   (i) $f'(x) > 0\ \forall\ x \in (a, b)$

   OR

   (ii) $x_1 > x_2 \Rightarrow f(x_1) > f(x_2),\ \forall x_1, x_2 \in (a, b)$

   OR

   (iii) $x_1 < x_2 \Rightarrow f(x_1) < f(x_2),\ \forall x_1, x_2 \in (a, b)$

NOTE

Non-decreasing Function: A function $f(x)$ is said to be a non-decreasing function on (a, b) if

$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$

$x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2)$

3. **Strictly Decreasing Function:** A function $f(x)$ is said to be a strictly decreasing function on (a, b) if:

   (i) $f'(x) < 0\ \forall x \in (a, b)$

   OR

   (ii) $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in (a, b)$

   OR

   (iii) $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall x_1, x_2 \in (a, b)$

NOTES

1. **Non increasing Function:** A function $f(x)$ is said to be a non increasing function on $(a. b)$ if

   $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2),\ x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$
2. **Non-Monotonic Function:**
   A function $f(x)$ is said to be non–monotonic on an interval (a, b) if it is increasing in some part of interval and decreasing in other or remaining part of the same interval.

4. **Concave up and concave down:**
   The graph of a differentiable function $y = f(x)$ is concave down on an interval (a, b) when $\frac{dy}{dx} = y'$, decreases i.e., $\frac{d^2y}{dx^2} < 0$ and concave up on interval where $\frac{dy}{dx}$ or y' increases i.e., $\frac{d^2y}{dx^2} > 0$.

Examples

1. Function $y = x^2$ is concave up on the entire x-axis because $y'' = \frac{d^2y}{dx^2} = 2 > 0.$
2. Function $y = \sin x$ is concave down on entire interval $(0, \pi)$ because $f'' = \frac{d^2y}{dx^2} = -\sin x < 0$ on its interval.

5. **Point of Inflection:** A point on a curve $y = f(x)$ where the concavity changes from up to down or vice versa is called a point of inflection. Thus a point of inflection on a twice differentiable curve is a point where y'' or $\frac{d^2y}{dx^2}$ is positive on one side and negative on the other,
   The points on the curve at which $\frac{d^2y}{dx^2} = 0$ (or $\frac{d^2y}{dx^2}$ does not exist) but $\frac{d^3y}{dx^3} \neq 0$ are known as the points of inflection.

LARGRANGE'S MEAN VALUE THEOREM (LMVT)

Let $f(x)$ be a function of x subject to the following conditions:

(i) $f(x)$ is a continuous function of x in the closed interval $a \le x \le b$.

(ii) $f(x)$ is differentiable for every point in the open interval (a, b) then there exists at least one value of x, say c such that a $< c < b$ where $f'(c) = \frac{f(b) - f(a)}{b - a}$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \ge 2$ for all $x \in [1, 6]$, then prove that $f(6) \ge 8$.

Solution

By lagrange's mean value theorem, there exist $\in (1, 6)$ such that $f'(c) = \frac{f(6) - f(1)}{6 - 1}$

$$\Rightarrow \quad \frac{f(6) + 2}{5} = f'(c) \ge 2$$

$[\because f'(x) \ge 2$ for all $x \in [1, 6]]$

$\Rightarrow \; f(6) + 2 \ge 10 \Rightarrow f(6) \ge 8$ **Proved.**

2. Using Lagrange'e mean value theorem, show that $\sin x < x$ for $x > 0$. ***[NCERT-Book]***

Solution

Consider the function $f(x) = x - \sin x$ defined on the interval $[0, x]$ where $x > 0$. Clearly, $f(x)$ is everywhere continuous and differentiable. So, it is continuous on $[0, x]$ and differentiable on $(0, x)$. Consequently, there exists $c \in (0, x)$ such that $f'(c) = \frac{f(x) - f(0)}{x - 0}$ [By Lagrange's mean value theorem]

$$\Rightarrow \quad 1 - \cos c = \frac{x - \sin x}{x}$$

$$\Rightarrow \quad \frac{x - \sin x}{x} > 0 \quad [\because 1 - \cos c > 0]$$

$$\Rightarrow \quad x - \sin x > 0 \quad [\because x > 0]$$

$$\Rightarrow \quad x > \sin x \quad \Rightarrow \sin x < x \text{ for all } x > 0.$$

Proved.

3. Using mean value theorem, prove that $\tan x > x$, for all $x \in (0, \pi/2)$.

Solution

Let x be any point in the interval $(0, \pi/2)$. Consider the function f given by $f(x) = \tan x - x$, where $x \in [0, x] \subset (0, \pi/2)$
Clearly, $f(x)$ is continuous on $[0, x]$ and differentiable on $(0, x)$.
So, there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \quad \sec^2 c - 1 = \frac{\tan x - x - 0}{x}$$

$$\Rightarrow \quad \frac{\tan x - x}{x} > 0$$

$[\because \sec^2 c > 1$ for all $c \in (0, x) \subset [0, \pi/2]]$

$$\Rightarrow \quad \tan x - x > 0$$

$[\because x > 0]$

$$\Rightarrow \quad \tan x > x \text{ for all } x \in [0, \pi/2]$$

4. Let f be a function defined on R (the set of all real numbers) such that
   $f'(x) = 2010(x - 2009)(x - 2010)^2 (x - 2011)^3 (x - 2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that

$f(x) = \ell n(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is ***[IIT(2)-2010]***

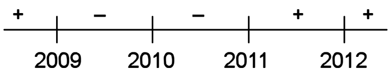
Solution

Here $f(x) = ing(x) \Rightarrow f'(x) = \dfrac{g'(x)}{g(x)}$

(Here g(x) > 0 to satisfy the domain of log)

$g'(x) = f'(x)g(x) = 2010(x - 2009)(x - 2010)^2 (x - 2011)^3(x - 2002)^4.g(x)$

$\therefore$ Number line of $g'(x)$ is



for local maximum, sign changes from positive to negative

$\therefore$ $x = 2009$ is point of point of local maxima

$\therefore$ number of points = 1

5. Let f be a real-valued differentiable function on R (the set of all real numbers) such that $f(1) = 1$. If the y-intercept of the tangent any point $P(x, y)$ on the curve $y = f(x)$ is equal to the cube of the abscissa of P, then the value of $f(-3)$ is equal to. ***[IIT(1)-2010]***

Solution

$Y - y = m(X - x)$ y-intercept $(x = 0)$

$y = y - mx$

Given that $y - mx = x^3 \Rightarrow x\dfrac{dy}{dx} - y = -x^3$

$\Rightarrow \quad \dfrac{dy}{dx} - \dfrac{y}{x} = -x^2$

Integrating factor $e^{-\int \frac{1}{x}dx} = \dfrac{1}{x}$

Solution $y.\dfrac{1}{x} = \int \dfrac{1}{x}.(-x^2)dx$

$\Rightarrow \quad f(x) = y = -\dfrac{x^3}{2} + cx$

Given $f(1) = 1 \Rightarrow c = \dfrac{3}{2}$

$\therefore \quad f(x) = -\dfrac{x^3}{2} + \dfrac{3x}{2} \Rightarrow f(-3) = 9$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. $f(x) = (x - a)^m (x - b)^n$ on $[a, b]$ where m, n are positive integers. Find the value of c. ***[PB-91C, 96, CBSE-91C, SP-2006]***

2. Verify Lagrange's Mean value Theorem for the following function $f(x) = x^2 + 2x + 3$, [4, 6] ***[CBSE-2006]***

3. Verify Lagrange's Mean value Theorem for the following $f(x) = 2 \sin x + \sin 2x$ on $[0, \pi]$

4. Verify Lagrange's Mean value Theorem for the following $f(x) = \begin{cases} 2 + x^3 & \text{if } x \le 1 \\ 3x & \text{if } x > 1 \end{cases}$ on $[-1, 2]$

5. Using Lagrange's mean value theorem prove that $\dfrac{b-a}{b} < \log\left(\dfrac{b}{a}\right) < \dfrac{b-a}{a}$ where $0 < a < b$.

6. Using Lagrange's Mean Value Theorem find a point p on the curve $y = \sqrt{x^2 - 4}$ defined in the interval [2, 4], where the tangent is parallel to the chord joining the end points on the curve. ***[CBSE-86, 90, 94, 2002; PSB-93 S, 98 C]***

Directions for Q 7 and 8: Find c of the Lagrange's Mean value Theorem for the following functions

7. $f(x) = x^2 - 3x - 2$; $x \in [-1, 2]$ ***[PB-91C]***

8. $f(x) = e^x$ on [0, 1]

9. Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in the interval (0, 1).

10. $f(x) = \log(1 + x) - \dfrac{x}{1+x}$. Or Prove that $\dfrac{x}{1+x} < \log(1 + x) < x$. for $x > 0$. ***[CBSE-2000C, 2002C; IIT-87]***

11. Determine the values of x for which $f(x) = x^x$, $x > 0$ is increasing or decreasing.

12. Separate the interval $[0, \pi/2]$ into sub–intervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or decreasing. ***[CBSE-2000 C]***

13. Find the values of k for which $f(x) = kx^3 - 9kx^2 + 9kx + 3$ is increasing on R.

14. Find the intervals in which the following function are increasing or decreasing $f(x) = (x-1)^3 (x-2)^2$

15. If $a < 0$, prove that the function $f(x) = e^{ax} + e^{-ax}$ is a monotonically decreasing function for values of x given by $x < 0$.

16. Find the least value of a such that the function $x^2 + ax + 1$ is increasing on $[1, 2]$. ***[HPSB-95, 2002; NCERT Book]***

17. Prove that the function e^x is an increasing function for every value of x. ***[MP-99; NCERET Book]***

18. Find the increasing and decreasing function of $f(x) = \dfrac{4x^2+1}{x}$ ***[CBSE-2004]***

19. Find the increasing and decreasing function of $f(x) = x^4 - \dfrac{x^3}{3}$ ***[CBSE-78, 80, 83, 92C]***

20. Find the intervals in which $f(x) = \sin x - \cos x$, where $0 < x < 2\pi$ is increasing or decreasing. ***[CBSE-2005; SP-2006]***

21. Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$ and strictly decreasing on $\left(\dfrac{\pi}{2}, \pi\right)$. ***[NCERT Book]***

22. Find the interval in which the function $f(x) = x^3 - 12x$ is increasing or decreasing. ***[CBSE-92]***

ANSWERS

1. $c = \dfrac{mb + na}{m+n}$

6. Point $(\sqrt{6}, \sqrt{2})$

7. $c = 1/2$

8. $c = \log(e-1)$

10. Increasing $(0, \infty)$, decreasing $(-1, 0)$

11. Increasing on $\left(\dfrac{1}{e}, \infty\right)$, decreasing on $\left(0, \dfrac{1}{e},\right)$

12. Increasing $[\pi/4, \pi/2]$, decreasing $[0, \pi/4]$

13. $f(x)$ is increasing on R, if $k \in (-\infty, 1/3)$

14. increasing on $(-\infty, 8/5) \cup (2, \infty)$ decreasing on $(8/5, 2)$

15. decreasing for $x < 0$.

16. -2

18. Increasing on $\left(-\infty, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$ decreasing on $\left(\dfrac{-1}{2}, 0\right) \cup \left(0, \dfrac{1}{2}\right)$

19. Increasing $\left(\dfrac{1}{4}, \infty\right)$ decreasing $(-\infty, 0)$ and $\left(0, \dfrac{1}{4}\right)$

20. Increasing on $\left(0, \dfrac{3\pi}{4}\right) \cup \left(\dfrac{7\pi}{4}, 2\pi\right)$ decreasing on $\left(\dfrac{3\pi}{4}, \dfrac{7\pi}{4}\right)$

22. Increasing $(\infty, -2) \cup (2, \infty)$ decreasing $(-2, 2)$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The function f defined by $f(x) = (x+2)e^{-x}$ is: ***[IIT-Screening-1994]***

(a) Decreasing for all x

(b) Decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$

(c) Increasing for all x
(d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$

Solution

(d) $f(x) = (x+2)e^{-x}$
$f'(x) = e^{-x} - e^{-x}(x+2)$
$f'(x) = -e^{-x}[x+1]$
For increasing, $-e^{-x}(x+1) > 0$ or $e^{-x}(x+1) < 0$
$\therefore$ $e^{-x} > 0$, $(x+1) < 0$
$x \in (-\infty, \infty)$ and $x \in (-\infty, -1)$
$\therefore$ $x \in (-\infty, -1)$
Hence, the function is increasing in $(-\infty, -1)$
For decreasing, $-e^{-x}(x+1) < 0$ or $-e^{-x}(x+1) > 0$, $x \in (-1, \infty)$
Hence the function is decreasing in $(-1, \infty)$.

2. If $f(x) = \dfrac{x}{\sin x}$ and $g(x) = \dfrac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval: **[IIT-1997 Re-Exam]**
(a) Both $f(x)$ and $g(x)$ are increasing functions
(b) Both $f(x)$ and $g(x)$ are decreasing functions
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function

Solution

(c) $f'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x} = \dfrac{\cos x(\tan x - x)}{\sin^2 x}$
$0 < x \leq 1 \Rightarrow x \in Q_1$
$\Rightarrow$ $\tan x > x$, $\cos x > 0$
$\therefore$ $f'(x) > 0$ for $0 < x \leq 1$
$\therefore$ $f(x)$ is an increasing function.

$$g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$$
$$= \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x}$$

$(\sin 2x - 2x)' = 2\cos 2x - 2 = 2[\cos 2x - 1] < 0$
$\Rightarrow$ $\sin 2x - 2x$ is decreasing
$\Rightarrow$ $\sin 2x - 2x < 0$
$\therefore$ $g'(x) < 0$
$\Rightarrow$ $g(x)$ is decreasing.

3. The function $f(x) = 1 - e^{-x^2/2}$ is: **[AMU-1999]**
(a) Decreasing for all x
(b) Increasing for all x
(c) Decreasing for $x < 0$ and increasing for $x > 0$
(d) Increasing for $x < 0$ and decreasing for $x > 0$

Solution

(c) $f(x) = 1 - e^{-x^2/2}$
$$f'(x) = -e^{-x^2/2}(-x) = xe^{-x^2/2}$$
For $f(x)$ to be increasing, $f'(x) > 0$
$\Rightarrow$ $xe^{-x^2/2} > 0 \Rightarrow x > 0$ and $f(x)$ to be decreasing for $x < 0$.

4. In the Mean – Value theorem $\dfrac{f(b) - f(a)}{b - a} = f'(c)$, if $a = 0$, $b = \dfrac{1}{2}$ and $f(x) = x(x-1)(x-2)$, the value of c is: **[MPPET-2003]**
(a) $1 - \dfrac{\sqrt{15}}{6}$ (b) $1 + \sqrt{15}$
(c) $1 - \dfrac{\sqrt{21}}{6}$ (d) $1 + \sqrt{21}$

Solution

(c) From mean value theorem
$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad a = 0, f(a) = 0$$
$\Rightarrow$ $b = \dfrac{1}{2}$, $f(b) = \dfrac{3}{8}$
$f'(x) = (x-1)(x-2) + x(x-2) + x(x-1)$
$f'(c) = (c-1)(c-2) + c(c-2) + c(c-1)$
$= c^2 - 3c + 2 + c^2 - 2c + c^2 - c$
$f'(c) = 3c^2 - 6c + 2$
According to mean value theorem,
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
$$\Rightarrow \quad 3c^2 - 6c + 2 = \frac{(3/8) - 0}{(1/2) - 0} = \frac{3}{4}$$
$$\Rightarrow \quad 3c^2 - 6c + \frac{5}{4} = 0$$
$$c = \frac{6 \pm \sqrt{36 - 15}}{2 \times 3} = \frac{6 \pm \sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}$$

5. The function $f(x) = \sin^4 x + \cos^4 x$ increases, if: **[IIT-1999; Pb. CET-2001]**
(a) $0 < x < \dfrac{\pi}{8}$ (b) $\dfrac{\pi}{4} < x < \dfrac{3\pi}{8}$
(c) $\dfrac{3\pi}{8} < x < \dfrac{5\pi}{8}$ (d) $\dfrac{5\pi}{8} < x < \dfrac{3\pi}{4}$

Solution

(b) $f(x) = \sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$$= 1 - \frac{4\sin^2 x\cos^2 x}{2} = 1 - \frac{\sin^2 2x}{2}$$

$$= 1 - \frac{1}{4}(2\sin^2 2x)$$

$$= 1 - \left(\frac{1-\cos 4x}{4}\right) = \frac{3}{4} + \frac{1}{4}\cos 4x$$

Hence function $f(x)$ is increasing when $f'(x) > 0$

$f'(x) = -\sin 4x > 0$

$\Rightarrow \quad \sin 4x < 0$

Hence $\pi < 4x < \frac{3\pi}{2}$

or $\quad \frac{\pi}{4} < x < \frac{3\pi}{8}$

6. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x. Then

[IIT-1998]

(a) h is increasing whenever f is increasing
(b) h is increasing whenever f is decreasing
(c) h is decreasing whenever f is decreasing
(d) Nothing can be said in general

Solution

(a, c) $h(x) = f(x) - (f(x))^2 + (f(x))^3$

$$h'(x) = f'(x) - 2f(x)f'(x) + 3[f(x)]^2 f'(x)$$
$$= f'(x)[1 - 2f(x) + 3[f(x)]^2]$$
$$= 3f'(x)\left\{\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right\}$$

$\therefore \quad h'(x)$ and $f'(x)$ have same sign.

7. In [0, 1] Lagrange's mean value theorem is NOT applicable to

[IIT Screening-2003]

(a) $f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$

(b) $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = x\,|x|$
(d) $f(x) = |x|$

Solution

(a) The function defined in option (a) is not differentiable at $x = 1/2$.

8. $f(x) = 2x^2 - \log|x|$ $(x \neq 0)$ is monotonic increasing in the interval:

[IIT-83]

(a) $(1/2, \infty)$
(b) $(-\infty, -1/2) \cup (1/2, \infty)$
(c) $(-\infty, -1/2) \cup (0, 1/2)$
(d) $(-1/2, 0) \cup (1/2, \infty)$

Solution

(d) $f'(x) = 4x - 1/x$ $f(x)$ is monotonic increasing when $f'(x) > 0$

$\Rightarrow \quad 4x - 1/x > 0 \Rightarrow \frac{4x^2 - 1}{x} > 0$

$\Rightarrow \quad \begin{cases} 4x^2 - 1 > 0 & \text{when} \quad x > 0 \\ 4x^2 - 1 < 0 & \text{when} \quad x < 0 \end{cases}$

But $x > 0$, $4x^2 - 1 > 0 \Rightarrow x^2 > 1/4$

$\Rightarrow \quad |x| > 1/2 \Rightarrow x \in (1/2, \infty)$

and $\quad x < 0$, $4x^2 - 1 < 0 \Rightarrow x^2 < 1/4 \Rightarrow |x| < 1/2$

$\Rightarrow \quad x \in (-1/2, 0) \therefore x \in (-1/2, 0) \cup (1/2, \infty)$

9. If $y = ax^3 + 3x^2 + (2a + 1)x + 1000$ is strictly increasing function for all real values of x, then:

[JEE (WB)-98]

(a) $-3/2 < a < 1$ (b) $a > 1$
(c) $a < -3/2$ (d) $a > 1$ or $a < -3/2$

Solution

(b) y is increasing function

$\Rightarrow \quad 3ax^2 + 6x + (2a + 1) > 0 \; \forall \; x$

$\Rightarrow \quad 36 - 4 \,.\, 3a(2a + 1) < 0$ and $a > 0$

$\Rightarrow \quad 3 - 2a^2 - a < 0$ and $a > 0$

$\Rightarrow \quad (2a + 3)(a - 1) > 0$

and $a > 0$

$\Rightarrow \quad a > 1$ or $a < -3/2$

and $a > 0 \Rightarrow a > 1$

10. Function $f(x) = 2x + \cot^{-1} x - \log\left(x + \sqrt{1 + x^2}\right)$ is increasing in:

[UPSEAT-98]

(a) $(-\infty, 0)$ (b) $(0, \infty)$
(c) $(-\infty, \infty)$ (d) no where

Solution

(c) $f'(x) > 0$

$\Rightarrow \quad 2 - \dfrac{1}{1+x^2} - \dfrac{1}{\sqrt{1+x^2}} > 0$

$\Rightarrow \quad \dfrac{2 + 2x^2 - 1 - \sqrt{1+x^2}}{1+x^2} > 0$

$\Rightarrow \quad (2x^2 + 1) - \sqrt{1+x^2} > 0 \quad [\because 1 + x^2 > 0]$

$\Rightarrow \quad 2x^2 + 1 > \sqrt{1+x^2}$ which is true $\forall\, x \in R$.

11. $y = [x(x-3)]^2$ is increasing when:

[Rookee (Screening)-93]

(a) $0 < x < 3/2$ (b) $0 < x < \infty$
(c) $-\infty < x < 0$ (d) $1 < x < 3$

Solution

(a) $\dfrac{dy}{dx} = 2x(x-3)(2x-3) > 0$

$\Rightarrow \quad 4x(x - 3/2)(x-3) > 0$

$\Rightarrow \quad x \in (0, 3/2) \cup (3, \infty)$

$\Rightarrow \quad 0 < x < 3/2$

12. If $a < 0$, the function $(e^{ax} + e^{-ax})$ is decreasing function for all values of x, where:

[Orissa JEE-2008]

(a) $x > 0$ (b) $x < 0$
(c) $x > 1$ (d) $x < 1$

Solution

(b) Let $f(x) = e^{ax} + e^{-ax}$

$f'(x) = \dfrac{d}{dx}(e^{ax} + e^{-ax}) = a(e^{ax} - e^{-ax})$

If $f(x)$ is decreasing then $f'(x) < 0$

$\Rightarrow \quad a(e^{ax} - e^{-ax}) < 0$

$\Rightarrow \quad e^{ax} > e^{-ax} \qquad (\because a < 0)$

$\Rightarrow \quad e^{2ax} > 1 \Rightarrow e^{2ax} > e^0$

$\Rightarrow \quad 2ax > 0 \Rightarrow x < 0 \qquad (\because a < 0)$

$\therefore \quad e^{ax} + e^{-ax}$ is decreasing when $x < 0$.

13. Match List I with List II and select the correct ANSWER using the code given below the lists:

[UPSC-2007]

	List I	**List II**
A	The function $x^3 - 6x^2 - 36x + 7$ increases when	1. $x = -2$
B	The function $x^3 - 6x^2 - 36x + 7$ is maximum at	2. $x = 6$
C	The function $x^3 - 6x^2 - 36x + 7$ is minimum at	3. $x < -2$ or $x > 6$
D	The function $x^3 - 6x^2 - 36x + 7$ decreases when	4. $-2 < x < 6$

Code:

	A	B	C	D
(a)	4	2	1	3
(b)	3	1	2	4
(c)	3	2	1	4
(d)	4	1	2	3

Solution

(b) $f(x) = x^3 - 6x^2 - 36x + 7$

$f'(x) = 3x^2 - 12x - 36 = 3(x^2 - 4x - 12)$
$= 3(x-6)(x+2)$

$f'(x) = 0 \Rightarrow x = -2, 6$

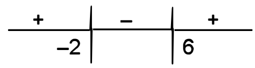
$f''(x) = 3(2x - 4) = 6(x - 2)$

$f''(-2) = 6(-2-2) = -24 < 0$

$f''(-6) = 6(6-2) = 6 \times 4 = 24 > 0$

$f(x)$ is max at $x = -2$ and min at $x = 6$

The sign of $f'(x) = 3(x-6)(x+2)$ is as follows.



$f(x)$ is increasing when $f'(x) > 0$ i.e., $x \in (-\infty, -2) \cup (6, \infty)$

$f(x)$ is decreasing when $f'(x) < 0$ i.e., $x \in (-2, 6)$

14. For function $f(x) = x\cos\dfrac{1}{x}, x \geq 1$, :

[IIT - 2009]

(a) for atleast one x is interval $[1, \infty)$, $f(x+2) - f(x) < 2$
(b) $\lim\limits_{x\to\infty} f'(x) = 1$
(c) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$
(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$

Solution

(b, c, d) For $f(x) = x\cos\dfrac{1}{x}, x \geq 1$,

$f'(x) = \cos\left(\dfrac{1}{x}\right) + \dfrac{1}{x}\sin\dfrac{1}{x} \to 1$ for $x \to \infty$

also

$$f'(x) = \frac{1}{x^2}\sin\left(\frac{1}{x}\right) - \frac{1}{x^2}\sin\left(\frac{1}{x}\right) - \frac{1}{x^3}\cos\left(\frac{1}{x}\right)$$

$$= -\frac{1}{x^3}\cos\left(\frac{1}{x}\right) < 0 \text{ for } x \geq 1$$

$\Rightarrow f'(x)$ is decreasing for $[1,\infty)$

$\Rightarrow f'(x+2) < f'(x)$.

Also

$$\lim_{x\to\infty} f(x+2) - f(x)$$

$$= \lim_{x\to\infty}\left[(x+2)\cos\frac{1}{x+2} - x\cos\frac{1}{x}\right] = 2$$

$\therefore f(x+2) - f(x) > 2 \forall x \geq 1$

15. The function $f(x) = x^2 - 2x$ increases for all: ***[NDA-2009]***

(a) $x > -1$ (b) $x < -1$ only
(c) $x > 1$ only (d) $x < 1$

Solution

(c) $\because f(x) = x^2 - 2x$
On differentiating w.r.t. x, we get
$f'(x) = 2x - 2$
$f(x)$ is increasing, if $2x - 2 > 0 \Rightarrow x > 1$

16. If $f(x) = 3x^2 + 6x - 9$, then
(a) $f(x)$ increasing in $(-1, 3)$
(b) $f(x)$ is decreasing in $(3, \infty)$
(c) $f(x)$ is increasing in $(-\infty, -1)$
(d) $f(x)$ is decreasing in $(-\infty, -1)$

Solution

(d) $\because f(x) = 3x^2 + 6x - 9$
On differentiating w.r.t.x, we get
$f'(x) = 6x + 6 \Rightarrow f'(x) < 0, \forall (-\infty, -1)$
$\therefore$ $f(x)$ is decreasing in $(-\infty, -1)$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. For all the real values of x, the increasing function is ***[PET-1996]***
(a) x^{-1} (b) x^2
(c) x^3 (d) x^4

2. The function $f(x) = x^3 - 3x^2 - 24x + 5$ is an increasing function in the interval given below ***[MP PET-1998]***
(a) $(-\infty, -2) \cup (4, \infty)$ (b) $(-2, \infty)$
(c) $(-2, 4)$ (d) $(-\infty, 4)$

3. If $f(x) = x^3 - 10x^2 + 200x - 10$ then ***[Kurukshetra CEE-1998]***
(a) $f(x)$ decreasing in$]-\infty, 10]$ and increasing in $[10, \infty)$
(b) $f(x)$, increasing in$]-\infty, -10]$ and decreasing in $[10, \infty[$
(c) $f(x)$ is increasing throughout real line
(d) $f(x)$ is decreasing throughout real line.

4. The function $f(x) = x + \cos x$ is ***[DCE-2002]***
(a) Always increasing
(b) Always decreasing
(c) Increasing for certain range of x
(d) None of these

5. If the function $f(x) = \cos|x| - 2ax + b$ increasing along entire number scale then the range of 'a' is: ***[EAMCET-1991]***
(a) $a \leq b$ (b) $a = b/2$
(c) $a \leq -1/2$ (d) $a \geq -3/2$

6. The function $f(x) = \log(1 + x) - \dfrac{2x}{2+x}$ is increasing on ***[EAMCET-2002]***
(a) $(-1, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty)$ (d) None of these

7. Let $f(x) = x^3 + bx^2 + cx + d$, $0 < b^2 < c$. Then f ***[IIT-Screening-04]***
(a) Is bounded (b) Increasing
(c) Stationary (d) Discontinuous

8. A function is matched an interval where it is supposed to be increasing which of the following pairs is incorrectly matched? ***[AIEEE-05]***

	Interval	Function
(a)	$(-\infty, -4)$	$x^3 + 6x^2 + 6$
(b)	$(-\infty, 1/3)$	$3x^2 - 2x + 1$
(c)	$[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(d)	$(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$

9. For all $x \in (0, 1)$ ***[IIT (Screening)-2000]***

(a) $e^x < 1 + x$ (b) $\log_e (1 + x) < x$
(c) $\sin x > x$ (d) $\log_e x > x$

10. If function $2x^2 + 3x - m \log x$ is monotonic decreasing in the interval (0, 1) then least value of the parameter 'm' is

(a) 7 (b) 15/2
(c) 31/4 (d) 4

11. Function $f(x) = \dfrac{\lambda \sin x + 6\cos x}{2\sin x + 3\cos x}$ is increasing if ***[MPPET-2001]***

(a) $\lambda > 1$ (b) $\lambda < 1$
(c) $\lambda < 4$ (d) $\lambda > 4$

12. The function $f(x) = \cos x - 2ax$ is monotonically decreasing for ***[RPET-1987; MPPET-2002]***

(a) $a < \dfrac{1}{2}$ (b) $a > \dfrac{1}{2}$
(c) $a < 2$ (d) $a > 2$

13. The length of a longest interval in which the function $3 \sin x - 4 \sin^3 x$ is increasing is ***[IIT (Screening-2002]***

(a) $\pi/3$ (b) $\pi/2$
(c) $3\pi/2$ (d) π

14. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is monotonically increasing in each interval, then

(a) $k < 3$ (b) $k \le 3$
(c) $k > 3$ (d) None of these

15. The function x^x is increasing, when ***[MPPET-2003]***

(a) $x > 1/e$ (b) $x < 1/e$
(c) $x < 0$ (d) For all real x

16. The function $f(x) = \tan^{-1} (\sin x + \cos x)$ is an increasing function in ***[AIEEE-2007]***

(a) $(\pi/4, \pi/2)$ (b) $(-\pi/2, \pi/4)$
(c) $(0, \pi/2)$ (d) $(-\pi/2, \pi/2)$

17. If $f(x) = \sin x - \cos x$, the function decreasing in $0 \le x \le 2\pi$ is ***[UPSEAT-2001]***

(a) $\left[\dfrac{5\pi}{6}, \dfrac{3\pi}{4}\right]$ (b) $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$
(c) $\left[\dfrac{3\pi}{2}, \dfrac{5\pi}{2}\right]$ (d) None of these

18. The value of a in order that $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ decreases for all real values of x, is given by

(a) $a < 1$ (b) $a \ge 1$
(c) $a \ge \sqrt{2}$ (d) $a < \sqrt{2}$

19. A value of c for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is: ***[AIEEE-2007]***

(a) $2 \log_3 e$ (b) $\dfrac{1}{2}\log_e 3$
(c) $\log_3 e$ (d) $\log_e 3$

20. If $f(x) = \cos x$, $0 \le x \le \dfrac{\pi}{2}$, then the real number 'c' for the mean value theorem is ***[MPPET-1994]***

(a) $\pi/6$ (b) $\pi/4$
(c) $\sin^{-1} (2/\pi)$ (d) $\cos^{-1} (2/\pi)$

21. From mean value theorem $f(b) - f(a) = (b - a) f'(x)$, $a < x_1 < b$ If $f(x) = 1/x$, then $x_1 =$ ***[MPPET-1992]***

(a) $\sqrt{ab}$ (b) $\dfrac{a+b}{2}$
(c) $\dfrac{2ab}{(a+b)}$ (d) $\dfrac{b-a}{b+a}$

22. The abscissas of the points of the curve $y = x^3$ in the interval [–2, 2] where the slope of tangents can be obtained by mean value theorem for the interval [–2, 2] are ***[MPPET-1993]***

(a) $\pm 2/\sqrt{3}$ (b) $\pm\sqrt{3}$
(c) $\pm\sqrt{3}/2$ (d) 0

23. If $f'(x) = g(x)(x - a)^2$ where $g(a) \ne 0$ and $g(x)$ is cotinuous at $x = a$. Then ***[Roorkee (Screening)-1999]***

(a) f is increasing in the nbd of a if $g(x) < 0$.
(b) f is decreasing in the nbd of 'a' if $g(a) > 0$
(c) f is increasing in nbd of 'a' if $g(a) > 0$
(d) None of these

24. Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 1$, such that $f(0) = 0$, $g(0) = 0$, $f(1) = 6$. Let there exist a real number c in (0, 1) such that $f'(c) = 2g'(c)$, then the value of $g(1)$ must be ***[Pb. CET-1991]***

(a) 1 (b) 3
(c) –2 (d) –1

SOLUTIONS

1. (c) Step 1 : Monotonicity: continuous function is monotonically increasing in (a, b)

$\therefore \quad f'(x) > 0, \forall x \in (a,b)$

$\therefore \quad \frac{d}{dx}(x^{-1}) = \frac{-1}{x^2} < 0; \frac{d}{dx}(x^2) = 2x > 0$

if $\quad x > 0$

Also $\frac{d}{dx}(x^3) = 3x^2 \geq 0$, for all real x

$\therefore$ Ans is (c)

2. (a) $f'(x) = 3x^2 - 6x - 24 > 0$,

$\because \quad f(x)$ is increasing function

$3(x^2 - 2x - 8) > 0$

$\Rightarrow \quad (x-4)(x+2) > 0$

$\therefore \quad x \in (-\infty, -2) \cup (4, \infty)$

3. (a) Step 1: For increasing function, $f'(x) > 0$

$3x^2 - 20x + 200 > 0$

But $D = B^2 - 4AC = 400 - 2400 < 0$

$\therefore$ expression is always positive.

$\therefore \quad f'(x) > 0$ or $f(x)$ is increasing for all $x \in R$

4. (a) $f(x) = x + \cos x$

$f'(x) = 1 - \sin x \geq 0$;

$\therefore \quad f(x)$ is increasing for all $x \in R$

5. (c) Step 1: $\cos x = \cos|x| = \cos(x)$

Step 2: $f(x) = \cos x - 2ax + b$

Step 3: For increasing function $f'(x) > 0$

$-\sin x - 2a > 0 \Rightarrow 2a < -\sin x$

$2a \leq -1 \Rightarrow a \leq \frac{-1}{2}$

6. (a) $f(x) = \log(1+x) - \frac{2x}{x+2}$;

Here $x > -1$

($\because$ log is defined for +ve values)

$$f'(x) = \frac{1}{1+x} - \frac{4}{(x+2)^2} \geq 0;$$

$$\frac{x^2 + 4x + 4 - 4(1+x)}{(x+2)^2(x+1)} \geq 0$$

$$\frac{x^2}{(x+2)^2(x+1)} \geq 0 \Rightarrow x \in (-1, \infty)$$

7. (b) $f(x) = x^3 + bx^2 + cx + d$;

$f'(x) = 3x^2 + 2bx + c$;

Here $D = (2b)^2 - 4.3.c = 4(b^2 - 3c)$

$\because \quad b^2 - c < 0 \Rightarrow b^2 - 3c < 0$

$\therefore \quad D < 0;\ f'(x) > 0 \quad \forall x \in R$

So, $f(x)$ is increasing $\forall x \in R$

8. (d)

(a) $f(x) = x^3 + 6x^2 + 6$;

for increasing $f'(x) > 0$

$3x^2 + 12x > 0 \Rightarrow 3x(x+4) > 0$

$x \in (-\infty, -4) \cup (0, \infty)$

(b) $f(x) = 3x^2 - 2x + 1$

$f'(x) = 6x - 2 > 0 \Rightarrow x > \frac{1}{3}$

(c) $f(x) = 2x^3 - 3x^2 - 12x + 6$

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

$= 6(x-2)(x+1)$

$\therefore \quad f'(x) > 0 \ \forall x \in (-1, 2)$

(d) $f(x) = x^3 - 3x^2 + 3x + 3$

$f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 \geq 0$

$\therefore \quad f(x)$ is increasing for $x \in R$

$\therefore$ (d) is correct.

9. (b) Step 1: Let $f(x) = e^x - (1+x)$ then $f'(x) = e^x - 1$

clearly $f'(x) > 0 \forall x \in (0,1)$

$\therefore \quad f(x)$ is increasing in $(0, 1)$

$f(x) > f(0)$

$e^x - (1+x) > e^0 - (1+0)$

$e^x > (1+x)$

Step 2: Let $f(x) = \log(1+x) - x$

$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x} < 0 \forall x \in (0,1)$

$\therefore$ $f(x)$ is decreasing in (0, 1)

So $x > 0$ or $f(x) < f(0)$

$\log(1+x) - x < 0 \Rightarrow \log(1+x) < x$

NOTE

Similarly, we can prove that (c) and (d) are not true.

10. (a) $f(x) = 2x^2 - 3x - m\log x$, Here $x > 0$

$f'(x) \le 0$; where $x \in (0, 1)$

$4x + 3 - \frac{m}{x} \le 0$; $4x^2 + 3x - m \le 0$

(x can be cross multiplied since $x > 0$)

$\therefore$ at $x = 0; m \ge 0$

at $x = 1\, m \ge 7 \Rightarrow m \ge 7$

min. value of $m = 7$

11. (d) Step 1: $f(x)$ is monotonically increasing in (a, b) if

$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

Step 2: $\frac{\pi}{2} > 0 \Rightarrow f\left(\frac{\pi}{2}\right) > f(0)$

$\frac{\lambda + 0}{2+0} > \frac{0+6}{0+3} \Rightarrow \lambda > 4$

12. (b) Step: For monotonically decreasing function

$f'(x) < 0 \Rightarrow -\sin x - 2a < 0$

$2a > -\sin x \Rightarrow 2a > 1$

or $a > \frac{1}{2}$ ($\because$ minimum value of $\sin x = -1$)

13. (a) $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

$f(x)$ is increasing $f'(x) > 0$

$3\cos 3x > 0 \Rightarrow \cos 3x > 0$

is possible if $-\frac{\pi}{2} < 3x < \frac{\pi}{2}$

$x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$

length of interval $= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

14. (d) Step 1: Monotonically increasing function is always strictly increasing function and function $f(x)$ is monotonically increasing function in interval (a, b) then $f'(x) > 0$

$3kx^2 - 18x + 9 > 0 \Rightarrow kx^2 - 6x + 3 > 0$

Step 2: $ax^2 + bx + c$ and a are of same sign.

If $b^2 \le 4ac$

Hence $k > 0, 36 \le 12K$

i.e., $k > 0$, $3 \le k \Rightarrow k \ge 3$

15. (a) Step 1: For increasing function $f'(x) > 0$

Step 2: $f'(x) = x^x(1 + \log x) > 0$

$\Rightarrow \log x > -1 \Rightarrow x > e^{-1}$

16. (b) $f(x) = \tan^{-1}(\sin x + \cos x)$

Here,

$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \times (\cos x - \sin x)$

For increasing function $f'(x) = 0$

$\cos x - \sin x > 0 \Rightarrow \cos x > \sin x$

$\therefore$ $x \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

17. (a) $f(x) = \sin x - \cos x$

$f'(x) \le 0 \Rightarrow \cos x + \sin x \le 0$

$\sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \le 0$

$\frac{\pi}{4} + x \in [\pi, 2\pi] \Rightarrow x \in \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$

$\therefore$ (a) is satisfied.

18. (b) For decreasing function, $f'(x) \le 0$

$\sqrt{3}\cos x + \sin x - 2a \le 0$

$2a \ge \sqrt{3}\cos x + \sin x$

$-2 \le \sqrt{3}\cos x + \sin x \le 2$

($\because -\sqrt{a^2 + b^2} \le a\sin x + b\cos x \le \sqrt{a^2 + b^2}$)

$\therefore$ $2a \ge 2 \Rightarrow a \ge 1$

19. (a) By mean value theorem

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\frac{1}{c} = \frac{\log 3 - \log 1}{2} \Rightarrow c = \frac{2}{\log_e 3}$$

$$c = 2\log_3 e$$

20. (c) Step 1: If a function $f(x)$ satisfies mean value theorem in $[a, b]$, then their exist a variable c for which $f'(c) = \frac{f(b) - f(a)}{b - a}$

Step 2: $f(x) = \cos x$

$$-\sin c = \frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0}$$

$$-\sin c = \frac{0 - 1}{\frac{\pi}{2}} \Rightarrow \sin c = \frac{2}{\pi} < 1$$

$$\because c = \sin^{-1}\left(\frac{2}{\pi}\right)$$

21. (a) Step 1: $f(x) = \frac{1}{x}, f'(x) = \frac{-1}{x^2}$

Step 2: $f(b) - f(a) = (b - a)\left(-\frac{1}{x_1^2}\right)$

$$\frac{1}{b} - \frac{1}{a} = (b - a)\left(\frac{-1}{x_1^2}\right)$$

$$x_1^2 = ab \Rightarrow x_1 = \sqrt{ab}$$

22. (a) Step 1: By mean value theorem

Slope of tangent $= f'(c) = \frac{f(b) - f(a)}{b - a}$

Step 2: $a = -2$, $b = 2$

Slope of tangent $= \frac{f(2) - f(-2)}{2 - (-2)}$

$$3x_1^2 = \frac{8 - (-8)}{4} = 4$$

$$x_1 = \pm\frac{2}{\sqrt{3}}$$

23. (c) Step 1: Any function $f(x)$ is increasing if $f(x) > 0$ and decreasing if $f'(x) < 0$

Step 2: $f'(x) = g(x)(x - a)^2$

Here $(x - a)^2$ is non negative

$\therefore$ if $g(x) > 0$; $f'(x) > 0$ or $f(x)$ is increasing

if $g(x) < 0, f'(x) < 0$ or $f(x)$ is decreasing.

24. (b) Step 1: If $f(x)$ and $g(x)$ are differentiable in $x \in [0,1]$,

$\therefore$ Both function satisfies mean value theorem

$$\therefore \quad f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$g'(c) = \frac{g(1) - g(0)}{1 - 0}$$

Step 2: $f'(c) = 2g'(c)$

$$\frac{f(1) - f(0)}{1 - 0} = 2\left[\frac{g(1) - g(0)}{1 - 0}\right]$$

$$\frac{6 - 0}{1} = 2[g(1) - 0] \Rightarrow g(1) = 3$$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. $f(x) = x^3 - 27x + 5$ is an increasing function, when

[MPPET-1995]

(a) $x < -3$ (b) $|x| > 3$
(c) $|x| \leq -3$ (d) $|x| < 3$

2. The function $f(x) = \tan x - x$ is

(a) Always increasing
(b) always decreasing
(c) Never decreasing
(d) Some time increased some time decreased

3. $2x^3 - 6x + 5$ is an increasing function if

[UPSEAT-2003]

(a) $0 < x < 1$
(b) $-1 < x < 1$
(c) $x < -1$ or $x > 1$
(d) $-1 < x < -1/2$

4. Function $f(x) = 2x^3 - 9x^2 + 12x + 29$ is monotonically decreasing, when:

[RPET-1996]

(a) $x < 2$ (b) $x > 2$
(c) $x > 1$ (d) $1 < x < 2$

5. The function $f(x) = \frac{\log x}{x}$ is increasing in the interval:

[UPSEAT-2001]

(a) $(1, 2e)$ (b) $(0, e)$
(c) $(2, 2e)$ (d) $(1/e, 2e)$

6. The function $f(x) = 1 - x^3 - x^5$ is decreasing for: ***[Kerala Engg.-2002]***

(a) $1 \le x \le 5$ (b) $x \le 1$
(c) $x \ge 1$ (d) All values of x

7. On the interval (1, 3) the function

$f(x) = 3x + \frac{2}{x}$ is: ***[AMU-99]***

(a) Strictly decreasing
(b) Strictly increasing
(c) Decreasing in (2, 3) only
(d) Neither increasing nor decreasing

8. If $f(x) = x^5 - 20x^3 + 240x$, then $f(x)$ satisfies which of the following:

[Kurukshetra EE-1996]

(a) It is monotonically decreasing every where
(b) It is monotonically decreasing only in $(0, \infty)$
(c) It is monotonically increasing every where
(d) It is monotonically every where

9. If $f(x) = \sin x - x/2$ is increasing function, then

[DCE-2002; MPPET-1987]

(a) $0 < x < \frac{\pi}{3}$ (b) $-\frac{\pi}{3} < x < 0$

(c) $-\frac{\pi}{3} < x < \frac{\pi}{3}$ (d) $x = \frac{\pi}{2}$

10. For the function $f(x) = e^x$, $a = 0$, $b = 1$ the value of c in mean value theorem will be:

(a) $\log x$ (b) $\log (e - 1)$
(c) 0 (d) 1

11. What is the interval in which the function $f(x) = \sqrt{9 - x^2}$ is increasing? $(f(x) > 0)$

[NDA-2007]

(a) $0 < x < 3$ (b) $-3 < x < 0$
(c) $0 < x < 9$ (d) $-3 < x < 3$

12. The function $f(x) = x^{1/x}$ is:

[AMU-2002]

(a) Increasing in $(1, \infty)$
(b) Decreasing in $(1, \infty)$
(c) Increasing in $(1, e)$ decreasing in (e, ∞)
(d) Decreasing in $(1, e)$, increasing in (e, ∞)

13. If $f(x) = \frac{1}{x+1} - \log(1 + x)$, $x > 0$, then f is:

[RPET-2002]
[Kerala (Engg.)-2002]

(a) An increasing function
(b) A decreasing function
(c) Both increasing and decreasing function
(d) None of these

14. If $f(x) = x^2 e^{-x}$ is monotonically increasing function then the interval of 'x' are:

[MNR-94; Kurukshetra CEE-98]

(a) $(-\infty, \infty)$ (b) $(-2, 0)$
(c) $(2, \infty)$ (d) $(0, 2)$

15. The function $\sin x - bx + c$ will be increasing in the interval $(-\infty, \infty)$ if:

(a) $b \le 1$ (b) $b \le 0$
(c) $b < -1$ (d) $b \ge 0$

16. Function $\frac{e^{2x} - 1}{e^{2x} + 1}$ is:

[Roorkee (Screening)-1998;
Orrisa JEE-2005]

(a) Increasing
(b) Decreasing
(c) Neither increasing nor decreasing
(d) Even function

17. If from mean value theorem

$f'(x_1) = \frac{f(b) - f(a)}{b - a}$ then

(a) $a < x_1 \le b$
(b) $a \le x_1 < b$
(c) $a < x_1 < b$
(d) $a \le x_1 \le b$

18. If for $f(x) = 2x - x^2$ Lagrange's theorem satisfies in [0, 1], then the value of $c \in [0, 1]$ is

(a) $c = 0$ (b) $c = 1/2$
(c) $c = 1/4$ (d) $c = 1$

19. The interval of the decreasing function $f(x) = x^3 - x^2 - x - 4$ is

(a) (1/3, 1) (b) (–4/3, 1)
(c) (–1/3, 1/3) (d) –1, –1/3)

20. The values of 'a' for which the function $(a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically throughout for all real x, are

[Kurukshetra CEE-2002]

(a) $a < -2$
(b) $a > -2$
(c) $-3 < a < 0$
(d) $-\infty < a \le -3$

21. The function $f(x) = x^3 + 6x^2 + (9 + 2k)x + 1$ Strictly increasing for all the value of 'x' then
(a) $k > 3/2$
(b) $k \ge 3/2$
(c) $k > 3/2$
(d) $k \le 3/2$

22. If $f(x) = \dfrac{\sin x + b\cos x}{\sin x + 4\cos x}$ is monotonic decreasing then:
(a) $b < 8$
(b) $b < 4$
(c) $b > 4$
(d) $b > 8$

23. Let $f(x)$ satisfy the requirements of Lagrange's Mean Value Theorem in [0, 2]. If $f(0) = 0$ and $|f'(x)| \le \frac{1}{2}$ for all x in [0, 2], then:
(a) $f(x) \le 2$
(b) $|f(x)| \le 1$
(c) $f(x) = 2x$
(d) $f(x) = 3$ for at least one x in [0, 2]

24. The function f defined by $f(x) = x^3 - 6x^2 - 36x + 7$ is increasing, if

[MPPET-2008]

(a) $x > 2$ and also $x > 6$
(b) $x > 2$ and also $x < 6$
(c) $x < -2$ and also $x < 6$
(d) $x < -2$ and also $x > 6$

25. In the interval $(-\infty, 0)$ function $f(x) = x^2$ is:

[MPPET-2007]

(a) increasing
(b) decreasing
(c) constant
(d) continuously increasing

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. For the function $f(x) = x + \frac{1}{x}, x \in x \in [1, 3]$, the value of '$c$' for the mean value theorem is: ***[MP PET-1997]***
   (a) 1 (b) $\sqrt{3}$
   (c) 2 (d) None of these

2. Which of the following statement is correct for the function $f(x) = \sin 2x$:
   (a) Function $f(x)$, increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{\pi}{2}, \pi\right)$
   (b) Function $f(x)$, decreasing in $\left(0, \frac{\pi}{2}\right)$ and increasing in $0 < x < \frac{\pi}{3}$
   (c) Function $f(x)$, increasing in $\left(0, \frac{\pi}{4}\right)$ and decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
   (d) option (a), (b) and (c) all are correct.

3. If $f(x) = x^3 - 6x^2 + 9x + 3$ be a decreasing function, then x lies in: ***[RPET-2002]***
   (a) $(-\infty, -1) \cap (3, \infty)$
   (b) $(1, 3)$
   (c) $(3, \infty)$
   (d) None of these

4. The function which is neither decreasing nor increasing in $(\pi/2, 3\pi/2)$ is: ***[MPPET-2000]***
   (a) cosec x (b) $\tan x$
   (c) x^2 (d) $|x - 1|$

5. The function $\frac{a\sin x + b\cos x}{c\sin x + d\cos x}$ is decreasing if: ***[RPET-1999]***
   (a) $ad - bc > 0$ (b) $ad - bc < 0$
   (c) $ab - cd > 0$ (d) $ab - cd < 0$

6. $2x^3 + 18x^2 - 96x + 45 = 0$ is an increasing function when: ***[RPET-1997]***
   (a) $x \le -8, x \ge 2$ (b) $x < -2, x \ge 8$
   (c) $x \le -2, x \ge 8$ (d) $0 \le x \le -2$

7. The least value of k for which the function $x^2 + kx + 1$ is an increasing in the interval $1 < x < 2$ is:
   (a) –4 (b) –3
   (c) –1 (d) –2

8. Let $f(x) = x^3 + 6x^2 + px + 2$, if the largest possible interval in which $f(x)$ is a decreasing function is $(-3, -1)$, then p equals: ***[MESRA-JEE-2002]***
   (a) 3 (b) 9
   (c) –2 (d) None of these

9. For which interval the given function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is decreasing: ***[PET-1993]***
   (a) $(-2, \infty)$
   (b) $(-2, -1)$
   (c) $(-\infty, -1)$
   (d) $(-\infty, -2)$ and $(-1, \infty)$

10. If the mean value theorem $f(b) - f(a) = (b - a) f'(c)$. If $a = 4$, $b = 9$ and $f(x) = \sqrt{x}$. Then the value of c is: ***[J and K-2005]***
    (a) 8.00 (b) 5.25
    (c) 4.00 (d) 6.25

11. If function $f(x) = x^2 + ax + 1$ is monotonic increasing in the interval [1, 2] then the minimum value of 'a' is: ***[VIT-2006]***
    (a) –1 (b) –2
    (c) 1 (d) 0

12. Let $f(x) = \frac{\log x}{x} + \log 51$, then $f(x)$ is:
    (a) increasing for $x > e$
    (b) decreasing for $x > e$
    (c) decreasing for $2 < x < e$
    (d) $f'(x) = 0$ for $x = 2e^3$.

13. If $y = 8x^3 - 60x^2 + 144x + 27$ is a decreasing function in the interval (a, b) then (a, b) is:

[CET-1996; Pb. CET-2002]

(a) $(-4, 2)$ (b) $(2, 3)$
(c) $(5, 6)$ (d) $(-4, 2)$

14. The function $f(x) = (9 - x^2)^2$ increases in:

[Kerala PET-2008]

(a) $(-3, 0) \cup (3, \infty)$
(b) $(-\infty, -3) \cup (3, \infty)$
(c) $(-\infty, -3) \cup (0, 3)$
(d) $(-\infty, 3)$

15. If the mean value theorem is $f(b) - f(a) = (b - a)f'(c)$. Then for the function $x^2 - 2x + 3$ in $\left[1, \frac{3}{2}\right]$, the value of c is:

[MP PET-2008]

(a) 6/5 (b) 5/4
(c) 4/3 (d) 7/6

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**6.** ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (b) Step 1: If $f(x)$ satisfies mean value theorem in $[a, b]$, then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Step 2: $f(x) = x + \dfrac{1}{x}, a = 1, b = 3$

$$1 - \frac{1}{c^2} = \frac{\frac{10}{3} - 2}{3 - 1} \quad 1 - \frac{1}{c^2} = \frac{2}{3} \Rightarrow c = \sqrt{3}$$

2. (c) Step 1: For increasing function $f'(x) > 0$ and for decreasing function $f'(x) < 0$

Step 2: $f(x) = \sin 2x \quad f'(x) = 2\cos 2x$

Here $f'(x) > 0$ if $2x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

or $\quad x \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

and $f'(x) < 0$

if $\quad 2x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right), x \in \left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$

$\therefore$ from options (c) is answer.

4. (a) Step 1: A function is neither increasing nor decreasing if $f'(x)$ contain both +ve and -ve sin in given interval.

Step 2: In interval $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

(a) $f(x) = \operatorname{cosec} x, f'(x) = -\operatorname{cosec} x \cot x$

$f'(x) = \dfrac{-\cos x}{\sin^2 x} > 0$ in given interval.

But $f(x) = \operatorname{cosec} x$ is discontinuous in $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$ or at π, $\therefore$ It is non monotonic.

11. (b) Step 1: If $f(x)$ is monotonically increasing in $[a, b]$, then $f'(x) \geq 0 \ \forall \ x \in [a, b]$

Step 2: $f'(x) = 2x + a \quad f'(1) \geq 0$

$\Rightarrow \quad 2 + a \geq 0$ or $a \geq -2$ ....(i)

$f'(2) \geq 0 \Rightarrow 4 + a \geq 0$

$a \geq -4$ ....(ii)

$\therefore$ combining (i) and (ii) $a \geq -2$

$\therefore$ min. value of $a = -2$

LECTURE

2

Rolle's Theorem and Rate of Change of Quantities

BASIC CONCEPTS

ROLLE'S THEOREM

Let $f(x)$ be a function of x subject to the following conditions:

(i) $f(x)$ is a continous function of x in the closed interval of $a \le x \le b$.

(ii) $f(x)$ is differentiable for every point in the open interval (a, b)

(iii) $f(a) = f(b)$

Then there exists at least one value of open interval (a, b) say c such that $f'(c) = 0$.

NOTE

Roll's Theorem is a special case of LMVT since $f(a) = f(b)$.

DERIVATIVES AS THE RATE OF CHANGE

1. $\frac{\delta y}{\delta x} = \frac{dy}{dx}$ i.e., $\delta y = \frac{dy}{dx}\delta x$ (approximately) i.e., corresponding to a small error δx in the value of x, the approximate error in the value of y is δy.

NOTES

1. δx is known as absolute error in x.
2. $\delta x/x$ is known as relative error in x.
3. $\frac{\delta x}{x} \times 100$ is known as percentage error in x.
4. δx and δy are known as small increments in x and y respectively.
5. $f(a + \delta x) = f(a) + \delta x f'(a)$ (approximately)
6. An error (δx or δy) is taken positive when the measured value is greater than the actual value and negative when it is less.

2. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\text{rate of change of } y}{\text{rate of change of } x}$ (The rate of change of one variable can be obtained when the rate of change of the other is given by $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$)

3. Velocity of a particle at P at time t: $V = \frac{ds}{dt}$
i.e., rate of displacement of the particle.

NOTE

ds/dt is positive or negative according as the particle is moving away from origin or towards origin.

4. Acceleration of a particle at P at time t:
Acceleration $= f = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}$

NOTE

The rate of increasing velocity is called acceleration and the rate of decreasing velocity is called retardation.

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. A particle is moving in a straight line such that its distance from a fixed point in straight line at any instant t is given by $S = 3t^n$. If at time t, velocity is v and acceleration f, then show that $v^2 = \dfrac{nfs}{n-1}$.

Solution

Given $S = 3\,t^n$ .... (i)

$$v = \frac{ds}{dt} = 3nt^{n-1} \quad \text{........... (ii)}$$

and $\quad f = 3n\,(n-1)\,t^{n-2}$ .... (iii)

Now R.H.S. $= \dfrac{nfs}{n-1} = \dfrac{n.3n(n-1)t^{n-2}3t^n}{(n-1)}$

$= 9n^2\,.\,t^{2n-2}$

$= (3n\,t^{n-1})^2 = v^2 =$ L.H.S. **Proved.**

2. A particle is moving in a straight line. Its displacement s at time t is given by the law $S = 3t - t^3$. Find the velocity and acceleration of the particle after half second. When it will come into rest?

Solution

Given $S = 3t - t^3 \;\therefore\; \dfrac{ds}{dt} = 3 - 3t^2$ .... (i)

Velocity of particle after $\dfrac{1}{2}$ second is

$$\frac{ds}{dt} = 3 - 3\left(\frac{1}{2}\right)^2 = 3 - \frac{3}{4} = 2.25 \text{ units/sec.}$$

Again differentiating equation (i) with respect to t, we get

$$\frac{d^2s}{dt^2} = 0 - 6t = -6t$$

Acceleration of particle after 1/2 second is

$$\frac{d^2s}{dt^2} = -6\times\frac{1}{2} = -3 \text{ units/sec}^2$$

Let the particle comes to rest after t second, then $\dfrac{ds}{dt} = 0$, from equation (i), we get $3 - 3t^2$

$= 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

neglecting negative sign, the required time is 1 second.

3. A particle according to the formula $S = 10 + 20\,t - t^2$, starts moving from a distance of 10 metres from a mark and moving forward in a straight line. How far from the mark does it go before it starts moving in the opposite direction?

Solution

Given $\;S = 10 + 20t - t^2$ .... (i)

$$v = \frac{ds}{dt} = 20 - 2t \quad \text{............ (ii)}$$

and $\dfrac{d^2s}{dt^2} = -2$ .... (iii)

when particle starts moving in opposite direction, at that time its velocity is zero, then from equation (ii) we get $20 - 2t = 0$

$\Rightarrow \quad t = 10$ sec

Therefore, particle starts moving in opposite direction after time $t = 10$ seconds with acceleration $= -2$

Putting $t = 10$ sec in equation (i), we get

$$S = 10 + 20 \times 10 - (10)^2$$
$$= 10 + 200 - 100 = 110 \text{ metres.}$$

4. A particle is moving on X-axis. The position of particle x and time t are related by $x = ae^{\alpha t} + be^{-\alpha t}$, where a, b and α are constants. Show that at any time, its acceleration is proportional to the displacement.

Solution

Given $\;x = ae^{\alpha t} + be^{-\alpha t}$ .... (i)

Differentiating equation (i), with respect to t, we get

Velocity $= \dfrac{dx}{dt} = ae^{\alpha t}.be^{-\alpha t}(-\alpha)$

$= \alpha(ae^{\alpha t} - be^{-\alpha t})$

and acceleration

$= \dfrac{d^2x}{dt^2} = \alpha[ae^{\alpha t}.\alpha - be^{-\alpha t}.(-\alpha)]$

$= \alpha^2[ae^{\alpha t} + be^{-\alpha t}]$

$\therefore \quad$ Acceleration $= \alpha^2 x$ [from equation (i)]

Proved.

5. If the distance S is given by $S = at^2 + bt + c$, where t is time and a, b, c are constants. Prove that $4a(s - c) = v^2 - b^2$, where v denotes the velocity. ***[CBSE-2000]***

Solution

Given $S = at^2 + bt + c$ .... (i)

differentiating equation (i) with respect to t, we get

Velocity $(v) = \frac{ds}{dt} = 2at + b$

Now R.H.S. $= v^2 - b^2$

$= (2at + b)^2 - b^2 = 4a^2 t^2 + b^2 + 4abt - b^2$

$= 4a[at^2 + bt] = 4a\,[at^2 + bt + c - c]$

$= 4a(s - c) =$ L.H.S. **Proved**.

6. The length x of a rectangle is decreasing at the rate of 2 cm/sec and the width y is increasing at the rate of 2 cm/sec. When x =12 cm and y = 5 cm. Find the rate of change of:
(i) the perimeter (ii) the area of rectangle. ***[NCERT-Book]***

Solution

Given $\frac{dx}{dt} = -2$ cm/s

and $\frac{dy}{dt} = 2$ cm/s

(i) Let p be the perimeter of the rectangle

Then, $p = 2x + 2y$

$$\Rightarrow \frac{dp}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt}$$

$$= 2\,(-2) + 2(2) = 0$$

Rate of change of perimeter $= \frac{dp}{dt} = 0$ cm/s

(ii) Let A be the area of the rectangle

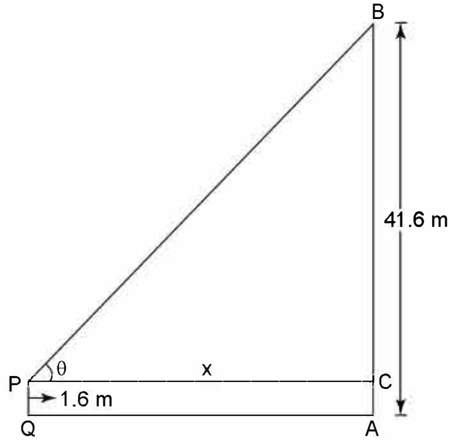
Then $A = xy \Rightarrow \frac{dA}{dt} = x.\frac{dy}{dt} + y\frac{dx}{dt} = 2x - 2y$

Rate of change of area when (x = 12 and y = 5)

$$\Rightarrow \frac{dA}{dt} = 2\times 12 - 2\times 5 = 14 \text{ cm}^2/\text{s}.$$

7. A man is moving away from a tower 41.6 m high at the rate of 2m/sec. Find the rate at which the angle of elevation of the top of tower is changing, when he is at a distance of 30 m from the foot of the tower. Assume that the eye level of the man is 1.6 m from the ground.

Solution



Let AB be the tower. Let at any time t, the man be at a distance of x metres from the tower AB and let θ be the angle of elevation at that time. Then

$$\tan\theta = \frac{BC}{PC} \Rightarrow \tan\theta = \frac{40}{x} \Rightarrow x = 40\cot\theta \quad \text{.......... (i)}$$

$$\Rightarrow \frac{dx}{dt} = -40\,\text{cosec}^2\,\theta.\frac{d\theta}{dt}$$

we are given that $\frac{dx}{dt} = 2$ m/sec

$$\therefore \quad 2 = -40\,\text{cosec}^2\theta.\frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = -\frac{1}{20\,\text{cosec}^2\,\theta} \quad \text{.............. (ii)}$$

when $x = 30$ we have from equation (i)

$$\cot\theta = \frac{30}{40} = \frac{3}{4}$$

$$\therefore \quad \text{cosec}^2\theta = 1 + \cot^2\theta = 1 + \frac{9}{16} = \frac{25}{16}$$

Substituting $\text{cosec}^2\,\theta = \frac{25}{16}$ in equation (ii),

we get $\frac{d\theta}{dt} = -\frac{1}{20\times\frac{25}{16}} = -\frac{4}{125}$ radians/sec.

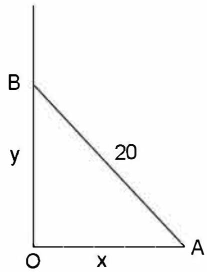
Thus the angle of elevation of the top of tower is changing at the rate of 4/125 radians/sec.

8. A ladder 20 ft long has one end on the ground and the other end in contact with a vertical wall. The lower end slips along the ground. Show that when the lower end of the ladder is 16 ft away from the wall, upper end is moving 4/3 times as fast as the lower end.

Solution

Let $OA = x$ and $OB = y$ at time t

Now in right angle ΔAOB



$$x^2 + y^2 = (20)^2 \qquad \text{....... (i)}$$

Put $x = 16$ in equation (i) we get $(16)^2 + y^2 = (20)^2$

$\Rightarrow \quad y^2 = 400 - 256 = 144 \Rightarrow y = 12$

Differentiating equation (i) with respect to t we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

Rate of change of upper end at (16, 12)

$$\Rightarrow \quad \left(\frac{dy}{dt}\right)_{(16,12)} = -\frac{16}{12}\frac{dx}{dt} = \frac{-4}{3}$$

(Rate of change of lower end)

Proved.

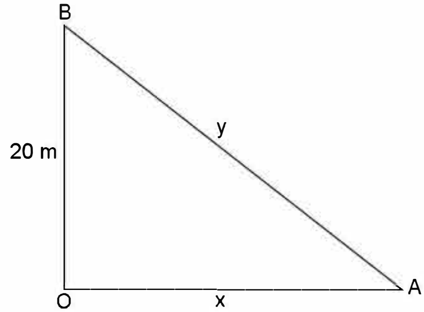
9. A man on a wharf 20 m above the water level, pulls a rope to which a boat is attached at the rate of 4m/s. At what rate is the boat approaching the shore, when there is still 25 m of rope out?

Solution

Let x be the horizontal distance and y be the length of rope out at time t. Given

$$\frac{dy}{dt} = -4\ m/s \qquad \text{............ (i)}$$

[$\because$ Length of rope out is decreasing]



Now in right angle Δ OAB $y^2 = x^2 + (20)^2$

$\Rightarrow \quad y^2 = x^2 + 400 \qquad$ .... (ii)

Differentiating both sides with respect to t

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{y}{x}\cdot\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{-4y}{x} \qquad \text{........ (iii)}$$

Put $y = 25$ in equation (ii)

we get $\quad x^2 = 625 - 400$

$\Rightarrow \quad x^2 = 225 \Rightarrow x = 15$

Rate of change of horizontal distance of boat

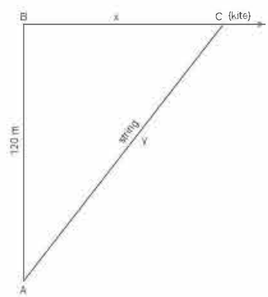
$$\text{at } (15, 25) = \left[\frac{dx}{dt}\right]_{(15,25)}$$

$$= \frac{-4 \times 25}{15} = \frac{-20}{3}\ m/\text{sec.}$$

10. A kite is 120 m high and 130 m of string is out. If the kite is moving away horizontally at the rate of 52 m/sec. Find the rate at which the string is being out.

Solution

We have $y^2 = x^2 + (120)^2$



$$\Rightarrow \quad 2y\frac{dy}{dt} = 2x.\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = 52\frac{x}{y} \left[\because \frac{dx}{dt} = 52\right]$$

Putting $y = 130$ in $y^2 = x^2 + (120)^2$

we get $x = 50$

$$\therefore \quad \frac{dy}{dt} = \frac{52 \times 50}{130} = 20 \text{ m/sec}$$

11. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the approximate change in y? ***[HPB-93C; PSB-95; CBSE-99C, 2002; NCERT-Book]***

Solution

Let $x = 2$, $x + \Delta x = 1.99$ Then $\Delta x = 1.99 - 2 = -0.01$

Let $dx = \Delta x = -0.01$ we have $y = x^4 - 10$

$$\Rightarrow \quad \frac{dy}{dx} = 4x^3 \Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = 4 \times (2)^3 = 32$$

Now $dy = \frac{dy}{dx}.dx$

$\Rightarrow \quad dy = 32\,(-0.01) = -0.32$

$\Rightarrow \quad \Delta y = -0.32$ approximately $[\because \Delta y \cong dy]$

when $x = 2$, we have $y = (2)^4 - 10 = 16 - 10 = 6$

So, changed value of $y = y + \Delta y$

$= 6 + (-4.32) = 5.68$

$\therefore \quad y$ changes from 6 to 5.68

12. A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate if the radius of the plate before heating is 10 cm. ***[NCERT-Book]***

Solution

Let at any time, x be the radius and y be the area of the plate

Then $y = \pi x^2$

Let Δx be the change in the radius and let Δy be the corresponding change in the area of the plate. Then

$$\frac{\Delta x}{x} \times 100 = 2 \text{ (given) when } x = 10,$$

$$\frac{\Delta x}{x} \times 100 = 2 \Rightarrow \frac{\Delta x}{10} \times 100 = 2 \Rightarrow \Delta x = \frac{2}{10}$$

$$\Rightarrow \quad dx = \frac{2}{10} \quad \text{......... (i)} \quad [\because dx \cong \Delta x]$$

Now, $y = \pi x^2 \Rightarrow \frac{dy}{dx} = 2\pi x$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=10} = 20\pi$$

$$\therefore \quad dy = \frac{dy}{dx} dx$$

$$\Rightarrow \quad dy = 20\pi \times \frac{2}{10} = 4\pi$$

$\Rightarrow \quad \Delta y = 4\pi$

$[\because dy \cong \Delta y]$

Hence, the approximate change in the area of the plate is 4π.

13. Calculate the approximate value of $\log_{10}(4.04)$, it is being given that $\log_{10} 4 = 0.6021$ and $\log_{10} e = 0.4343$.

Solution

Let $\quad f(x) = \log_{10} 4.04 = \frac{\log 4.04}{\log 10}$

$f(x + h) = \log_{10}(x + h)$

$$\Rightarrow \quad f'(x) = \frac{1}{\log 10}.\frac{1}{x} = \frac{1}{x}\log_{10} e$$

Taking $x = 4$ and $h = 0.4$

we know that $f(x + h) = h f'(x) + f(x)$

$$\Rightarrow \quad \log_{10}(x + h) = h\left(\frac{1}{x}\log_{10} e\right) + \log_{10} 4.04$$

$$\Rightarrow \quad \log_{10}(4 + 0.04) = \frac{.04}{4}(0.4343) + 0.6021$$

$$\Rightarrow \quad \log_{10}(4.04) = .004343 + 0.6021$$

$= .606443$

14. The radius of a sphere is measured as 4 cm with an error possibly as large as 0.01 cm. Find the approximation to the greatest possible error and the percentage error in the computed volume.

Solution

Let r be the radius and v be the volume of the sphere

Given $\Delta r = \pm 0.01$ .... (i)

$$\text{Now, } v = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4)^3 = \frac{256\pi}{3} cm^3 \quad \text{... (ii)}$$

$$\Rightarrow \quad \frac{dv}{dr} = \frac{4}{3}(3\pi r^2) = 4\pi r^2$$

Greatest possible error in v

$$= \Delta v = \left(\frac{dv}{dr}\right)\Delta r = (4\pi r^2)\Delta r$$

$$= [4\pi\,(4)^2]\,(\pm 0.01) = \pm 0.64\pi \text{ cm}^3$$

Relative error in $v = \dfrac{\Delta v}{v} = \dfrac{\pm 0.64\pi}{\frac{256\pi}{3}} = \pm 0.0075$

Percentage error in $v = \dfrac{\Delta v}{v} \times 100 = \pm 0.75\%$

15. Calculate the approximate values of cos 61°, it is being given that 1° = 0.01745 radians.

Solution

Let $f(x) = \cos x$, $f(x + h) = \cos (x + h)$, $f'(x) = -\sin x$

Taking $x = 60°$ and $h = 1° = 0.01745$ radians

we know $f(x + h) = h f'(x) + f(x)$

$\Rightarrow \quad \cos(x + h) = h(-\sin x) + \cos x$

$\Rightarrow \quad \cos(60° + 1°) = (0.01745)(-\sin 60°) + \cos 60°$

$\Rightarrow \quad \cos 61° = (0.01745)\left(\dfrac{-\sqrt{3}}{2}\right) + \dfrac{1}{2}$

$\Rightarrow \quad \cos 61° = \dfrac{(0.01745)(-1.73) + 1}{2}$

$\Rightarrow \quad \cos 61° = \dfrac{-0.3022 + 1}{2}$

$= 0.4849.$

16. Apply Rolle's Theorem to find the position of zeroes of $f'(x)$ where $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$.

Solution

f being polynomial is derivable and therefore continuous $\forall\ x \in R$ and $f(1) = f(2) = f(3) = f(4) = 0$

Thus f satisfies all conditions of Rolle's theorem in [1, 2], [2, 3], [3, 4].

Hence $\exists$ at least one c in each of the intervals (1, 2), (2, 3), (3, 4) where f' is zero.

So, f' has zeros between 1 and 2, between 2 and 3 and between 3 and 4.

17. Find the intervals in which the following functions are increasing or decreasing $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

[HSB-93, CBSE-84C]

Solution

We must have $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

$\Rightarrow \quad f'(x) = 4x^3 - 24x^2 + 44x - 24$

$\Rightarrow \quad f'(x) = 4\,(x^3 - 6x^2 + 11x - 6)$

$\Rightarrow \quad f'(x) = 4(x - 1)(x^2 - 5x + 6)$

$\Rightarrow \quad f'(x) = 4\,(x - 1)(x - 2)(x - 3)$

For $f(x)$ to be increasing, we must have $f'(x) > 0$

$\Rightarrow \quad 4\,(x - 1)(x - 2)(x - 3) > 0$

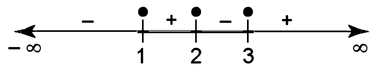
$\Rightarrow \quad (x - 1)(x - 2)(x - 3) > 0$

$[\because 4 > 0]$

$\Rightarrow \quad 1 < x < 2$ or $3 < x < \infty$

$\Rightarrow \quad x \in (1, 2) \cup (3, \infty)$

So, $f(x)$ is increasing on $(1, 2) \cup (3, \infty)$



For $f(x)$ to be decreasing, we must have $f'(x) < 0$

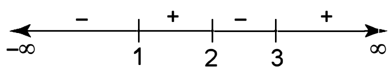
$\Rightarrow \quad 4(x - 1)(x - 2)(x - 3) < 0$

$\Rightarrow \quad (x - 1)(x - 2)(x - 3) < 0$

$[\because 4 > 0]$

$\Rightarrow \quad 2 < x < 3$ or $x < 1$

$\Rightarrow \quad x \in (2, 3) \cup (-\infty, 1)$



So, $f(x)$ is decreasing on $(2, 3) \cup (-\infty, 1)$

18. Sand is poured from a pipe at the rate of 12 cm³/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one sixth of the radius of base. Find how fast the height of the sand cone is increasing when the height is 4 cm.

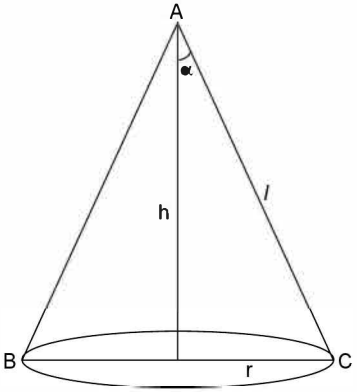
[HPSB-98; NCERT-Book]

Solution

Let r be the radius and h be the height of the sand-cone at any time t sec. Then $h = \dfrac{r}{6}$ .... (i)

Let V be the volume of cone, then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6h)^2 h = 12\pi h^3.$$



$$\therefore \quad \frac{dv}{dt} = 12\pi \times 3h^2 . \frac{dh}{dt} \Rightarrow \frac{dv}{dt} = 36\pi h^2 \frac{dh}{dt}$$

.... (ii)

Given that, $\frac{dv}{dt}$ = rate at which sand is pouring = 12 cm^3/sec.

$\therefore$ from equation (ii), we get

$$12 = 36\pi h^2 . \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36\pi h^2} = \frac{1}{3\pi h^2}$$

but at h = 4 cm, the rate of increasing of height

$$\left(\frac{dh}{dt}\right)_{h=4} = \frac{1}{3\pi(4)^2} = \frac{1}{48\pi} \text{ cm/sec.}$$

19. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 3cm/sec. How fast is its height on the wall decreasing. When the foot of the ladder is 4 m away from the wall?

Solution

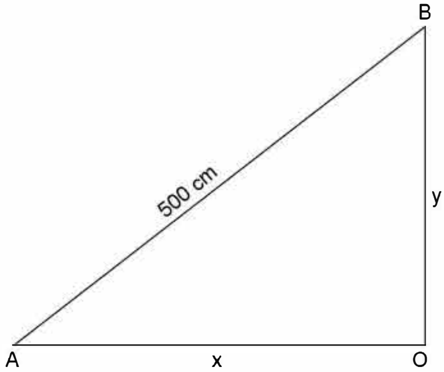
Let at any time t, the bottom of the ladder be at a distance x cm from the wall and the height of the wall be y cm.

Then $x^2 + y^2 = (500)^2$ .... (i)

Differentiating with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{......... (ii)}$$

Given $\frac{dx}{dt} = 3$ cm/sec



from equation (ii) $2y\frac{dy}{dt} = -2x\frac{dx}{dt}$

$$\Rightarrow \quad y\frac{dy}{dt} = -x \times 3$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-3x}{y} \quad \text{....... (iii)}$$

when = 4m = 400 cm, from equation (i), we get

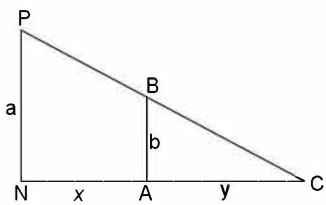
$$y^2 = 250000 - 160000$$

$\Rightarrow \quad y^2 = 90000 \Rightarrow y = 300$ cm

Putting x = 400 cm & y = 300 cm in equation (iii) we get $\frac{dy}{dt} = -\frac{3 \times 400}{300} = -4$ cm/sec.

20. A source of light is hung a metres directly above a straight horizontal path on which a boy b metres in height is walking. How fast is the shadow lengthening when he is walking away from the light at the rate of C metres/minute?

Solution



Let P be the source of light and AB is boy, we have

$$PN = a, AB = b$$

further, Let $NA = x$ and shadow $AC = y$ then $\frac{dx}{dt} = c$ metres/min

We have to find $\frac{dy}{dt}$, the rate of change of shadow.

Consider the similar triangles PNC and BAC

$$\frac{PN}{AB} = \frac{NC}{AC} = \frac{NA + AC}{AC} \Rightarrow \frac{a}{b} = \frac{x+y}{y}$$

$$\Rightarrow \quad y\frac{a}{b} = x + y$$

$$\Rightarrow \quad y\left(\frac{a}{b} - 1\right) = x \Rightarrow y = \frac{b}{a-b}x \quad \text{........ (i)}$$

Differentiating equation (i) with respect to t, we get

$$\frac{dy}{dt} = \frac{b}{a-b}.\frac{dx}{dt} = \frac{b}{a-b}.c = \frac{bc}{a-b} \text{ metres/min.}$$

21. An air force plane is ascending vertically at the rate of 100 km/hr. If the radius of the earth is r km. How fast is the area of the earth visible from the plane increasing at 3 minutes after it started ascending. (Visible area $A = \frac{2\pi r^2 h}{r+h}$, where h is the height of the plane above the earth) ***[NCERT Book]***

Solution

Rate of ascending the plane = 100 km/hr

Height of plane after 3 minutes $= \frac{100 \times 3}{60} = 5$ km

5 km visible area $A = \frac{2\pi r^2 h}{r+h}$ .... (i)

where h is the height of plane above the earth

and $\frac{dh}{dt} = 100$ km/hr

Differentiating equation (i) with respect to t, we get

$$\frac{dA}{dt} = 2\pi r^2\left[\frac{r+h-h}{(r+h)^2}\right]\frac{dh}{dt}$$

$$= \frac{2\pi r^3}{(r+h)^2}.\frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{2\pi r^3}{(r+5)^2}\times 100 = \frac{200\pi r^3}{(r+5)^2} \text{ km}^2\text{/hr}$$

22. The radius x of a cylinder is increasing at the rate of 2 ft/sec. and its height y is decreasing at the rate of 3 ft/sec. Find the rate of change of its volume when $x = 3$ ft, $y = 5$ ft. ***[NMOC-98]***

Solution

Given at any time t, $\frac{dx}{dt} = 2$ ft/sec

and $\frac{dy}{dt} = -3$ ft/sec.

If V be the volume at time t, we get $V = \pi x^2 y$ ...... (i)

Differentiating equation (i) with respect to t, we get

$$\frac{dV}{dt} = \pi\frac{d}{dt}(x^2 y) = \pi\left[y.2x\frac{dx}{dt} + x^2\frac{dy}{dt}\right]$$

$$\Rightarrow \quad \frac{dV}{dt} = \pi\left[2xy\frac{dx}{dt} + x^2\frac{dy}{dt}\right]$$

$$= \pi[2xy.2 + x^2(-3)] = \pi[4xy - 3x^2]$$

when $x = 3$ ft, $y = 5$ ft, then the rate of change of volume is

$\frac{dV}{dt} = \pi\,[4 \times 3 \times 5 - 3 \times 3^2] = \pi[60 - 27] = 33\pi$ ft³/sec.

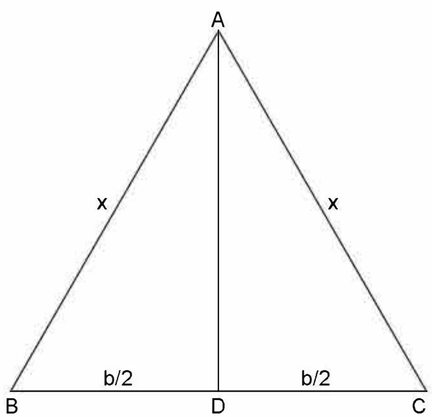
23. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/sec. How fast is the area decreasing when the two equal sides are equal to the base?

Solution

Let at any time t, the length of each equal side be x cm and area of triangle be A. Then

$$A = \frac{1}{2}BC \times AD$$

$$\Rightarrow \quad A = \frac{1}{2}\times b \times \sqrt{x^2 - \frac{b^2}{4}}$$



$$\Rightarrow \quad A = \frac{b}{4}\sqrt{4x^2 - b^2}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{b}{4} \times \frac{1}{2\sqrt{4x^2 - b^2}}\frac{d}{dt}(4x^2 - b^2)$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{b \times 4}{8\sqrt{4x^2 - b^2}} \times 2x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{bx}{\sqrt{4x^2 - b^2}}\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{3bx}{\sqrt{4x^2 - b^2}} \quad \left[\because \frac{dx}{dt} = 3 \text{ cm/sec}\right]$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{x=b} = \frac{3b^2}{\sqrt{4b^2 - b^2}} = \sqrt{3}b \text{ cm}^2/\text{sec.}$$

24. The bottom of a rectangular swimming tank is 25 m × 40 m. Water is pumped into the tank at the rate of 500 m^3/h. Find the rate at which level of water in the tank is rising.

Solution

Let h be the height and v be the volume of the water at time t

Given $\frac{dv}{dt} = 500 \text{ m}^3/\text{h}$

$$\Rightarrow \quad \frac{d}{dt}(25 \times 40 \times h) = 500$$

$$\Rightarrow \quad 1000\frac{dh}{dt} = 500$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{2} \text{ m/h}$$

Hence water level is rising at the rate of $\frac{1}{2}$ m/h.

25. A hemisphere is constructed on a circular base. If radius of base is increasing at the rate of 0.5 cm/s, find the rate at which volume of hemisphere is increasing when radius is 10 cm.

Solution

Let r be the radius and V be the volume of hemisphere at time t. Given $\frac{dr}{dt} = 0.5$ cm/sec

Now $V = \frac{2}{3}\pi r^3 \Rightarrow \frac{dv}{dt} = \frac{2}{3}\pi \times 3r^2\frac{dr}{dt}$

$$\Rightarrow \quad \frac{dv}{dt} = 2\pi r^2 \times 0.5 = \pi r^2.$$

Rate of change of volume when r = 10 cm

$$\left(\frac{dv}{dt}\right)_{r=10\text{ cm}} = \pi \times 100 = 100\pi \text{ cc/sec}$$

26. Use differential to approximate $\sqrt{36.6}$:

Step 1: Let $y + \sqrt{y} = (x + \sqrt{x}) = \sqrt{36 + .6}$ clearly

$f(x) = y \ \sqrt{x}, x = 36, \sqrt{x} = .6$

Step 2: $\sqrt{y} = \frac{dy}{dx}\sqrt{x}$

$$= \frac{1}{2\sqrt{x}} \times .6 = \frac{1}{12} \times .6 = \frac{1}{20} = .05$$

Step 3: $y + \sqrt{y} = \sqrt{36} + .05 = 6.05$

Thus the approximate value of

$\sqrt{36.6}$ is 6.05.

27. Find the approximate Change in the volume V of a cube of side x meters caused by increasing the side by 2%.
$V = x^3$ (Note that).
Step: Clearly using the following formula
$\delta \upsilon = \frac{dv}{dx}\sqrt{x}$

We get $\sqrt{v} = 3x^2\sqrt{x} = 3 \times x^2 \times \left(\frac{2x}{100}\right)$

$$= \frac{6x^3}{100} = .006x^3m^3$$

as 2% of x is $\frac{2}{100} \times x = .02x°$
Thus the approximate change in volume is $0.06\,x^3$ m^3

28. If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume:
Solutions
Step 1: Note that volume of the sphere
$v = \frac{4}{3}\pi r^3$

∴ Formula for the approximate error is

$$\therefore \quad \sqrt{v} = \frac{dv}{dr} - \sqrt{r} = \frac{4}{3}(\pi)(3r^2)\sqrt{r}$$

$$= \pi r^2\sqrt{r} = 4\pi \times (9)^2 \times (.03)^2 = 9.72\ \pi \text{ cm}^3$$

= approximate error in calculating the volume:

[redacted]
TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. State Rolle's theorem *[MP-2007]*

2. Verify Rolle's Theorem for the following functions:

(i) $x(x-3)^2$ on $[0, 3]$ *[CBSE-94 C]*

(ii) $f(x) = (x-2)(x-3)(x-4)$ on $[2, 4]$ *[CBSE-94 C]*

(iii) $f(x) = e^x \cos x$ on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ *[PB-97 (S)]*

(iv) $f(x) = x^2 - 4x + 3$ on $[1, 3]$ *[CBSE-2007]*

3. Discuss the applicability of Rolle's theorem for the following functions. *[CBSE-99]*

(i) $f(x) = x^{2/3}$ on $[-1, 1]$

(ii) $f(x) = 3 + (x-2)^{2/3}$ on $[1, 3]$ *[HSB-1993]*

4. The Rolle's Theorem holds for the function $f(x) = x^3 + bx^2 + ax + 5$ on $[1, 3]$ with $c = 2 + \frac{1}{\sqrt{3}}$ find the values of a and b. *[NCERT-Book; CBSE-SP-2006]*

5. A particle is moving in a straight line. Its distance S in metres at time t seconds is given by the law $S = 5 + 2t + t^3$ find:

(i) velocity of the particle after $2\frac{1}{2}$ seconds.

(ii) acceleration after $3\frac{1}{2}$ seconds

(iii) Distance travelled by the particle in fourth second.

6. A particle starts from rest and moves according to the law $x = a \cos \omega t + b \sin \omega t$, where a, b, ω are constants. Show that acceleration at time t is proportional to x and is always opposed to x.

7. A particle is moving in a straight line. Its displacement s at time t is given by the law of motion $S = 2t^3 - 5t^2 + 4t - 3$ find

(i) The time when the acceleration wil become 14

(ii) Velocity and displacement at that time. *[MP-99]*

8. A particle is moving in a straight line such that its position x after t seconds is given by the law $x = 3t + \cos 3t$. Show that its velocity and acceleration become simultaneously zero. *[MP-2005, 2008]*

9. A particle is moving in a straight line. The distance x at time t is given by the law $x = 4t^2 + 2t$. Find the velocity and acceleration of the particle at time $t = \frac{1}{2}$ sec. *[CBSE-81; SP-2006; MP-2005]*

10. If a particle moves in a straight line according to the law $S = t^3 - 6t^2 - 15t$ find the time interval during which the velocity is negative and acceleration is positive.

11. A particle is thrown vertically upwards. The law of motion is $S = ut - 4.9t^2$. Find the initial velocity of the particle to reach the height of 20 metres. *[MP-98]*

12. A man standing on the 9.8 metre high pole throws a stone vertically upwards in the direction of the pole. The stone after reaching a height come back to the ground. If the equation of motion is given by $S = 19.6t - 4.9t^2$ where S is measured in metres and t in seconds, find what time is taken by the stone for the vertical motion.

13. The radius of a balloon is increasing at the rate of 10 cm per second. At what rate is the surface area of the balloon increasing when its radius is 15 cm. *[HSB-86; CBSE-94C; PSB-2002]*

14. The radius of the base of a certain cone is increasing at the rate of 3 cm/min and altitude is decreasing at the rate of 4 cm/min. Find the rate of change of total surface of cone when radius is 7 cm and altitude 24 cm.

15. Using differentials, find the approximate value of $(82)^{1/4}$ upto three places of decimal.

[CBSE-2005]

16. Height of a tank in the form of an inverted cone is 10 metres and radius of its circular base is 2 metres. The tank contains water and it is leaking through a hole at its vertex at the rate of 0.02 m^3/sec. Find the rate at which the water level changes and the rate at which the radius of water surface changes when height water level is 5 metres.

17. A man is walking at the rate of 8 km/hr. towards the foot of a tower of 60 metres high. At what rate is he approaching the top of the tower, when he is 80 metres away from the foot of the tower?

18. The edge of a variable cube is increasing at the rate of 10 cm/sec. How fast the volume of the cube is increasing when the edge is 5 cm long? ***[CBSE-95C; HSB-2002]***

19. A body moves along a horizontal line according to the law $S = t^3 - 9t^2 + 24t$

(i) When is S increasing and when decreasing?

(ii) When is V increasing and when decreasing?

(iii) When is the speed of the body increasing and when decreasing.

20. A stone is dropped into a quite lake and waves moves in circles at speed of 4 cm per second. At the instant when the radius of the circular wave is 10 cm. How fast is the enclosed area increasing? ***[MP-2008]***

21. A ladder 5 metres long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the ball, at the rate of 2 metres per second. How fast is its height on the wall decreasing when the foot of the ladder is 4 metres away from the wall?

[MP-98, NCERT BOOK]

22. A balloon which always remains spherical is being in flated by pumping in 900 cc of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm. Verify Rolle's Theorem for the following functions ***[CBSE-2003]***

ANSWERS

3. (i) not applicable.
(ii) not applicable

4. $a = 11$ and $b = -6$

5. (i) $\frac{83}{4}$ m/sec
(ii) 21 m/sec^2
(iii) 39 metres

7. (i) 2 sec
(ii) 8 units/sec, 1 unit

9. 6 units/sec, 8 units/sec^2

10. $2 < t < 5$

11. $14\sqrt{2}$ m/s.

12. $t = 2$ seconds.

13. $\frac{dA}{dt} = 1200\pi$ cm^2/sec.

14. 96π cm^2/min

15. 3.0093

16. $\frac{0.02}{\pi}$ m/sec.

17. 6.4 km/hr.

18. 750 cm^3/sec

19. (i) $t < 2$ or $t > 4$ and $2 < t < 4$
(ii) $t > 3$ and $t < 3$
(iii) $t > 4$, $2 < t < 3$ and $0 < t < 2$, $3 < t < 4$

20. 80π cm^2/sec

21. $\frac{-8}{3}$ m/s

22. $\frac{dr}{dt} = \frac{7}{22}$ cm/sec.

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. A particle is moving on a straight line, where its position s (in metre) is a function of time t (in seconds) given by $s = at^2 + bt + 6, t \geq 0$. It it is known that the particle comes to rest after 4 seconds at a distance of 16 metre from the starting position ($t = 0$), then the retardation in its motion is ***[MPPET-1993]***

(a) -1m/sec^2 (b) $\frac{5}{4}\text{ m/sec}^2$

(c) $-\frac{1}{2}\text{ m/sec}^2$ (d) $-\frac{5}{4}\text{ m/sec}^2$

Solution

(b) Given equation $s = at^2 + bt + 6$ ..... (i)
Differentiating with respect to time, we get velocity $(v) = 2at + b$ ..... (ii)

After 4 sec, $v = 0$ and distance $s = 16$ metres

$\therefore$ $0 = 2a \times 4 + b \Rightarrow 8a + b = 0$ ...... (iii)
and $16 = 16a + 4b + 6 \Rightarrow 16 = 16a + 4(-8a) + 6$

$\therefore$ $a = -\frac{5}{8}$

But retardation in its motion is, $2a = \frac{-5}{4}\text{ m/sec}^2$

$\therefore$ Retardation $= \frac{5}{4}\text{ m/s}^2$. (Retardation itself means – ve)

2. A ladder 5 m in length is resting against vertical wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 1.5 m/sec. The length of the highest point of the ladder when the foot of the ladder 4.0 m away from the wall decreases at the rate of ***[Krukshetra CEE-1996]***

(a) 2 m/sec (b) 3m/sec

(c) 2.5 m/sec (d) 1.5 m/sec

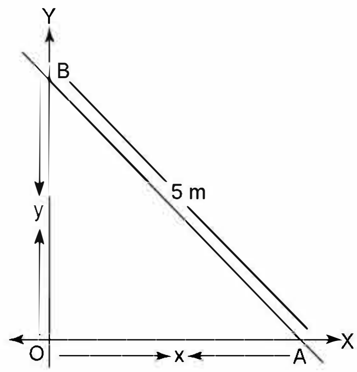
Solution

(a) According to figure $x^2 + y^2 = 25$ ..... (i)

Differentiate (i) with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{..... (ii)}$$

Here $x = 4$ and $\frac{dx}{dt} = 1.5$ From (i), $4^2 + y^2 = 25 \Rightarrow y = 3$



$\therefore$ From (ii), $2(4)(1.5) + 2(3)\frac{dy}{dt} = 0$ So,

$$\frac{dy}{dt} = -2\text{ m/sec}$$

Hence, length of the highest point decreases at the rate of 2m/sec.

3. If the rate of increase of area of a circle is not constant but the rate of increase of perimeter is constant, then the rate of increase of area varies ***[SCRA-1996]***

(a) As the square of the perimeter
(b) Inversely as the perimeter
(c) As the radius
(d) Inversely as the radius

Solution

(c) Perimeter $P = 2\pi r$

$$\frac{dp}{dt} = 2\pi.\frac{dr}{dt} \text{ and } A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \frac{dP}{dt} \times r \Rightarrow \frac{dA}{dt} \propto r$$

4. Moving along the x-axis are two points with $x = 10 + 6t$; $x = 3 + t^2$. The speed with which they are reaching from each other at the time of encounter is (x is in cm and t is in seconds) ***[MPPET-2003]***

(a) 16 cm/sec (b) 20 cm/sec

(c) 8 cm/sec (d) 12 cm/sec

Solution

(c) Time of encounter $10 + 6t = 3 + t^2$

$\Rightarrow$ $t^2 - 6t - 7 = 0, t = 7$ sec.

At $t = 7$ sec., $v_1 = \frac{d}{dt}(10 + 6t) = 6$ cm/sec

At $t = 7$ sec, $v_2 = \frac{d}{dt}(3 + t^2) = 2t = 2 \times 7 =$ 14 cm/sec

$\therefore$ Resultant velocity $= v_2 - v_1 = 14 - 6 =$ 8 cm/sec.

5. The position of a point in time 't' is given by $x = a + bt - ct^2, y = at + bt^2$. Its acceleration at time 't' is **[MPPET-2003]**

(a) $b - c$ (b) $b + c$
(c) $2b - 2c$ (d) $2\sqrt{b^2 + c^2}$

Solution

(d) Acceleration in direction of x – axis $= \frac{d^2x}{dt^2} = -2c$ and acceleration in direction of y – axis $= \frac{d^2y}{dt^2} = 2b$

Resultant acceleration is

$= \sqrt{(-2c)^2 + (2b)^2} = 2\sqrt{b^2 + c^2}$

6. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is **[AIEEE-2004]**

(a) $\left(\frac{9}{8}, \frac{9}{2}\right)$ (b) $(2, -4)$
(c) $\left(\frac{-9}{8}, \frac{9}{2}\right)$ (d) $(2, 4)$

Solution

(a) $y^2 = 18x$ Differentiate both sides with respect to t

$$2y\left(\frac{dy}{dt}\right) = 18\left(\frac{dx}{dt}\right)$$

$$\Rightarrow \quad 2y\left(2\frac{dx}{dt}\right) = 18\left(\frac{dx}{dt}\right), \left(\because \frac{dy}{dt} = 2\frac{dx}{dt}\right)$$

$\therefore$ $4y = 18$ or $y = \frac{9}{2}$ and $x = \frac{y^2}{18} = \frac{9}{8}$

Hence the required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$

7. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm³/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is **[AIEEE-2005]**

(a) $\frac{1}{54\pi}$ cm/min (b) $\frac{5}{6\pi}$ cm/min
(c) $\frac{1}{36\pi}$ cm/min (d) $\frac{1}{18\pi}$ cm/min

Solution

(d) $V = \frac{4}{3}\pi(x + 10)^3$ where x is thickness of ice.

$\therefore$ $\frac{dV}{dt} = 4\pi(10 + x)^2\frac{dx}{dt}$

$\therefore$ At $x = 5, \left(\frac{dx}{dt}\right) = \frac{1}{18\pi}$ cm/min.

8. A spherical balloon is being inflated at the rate of 35 cc/min. The rate of increase of the surface area of the balloon when its diameter is 14 cm is **[Karnataka CET-2005]**

(a) 7 cm²/min (b) 10 cm²/min
(c) 17.5 cm²/min (d) 28 cm²/min

Solution

(b) Volume $= V = \frac{4}{3}\pi r^3$

$\Rightarrow$ $\frac{dV}{dt} = 4\pi r^2.\frac{dr}{dt}$ at $r = 7$ cm

35 cm/min $= 4\pi\,(7)^2\,\frac{dr}{dt}$

$\Rightarrow$ $\frac{dr}{dt} = \frac{5}{28\pi}$

Surface area, $S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r\frac{dr}{dt} = \frac{8\pi.7.5}{28\pi} = 10$ cm²/min.

9. If a particle moves such that the displacement is proportional to the square of the velocity acquired, then its acceleration is **[Kerala (Engg.)-2005]**

(a) Proportion to s^2
(b) Proportional to $1/s^2$
(c) Proportional to s
(d) A constant

Solution

(d) If diplacement $\propto$ (velocity)² $s \propto v^2$

$\Rightarrow$ $v^2 = 2as$.

Hence a is constant.

10. The radius of a cylinder is increasing at the rate of 3m/sec and its altitude is decreasing at the rate of 4m/sec. The rate of change of volume when radius is 4 metres and altitude is 6 metres is

[Kerala (Engg.)-2005]

(a) 80π m³/sec
(b) 144π m³/sec
(c) 80 m³/sec
(d) 64 m³/sec

Solution

(a) $V = \pi r^2 h; \dfrac{dV}{dt} = \pi\left[2r\dfrac{dr}{dt}h + r^2\dfrac{dh}{dt}\right]$

$\dfrac{dV}{dt} = \pi\,[2(4)(3)(6) + (4)^2(-4)] = \pi$

$[144 - 64] = 80\pi$ m²/sec

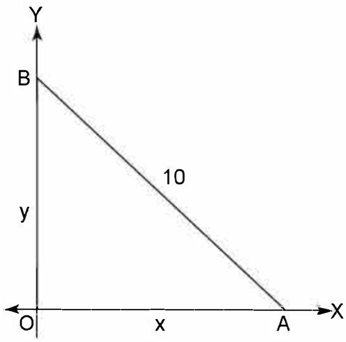
11. A ladder 10 m long rests against a vertical wall with the lower end on the horizontal ground. The lower end of the ladder is pulled along the ground away from the wall at the rate of 3 cm/sec. The height of the upper end while it is descending at the rate of 4 cm/sec is

[Kerala (Engg.)-2005]

(a) $4\sqrt{3}$ m (b) 6 m
(c) $5\sqrt{2}$ m (d) 8 m

Solution

(b) We have $x^2 + y^2 = 10^2$



$\Rightarrow \quad 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$\Rightarrow \quad x \,.\, 3 + y \,.\, (-4) = 0$

$x = \dfrac{4}{3}y$. Thus $\left(\dfrac{4}{3}y\right)^2 + y^2 = 10^2$

$\Rightarrow \quad y = 6$ m.

12. Consider the function $f(x) = e^{-2x} \sin 2x$ over the interval $\left(0, \dfrac{\pi}{2}\right)$. A real number $c \in \left(0, \dfrac{\pi}{2}\right)$, as guaranteed by Rolle's theorem, such that $f'(c) = 0$ is ***[AMU-1999]***

(a) $\pi/8$ (b) $\pi/6$
(c) $\pi/4$ (d) $\pi/3$

Solution

(a) $f(x) = e^{-2x} \sin 2x$

$\Rightarrow \quad f'(x) = 2e^{-2x}(\cos 2x - \sin 2x)$

Now, $f'(c) = 0$

$\Rightarrow \quad \cos 2c - \sin 2c = 0$

$\Rightarrow \quad \tan 2c = 1 \Rightarrow c = \dfrac{\pi}{8}$.

13. Let $f(x) = \begin{Bmatrix} x^\alpha \ln x, & x > 0 \\ 0, & x = 0 \end{Bmatrix}$, Rolle's theorem is applicable to f for $x \in [0, 1]$, if α =

[IIT Screening-2004]

(a) –2 (b) –1
(c) 0 (d) 1/2

Solution

(d) For Rolle's theorem to be applicable to f, for $x \in [0, 1]$, we should have

(i) $f(1) = f(0)$,
(ii) f is continuous for $x \in [0, 1]$ and f is differentiable for $x \in (0, 1)$

Form (i), $f(1) = 0$, which is true

Form (ii), $0 = f(0) = f(0_+) = \lim\limits_{x \to 0_+} x^\alpha \ln x$

Which is true only for positive values of α, thus (d) is correct.

14. If a path of a moving point is the curve $x = at$, $y = b$ sin at Then its acceleration at any instant

(a) Is constant
(b) Varies as the distance from the axis of x
(c) Varies as the distance from axis of y
(d) Varies as the distance of the point from the origin

Solution

(c) $\dfrac{dx}{dt} = V_x = a \Rightarrow \dfrac{d^2x}{dt^2} = 0 = a_x$, a_x is acceleration along x-axis, $\dfrac{d^2x}{dt^2} = -ba^2$ sin at

$\Rightarrow ay = -a^2y$

Hence, a_y changes as y changes.

15. The length of the normal at point 't' of the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is **[REPET-2001]**

(a) $a \sin t$
(b) $2a \sin 3(t/2) \sec(t/2)$
(c) $2a \sin(t/2) \tan(t/2)$
(d) $2a \sin(t/2)$

Solution

(c) $x = a(t + \sin t), y = a(1 - \cos t)$,

$$\frac{dx}{dt} = \frac{dy/dt}{dx/at} = \frac{a(\sin t)}{a(1+\cos t)} = \tan\frac{t}{2}$$

$$\text{Length of normal } = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= a(1-\cos t)\sqrt{1+\tan^2\left(\frac{t}{2}\right)}$$

$= a(1 - \cos t) \sec(t/2)$

$= 2a \sin^2 (t/2) \sec(t/2)$

$= 2a \sin (t/2) \tan (t/2)$

16. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies the conditions of Rolle's theorem for the interval $[1, 3]$ and $f'(2+1/\sqrt{3}) = 0$, then the values of a and b are respectively **[UPSEAT-2001]**

(a) 1, –6 (b) –2, 1
(c) –1, 1/2 (d) –1, 6

Solution

(a) Step 1: from conditions (3) of Rolle's theorem $f(1) = f(3)$

$\therefore$ $f(1) = f(3)$

$\Rightarrow$ $a + b + 5 = 27a + 9b + 27$

$\Rightarrow$ $26a + 8b + 22 = 0$ ....(1)

$$\text{Step 2: } f'\left(2+1\frac{1}{\sqrt{3}}\right) = 3ax^2 + 2bx + 11$$

$$\text{at} \quad x = 2+\frac{1}{\sqrt{3}} = 0$$

$$3a\left(4+\frac{1}{\sqrt{3}}+\frac{2\times 2}{\sqrt{3}}\right)+2b\left(2+\frac{1}{\sqrt{3}}\right)+11 = 0$$

$$12a+\frac{12a}{\sqrt{3}}+4b+\frac{2b}{\sqrt{3}}+11 = 0$$

$$\Rightarrow \quad 26a+\frac{24a}{\sqrt{3}}+8b+\frac{4b}{\sqrt{3}}+22 = 0 \quad \text{...... (2)}$$

from (1) and (2)

$$\frac{24a}{\sqrt{3}}+\frac{4b}{\sqrt{3}} = 0 \Rightarrow b = -6a \quad \text{......... (3)}$$

from equation (1) & (3) we get

$a = 1, b = -6$

Quiker Method: (Alternative)

Given that

$$f'\left(2+\frac{1}{\sqrt{3}}\right) = 3ax^2+2bx+11\bigg|_{x=2+\frac{1}{\sqrt{3}}}$$

$$= \frac{24a}{\sqrt{3}}+\frac{4b}{\sqrt{3}} = 0$$

Which is satisfied the option (a) only

17. Find the percentage error in calculating the volume of a cubical box if an error of 1% is made in measuring the length of edges of the cube.

$v = x^3$

(a) 1% (b) 2%
(c) 3% (d) 4%

Solution

(c)

$$\log\ v = 3\log x,\ \frac{1}{v}\delta v = \frac{3}{x}\delta x$$

$$\frac{1}{v}\delta v\times 100 = \frac{3}{x}\delta x\times 100$$

$$= 3\left(\frac{\delta x}{x}\times 100\right) = 3\times 1;\ \frac{\delta v}{v}\times 100 = 3$$

so there is 3% error in calculating the volume of the cube.

18. A stone thrown vertically upward satisfies the equation s = $64t - 16t^2$, where s is in meter and t is in second. What is the time required to reach the maximum hight? **[NDA-2009]**

(a) 1s (b) 2s
(c) 3s (d) 4s

Solution

(b) $\because s = 64t - 16t^2$

$\therefore$ On differentiating w.r.t. t, we get

$$\frac{ds}{dt} = 64 - 32t$$

Put $\frac{ds}{dt} = 0$ for maximum height

$64 - 32t = 0 \Rightarrow t = 2 \Rightarrow t = 2$

$$\therefore \left(\frac{d^2s}{dt^2}\right)_{t=2} < 0$$

Hence, required time = $2s$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. For which interval the function $F(x) = \dfrac{x^2 - 3x}{x - 1}$ satisfies all the conditions of Rolle's Theorem ***[MPPET-93]***

(a) $[0, 3]$ (b) $[-3, 0]$
(c) $[1.5, 3]$ (d) For no-interval

2. Rolle's theorem is true for the function $f(x) = x^2 - 4$ in the interval

(a) $[-2, 0]$ (b) $[-2, 2]$
(c) $[0, 1/2]$ (d) $[0, 2]$

3. A body moves according to the formula $V = 1 + t^2$ where V is the velocity at time t. The acceleration after 3 sec. will be (V gm cm/sec) ***[PET-1988]***

(a) 24 cm/sec^2 (b) 12 cm/sec^2
(c) 6 cm/sec^2 (d) None of these

4. The law of motion in a straight line is $S = \dfrac{1}{2}vt$, the acceleration is ***[MPPET-1991]***

(a) Constant (b) Proportional to t
(c) Proportional to v (d) Proportional to S

5. A particle moves in a straight line in such a way that its velocity at any point is given by $V^2 = 2 - 3x$, where x is measured from fixed point. The acceleration is ***[MPPET-1992]***

(a) uniform (b) zero
(c) non-uniform (d) Indeterminate

6. A particle moves so that $S = 6 + 48t - t^3$. The direction of motion reverses after moving a distance of ***[KCET-1998]***

(a) 63 (b) 104
(c) 134 (d) 288

7. A particle moves in a straight line so that $S = \sqrt{t}$ then its acceleration is porportional to

(a) velocity (b) (velocity)$^{3/2}$
(c) (velocity)^3 (d) (velocity)^2

8. A point moves in a straight line during the $t = 0$ to $t = 3$ according to the law $S = 15t - 2t^2$. The avg-velocity is ***[PET-1992]***

(a) 3 (b) 9
(c) 15 (d) 27

9. A particle moves along the curve $6y = x^3 + 2$, Points on the curve at which y co-ordinate is changiing 8 times as fast as the x-coordinate are

(a) $(4, 11)$ and $(-4, -31/3)$
(b) $(4, -11)$ and $(-4, 31/3)$
(c) $(-4, 11)$ and $(4, 31/3)$
(d) $(-4, -11)$ and $(4, 31/3)$

10. A particle is moving on a straight line where its position 'S' in meters is a function of time, t in seconds given by $S = t^3 + at^2 + bt + c$, where a, b, c are constants. It is known that at time $t = 1$ sec. The position of the particle is given by $S = 7$mt, Velocity is 7 m/sec and acceleration is 12m/sec^2, find the values of a, b and c = ?

(a) $a = 1, b = 2, c = -1$
(b) $a = 3, b = -2, c = 5$
(c) $a = -1, b = 2, c = 1$
(d) None of these

11. If the path of a moving point is the curve $x = at$, $y = b \sin at$ then its acceleration at any instant ***[SCRA-96]***

(a) Is constant
(b) varies as the distance from axis of x.
(c) varies as the distance from axis of y
(d) varies as distance of point from the origin.

12. If $t = \dfrac{v^2}{2}$, then $\left(-\dfrac{df}{dt}\right)$ is equal to (where f is acceleration) ***[PET-1997]***

(a) f^2 (b) f^3
(c) $-f^3$ (d) $-f^2$

13. A point is moving along the parabola $y^2 = 12x$ at the rate of 10cm/sec. Component velocity parallel to x-axis when it is at the point (3, 6)

(a) $5\sqrt{2}$ (b) $3\sqrt{2}$
(c) $10\sqrt{2}$ (d) $8\sqrt{2}$

14. A stone thrown vertically upwards from the surface of the moon at a velocity of 24 cm/sec reaches a height of $S = 24t - 0.8t^2$ meter after 1 sec. The acceleration due to gravity in m/sec^2 at surface of moon is ***[PET-92]***

(a) 0.8 (b) 1.6
(c) 2.4 (d) 4.9

15. The volume of a spherical balloon is increasing at the rate of 900 cm^3/sec then the rate of change of radius of balloon at instant when radius is 15 cm (in cm/sec) ***[RET-1996]***
(a) 22/7 (b) 22
(c) 7/22 (d) None of these

16. The sides of an equilateral Δ's are increasing at the of 2cm/sec. The rate at which the area increases when the side is 10 cm is:
(a) $\sqrt{3}$ sq unit/sec (b) 10 sq unit/sec
(c) $10\sqrt{3}$ sq unit/sec (d) $10/\sqrt{3}$ sq unit/sec

17. A man 1.80 m high moves directly away from a lamp post of 4.5 metres hight at the rate of 1.2 m/sec. How fast does the length of his shadow increase ***[MPPET-1989]***
(a) 0.4 m/sec (b) 0.8 m/sec
(c) 1.2 m/sec (d) None of these

18. A 10 cm long rod AB moves with it's ends on two mutually perpendicular straight line OX and OY. If the end A be moving at the rate of 2 cm/sec, and when the distance of A from O is 8 cm the rate at which the end B is moving is

[SCRA-1996]

(a) $\frac{8}{3}$ cm/sec (b) $\frac{4}{3}$ cm/sec
(c) $\frac{2}{9}$ cm/sec (d) None of these

19. On dropping a stone in stationary water circular ripples are observed. Rate of flow of ripples is 6 cm/sec. When radius of circle is 10 cm, then fluid rate of increased in area.

[RPET-1996]

(a) 120 sq cm/sec
(b) –120 sq cm/sec
(c) π sq cm/sec
(d) 120π sq cm/sec

20. If $y = \sin x$ and x changes from π/2 to 22/14. Then approximate change in y when $\pi = \frac{355}{113}$ is
(a) 0 (b) π/6
(c) π/15 (d) π/12

21. The time T of a complete oscillation of a simple pendulum of length l is given by the equation $T = 2\pi\sqrt{\frac{1}{g}}$ where g is constant then percentage error in T when l is increased by 1%
(a) 2% (b) 1/3%
(c) 1/2% (d) 3%

22. Cube root of 127 by differential is
(a) $5+\frac{2}{75}$ (b) $5+\frac{1}{75}$
(c) $5-\frac{2}{75}$ (d) $5-\frac{1}{75}$

23. A ladder is resting with the wall at an angle of 30° A man is ascending the ladder at the rate of 3 ft/sec, his rate of approaching the wall is
(a) 3 ft/sec (b) 3/2 ft./sec
(c) 3/4 ft/sec (d) $3/\sqrt{2}$ ft./sec

24. The velocity of a particle at time t is given by the relation $v = 6t - \frac{t^2}{6}$. The distance travelled in 3 sec is
(a) 39/2 (b) 57/2
(c) 51/2 (d) 33/2

25. A spherical iron ball of radius 10 cm, coated with a layer of ice of uniform thickness, melts at a rate of 100π cm^3/min. The rate at which the thickness of ice decreases when the thickness of ice is 5 cm, is ***[Kerala PET-2008]***
(a) $\frac{1}{54}$ cm/min (b) $\frac{1}{9\pi}$ cm/min
(c) $\frac{1}{36}$ cm/min (d) $\frac{1}{9}$ cm/min

SOLUTIONS

1. (d) Step 1: If $f(x)$ satisfies Rolle's them in $[a, b]$ then

$f(x)$ must be continuous in $[a, b]$ and differentiable in (a, b) Also $f(a) = f(b)$

Step 2: $f(x) = \dfrac{x^2 - 3x}{x - 1}$

(a) [0, 3]

$f(0) = 0$

$$f(3) = \frac{3^2 - 3.3}{3-1} = 0$$

$$f(0) = f(3)$$

But $f(x)$ is discontinuous at $x = 1$

(b) and (c) also don't satisfy Rolle's theorem ($f(a) \neq f(b)$)

(d) is Ans.

2. (b) $f(x) = x^2 - 4$ satisfies rolle's theorem in $[a, b]$
if $f(a) = f(b)$, and $f(x)$ is continuous and differentiable in (a, b)

$\therefore$ option (b) is correct $f(2) = f(-2)$.

3. (c) Step 1: acceleration $= \frac{dv}{dt}$, v = velocity

Step 2: $v = 1 + t^2 \Rightarrow \frac{dv}{dt} = 2t$

at $t = 3$, acceleration $= \frac{dv}{dt} = 6\ cm/\sec^2$

4. (a) Step 1: Given $S = \frac{1}{2}vt$

differentiating w.r.t. t, $\frac{ds}{dt} = \frac{1}{2}\left(v + t\frac{dv}{dt}\right)$

$$v = \frac{v}{2} + t\frac{dv}{dt} \Rightarrow \frac{-v}{2t} = \frac{dv}{dt} = \text{acceleration } (f)$$

Step 2: $\frac{df}{dt} = \frac{t\left(-\frac{dv}{dt}\right) + v}{2t^2} = \frac{-f}{2t} + \frac{v}{2t^2}$

$$\frac{df}{dt} = \frac{-f}{2t} - \frac{ft}{2t^2} = \frac{-f}{2t} + \frac{f}{2t} = 0$$

$\therefore$ f is constant

5. (a) Step 1: Acceleration $(f) = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}$

$\therefore \quad f = v\frac{dv}{dx} \left(\because v = \frac{ds}{dt}\right)$

Step 2: Given $v^2 = 2 - 3x$

differentiating w.r.t. x, $2v\frac{dv}{dx} = -3$

$$f = v\frac{dv}{dx} = \frac{-3}{2}$$

6. (c) Step 1: Direction of the motion reverses if velocity of the particle = 0 and $f \neq 0$

Step 2: Given $S = 6 + 48t - t^3$

$\frac{ds}{dt} = 48 - 3t^2 = 0 \Rightarrow t = 4\sec$

$\therefore \quad S = 6 + 48.4 - 4^3 = 134$

7. (c) $S = \sqrt{t}; v = \frac{ds}{dt} = \frac{1}{2\sqrt{t}}$

Also $\frac{dv}{dt} = -\frac{1}{4}t^{-3/2}$ = acceleration

acceleration $\propto v^3$

8. (b) Step 1: Velocity $= \frac{ds}{dt}$

So differentiating with respect to t

$V = \frac{ds}{dt} = 15 - 4t$

Step 2: Now Put $t = 0$, $V = 15$
and putting $t = 4$, $V = 3$

Step 3: So average velocity $= \frac{15 + 3}{2} = 9$

9. (a) Step 1: Here $\frac{dy}{dt} = 8\frac{dx}{dt}$

Step 2: Given curve $6y = x + 2$
differentiating w.r.t. t,

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

$$2\left(8\frac{dx}{dt}\right) = x^2\frac{dx}{dt} \Rightarrow x = \pm 4$$

When $x = 4$, $6y = 4^3 + 2 \Rightarrow y = 11$

When $x = -4$, $6y = -4^3 + 2 \Rightarrow y = \frac{-31}{3}$

$\therefore \quad$ Points are $(4, 11)$ $\left(-4, \frac{-31}{3}\right)$

10. (b) at $t = 1$; $S = 7$

$\therefore \quad 1 + a + b + c = 7 \Rightarrow a + b + c = 6$

at $\quad t = 1, v = 7 = \frac{ds}{dt}$

$$3 + 2a + b = 7$$

$$2a + b = 4$$

$$\text{acceleration} = \frac{dv}{dt} = 12$$

$$6 + 2a = 12 \Rightarrow a = 3, b = -2, c = 5$$

11. (b) acceleration along x axis $a_x = \dfrac{d^2x}{dt^2} = 0$

acceleration along

$$y \text{ axis } a_y = \frac{d^2y}{dt^2} = -a^2 b \sin at$$

$$\therefore \quad \text{acceleration} = \sqrt{a_x^2 + a_y^2} = a^2 b \sin at$$

$$= a^2 y = a^2 \text{ (distance from axis of } x)$$

12. (c) Step 1: $f = v\dfrac{dv}{ds}, \dfrac{ds}{dt} = v$

Here, f is acceleration

$$\text{Step 2: } t = \frac{v^2}{2} \Rightarrow \frac{dt}{ds} = \frac{2v}{2}\frac{dv}{ds}$$

$$f = \frac{1}{\frac{ds}{dt}} = f \Rightarrow f = \frac{1}{v}$$

$$\frac{df}{dt} = \frac{-1}{v^2} \times \frac{dv}{dt} = -f^2.f$$

$$\frac{df}{dt} = -f^3$$

13. (a) Here $2y\dfrac{dy}{dt} = 12\dfrac{dx}{dt} \Rightarrow y\dfrac{dy}{dt} = \dfrac{6dx}{dt}$

$$v_y = \frac{6}{y}v_x \left(\because v_y = \frac{dy}{dx}, v_x = \frac{dx}{dt} \right)$$

$$\text{Also } \sqrt{v_x^2 + v_y^2} = 10$$

$$v_x^2 + \left(\frac{6}{y}v_x\right)^2 = 10^2 \Rightarrow v_x = \frac{10y}{\sqrt{y^2 + 36}}$$

$$\text{at point } (3, 6) \; v_x = \frac{10 \times 6}{\sqrt{6^2 + 36}} = 5\sqrt{2} = v_y$$

14. (b) Step 1: Here $f = \dfrac{d^2s}{dt^2}$

$$\text{Step 2: } S = 24t - 0.8t^2$$

$$\frac{ds}{dt} = 24 - 1.6t \Rightarrow \frac{d^2s}{dt^2} = -1.6$$

$$f = -1.6 \; cm/\sec^2$$

(acceleration is –ve, since stone is thrown upwards)

15. (c) Step 1: Volume of spherical balloon

$$= \frac{4}{3}\pi r^3$$

Step 2: Given $\dfrac{dv}{dt} = 900$ (V = volume)

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 900 \Rightarrow 4\pi r^2 \frac{dr}{dt} = 900$$

$$4\pi \times 15^2 \frac{dr}{dt} = 900 \Rightarrow \frac{dr}{dt} = \frac{1}{\pi} = \frac{7}{22}$$

16. (c) Step 1: Area of equilateral $\Delta = \dfrac{\sqrt{3}}{4}x^2 = A$

Where x = length of side of Δ

$$\text{Step 2: } A = \frac{\sqrt{3}}{4}x^2$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4}\left(2x\frac{dx}{dt}\right)$$

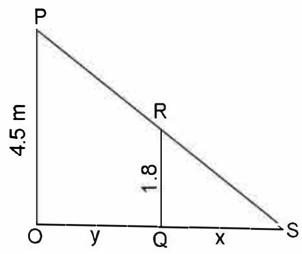
$$\text{Given } \frac{dx}{dt} = 2; \frac{dA}{dx} = \frac{\sqrt{3}}{4}.20.2$$

$$\frac{dA}{dt} = 10\sqrt{3} \text{ sq. unit/sec.}$$

17. (b) Step 1: Here OP = Lamp Post = 4.5 m

QR = man =1.8

QS = Shadow of man = x(Let)



$$\text{Also } \frac{dy}{dt} = 1.2 \; m/s$$

Step 2: ΔOPS and ΔQRS are similar.

$$\therefore \quad \frac{4.5}{1.8} = \frac{x + y}{x} \qquad \text{or} \quad 1 + \frac{y}{x} = \frac{5}{2}$$

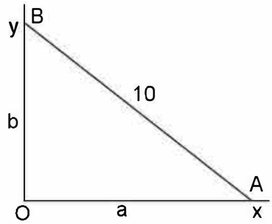
$$\frac{dy}{dt} = \frac{2 \times (1.2)}{3} = 0.8 \; m/s$$

18. (a) Step 1: Here $a^2 + b^2 = 10^2$

Also $\dfrac{d(OX)}{dt} = \dfrac{da}{dt} = 2cm$

$\dfrac{db}{dt}$ is to be calculate.

Step 2: $a^2 + b^2 = 100$



differentiating w.r.t. t, $a\dfrac{da}{dt} + b\dfrac{db}{dt} = 0$

also when $a = 8$, $b^2 = \sqrt{100 - a^2} = 6$

Put in equation, $8.2 + 6.\dfrac{db}{dt} = 0$

$\dfrac{db}{dt} = -\dfrac{8}{3} cm/\text{sec}$

19. (d) $\dfrac{dA}{dt} = \dfrac{d}{dt}(\pi r^2)$

Here $\dfrac{dr}{dt} = 6; \dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$

$\dfrac{dA}{dt} = 2\pi \times 10 \times 6 = 120\pi$ sq. cm./sec.

20. (a) $y = \sin x \Rightarrow \Delta y = \cos x \Delta x$

$\Delta y = \cos\dfrac{\pi}{2}\left(\dfrac{22}{14} - \dfrac{\pi}{2}\right)$

$\Delta y = 0$

21. (c) $T = 2\pi\sqrt{\dfrac{l}{g}}$, taking log on both sides

$\log T = \log 2\pi + \dfrac{1}{2}\log l - \dfrac{1}{2}\log g$

differentiating we get

$\dfrac{\Delta T}{T} = \dfrac{1}{2}\dfrac{\Delta l}{l}$

$\Rightarrow \dfrac{\Delta T}{T}\% = \dfrac{1}{2} \times 1\% = 1/2\ \%$

22. (a) $y = x^{1/3}$ be function

$\log y = \dfrac{1}{3}\log x \Rightarrow$ differentiating we get

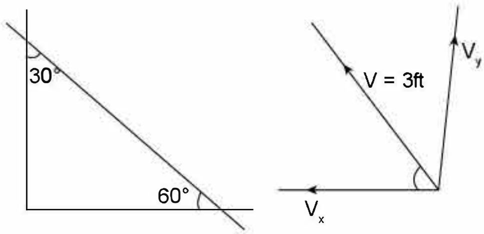
$\dfrac{\Delta y}{y} = \dfrac{1}{3}\dfrac{\Delta x}{x}$

$\Rightarrow \dfrac{\Delta y}{5} = \dfrac{1}{3} \times \dfrac{2}{125} \qquad \Delta y = \dfrac{2}{75}.$

$5 + \dfrac{2}{75}$

(Intially $x = 125$; $\therefore\ y = (125)^{\frac{1}{3}} = 5$)

23. (b)



Rate of approaching the wall

$= v_x = 3\cos 60° = 3/2$

24. (c) $\dfrac{dx}{dt} = 6t - \dfrac{t^2}{6}$

$\int_0^x dx = \int_0^x \left(6t - \dfrac{t^2}{6}\right)dt$

$x = 3t^2\Big|_0^3 - \dfrac{t^3}{18}\Big|_0^3 = 27 - \dfrac{27}{18}$

$= 27 - \dfrac{3}{2} = \dfrac{51}{2}$

25. (d) Let x be thickness of ice

$\dfrac{d}{dt}\left[\dfrac{4}{3}\pi((10 + x)^3 - 10^3)\right] = -100\pi$

$\dfrac{4}{3} \times \pi \times 3(10 + x)^2 \times \dfrac{dx}{dt} = 100\pi$

$(10 + x)^2\dfrac{dx}{dt} = 25$

at $x = 5$, $\dfrac{dx}{dt} = \dfrac{25}{225} = \dfrac{1}{9}$ cm/min.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. Rolle's theorem holds for the function $x^3 + bx^2 + cx$, $1 \le x \le 2$ at the point 4/3, the value of b and c are

[Pb. CET-1999, Him CET-2004]

(a) $b = 8, c = -5$ (b) $b = -5, c = 8$
(c) $b = 5, c = -8$ (d) $b = -5, c = -8$

2. A particle moves in a straight line so that. Its velocity at any point is given by $v^2 = a + bx$, where $a, b \neq 0$ are constants. Then acceleration is ***[PET-89]***

(a) Zero (b) Uniform
(c) Non-uniform (d) Indeterminate

3. If $2t = v^2$, then $\dfrac{dv}{dt}$ is equal to

[PET-1992]

(a) 0 (b) $\dfrac{1}{4}$
(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{v}$

4. A particle is moving in a straight line according to formula $S = t^3 - 9t^2 + 3t + 1$ where 'S' is measured in meters and t in seconds when velocity is –24m/sec then their acceleration is

(a) 0 (b) 1
(c) 2 (d) 3

5. If the distance travelled (S) by a particle in time 't' is $S = a \sin t + b \cos 2t$; then the acceleration at $t = 0$ is

(a) a (b) $-a$
(c) $4b$ (d) $-4b$

6. A particle moves along a straight line so that it's distance S in time t sec is $S = t + 6t^2 - t^3$. After what time is the acceleration zero.

[AMU-1999]

(a) 2 sec (b) 3 sec
(c) 4 sec (d) 6 sec

7. The distance travelled s (in metre) by a particle in t seconds in given by $s = t^3 + 2t^2 + t$. The speed of the particle after 1 second will be ***[UPSEAT-03]***

(a) 8 cm/sec (b) 6 cm/sec
(c) 2 cm/sec (d) None of these

8. A particle is moving along the curve $x = at^2 + bt + c$. If $ac = b^2$, then the particle would be moving with uniform

[Orissa-JEE-2003]

(a) Rotation (b) Velocity
(c) Acceleration (d) Retardation

9. The distance S meter covered by a body in t seconds is given by $S = 3t^2 - 8t + 5$, the body will stop after

(a) 1 sec (b) 3/4 sec
(c) 4/3 sec (d) 4 sec

10. A ball thrown vertically upwards falls back on the ground after 6 second. Assuming that the equation of motion is of the form $S = 4t - 4.9\,t^2$, where S is in metres and t is in second find the velocity at $t = 0$

(a) 0 m/s (b) 1 m/s
(c) 29.4 m/s (d) None of these

11. The radius of a cylinder is increasing at the rate of 2 m/sec and its height is decreasing at the rate of 3 m/sec. At what rate is the volume of the cylinder is changing when radius of cylinder is 3 m and height is 5 m

[Kerala (Engg.)-2005]

(a) $87\,\pi$ m³/sec (b) 33π m³/sec
(c) $27\,\pi$ m³/sec (d) $15\,\pi$ m³/sec

12. A particle is moving in a straight line. It's displacement at time 't' is given by $S = -4t^2 + 2t$, then it's velocity and acceleration at time $t = 1/2$ sec are

[AISSE-1981]

(a) –2, –8 (b) 2, 6
(c) –2, 8 (d) 2, 8

13. Gas is being pumped into a spherical balloon at the rate of 30 ft³/min. Then the rate at which the radius increases when it reaches the value 15 ft is

[EAMCET-2003; RPET-2000]

(a) $\dfrac{1}{30\pi}$ ft/min (b) $\dfrac{1}{15\pi}$ ft/min
(c) $\dfrac{1}{20}$ ft/min (d) $\dfrac{1}{25}$ ft/min

14. If by dropping a stone in a quiet lake a wave moves in circle at a speed of 3.5cm/sec, then the rate of increase of the enclosed circular region when the radius of the circular wave is 10 cm is

[MPPET-1998]

(a) 220 cm^2/sec (b) 110 cm^2/sec
(c) 35 cm^2/sec (d) 350 cm^2/sec

[When value of π is 22/7]

15. A particle is moving in a straight line according as $S = \sqrt{1+t}$, then the relation between its acceleration (a) and velocity (v) is

[MPPET-2004]

(a) $a \propto v^2$ (b) $a \propto v^3$
(c) $a \propto \frac{1}{v^3}$ (d) $a \propto v$

16. A spotlight on the ground shines on a wall 12 m away from the light. If a man of 2 m height walks from the spotlight towards the wall at a speed of $\frac{1}{2}$ m per second, then the rate at which the length of his shadow on the wall is decreasing at the instant when he is 8 m from the wall is *[Kerala PET-2008]*

(a) $\frac{3}{4}$ m/s (b) $\frac{5}{4}$ m/s
(c) $\frac{3}{8}$ m/s (d) $\frac{5}{8}$ m/s

17. If $y = 3x^2 + 2$ and if x changes from 10 to 10.1, then the approximate change in y will be

[NDA-2003]

(a) 4 (b) 5
(c) 6 (d) 8

18. Using differentials find the approximate value of $\sqrt{401}$

(a) 20.100 (b) 20.025
(c) 20.030 (d) 20.125

19. A man 2 meters high walks at a uniform speed 5 metres per hour away from a lamp post 6 metres high. The rate at which the length of his shadow increases is

(a) 5 m/h (b) 5/2 m/h
(c) 5/3 m/h (d) 5/4 m/h

20. If the function $f(x) = x^3 - 6x^2 + ax + b$ satisfied and Rolle's theorem in the interval [1, 3] and if $f'\left(\frac{2\sqrt{3}+1}{\sqrt{3}}\right) = 0$ then

[PET-2000]

(a) $a = 11$ (b) $a = -6$
(c) $a = 6$ (d) $a = 1$

21. The edge of a cube is increasing at the rate of 5 cm/sec. How fast is the volume of the cube increasing when the edge is 12 cm long

(a) 432 cm^3/sec
(b) 2160 cm^3/sec
(c) 180 cm^3/sec
(d) None of these

22. The rate of change of the surface area of a sphere of radius r when the radius is increasing at the rate of 2cm/sec is proportional to

[Karnataka CET-2003]

(a) $1/r$ (b) $1/r^2$
(c) r (d) r^2

23. The law of rectilinear motion of a particle is $x = a\cos(nt + k)$, then acceleration of the body towards the origin is proportional to

(a) x (b) $-x$
(c) x^2 (d) $1/x^2$

24. A spherical balloon is expanding. If the radius is increasing at the rate of 2 centimetres per minute, the rate at which the volume increases (in cubic centimetres per minute) when the radius is 5 centimetres is

[VITEEE-2008]

(a) 10π (b) 100π
(c) 200π (d) 50π

25. For $f(x) = x^2 - 5x - 6$ Rolle's theorem is applicable at

[MPPET-2007]

(a) 5/3 (b) -1
(c) 6 (d) 5/2

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The Answer sheet is immediately below the worksheet.
2. The test is of 16 minutes.
3. The worksheet consists of 16 questions. The maximum marks are 48.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. The value of c in Rolle's theorem when $f(x) = 2x^3 - 5x^2 - 4x + 3,\ x \in \left[\frac{1}{2}, 3\right]$ is

(a) 2 (b) $-1/3$
(c) -2 (d) $2/3$

2. The speed v of a particle moving along a straight line is gives by $a + bv^2 = x^2$ (where x is its distance from the origin) the acceleration of the particle is *[MPPET-2002]*

(a) bx (b) x/a
(c) x/b (d) x/ab

3. If the edge of a cube increases at the rate of 60 cm per second, at what rate the volume is increasing when the edge is 90 cm

(a) 486000 cm^3/sec
(b) 1458000 cm^3/sec
(c) 43740000 cm^3/sec
(d) None of these

4. If the distance travelled by a point in time t is $S = 180t - 16t^2$, then the rate of change in velocity is *[MPPET-1995]*

(a) -16 unit (b) 48 unit
(c) -32 unit (d) None of these

5. The motion of stone thrown up vertically is given by $S = 13.8\ t - 4.9\ t^2$, where S is in metres and t is in seconds. Then its velocity at $t = 1$ second is

(a) 3 m/s (b) 5 m/sec
(c) 4 m/sec (d) None of these

6. Radius of a circle is increasing uniformaly at the rate of 3m/sec. The rate of increasing of area when radius is 10cm, will be:

(a) π cm^2/s (b) 2π cm^2/s
(c) 10π cm^2/s (d) None of these

7. A particle is moving in a straight line according as $S = 45t + 11t^2 - t^3$, then the time when it will come to rest is

(a) -9 seconds (b) 5/3 seconds
(c) 9 seconds (d) $-5/3$ seconds

8. The law of motion of a particle in a straight line is $S = e^t(\sin t - \cos t)$, then acceleration at the end of time t is

(a) $e^t(\cos t + \sin t)$ (b) $e^t(\cos t - \sin t)$
(c) $2e^t(\cos t - \sin t)$ (d) $2e^t(\cos t + \sin t)$

9. The radius of a circle is increasing at the rate of 3cm/sec. At what rate is the area increasing when the radius of the circle is 10 cm

(a) 60π (b) 60π cm^2/sec
(c) 60π cm/sec (d) 60π cm^3/sec

10. A function f is defined by $f(x) = 2 + (x-1)^{2/3}$ in $[0, 2]$. Which of the following is not correct?

(a) f is not derivable in $(0, 2)$
(b) f is continuous in $[0, 2]$
(c) $f(0) = f(2)$
(d) Rolle's theorem is true in $[0, 2]$

11. The radius of a shpere is increasing at the rate of 0.1%. At what rate is the volume increasing

(a) 0.3 (b) 0.1
(c) 0.2 (d) 0.5

12. Which one of the following statements is correct in respect of the curve $4y - x^2 - 8 = 0$?

(a) The curve is increasing in $(-4, 4)$
(b) The curve is increasing in $(-4, 0)$
(c) The curve is increasing in $(0, 4)$
(d) The curve is decreasing in $(-4, 4)$

13. Rolle's Theorem is applicable in case of $\phi(x) = \alpha^{\sin x}$ in

(a) any interval
(b) the interval $[0, \pi]$
(c) the interval $\left(0, \frac{\pi}{2}\right)$
(d) None of these

14. What is the smallest value of m for which $f(x) = x^2 + mx + 5$ is increasing in the interval $1 \le x \le 2$? *[NDA-2008]*

(a) $m = 0$ (b) $m = -1$
(c) $m = -2$ (d) $m = -3$

15. A particle moves in a straight line so that it covered a distance $at^3 + bt + 5$ metre in t seconds. If its acceleration after 4 seconds is 48 metre/(sec)^2, then a is equal to **[MPPET-2008]**

(a) 1 (b) 2
(c) 3 (d) 4

16. The volume of a spherical balloon is increasing at the rate of 40 cubic centimeters per minute. The rate of change of the surface of the balloon at the instant when its radius is 8 centimetres is **[Roorkee-1983]**

(a) 5/2 square cm^2/minute
(b) 5 square cm^2/minute
(c) 10 square cm^2/minute
(d) 20 square cm^2/minute

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	7. ⓐ ⓑ ⓒ ⓓ	13. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	8. ⓐ ⓑ ⓒ ⓓ	14. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	9. ⓐ ⓑ ⓒ ⓓ	15. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	10. ⓐ ⓑ ⓒ ⓓ	16. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	11. ⓐ ⓑ ⓒ ⓓ	
6. ⓐ ⓑ ⓒ ⓓ	12. ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

2. (c) Step 1: Acceleration of a particle $f = v\dfrac{dv}{dx}$

Step 2: $a + bv^2 = x^2$

diff. w.r.t. x, $b\left(2v\dfrac{dv}{dx}\right) = 2x$

So $v\dfrac{dv}{dx} = \dfrac{x}{b}$

$\therefore \quad bf = x \Rightarrow f = \dfrac{x}{b}$

10. (d) Given $f(x) = 2 + (x-1)^{\frac{2}{3}}$ in $[0, 2]$

$f'(x) = \dfrac{2}{3(x-1)^{1/3}}$ doesn't exist at $x = 1$

$\therefore \quad f(x)$ is not differentiable at $x = 1$

But $f(x)$ is continuous everywhere

Also $f(0) = f(2)$

and so Rolle's theorem is not applicable in $[0, 2]$

16. (c) Given $\dfrac{dv}{dt} = 40$

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 40 \Rightarrow \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 40$$

Here $r = 8\ cm$; $4\pi \times 8^2\dfrac{dr}{dt} = 40$

$$\frac{dr}{dt} = \frac{5}{32\pi}$$

Rate of change of surface area $= \dfrac{d}{dt}(4\pi r^2)$

$$= 8\pi r\frac{dr}{dt} = 8\pi \times 8 \times \frac{5}{32\pi} = 10\text{cm}^2/\text{min}$$

LECTURE

3

Tangent and Normal

BASIC CONCEPTS

1. Geometrical meaning of $\frac{dy}{dx}$: The value of $\frac{dy}{dx}$ at any point $P(x_1, y_1)$ of a curve represents the gradient (or slope) of the tangent at the point P of the curve.

2. **Equation of Tangent** The equation of tangent to the curve $y = f(x)$ at the point $P\ (x_1, y_1)$ is:

$$y - y_1 = \left.\frac{dy}{dx}\right|_{(x_1,\, y_1)} (x - x_1)$$

where $\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \tan\theta$ is the value of $\frac{dy}{dx}$ at (x_1, y_1) and θ is the inclination of the tangent at P with positive direction of x-axis.

NOTES

1. The equation of tangent at any point $t(x = f(t),\ y = g(t))$ on the curve is given by $Y - g(t) = \frac{g'(t)}{f'(t)}(X - f(t))$
2. If the tangent is parallel to the x-axis then $\frac{dy}{dx} = 0$
3. If the tangent is perpendicular to the axis of X i.e., parallel to the axis of y then $\frac{dy}{dx} \to \infty$ or $\frac{dx}{dy} = 0$
4. If the tangent at any point on the curve is equally inclined to both the axes then $\frac{dy}{dx} = \pm 1$.
5. The equations of the tangent (or tangents) passing through the origin is obtained by equating the lowest degree terms in the equation of the curve to zero.
6. For equation of tangent at (x_1, y_1), substitute xx_1 for x^2, yy_1 for y^2, $\frac{x + x_1}{2}$ for x, $\frac{y + y_1}{2}$ for y and $\frac{xy_1 + x_1y}{2}$ for xy and keep the constant as such.
This method is applicable only for second degree conics,
i.e., $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

3. The equation of normal to the curve $y=f(x)$ at the point (x_1, y_1) is $(y - y_1) = \dfrac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}}(x - x_1)$

OR

$$(y - y_1)\left.\frac{dy}{dx}\right|_{(x_1,\, y_1)} + (x - x_1) = 0$$

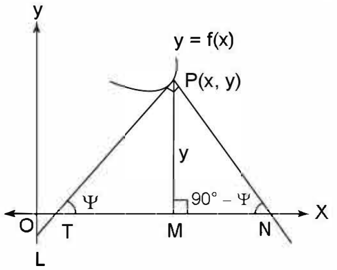
4. If the normal is parallel to the axis of y, then $\frac{dy}{dx} = 0$

5. If the normal is parallel to the axis of x, then $\frac{dx}{dy} = 0$.

6. Quicker method to find normal at (x_1, y_1) of second degree conics $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is

$$\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}$$

7. **Length of Sub-tangent: TM** $= \dfrac{y_1}{\left.\dfrac{dy}{dx}\right|_{(x_1, y_1)}} = \dfrac{y_1}{m}$



8. **Length of Sub-Normal:**

$$MN = y_1 \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = y_1 m$$

9. **Length of Tangent:**

$$PT = \frac{y_1}{m}\sqrt{1 + m^2},\ m = \left.\left(\frac{dy}{dx}\right)\right|_{(x_1, y_1)}$$

10. **Length of Normal:**

$$PN = y_1\sqrt{1 + m^2},\ m = \left.\frac{dy}{dx}\right|_{(x_1, y_1)}$$

11. $\dfrac{\text{sub} - \text{normal}}{\text{sub} - \text{tangent}} = \left(\dfrac{\text{Length of normal}}{\text{Length of tangent}}\right)^2$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. Find a point on $y = (x - 2)^2$ where the tangent is parallel to the chord joining (2, 0) and (4, 4). ***[CBSE-93]***

Solution

We apply Lagrange's mean value theorem for the functions

$f(x) = (x - 2)^2$ on [2, 4]

(a) $f(x)$ being a polynomial in x, it is continuous on [2, 4].

(b) $f'(x) = 2(x - 2)$, which exists for all values of $x \in (2, 4)$.

Therefore, $f(x)$ is differentiable in (2, 4)

Thus, both the conditions of Lagrange's mean value theorem are satisfied consequently there exists at least one $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2}$$

$$\Rightarrow \quad \cancel{2(c-2)} = \frac{(4-2)^2 - (2-2)^2}{2} = 2$$

$\Rightarrow \quad c - 2 = 1 \Rightarrow c = 3$

and $f(c) = (3 - 2)^2 = 1$

Hence (3, 1) is the point on the curve such that the tangent at it is parallel to chord joining the points (2, 0) and (4, 4).

2. Find a point on the curve $y = (x - 3)^2$ where the tangent is parallel to the line joining (4, 1) and (3, 0).

[CBSE-91, 93C, HPSB-2002]

Solution

Let the required point be $P(x_1, y_1)$. The equation of the given curve is $y = (x - 3)^2$ ..... (i)

$$\Rightarrow \quad \frac{dy}{dx} = 2(x - 3)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2(x_1 - 3)$$

Since the tangent at P is parallel to the line joining (4, 1) and (3, 0). Therefore slope of the tangent at P = slope of the line joining (4, 1) and (3, 0).

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{0 - 1}{3 - 4} \Rightarrow 2(x_1 - 3) = 1$$

$\Rightarrow \quad x_1 = 7/2$

Since the point $P(x_1, y_1)$ lies on (i). Therefore $y_1 = (x_1 - 3)^2$

$$\therefore \quad x_1 = \frac{7}{2} \Rightarrow y_1 = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$$

Thus, the required point is (7/2, 1/4)

3. Find the equation of the tangent to the curve $y = -5x^2 + 6x + 7$ at the point (1/2, 35/4).

[CBSE-1994]

Solution

The equation of the given curve is $y = -5x^2 + 6x + 7$

$$\Rightarrow \quad \frac{dy}{dx} = -10x + 6$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}, \frac{35}{4}\right)} = -10 \times \frac{1}{2} + 6 = 1$$

The required equation of the tangent at (1/2, 35/4) is

$$y - \frac{35}{4} = \left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}, \frac{35}{4}\right)} \left(x - \frac{1}{2}\right)$$

$$\Rightarrow \quad y - \frac{35}{4} = 1\left(x - \frac{1}{2}\right) \Rightarrow y = x + \frac{33}{4}$$

4. Find the equation of the tangent line to the curve $x = 1 - \cos\theta, y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$.

[CBSE-93, 2004; PSB-97, 2001S, 02]

Solution

When $\theta = \frac{\pi}{4}$, we have $x = 1 - \cos\theta$

$$= 1 - \cos\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}} \text{ and}$$

$$y = \theta - \sin\theta = \frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{\sqrt{2}}$$

So, co-ordinates of the point of contact are

$$\left(1 - \frac{1}{\sqrt{2}}, \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$$

Now $x = 1 - \cos\theta$ and $y = \theta - \sin\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = \sin\theta \text{ and } \frac{dy}{d\theta} = 1 - \cos\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1 - \cos\theta}{\sin\theta}$$ at $\theta = \frac{\pi}{4}$, we have

$$\frac{dy}{dx} = \frac{1 - \cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1 - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2} - 1$$

So the equation of tangent line at $\theta = \frac{\pi}{4}$ is

$$y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = (\sqrt{2} - 1)\left(x - 1 + \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad (\sqrt{2} - 1)x - y = 2(\sqrt{2} - 1) - \frac{\pi}{4}$$

5. Show that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where it crosses the y-axis.

[CBSE-2005; MNR-81; Karnataka CET-2002]

Solution

The equation of the given curve is $y = be^{-x/a}$ ...... (i)

It crosses y-axis at the point, where $x = 0$

Putting $x = 0$ in equation (i), we get $y = be^0 = b$

So, the point of contact is $(0, b)$

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = be^{-x/a} \frac{d}{dx}\left(\frac{-x}{a}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-b}{a} e^{-x/a}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(0, b)} = -\frac{b}{a} e^0 = \frac{-b}{a}$$

The equation of the tangent at $(0, b)$ is

$$y - b = -\frac{b}{a}(x - 0)$$

$$\Rightarrow \quad ay - ab = -bx \Rightarrow bx + ay = ab$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

Hence, $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where it crosses the axis of y.

6. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

[CBSE-2002]

Solution

The given curves are $xy = a^2$ .... (i)

$x^2 + y^2 = 2a^2$ .... (ii)

Substituting the value of y obtained from (i) in equation (ii),

we get $x^2 + (a^2/x)^2 = 2a^2$

$\Rightarrow \quad x^4 - 2a^2 x^2 + a^4 = 0$

$\Rightarrow \quad (x^2 - a^2)^2 = 0 \Rightarrow x = \pm a$

from (i), we have $y = a$ for $x = a$ and $y = -a$ and $x = -a$

Thus, the two curves intersect at $P(a, a)$ and $Q(-a, -a)$

Now $xy = a^2 \Rightarrow x\dfrac{dy}{dx} + y = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{-y}{x}$

and $x^2 + y^2 = a^2 \Rightarrow 2x + 2y\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}$$

At $P(a, a)$, we have

$$\left(\frac{dy}{dx}\right)_{c_1} = \frac{-a}{a} = -1; \left(\frac{dy}{dx}\right)_{c_2} = \frac{-a}{a} = -1$$

Clearly $\left(\dfrac{dy}{dx}\right)_{c_1} = \left(\dfrac{dy}{dx}\right)_{c_2}$ at P

So, the two curves touch each other at P. Similarly, it can be seen that the two curve touch each other at Q. **Proved.**

7. Determine the values of x for which the function $f(x) = x^2 - 6x + 9$ is increasing or decreasing. Also find the co-ordinates of the points on the curve $y = x^2 - 6x + 9$ where normal is parallel to the line $y = x + 5$.

[CBSE-94 C]

Solution

Given $f(x) = x^2 - 6x + 9 \Rightarrow f'(x) = 2x - 6$

$f'(x) > 0$ when $\Rightarrow 2x - 6 > 0 \Rightarrow 2x > 6 \Rightarrow x > 3$

Hence $f(x)$ is increasing on $(3, \infty)$

$f'(x) < 0$ when $\Rightarrow 2x - 6 < 0 \Rightarrow 2x < 6 \Rightarrow x < 3$

Hence $f(x)$ is decreasing on $(-\infty, 3)$

Given equation of the curve is $y = x^2 - 6x + 9$ .... (ii)

Let (x_1, y_1) be foot of normal and we know foot of normal lies on the curve $\therefore y_1 = x_1^2 - 6x_1 + 9$ .... (iii)

Differentiating equation (ii) we get

$$\Rightarrow \quad \frac{dy}{dx} = 2x - 6$$

Slope of normal at $(x_1, y_1) = -\left(\dfrac{dx}{dy}\right)_{(x_1, y_1)}$

$$= \frac{-1}{2x_1 - 6}$$

Since the normal is parallel to the line $y = x + 5$ whose slope is 1.

$$\therefore \quad \frac{-1}{2x_1 - 6} = 1 \Rightarrow 2x_1 - 6 = -1 \Rightarrow x_1 = \frac{5}{2}$$

Put $x_1 = \dfrac{5}{2}$ in equation (iii), we get

$$y_1\left(\frac{5}{2}\right)^2 - \frac{6 \times 5}{2} + 9 = \frac{25}{4} - 15 + 9 = \frac{1}{4}$$

Hence, foot of normal is $\left(\dfrac{5}{2}, \dfrac{1}{4}\right)$

8. Using Rolle's Theorem, find points on the curve $y = 16 - x^2$, $x \in [-1, 1]$, where tangent is parallel to x-axis. ***[CBSE-2000]***

Solution

Here $y = f(x) = 16 - x^2$; $x \in [-1, 1]$ .... (i)

Being a polynomial, $f(x)$ is continuous on $[-1, 1]$ and it is differentiable in $]-1, 1[$.

Also $f(1) = 15 = f(-1)$

$\therefore$ conditions of Rolle's Theorem are satisfied.

So, there must exist at least one real value of c in $]-1, 1[$ such that $f'(c) = 0$

$-2c = 0 \Rightarrow c = 0 \in]-1, 1[$

Hence Rolle's theorem is verified. As we know the value of c obtained in Rolle's theorem is nothing but the x-coordinate of that point on the curve where the tangent is parallel to x-axis.

$\therefore$ Put $x = 0$ in equation (i), we get $y = 16$

Hence the required point is $(0, 16)$.

9. For which values of x, the function $f(x) = \dfrac{x}{x^2 + 1}$ is increasing and for which values of x, it is decreasing. Find also the points on the graph of the function at which tangents are parallel to the x-axis.

[CBSE-2003]

Solution

We have $f(x) = \dfrac{x}{x^2 + 1}$

$$\Rightarrow \quad f'(x) = \frac{(x^2 + 1).1 - x(2x + 0)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

For $f(x)$ to be increasing, we must have $f'(x) > 0$

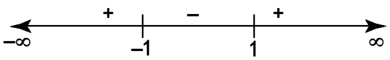
$\Rightarrow \quad \dfrac{1-x^2}{(x^2+1)^2} > 0$

$\Rightarrow \quad (1-x^2) > 0 \quad [\because \quad (x^2+1)^2 > 0]$

$\Rightarrow \quad -(x^2-1) > 0 \Rightarrow (x^2-1) < 0$

$\Rightarrow \quad (x-1)(x+1) < 0$

$\Rightarrow \quad -1 < x < 1$



$\Rightarrow \quad x \in (-1, 1)$

So, $f(x)$ is increasing on $(-1, 1)$

For $f(x)$ to be decreasing, we must have $f'(x) < 0$

$\Rightarrow \quad \dfrac{1-x^2}{(x^2+1)^2} < 0$

$\Rightarrow \quad (1-x^2) < 0 \quad [\because (x^2+1)^2 > 0]$

$\Rightarrow \quad x < -1$ or $x > 1$

So, $f(x)$ is decreasing on $(-\infty, -1) \cup (1, \infty)$

For tangent parallel to x-axis $f'(x) = 0$

$\Rightarrow \quad \dfrac{(1+x)(1-x)}{(x^2+1)^2} = 0$

$\Rightarrow x = -1$ or $x = 1$

Let $y = f(x)$ Then $y = \dfrac{x}{x^2+1}$ .... (i)

Put $x = -1$ in equation (i), we get $y = \dfrac{-1}{(-1)^2+1} = \dfrac{-1}{2}$

Point is $(-4, -1/2)$

Put $x = 1$ in equation (i), we get $y = \dfrac{1}{(1)^2+1} = \dfrac{1}{2}$

Point is $(1, 1/2)$

Hence $(-4, -1/2)$ and $(1, 1/2)$ are two points at which tangents are parallel to x-axis.

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Find the point on the curve $y = \cos x - 1$, $x \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$ at which the tangent is parallel to the x-axis.

2. Determine the values of x for which the function $f(x) = x^2 + 2x - 3$ is increasing or decreasing. Also find the co-ordinate of the points on the curve $y = x^2 + 2x - 3$ which normal is parallel to the line $x - 4y + 7 = 0$. ***[CBSE-94C]***

3. Find a point on the curve $y = x^3 - 3x$ where the tangent is parallel to the chord joining $(1, -2)$ and $(2, 2)$. ***[CBSE-93]***

4. Find the equations of all lines of slope -1 that are tangents to the curve $y = \dfrac{1}{x-1}$.

5. Find the slope of normal at the point (am^3, am^2) to the curve $ax^2 = y^3$. ***[CBSE-1991]***

6. Find the equation of normal to the curve $x^2 = 4y$ which passes through the point $(1, 2)$. ***[NCERT-Book]***

7. Show that the curves $4x = y^2$ and $4xy = k$ cut at right angles, if $k^2 = 512$. ***[CBSE-96, 2002]***

8. For the curve $y = 4x^3 - 2x^5$. Find all points at which the tangent passes through the origin. ***[NCERT-Book]***

9. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-4, 0)$ and cuts the y-axis at the point Q where its gradient is 3. Find the equation of the curve completely.

10. Show that the condition that the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ should intersect orthogonally is that $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a'} - \dfrac{1}{b'}$

11. Find the equations of tangent and normal to the curve

    $x = a\cos t + at\sin t$
    $y = a\sin t - at\cos t$

    at any point t. Also show that the normal to the curve is at a constant distance from origin. ***[CBSE (SP)-2006, NCERT-Book]***

12. Prove that $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at (a, b) for all n.

13. Prove that no angle of intersection exists for the curves $x^2 + y^2 = a^2$ and $x^2 + y^2 = 4a^2$

14. Find the equations of tangent and normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).
[NCERT-Book, J & K-2004]

15. Find the equation of the tangent line to the curve $y = \sqrt{5x-3} - 2$ which is parallel to the line $4x - 2y + 3 = 0$
[CBSE-83, HSB-89]

16. Find the point on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to x-axis.
[CBSE-94, 95]

17. Find the equations of the tangent and the normal to the curve $y = x^2 + 4x + 1$ at $x = 3$.
[CBSE-2004]

18. Show that the curves $x = y^2$ and $xy = k$ cut at right angles, if $8k^2 = 1$
[CBSE-96, 2003, 2004, 2005, CHSB-2001, HPSB-2002S]

19. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. *[CBSE-2005]*

20. Find the equation of tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$.
[CBSE-1994]

21. Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).
[NCERT BOOK, HPSB-2000, 2001, 2003]

22. Find the point on the curve $y = x^3$ at which tangent is horizontal.

23. If the straight line $x \cos \alpha + y \sin \alpha = \rho$ touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $\rho^2 = a^2\cos^2 \alpha + b^2 \sin^2 \alpha$

24. Find a point on the parabola $y = (x - 2)^2$ whose tangent is parallel to the line joining (2, 0) and (4, 4)
[CBSE-93]

25. Find the equation of tangent to the curve $x^2 - 2y^2 = 8$, which are perpendicular to the line $x - y + 29 = 0$.

26. At what point on the curve $y = x^2$ on $[-2, 3]$ is the tangent parallel to x-axis?

27. Using Rolle's Theorem find the points on the curve $y = x^2$, $x \in [-2, 2]$ where the tangent is parallel to x-axis.
[CBSE-2000, 2001]

ANSWERS

1. $(\pi, -2)$
2. Increasing on $(-1, \infty)$; decreasing on $(-\infty, -1)$ Required point $(-3, 0)$
3. $\left(\pm\sqrt{\frac{7}{3}}, \mp\frac{2}{3}\sqrt{\frac{7}{3}}\right)$
4. $x + y + 1 = 0$ and $x + y - 3 = 0$
5. $\frac{-3}{2}$ m
6. $x + y - 3 = 0$
8. Points are (0, 0), (1, 2) and (–1, –2)
9. $y = -\frac{1}{2}x^3 - \frac{3}{4}x^2 + 3x + 5$
11. Equation of tangentis $y - (a \sin t - at \cos t) = \tan t\,[x - (a \cos t + at \sin t)]$; equation of normal $x \cos t + y \sin t - a = 0$
14. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$; $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$
15. $80x - 40y - 103 = 0$
16. $\left(\frac{3}{2}, \frac{-17}{2}\right)$
17. $10x - y - 8 = 0$ and $x + 10y - 223 = 0$

19. $48x - 24y = 23$

20. $2x + 2y = a^2$

21. $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$ and

$$\frac{y - y_0}{a^2 y_0} - \frac{x - x_0}{b^2 x_0} = 0$$

22. (0, 0)

24. (3, 1)

25. $x + y \pm 2 = 0$

26. (0, 0)

27. (0, 0)

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. Co-ordinates of a point on the curve $y = x \log x$ at which the normal is parallel to the line $2x - 2y = 3$ are:

[RPET-2000]

(a) (0, 0) (b) (e, e)
(c) $(e^2, 2e^2)$ (d) $e^{-2} - 2e^{-2})$

Solution

(d) $y = x \log x \Rightarrow \frac{dy}{dx} = 1 + \log x$

$$\text{The slope of the normal} = -\frac{1}{(dy/dx)} = \frac{-1}{1+\log x}$$

The slope of the line $2x - 2y = 3$ is 1.

$\therefore \quad \frac{-1}{1+\log x} = 1 \Rightarrow \log x = -2 \Rightarrow x = e^{-2}$

$\therefore \quad y = -2e^{-2}$

$\therefore$ Co-ordinate of the point is $(e^{-2}, -2e^{-2})$

2. If the normal to the curve $y^2 = 5x - 1$, at the point (1, –2) is of the form $ax - 5y + b = 0$, then a and b are: ***[Pb. CET-2001]***

(a) 4, –14 (b) 4, 14
(c) –4, 14 (d) –4, –14

Solution

(a) We have $y^2 = 5x - 1$ .... (i)

At (1, –2); $\frac{dy}{dx} = \left[\frac{5}{2y}\right]_{(1,-2)} = \frac{-5}{4}$

$\therefore$ Equation of normal at the point (1, –2) is, $\frac{4}{5}$

$y - (-2) = (x - 1)$ ($\because$ Slope of normal $= \frac{4}{5}$)

$\therefore \quad 4x - 5y - 14 = 0$ ....(ii)

As the normal is of the form $ax - 5y + b = 0$, comparing this with (ii), we get $a = 4$ and $b = -14$.

3. The normal to the curve $x = a(1 + \cos\theta)$, $y = a \sin\theta$ at 'θ' always passes through the fixed point ***[AIEEE-2004]***

(a) (a, a) (b) $(0, a)$
(c) (0, 0) (d) $(a, 0)$

Solution

(d) Slope of normal

$$= \frac{-dx}{dy} = \frac{\frac{-d\{a(1+\cos\theta)\}}{d\theta}}{\frac{d(a\sin\theta)}{d\theta}} = \frac{a\sin\theta}{a\cos\theta} = \tan\theta$$

Now, the equation of normal at θ is,
$y - a\sin\theta = \tan\theta\,[x - a(1 + \cos\theta)]$
Clearly, this line passes through $(a, 0)$.

4. If ST and SN are the lengths of the subtangent and the subnormal at the point $\theta = \frac{\pi}{2}$ on the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, $a \neq 1$, then:

[Karnataka CET-2005]

(a) $ST = SN$ (b) $ST = 2SN$
(c) $ST^2 = aSN^3$ (d) $ST^3 = aSN$

Solution

(a) $\frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a(\sin\theta)$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = 1, \left. y\right|_{\theta=\frac{\pi}{2}} = a$$

Length of sub-tangent $ST = \frac{y}{dy/dx} = \frac{a}{1} = a$

and length of sub-normal $SN = y\frac{dy}{dx} = a\,.1 = a$

Hence $ST = SN$

5. The equation of the tangent to the curve $x = 2\cos^3\theta$ and $y = 3\sin^3\theta$ at the point $\theta = \pi/4$ is: ***[J & K-2005]***

(a) $2x + 3y = 3\sqrt{2}$ (b) $2x - 3y = 3\sqrt{2}$

(c) $3x + 2y = 3\sqrt{2}$ (d) $3x - 2y = 3\sqrt{2}$

Solution

(c) $x\big|_{\theta=\frac{\pi}{4}} = \dfrac{2}{2\sqrt{2}} = \dfrac{1}{\sqrt{2}}$,

$$y\big|_{\theta=\frac{\pi}{4}} = \frac{3}{2\sqrt{2}}, \frac{dy}{dx}\bigg|_{\theta=\frac{\pi}{4}} = \frac{9\sin^2\theta\cos\theta}{-6\cos^2\theta\sin\theta}\bigg|_{\theta=\frac{\pi}{4}} = \frac{-3}{2}$$

$\therefore$ Equation of tangent is

$$\left(y - \frac{3}{2\sqrt{2}}\right) = \frac{-3}{2}\left(x - \frac{1}{\sqrt{2}}\right)$$

$\Rightarrow$ $3\sqrt{2}x + 2\sqrt{2}y = 6 \Rightarrow 3x + 2y = 3\sqrt{2}$

6. The curve given by $x + y = e^{xy}$ has a tangent parallel to the y-axis at the point ***[AMU-2005]***

(a) (0, 1) (b) (1, 0)

(c) (1, 1) (d) (–1, –1)

Solution

(b) Curve $x + y = e^{xy}$. Differentiating with respect to x

$$1 + \frac{dy}{dx} = e^{xy}\left(y + x\frac{dy}{dx}\right)$$

or $$\frac{dy}{dx} = \frac{ye^{xy} - 1}{1 - xe^{xy}}$$

$$\frac{dy}{dx} = \infty$$

$\Rightarrow$ $1 - xe^{xy} = 0 \Rightarrow 1 - x(x + y) = 0$

This hold for $x = 1, y = 0$

7. The function $f(x) = (x - 3)^2$ satisfies all the conditions of mean value theorem in [3, 4]. A point on $y = (x - 3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1) is:

(a) (7/2, 1/2) (b) (7/2, 1/4)

(c) (1, 4) (d) (4, 1)

Solution

(b) Let the point be (x_1, y_1). Therefore $y_1 = (x_1 - 3)^2$ .... (i)

Now slope of the tangent at (x_1, y_1) is $2(x_1 - 3)$, but it is equal to 1. Therefore, $2(x_1 - 3) = 1 \Rightarrow x_1 = 7/2$

$\therefore$ $y_1 = \left(\dfrac{7}{2} - 3\right)^2 = \dfrac{1}{4}$. Hence the point is $\left(\dfrac{7}{2}, \dfrac{1}{4}\right)$

8. At what points of the curve $y = \dfrac{2}{3}x^3 + \dfrac{1}{2}x^2$, tangent makes the equal angle with axis ***[UPSEAT-1999]***

(a) $\left(\dfrac{1}{2}, \dfrac{5}{24}\right)$ and $\left(-1, -\dfrac{1}{6}\right)$

(b) $\left(\dfrac{1}{2}, \dfrac{4}{9}\right)$ and (–1, 0)

(c) $\left(\dfrac{1}{3}, \dfrac{1}{7}\right)$ and $\left(-3, \dfrac{1}{2}\right)$

(d) $\left(\dfrac{1}{3}, \dfrac{4}{47}\right)$ and $\left(-1, -\dfrac{1}{3}\right)$

Solution

(a) $y = \dfrac{2}{3}x^3 + \dfrac{1}{2}x^2 \Rightarrow \dfrac{dy}{dx} = 2x^2 + x$ ... (i)

Now tangent makes equal angle with axis

$\therefore$ $y = 45°$ or $-45°$

$\therefore$ $\dfrac{dy}{dx} = \tan(\pm 45°) = \pm\tan(45°) = \pm 1$

$\therefore$ From equation (i), $2x^2 + x = 1$ (taking + ve sign)

$\Rightarrow$ $2x^2 + x - 1 = 0$

$\Rightarrow$ $(2x - 1)(x + 1) = 0$

$\therefore$ $x = 1/2, -1$

From the given curve, when $x = \dfrac{1}{2}$,

$$y = \frac{2}{3}.\frac{1}{8} + \frac{1}{2}.\frac{1}{4} = \frac{1}{12} + \frac{1}{8} = \frac{5}{24} \text{ and when } x = -1,$$

$$y = \frac{2}{3}(-1) + \frac{1}{2}.1 = -\frac{2}{3} + \frac{1}{2} = -\frac{1}{6}$$

Therefore, required points are $\left(\dfrac{1}{2}, \dfrac{5}{24}\right)$ and $\left(-1, -\dfrac{1}{6}\right)$

9. The length of the subtangent to the curve $x^2y^2 = a^4$ at the point $(-a, a)$ is: ***[Karnataka CEE-2001]***

(a) $3a$ (b) $2a$

(c) a (d) $4a$

Solution

(c) Equation of curve $x^2y^2 = a^4$

Differentiating the given equation

$$x^2y^2\frac{dy}{dx} + y^2 2x = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(-a,a)} = -\left(\frac{a}{-1}\right) = 1$$

Therefore Sub-tangent $= \dfrac{y}{\left(\dfrac{dy}{dx}\right)} = a.$

10. An equation of tangent to the curve $y = x^4$ from the point (2, 0) not on the curve is: ***[RPET 2000]***

(a) $y = 0$ (b) $x = 0$
(c) $x + y = 0$ (d) None of these

Solution

(a) Let the point of contact be (h, k) where $k = h^4$

Tangent is $y - k = 4h^3 (x - y)$ $\left[\because \dfrac{dy}{dx} = 4x^3\right]$

It passes through (2, 0) $\therefore$ $-k = 4h^3 (2 - h)$

$\Rightarrow$ $h = 0$ or $8/3$

$\therefore$ $k = 0$ or $(8/3)^4$

$\therefore$ points of contact are (0, 0) and

$$\left(\frac{8}{3}, \left(\frac{8}{3}\right)^4\right)$$

$\therefore$ Equation of tangents are $y = 0$ and

$$y - \left(\frac{8}{3}\right)^4 = 4\left(\frac{8}{3}\right)^3\left(x - \frac{8}{3}\right) \text{ i.e., } y = 0.$$

11. The sum of intercept on the co-ordinates axes made by tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is ***[RPET-1999]***

(a) a (b) $2a$
(c) $2\sqrt{a}$ (d) None of these

Solution

(a) $\sqrt{x} + \sqrt{y} = \sqrt{a} = a;$ $\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$, Hence tangent at (x, y) is

$$Y - y = -\frac{\sqrt{y}}{\sqrt{x}}(X - x)$$

or $X\sqrt{y} + Y\sqrt{x} = \sqrt{xy}(\sqrt{x} + \sqrt{y}) = \sqrt{axy}$

or $\dfrac{X}{\sqrt{x}\sqrt{x}} + \dfrac{Y}{\sqrt{a}\sqrt{y}} = 1$, Clearly, its intecepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$

Sum of intercept
$= \sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{a} \times \sqrt{a} = a$

12. If tangent to the curve $y^2 = x^3$ at point (m^2, m^3) is also a normal to the curve at point (M^2, M^3), then mM is equal to ***[NDA-05]***

(a) $-1/9$ (b) $-2/9$
(c) $-1/3$ (d) $-4/9$

Solution

(d) On differentiating $2y\dfrac{dy}{dx} = 3x^2$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$$

$\therefore$ slope of tangent at point (m^2, m^3) $= \dfrac{3m^4}{2m^3} = \dfrac{3}{2}m$ and slope of the normal at point (M^2, M^3)

$$= -\left(\frac{2y}{3x^2}\right)_M = -\frac{2M^3}{3M^4} = -\frac{2}{3M}$$

$$\therefore \frac{3}{2}m = -\frac{2}{3M}$$

$$\Rightarrow mM = -\frac{4}{9}$$

13. The equation of the tangents to the curve $y = (x^3 - 1)(x - 2)$ at the points where it meets x-axis are ***[Roorkee-80]***

(a) $y + 3x = 3, y - 7x - 14 = 0$
(b) $y - 3x = 3, y - 7x + 14 = 0$
(c) $y + 3x = 3, y - 7x + 14 = 0$
(d) None of these

Solution

(c) $y = 0 \Rightarrow x = 1, 2$. Hence given points are (1, 0) and (2, 0).

Now find equation of tangent at two points.

$$\frac{dy}{dx} = 3x^2(x - 2) + x^3 - 1$$

slope of tangents at (1, 0); $m_1 = -3$
at (2, 0); $m_2 = 7$
$\therefore$ eqn are $y - 0 = -3(x - 1); y - 0 = 7(x - 2)$

14. If the sum of the squares of the intercepts made by the tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$

($a > 0$) at point (a/8, a/8) on coordinate axes is 2, then the value of a is: *[DCE-94]*

(a) 1 (b) 2
(c) 4 (d) 8

Solution

(c) Equation of tangent (x_1, y_1) is

$$\frac{x}{x_1^{2/3}} + \frac{y}{y_1^{2/3}} = a^{1/3} \qquad \text{..... (i)}$$

Sum of squares of intersects

$x_1^{4/3}\, a^{2/3} + y_1^{4/3}\, a^{2/3} = a^{2/3}$

$$\left[\left(\frac{a}{8}\right)^{4/3} + \left(\frac{a}{8}\right)^{4/3}\right] = a\left(\frac{2\times a^{4/3}}{16}\right) = \frac{a^2}{16} = 1$$

$\Rightarrow \quad a = 4$

15. The point on the curve $y^2 = 2x^3$ where tangent to the curve is perpendicular to the line $4x - 3y = 0$ is: *[MNR-98]*

(a) (2, 4) (b) $(1, \sqrt{2})$
(c) (1/2, 1/2) (d) (1/8, –1/16)

Solution

(d) $\frac{dy}{dx} = \frac{3x^2}{y}$. At required point $\frac{dy}{dx} = -\frac{3}{4}$.

$\Rightarrow \quad \frac{3x^2}{y} = -\frac{3}{4} \Rightarrow y = -4x^2$

Solving this with the equation of the curve, $x = 1/8, y = -1/6$

16. One angle of intersection between the curves $y^2 = 2x/\pi$ and $y = \sin x$ is:

(a) $\tan^{-1}(-1/\pi)$ (b) $\cot^{-1}(-1/\pi)$
(c) $\tan^{-1}\pi$ (d) $\cot^{-1}\pi$

Solution

(d) For point of intersection $\sin^2 x = 2x/\pi$

$\Rightarrow \quad x = 0, \pi/2 \Rightarrow y = 0, 1$

$\Rightarrow$ points of intersection are (0, 0), (π/2, 1)

Now $y^2 = \frac{2x}{\pi} \Rightarrow \frac{dy}{dx} = \frac{1}{\pi y}$

$y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

At (π/2, 1), $\left(\frac{dy}{dx}\right)_1 = \frac{1}{\pi}, \left(\frac{dy}{dx}\right)_2 = 0$

$\therefore$ Angle of intersection $= \tan^{-1}\left(\frac{1/\pi - 0}{1 + 1/\pi.0}\right)$

$= \tan^{-1}(1/\pi) = \cot^{-1}\pi$

17. The length of the tangent at any point to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ which is intercepted between the axes, is *[EAMCET-91, PET (Raj.)-2002]*

(a) a (b) $2a$
(c) $\sqrt{a}$ (d) $a/2$

Solution

(a) Any point on the curve is $(a\cos^3\theta, a\sin^3\theta)$

$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = \frac{\sin\theta}{\cos\theta}$ $\therefore$ equation of the tangent at this point is

$$y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$$

$\Rightarrow \quad \frac{x}{\cos\theta} + \frac{y}{\sin\theta} = a$

Its length intercepted between axes

$= \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = a$

18. The angle of intersection between curves $x^2 + 4y + 1 = 0$ and $x^2 - 4y - 1 = 0$ is: *[UPSEAT-2002; Haryana (CEET)-2004]*

(a) $\pi/2$ (b) $\pi/4$
(c) $\pi/3$ (d) 0

Solution

(d) $\left(\frac{dy}{dx}\right)_1 = -\frac{x}{2}, \left(\frac{dy}{dx}\right)_2 = \frac{x}{2}$ Point of intersection = (0, –1/4)

At this point $\left(\frac{dy}{dx}\right)_1 = 0, \left(\frac{dy}{dx}\right)_2 = 0$

$\Rightarrow$ angle of intersection = 0^c.

19. The equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2) is: *[IIT-84; PET-92]*

(a) $x - y - 3 = 0$ (b) $x + y + 3 = 0$
(c) $x + y - 3 = 0$ (d) $x - y + 3 = 0$

Solution

(c) $\frac{dy}{dx} = \frac{x}{2}$. Let normal point be (x_1, y_1). Then slope of the normal $= -2/x_1$. Hence equation of normal at (x_1, y_1) is:

$$y - y_1 = -\frac{2}{x_1}(1 - x_1) \qquad \text{........ (1)}$$

$\because$ It passes through given point (1, 2), so

$$2 - y_1 = -\frac{2}{x_1}(1 - x_1)$$

$\Rightarrow \quad x_1 y_1 = 2 \quad \ldots (2)$

Also (x_1, y_1) lies on the given curve, so $x_1^2 = 4y_1$ ....(3)

(2), (3) $\Rightarrow x_1 = 2, y_1 = 1$

Hence from (1), equation of normal will be $y - 1 = -(x - 2)$

$\Rightarrow \quad x + y = 3$

20. If the function $f(x) = x^3 - 6ax^2 + 5x$ satisfies the conditions of Lagrange's mean value theorem for the interval [1, 2] and the tangent to the curve $y = f(x)$ at $x = \frac{7}{4}$ is parallel to the chord that joins the points of intersection of the curve with the ordinates $x = 1$ and $x = 2$. Then the value of a is

[MPPET-1998]

(a) $\frac{35}{16}$ (b) $\frac{35}{48}$

(c) $\frac{7}{16}$ (d) $\frac{5}{16}$

Solution

(b) $f(b) = f(2) = 8 - 24a + 10 = 18 - 24a$

$f(a) = f(1) = 1 - 6a + 5 = 6 - 6a$

$f'(x) = 3x^2 - 12ax + 5$

From Lagrange's mean value theorem,

$$f'(x) = \frac{f(b) - f(a)}{b - a} = \frac{18 - 24a - 6 + 6a}{2 - 1}$$

$\therefore \quad f'(x) = 12 - 18a$

At $x = \frac{7}{4}, 3 \times \frac{49}{16} - 12a \times \frac{7}{4} + 5 = 12 - 18a$

$$\Rightarrow \quad 3a = \frac{147}{16} - 7 \Rightarrow 3a = \frac{35}{16} \Rightarrow a = \frac{35}{48}$$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The slope of the normal to the curve $y = \cos 2x$ at $\pi/6$ is

(a) $\sqrt{3}$ (b) $-1/\sqrt{3}$

(c) $2/\sqrt{3}$ (d) None of these

2. The tangent to the curve $x^2 + y^2 - 2x - 3 = 0$ is parallel to x-axis at the point

(a) $(2, \pm\sqrt{3})$ (b) $(1, \pm 2)$

(c) $(\pm 1, 2)$ (d) $(\pm 3, 0)$

3. The tangent to the curve $y = 2x^2 - x + 1$ is parallel to the line $y = 3x + 9$ at the point whose co-ordinate are

(a) $(-1, 5)$ (b) $(1, 2)$

(c) $(2, 7)$ (d) $(3, 16)$

4. The equation of normal to the curve $y = x + \sin x \cos x$ at $x = \pi/2$ is

(a) $x = 2$ (b) $x = -\pi$

(c) $x = \pi$ (d) $x = \pi/2$

5. If $y = 4x - 5$ is tangent to the curve $y^2 = px^3 + q$ at (2, 3) then

(a) $p = 2, q = -7$ (b) $p = -2, q = 7$

(c) $p = -2, q = -7$ (d) $p = 2, q = 7$

6. The equation of tangent to the curve $y = 2 \cos x$ at $x = \pi/4$ is

(a) $y - \sqrt{2} = 2\sqrt{2}\left(x - \frac{\pi}{4}\right)$

(b) $y + \sqrt{2} = \sqrt{2}\left(x + \frac{\pi}{4}\right)$

(c) $y - \sqrt{2} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)$

(d) $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$

7. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$ $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that

[IIT-1983; DCE-2000; AIEEE-2005]

(a) It makes a constant angle with x-axis

(b) It pass through the origin

(c) It has constant distance from the origin

(d) None of these

8. If the parametric equation of the curve is given by $x = e^t \cos t, y = e^t \sin t$, then the angle made by tangent to the curve at the point $t = \pi/4$ with axis of x:

(a) π (b) $\pi/2$
(c) 2π (d) $\pi/4$

9. The abscissa of the point on the curve $y = a(e^{x/a} + e^{-x/a})$ where the tangent is parallel to the x-axis is: ***[MPPET-2008]***

(a) 0 (b) a
(c) $2a$ (d) $-2a$

10. The line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point

(a) $(a, b/a)$ (b) $(-a, b/a)$
(c) $(a, a/b)$ (d) None of these

11. The co-ordinates of the point on the curve $y = x^2 + 3x + 4$. The points at which tangent passes through the origin are:

(a) (2, 2); (–4, 14) (b) (–2, 2); (2, 14)
(c) (–2, –2); (2, –14) (d) None of these

12. The length of subtangent to the curve $\sqrt{x} + \sqrt{y} = 3$ at the point (4, 1) is:

(a) –2 (b) 1/2
(c) –3 (d) 4

13. If $ax + by + c = 0$ is a normal to the curve $xy = 1$ then: ***[IIT-1986; UPSEAT-2005]***

(a) $a < 0, b > 0$
(b) $a < 0, b < 0$
(c) $a > 0, b < 0$ or $a < 0, b > 0$
(d) $a, b \in R$

14. If at any point S of the curve $by^2 = (x + a)^3$, the relation between subnormal SN and subtangent ST be $p(SN) = q(ST)^2$ then p/q is equal to ***[PET (Raj.)-99; EAMCET-1991]***

(a) $8b/27$ (b) $8a/27$
(c) b/a (d) None of these

15. The point (S) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical (Parallel to y-axis), is (are): ***[IIT Screening-2002]***

(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\frac{\sqrt{11}}{3}, 1\right)$
(c) $(0, 0)$ (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

16. For the curve $by^2 = (x + a)^3$ the square of sub-tangent is proportional to: ***[RPET-1999]***

(a) $(\text{subnormal})^{1/2}$ (b) subnormal
(c) $(\text{subnormal})^{3/2}$ (d) None of these

17. The area of the triangle formed by the co-ordinate axes and the normal to the curve $y = e^{2x} + x^2$ at the point (0, 1) is: ***[Kerala PET-2008]***

(a) 0 (b) 1
(c) 1/2 (d) 2

18. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets the x-axis at: ***[RPET-2002]***

(a) (0, 0) (b) (2, 0)
(c) (–1/2, 0) (d) None of these

19. The slope of the tangent curve represented by $x = t^2 + 3t - 8y = 2t^2 - 2t - 5$ at the point $M(2, -1)$ is: ***[MPPET-2008]***

(a) 7/6 (b) 2/3
(c) 3/2 (d) 6/7

20. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x-axis then $f'(3)$ is equal to: ***[IIT Screening-2000; DCE-2001]***

(a) –1 (b) –3/4
(c) 4/3 (d) 1

21. The slope of normal at the point $(at^2, 2at)$ of parabola $y^2 = 4ax$ is:

(a) $1/t$ (b) t
(c) $-t$ (d) $-1/t$

22. The length of tangent to the curve $x = a(\cos t + \log \tan t/2)$ $y = a \sin t$ is:

(a) ax (b) ay
(c) a (d) xy

23. For the curve $xy = c^2$ the subnormal at any point varies as: ***[Karnataka-CET-2003]***

(a) x^2 (b) x^3
(c) y^2 (d) y^3

SOLUTIONS

1. (d) Slope of normal $= \dfrac{-1}{\frac{dy}{dx}} = \dfrac{-1}{-2\sin 2x}$ at

$x = \dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$

2. (b) $\because$ tangent is parallel to x axis $\dfrac{dy}{dx} = 0$;

Given equation is $y^2 = -x^2 + 2x + 3$

differentiating, $2y\dfrac{dy}{dx} = -2x + 2 = 0$

$x = 1$, put in equation $y^2 = -1 + 2 + 3$

$y = \pm 2$; point is $(1, \pm 2)$.

4. (d) Given curve $y = x + \sin x \cos x$

at $x = \pi/2$; $y = \dfrac{\pi}{2} + \sin\dfrac{\pi}{2}\cos\dfrac{\pi}{2} \Rightarrow y = \dfrac{\pi}{2}$

Also $\dfrac{dy}{dx} = 1 + \dfrac{d}{dx}\left(\dfrac{1}{2}\sin 2x\right)$ at $x = \dfrac{\pi}{2}$

$= 1 + \cos 2x$ at $x = \dfrac{\pi}{2} = 1 - 1 = 0$

$\therefore$ Equation is $y - \dfrac{\pi}{2} = 0\left(x - \dfrac{\pi}{2}\right) \Rightarrow y = \dfrac{\pi}{2}$

5. (a) Step 1: The equation of tangent to the curve $y = f(x)$ at the point $P(x_1, y_1)$ is

$$y - y_1 = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} (x - x_1)$$

So from given curve

$$2y\frac{dy}{dx} = 3px^2$$

$$\left.\frac{dy}{dx}\right|_{(2,3)} = \left.\frac{3px^2}{2y}\right|_{(2,3)} = \frac{12p}{6} = 2p$$

$\therefore$ The equation of tangent is

$$y - 3 = 2p(x - 2) \Rightarrow y - 3 = 2px - 4p$$

$\Rightarrow \quad y = 2px - 4p + 3$ ....(1)

$y = 4x - 5$ (given) ....(2)

Equation (1) and (2) are coinciding each other if $2p = 4 \Rightarrow p = 2$

Step 2: Putting $x = 2, y = 3$ and $p = 2$ in curve

$9 = 16 + q \Rightarrow q = -7$

6. (c) Given $y = 2\cos x$ is curve

If $x = \dfrac{\pi}{4}$; $y = 2\cos\dfrac{\pi}{4} = 2.\dfrac{1}{\sqrt{2}} = \sqrt{2}$

Equation is $y - \sqrt{2} = \dfrac{dy}{dx}\left(x - \dfrac{\pi}{4}\right)$

$$\frac{dy}{dx} = -2\sin x$$

at $\quad x = \dfrac{\pi}{4} = -\sqrt{2}$

$\therefore$ Equation is $y - \sqrt{2} = -\sqrt{2}\left(x - \dfrac{1}{\sqrt{2}}\right)$

7. (c) Step 1: $x = a(\cos\theta + \theta\sin\theta)$

$\Rightarrow \dfrac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta)$

$\Rightarrow \dfrac{dx}{d\theta} = a\theta\cos\theta$ .... (1)

and $y = a(\sin\theta - \theta\cos\theta)$

$\dfrac{dy}{d\theta} = a[\cos\theta - \cos\theta + \theta\sin\theta]$

$\Rightarrow \dfrac{dy}{d\theta} = a\theta\sin\theta$ .... (2)

Step 2: From equation (1) and (2) we get

$\dfrac{dy}{dx} = \tan\theta \Rightarrow$ slope of normal $= -\cot\theta$

Equation of normal at 'θ' is

$y - a(\sin x\ \theta - \theta\cos\theta)$

$= \cot\theta(x - a(\cos\theta + \theta\sin\theta))$

$\Rightarrow \quad y\sin\theta - a\sin^2\theta + a\,\theta\cos\theta\sin\theta$

$= -x\cos\theta + a\cos^2\theta + a\,\theta\sin\theta\cos\theta$

$\Rightarrow \quad x\cos\theta + y\sin\theta = a$

Clearly this is an equation of straight line which is at a constant distance 'a' from origin.

8. (b) $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{1}{\dfrac{dx}{dt}} = \dfrac{-e^t(\cos t + \sin t)}{e^t(\cos t - \sin t)}$

at $t = \dfrac{\pi}{4}; \dfrac{dy}{dx} = \dfrac{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}}{\cos\frac{\pi}{4} - \sin\frac{\pi}{4}}$ = Not defined,

$\theta = 90°$

with x-axis.

9. (a) $\dfrac{dy}{dx} = a\left(e^{x/a} - e^{-x/a}\right).\dfrac{1}{a} = 0$

$e^{x/a} = e^{-x/a} \Rightarrow x = 0$

NOTE

If a line is parallel to x-axis, slope = 0

10. (d) Let point of contact be (x_1, y_1)

$\therefore \quad \dfrac{dy}{dx} = be^{-x_1/a}\left(\dfrac{-1}{a}\right) = \dfrac{-b}{a}$

$e^{-x_1/a} = 1 \Rightarrow x_1 = 0$; put in $\dfrac{x}{a} + \dfrac{y}{b} = 1$

to get $y = b$,

$\therefore$ Point is $(0, b)$

11. (b) Step 1: The given equation of the curve is

$y = x^2 + 3x + 4$ ....(1)

Differentiating (1) w.r.t. x, we get

$\dfrac{dy}{dx} = 2x + 3$

Let the tangent at (x_1, y_1) to the curve (1) passes through the origin

$\therefore \quad \left(\dfrac{dy}{dx}\right)\Bigg|_{(x_1, y_1)} = 2x_1 + 3$

Step 2: The equation of tangent to (1) at (x_1, y_1) is

$y - y_1 = \left(\dfrac{dy}{dx}\right)\Bigg|_{(x_1, y_1)}(x - x_1)$

or $\quad y - y_1 = (2x_1 + 3)(x - x_1)$

But it passes through the origin (0, 0)

$0 - y_1 = (2x_1 + 3)(0 - x_1)$

or $\quad -y_1 = -2x_1^2 - 3x_1$

$\therefore \quad y_1 = 2x_1^2 + 3x_1$ .... (2)

But (x_1, y_1) is on (1)

$\therefore \quad y_1 = x_1^2 + 3x_1 + 4$ .... (3)

From (2) and (3),

Step 3: $x_1^2 = 4 \Rightarrow x_1 = \pm 2$

when $x_1 = 2$, from (2),

$y_1 = 2.(-2)^2 + 3.(-2) = 2$

Hence the required points are (2, 14) and (–2, 2).

12. (a) Consider $\sqrt{x} + \sqrt{y} = 3$

differentiating w.r.t. x

$\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}\Bigg|_{(4,1)}$

$\dfrac{dy}{dx} = -\dfrac{1}{2}$

$\because$ Length of subtangent $= \dfrac{y_1}{\dfrac{dy}{dx}\Big|_{(x_1, y_1)}}$

$= \dfrac{y_1}{m} = -2$

13. (a, c) Slope of normal $= \dfrac{-a}{b}$

$\dfrac{-1}{\dfrac{dy}{dx}} = \dfrac{-a}{b} \Rightarrow \dfrac{dy}{dx} = b/a$ where $y = \dfrac{1}{x}$

$\dfrac{-1}{x^2} = b/a; \therefore \dfrac{b}{a} < 0$

i.e., $a > 0$ then $b < 0$ or $a < 0$ then $b > 0$

14. (a) Given $by^2 = (x + a)^3$ ....(i)

$2by\dfrac{dy}{dx} = 3(x + a)^2 \Rightarrow \dfrac{dy}{dx} = \dfrac{3(x + a)^2}{2by}$

$SN = y \times m = \dfrac{3(x + a)^2}{2b}$

$ST = \dfrac{y}{m} = \dfrac{2by^2}{3(x + a)^2}$

$= \dfrac{2(x + a)^3}{3(x + a)^2} = \dfrac{2(x + a)}{3}$

(using ....(i))

$\dfrac{p}{q} = \dfrac{(ST)^2}{SN} = \dfrac{4}{9}(x + a)^2 \times \dfrac{2b}{3(x + a)^2} = \dfrac{8b}{27}$

15. (d) Step 1: For vertical tangent $\frac{dx}{dy} = 0$

$\therefore$ Differentiating w.r.t. y

$$3y^2 + 6x\frac{dy}{dx} = 12 \Rightarrow \frac{dx}{dy} = \frac{12 - 3y^2}{6x} = 0$$

$\Rightarrow$ $3y^2 = 12 \Rightarrow y = \pm 2$

Putting $y = -2$, $3x^2 = -24 + 8$

$$x^2 = \frac{-16}{3} \text{ not possible}$$

$\therefore$ only possible values is $y = 2$, which is in option (D)

NOTE

For real x, $x^2 \geq 0$ i.e.,
Square of real value is positive or zero.

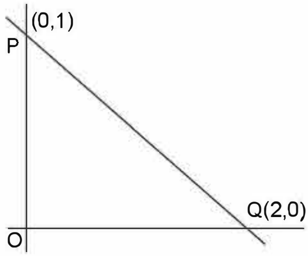
16. (b) From Question 14;

$\frac{(ST)^2}{SN} = \text{const} \Rightarrow \therefore (ST)^2$ is proportional to SN.

17. (b) Slope of normal $= \frac{-1}{\frac{dy}{dx}} = -\frac{1}{2e^{2x} + 2x}$

at $x = 0$ slope $= -\frac{1}{2}$

$\therefore$ Equation of normal, $y - 1 = -\frac{1}{2}(x - 0)$



$$2y - 2 = -x \Rightarrow x + 2y = 2$$

$$\frac{x}{2} + \frac{y}{1} = 1,$$

OPQ is required triangle

Area $= \frac{1}{2} \times OP \times OQ = \frac{1}{2} \times 1 \times 2 = 1$ sq.unit.

18. (c) $y = e^{2x}$ is given curve.

Equation of tangent; $y - 1 = m(x - 0)$

$$m = \frac{dy}{dx} = 2e^{2x} \text{ at } x = 0, m = 2$$

$\therefore$ $y - 1 = 2x \Rightarrow -2x + y = 1$

$\frac{x}{-\frac{1}{2}} + \frac{y}{1} = 1$, x intercept is $\left(-\frac{1}{2}, 0\right)$

19. (a) Here $x = t^2 + 3t - 8 = 2$

$$t^2 + 3t - 10 = 0 \Rightarrow (t + 5)(t - 2) = 0$$

$\therefore$ $t = -5$ or $t = 2$ ....(i)

Also

$y = 2t^2 - 2t - 5 = -1 \Rightarrow t^2 - t - 2 = 0$

$(t - 2)(t + 1) = 0$ or $t = -1, 2$ .... (ii)

Taking common value from (i) and (ii) $t = 2$

$$\text{Slope} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + 3}{4t - 2} \text{ at } t = 2$$

$$= \frac{7}{6}$$

20. (d) Step 1: Slope of normal at a point (3, 4).

$$= \frac{-1}{\left.\frac{dy}{dx}\right|_{(3,4)}} = -\frac{1}{f'(3)}$$

Step 2: By definition; slope of normal $= \tan\theta$ where θ is angle between positive direction of x-axis and normal.

$$\tan\theta = \tan 135 = -1$$

Step 3: $-\frac{1}{f'(3)} = -1 \Rightarrow f'(3) = 1$

21. (c) Slope of normal $= \frac{-1}{\frac{dy}{dx}} = \frac{-dx}{dy}$

Given curve $y^2 = 4ax$

differentiating, $2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{2a}{y}$; m (normal) $= \frac{-y}{2a}$ at $y = 2at$

$\therefore M$ (normal) $= -t$

22. (c) Slope of tangent $= m = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$m = \frac{a\cos t}{a\left(-\sin t + \dfrac{1}{\tan\frac{t}{2}}\cdot\sec^2\dfrac{t}{2}\cdot\dfrac{1}{2}\right)}$$

$$m = \frac{\cos t}{-\sin t + \dfrac{1}{\sin t}} = \frac{\cos t\sin t}{1-\sin^2 t} = \frac{\cos t\sin t}{\cos^2 t}$$

$= \tan t$

length of tangent $= y\sqrt{1+\dfrac{1}{m^2}}$

$$= y\sqrt{1+\cot^2 t} = y\times\operatorname{cosec} t = a\sin t.\frac{1}{\sin t}$$

$y = a$

23. (d) subnormal $= y \times m$; m = slope of tangent

$$m = \frac{dy}{dx} = \frac{-c^2}{x^2} \qquad \left(\because y = \frac{c^2}{x}\right)$$

$$\text{Subnormal} = \frac{c^2}{x}\cdot\left(\frac{-c^2}{x^2}\right) = \frac{-c^4}{x^3}$$

$$= -c^4 \times \left(\frac{y}{c^2}\right)^3 = \frac{-y^3}{c^2}$$

$\therefore$ Subnormal varies as y^3.

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The slope of tangent to the curve $y = (x+1)(x-3)$ at the points when it cut the axis of x are:

(a) 1, 3 (b) −1, 3
(c) 4, −4 (d) None of these

2. For the curve $x = t^2 - 1, y = t^2 - t$. The tangent line is perpendicular to the x-axis where:

(a) $t = 0$ (b) $t = \infty$
(c) $t = 1/\sqrt{3}$ (d) $t = -1/\sqrt{3}$

3. The straight line $x + y = a$ will be tangent to the curve $\dfrac{x^2}{9}+\dfrac{y^2}{16} = 1$ at $a =$

(a) 8 (b) ± 5
(c) ± 10 (d) ± 6

4. The points on the curve $y = 12x - x^3$ at which the gradient is zero are

(a) (0, 2), (2, 16)
(b) (0, −2), (2, 16)
(c) (2, −16), (−2, 16)
(d) (2, 16), (−2, −16)

5. The point of curve $y^2 = 2(x-3)$ at which the normal is parallel to line $y - 2x + 1 = 0$ is:

[MPPET-1998]

(a) (5, 2) (b) (−1/2, −2)
(c) (5, −2) (d) (3/2, 2)

6. If the equation of the curve remain unchanged by replacing x and y from y and x respectively then the curve is:

(a) Symmetric along x-axis
(b) Symmetric along y-axis
(c) Symmetric along the line y-axis
(d) Symmetric along line $y = x$.

7. The slope of the tangent to the curve $x = 3t^2 + 1, y = t^3 - 1$ at $x = 1$ is:

[Karnataka-CET-2003]

(a) 0 (b) 1/2
(c) ∞ (d) −2

8. If the line $y = 2x + k$ is a tangent to the curve $x^2 = 4y$, then k is equal to

(a) 4 (b) 1/2
(c) −4 (d) −1/2

9. At what point on the curve $x^3 - 8a^2y = 0$, the slope of the normal is −2/3

[RPET-2002]

(a) (a, a) (b) $(2a, -a)$
(c) $(2a, a)$ (d) None of these

10. The equation of tangent at (+4, −4) on the curve $x^2 = -4y$ is

[Karnataka CET-2001; Pb. CET-2000]

(a) $2x + y + 4 = 0$ (b) $2x - y - 12 = 0$
(c) $2x + y - 4 = 0$ (d) $2x - y + 4 = 0$

11. The abscissa of the points where the tangent to curve $y = x^3 - 3x^2 - 9x + 5$ is parallel to x-axis, are ***[Karnataka CET-2001]***
(a) 0 and 0 (b) $x = 1$ and -1
(c) $x = 1$ and -3 (d) $x = -1$ and 3

12. If normal to the curve $y = f(x)$ is parallel to x-axis, then correct statement is ***[RPET-2000]***
(a) $\frac{dy}{dx} = 0$ (b) $\frac{dy}{dx} = 1$
(c) $\frac{dx}{dy} = 0$ (d) None of these

13. The equation of the normal to the curve $y = \sin\frac{\pi x}{2}$ at (1, 1) is
(a) $y = 1$ (b) $x = 1$
(c) $y = x$ (d) $y - 1 = \frac{-2}{\pi}(x - 1)$
[AMU-99]

14. The tangent to the curve $y = ax^2 + bx$ at (2, –8) is parallel to x-axis. Then ***[AMU-1999]***
(a) $a = 2, b = -2$ (b) $a = 2, b = -4$
(c) $a = 2, b = -8$ (d) $a = 4, b = -4$

15. The abscissa of the points of curve $y = x(x - 2)(x - 4)$ where tangents are parallel to x-axis is obtained as: ***[UPSEAT-1999]***
(a) $x = 2 \pm \frac{2}{\sqrt{3}}$ (b) $x = 1 \pm \frac{1}{\sqrt{3}}$
(c) $x = 2 \pm \frac{1}{\sqrt{3}}$ (d) $x = \pm 1$

16. The length of normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at the point $\theta = \pi/2$ is ***[RPET-1999]***
(a) $2a$ (b) $a/2$
(c) $\sqrt{2}a$ (d) $a/\sqrt{2}$

17. The line $x + y = 2$ is tangent to the curve $x^2 = 3 - 2y$ at its point ***[MPPET-1998]***
(a) (1, 1) (b) (–1, 1)
(c) $(\sqrt{3}, 0)$ (d) (3, –3)

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 17 minutes.
3. The worksheet consists of 17 questions. The maximum marks are 51.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. What is the x-coordinate of the point on the curve $f(x) = \sqrt{x}(7x - 6)$, where the tangent is parallel to x-axis? ***[NDA-2007]***
(a) $-1/3$ (b) $2/7$
(c) $6/7$ (d) $1/3$

2. The tangent to the curve $y = 2x^2 - x + 1$ at a point P is parallel to $y = 3x + 4$, the co-ordinates of P are ***[RPET-2003]***
(a) (2, 1) (b) (1, 2)
(c) (–1, 2) (d) (2, –1)

3. The equation of the tangent to the curve $(1 + x^2)y = 2 - x$, where it crosses the x-axis, is ***[Kerala (Engg.)-2002]***
(a) $x + 5y = 2$ (b) $x - 5y = 2$
(c) $5x - y = 2$ (d) $5x + y - 2 = 0$

4. The point at which the tangent to the curve $y = 2x^2 - x + 1$ is parallel to $y = 3x + 9$ will be ***[Kurukshetra CEE-2001]***
(a) (2, 1) (b) (1, 2)
(c) (3, 9) (d) (–2, 1)

5. Equation of the normal line to the curve $y = x \log x$ parallel to $2x - 2y + 3 = 0$ is
(a) $x - y = 3e^{-2}$ (b) $x - y = 6e^{-2}$
(c) $x - y = 3e^2$ (d) None of these

6. The points on the curve $9y^2 = x^3$ where the normal to the curve makes equal intercepts with the axis is
(a) $\left(4, \frac{8}{3}\right)$ or $\left(4, -\frac{8}{3}\right)$ (b) $\left(-4, \frac{8}{3}\right)$
(c) $\left(-4, -\frac{8}{3}\right)$ (d) None of these

7. The slope of the tangent to the curve $y = 16 - x^2$ at $x = 0$ is
(a) 0 (b) –2
(c) 2 (d) 16

8. The tangent to the parabola $x^2 = 2y$ at the point $\left(1, \frac{1}{2}\right)$ makes with x-axis an angle
(a) 0° (b) 45°
(c) 30° (d) 60°

9. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets the x-axis at
(a) (0, 0) (b) (2, 0)
(c) $\left(-\frac{1}{2}, 0\right)$ (d) None of these

10. The equation to the normal to the curve $y = \sin x$ at (0, 0) is
(a) $x = 0$ (b) $y = 0$
(c) $x + y = 0$ (d) $x - y = 0$

11. The length of the subtangent to the curve $x^2 + xy + y^2 = 7$ at (1, –3) is
(a) 3 (b) 5
(c) 15 (d) 3/5

12. Equation of normal to the curve $y = x(2 - x)$ at the point (2, 0) is
(a) $x - 2y = 2$ (b) $2x + y = 4$
(c) $x - 2y + 2 = 0$ (d) None of these

13. The subtangent, ordinate and subnormal to the parabola $y^2 = 4ax$ at a point (different from the origin) are in
(a) G.P. (b) A.P.
(c) H.P. (d) None of these

14. The points at which the tangent to the curve $y = x^3 + 5$ is perpendicular to the line $x + 3y = 2$ are:
(a) (6, 1), (–4, 4) (b) (1, 6), (1, 4)
(c) (6, 1), (4, –1) (d) (1, 6), (–1, 4)

15. If $x = t^2$ and $y = 2t$ then equation of normal at $t = 1$ is ***[RPET-1996]***
(a) $x + y = 3$ (b) $x + y = 1$
(c) $x + y + 1 = 0$ (d) $x + y + 3 = 0$

16. What is the slope of tangent to the circle $x^2 + y^2 = 2$ at (1, 1)? ***[Gurajrat CET-2007]***
(a) 0 (b) 2
(c) 1 (d) –1

17. What is the equation of tangent to the curve $x = \cos t, y = \sin t$ at $t = \pi/4$ ***[Gujrarat CET-2007]***

(a) $x + y = \sqrt{2}$ (b) $x - y = 2$

(c) $x - y = \sqrt{2}$ (d) $x + y = 2$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	**7.** ⓐ ⓑ ⓒ ⓓ	**13.** ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	**8.** ⓐ ⓑ ⓒ ⓓ	**14.** ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	**9.** ⓐ ⓑ ⓒ ⓓ	**15.** ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	**10.** ⓐ ⓑ ⓒ ⓓ	**16.** ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	**11.** ⓐ ⓑ ⓒ ⓓ	**17.** ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ	**12.** ⓐ ⓑ ⓒ ⓓ	

HINTS AND EXPLANATIONS

3. (a) Step 1: The equation of tangent to the curve at the point (x_1, y_1) is:

$$y - y_1 = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} (x - x_1)$$

Step 2:

$$(1 + \cancel{x^2})\frac{dy}{dx} + 2xy = -1 \Rightarrow \frac{dy}{dx} = -\frac{(1 + 2xy)}{1 + x^2}$$

For abscissa of required point $y = 0$ (x-axis) in equation of curve.

$\therefore \quad 0 = 2 - x \Rightarrow x = 2$

$\Rightarrow \quad$ Point of curve is (2, 0)

$$\therefore \quad \left.\frac{dy}{dx}\right|_{(2,\,0)} = -\left(\frac{1}{5}\right)$$

So, the equation of tangent from step 1

$$y - 0 = -\frac{1}{5}(x - 2)$$

or $\quad x + 5y = 2$

6. (a) $\because$ Normal makes equal intercept

$$(M_N) \text{ slope} = \pm 1 = \frac{-1}{\cancel{dy/dx}}$$

Given curve is $9y^2 = x^3$

Differentiating,

$$18y\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{x^2}{\cancel{6y}} = \pm 1$$

Squaring $x^4 = 36y^2$

$$\Rightarrow \quad x^4 = 36\left(\frac{x^3}{9}\right) \left(\because y^2 = \frac{x^3}{9}\right)$$

$$x = 4;\ y^2 = \frac{4^3}{9} = \frac{64}{9} \Rightarrow y = \pm\frac{8}{3}$$

$\therefore$ Points are $\left(4, \pm\frac{8}{3}\right)$

10. (c) Step 1: $M = \left.\frac{dy}{dx}\right|_{(0,\,0)} = \cos x|_{(0,\,0)} = 1 =$

Slope of tangent

$\therefore \quad$ Slope of normal $= -1$

Step 2: The equation of normal

$$y - 0 = -1(x - 0)$$

$$x + y = 0$$

15. (a) Step 1: The equation of normal to the curve $y = f(x)$ at point (x_1, y_1) is

$$y - y_1 = \frac{-1}{\frac{dy}{dx}}(x - x_1)$$

Step 2: The point $(x, y) = (1, 2)$ is defined at $t = 1$

$$\frac{dy}{dx} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right| = \left.\frac{2}{2t}\right|_{t=1} = 1$$

$\therefore$ The equation is

$$y - 2 = -\frac{1}{1}(x - 1) \Rightarrow y - 2 = -x + 1$$

$\Rightarrow x + y = 3$

LECTURE

4

Maxima and Minima 1

BASIC CONCEPTS

1. A function $f(x)$ has a maximum value at $x = a$, if $f(a)$ is greater than any other value of $f(x)$ in some interval $(a - h, a + h)$ where h is small. This is also called local or relative maxima.
2. A function $f(x)$ has a minimum value at $x = a$, if $f(a)$ is less than any other value of $f(x)$ is some interval $(a - h, a + h)$ where h is small.
3. The necessary condition for a function $y = f(x)$ to have a maximum or minimum value at a point is that $\frac{dy}{dx} = 0$ or $f'(x) = 0$ at this point. But the converse of the above theorem is not ture i.e., If $f'(a) = 0$, then $f(a)$ may not be the extreme value for example: $f(x) = x^3$ at $x = 0$.
   Here $f'(x) = 3x^2$; $f'(0) = 0$.
   But 0 is neither maximum nor minimum value of $f(x)$.
4. **Critical Point** If f is defined at c, then c is called a critical point of f if $f'(c) = 0$ or if f' is undefined at c.
5. **First Derivative Test** If a function has a maximum value at $x = a$, then $f'(x)$.
   (i) Changes sign from +ve to –ve as x increases from values slightly less than a to those which are slightly greater then a.
   OR
   If $f(x)$ is maximum if $f'(x) = 0$ and $f'(x)$ changes sign from +ve to –ve.
   (ii) If function $f(x)$ has a minimum value at $x = a$, then $f'(x)$ changes sign from –ve to +ve as x increases from values slightly less then a to those which are slightly greater than a.
   OR
   If $f(x)$ is minimum if $f'(x) = 0$ and $f'(x)$ changes sign from – ve to + ve.
   Example: $f(x) = 2x^3 - 9x^2 + 12x - 3$, $x = 1$ point of maxima.
6. **Use of Second Derivative Test**
   1. A function $f(x)$ is maximum at $x = a$ if $f'(a) = 0$ and $f''(a) < 0$.
   2. A function $f(x)$ is minimum at $x = a$ if $f'(a) = 0$ and $f''(a) > 0$.

   Working Rule when $f''(a) = 0$
   (i) $f'''(a) \neq 0$, $x = a$ is a point of inflection.
   (ii) If $f'''(a) = 0$ and $f^{iv}(a) \neq 0$, then
   (1) $f(a)$ is maximum at $x = a$ if $f^{iv}(a) < 0$
   (2) Minimum at $x = a$ if $f^{iv}(a) > 0$ and so on.

NOTES

1. If $f'(a) = f''(a) = \ldots\ldots = f^{n-1}(a) = 0$ and $f^n(a) \neq 0$. If function $f(x)$ has maximum or minimum value, then n is even .
   (i) $f(a)$ is minimum at $x = a$; if $f^n(a) > 0$ and n is even.
   (ii) $f(a)$ is maximum at $x = a$ if $f^n(a) < 0$ and n is even. If n is odd then function has neither maximum nor minimum at $x = a$.
2. In an open interval, a continuous function may not have a maximum or minimum value but in closed interval it always has a maximum and a minimum value. (Every continuous

function has a maximum and a minimum value in closed interval).

3. Maximum and minimum values of a function $f(x)$ are called extremum or turning or stationary or critical values and a point which is either a point of local maxima or local minima is called an extreme point.
4. By maximum values of a function $f(x)$ at $x = a$ we do not mean the greatest value of the function, of course, it is the greatest only in the neighourhood of a i.e., in the interval $(a - h, a + h)$. A function can have several maximum values and similarly it can have several minimum values.
5. The rate of change of function at stationary point is zero.
6. **General properties of maximum and minimum values of a function:**
   1. Between two equal values of $f(x)$, there must lie at least one maximum or minimum value.
   2. Maximum and minimum values of a function occur alternately i.e., if there are three successive critical values of $f(x)$ then if 1st gives maximum, 2nd will give minimum and 3rd maximum and so on.
7. The largest (smallest) values of a function $f(x)$ in an closed interval value $[a, b]$ is either a maximum (minimum) value of $f(x)$ at a point inside the interval or the value of $f(x)$ at an end points of the interval i.e., at $x = a$ or $x = b$.

7. **Angle of Intersection of two Curves** The angle of intersection of two curves is defined as the angle between the tangents to the two curves at their point of intersection. Let two curves be $y = f(x)$ and $y = g(x)$ and if point of intersection is (x_1, y_1) and let

$$m_1 = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = f'(x)$$

and $$m_2 = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = g'(x)$$

If angle between the two curves is θ, then $\tan \theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

8. If two curves cut each other orthogonally $\left(\theta = \dfrac{\pi}{2}\right)$ then $m_1 m_2 = -1$.
9. If the two curves touch each other, then $m_1 = m_2$
10. Points of intersection of parabolas $y^2 = 4ax$ and $x^2 = 4by$ are $V(0, 0)$ and $P(4a^{1/3} b^{2/3}, 4a^{2/3}, b^{1/3})$ and angle between two parabolas at
    (i) $V(0, 0)$ is 90°
    (ii) $P(4a^{1/3}b^{2/3}, 4a^{2/3}, b^{1/3})$ is

$$\tan^{-1} \frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}$$

SOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): FOR BETTER UNDERSTANDING AND CONCEPT BUILDING OF THE TOPIC

1. The side of a square sheet is increasing at the rate of 4 cm per minute. At what rate is the area increasing when the side is 8 cm long? ***[HSB-92]***

Solution

Let a be the side of square sheet.

Let A be its area at time t, then $A = a^2$ ..... (i)

Rate of change of its side with respect to

$t, \dfrac{da}{dt} = 4$ cm/min .... (ii)

and the rate of change of its area with respect to

$$t, \frac{dA}{dt} = 2a.\frac{da}{dt}$$

$= 2a \times 4 = 8a$ (form equation (ii)) .... (iii)

If $a = 8$ cm, then by equation (iii),

we get $\dfrac{dA}{dt} = 8 \times 8 = 64$ cm²/min

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)): TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. If the function $f(x) = x^4 - 62x^2 + ax + 9$ attains the maximum value at $x = 1$ in the interval $[0, 2]$, find the value of a.
[NCERT-Book]

2. Find the points of local maxima and local minima of the function $f(x) = (x - 1)^3 (x + 1)^2$. Find also the local maximum and local minimum values.
[CBSE-91; PB-89]

3. A ship of an enemy is moving along the curve $y = x^2 + 2$. A soldier is at the point (3, 2). Find the minimum distance between the soldier and the ship. ***[NCERT-Book]***

4. The combined resistance R of two resistors R_1 and R_2 $(R_1, R_2 > 0)$ is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ If $R_1 + R_2 = C$ (a constant), show that the maximum resistance R is obtained by choosing $R_1 = R_2$
[HPSB-2001C; CBSE-97 C]

5. A point on the hypotenuse of a right triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.
[NCERT-Book]

6. The perimeter of a triangle is 16 cm. If one side is 6 cm and area of triangle is maximum, find the other two sides.
[CBSE-(SP)-2006]

7. Find the largest term in the sequence $a_n = \frac{n^2}{n^3 + 200}$.

8. N characters of information are held on magnetic tape, in batches of x characters each, the batch processing time is $\alpha + \beta x^2$ seconds; α and β are constants. Find the optimal value of x for fast processing.
[MNR-1986]

For what value of x, the following functions are maximum or minimum

9. $y = x(5 - x)$ ***[MP-94, 98]***

10. Find the maximum value of $\sqrt{3}\sin x + 3\cos x$.
[MP-1999]

11. A square piece of tim of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.
[CBSE-95]

12. Show that the height of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height h is $\frac{h}{3}$.
[HSB-99, 2001, 2002; PB-2003; CBSE-2000'S]

13. Show that of all the rectangles inscribed in a given circle, the square has the maximum area. ***[CBSE-91, 93C, 2002; PSB-99]***

14. Divide a number 4 into two positive numbers such that the sum of the square of first and the cube of the second is a minimum. ***[CBSE-88]***

15. If $y = \sin x(1 + \cos x)$, $0 < x < \frac{\pi}{2}$ find the values of x where local maxima and local minima occur. Also find the local maximum and local minimum values of y. ***[CBSE-89]***

16. Show that all the rectangles with a given perimeter the square has the largets area.
[PSB-91C, 99; HSB-96; HPSB-98; CBSE-81, 90, 92]

17. Prove that the maximum value of $\sin x + \cos x$ is $\sqrt{2}$. ***[MP-93, 98]***

18. For what value of x, $y = x(5 - x)$ is maximum or minimum? ***[MP-2007]***

19. A wire of length 28 m is to be cut into two pieces. One of the two pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of these is minimum?
[CBSE-90, 91, 2007]

20. A ball thrown vertically upwards, moves according to the formula $S = 13.8\ t - 4.9\ t^2$, where S is in metres and t is in seconds. Find

(i) its velocity at $t = 1$ second
(ii) its acceleration at $t = 1$ second
(iii) The maximum height reached by the ball.

ANSWERS

1. maximum at $x = 1$ and $a = 120$

2. $x = 1, -1, -1/5; 0; -3456/3125$

3. $\sqrt{5}$ units.

6. Two sides of the triangle are 5 cm and 5 cm

7. $a_7 = \dfrac{49}{543}$ is the greatest term

8. $x = \sqrt{\dfrac{\alpha}{\beta}}$

9. Maximum value $= \dfrac{25}{4}$ at $x = \dfrac{5}{2}$

10. $\sqrt{12}$ at $x = \dfrac{\pi}{6}$

11. Maximum volume $= 1024\ \text{cm}^3$

14. 8/3, 4/3

15. $x = \dfrac{\pi}{3}$ is a point of local maxima and the local max. value $y = \dfrac{3\sqrt{3}}{4}$

18. $x = 5/2$ is point of maxima

19. Length of square $= \dfrac{112}{4+\pi}$ circumference of circle $= \dfrac{28\pi}{4+\pi}$

20. (i) 4 m/sec
(ii) $-9.8\ \text{m/sec}^2$
(iii) 9.716 m

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. The value of a so that the sum of the squares of the roots of the equation $x^2 - (a-2)\,x - a + 1 = 0$ assume the least value, is

[RPET-2000; AIEEE-2005]

(a) 2 (b) 1
(c) 3 (d) 0

Solution

(b) Let α, β be the roots of the equation

$x^2 - (a-2)\,x - a + 1 = 0,$

then $\alpha + \beta = a - 2$, $\alpha\beta = -a + 1$

$\therefore z = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (a-2)^2 + 2\,(a-1) = a^2 - 2a + 2$

$\dfrac{dz}{da} = 2a - 2 = 0 \Rightarrow a = 1$

$\dfrac{d^2z}{da^2} = 2 > 0,$ so z has minima at $a = 1$

So $\alpha^2 + \beta^2$ has least value for $a = 1$. This is because we have only one stationary value at which we have minima.

Hence $a = 1$.

2. A minimum value of $\int_0^x te^{-t^2}\,dt$ is

[EAMCET-2003]

(a) 1 (b) 2
(c) 3 (d) 0

Solution

(d) $f(x) = \int_0^x te^{-t^2}dt$

$\Rightarrow f'(x) = xe^{-x^2} = 0 \Rightarrow x = 0$

$f''(x) = e^{-x^2}(1 - 2x^2);\ f''(0) = 1 > 0$

$\therefore$ Minimum value $f(0) = 0$.

3. The minimum value of $[(5 + x)\,(2 + x)]/[1 + x]$ for non-negative real x is

[Kurukshetra CEE-1998]

(a) 12 (b) 1
(c) 9 (d) 8

Solution

(c) Given $f(x) = \frac{[(5+x)(2+x)]}{[1+x]}$

$$f(x) = 1 + \frac{4}{1+x} + (5+x) = (6+x) + \frac{4}{1+x}$$

$\Rightarrow \quad f'(x) = 1 - \frac{4}{(1+x)^2} = 0;\ x^2 + 2x - 3 = 0$

$\Rightarrow \quad x = -3, 1$

Now $\quad f''(x) = \frac{8}{(1+x)^3} = 0,$

$$f''(-3) = -\text{ve},\ f''(1) = +\text{ve}$$

Hence minimum value at $x = 1$

$$f(1) = \frac{(5+1)(2+1)}{(1+1)} = \frac{6\times 3}{2} = 9$$

4. One maximum point of $\sin^p x \cos^q x$ is

[RPET-1997; AMU-2000]

(a) $x = \tan^{-1}\sqrt{(p/q)}$

(b) $x = \tan^{-1}\sqrt{(q/p)}$

(c) $x = \tan^{-1}(p/q)$

(d) $x = \tan^{-1}(q/p)$

Solution

(a) Let $y = \sin^p x \,.\, \cos^q x$

$\frac{dy}{dx} = p \sin^{p-1} x \,.\, \cos x \,.\, \cos^q x + q \cos^{q-1}$
$x \,.\, (-\sin x) \sin^p x$

$\frac{dy}{dx} = p \sin^{p-1} x \,.\, \cos^{q+1} x - q \cos^{q-1} x \,.\, \sin^{p+1} x$

Put $\frac{dy}{dx} = 0, \ \therefore\ \tan^2 x = \frac{p}{q}$

$\Rightarrow \quad \tan x = \pm\sqrt{\frac{p}{q}}$

$\therefore \quad$ Point of maxima $x = \tan^{-1}\sqrt{p/q}$.

5. The minimum value of $e^{(2x^2-2x+1)\sin^2 x}$ is

[Roorkee Qualifying-1998]

(a) e (b) $1/e$

(c) 1 (d) 0

Solution

(c) Given $e^{(2x^2-2x+1)\sin^2 x}$

For minima or maxima, $\frac{dy}{dx} = 0$

$\therefore \quad e^{(2x^2-2x+1)\sin^2 x}[(4x-2)\sin^2 x + 2(2x^2 - 2x + 1)\sin x \cos x] = 0$

$\Rightarrow \quad [(4x-2)\sin^2 x + 2(2x^2 - 2x + 1)\sin x \cos x] = 0$

$\Rightarrow \quad 2\sin x\,[(2x-1)\sin x + (2x^2 - 2x + 1)\cos x] = 0$

$\Rightarrow \quad \sin x = 0$

$\therefore \quad y$ is minimum for $\sin x = 0$

Thus minimum value of

$y = e^{(2x^2-2x+1)(0)} = e^0 = 1.$

6. What are the minimum and maximum values of the function $x^5 - 5x^4 + 5x^3 - 10$.

[DCE-1999]

(a) $-37, -9$

(b) $10, 0$

(c) It has 2 min. and 1 max. values

(d) It has 2 max. and 1 min. values

Solution

(a) $y = x^5 - 5x^4 + 5x^3 - 10$

$\therefore \quad \frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3)$

$= 5x^2(x-3)(x-1)$

$\frac{dy}{dx} = 0,$ gives $x = 0, 1, 3$

Now, $\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x$

$= 10x(2x^2 - 6x + 3)$

and $\frac{d^3y}{dx^3} = 10(6x^2 - 12x + 3),$

For $x = 0$: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} \neq 0$

$\therefore \quad$ Neither minimum nor maximum

For $x = 1, \frac{d^2y}{dx^2} = -10 =$ negative

$\therefore \quad$ Maximum value $y_{\text{max.}} = -9$

For $x = 3, \frac{d^2y}{dx^2} = 90 =$ positive

$\therefore \quad$ Minimum value $y_{\text{min.}} = -37$

7. The maximum value of $\sin x(1 + \cos x)$ will be at the ***[UPSEAT-1999]***

(a) $x = \frac{\pi}{2}$ (b) $x = \frac{\pi}{6}$

(c) $x = \frac{\pi}{3}$ (d) $x = \pi$

Solution

(c) $y = \sin x\,(1 + \cos x) = \sin x + \frac{1}{2}\sin 2x$

$\therefore \quad \frac{dy}{dx} = \cos x + \cos 2x$

and $\frac{d^2y}{dx^2} = -\sin x - 2\sin 2x$

On putting $\frac{dy}{dx} = 0$, $\cos x + \cos 2x = 0$

$\Rightarrow \quad \cos x = -\cos 2x = \cos(\pi - 2x)$

$\Rightarrow \quad x = \pi - 2x$

$\therefore \quad x = \frac{\pi}{3}; \quad \therefore \left(\frac{d^2y}{dx^2}\right)_{x=\pi/3}$

$= -\sin\left(\frac{1}{3}\pi\right) - 2\sin\left(\frac{2}{3}\pi\right)$

$= \frac{-\sqrt{3}}{2} - 2.\frac{\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2},$

which is negative.

$\therefore$ At $x = \frac{\pi}{3}$ the function is maximum.

8. $\frac{x}{1 + x\tan x}$ is maxima at **[UPSEAT-1999]**

(a) $x = \sin x$ (b) $x = \cos x$

(c) $x = \frac{\pi}{3}$ (d) $x = \tan x$

Solution

(b) If $\frac{x}{1 + x\tan x}$ is maxima, then its reciprocal $\frac{1 + x\tan x}{x}$ will be minima.

Let $y = \frac{1 + x\tan x}{x} = \frac{1}{x} + \tan x$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{x^2} + \sec^2 x,$

$\frac{d^2y}{dx^2} = \frac{2}{x^3} + 2\sec x \sec x \tan x$

On putting $\frac{dy}{dx} = 0, -\frac{1}{x^2} + \sec^2 x = 0$

$\Rightarrow \quad \sec^2 x = \frac{1}{x^2}$

$\Rightarrow \quad x^2 = \cos^2 x$

$\Rightarrow \quad x = \cos x$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{2}{\cos^3 x} + 2\sec^2 x \tan x$

$= 2\sec^2 x\,(\sec x + \tan x),$

which is positive.

At $x = \cos x$, $\frac{1 + x\tan x}{x}$ is minimum

So $\frac{x}{1 + x\tan x}$ will be maximum.

9. The maximum value of $x^4e^{-x^2}$ is **[AMU-1999]**

(a) e^2 (b) e^{-2}

(c) $12e^{-2}$ (d) $4e^{-2}$

Solution

(d) $f(x) = x^4e^{-x^2}$

$\Rightarrow \quad f'(x) = 4x^3e^{-x^2} + x^4e^{-x^2}(-2x)$

For max. $f'(x) = 0 \Rightarrow 4x^3e^{-x^2} - 2x^5e^{-x^2} = 0$

$\Rightarrow \quad x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

$f''(x) = 12x^2e^{-x^2} + 4x^3e^{-x^2}(-2x)$

$-10x^4e^{-x^2} - 2x^5e^{-x^2}(-2x)$

$\Rightarrow \quad f''(\sqrt{2}) = 24e^{-2} - 32e^{-2} - 40e^{-2} + 32e^{-2}$

$= -ve$

Hence, $f(x)$ is maximum at $x = \sqrt{2}$

$\therefore$ Maximum value $= 4\,e^{-2}$.

10. If $a^2x^4 + b^2y^4 = c^6$, then maximum value of xy is **[RPET-2001]**

(a) $\frac{c^2}{\sqrt{ab}}$ (b) $\frac{c^3}{ab}$

(c) $\frac{c^3}{\sqrt{2ab}}$ (d) $\frac{c^3}{2ab}$

Solution

(c) $a^2x^4 + b^2y^4 = c^6 \Rightarrow y = \left(\frac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

Hence $f(x) = xy = x\left(\frac{c^6 - a^2x^4}{b^2}\right)^{1/4}$

$\Rightarrow \quad f(x) = \left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{1/4}$

differentiate $f(x)$ with respect to x, then

$$f'(x) = \frac{1}{4}\left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{-3/4}\left(\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2}\right)$$

Put $f'(x) = 0, \frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2} = 0$

$\Rightarrow \quad x^4 = \frac{c^6}{2a^2} \Rightarrow x = \pm\frac{c^{3/2}}{2^{1/4}\sqrt{a}}$

At $x = \frac{c^{3/2}}{2^{1/4}\sqrt{a}}$ the $f(x)$ will be maximum, so

$$f\left(\frac{c^{3/2}}{2^{1/4}\sqrt{a}}\right) = \left(\frac{c^{12}}{2a^2b^2} - \frac{c^{12}}{4a^2b^2}\right)^{1/4}$$

$$= \left(\frac{c^{12}}{4a^2b^2}\right)^{1/4} = \frac{c^3}{\sqrt{2ab}}$$

11. On $[1, e]$ the greatest value of $x^2 \log x$ ***[AMU-2002]***

(a) e^2 (b) $\frac{1}{e}\log\frac{1}{\sqrt{e}}$

(c) $e^2 \log\sqrt{e}$ (d) None of these

Solution

(a) $f(x) = x^2 \log x \Rightarrow f'(x) = (2 \log x + 1)\, x$

Now $f'(x) = 0 \Rightarrow x = e^{-1/2}, 0 \because 0 < e^{-1/2} < 1$

$\because$ None of these critical points lies in the interval $[1, e]$

$\therefore$ So we only find the value of $f(x)$ at the end points 1 and e. We have $f(1) = 0$, $f(e) = e^2$

$\therefore$ Greatest value $= e^2$.

12. The minimum value of $4e^{2x} + 9e^{-2x}$ is ***[J & K-2005]***

(a) 11 (b) 12

(c) 10 (d) 14

Solution

(b) Let $f(x) = 4e^{2x} + 9e^{-2x}$

$\therefore \quad f'(x) = 8e^{2x} - 18e^{-2x}$

Put $f'(x) = 0$

$\Rightarrow \quad 8e^{2x} - 18e^{-2x} = 0$

$e^{2x} = 3/2 \Rightarrow x = \log(3/2)^{1/2}$

Again $f''(x) = 16\, e^{2x} + 36\, e^{-2x} > 0$

Now $f(\log(3/2)^{1/2}) = 4e^{2.(\log(3/2)^{1/2})}$

$+ 9e^{-2(\log(3/2)^{1/2})}$

$$= 4\times\frac{3}{2} + 9\times\frac{2}{3} = 6 + 6 = 12$$

Hence minimum value = 12.

13. The minimum value of $f(a) = (2a^2 - 3) + 3(3 - a) + 4$ is ***[DCE-2005]***

(a) 15/2 (b) 11/2

(c) −13/2 (d) 71/8

Solution

(d) $f(a) = 2a^2 - 3a + 10$

$f'(a) = 4a - 3, f''(a) = 4 > 0$

for exteremum, $f'(a) = 0 \Rightarrow a = \frac{3}{4}$

$\therefore f(a)$ is minimum at $a = \frac{3}{4}$

$$f(a)_{\min} = 2\times\left(\frac{3}{4}\right)^2 - 3\times\left(\frac{3}{4}\right) + 10 = \frac{71}{8}$$

14. Let $f(x) = 1 + 2x^2 + 2^2x^4 + ... + 2^{10}x^{20}$, then $f(x)$ has ***[AMU-2005]***

(a) More than one minimum

(b) Exactly one minimum

(c) At least one maximum

(d) None of these

Solution

(b) $f(x) = 1 + 2x^2 + 2^2x^4 + 2^3x^6 + ...\ 2^{10}x^{20}$

$f'(x) = x(4 + 4.2^2x^2 + ... + 20.2^{10}x^{18})$

$\therefore \quad f'(x) = 0 \Rightarrow x = 0$ only. Also $f''(0) > 0$.

15. On the interval $[0, 1]$, the function $x^{25}(1 - x)^{75}$ takes its maximum value at the point ***[IIT-1995]***

(a) 0 (b) 1/2

(c) 1/3 (d) 1/4

Solution

(d) $f(x) = x^{25}(1 - x)^{75}$

$f'(x) = x^{25}(75)(1 - x)^{74}(-1) + 25x^{24}(1 - x)^{75}$

For maxima and minima.

$-75x^{25}(1 - x)^{74} + 25x^{24}(1 - x)^{75} = 0$

$\Rightarrow \quad 25x^{24}(1 - x)^{74}[(1 - x) - 3x] = 0$

$\Rightarrow \quad$ Either $x = 0$ or $x = 1$ or $x = 1/4$

At $\quad x = \frac{1}{4}, f'\left(\frac{1}{4} - h\right) > 0$

and $\quad f'\left(\frac{1}{4} + h\right) < 0$

$\therefore \quad f(x)$ is maximum at $x = 1/4$.

16. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$, where $a > 0$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals ***[AIEEE-2003]***

(a) 3 (b) 1
(c) 2 (d) 1/2

Solution

(c) $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$

$f'(x) = 6x^2 - 18ax + 12a^2$; $f''(x) = 12x - 18a$

For maximum and minimum,

$6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow \quad x^2 - 3ax + 2a^2 = 0$

$x = a$ or $x = 2a$, at $x = a$ maximum and at $x = 2a$ minimum.

$\because \quad p^2 = q$

$\therefore \quad a^2 = 2a$

$\Rightarrow \quad a = 2$ or $a = 0$

But $a > 0$, therefore $a = 2$.

17. The interval in which the function x^3 increases less rapidly then $6x^2 + 15x + 5$ is

(a) $(-\infty, -1)$ (b) $(-5, 1)$
(c) $(-1, 5)$ (d) $(5, \infty)$

Solution

(c) The function $f(x) = x^3$ increases for all x and the function $g(x) = 6x^2 + 15x + 5$ increases

If $\quad g'(x) > 0 \Rightarrow 12x + 15 > 0 \Rightarrow x > \dfrac{-5}{4}$

Thus, $f(x)$ and $g(x)$ both increases for $x > \dfrac{-5}{4}$

It is given that $f(x)$ increases less rapidly than $g(x)$, Therefore, the function $\phi(x) = f(x) - g(x)$ is decreasing function, which implies that $\phi'(x) < 0 \Rightarrow 3x^2 - 12x - 5 < 0$

$x \in (-1, 5)$

18. The points of extrema of $f(x) = \int_0^x \frac{\sin t}{t} dt$ in the domain $x > 0$ are ***[Orissa JEE-2002]***

(a) $(2n+1)\dfrac{\pi}{2},\ n = 1, 2$ ....

(b) $(4n+1)\dfrac{\pi}{2},\ n = 1, 2$ ....

(c) $(2n+1)\dfrac{\pi}{4},\ n = 1, 2$ ....

(d) $n\pi, n = 1, 2$ .....

Solution

(d) $\quad f(x) = \int_0^x \frac{\sin t}{t} dt \quad f'(x) = \dfrac{\sin x}{x}$

put $\quad f'(x) = 0 \Rightarrow \dfrac{\sin x}{x} = 0 \Rightarrow \sin x = 0$

$x = n\pi$, $n = 1, 2, 3$ ....

19. If $xy = c^2$, then minimum value of $ax + by$ is ***[MPPET-01]***

(a) $c\sqrt{ab}$ (b) $2c\sqrt{ab}$
(c) $-c\sqrt{ab}$ (d) $-2c\sqrt{ab}$

Solution

(b) $\quad xy = c^2 \Rightarrow y = \dfrac{C^2}{x} \Rightarrow f(x) = ax + by,$

$f(x) = ax + \dfrac{bc^2}{x}$

Differentiatte with respect to x

$f'(x) = a - \dfrac{bc^2}{x^2}$

Put $f'(x) = 0 \Rightarrow ax^2 - bc^2 = 0 \Rightarrow x^2 = \dfrac{bc^2}{a}$

$\Rightarrow \quad x = \pm c\sqrt{b/a}$,

Here $x = -c\sqrt{\dfrac{b}{a}}$ is pt of minima

$f(-c\sqrt{b/a}) = -(c\sqrt{b/a})\, a + b(-c\sqrt{a/b})$

$= -2c\sqrt{ab}$

[Here at $x = -c\sqrt{b/a}$ $y = -c\sqrt{a/b}$]

20. The maximum and minimum values of $\sin 2x - x$ are

(a) 1, –1

(b) $\dfrac{3\sqrt{3}-\pi}{6}, \dfrac{\pi - 3\sqrt{3}}{6}$

(c) $\dfrac{\pi - 3\sqrt{3}}{6}, \dfrac{3\sqrt{3}-\pi}{6}$

(d) do not exist

Solution

(b) $\quad f(x) = \sin 2x - x; f'(x) = 2\cos 2x - 1; f''(x) = -4 \sin 2x$

Now $f'(x) = 0 \Rightarrow 2 \cos 2x - 1 = 0$

$\Rightarrow \quad x = n\pi \pm \pi/6, \qquad n = 0, 1, 2,$ .....

$\Rightarrow \quad x = \pi/6, 5\pi/6, 7\pi/6, -\pi/6,$ ....

But $f''(\pi/6) = -2\sqrt{3} < 0$

$\Rightarrow$ $x = \pi/6$ is a maximum point

Also $f''(5\pi/6) = 2\sqrt{3} > 0$

$\Rightarrow$ $x = 5\pi/6$ is a minimum point

Hence one maximum value

$= f(\pi/6) = \dfrac{3\sqrt{3}-\pi}{6}$

one minimum value $= f(5\pi/6) = \dfrac{3\sqrt{3}+5\pi}{6}$.

But it is not there in given alternatives. Hence by alternate position another minimum point is $-\pi/6$ so one minimum value

$= f(-\pi/6) = \dfrac{\pi - 3\sqrt{3}}{6}$

21. $\sin x + \cos 2x$ $(x > 0)$ is minimum when x equals ***[PET (Raj.)-90; MNR-98]***

(a) $\dfrac{n\pi}{2}$ (b) $\dfrac{3(n+1)\pi}{2}$

(c) $\dfrac{(2n+1)\pi}{2}$ (d) None of these

Solution

(c) Let $f(x) = \sin x + \cos 2x$

$\Rightarrow$ $f'(x) = \cos x - 2\sin 2x$

$f''(x) = -\sin x - 4\cos 2x$

Now $f'(x) = 0 \Rightarrow \cos x\,(1 - 4\sin x) = 0$

$\Rightarrow$ $\cos x = 0$, $\sin x = 1/4$

$\Rightarrow$ $x = \pi/2, 3\pi/2, 5\pi/2, \ldots, \sin^{-1}(1/4), \ldots$

Also $f''(\pi/2) = 3 > 0, f''(3\pi/2) = 5 > 0, \ldots\ldots\ldots\ldots$

$\Rightarrow$ $f(x)$ is minimum when $x = \dfrac{(2n+1)\pi}{2}$

22. The minimum value of $64\sec x + 27\,\text{cosec}\,x$, $0 < x < \dfrac{\pi}{2}$ is ***[CET-1992]***

(a) 91 (b) 25

(c) 125 (d) None of these

Solution

(c) Let $y = 64\sec x + 27\,\text{cosec}\,x$

$\Rightarrow$ $\dfrac{dy}{dx} = 64\sec x\tan x - 27\,\text{cosec}\,x\cot x$

$\dfrac{d^2y}{dx^2} = 64\sec^3 x + 64\sec x\tan^2 x + 27\text{cosec}^3 x + \text{cosec}\,x\cot^2 x$

Now $\dfrac{dy}{dx} = 0 \Rightarrow 64\sec x\tan x = 27\,\text{cosec}\,x\cot x$

$\Rightarrow$ $\tan^3 x = 27/64 \Rightarrow \tan x = 3/4$

Also then $\dfrac{d^2y}{dx^2} > 0$ $[\because 0 < x < \pi/2]$

So y is minimum when $x = \tan^{-1}(3/4)$ and its minimum value $= 64\,(5/4) + 27\,(5/3) = 125$

23. Maximum value of the function $f(x) = \sum_{k=1}^{5}(x-k)^2$ is at ***[CET (Pb.)-91]***

(a) $x = 2$ (b) $x = 5/2$

(c) $x = 3$ (d) $x = 5$

Solution

(c) $f(x) = (x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + (x-5)^2$

$f(x) = 2\,[(x-1) + (x-2) + (x-3) + (x-4) + (x-5)] = 2\,[5x - 15]$

$f''(x) = 10 > 0$, when $f'(x) = 0 \Rightarrow x = 3$ when $f''(x) > 0$

24. The rate of change of the function $f(x) = 3x^5 - 5x^3 + 5x - 7$ is minimum when

(a) $x = 0$ (b) $x = 1/\sqrt{2}$

(c) $x = -1/\sqrt{2}$ (d) $x = \pm 1/\sqrt{2}$

Solution

(b) If v is the rate of change of the given function, then

$v = f'(x) = 15x^4 - 15x^2 + 5$

$\Rightarrow$ $\dfrac{dv}{dx} = 60x^3 - 3x,\ \dfrac{d^2v}{dx^2} = 180x^2 - 30$

$\therefore$ $\dfrac{dv}{dx} = 0 \Rightarrow x = 0 \pm \dfrac{1}{\sqrt{2}}$

At $x = 0,\ \dfrac{d^2v}{dx^2} = -30 < 0$

At $x = \pm\dfrac{1}{\sqrt{2}}\ \dfrac{d^2v}{dx^2} = 60 > 0$

Hence v is minimum at $x = \pm\dfrac{1}{\sqrt{2}}$

25. $f(x) = 1 + 2\sin x + 2\cos^2 x, 0 \le x \le \pi/2$ is maximum at

(a) $x = \pi/2$ (b) $x = \pi/6$

(c) $x = \pi/3$ (d) No where

Solution

(b) $f'(x) = 2\cos x - 4\cos x \sin x = 2\cos x - 2\sin 2x$

$f''(x) = -2\sin x - 4\cos 2x$

$f'(x) = 0 \Rightarrow \cos x\,(1 - 2\sin x) = 0$

$\Rightarrow \quad x = \pi/6, \pi/2$

$[\because 0 \le x \le \pi/2]$

But $f''(\pi/6) = -1 - 2 < 0$, Hence $x = \pi/6$ is a maxima.

26. A point on the curve $xy^2 = 1$ which is at minimum distance from the origin is **[Kerala (CEE)-2003]**

(a) (1, 1) (b) (1/4, 2)
(c) $(2^{1/6}, 2^{-1/3})$ (d) $(2^{-1/3}, 2^{1/6})$

Solution

(d) Let $(t^2, 1/t)$ be a point on the curve. If its distance from origin is λ, then

$$\lambda^2 = t^4 + \frac{1}{t^2}$$

$$\Rightarrow \quad \frac{d}{dt}(\lambda^2) = 4t^3 - \frac{2}{t^3},$$

$$\frac{d^2}{dt^2}(\lambda^2) = 12t^2 + \frac{6}{t^4} > 0$$

Now $\frac{d}{dt}(\lambda^2) = 0 \Rightarrow t = 2^{-1/6}$

where $\frac{d^2}{dt^2}(\lambda^2) > 0$

so required point $= (2^{-1/3}, 2^{1/6})$

27. $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t}\,dt$ has

(a) two maxima and two minima
(b) two maxima and three minima
(c) three maxima and two minima
(d) one maxima and one minima

Solution

(b) $$f'(x) = \frac{x^4 - 5x^2 + 4}{2 + e^{x^2}}(2x)$$

$$= \frac{2x(x^2 - 4)(x^2 - 1)}{2 + e^{x^2}}$$

$f'(x) = 0 \Rightarrow x = 0, \pm 1, \pm 2$

Now observing the change in the sign of $f'(x)$ at these points we find that $x = -1, 1$ are two maxima and $x = -2, 0, 2$ are three minima.

28. The function $f(x) = \cos|x| - 2ax + b$ increases along the entire number scale. The range of 'a' is given by **[CET-91]**

(a) $a = b$ (b) $a = b/2$
(c) $a \le -1/2$ (d) $a > -3/2$

Solution

(c) $f'(x) = -\sin x - 2a\left(\frac{d}{dx}\cos|x| = -\sin x\right)$

Hence $f'(x) > 0 \Rightarrow -\sin x - 2a > 0$

$$\Rightarrow a < -\frac{1}{2}\sin x \Rightarrow a \le -\frac{1}{2}$$

$$\left[\because \text{min of}\left(-\frac{1}{2}\sin x\right) = -\frac{1}{2}\right]$$

29. Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds? **[AIEEE-2008]**

(a) The cubic has maxima at both $\sqrt{p/3}$ and $-\sqrt{p/3}$
(b) The cubic has minima at $\sqrt{p/3}$ and maxima at $-\sqrt{p/3}$
(c) The cubic has minima at $-\sqrt{p/3}$ and maxima at $\sqrt{p/3}$
(d) The cubic has minima at both $\sqrt{p/3}$ and $-\sqrt{p/3}$

Solution

(b) Denote $x^3 - px + q$ by $f(x)$
i.e., $f(x) = x^3 - px + q$

Now for expression, $f'(x) = 0$. i.e., $3x^2 - p = 0$

$$x = -\sqrt{\frac{p}{3}}, \sqrt{\frac{p}{3}}$$

$f''(x) = 6x$

$$f''\left(-\sqrt{\frac{p}{3}}\right) < 0$$

$$f''\left(\sqrt{\frac{p}{3}}\right) > 0$$

Thus maxima at $-\sqrt{p/3}$ and minima at $\sqrt{p/3}$.

30. The angle between curves $y^2 = 4x$ and $x^2 + y^2 = 5$ at (1, 2) is **[Karnataka CET-1999]**

(a) $\tan^{-1}(3)$ (b) $\tan^{-1}(2)$
(c) $\pi/2$ (d) $\pi/4$

Solution

(a) For curve $y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{4}{2y}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(1,2)} = 1$ and for curve $x^2 + y^2 = 5$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{y}$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{-1}{2}$

$\therefore$ Angle between the curves is

$$\theta = \tan^{-1}\left|\frac{\frac{-1}{2} - 1}{1 + \left(\frac{-1}{2}\right)}\right| = \tan^{-1}(3)$$

31. The greatest value of $f(x) = (x + 1)^{1/3} - (x - 1)^{1/3}$ on $[0, 1]$ is ***[Orissa JEE-2008]***

(a) 1 (b) 2
(c) 3 (d) 4

Solution

(b) $f(x) = (x + 1)^{1/3} - (x - 1)^{1/3}$

$$f'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}} - \frac{1}{3}(x-1)^{-\frac{2}{3}}$$

To be maximum or minimum $f'(x) = 0$

$$\Rightarrow \left[\frac{1}{3}(x+1)^{-\frac{2}{3}} - \frac{1}{3}(x-1)^{-\frac{2}{3}}\right] = 0 \Rightarrow x = 0$$

$\therefore \quad f(0) = (0 + 1)^{1/3} - (0 - 1)^{1/3}$
$= 1 - (-1) = 2$
$f(1) = (1 + 1)^{1/3} - (1 - 1)^{1/3} = (2)^{1/3}$.

$\therefore$ greatest value in $f(0)$ and $f(1)$ is 2.

32. The profit function, in rupees, of a firm selling x items ($x \geq 0$) per week is given by $P(x) = -3500 + (400 - x)x$. How many items should the firm sell so that the firm has maximum profit? ***[NDA-2009]***

(a) 400 (b) 300
(c) 200 (d) 100

Solution

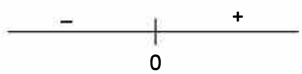
(c) $P(x) = -3500 + (400 - x)x$
On differentiating w.r.t. x, we
$P'(x) = 400 - 2x$
Put $P'(x) = 0$ for maximum
$400 - 2x = 0 \Rightarrow x = 200$
Now, $P''(x) = -2x$
$\Rightarrow P''(200) = -400 < 0$
$\therefore$ $P(x)$ is maximum at $x = 200$
Hence, required number of items = 200

33. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$: ***[AIEEE-2009]***

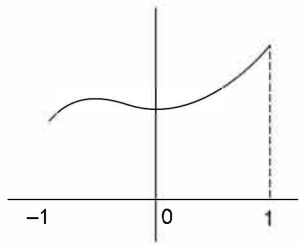
(a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
(c) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P
(d) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

Solution

$p(x) = x^4 + ax^3 + bx^2 + cx + d$; $p'(x) = 0$ has only are root i.e., $x = 0$
$\therefore$ $x = 0$ is minima



$\therefore$ Graph of $y = p(x)$ is



Here $p(-1) < p(1)$
$\therefore$ $p(0)$ minimum and $p(1)$ is maximum.

34. The maximum value of $f(x) = \sin x.(1 + \cos x)$ is: ***[MPPET-2009]***

(a) $\frac{3\sqrt{3}}{4}$ (b) $\frac{3\sqrt{3}}{2}$
(c) $3\sqrt{3}$ (d) $\sqrt{3}$

Solution

(a) Given, $f(x) = \sin x(1 + \cos x)$

It is maximum at $x = \frac{\pi}{3}$

$$\therefore f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\left(1 + \cos\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$$

35. The local minimum of the function $f(x) = \frac{x}{2} + \frac{2}{x}$ is: *[MPPET-2009]*

(a) at $x = 2$ (b) at $x = -2$

(c) at $x = 0$ (d) at $x = 1$

Solution

(a) Given, $f(x) = \frac{x}{2} + \frac{2}{x} = \frac{x^2 + 4}{2x}$

$\Rightarrow \quad f'(x) = \frac{2x(2x) - (x^2 + 4).2}{(2x)^2}$.

For local minimum, put $f'(x) = 0$

$\Rightarrow \frac{2x^2 - 8}{(2x)^2} = 0 \Rightarrow x = \pm 2$

Now, $f''(x) = \frac{4x^2(4x) - (2x^2 - 8)8x}{(4x^2)^2}$

$= \frac{16x^3 - 16x^3 + 64x}{16x^4} = \frac{4}{x^3}$

At $x = 2$, $f''(x) = \frac{4}{8} > 0$

Hence, it is local minimum at $x = 2$

36. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively and $p^2 = q$, then a equals: *[MPPET-2009]*

(a) 1 (b) 2

(c) 3 (d) ½

Solution

(b) Given $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$

$\Rightarrow \quad f'(x) = 6x^2 - 18ax + 12a^2$

Put $f'(x) = 0$

$\Rightarrow \quad 6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow \quad (x - a)(6x - 12a) = 0$

$\Rightarrow \quad x = a, 2a$

Now, $f''(x) = 12x - 18a$

At $x = a$, $f''(x) = 12a - 18a = -6a < 0$, maxima

At $x = 2a$, $f''(s) = 24a - 18a = 6a > 0$, minimum

$\therefore \quad p = a, q = 2a$

Also, $p^2 = q$

$\Rightarrow \quad a^2 = 2a$

$\Rightarrow \quad a = 2$

37. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 18$ on the set $A = \{x | x^2 + 20 \le 9x|\}$ is: *[IIT-2009]*

(a) 3 (b) 5

(c) 7 (d) None of these

Solution

(c) $f'(x) = 6(x - 2)(x - 3)$

so $f(x)$ is increasing in $(3, \infty)$

Also $A = \{4 \le x \le 5\} \therefore f_{\max} = f(5) = 7$.

38. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x\to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2$.

Then the value of $p(2)$ is: *[IIT-2009]*

(a) 0 (b) 1

(c) 1/2 (d) None of these

Solution

(a) Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$P'(1) = P'(2) = 0$

$\lim_{x\to 0}\left(\frac{x^2 + P(x)}{x^2}\right) = 2 \Rightarrow P(0) = 0 \Rightarrow e = 0$

$\lim_{x\to 0}\left(\frac{2x + P'(x)}{2x}\right) = 2 \Rightarrow P'(0) = 0 \Rightarrow d = 0$

$\lim_{x\to 0}\left(\frac{2 + P''(x)}{2}\right) = 2 \Rightarrow c = 1$

On solving, $a = 1/4$, $b = -1$

So, $P(x) = \frac{x^4}{4} - x^3 + x^2 \Rightarrow P(2) = 0$

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$ Let's be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

39. The real number s lies in the interval: *[IIT(2)-2010]*

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, -\frac{1}{4}\right)$

Solution

Given $f(x) = 1 + 2x + 3x^2 + 4x^3$, $f'(x) = 2 + 6x + 12x^2$

$\because$ Discriminant $= 6^2 - 4.12 < 0$, $\therefore f'(x) > 0 \forall x \in R$

$\therefore f(x)$ is strictly increasing function $\forall x \in R$

No. of real root = 1

If $f(x)$ has root in (a, b) $f(a) + f(b) < 0$

40. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval: ***[IIT(2)-2010]***

(a) $\left(\frac{3}{4}, 3\right)$ (b) $\left(\frac{21}{64}, \frac{11}{16}\right)$

(c) (9, 10) (d) $\left(0, \frac{21}{64}\right)$

Solution

(a) By estimation of integration

$$\int_0^{1/2} t(x)dx < \int_0^{t} f(x)dx < \int_0^{3/4} f(x)dx$$

$$\Rightarrow \frac{15}{16} < \int_0^1 f(x)\,dx < \frac{525}{256}$$

Hence option (a) is correct.

41. The function $f'(x)$ is: ***[IIT(2)-2010]***

(a) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(b) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(c) increasing in $(-t, t)$

(d) decreasing in $(-t, t)$

Solution

(b) $f'(x) = 2 + 6x + 12x^2$

$\Rightarrow f''(x) = 6 + 24x$

$\Rightarrow f''(x) = 6(4x + 1) > 0 \Rightarrow x > -\frac{1}{4}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. The angle of intersection of the curve $y = x^2$ and $6y = 7 - x^3$ at (1, 1) is ***[Kurukshetra CEE-2002]***

(a) π (b) $\pi/2$

(c) 2π (d) None of these

2. The angle of intersection between the curves $x^2 = 32y$ and $y^2 = 4x$ at the point (16, 8) is ***[PET (Raj.)-1986]***

(a) 60° (b) 90°

(c) $\tan^{-1}(3/5)$ (d) $\tan^{-1}(4/3)$

3. If $f(x) = (x + 1)^{2/3}$; $0 \le x \le 8$, then the maximum value and minimum value of $f(x)$ are

(a) $3\sqrt{3}, 1$ (b) $3\sqrt[3]{3}, 1$

(c) $\sqrt{3}, 1$ (d) $\sqrt[5]{3}, 1$

4. The minimum value of $2x + 3y$, when $xy = 6$ is ***[MP PET-2003]***

(a) 12 (b) 9

(c) 8 (d) 6

5. The points of local maxima and local minima. If any of the function $f(x) = (x - 1)(x + 2)^2$

(a) 0, 2 (b) 2, 0

(c) 0, –2 (d) None of these

6. Find the points of local maxima and local minima. If any of the function $y = \sin 2x - x$ when $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(a) $\pi/6, -\pi/6$ (b) $\pi/2, -\pi/2$

(c) $\pi, -\pi$ (d) None of these

7. Local maximum value of the function $\frac{\log x}{x}$ is ***[MNR-1984; RPET-1997, 2002; DCE-2002; Karnataka CET-2000, 08; UPSEAT-2001; MPPET-2002, 08]***

(a) e (b) 1

(c) $1/e$ (d) $2e$

8. Which of the following is not true

(a) Every differentiable function is continuous.

(b) If derivative of a function is zero at all points then the function is constant.

(c) If a function has maxima or minima at a point, then its derivatives is zero.

(d) If a function is constant then its derivatives is zero at all points.

9. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

[MPPET-2001]

(a) 0 (b) 12
(c) 16 (d) 32

10. The minimum value of $\dfrac{40}{3x^4 + 8x^3 - 18x^2 + 60}$ is ***[MNR-88]***

(a) 2/3 (b) 3/2
(c) 40/53 (d) None of these

11. The points of maximum and minimum of the curve $y = (x-1)^{1/3}(x-2)$; $1 \le x \le 9$ are

(a) (9, 5/4) (b) (–9, 5)
(c) (9, –5/4) (d) (9, 5)

12. x and y be two variables such that $x > 0$ and $xy = 1$. Then the minimum value of $x + y$ is

[Kurukshetra CEE-1998; MPPET-2002]

(a) 2 (b) 3
(c) 4 (d) 0

13. If $f(x) = 2x^3 - 21x^2 + 36x - 30$, then which one of the following is correct

(a) $f(x)$ has maximum at $x = 1$
(b) $f(x)$ has minimum at $x = 6$
(c) $f(x)$ has maxima at $x = 1$
(d) $f(x)$ has no maxima or minima

14. The function $f(x) = x^{-x}$, $(x \in R)$ attains a maximum value at $x =$

[EAMCET-2002]

(a) 2 (b) 3
(c) $1/e$ (d) 1

15. The function $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$ has a local minimum at ***[AIEEE-2006]***

(a) $x = 1$ (b) $x = 2$
(c) $x = -2$ (d) $x = 0$

16. The maximum value of $2x^3 - 3x^2 - 12x + 5$ for $-2 \le x \le 4$ occures when

[MNR-93; MP-2000]

(a) $x = -2$ (b) $x = -1$
(c) $x = 2$ (d) $x = 4$

17. If sum of two numbers is 3 then maximum value of the product of first and the square of second is

[MPPET-1996]

(a) 4 (b) 2
(c) $1/\sqrt{2}$ (d) $\sqrt{2}$

18. The minimum value of the function $2\cos 2x - \cos 4x$ in $0 \le x \le \pi$ is

(a) 0 (b) 1
(c) 3/2 (d) –3

19. The real number which most exceeds its cube is ***[MPPET-2000]***

(a) 1/2 (b) $1/\sqrt{3}$
(c) $1/\sqrt{2}$ (d) None of these

20. In the graph of the function $\sqrt{3}\sin x + \cos x$. The maximum distance of a point from x-axis is

(a) 4 (b) 2
(c) 1 (d) $\sqrt{3}$

21. For real values of x, the least value of $f(x) = 2^{x-1} + \dfrac{1}{2^{x-1}}$ is ***[KCET-1997]***

(a) –1 (b) –2
(c) 1 (d) 2

22. If $y = a\log|x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then

[IIT-1981]

(a) $a = 2, b = -1$ (b) $a = 2, b = -1/2$
(c) $a = -2, b = 1/2$ (d) None of these

23. If $ax^2 + bx + 4$ attains its minimum value –1 at $x = 1$, then the values of a and b are respectively.

[Kerala PET-2008]

(a) 5, –10 (b) 5, –5
(c) 5, 5 (d) 10, –5

24. The angle between the curves $x^2 = 4y$ and $x^2 + y^2 = 5$ at the point (–2, 1) is

(a) π (b) $\pi/2$
(c) 2π (d) None of these

25. The angle of intersection between the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ is

[UPSEAT 2002; AIEEE-2002]

(a) 45° (b) 90°
(c) 60° (d) 30°

26. If the curve $y = a^x$ and $y = b^x$ intersect at angle α, then $\tan\alpha =$

[MPPET-2001]

(a) $\dfrac{a-b}{1+ab}$ (b) $\dfrac{\log a - \log b}{1 + \log a \log b}$

(c) $\dfrac{a+b}{1-ab}$ (d) $\dfrac{\log a + \log b}{1 - \log a \log b}$

27. The angle of the intersection of the curves $y = x^2$ and $x = y^2$ at (1, 1) is

[Roorkee-2000; Karnataka CET-2001]

(a) $\tan^{-1}(4/3)$ (b) $\tan^{-1}(1)$
(c) 90° (d) $\tan^{-1}(3/4)$

28. The curves $ax^2 + by^2 = 1$ and $a' x^2 + b' y^2 = 1$ intersect orthogonally if

(a) $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$ (b) $\frac{1}{a} + \frac{1}{b} = \frac{1}{a'} + \frac{1}{b'}$
(c) $\frac{1}{a} + \frac{1}{a'} = \frac{1}{b} + \frac{1}{b'}$ (d) None of these

29. Let $f(x) = x^2 e^{-2x}$; $x > 0$. Then, the maximum value of $f(x)$ is

[VIT-2007]

(a) $\frac{1}{e}$
(b) $\frac{1}{2e}$
(c) $\frac{1}{e^2}$
(d) $\frac{4}{e^4}$

SOLUTIONS

1. (b) Let angle of intersection be θ

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Given curves $y = x^2, 6y = 7 - x^3$

Differentiating both curve $m_1 = 2x$ at $x = 1$, $m_1 = 2$

$$m_2 = \frac{-3x^2}{6} = \frac{-1}{2}.1^2 = -\frac{1}{2} \ (\because x_1 = 0)$$

∴ Angle = 90°; $(\because m_1 m_2 = -1)$

2. (c) $x^2 = 32y; m_1 = \frac{dy}{dx} = \frac{2x}{32} = 1$

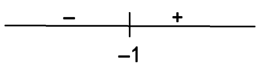
$(\because x_1 = 16)$

$$y^2 = 4x; m_2 = \frac{2}{y} = \frac{2}{8} = \frac{1}{4} (\because y_1 = 8)$$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}} = \frac{3}{5}$$

$$\theta = \tan^{-1}\left(\frac{3}{5}\right)$$

3. (b) $f(x) = (x+1)^{2/3}, f'(x) = \frac{2}{3}.\frac{1}{(x+1)^{1/3}}$



∴ –1 is point of minima.

But $x \in [0, 8]$,

∴ $f(0) = 1, f(8) = 9^{2/3} = 3\sqrt[3]{3}$

∴ Max value = $3\sqrt[3]{3}$, min. value = 1

4. (a) $f(x) = 2x + 3y$, $xy = 6$

or $y = \frac{6}{x}$

$f(x) = 2x + \frac{18}{x}$, for minimum value

$f'(x) = 0$; $f''(x) > 0$

$$2 - \frac{18}{x^2} = 0 \Rightarrow x = 3, -3$$

$\frac{d^2y}{dx^2} = \frac{36}{x^3} > 0$ at $x = 3$,

∴ 3 is point of minima, $y = \frac{6}{x} = 2$

∴ $f(x)_{\min} = 2x + 3y = 2.3 + 3.2 = 12$

5. (c) Here $f(x) = (x-1)(x+2)^2$

$$f'(x) = (x+2)^2 + (x-1)(2(x+2))$$

$$= (x+2)[x + 2 + 2x - 2) = (x+2)(3x)$$

For max. or min. $f'(x) = 0$

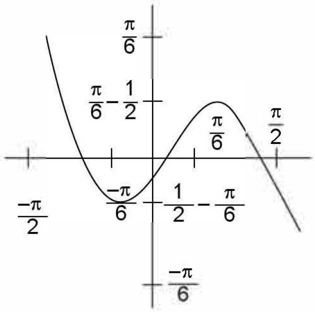
$$x = 0, -2$$

6. (b) $y = \sin 2x - x; -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

$$\frac{dy}{dx} = 0 \Rightarrow 2\cos 2x - 1 = 0 \Rightarrow \cos 2x = \frac{1}{2}$$

$$2x = \frac{\pi}{3}, \frac{-\pi}{3} \Rightarrow x = \frac{-\pi}{6}, \frac{\pi}{6}$$

at $x = \dfrac{-\pi}{6}$; $y = -\sin\dfrac{\pi}{3} + \dfrac{\pi}{6} = \dfrac{\pi}{6} - \dfrac{1}{2}$



$$x = \frac{\pi}{6}, y = \sin\frac{\pi}{6} - \frac{\pi}{6} = \frac{1}{2} - \frac{\pi}{6}$$

at $\quad x = \dfrac{\pi}{2}; y = \dfrac{-\pi}{2}$

at $\quad x = \dfrac{-\pi}{2}; y = \dfrac{\pi}{2}$

$\therefore \quad y$ is max and min. at $\dfrac{\pi}{2}$ and $-\dfrac{\pi}{2}$

7. (c) For maximum value

$f'(x) = 0; f''(x) < 0$

$$f'(x) = \frac{x \times \frac{1}{x} - \log x \times 1}{x^2} = 0 \Rightarrow \log x = 1$$

$\Rightarrow x = e \Rightarrow$ Number of stationary point = 1

step 2: $f'(x) = \dfrac{1 - \log x}{x^2}$

$$\therefore f''(x)\Big|_{x=e} = \left.\frac{(x^2)\left(\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}\right|_{x=e}$$

$$= \frac{-e}{e^4} = -\frac{1}{e^3} < 0$$

Function has maximum value at $x = e$

or $\therefore x = \mathrm{e}$ is *a* point of maxima

Step 3: If function $f(x)$ is continuous at close interval $[a, b]$, then maximum or minimum value at stationary point and as well as at end extreme points

$$f(e) = \frac{\log e}{e} = \frac{1}{e}, f(2) = \frac{\log 2}{2}$$

$$f(e) > f(2)$$

8. (c) By definition

9. (b) Slope $= \dfrac{dy}{dx} = -3x^2 + 6x + 9 = m$

For maximum slope, $\dfrac{dm}{dx} = -6x + 6 = 0$

$$x = 1, m = -3 + 6 + 9 = 12$$

10. (a) $y = \dfrac{40}{3x^4 + 8x^3 - 18x^2 + 60}$

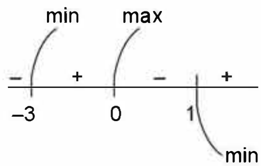
For min. value, consider

$g(x) = 3x^4 + 8x^2 - 18x^2 + 60$

must be maximum,

$g'(x) = 12x^3 + 24x^2 - 36x$

$g'(x) = 12x(x^2 + 2x - 3) = 12x(x + 3)(x - 1)$



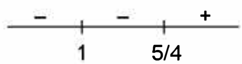
Critical points are

$\therefore \quad g(x)$ is maximum at $x = 0$

$$y_{\min} = \frac{1}{g(x)} = \frac{40}{60} = \frac{2}{3}$$

11. (a) $\dfrac{dy}{dx} = \dfrac{1}{3}(x-1)^{-2/3}(x-2) + (x-1)^{1/3}$

$$= \frac{x - 2 + 3(x-1)}{3(x-1)^{2/3}} = \frac{4x - 5}{3(x-1)^{2/3}}$$



But $1 \le x \le 9$

$\therefore \quad f(1) = 0, f(9) = 8^{1/3}.7 = 14$

$$f(5/4) = \left(\frac{5}{4} - 1\right)^{1/3}\left(\frac{5}{4} - 2\right) = \frac{-3}{2^{2/3}.4}$$

$\therefore\ f(x)$ is maximum at $x = 9$

and minimum at $x = \dfrac{5}{4}$

$\therefore$ points are $\left(9, \dfrac{5}{4}\right)$

12. (a) Step1: From given conditions

$x + y = x + \dfrac{1}{x}$

$\therefore\ f(x) = x + \dfrac{1}{x}$

Step 2: $f'(x) = 0, f''(x) > 0$

$\therefore\ f'(x) = 1 - \dfrac{1}{x^2} = 0 \Rightarrow x = \pm 1$

and $f''(1) = \left.\dfrac{2}{x^3}\right|_{x=1} = 2 > 0$

where $f'(1) = 0, f''(1) > 0$

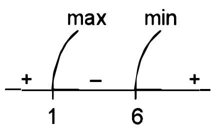
$\therefore$ minimum value, put $x = 1$ in function

$f(1) = 1 + 1 = 2$

13. (a) $f(x) = 2x^3 - 21x^2 + 36x - 30$

$f'(x) = 6x^2 - 42x + 36$

$= 6(x^2 - 7x + 6) = 6(x-1)(x-6)$



$\therefore$ $f(x)$ is minima at $x = 6$ and maxima at $x = 1$

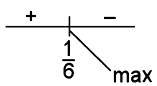
14. (c) $f(x) = x^{-x}$

for max. points, $f'(x) = 0$

$x^{-x}\left(\dfrac{d}{dx}(-x\log x)\right) = 0 \Rightarrow 1 + \log x = 0$

$x = \dfrac{1}{e}$,

Consider $f'(x) = -x^{-x}(1 + \log x)$



$\therefore\ x = \dfrac{1}{e}$ is point of max.

15. (b) $f'(x) = 0 \Rightarrow \dfrac{1}{2} - \dfrac{2}{x^2} = 0$

$x = \pm 2$, Now $f''(x) = \dfrac{4}{x^3}$

$f''(x)$ is > 0 at $x = 2; \therefore\ x = 2$ is min.

16. (b) Step 1: $f'(x) = 0$ For maxima $f'(x) = 0, f''(x) < 0$

$\Rightarrow 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x = 2, -1$

Step 2:

$\Rightarrow\ f''(x) = 12x - 6, f''(2)$

$= 18 > 0, f''(-1) = -18 < 0$

$x = -1$ is point of maxima

17. (a) Step 1: According to question,
function $f(x, y) = xy^2$ where (given)
$x + y = 3$ = First number + Second number

$\therefore\ f(y) = (3 - y)y^2 = 3y^2 - y^2$ .... (1)

Step 2: $f'(y) = 0, f''(y) < 0$

$f'(y) = 6y - 3y^2 = 0 \Rightarrow y = 0, 2$

$f''(y) = 6 - 6y \Rightarrow f''(2) = -6 < 0$

$\therefore$ $y = 2$ is point of maximum and minimum value

$= 3(2)^2 - (2)^3 = 12 - 8 = 4$

18. (b) $f(x) = 2\cos 2x - \cos 4x$

$f'(x) = -4\sin 2x + 4\sin 4x$

$= 4(\sin 4x - \sin 2x) = 8\sin x\cos 3x$

For extreme values, $\sin x = 0$ or $\cos 3x = 0$

For $0 \le x \le \pi$; $x = 0, \pi, \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$

$f''(x) = \dfrac{d}{dx}4(\sin 4x - \sin 2x)$

$= 16\cos 4x - 8\cos 2x$

$f''(x) > 0$ for $x = 0, \pi, \dfrac{\pi}{2}$

$\therefore$ at $f(0) = 1, f(\pi) = 1, f\left(\dfrac{\pi}{2}\right) = 3$

$\therefore$ min. value = 1

19. (b) Step1: Let number = x and its cube = x^3 and $f(x) = x - x^3$.

Step 2:

$f'(x) = 1 - 3x^2 = 0, x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

and $f''\left(\frac{1}{x}\right) = -6x, f''\left(\frac{1}{\sqrt{3}}\right) = \frac{-6}{\sqrt{3}} < 0$

Step 3: $f'\left(\frac{1}{\sqrt{3}}\right) = 0, f''\left(\frac{1}{\sqrt{3}}\right) < 0$

$\therefore x = \frac{1}{\sqrt{3}}$

20. (b) Distance of point (x, y) from x axis is $= y$

Where $y = \sqrt{3}\sin x + \cos x$

$\therefore \quad \text{Max } y = \sqrt{\left(\sqrt{3}^2\right) + 1^2} = 2$

NOTE

Max. value of $a\sin x + b\cos x = \sqrt{a^2 + b^2}$

21. (d) $f(x) = 2^{x-1} + \frac{1}{2^x - 1} = 2^{x-1} + 2^{-x+1}$

For least value of x, $f'(x) = 0, f''(x) > 0$

$f'(x) = (2^{x-1})\log 2 + 2^{-x+1}\log 2(-1) = 0$

$2^{x-1} = 2^{-x+1} \Rightarrow x - 1 = -x + 1$

$x = 1$, Also $f''(1) = +ve$

$\therefore f(1) = 2$ is minimum.

22. (b) Step 1: For maximum or minimum values extreme values can also be use.

Step 2: $\left.\frac{dy}{dx}\right|_{\substack{x=-1\\x=2}} = 0 \Rightarrow \left.\frac{a}{x} + 2bx + 1\right|_{\substack{x=-1\\x=2}} = 0$

$$\therefore \left.\begin{aligned} -a - 2b + 1 = 0 \\ \frac{a}{2} + 4b + 1 = 0 \end{aligned}\right\} \Rightarrow a = 2, b = -\frac{1}{2}$$

23. (a) $f(x)$ has minimum value -1 at $x = 1$

$\therefore \quad f(1) = -1, f'(1) = 0$

$a + b + 4 = -1$

or $\quad (2ax + b)_{x=1} = 0$

$b = -2a; \therefore a = 5, b = -10$

24. (d) $C_1: x^2 = 4y, \frac{dy}{dx} = \frac{x_1}{2} = m_1$

$C_2: x^2 + y^2 = 5, \left(\frac{dy}{dx}\right) = \frac{-x_1}{y_1} = m_2$

Where (x_1, y_1) is $(-2, 1)$

$m_1 = -1; m_2 = 2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = 3 \Rightarrow \theta = \tan^{-1} 3$

25. (b) Let (x_1, y_1) be point of intersection

$C_1: x^3 - 3xy^2 + 2 = 0$

$\Rightarrow \quad 3x_1^2 - 3y_1^2 - 6x_1y_1\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{x_1^2 - y_1^2}{2x_1y_1} = m_1$

$C_2: 3x^2y - y^3 - 2 = 0$

$\frac{dy}{dx} = \frac{-2x_1y_1}{x_1^2 - y_1^2} = m_2$

$\therefore \quad m_1m_2 = -1$, $\therefore$ Angle between them is $90°$

26. (b) Point of intersection of $y = a^x$ and $y = b^x$

$a^x = b^x \Rightarrow \therefore x = 0 \quad (\because a \neq b)$

at $x = 0, y = 1$; $(0, 1)$ is point

$C_1: y = a^x; \frac{dy}{dx} = a^x \log a$

$\therefore \quad m_1 = \log a$

Similarly, $m_2 = \log b$

$\tan\alpha = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{\log a - \log b}{1 + \log a \log b}$

27. (d) $y = x^2, \frac{dy}{dx} = 2x, m_1 = 2$ at $(1, 1)$

$x = y^2, \frac{dy}{dx} = \frac{1}{2y},$

$m_2 = \frac{1}{2}$ at $(1, 1)$

$\therefore$ Angle between them $= \tan^{-1}\left(\dfrac{m_1 - m_2}{1 + m_2 m_3}\right)$

$$= \tan^{-1}\left(\frac{2 - \frac{1}{2}}{1 + 2.\frac{1}{2}}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

28. (a) $C_1 : ax^2 + by^2 = 1$

Differentiating $2ax + 2by\dfrac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-ax}{by} \Rightarrow m_1 = \frac{-ax_1}{by_1}$$

where (x_1, y_1) is point of intersection

$$\therefore \quad m_2 = \frac{-a'x_1}{b'y_1},$$

For orthogonal curve $m_1 m_2 = -1$

$$\frac{aa'x_1^2}{bb'y_1^2} = -1 \Rightarrow \frac{x_1^2}{y_1^2} = \frac{-bb'}{aa'} \qquad \text{.....................(i)}$$

Also $ax_1^2 + by_1^2 = 1$

$a'x_1^2 + b'y_1^2 = 1$

By cross multiplication method:

$$\frac{x_1^2}{b - b'} = \frac{y_1^2}{a' - a} = \frac{-1}{ab' - a'b}$$

$$\therefore \frac{x_1^2}{y_1^2} = \frac{b - b'}{a' - a} = \frac{-bb'}{aa'} \text{ .. using (i)}$$

$$\frac{b - b'}{-bb'} = \frac{a' - a}{aa'}$$

$$\frac{-1}{b'} + \frac{1}{b} = \frac{1}{a} - \frac{1}{a'} \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$$

29. (c) $f(x) = x^2 e^{-2x}$ .... $(1), x > 0$

for maxima; $f'(x) = 0$; $f''(x) < 0$

$f'(x) = -2x^2 e^{-2x} + 2xe^{-2x} = 0$

$\Rightarrow 2xe^{-2x}(1 - x) = 0$

and

$f''(x) = (-4e^{-2x}x + 2e^{-2x})(1 - x) - 2xe^{-2x}$

$= -2e^{-2} < 0$

So $x = 1$ is a point of maxima; then put it in equation (1)

$$f(x) = \frac{1}{e^2}$$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. The maximum value of $x(1 - x)^2$ when $0 \le x \le 2$ is
(a) 2 (b) 4/27
(c) 5 (d) 0

2. The maximum value of $2x^3 - 24x + 107$ in the interval $[-3, 3]$ is
(a) 75 (b) 89
(c) 125 (d) 139

3. The function $f(x) = 2x^3 - 15x^2 + 36x + 4$ has maximum value at ***[KCET-2001]***
(a) $x = 4$ (b) $x = 2$
(c) $x = 3$ (d) $x = 0$

4. The difference between maximum and minimum values of the function $a \sin x + b \cos x$ is ***[ICS-2001]***
(a) $2\sqrt{a^2 + b^2}$
(b) $2(a^2 + b^2)$
(c) $\sqrt{a^2 + b^2}$
(d) $-\sqrt{a^2 + b^2}$

5. If $x - 2y = 4$, the minimum value of xy is ***[UPSEAT-2003]***
(a) -2 (b) 2
(c) 0 (d) -3

6. The minimum value of $x^2 + \dfrac{1}{1 + x^2}$ is at ***[UPSEAT-2003]***
(a) $x = 0$ (b) $x = 1$
(c) $x = 4$ (d) $x = 3$

7. The minimum value of $\left(x^2 + \frac{250}{x}\right)$ is **[Kurukshetra CEE-02]**

(a) 75 (b) 50
(c) 25 (d) 55

8. If $f(x) = x + \frac{1}{x}, x > 0,$ then its greatest value is **[RPET-2002]**

(a) –2 (b) 0
(c) 3 (d) None of these

9. If $f(x) = \frac{1}{4x^2 + 2x + 1},$ then its maximum value is **[RPET-2002]**

(a) 4/3 (b) 2/3
(c) 1 (d) 3/4

10. If $A + B = \frac{\pi}{2},$ the maximum value of cos$A$ cos$B$ is **[AMU-99]**

(a) 1/2 (b) 3/4
(c) 1 (d) 4/3

11. 20 is divided into two parts so that product of cube of one quantity and square of the other quantity is maximum. The parts are **[RPET-1997]**

(a) 10, 10 (b) 16, 4
(c) 8, 12 (d) 12, 8

12. The maximum value of function $x^3 - 12x^2 + 36x + 17$ in the interval [1, 10] is

(a) 17 (b) 177
(c) 77 (d) None of these

13. The function $y = a(1 - \cos x)$ is maximum when x = **[Kerala (Engg.)-2002]**

(a) π (b) $\pi/2$
(c) $-\pi/2$ (d) $-\pi/6$

14. The angle of intersection of curves $y = 4 - x^2$ and $y = x^2$ is **[RPET-89; MNR-78]**

(a) $\pi/2$ (b) $\tan^{-1}(4/3)$
(c) $\tan^{-1}(4\sqrt{2}/7)$ (d) None of these

15. If curves $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then values of 'a' is

(a) 1/2 (b) 1/3
(c) 2 (d) 3

16. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is **[AIEEE-2006]**

(a) $\pi/4$ (b) $\pi/3$
(c) $\pi/2$ (d) $\pi/6$

17. The angle between the curve $y = \sin x$ and $y = \cos x$ is **[EAMCET-2003]**

(a) $\tan^{-1}(2\sqrt{2})$ (b) $\tan^{-1}(3\sqrt{2})$
(c) $\tan^{-1}(3\sqrt{3})$ (d) $\tan^{-1}(5\sqrt{2})$

18. What is the maximum slope of the curve $y = -x^3 + 3x^2 + 2x - 27$? **[NDA-2007]**

(a) 1 (b) 2
(c) 5 (d) – 23

19. If $y = a\log x + bx^2 + x$ has its extremum value at $x = 1$ and $x = 2$, then (a, b) = **[UPSEAT-2002]**

(a) $\left(1, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 2\right)$
(c) $\left(2, \frac{-1}{2}\right)$ (d) $\left(\frac{-2}{3}, \frac{-1}{6}\right)$

20. What is the maximum value of $y = \sin^3 x \cos x, 0 < x < \pi$? **[UPSC-2007]**

(a) $-3\sqrt{3}/16$ (b) $3\sqrt{3}/4$
(c) – 3/16 (d) $3\sqrt{3}/16$

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 16 minutes.
3. The worksheet consists of 16 questions. The maximum marks are 48.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. Function $2x^3 - 3x^2 - 12x + 4$ has *[KCET-96; DCE-2002]*
(a) Two maximum points
(b) Two minimum points
(c) No maximum, no-minimum point
(d) One maximum and one minimum point

2. The maximum value of the function $f(x) = x^3 - x^2 - 5x + 6$ is at *[Ranchi-2001]*
(a) $x = -1$ (b) $x = 5/3$
(c) $x = 2$ (d) No-where

3. What is the product of two parts of 20, such that the product of square of one part and the cube of the other is maximum? *[NDA-2007]*
(a) 75 (b) 91
(c) 84 (d) 96

4. The real number x when added to its inverse gives the minimum value of the sum at x equal to *[RPET-2000; AIEEE-2003]*
(a) -2 (b) 2
(c) 1 (d) -1

5. The sum of two non-zero numbers in 4. The minimum value of the sum of the reciprocals is *[Kurukshetra CEE-1998]*
(a) 3/4 (b) 6/5
(c) 1 (d) None of these

6. The minimum value of $2x^2 + x - 1$ is *[EAMCET-2003]*
(a) $-1/4$ (b) 3/2
(c) $-9/8$ (d) 9/4

7. The value of the function $(x - 1)(x - 2)^2$ at its maxima is
(a) 1 (b) 2
(c) 0 (d) 4/27

8. The sum of two numbers is fixed. Then its multiplication is maximum, when
(a) Each number is half of the sum
(b) Each number is 1/3 and 2/3 respectively of the sum
(c) Each number is 1/4 and 3/4 respectively of the sum
(d) None of these

9. The function $x^5 - 5x^4 + 5x^3 - 1$ is *[MPPET-1993]*
(a) Maximum at $x = 3$ and minimum at $x = 1$
(b) minimum at $x = 1$
(c) Neither maximum nor minimum at $x = 0$
(d) maximum at $x = 0$

10. The maximum value of the function $x^3 + x^2 + x - 4$ is
(a) 127
(b) 4
(c) does not have a maximum value
(d) None of these

11. The point for the curve $y = xe^x$ *[MNR-90]*
(a) $x = -1$ is minimum
(b) $x = 0$ is minimum
(c) $x = -1$ is maximum
(d) $x = 0$ is maximum

12. The height of the circular cylinder of maximum volume that can be inscribed in a given right circular cone of height $6h$ is
(a) h (b) $3h$
(c) $2h$ (d) $4h$

13. The curves $y^2 = 16x$ and $9x^2 + by^2 = 16$ are orthogonal, then $b =$
(a) 2 (b) 4
(c) 9/2 (d) None of these

14. The angle of intersection between the curves $x^2 = 8y$ and $y^2 = 8x$ at origin is
(a) $\pi/4$ (b) $\pi/3$
(c) $\pi/6$ (d) $\pi/2$

15. The curves $4x^2 + 9y^2 = 72$ and $x^2 - y^2 = 5$ at $(3, 2)$

[Kerala PET-2001]

(a) touch each other
(b) cut orthogonally
(c) intersect at 45°
(d) intersect at 60°

16. The least value of $2 \sin^2 \theta + 3 \cos^2 \theta$ is
(a) 1 (b) 2
(c) 3 (d) 5

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ
6. ⓐ ⓑ ⓒ ⓓ
7. ⓐ ⓑ ⓒ ⓓ
8. ⓐ ⓑ ⓒ ⓓ
9. ⓐ ⓑ ⓒ ⓓ
10. ⓐ ⓑ ⓒ ⓓ
11. ⓐ ⓑ ⓒ ⓓ
12. ⓐ ⓑ ⓒ ⓓ
13. ⓐ ⓑ ⓒ ⓓ
14. ⓐ ⓑ ⓒ ⓓ
15. ⓐ ⓑ ⓒ ⓓ
16. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

1. (c) Step 1: For maxima or minima $f'(x) = 0$

$f'(x) = 6x^2 - 6x - 12 = 0 \Rightarrow x = -1, 2$

$f''(x) = 12x - 6$ and $f''(-1) = -18 < 0$

$f''(2) = 6 > 0$

Step 2: $f'(-1) = 0, f''(-1) < 0 \Rightarrow x = -1$ point of maxima

and $f'(2) = 0, f''(2) > 0 \Rightarrow x = 2$ point of minima

3. (d) Let $20 = x + y$ and

$P = x^3y^2 = x^3(20 - x)$

$\therefore$ For maxima $\dfrac{dp}{dx} = 0; \dfrac{d^2p}{dx^2} < 0$

$P = x^3(400 + x^2 - 40x)$

$= 400x^3 + x^5 - 40x^4$, then

$\dfrac{dp}{dx} = 1200x^2 + 5x^4 - 160x^5 = 0$

$\Rightarrow x^2(1200 + 5x^2 - 160x) = 0$

$\Rightarrow x = 0, x^2 - 32x + 240 = 0$

$\Rightarrow x = 12, 20$

$\left.\dfrac{d^2p}{dx^2}\right|_{x=12} = 2400x + 20x^3 - 40x^2$

$= 20(x^3 + 120x - 24x^2)\big|_{x=12} < 0$

$\therefore$ $x = 12$ and $y = 8$ and then product of two parts
$= 12 \times 8$
$= 96$

4. (b) Step 1: Consider $f(x) = x + \dfrac{1}{x}; x > 0$

Step 2: $f'(x) = 1 - \dfrac{1}{x^2} = 0 \Rightarrow x = 1, -1$

$f''(x) = \left.\dfrac{2}{x^3}\right|_{x=1} = 2 > 0$

$\therefore$ minimum value 2 of function $f(x) = x + \dfrac{1}{x}$

5. (c) Step 1: Given $x + y = 4, R = \frac{1}{x} + \frac{1}{y}$

$\therefore \quad R = \frac{1}{x} + \frac{1}{4 - x}$

Step 2: For minimum R: $\frac{dR}{dx} = 0, \frac{d^2R}{dx^2} > 0$

$\therefore \quad \frac{dR}{dx} = -\frac{1}{x^2} + \frac{1}{(4-x)^2} = 0$

$\Rightarrow \frac{1}{(4-x)^2} = \frac{1}{x^2}$

$\Rightarrow x = \pm(4 - x) \Rightarrow 2x = 4 \Rightarrow x = 2$

$\therefore \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{2} + \frac{1}{2} = 1$

10. (d) Step 1:

(i) Function $y = f(x)$ will have a maxima or minima at point $x = a$ if $f'(a) = 0$.

(ii) Function $f(x)$ will have a maxima at point $x = a$ if $f'(a) = 0$ and $f''(a) < 0$

(iii) Function $f(x)$ will have a minima at point $x = a$ if $f'(a) = 0$ and $f''(a) > 0$

(iv) Function $f(x)$ will have neither maxima nor minima at point $x = a$ if $f'''(a) \neq 0$

Step 2: $f'(x) = 5x^4 - 20x^3 + 15x^2 = 0$

$\Rightarrow x^2(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3$

$f''(x) = 20x^3 - 60x^2 + 30x$

$f'''(x) = 60x^2 - 120x + 30, f'''(0) = 30$

12. (a) Step 1:

$\frac{dy}{dx} = e^x + xe^x = e^x(x+1) = 0 \Rightarrow x = -1$

$\frac{d^2y}{dx^2} = e^x(x+1) + e^x = e^x(x+2)$

Step 2: y for minima $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$

$\left.\frac{d^2y}{dx^2}\right|_{x=-1} = e^{-1}(1) = \frac{1}{e} > 0$

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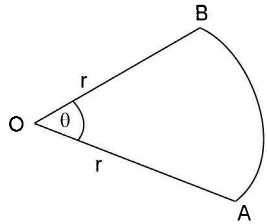
LECTURE

5

Maxima and Minima 2

BASIC CONCEPTS

POINTS TO REMEMBER

1. 

If centre is at 0 and radius is r, then

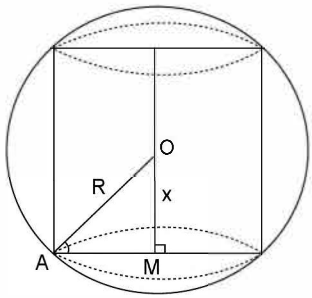
(i) Area of sector OAB of circle $= \frac{1}{2}r^2\theta$ where θ is in radian.

(ii) Perimeter of this sector $= 2r + r\theta$.

2. Volume (V) of a right circular cylinder $= \pi r^2 h$
3. Curved surface area (S) of cylinder $S = 2\pi rh$
4. Total surface area (T.S.) of cylinder $= 2\pi rh + 2\pi r^2$
5. Volume (V) of sphere $= \frac{4}{3}\pi r^3$
6. Surface area (S) of a sphere $S = 4\pi r^2$
7. Volume of the right circular cone $V = \frac{1}{3}\pi r^2 h$.
8. Curved surface area of cone $S = \pi rl$ (l = slant height)
9. Total surfce area of right circular cone T.S. $= \pi rl + \pi r^2$.
10. Right circular cylinder in sphere:

$\therefore \quad AM = \sqrt{(R^2 - x^2)}$

V(Volume of the cylinder)



$= \pi r^2 h = \pi(AM)^2 . 2x$

$= \pi (R^2 - x^2) . 2x = 2\pi x (R^2 - x^2)$

S (curved surface area of cylinder)

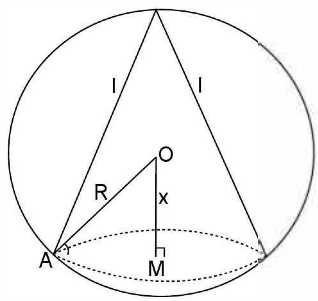
$= 2\pi rh = 2\pi (AM) . 2x = 4\pi x \sqrt{(R^2 - x^2)}$

T.S. (Total surface area of cylinder)

$= 2\pi rh + 2\pi r^2 = 4\pi x \sqrt{(R^2 - x^2)} + 2\pi (R^2 - x^2)$

11. **Right Circular cone in a sphere:**

$\therefore \quad AM = \sqrt{(R^2 - x^2)}$



V (Volume of the cone)

$$= \frac{1}{3}\pi r^2 h = \frac{2\pi x}{3}(R^2 - x^2)$$

S (Curved surface area of cone)

$$= \pi rl = \pi(AM)\sqrt{(AM)^2 + (x + R^2)}$$

T.S. (Total surface area of cone) = $\pi rl + \pi r^2$

$$= \pi(AM)\sqrt{(AM)^2 + (x + R^2)} + \pi(AM)^2$$

$$= \pi rl = \pi(AM)\sqrt{(AM)^2 + (x + R^2)}$$

12. **Right circular Cylinder in cone:** Δ, sAOB and $AO'\,B'$ are similar

$$\frac{AO'}{AO} = \frac{O'B'}{OB}$$

$$\Rightarrow \quad \frac{H - h}{H} = \frac{r}{R}$$

$$\Rightarrow \quad h = h\left(1 - \frac{r}{R}\right)$$

V (Volume of the cylinder)

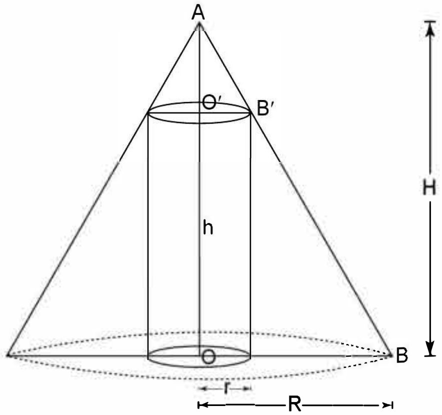
$$= \pi r^2 h = \pi r^2 \times H\left(1 - \frac{r}{R}\right)$$

S(Curved surface area of cylinder)

$$= 2\pi\, rh = 2\pi r \times H\left(1 - \frac{r}{R}\right)$$

T.S. (Total surface area of cylinder) = $2\pi rh + 2\pi r^2$

$$= 2\pi r \times H\left(1 - \frac{r}{R}\right) + 2\pi r^2$$



13. **Cone around sphere:**

Height of cone $h = OB + OA = R + R$ cosec θ

Radius = $r = n \tan\theta$

$$= R\,(1 + \text{cosec}\ \theta)\tan\theta$$

$\therefore$ Volume of cone $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi R^3(1 + \text{cosec}\ \theta)^3.\ \tan^2\theta$$

S (Curved surface area of cone) = πrl

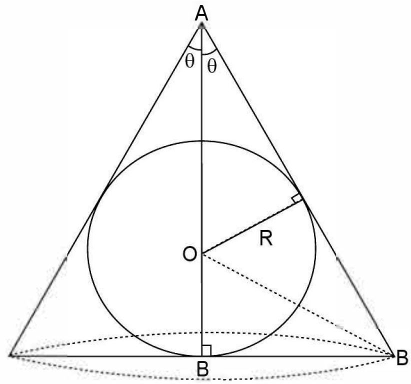
$= \pi r\ .\ r$ cosec π

$= \pi R^2\,(1 + \text{cosec}\ \pi)^2$ cosec θ

T.S. (Total surface area of cone) = $\pi rl + \pi r^2$

$= \pi R^2\ (1 + \text{cosec}\ \theta)^2$ cosec $\theta + \pi R^2\ (1 + \text{cosec}\ \theta)^2$

$= \pi R^2(1 + \text{cosec}\ \theta)^3$



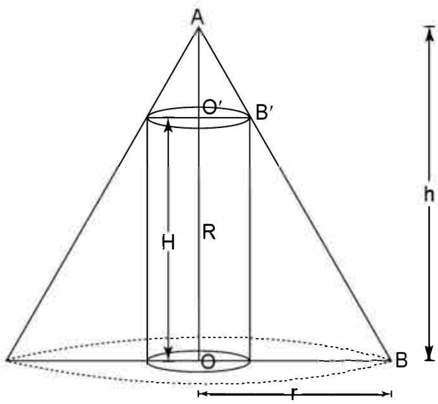
14. **Cone around cylinder:**

$\because$ Δ's AOB and $AO'B'$ are similar

$$\therefore \quad \frac{AO'}{AO} = \frac{O'B'}{OB}$$

$$\Rightarrow \quad \frac{h - H}{h} = \frac{R}{r} \quad \therefore \quad h = \frac{rH}{r - R}$$

$$V\text{ (Volume of cone) } = \frac{1}{3}\pi r^2 h = \frac{1}{3}\frac{\pi r^3 H}{(r - R)}$$



(Now differentiate with respect to x) S(curved surface of cone)

$$= \pi rl = \pi r\sqrt{(r^2+h^2)}$$

$$= \pi r\sqrt{\left(r^2+\frac{r^2H^2}{(r-R)^2}\right)}$$

$$= \frac{\pi r^2}{(r-R)}\sqrt{\{(r-R)^2+H^2\}}$$

T.S. (Total surface area of cone)

$$= \pi r\, l + \pi r^2 = \pi r^2\left\{\frac{\sqrt{\{(r-R)^2+H^2\}}}{(r-R)}\right\}$$

15. The triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.

16. The volume of the greatest cylinder which can be inscribed in a cone of height h and semivertical angle α is.

Volume of greatest cylinder $= \frac{4}{27}\pi h^3 \tan^2\alpha$

height of the greatest cylinder $H = \frac{h}{3} = \frac{1}{3}$.

height of cone

$\frac{R}{r} = \frac{2}{3}$, $R =$ Radius of the greatest cylinder

17. Find the maximum value of the volume of a right circular cylinder situated symmetrically inside a sphere of radius a . Height of the cylinder of greatest volume $= \frac{2a}{\sqrt{3}}$

Volume of the cylinder $r = \pi r^2 h$

Maximum value $= \frac{4\pi a^3}{3\sqrt{3}}$.

18. The cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}rd$ the diameter of the sphere.

19. For any closed right circular cylinder of given total surface area the maximum volume is such that height is equal to the diameter of the base. i.e., $h = 2r$.

20. The semivertical angle of the cone of maximum volume and of given slant height is $\alpha = \tan^{-1}\sqrt{2}$

21. The semivertical angle of the right cone of given total surface (including area of base) and maximum volume is $\alpha = \sin^{-1}(1/3)$

[redacted]

1. Prove that the area of a right angled triangle of given hypotenuse is maximum when the triangle is isosceles.

[MP-98, 2003; HB-87, 2003' CBSE-88' PSB-91, 97, 99, 2002]

Solution

Let ΔABC be right angled triangle.

Again, let $AC = y$

$AB = y \sin\theta$

$BC = y\cos\theta$

Area $= \frac{1}{2}y\sin\theta \;\; y\cos\theta = \frac{y^2\sin 2\theta}{4}$

area will be maximum if $\sin 2\theta = 1$

$\therefore \quad 2\theta = 90°$

$\Rightarrow \quad \theta = 45°$

Hence area is maximum when Δ is isosceles right Δ.

2. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when depth of the tank is half of its width.

[CBSE-99, 2003; HSB-2001]

Solution

Let the length, width and height of the open tank be x, x and y units respectively. Then its volume is x^2y and the total surface area is $x^2 + 4xy$.

It is given that the tank can be hold a given quantity of water.

This means that its volume is constant. Let it be V.

$\therefore \quad V = x^2 y$ .... (i)

The cost of the material will be least if the total surface area is least. Let S denote the total surface area. Then

$$S = x^2 + 4xy \quad \text{............. (ii)}$$

We have to minimize S subject to the condition that the volume is constant. Now, $S = x^2 + 4xy$

$$\Rightarrow \quad S = x^2 + \frac{4V}{x}$$

$$\Rightarrow \quad \frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

and $\dfrac{d^2S}{dx^2} = 2 + \dfrac{8V}{x^3}$

For maximum or minimum values of S, we must have

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad 2x - \frac{4V}{x^2} = 0 \Rightarrow 2x^3 = 4V$$

$$\Rightarrow \quad 2x^3 = 4x^2y$$

$$\Rightarrow \quad x = 2y$$

Clearly, $\dfrac{d^2S}{dx^2} = 2 + \dfrac{8V}{x^3} > 0$ for all x

Hence, S is minimum when $x = 2y$ i.e., the depth (height) of the tank is half of its width.

Proved.

3. If the sum of the lengths of the hypoytenuse and a side of a right angled triangle is given, show that area of triangle is maximum when the angle between them is $\pi/3$.

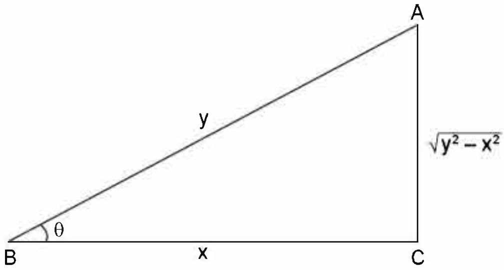
[CBSE-90, 2005 (Foreign)]

Solution

Let ABC be a right angled triangle with Base $BC = x$ and hypotenuse $AC = y$ such that $x + y = k$, where k is a constant.

Let θ be the angle between the base and hypotenuse.

Let A be the area of the triangle. Then



$$A = \frac{1}{2}BC \times AC = \frac{1}{2}x\sqrt{y^2 - x^2}$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}(y^2 - x^2)$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}\{(k-x)^2 - x^2\} \quad [\because x + y = k]$$

$$\Rightarrow \quad A^2 = \frac{k^2x^2 - 2kx^3}{4} \quad \text{................ (i)}$$

Differentiating with respect to x, we get

$$2A\frac{dA}{dx} = \frac{2k^2x - 6kx^2}{4} \quad \text{....... (ii)}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A}$$

For maximum or minimum, we have

$$\frac{dA}{dx} = 0 \Rightarrow \frac{k^2x - 3kx^2}{4A} = 0 \Rightarrow x = \frac{k}{3}$$

Now, differentiating (ii) with respect to x, we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A\frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4} \quad \text{...... (iii)}$$

Putting $\dfrac{dA}{dx} = 0$ and $x = \dfrac{k}{3}$ in (iii), we get

$$\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$$

Thus, A is maximum, when $x = \dfrac{k}{3}$

Now $x = \dfrac{k}{3} \Rightarrow y = k - \dfrac{k}{3} = \dfrac{2k}{3}$ $(\because x + y = k)$

$$\therefore \quad \frac{x}{y} = \cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{k/3}{2k/3} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Proved.

4. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

[CBSE-90, 98, 2005]

Solution

Let V be the fixed volume of a closed cuboid with length x, breadth x and height y. Let S be the surface area of the cuboid.

Then $V = x^2 y$ and $S = 2\,(x^2 + xy + xy) = 2x^2 + 4xy$

Now $V = x^2 y$......... (i) (Given condition) and $S = 2x^2 + 4xy$......... (ii) (To be minimized)

Now, $S = 2x^2 + 4xy \Rightarrow S = 2x^2 + 4x.\dfrac{V}{x^2}$

$$\Rightarrow \quad S = 2x^2 + \frac{4V}{x}$$

$$\Rightarrow \quad \frac{dS}{dx} = 4x - \frac{4V}{x^2} \qquad \text{... (iii)}$$

For maximum or minimum $\dfrac{dS}{dx} = 0$

$$\Rightarrow \quad 4x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \quad V = x^3$$

$$\Rightarrow \quad x^2 y = x^3 \Rightarrow x = y$$

Differentiating (iii) with respect to x, we get

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} = 4 + \frac{8x^2y}{x^3} = 4 + \frac{8y}{x}$$

$$\Rightarrow \quad \left(\frac{d^2S}{dx^2}\right)_{y=x} = 12 > 0$$

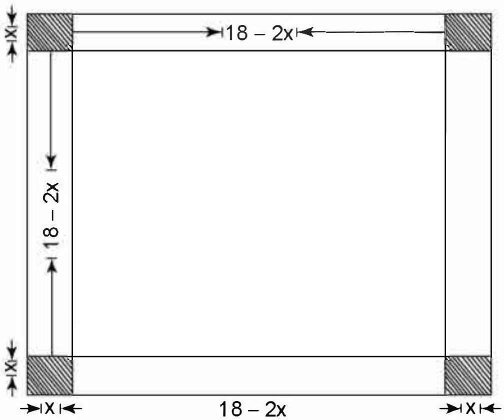
Hence, S is minimum when length $= x$, breadth $= x$ and height $= x$ i.e., when it is a cube.

5. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum?
Also, find this maximum volume.

[CBSE-91, 95, 2003; HPSB-2002S]

Solution

Let x cm be the length of a side of the square which is cut-off from each corner of the plate. Then sides of the box as shown in figure are $18 - 2x$, $18 - 2x$ and x.



Let V be the volume of the box. Then
$V = (18 - 2x)^2\, x = (324 + 4x^2 - 72x)\, x$
$= 4x^3 - 72x^2 + 324\, x$

$$\Rightarrow \quad \frac{dv}{dx} = 12x^2 - 144x + 324 \text{ and}$$

$$\frac{d^2v}{dx^2} = 24x - 144$$

For maximum or minimum v, $\dfrac{dv}{dx} = 0$

$$\therefore \quad \frac{dv}{dx} = 0 \Rightarrow 12x^2 - 144\,x + 324 = 0$$

$$\Rightarrow x^2 - 12x + 27 = 0$$

$$\Rightarrow \quad (x - 3)\,(x - 9) = 0 \Rightarrow x = 3, 9$$

But $x = 9$ is not possible, Therefore $x = 3$

Now, $\left(\dfrac{d^2v}{dx^2}\right)_{x=3} = 24 \times 3 - 144 < 0.$

Thus, v is maximum when $x = 3$.

Hence, the volume of box is maximum when the side of the square is 3 cm.

The maximum volume is $v = (18 - 6)^2 \times 3 = 144 \times 3 = 432$ cm^3

6. Show that the height of the closed cylinder of given surface and maximum volume is equal to the diameter of its base.

[PSB-99, 2002, 2003; HSB-93, 97; CBSE-92, 92C, 99, 2003]

Solution

Let r be the radius of the base and h be the height of a closed cylinder of given surface area S. Then

$$S = 2\pi r^2 + 2\pi rh \Rightarrow h = \frac{S - 2\pi r^2}{2\pi r} \qquad \text{.... (i)}$$

Let V be the volume of the cylinder. then

$$V = \pi r^2 h \Rightarrow V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right) \quad \text{[using (i)]}$$

$$\Rightarrow \quad V = \left(\frac{rS - 2\pi r^3}{2}\right)$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{S}{2} - 3\pi r^2 \qquad \text{.......... (ii)}$$

For maximum or minimum, we have $\frac{dV}{dr} = 0$

$$\Rightarrow \quad \frac{S}{2} - 3\pi r^2 = 0 \Rightarrow S = 6\pi r^2$$

$$\Rightarrow \quad 2\pi r^2 + 2\pi rh = 6\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 = 2\pi rh \Rightarrow h = 2r$$

Differentiating (ii) with respect to r, we obtain

$$\frac{d^2V}{dr^2} = -6\pi r < 0$$

Hence, V is maximum when $h = 2r$ i.e., when the height of the cylinder is equal to the diameter of the Base.

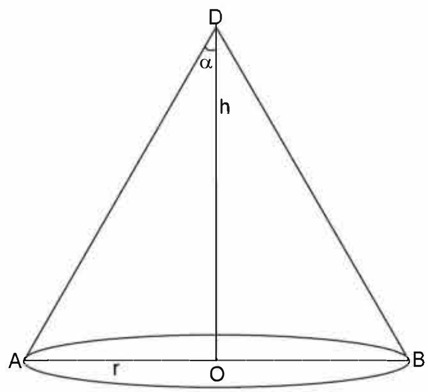
7. Show that the semi-vertical angle of a right circular cone of given surface area and maximum volume is $\sin^{-1}(1/3)$.

[CBSE-2004]

Solution

Let r be radius, l be the slant height and h be the height of the cone of given surface are S. Then

$$S = \pi r^2 + \pi rl \Rightarrow l = \frac{S - \pi r^2}{\pi r} \qquad \text{....... (i)}$$



Let V be the volume of the cone. Then

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^2 h^2$$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2)$$

$$\Rightarrow \quad V^2 = \frac{\pi^2}{9} r^4\left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right]$$

[using (i)]

$$\Rightarrow \quad V^2 = \frac{\pi^2 r^4}{9}\left[\frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}\right]$$

$$\Rightarrow \quad V^2 = \frac{1}{9}S(Sr^2 - 2\pi r^4)$$

Let $Z = V^2$. Then V is maximum or minimum according as Z is maximum or minimum.

$$\therefore \quad Z = \frac{1}{9}S(Sr^2 - 2\pi r^4)$$

$$\Rightarrow \quad \frac{dZ}{dr} = \frac{1}{9}\, S\,(2r\,S - 8\pi r^3) \qquad \text{......... (ii)}$$

$\Rightarrow$ For maximum or minimum $\frac{dZ}{dr} = 0$

$$\Rightarrow \quad \frac{1}{9}\,(S\,2r - 8\pi r^3)\,S = 0 \Rightarrow \quad 2Sr - 8\pi r^3 = 0$$

$$\Rightarrow \quad S = 4\pi r^2 \qquad \text{...(iii)}$$

Differentiating (ii) with respect to r, we get

$$\frac{d^2Z}{dr^2} = \frac{1}{9}\, S(2S - 24\pi r^2) = \frac{S}{9}\left[2S - 24\pi.\frac{S}{4\pi}\right]$$

$$\left[\because \text{Putting } r^2 = \frac{S}{4\pi} \text{ from (iii)}\right]$$

$$= \frac{-4S^2}{9} < 0$$

So, Z is maximum when $S = 4\pi r^2$.

Hence V is maximum when $S = 4\pi r^2$.

Now $S = 4\pi r^2 \Rightarrow \pi rl + \pi r^2 = 4\pi r^2$

$$\Rightarrow \quad \pi rl = 3\pi r^2 \Rightarrow l = 3r$$

$$\therefore \quad \sin\alpha = \frac{r}{l} = \frac{r}{3r} = \frac{1}{3}$$

Hence, V is maximum when $\alpha = \sin^{-1}(1/3)$

8. An open box with a square base is to be made out of a given quantity of card board of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

[CBSE-2001C; CBSE-2006]

Solution

Let the length, breadth and height of the box x, x and y units respectively. Then

$x^2 + 4xy = c^2$ .... (i) (Given condition)

Let V be the volume of the box. Then

$V = x^2y$ ..... (ii) [To be maximized]

$\Rightarrow \quad V = x^2\left(\dfrac{c^2 - x^2}{4x}\right)$ [using (i)]

$\Rightarrow \quad V = \dfrac{c^2}{4}x - \dfrac{x^3}{4}$

$\Rightarrow \quad \dfrac{dv}{dx} = \dfrac{c^2}{4} - \dfrac{3x^2}{4}$ and $\dfrac{d^2v}{dx^2} = \dfrac{-3x}{2}$

For maximum or minimum $\dfrac{dv}{dx} = 0$

$\Rightarrow \quad \dfrac{c^2}{4} - \dfrac{3x^2}{4} = 0 \Rightarrow x = \dfrac{c}{\sqrt{3}}$

Now $\left(\dfrac{d^2v}{dx^2}\right)_{x=\frac{c}{\sqrt{3}}} = \dfrac{-3\times c}{2\sqrt{3}} < 0$. Thus V is maximum when $x = \dfrac{c}{\sqrt{3}}$

Putting $x = \dfrac{c}{\sqrt{3}}$ in (i), we obtain $y = \dfrac{c}{2\sqrt{3}}$.

The maximum volume of the box is given by

$V = x^2y = \dfrac{c^2}{3} \times \dfrac{c}{2\sqrt{3}} = \dfrac{c^3}{6\sqrt{3}}$ cubic units.

9. A wire of length 36 cm is cut into two pieces. One of the pieces is turned in the form of a square and the other in the form of an equilateral triangle. Find the length of each piece so that the sum of the areas of the two be minimum. ***[CBSE-2005]***

Solution

Let the piece forming square be $4x$ cm long and the piece forming equilateral triangle be $3y$ cm long. Then length of each side of the square = x cm length of each side of the equilateral triangle = y cm

$\therefore \quad 4x + 3y = 36$ .... (i)

Let A = sum of their areas $= x^2 + \dfrac{\sqrt{3}}{4}y^2$

$= x^2 + \dfrac{\sqrt{3}}{4}\left(\dfrac{36 - 4x}{3}\right)^2 = x^2 + \dfrac{\sqrt{3}}{36}(36 - 4x)^2$

$\Rightarrow \quad \dfrac{dA}{dx} = 2x + \dfrac{\sqrt{3}}{36}.2(36 - 4x)(-4)$

$= 2x - \dfrac{2\sqrt{3}}{9}(36 - 4x)$

and $\dfrac{d^2A}{dx^2} = 2 - \dfrac{2\sqrt{3}}{9}(-4) = 2 + \dfrac{8}{9}\sqrt{3}$

For maximum or minimum $\dfrac{dA}{dx} = 0$

$\Rightarrow \quad 2x - \dfrac{2\sqrt{3}}{9}(36 - 4x) = 0$

$\Rightarrow \quad 18x + 8\sqrt{3}x - 72\sqrt{3} = 0$

$\Rightarrow \quad (9 + 4\sqrt{3})x = 36\sqrt{3}$

$\Rightarrow \quad x = \dfrac{36\sqrt{3}}{9 + 4\sqrt{3}}$

For this value of x, clearly $\dfrac{d^2A}{dx^2} > 0$

$\therefore$ Area A is minimum. Now piece forming the square $= 4x = \dfrac{144\sqrt{3}}{9 + 4\sqrt{3}}$ cm and piece forming the equilateral triangle = $3y$

$= 36 - 4x = 36 - \dfrac{144\sqrt{3}}{9 + 4\sqrt{3}} = \dfrac{324}{9 + 4\sqrt{3}}$ cm

10. An open box with a square base is to be made out of a given iron sheet of area 27 m^2. Show that the maximum volume of the box is 13.5 m^3. ***[CBSE-2005]***

Solution

Let the open box with a square have dimensions $a \times a \times b$ in metres. Surface area of the box $a^2 + 4ab = 27$ (Given)

$\Rightarrow \quad b = \dfrac{27 - a^2}{4a}$

Also V = volume of the box = $a^2 b$

$= a^2\left(\dfrac{27 - a^2}{4a}\right) = \dfrac{1}{4}(27a - a^3)$

$\Rightarrow \quad \dfrac{dv}{da} = \dfrac{1}{4}(27 - 3a^2)$

and $\dfrac{d^2v}{da^2} = \dfrac{1}{4}(-6a)$. For maximum or minimum $\dfrac{dv}{da} = 0$

$\Rightarrow \quad 27 - 3a^2 = 0 \Rightarrow a^2 = 9 \Rightarrow a = 3m \ (a \neq -3)$

For this value of a $\frac{d^2v}{da^2} = \frac{1}{4}(-6\times 3) < 0$

$\therefore$ v is maximum, when a = 3m from (i),

$$b = \frac{27-9}{12} = \frac{3}{2}\text{ m}$$

$\therefore$ Maximum volume

$$= (3)^2 \times \left(\frac{3}{2}\right) = \frac{27}{2} = 13.5\text{ m}^3$$

11. Given the sum of the perimeters of a circle and a square show that the sum of their areas is least when the diameter of the circle is equal to the side of the square.

[CBSE (foreign)-2005]

Solution

Let x units and y units be the side of the square and radius of circle respectively and p be the perimeter of square and circle. $P = 4x + 2\pi y$

$$\therefore \quad y = \frac{P-4x}{2\pi} \qquad ...(i)$$

Let A be their combined area $\therefore$ $A = x^2 + \pi y^2$

$$= x^2 + \pi\left(\frac{P-4x}{2\pi}\right)^2 = x^2 + \frac{\pi}{4\pi^2}(P-4x)^2$$

$$\therefore \quad \frac{dA}{dx} = 2x + \frac{1}{4\pi}.2(P-4x)(-4)$$

$$= 2x - \frac{2}{\pi}(P-4x)$$

A to be maximum or minimum $\frac{dA}{dx} = 0$

$$\therefore \quad 2x - \frac{2}{\pi}(P-4x) = 0$$

$$\Rightarrow \quad 2x = \frac{2}{\pi}(P-4x)$$

$$\Rightarrow \quad \pi x = P - 4x$$

$$\Rightarrow \quad (4+\pi)x = P \Rightarrow \quad x = \frac{P}{4+\pi}$$

and $\frac{d^2A}{dx^2} = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi} > 0$

$\therefore$ A is minimum when $x = \frac{P}{4+\pi}$

Put the value of x in (i) $y = \frac{x}{2}$ $\therefore$ $x = 2y$

$\therefore$ Combined area is minimum when side of the square equals to the diameter of the circle.

12. The sum of the surface area of a rectangular parallelopiped with sides x, $2x$ and $x/3$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum if x is equal to three times the radius of the sphere.

[CBSE-2002 C]

Solution

Let r be the radius of the sphere.

Then surface area of sphere is $4\pi r^2$.

Surface area of parallelopiped is

$$2\left(2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right).$$

Total surface area (which is given to be constant)

$$S = 4\pi r^2 + 6x^2$$

$$\Rightarrow \quad x^2 = \frac{S - 4\pi r^2}{6}$$

$$\Rightarrow \quad x = \left(\frac{S-4\pi r^2}{6}\right)^{1/2} \qquad ...(i)$$

Total volume of parallelopiped and shpere,

$$V = x \times 2x \times \frac{x}{3} + \frac{4}{3}\pi r^3 = \frac{2x^3}{3} + \frac{4}{3}\pi r^3$$

$$= \frac{2}{3\times 6\sqrt{6}}(S - 4\pi r^2)^{3/2} + \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \frac{dv}{dr} = \frac{1}{6\sqrt{6}}(S-4\pi r^2)^{1/2}(-8\pi r) + 4\pi r^2$$

For maxima or minima $\frac{dv}{dr} = 0$

$$\Rightarrow \quad \frac{4}{3\sqrt{6}}(S-4\pi r^2)^{1/2}\pi r = 4\pi r^2$$

$$\Rightarrow \quad \frac{1}{3}\left(\frac{S-4\pi r^2}{6}\right)^{1/2} = r \Rightarrow \frac{1}{3}x = r$$

[using (i)]

$$\Rightarrow \quad x = 3r$$

$$\text{Now, } \frac{d^2v}{dr^2} = \frac{-4\pi}{3\sqrt{6}}(S-4\pi r^2)^{1/2}$$

$$\frac{-2\pi r(-8\pi r)}{3\sqrt{6}(S-4\pi r^2)^{1/2}} + 8\pi r$$

$$\left(\frac{d^2v}{dr^2}\right)_{r=\frac{1}{3}x} = -\frac{4\pi}{3}x + \frac{16\pi^2.\frac{x^2}{9}}{18x} + 8\pi.\frac{x}{3}$$

$$= \left(\frac{-4\pi}{3} + \frac{16\pi^2}{162} + \frac{8\pi}{3} \right) x$$

$$= \left(\frac{4\pi}{3} + \frac{16\pi^2}{162} \right) x > 0$$

Hence, total volume is minimum when $x = 3r$.

13. A closed cylinder has volume 2156 cm^3. What will be the radius of its base so that its total surface area is minimum.

[CBSE-90, 2000 C]

Solution

Let r be the radius, h be the height, v be the volume and s be the surface area of the closed right circular cylinder.
Given $v = 2156$

$$\Rightarrow \quad \pi r^2 h = 2156 \quad \Rightarrow \quad h = \frac{2156}{\pi r^2} \qquad \text{........ (i)}$$

Now $s = 2\pi r^2 + 2\pi r h$

$$\Rightarrow \quad s(r) = 2\pi r^2 + 2\pi r \left(\frac{2156}{\pi r^2} \right)$$

$$\Rightarrow \quad s(r) = 2\pi r^2 + \frac{4312}{r}$$

$$\Rightarrow \quad s'(r) = 4\pi r - \frac{4312}{r^2}$$

$$\Rightarrow \quad s''(r) = 4\pi + \frac{4312 \times 2}{r^3}$$

For maxima or minima

$$s'(r) = 0 \Rightarrow \; 4\pi r - \frac{4312}{r^2} = 0$$

$$\Rightarrow \; 4\pi r = \frac{4312}{r^2} \;\Rightarrow r^3 = \frac{4312}{4\pi} \;\Rightarrow r^3 = \frac{1078}{\pi}$$

$$\Rightarrow \quad r^3 = \frac{1078 \times 7}{22} = 343 \;\Rightarrow\; r = 7$$

Now $s''(7) = 4\pi + \dfrac{8624}{(7)^3} > 0$

Hence surface area is minimum when $r = 7$.

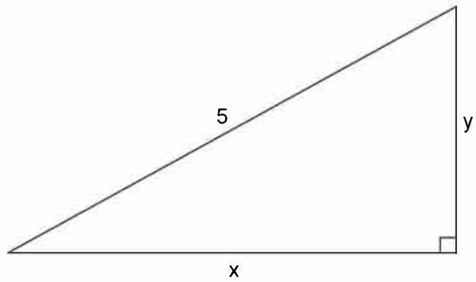
14. Find the largest possible area of a right angled triangle whose hypotenuse is 5 cm long?

[CBSE-2000]

Solution

Let x and y be the sides of the right angled triangle
Then $x^2 + y^2 = 25$

$$y = \sqrt{25 - x^2}$$



Now area

$$(A) \quad = \frac{1}{2} xy$$

$$\Rightarrow \quad A = \frac{1}{2} x \sqrt{25 - x^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2} \left[\sqrt{25 - x^2} - \frac{x^2}{\sqrt{25 - x^2}} \right]$$

For maxima or minima $\dfrac{dA}{dx} = 0$

$$\Rightarrow \quad \sqrt{25 - x^2} - \frac{x^2}{\sqrt{25 - x^2}} = 0$$

$$\Rightarrow \quad 25 - 2x^2 = 0 \Rightarrow x^2 = 25/2$$

$$\Rightarrow \quad x = 5/\sqrt{2}$$

Again $\dfrac{d^2A}{dx^2}$

$$= \frac{1}{2} \left[\frac{-x}{\sqrt{25 - x^2}} - \frac{2x}{\sqrt{25 - x^2}} - \frac{x^3}{(25 - x^2)^{3/2}} \right]$$

Clearly, for $x = \dfrac{5}{\sqrt{2}}, \dfrac{d^2A}{dx^2} < 0$

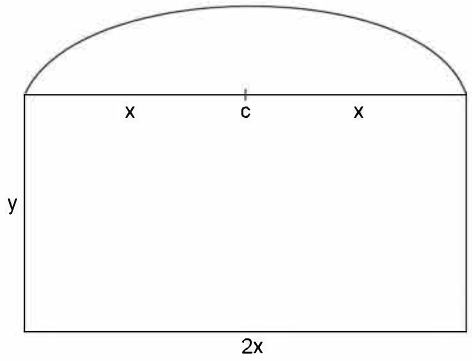
$\therefore$ Area (A) is maximum and its maximum value is $= \dfrac{1}{2} \times \dfrac{5}{\sqrt{2}} \sqrt{25 - \dfrac{25}{2}} = \dfrac{1}{2} \dfrac{5}{\sqrt{2}} \times \dfrac{5}{\sqrt{2}} = \dfrac{25}{4}$ cm^2

15. A window is in the form of a rectangle surmounted by a semicircular opening. If the perimeter of the window is 20m, find the dimensions of the window so that the maximum possible light is admitted through the whole opening.

[CBSE-2000]

Solution

Let the length of window be $2x$ and breadth by y.



Given perimeter = 20

$$2x + y + y + \frac{1}{2}(2\pi x) = 20$$

$\Rightarrow \quad 2x + 2y + \pi x = 20$

$\Rightarrow \quad (\pi + 2)\,x + 2y = 20$

$\Rightarrow \quad y = \dfrac{20 - (\pi + 2)x}{2}$ .... (i)

Now A = Area of the window

$$= 2x \times y + \frac{1}{2}\pi t^2$$

$$= 2x\left[\frac{20 - (\pi + 2)x}{2}\right] + \frac{1}{2}x^2\pi$$

$$= 20x - \pi x^2 - 2x^2 + \frac{1}{2}x^2\pi$$

$$= 20x - \left(\frac{\pi}{2} + 2\right)x^2$$

$\therefore \quad \dfrac{dA}{dx} = 20 - (\pi + 4)x$

For maxima or Minima

$$\frac{dA}{dx} = 0$$

$\Rightarrow \quad 20 - (\pi + 4)\,x = 0$

$\Rightarrow \quad x = \dfrac{20}{\pi + 4}$

Also $\dfrac{d^2A}{dx^2} = -(\pi + 4) < 0$

For maximum possible light, the dimensions of the window are $2x = \dfrac{40}{\pi + 4}$

and $y = \dfrac{20 - \dfrac{(\pi + 2).20}{\pi + 4}}{2}$

$$= \frac{20\pi + 80 - 20\pi - 40}{2(\pi + 4)}$$

$$= \frac{20}{\pi + 4}$$

Hence dimensions of the window are

$\dfrac{40}{\pi + 4}$ and $\dfrac{20}{\pi + 4}$

UNSOLVED SUBJECTIVE PROBLEMS (XII BOARD (C.B.S.E./STATE)):
TO GRASP THE TOPIC SOLVE THESE PROBLEMS

1. Find the point on the curve $x^2 = 4y$ which is nearest to the point (–4, 2)

   [CBSE-2007]

2. Find the volume of the largest cylinder that can be inscribed in a sphere of radius r cm.

   [CBSE-95C, 2001]

3. Find the shortest distance between the line $y - x = 1$ and the curve $x = y^2$.

4. Show that a cylinder of a given volume which is open at the top, has minimum total surface area, provided its height is equal to the radius of its base.

   [CBSE-87, 92C; HSB-96]

5. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\dfrac{8}{27}$ of the volume of the sphere.

   [CBSE (SP)-2006]

6. A straight line AB of length 8 cm is divide into two parts AP and PB by a point P. Show that P is a mid-point if $AP^2 + BP^2$ is minimum.

7. A rectangle is inscribed in a semicircle of radius r with one of its sides on diameter of semicircle. Find the dimensions of the rectangle so that its area is maximum. Find also the area.

[CBSE-88, 98; SP-2006]

8. Show tht the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is 500 π cm^3.

[CBSE-04]

9. A wire of length 25 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the square and the circle is minimum.

[CBSE-96C, 99]

10. A window in the form of a rectangle is surmounted by a semi-circular opening. The total perimeter of the window is 10m. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.

[CBSE-96C, 2000, 2002]

11. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}\,a$.

[CBSE-92, 2002]

12. Show that the semivertical angle of a cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.

[HSB-97, 2003; PSB-96, 98, 2003, CBSE-92]

13. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semivertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$

[CBSE-2001C, 2004; HB-2003; CET-98]

14. Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm. *[CBSE-2005]*

15. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $\frac{2a}{\sqrt{3}}$.

[CBSE-2001; Karnataka CET-2002]

ANSWERS

2. $\dfrac{4\pi r^3}{3\sqrt{3}}$

3. $3\sqrt{2}/8$

7. $\alpha = \pi/4$; r^2.

9. $\dfrac{100}{\pi+4}, \dfrac{25\pi}{\pi+4}$

10. length $= \dfrac{20}{\pi+4}$, breadth $= \dfrac{10}{\pi+4}$

SOLVED OBJECTIVE PROBLEMS: HELPING HAND

1. A population $p(t)$ of 1000 bacteria introduced into nutrient medium grows according to the relation $p(t) = 1000 + \dfrac{1000t}{100+t^2}$. The maximum size if this bacterial population is:

[Karnataka CET-2005]

(a) 1100 (b) 1250
(c) 1050 (d) 5250

Solution

(c) $p(t) = 1000 + \dfrac{1000t}{100+t^2}$

$$\frac{dp}{dt} = \frac{(100+t^2)1000 - 1000t.2t}{(100+t^2)^2}$$

$$= \frac{1000(100 - t^2)}{(100 + t^2)^2}$$

For extremum, $\frac{dp}{dt} = 0 \Rightarrow t = 10$

Now $\left.\frac{dp}{dt}\right|_{t<10} > 0$ and $\left.\frac{dp}{dt}\right|_{t>10} < 0$

$\therefore$ At $t = 10$, $\frac{dp}{dt}$ change from positive to negative.

$\therefore$ p is maximum at $t = 10$.

$\therefore$ $p_{max} = p(10)$

$$= 1000 + \frac{1000.10}{100 + 10^2} = 1050$$

2. A cone of maximum volume is inscribed in a given sphere, then ratio of the height of the cone to diameter of the sphere is

[MNR-1985; UPSEAT-2000; Orissa JEE-2004]

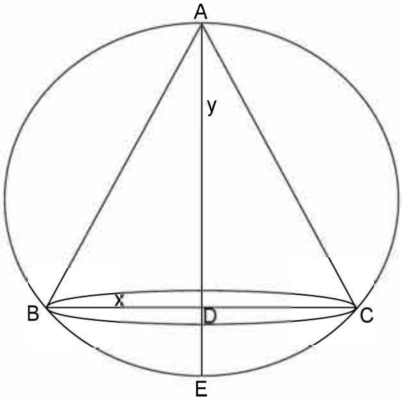
(a) 2/3 (b) 3/4
(c) 1/3 (d) 1/4

Solution

(a) Let diameter of sphere $AE = 2r$

Let radius of cone is x and height is y

$\therefore$ $AD = y$, since $BD^2 = AD\ .\ DE$ or $x^2 = y(2r - y)$ .... (i)



Volume of cone $V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi y(2r - y)y$

$$= \frac{1}{3}\pi(2ry^2 - y^3)$$

$$\Rightarrow \frac{dV}{dy} = \frac{1}{3}\pi(4ry - 3y^2) \Rightarrow \frac{dV}{dy} = 0$$

$$\Rightarrow \frac{1}{3}\pi(4ry - 3y^2) = 0$$

$$\Rightarrow y(4r - 3y) = 0$$

$$\Rightarrow y = \frac{4}{3}r, 0$$

Now $\frac{d^2y}{dy^2} = \frac{1}{3}\pi(4r - 6y)$,

put $y = \frac{4}{3}r$ $\frac{d^2y}{dy^2} = \frac{1}{3}\pi\left(4r - 6 \times \frac{4}{3}r\right) =$ negative value

So, volume of cone is maximum at $y = \frac{4}{3}r$

$$\Rightarrow \frac{\text{Height}}{\text{Diameter}} = \frac{y}{2r} = \frac{2}{3}$$

3. If PQ and PR are the two sides of a triangle, then the angle between them which gives maximum area of the triangle is

[Kerala (Engg.)-2002]

(a) π (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/2$

Solution

(d) Let $PQ = a$ and $PR = b$, then

$\Delta = \frac{1}{2}ab \sin\theta$

$\because$ $-1 \le \sin\theta \le 1$

Since, area is maximum when $\sin\theta = 1$

$$\Rightarrow \theta = \frac{\pi}{2}$$

4. If $P = (1, 1)$, $Q = (3, 2)$ and R is a point on x-axis then the value of $PR + RQ$ will be minimum at:

[AMU-2005]

(a) (5/3, 0) (b) (1/3, 0)
(c) (3, 0) (d) (1, 0)

Solution

(a) Let co-ordinate of R $(x, 0)$. Given $P(1, 1)$ and $Q(3, 2)$

$$PR + RQ = \sqrt{(x-1)^2 + (0-1)^2}$$

$$+ \sqrt{(x-3)^2 + (0-2)^2}$$

$$= \sqrt{x^2 - 2x + 2} + \sqrt{x^2 - 6x + 13}$$

For minimum value of

$$PR + RQ, \frac{d}{dx}(PR + PQ) = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left(\sqrt{x^2 - 2x + 2}\right) + \frac{d}{dx}\left(\sqrt{x^2 - 6x + 13}\right) = 0$$

$$\Rightarrow \quad \frac{(x-1)}{\sqrt{x^2 - 2x + 2}} = -\frac{(x-3)}{\sqrt{x^2 - 6x + 13}}$$

Squaring both sides,

$$\frac{(x-1)^2}{(x^2 - 2x + 2)} = \frac{(x-3)^2}{x^2 - 6x + 13}$$

$\Rightarrow \quad 3x^2 - 2x - 5 = 0$

$\Rightarrow \quad (3x - 5)(x + 1) = 0,$

$$x = \frac{5}{3}, -1$$

Also $1 < x < 3$.

$\therefore \; R = (5/3, 0)$.

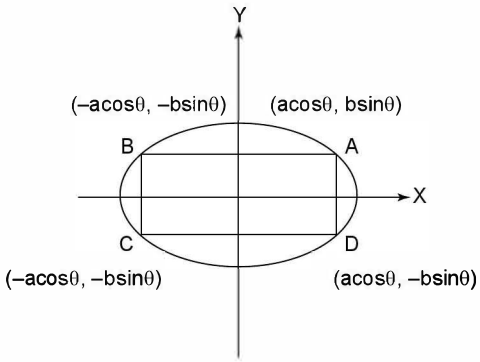
5. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is:

[AIEEE-2005]

(a) $\sqrt{ab}$ (b) a/b
(c) $2ab$ (d) ab

Solution

(c)



Area of rectangle $ABCD = (2a\cos\theta).(2b\sin\theta) = 2ab\sin 2\theta$

Hence, area of greatest rectangle is equal to $2ab$, when $\sin 2\theta = 1$.

6. The volume of a spherical balloon is increasing at the rate of 40 cubic centimetre per minute. The rate of change of the surface of the balloon at the instant when its radius is 8 centrimetre, is:

[Roorkee-1983]

(a) $\frac{5}{2}$ cm²/min (b) 5 cm²/min
(c) 10 cm²/min (d) 20 cm²/min

Solution

(c) Here $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$$\Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{40}{4\pi r^2} = \frac{5}{32\pi}$$

$$\therefore \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \times 8 \times \frac{5}{32\pi} = 10$$

7. The radius of the cylinder of maximum volume, which can be inscribed in a sphere of radius R is: ***[AMU-1999]***

(a) $\frac{2}{3}R$ (b) $\sqrt{\frac{2}{3}}R$

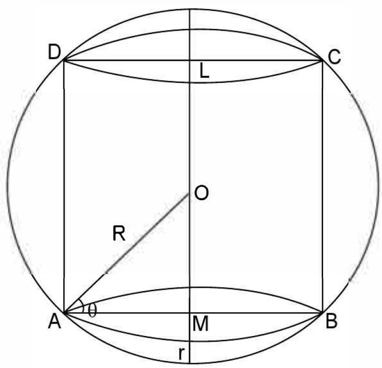
(c) $\frac{3}{4}R$ (d) $\sqrt{\frac{3}{4}}R$

Solution

(b) If r be the radius and h the height, then from the figure,

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow h^2 = 4(R^2 - r^2)$$

Now, $V = \pi r^2 h = 2\pi r^2\sqrt{R^2 - r^2}$



$$\therefore \quad \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + 2\pi r^2 . \frac{1}{2}\frac{(-2r)}{\sqrt{R^2 - r^2}}$$

For maximum of minimum, $\frac{dV}{dr} = 0$

$\Rightarrow \quad 4\pi r\sqrt{R^2 - r^2} = \dfrac{2\pi r^3}{\sqrt{R^2 - r^2}}$

$\Rightarrow \quad 2(R^2 - r^2) = r^2$

$\Rightarrow \quad 2R^2 = 3r^2$

$\Rightarrow \quad r = \sqrt{\dfrac{2}{3}}R \Rightarrow \dfrac{d^2V}{dr^2} = -\text{ve}$

Hence V is maximum, when $r = \sqrt{\dfrac{2}{3}}R$

8. Let $f(x) = \int_0^x \dfrac{\cos t}{t}dt, x > 0$ then $f(x)$ has:

[Kurukshetra CEE-2002]

(a) Maxima when $n = -2, -4, -6, \ldots\ldots\ldots\ldots$
(b) Maxima when $n = -1, -3, -5, \ldots\ldots\ldots\ldots$
(c) Minima when $n = 0, 2, 4, \ldots\ldots$
(d) Minima when $n = 1, 3, 5, \ldots\ldots$

Solution

(b, d) $f(x) = \int_0^x \dfrac{\cos t}{t}dt, x > 0$

$\Rightarrow \quad f'(x) = \dfrac{\cos x}{x}, x > 0$

$\Rightarrow \quad f'(x) = 0 \Rightarrow \dfrac{\cos x}{x} = 0$

$\Rightarrow \quad x = (2n+1)\dfrac{\pi}{2}$ for $n \in z$.

Now $f''(x) = \dfrac{-x\sin x - \cos x}{x^2}$

$\therefore \quad f''[(2n+1)\pi/2] = \dfrac{-2}{(2n+1)\pi}(-1)^n$

$= \dfrac{2(-1)^{n+1}}{(2n+1)\pi}$

Thus $f'(x) > 0$ $n = -2, -4, -6, \ldots\ldots\ldots\ldots$
$f''(x) < 0$ $n = 0, 2, 4, \ldots\ldots\ldots\ldots$
$f'(x) > 0$ $n = 1, 3, 5 \ldots\ldots\ldots$
$f'(x) < 0$ $n = -1, -3, -5 \ldots\ldots\ldots\ldots$

Thus $f(x)$ attain maximum for $n = -1, -3, -5, \ldots\ldots\ldots$ and minimum for $n = 1, 3, 5, \ldots\ldots\ldots$

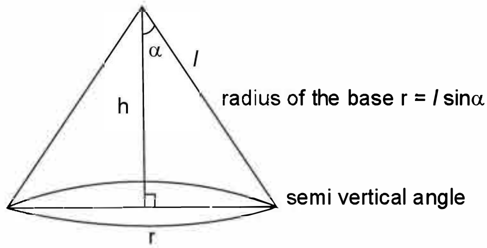
9. The semi vertical angle of a right circular cone of given slant height and maximum volume is:

[PET (Raj.)-1996]

(a) $\tan^{-1} 2$ (b) $\tan^{-1}\sqrt{2}$
(c) $\tan^{-1} 1/2$ (d) $\tan^{-1} 1/\sqrt{2}$

Solution

Step 1:
Let l be given slant height and α be semi-vertical angle of the cone. Then its height $h = l\cos\alpha$



Step 2: For maximum, $\dfrac{dv}{dx} = 0, \dfrac{d^2v}{dx^2}, co$

Volume $V = \dfrac{1}{3}\pi r^2 h = \dfrac{1}{3}\pi l^3 \sin^3\alpha\cos\alpha$

$\Rightarrow \quad \dfrac{dV}{d\alpha} = \dfrac{1}{3}\pi l^3\,[2\sin\alpha\cos^2\alpha - \sin^3\alpha]$

$\dfrac{d^2V}{d\alpha^2} = \dfrac{1}{3}\pi l^3\,[2\cos^3\alpha - 7\sin^2\alpha\cos\alpha]$

Now $\dfrac{dV}{d\alpha} = 0 \Rightarrow \sin\alpha = 0$ or $\tan\alpha = \sqrt{2}$

But $\sin\alpha \neq 0$ so $\alpha = \tan^{-1}\sqrt{2}$

Also then $\dfrac{d^2V}{d\alpha^2} < 0$, hence when

$\alpha = \tan^{-1}\sqrt{2}$, volume is maximum.

10. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ then difference between the greatest and least values of u^2 is:

[AIEEE-2004]

(a) $(a+b)^2$ (b) $2\sqrt{a^2+b^2}$
(c) $2(a^2+b^2)$ (d) $(a-b)^2$

Solution

(d) $u^2 = a^2 + b^2 + 2\sqrt{a^2\cos^2\theta + b^2\sin^2\theta} \times \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$

$= a^2 + b^2 + 2\sqrt{(a^4+b^4)\sin^2\theta\cos^2\theta + a^2b^2(\sin^4\theta + \cos^4\theta)}$

$= a^2 + b^2 + 2\sqrt{(a^4+b^4)\sin^2\theta\cos^2\theta + a^2b^2(1 - 2\sin^2\theta\cos^2\theta)}$

$= (a^2 + b^2) + \sqrt{4a^2b^2 + (a^2 - b^2)^2 \sin^2 2\theta}$

$\therefore$ max. $(u^2) = (a^2 + b^2) + (a^2 + b^2)$
$= 2(a^2 + b^2)$
(when $\sin^2 2\theta$ is maximum i.e., 1)
min. $(u^2) = (a^2 + b^2) + 2ab$ (when $\sin^2 2\theta$ is minimum i.e., 0)

Hence required difference $= a^2 + b^2 - 2ab = (a - b)^2$.

11. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. the maximum area enclosed by the park is

[AIEEE-2006]

(a) πx^2 (b) $\frac{3}{2}x^2$

(c) $\frac{1}{2}x^2$ (d) $\sqrt{\frac{x^3}{8}}$

Solution

(c) Let ABC be given park with two sides AB and AC equal to x, Let $\angle ACB = \theta$ If AD to perpendicular on BC, then $AD = x \text{ Sin } \theta$ and $DC = x \cos \theta$

$\therefore$ area of the park $\Delta = (AD)(DC)$

$x^2 \sin\theta \cos\theta = \frac{x^2}{2} \sin 2\theta$

It is maximum when $\sin 2\theta = 1$.

Hence maximum area enclosed $= \frac{1}{2}x^2$

12. A line is drawn through a fixed point (a, b), $(a > 0, b > 0)$ to meet the positive direction of the coordinate axes in P, Q respectively. The minimum value of $OP + OQ$ is:

(a) $\sqrt{a} + \sqrt{b}$ (b) $(\sqrt{a} + \sqrt{b})^2$

(c) $(\sqrt{a} + \sqrt{b})^3$ (d) None of these

Solution

(b) Let equation of the line be $\frac{x}{p} + \frac{y}{q} = 1$.

Since it passes through (a, b), so we have $\frac{a}{p} + \frac{b}{q} = 1$

$\Rightarrow \quad \frac{a}{p} = \frac{q-b}{q} \Rightarrow p = \frac{aq}{q-b}$

Now $S = OP + OQ = p + q = q + \frac{aq}{q-b}$

$\frac{dS}{dq} = 1 + \frac{a[(q-b) - q]}{(q-b)^2} = 1 - \frac{ab}{(q-b)^2}$

$\therefore \quad \frac{dS}{dq} = 0 \Rightarrow q - b = \sqrt{ab}$

$\Rightarrow \quad q = b + \sqrt{ab}, p = a + \sqrt{ab}$

Obviously $\frac{d^2S}{dq^2} > 0$, so min. $(p + q) = a + b + 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2$

13. The ratio between two sides of a rectangle of given area with minimum perimeter is.

(a) 2 : 1 (b) 3 : 2
(c) 1 : 1 (d) 4 : 3

Solution

(c) Let x, y be length and breadth of the rectangle. Then its area $= xy = \lambda$ (constant)

Perimeter $S = 2x + 2y = 2x + \frac{2\lambda}{x} = f(x)$

Now $\frac{ds}{dx} = 2 - \frac{2\lambda}{x^2} = 0 \Rightarrow x = \sqrt{\lambda}$ then

$y = \sqrt{\lambda}$

$\therefore \quad x : y = 1 : 1$

14. The lengths of the sides of the rectangle of greatest area drawn in the ellipse $x^2 + 2y^2 = 8$ are:

(a) $2, \sqrt{2}$ (b) $2\sqrt{2}, 2$
(c) $4, 2\sqrt{2}$ (d) None of these

Solution

(c) Let $(2\sqrt{2}\cos\theta, 2\sin\theta)$ be a point on the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$. If it is one vertex of the rectangle, then sides of the rectangle are: $4\sqrt{2}\cos\theta, 4\sin\theta$

$\therefore$ Area of the rectangle

$A = 16\sqrt{2}\sin\theta\cos\theta = 8\sqrt{2}\sin 2\theta$

$\Rightarrow \quad \frac{dA}{d\theta} = 4\sqrt{2}\cos 2\theta, \frac{d^2A}{d\theta^2} = -8\sqrt{2}\sin 2\theta$

Now $\frac{dA}{d\theta} = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow \theta = \pi/4$ where

$\frac{d^2A}{d\theta^2} < 0$

Hence sides of the rectangle of greatest area are $4, 2\sqrt{2}$.

15. A circle of radius r is drawn in an isosceles triangle. The least perimeter of such a triangle is:

[Hayana (CEE)-91]

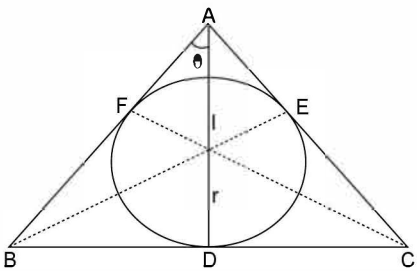
(a) $2\sqrt{3}r$ (b) $4\sqrt{3}r$

(c) $6\sqrt{3}r$ (d) $8\sqrt{3}r$

Solution

(c) Let ABC be the required isosceles triangle and $AB = AC$.

If I be its incentre, then AD will be the bisector of $\angle A$; where D is the mid point of BC. Let $\angle BAD = \theta$, then perimeter of triangle.



$s = AB + AC + BC = 2AB + 2BD$

$= 2(AB + BD) = 2(AF + BF + BD)$

$= 2(AF + 2BD)$

$= 2[r \cot\theta + 2(r \operatorname{cosec}\theta + r)\tan\theta]$

$= 2r(\cot\theta + 2\sec\theta + 2\tan\theta)$

Now, $\frac{ds}{d\theta} = 2r\ (-\operatorname{cosec}^2\theta + 2\sec\theta\tan\theta + 2\sec^2\theta)$

$\therefore\quad \frac{ds}{d\theta} = 0 \Rightarrow 2\sin^3\theta + 3\sin^2\theta = 1$

It is satisfied by $\sin\theta = 1/2 \Rightarrow \theta = \pi/6$

Hence $\Delta\, ABC$ is equilateral and its perimeter $= 6\sqrt{3}r$

16. The point (0, 5) is closest to the curve $x^2 = 2y$ at ***[MNR-1983]***

(a) $(2\sqrt{2}, 0)$ (b) (0, 0)

(c) (2, 2) (d) None of these

Solution

(d) Let a point on the curve by (h, k) Then $h^2 = 2k$ .... (i)

Distance $= D = \sqrt{h^2 + (k-5)^2}$

By (i); $D = \sqrt{2k + (k-5)^2}$

$$\frac{dD}{dk} = \frac{1}{2\sqrt{2k+(k-5)^2}} \times 2(k-5) + 2 = 0$$

$\Rightarrow\quad k = 4$

So, at $k = 4$ function D must be minimum

Then point will be $(\pm 2\sqrt{2}, 4)$

17. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots\ldots + a_n x^{2n}$ is a polynomial in x and $0 < a_0 < a_1 < \ldots\ldots < a_n$; then $p(x)$ has:

[Delhi (EEE)-98; IIT-86]

(a) only one maxima

(b) only one minima

(c) no maxima and no minima

(d) none of these

Solution

(b) $P'(x) = 2a_1x + 4a_2x^3 + \ldots\ldots + 2na_n x^{2n-1}$

$P''(x) = 2a_1 + 12a_2x^2 + \ldots\ldots\ldots + 2n(2n-1)a_n x^{2n-2}$

Now $P'(x) = 0 \Rightarrow 2x(a_1 + 2a_2x^2 + \ldots + n\,a_n x^{2n-2}) = 0$

$\Rightarrow\quad x = 0$

($\because$ other factor $\neq 0$ as $0 < a_1 < a_2 < \ldots\ldots\ldots < a_n$)

Also $P''(0) = 2a_1 > 0$, which shows that $x = 0$ is a minimum point of $P(x)$ and it is the only minimum point.

18. If $0 \leq c \leq 5$, then the minimum distance of the point $(0, c)$ from parabola $y = x^2$ is:

[IIT-82]

(a) $\sqrt{c-4}$ (b) $\sqrt{c-1/4}$

(c) $\sqrt{c+1/4}$ (d) None of these

Solution

(b) Let $(\sqrt{t}, t)$ be a point on the parabola whose distance from $(0, c)$ be d. Then $z = d^2 = t + (t-c)^2 = t^2 + t(1-2c) + c^2$

$\Rightarrow\quad \frac{dz}{dt} = 2t + 1 - 2c,\ \frac{d^2z}{dt^2} = 2 > 0$

Now $\frac{dz}{dt} = 0$

$\Rightarrow\quad t = c - 1/2$

Which gives the minimum distance, So minimum distance

$$= \sqrt{(c-1/2)+(-1/2)^2} = \sqrt{c-1/4}$$

19. A point is motion along a line and at time t its distance s is given by $x = t^4/4 - 2t^3 + 4t^2 - 7$. Its acceleration will be minimum when:

(a) $t = 1$ (b) $t = 2$
(c) $dt = 3$ (d) $t = 4$

Solution

(b) Acceleration $f = \dfrac{d^2s}{dt^2} = 3t^2 - 12t + 8$

$$\Rightarrow \quad \frac{df}{dt} = 6t - 12, \frac{d^2f}{dt^2} = 6$$

$$\frac{df}{dt} = 0 \Rightarrow t = 2.$$

20. The value of 'a' for which the function $(a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically throughout for all real x, are:

[Kurukshetra CEE-2002]

(a) $a < -2$ (b) $a > -2$
(c) $-3 < a < 0$ (d) $-\infty < a \le -3$

Solution

(d) If $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in$ R, then $f'(x) \le 0$ for all $x \in R$

$\Rightarrow$ $3(a + 2)x^2 - 6ax + 9a \le 0$ for all $x \in R$

$\Rightarrow$ $(a + 2)x^2 - 2ax + 3a \le 0$ for all $x \in R$

$\Rightarrow$ $a + 2 < 0$ and Discriminant ≤ 0

$\Rightarrow$ $a < -2, -8a^2 - 24a \le 0$

$\Rightarrow$ $a < -2$ and $a(a + 3) \ge 0$

$\Rightarrow$ $a < -2, a \le -3$ or $a \ge 0$

$\Rightarrow$ $a \le -3 \Rightarrow -\infty < a \le -3$.

21. Consider the following statements S and R

S: Both $\sin x$ and $\cos x$ are decreasing functions in $\left(\dfrac{\pi}{2}, \pi\right)$

R: If a differentiable function decreases in (a, b) then its derivative also decreases in (a, b).

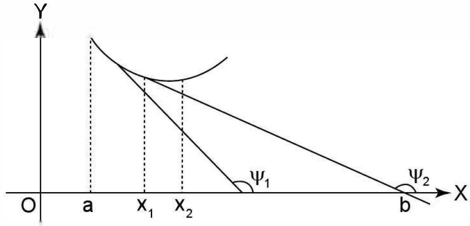
Which of the following is true

[IIT Screening-2000]

(a) Both S and R are wrong
(b) Both S and R are correct but R is not the correct explanation for S
(c) S is correct and R is the correct explanation for S
(d) S is correct and R is wrong

Solution

(d) From the trend of value of $\sin x$ and $\cos x$ we know $\sin x$ and $\cos x$ decrease in $\dfrac{\pi}{2} < x < \pi$. So, the statement S is correct.



The statement R is incorrect which is clear from graph.

Clearly $f(x)$ is differntiable in (a, b).

Also, $a < x_1 < x_2 < b$.

But $f'(x_1) = \tan\phi_1 < \tan\phi_2 = f'(x_2)$

22. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is:

[RPET-95; IIT Screening-2001]

(a) Increasing on $\left[-\dfrac{1}{2}, 1\right]$
(b) Decreasing on R
(c) Increasing on R
(d) Decreasing on $\left[-\dfrac{1}{2}, 1\right]$

Solution

(a) $f'(x) = e^{x(1-x)} + x.e^{x(1-x)}.(1-2x)$

$$= e^{x(1-x)}\{1 + x(1-2x)\}$$

$$= e^{x(1-x)}.(-2x^2 + x + 1)$$

Now by the sign-scheme for $-2x^2 + x + 1$

$$\begin{array}{ccccc} - & & + & & - \\ \hline & -1/2 & & 1 & \end{array}$$

$f'(x) \ge 0$ if $x \in \left[-\dfrac{1}{2}, 1\right]$, because $e^{x(1-x)}$ is always positive.

So, $f(x)$ is increasing on $\left[-\dfrac{1}{2}, 1\right]$.

23. Let $f(x) = x^3 + bx^2 + cx + d$, $0 < b^2 < c$. Then f:

[IIT Screening-2004]

(a) Is bounded
(b) Has a local maxima
(c) Has a local minima
(d) Is strictly increasing

Solution

(d) Given $f(x) = x^3 + bx^2 + cx + d$

$\therefore \quad f'(x) = 3x^2 + 2bx + c$

Now its discriminant $= 4(b^2 - 3c)$

$\Rightarrow \quad 4(b^2 - c) - 8c < 0$. as $b^2 < c$ and $c > 0$

Therefore, $f'(x) > 0$ for all $x \in R$,

Hence f is strictly increasing.

24. The function $f(x) = \tan^{-1}(\sin x + \cos x)$, $x > 0$ is always an increasing function on the interval

[Kerala (Engg.)-2005]

(a) $(0, \pi)$ (b) $(0, \pi/2)$
(c) $(0, \pi/4)$ (d) $(0, 3\pi/4)$

Solution

(c) $f(x) = y = \tan^{-1}\left(\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right)$

$\Rightarrow \quad \tan y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

$\Rightarrow \quad \sec^2 y \frac{dy}{dx} = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

$\therefore \quad \frac{dy}{dx} > 0 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) > 0.$

$\therefore \quad x \in \left(0, \frac{\pi}{4}\right).$

25. The function $f(x) = x(x + 3)\, e^{-(1/2)x}$ satisfies all the conditions of Rolle's theorem in $[-3, 0]$. The value of c is:

(a) 0 (b) -1
(c) -2 (d) -3

Solution

(c) To determine 'c' in Rolle's theorem, $f'(c) = 0$

Her $f'(x) = (x^2 + 3x)\, e^{-(1/2)x}$.

$\left(-\frac{1}{2}\right) + (2x + 3)\, e^{-(1/2)x}$

$= e^{-(1/2)x}\left\{-\frac{1}{2}(x^2 + 3x) + 2x + 3\right\}$

$= -\frac{1}{2}e^{-(x/2)}\{x^2 - x - 6\}$

$\therefore \quad f'(c) = 0 \Rightarrow c^2 - c - 6 = 0 \Rightarrow c = 3, -2$

But $c = 3 \notin [-3, 0]$.

26. Let $f(x)$ satisfy all the conditions of mean value theorem in $[0, 2]$. If $f(0) = 0$ and $|f'(x)| \le \frac{1}{2}$ for all x, in $[0, 2]$ then:

[Punjab CET-1988]

(a) $f(x) \le 2$
(b) $|f(x)| \le 1$
(c) $f(x) = 2x$
(d) $f(x) = 3$ for atleast one x in $[0, 2]$

Solution

(b) $\frac{f(2) - f(0)}{2 - 0} = f'(x)$

$\Rightarrow \quad \frac{f(2) - 0}{2} = f'(x)$

$\Rightarrow \quad \frac{df(x)}{dx} = \frac{f(2)}{2}$

$\Rightarrow \quad f(x) = \frac{f(2)}{2}x + c$

$\therefore \quad f(0) = 0 \Rightarrow c = 0;$

$\therefore \quad f(x) = \frac{f(2)}{2}x \qquad$ ..(i)

Given $|f'(x)| \le \frac{1}{2} \Rightarrow \left|\frac{f(2)}{2}\right| \le \frac{1}{2} \qquad$ ....(ii)

(i) $\Rightarrow |f(x)| = \left|\frac{f(2)}{2}x\right| = \left|\frac{f(2)}{2}\right||x| \le \frac{1}{2}|x|$

[from (ii)]

In $[0, 2]$, for maximum x $(x = 2)$

$$|f(x)| \le \frac{1}{2}.2 \Rightarrow |f(x)| \le 1$$

27. The function

$f(x) = \int_{-1}^{x} t(e^t - 1)(t - 1)(t - 2)^3(t - 3)^5\, dt$ has a local minimum at $x =$

[IIT-1999]

(a) 0 (b) 1
(c) 2 (d) 3

Solution

(b, d) $f(x)=\int_{-1}^{x} t(e^t-1)(t-1)(t-2)^3(t-3)^5\,dt$

$\therefore$ $f'(x) = x\,(e^x - 1)(x - 1)\,(x - 2)^3\,(x - 3)^5$

For local minima, slope i.e., $f'(x)$ should change sign from –ve to +ve.

$f'(x) = 0 \Rightarrow x = 0, 1, 2, 3,$

If $x = 0 - h$, where h is a very small number, then

$f'(x) = (-)\,(-)\,(-1)\,(-1)\,(-1) = -$ve

If $x = 0 + h$; $f'(x) = (+)\,(+)\,(-)\,(-1)\,(-1) = -$ve

Hence at $x = 0$ neither maxima nor minima.

If $x = 1 - h$; $f'(x) = (+)\,(+)(-)\,(-1)\,(-1) = -$ve

If $x = 1 + h$; $f'(x) = (+)\,(+)\,(+)\,(-1)\,(-1) = +$ve

Hence at $x = 1$ there is a local minima.

If $x = 2 - h$; $f'(x) = (+)\,(+1)\,(+)\,(-)\,(-) = +$ve

If $x = 2 + h$; $f'(x) = (+)\,(+)\,(+)\,(+)\,(-1) = -$ve

Hence at $x = 2$ there is a local maxima.

If $x = 3 - h$; $f'(x) = (+)\,(+)\,(+)\,(+)\,(-) = -$ve

If $x = 3 + h$; $f'(x) = (+)\,(+)\,(+)\,(+)(+) = +$ve

Hence at $x = 3$ there is a local minima.

28. The function $f(x)=\dfrac{\ln(\pi+x)}{\ln(e+x)}$ is:

[IIT-1995]

(a) Increasing on $[0, \infty)$

(b) Decreasing on $[0, \infty)$

(c) Decreasing on $[0, \pi/e)$ and increasing on $[\pi/e, \infty)$

(d) Increasing on $[0, \pi/e)$ and decreasing on $[\pi/e, \infty)$

Solution

(b) Let $f(x)=\dfrac{\ln(\pi+x)}{\ln(e+x)}$

$$\therefore \quad f'(x)=\frac{\ln(e+x)\times\dfrac{1}{\pi+x}-\ln(\pi+x)\dfrac{1}{e+x}}{\ln^2(e+x)}$$

$$=\frac{(e+x)\ln(e+x)-(\pi+x)\ln(\pi+x)}{\ln^2(e+x)\times(e+x)(\pi+x)}$$

$\Rightarrow$ $f'(x) < 0$ for all $x \geq 0$, $\{\because \pi > e\}$

Hence $f(x)$ is decreasing in $[0, \infty)$

29. The area of the tringle formed by coordinate axes and the tangent to curve $f(x) = x^2 + bx - b$ at point (1, 1) is 2 units if this triangle lies in first quardrant, then b is equal to

[IIT–Screening 2001]

(a) –1 (b) 3

(c) –3 (d) 1

Solution

(c) $f(x) = x^2 + \text{b}x - b$

$\Rightarrow$ $f'(x) = 2x + b$

$\Rightarrow$ $f'(1) = 2 + b =$ slope of the tengent at (1, 1)

$\therefore$ equation of the tengent at (1, 1) is $y - 1 = (2 + b)(x - 1)$

$\Rightarrow$ $(2 + b)\,x - y = 1 + b$

$$\Rightarrow \quad \frac{x}{\left(\dfrac{1+b}{2+b}\right)}+\frac{y}{-(1+b)}=1$$

Area of given triangle = 2

$$\Rightarrow \quad \left|\frac{1}{2}\left(\frac{1+b}{2+b}\right)\{-(1+b)\}\right|=2$$

$\Rightarrow$ $-(1 + b)^2 = 8 + 4b$ (taking '+' sign)

$\Rightarrow$ $b^2 + 6b + 9 = 0$

$\Rightarrow$ $b = -3$

30. If $p(x)$ be a polynomial of degree 3 satisfying $p(-1) = 10$, $p(1) = -6$.
If $p(x)$ has maxima at $x = -1$ and $p(x)$ has minima at $x=1$, then distance between the local maximum and local minimum of the curve is:

[IIT (Main)-2005]

(a) $4\sqrt{65}$ (b) $3\sqrt{65}$

(c) $2\sqrt{65}$ (d) $\sqrt{65}$

Solution

(d) Let $p(x) = ax^3 + bx^2 + cx + d$. As given

$p(-1) = 10 \Rightarrow -a + b - c + d = 10$ ...(1)

$p(1) = 6 \Rightarrow -a + b + c + d = -6$ ....(2)

Also $p(x)$ has maxima at

$x = -1 \Rightarrow p1(-1) = 0 \Rightarrow 3a - 2b + c = 0$

and $p(x)$ has minima at

$x = 1 \Rightarrow p''(1) = 0 \Rightarrow 6a + 2b = 0$

$\Rightarrow$ $3a + b = 0$ ....(4)

(1), (2), (3), (4) $\Rightarrow$ $a = 1, b = -3, c = -9, d = 5$

$\therefore$ $p(x) = x^3 - 3x^2 - 9x - 5 = 3(x + 1)(x - 3)$

$p''(x) = 6x - 6$

Now

$p'(x) = 0 \Rightarrow x = -1, 3$. Also $p''(-1) = -12 < 0$, $p''(3) = 12 > 0$

$\Rightarrow$ $x = -1$ is maxima and $x = 3$ is minima. Hence local maximum and minimum points are $(-1, 10)$ and $(3, -22)$

$\therefore$ required distance $= \sqrt{16+1024} = 4\sqrt{65}$

31. If $f(x) = \begin{cases} e^x & , \; 0 \le x \le 1 \\ 2 - e^{x-1} & , \; 1 < x \le 2 \\ x - e & , \; 2 < x \le 3 \end{cases}$

and $g(x) = \int_0^x f(t)dt, x \in [1, 3]$ Then $g(x)$ has

[IIT-JEE-2006]

(a) Local maxima at $x = 2$ and local minima at $x = 1$
(b) Local maxima at $x = 1 + \ln 2$ and local minima at $x = e$
(c) No local maxima
(d) No local minima

Solution

(b) $g'(x) = f(x) = \begin{cases} e^x & , \; 0 \le x \le 1 \\ 2 - e^{x-1} & , \; 1 < x \le 2 \\ x - e & , \; 2 < x \le 3 \end{cases}$

$\therefore$ $g'(x) = 0 \quad \Rightarrow x = e, 1 + \ln 2$ now

$g''(x) = \begin{cases} -e^{x-1} & , \; 1 < x \le 2 \\ 1 & , \; 2 < x \le 3 \end{cases}$

$\therefore$ $g''(1+ \ln 2) = -e^{ln\, 2} = -2 < 0 \Rightarrow x = 1 +$ in 2 is a local maxima. Also $g''(e) = 1 > 0$ $\Rightarrow e$ is a local manima.

Hence (b) is the correct answers.

32. Cosine of the angle of intersection of curves $y = 3^{x-1} \log x$ and $y = x^x - 1$ is:

[IIT-JEE-2006]

(a) 0 (b) 1
(c) 1/2 (d) 1/3

Solution

(b) Given curves meet at $x = 1$

For first curve $\dfrac{dy}{dx} = \dfrac{3^{x-1}}{x} + 3^{x-1} \log x = 1$ (at $x = 1$)

For second curve $\dfrac{dy}{dx} = x^x \, (1 + \log x) = 1$ (at $x = 1$)

If θ be the angle between these curves at $x = 1$, then $\tan\theta = 0 \Rightarrow \cos\theta = 1$

33. Curve $y = ax^3 + bx^2 + cx + 5$ touches x-axis at point $A(-4, 0)$ and intersects y-axis at point B where its slope is 3. Then

[IIT-84]

(a) $a = 1/2, b = 3/4, c = 3$
(b) $a = 1/2, b = -3/4, c = 3$
(c) $a = -1/2, b = -3/4, c = 3$
(d) $a = 1/2, b = 3/4, c = -3$

Solution

(c) $\dfrac{dy}{dx} = 3ax^2 + 2bx + c$

$\because$ Given curve touches x-axis at $(-2, 0)$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{(-2,0)} = 0$

$\Rightarrow$ $12a - 4b + c = 0$ .... (1)

Also $(-2, 0)$ lies on the given curve,

so $-8a + 4b - 2c + 5 = 0$ ......(2)

As given at $x = 0$, $\dfrac{dy}{dx} = 3$

$\Rightarrow$ $c = 3$ .... (3)

(1), (2), (3) $\Rightarrow a = -1/2, b = -3/4, c = 3$

34. The total number of local maxima and local minima of the function

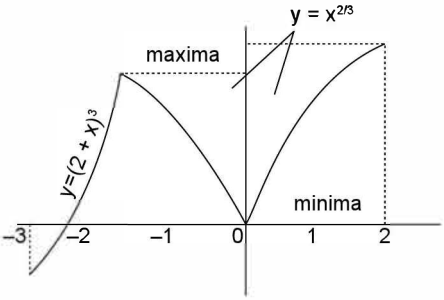
$f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is:

[IIT JEE-2008]

(a) 0 (b) 1
(c) 2 (d) 3

Solution

(c) $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$



One can easily graph the function. It is an example of a piecewise continuous function. The function has a local maxima at $x = -1$ and a local minima at $x = 0$. Thus the total number of local maxima and local minima of the function is 2.

35. Let the function $g: (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is: ***[IIT JEE-2008]***

(a) even and is strictly increasing in $(0, \infty)$
(b) odd and is strictly decreasing in $(-\infty, \infty)$
(c) odd and is strictly increasing in $(-\infty, \infty)$
(d) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

Solution

(c) $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$

$= 2\tan^{-1}(e^u) - (\tan^{-1}(e^u) + \cot^{-1}(e^u))$

$= \tan^{-1}(e^u) - \cot^{-1}(e^u)$

thus $g(-u) = \tan^{-1}(e^{-u}) - \cot^{-1}(e^{-u})$ which can, after transformation, be seen to be $g(-u) = g(u)$

Hence g is odd

Also $g'(u) > 0$, which means g is increasing.

36. Consider the function $f: (-\infty, \infty) \to (-\infty, \infty)$ defined by $f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2.$

Which of the following is true? ***[IIT JEE-2008]***

(a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x = 1$
(b) $f(x)$ is increasing on $(-1, 1)$ and has a local maximum at $x = 1$.
(c) $f(x)$ is increasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$
(d) $f(x)$ is decreasing on $(-1, 1)$ but has neither a local maximum nor a local minimum at $x = 1$.

Solution

(a) $f'(x) = 2a\dfrac{(x^2 - 1)}{(x^2 + ax + 1)^2}$

$= 2a\dfrac{(x-1)(x-1)}{(x^2 + ax + 1)^2}$

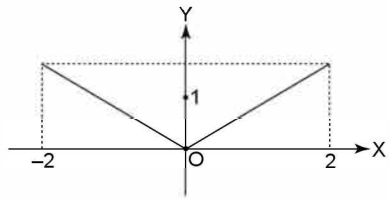
It is easily seen that $f(x)$ decreases on $(-1, 1)$, and has a local minimum at $x = 1$, because the derivative changes its sign from negative to positive.

37. Let $f(x) = \begin{cases} |x| & \text{if} \quad 0 |x| \le 2 \\ 1 & \text{if} \quad x = 0 \end{cases}$ thet at $x = 0, f$ has ***[IIT (Screening)-2000]***

(a) a local maximum
(b) not a local maximum
(c) a local minimum
(d) no extreme value

Solution

(a) $f(x) = \begin{cases} -x \ , & -2 \le x < 0 \\ 1 \ , & x = 0 \\ x \ , & 0 < x \le 2 \end{cases}$



Obviously $f(x)$ is discontinuous at $x = 0$. Also it can be easily observed from its graph that $f(0) = 1 > f(x)\ \forall\ x \in (-h + 0, 0 + h)$

$\therefore$ $f(x)$ is locally maximum at $x = 1$.

38. Let $f(x) = (x - a)(x - b)(x - c)$ be a real valued function where $a < b < c$ $(a, b, c \in R)$ such that $f''(\alpha) = 0$. Then if $\alpha \in (c_1, c_2)$, which one of the following is correct? ***[ICS-2006]***

(a) $a < c_1 < b$ and $b < c_2 < c$
(b) $a < c_1, c_2 < b$
(c) $b < c_1, c_2 < c$
(d) None of these

Solution

(a) $f(x)$ is continuous and differentiable. Also $f(x) = 0$ at $x = a, b, c$ (three points), so by Rolle's theorem $f'(x) = 0$ atleast at two points say c_1 and c_2 such that $a < c_1 < b$ and $b < c_2 < c$. Then $f''(x) = 0$ atleast at one point α lying between c_1 and c_2. Hence (a) is correct.

39. If $f(x)$ is twice differentiable and $f(1) = 1, f(2) = 4, f(3) = 9$, then ***[IIT (Screening)-2005]***

(a) $f''(x) = 2\ \forall\ x \in (1, 3)$
(b) $f''(x) = 2$ for atleast one $x \in (1, 3)$
(c) $f''(x) = 3\forall\ x \in (1, 3)$
(d) At any $x \in (1, 3), f''(x) = f'(x) = 5$

Solution

(b) Let $g(x) = f(x) - x^2$. Then $g(x)$ is also twice differentiable and so continuous also. Further $g(1) = f(1) - 1 = 0$, $g(2) = f(2) - 4 = 0$, $g(3) = f(3) - 9 = 0$

Hence $g(x)$ satisfies Rolle's theorem conditions in [1, 2] and [2, 3]. Consequently there exist atleast one $a \in (1, 2)$ and one $b \in (2, 3)$ such that $g'(a) = 0$ and $g'(b) = 0$

Further $g'(x)$ satisfies Rolle's theorem in $[a, b]$, so there exists atleast one $x \in (a, b)$ i.e., $x \in (1, 3)$ such that $g''(x) = 0$

$\Rightarrow \quad f''(x) - 2 = 0 \Rightarrow f''(x) = 2$

40. If $f(x)$ is twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, where $a < b < c < d < e$; then minimum number of zeroes of $g(x) = (f'(x))^2 + f''(x) f(x)$ in the interval $[a, e]$ is:

[IIT-JEE-2006]

(a) 4 (b) 5
(c) 6 (d) 7

Solution

(c) Obviously $f(x)$ is continuous and $f(a) = 0 = f(e), f(b) > 0, f(c) < 0, f(d) > 0$, so $f(x)$ meets x-axis atleast at 4 points in $[a, e]$

Now $g(x) = \frac{d}{dx}(f(x).f'(x))$

Let $h(x) = f(x)f'(x)$, then $g(x) = h'(x)$

$\because$ $f(x)$ is zero at least four places in $[a, e]$

$\Rightarrow$ $f'(x)$ is zero at atleast three places in $[a, e]$ [by Rolle's theorem]

$\Rightarrow$ $h(x)$ is zero at atleast seven places in $[a, e]$

$\Rightarrow$ $h'(x)$ is zero at atleast six places in $[a, e]$

$\therefore$ $g(x)$ has minimum six zeroes in $[a, e]$

41. What is the maximum area of the rectangle whose sides pass through the angular points of a given rectangle of sides a and b

[UPSC-2007]

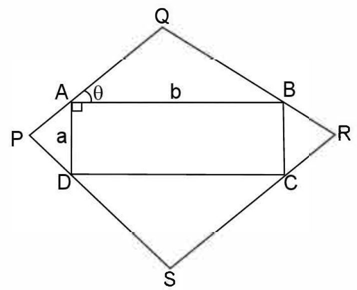
(a) $(a + b)^2/2$ (b) $(a + b)^2$
(c) $(a^2 + b^2)/2$ (d) $(a^2 + b^2)$

Solution

(a) Let $ABCD$ be the given rectangle with sides a and b. Also let $PQRS$ be the rectangle passing through the angular points of the given rectangle.

$PQ = a\cos(90° - \theta) + b\cos\theta$
$= a\sin\theta + b\cos\theta$

$PS = a\cos\theta + b\cos(90° - \theta)$
$= a\cos\theta + b\sin\theta$



$PQ \,.\, PS = (a\sin\theta + b\cos\theta)(a\cos\theta + b\sin\theta)$

$= a^2\sin\theta\cos\theta + ab\sin^2\theta + ab\cos^2\theta + b^2\sin\theta\cos\theta$

$= ab + (a^2 + b^2)\sin\theta\cos\theta = ab + (a^2 + b^2)\frac{\sin 2\theta}{2}$

Maximum area is obtained when $\sin 2\theta = 1$, and then the area is $ab + \frac{(a^2 + b^2)}{2}$

$= \frac{a^2 + b^2 + 2ab}{2} = \frac{(a+b)^2}{2}$

OBJECTIVE PROBLEMS: IMPORTANT QUESTIONS WITH SOLUTIONS

1. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is: ***MPPET-2010]***

(a) $\frac{a}{b}$ (b) $\sqrt{ab}$
(c) ab (d) $2ab$

2. If axis of the parabola $y = f(x)$ is parallel to y-axis and it touches the line $y = x$ at (1, 1), then:

[MPPET-2010]

(a) $f'(0) - 2f(0) + 1 = 0$ (b) $f'(0) + 2f(0) = 1$
(c) $2f'(0) + f(0) = 1$ (d) $2f'(0) - f(0') + 1 = 0$

3. Let $f: R \rightarrow R$ be defined by
$f(x)=\begin{cases} k-2x, & \text{if } x \le -1 \\ 2x+3, & \text{if } x > -1 \end{cases}$. If f has a local minimum at $x = -1$, then a possible value of k is: ***[AIEEE-2010]***

(a) 0 (b) $-\frac{1}{2}$
(c) -1 (d) 1

4. The volume of the greatest cylinder which can be inscribed in a cone of height 30 cm and the semi-vertical angle 30°. Is

(a) $\frac{4}{3}\pi\ (10)^3$ cubic cm
(b) $4\pi(10)^3$ cubic cm
(c) $\frac{4}{3}\pi\ (15)^3$ cubic cm
(d) None of these

5. The height of a cylinder if maximum volume inscribed in a sphere of radius 'a' is

(a) $a/\sqrt{3}$ (b) $2a/\sqrt{3}$
(c) $\sqrt{3}a$ (d) $2\sqrt{3}a$

6. A wire 34 cm long is to be bent in the form of a quadrilateral of which each angle is 90°. What is the maximum area which can be enclosed inside the quadrilateral? ***[NDA-2007]***

(a) 68 cm^2 (b) 70 cm^2
(c) 71.25 cm^2 (d) 72.25 cm^2

7. The number of values of x where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum is ***[IIT-1998; DCE-2001, 2005]***

(a) 0 (b) 1
(c) 3 (d) infinite

8. If from a wire of length 36 metre a rectangle of greatest area is made, then its two adjacent sides in metre are ***[MPPET-1998]***

(a) 6, 12 (b) 9, 9
(c) 10, 8 (d) 13, 5

9. The maximum area of a rectangle of perimeter 176 cms is ***[PET Raj.-90, 98]***

(a) 1936 cm^2 (b) 1854 cm^2
(c) 2110 cm^2 (d) None of these

10. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that min $f(x) >$ max $g(x)$, then the relation between b and c is ***[IIT JEE-2003]***

(a) no real vlaue of b and c
(b) $0 < c < b\sqrt{2}$
(c) $|c| < |b|\sqrt{2}$
(d) $|c| > |b|\sqrt{2}$

11. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is ***[IIT Screening-2001]***

(a) [0, 1] (b) $\left(0, \frac{1}{2}\right]$
(c) $\left[\frac{1}{2}, 1\right]$ (d) (0, 1]

12. The minimum value of
$|x| + \left|x+\frac{1}{2}\right| + |x-3| + \left|x-\frac{5}{2}\right|$ is

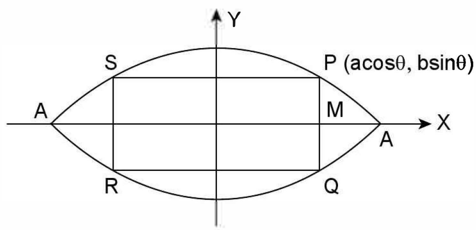
(a) 0 (b) 2
(c) 4 (d) 6

13. The greatest distance from the point $(\sqrt{7}, 0)$ to the curve $9x^2 + 16y^2 = 144$ is ***[Kerala PET-2008]***

(a) 4 (b) $\sqrt{7}$
(c) $2+\sqrt{7}$ (d) $4+\sqrt{7}$

SOLUTIONS

1. (d) Step 1:
Clearly are a of the rectangle = $PQ \times RQ$
$= (2b \sin\theta)(2a\cos\theta)$
$A = 2ab \sin 2\theta$
Step 2: For maximum area:
$\frac{dA}{d\theta} = 0$ and $\frac{d^2A}{d\theta^2} < 0$



i.e. $2ab \times 2\cos 2\theta = 0$

$\Rightarrow \quad \theta = 45°$

$\therefore \quad$ maximum area = $2ab$

2. (b) Let equation of parabola be $f(x) = ax^2 + bx + c$

Line $y = x$ touches curve at (1, 1); $f(1) = 1$

$\therefore \quad 1 = a + b + c$ .....(i)

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \text{slope of tangent} = 1$$

$f'(1) = 1 \Rightarrow 2a + b = 1$ ......(ii)

Also $f(0) = c$; $f'(0) = b$

Now 2(i) - (ii) given

$b + 2c = 1$

$f'(0) + 2f(0) = 1$

3. (b) $f(x) = \begin{cases} k - 2x; & \text{if } x \le -1 \\ 2x + 3; & \text{if } x > -1 \end{cases}$

if $x = -1$ is a local minima then

$f(-1 - h) \ge f(-1) \le f(-1 + h)$

$\therefore$ Consider $f(-1) \le f(-1 + h)$

$k + 2 \le 2(-1) + 3$

$k \le -1$

$\therefore$ (b) From options $k = -1$ is only possible

4. (a) Step 1:

Given height of the cone $OC = 30$ cm, $\alpha = 30$ cm

Using formula, maximum volume

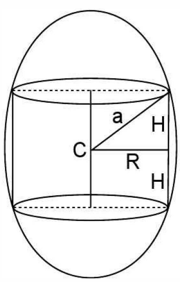
$$= V = \frac{4\pi h^3 \tan^2 \alpha}{27}$$

Step 2: Volume

$$= \frac{4}{27}\pi \times (30)^3 \times \frac{1}{3} = \frac{4\pi}{3}\left(\frac{30}{3}\right)^3$$

$$= \frac{4\pi}{3}(10)^3 \text{ cubic cm}$$

5. (b) Step 1: Let R be the radius and 2A be the height then from the figure.



clearly from the figure $R^2 + H^2 = a^2$ .... (i)

2H = ?? or

Step 2: Now for maximum volume

$V = \pi(\alpha^2 - H^2)(2H)$ .... (ii)

we have $\frac{dV}{dH} = 0$ and $\frac{d^2V}{dH^2} < 0$

Solving (ii) we find

$$\frac{dV}{dH} = 2\pi r^2 - 2\pi(3H^2) = 0$$

$\alpha^2 = 3H^2 \Rightarrow a = H\sqrt{3}$ .... (iii)

from (i) and (iii) we get $H = \frac{\alpha}{\sqrt{3}}$

or $2H = \frac{2a}{\sqrt{3}}$ = height of the cylinder of maximum volume.

6. (d) Step 1: We know that maximum area which can be inclosed as the quadrilateral must be square.

Therefore $4 \times a = 34$ cm.

or $a = \frac{17}{2}$cm = each side of the square

Step 2: Maximum Area

$$= (\text{side})^2 = \frac{289}{4} = 72.25\ (\text{cm})^2$$

7. (b) Step 1: For maximum value

$f'(x) = 0, f''(x) < 0,$

$f'(x) = -\sin x - \sqrt{2}\sin\sqrt{2}x = 0$

$\Rightarrow$ Hence $x = 0$ is only value as $\sqrt{2}$ is a irrational number

Step 2:

$$\left.f''(x)\right|_{x=0} = -\cos x - 2\cos\left.\sqrt{2}x\right|_{x=0} = -3 < 0$$

8. (b) Step 1: Perimeter of rectangle = $2(x + y)$ = Sum of four sides = 36

$\therefore \quad x + y = 18, y = 18 - x$

Step 2: For maximum area of $f(x)$

$f(x)$, $x = a$ is a point of maxima if $f'(a) = 0$ and $f''(a) < 0$

$\frac{dA}{dx} = 0, \frac{d^2A}{dx^2} < 0$, where

$A = xy = x(18 - x)$

$\therefore \dfrac{dA}{dx} = 18 - 2x = 0 \Rightarrow x = 9, y = 9$

9. (a) The Area of the rectangle is maximum when it is square.

Also given = $4a = 176$

$a = 44$

Step 2: Maximum area

$= (44)^2 = 44 \times 44 = 1936 \text{ cm}^2$.

10. (d) Step 1: The value $\dfrac{4ac - b^2}{4a}$ of the quadratic $ax^2 + bx + c$ is maximum or minimum according as a is negative or positive respectively.

Given, for the quadratic

$f(x) = x^2 + 2bx + 2c^2, a = 1 > 0$

and quadratic

$g(x) = -x^2 - 2cx + b^2, a = -1 < 0$

Step 2: min.

$$f(x) = \frac{4 \times 1 \times 2c^2 - 4b^2}{4 \times 1} = 2c^2 - b^2$$

Also max.

$$g(x) = \frac{4 \times (-1)(b^2) - 4c^2}{4 \times (-1)} = b^2 + c^2$$

Step 3: Given:

$\min f(x) > \max g(x)$

$2c^2 - b^2 > b^2 + c^2$

$c^2 > 2b^2$ or $|c| > |b|\sqrt{2}$

11. (d) Step 1: Clearly co-efficient of $x^2 = 1 + b^2 > 0$

Hence minimum value of the given quadratic $f(x)$

$$= \frac{4(1 + b^2)(1) - (2b)^2}{4(1 + b^2)} = \frac{1}{1 + b^2} > 0$$

Step 2: Clearly $1 + b^2 \geq 1$

$\therefore \quad \dfrac{1}{1 + b^2} \leq 1$

$\therefore$ The range of $m(b) = (0, 1]$

12. (d) Step 1:

Let $|x| + |x + \frac{1}{2}| + |x - 3| + |x - \frac{5}{2}|$

Since absolute value of a function is never negative therefore minimum value of the given expression can be obtained on equating each of $|x|$, $\left|x + \frac{1}{2}\right|$, $|x - 3|$ and $\left|x - \frac{5}{2}\right|$ to zero separately and comparing the resulting values, thus obtained as follows.

Step 2: If (i) $|x| = 0$ then $x = 0$ and

$|x| + |x + \frac{1}{2}| + |x - 3| + |x - \frac{5}{2}| = 6$

(ii) If $|x + \frac{1}{2}| = 0$ then $x = -\frac{1}{2}$ and

$|x| + |x + \frac{1}{2}| + |x - 3| + |x - \frac{5}{2}| = 7$

(iii) If $|x - 3| = 0$ then $x = 3$ and

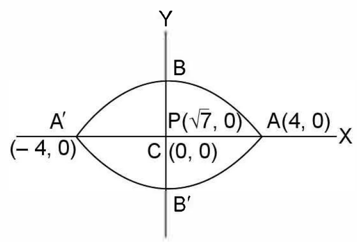
$|x| + |x + \frac{1}{2}| + |x - 3| + |x - \frac{5}{2}| = 7$

(iv) If $|x - \frac{5}{2}| = 0$ then $x = \frac{5}{2}$ and

$|x| + |x + \frac{1}{2}| + |x - 3| + |x - \frac{5}{2}| = 6$

$\therefore$ minimum value of $f(x) = 6$.

13. (d) Step 1: Given curve is $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$



Clearly from figure the greatest distance of the point $P(\sqrt{7}, 0)$ on x-axis from the curve is $PA' = CA' + CP = 4 + \sqrt{7}$

UNSOLVED OBJECTIVE PROBLEMS (IDENTICAL PROBLEMS FOR PRACTICE): FOR IMPROVING SPEED WITH ACCURACY

1. Find the dimensions of the rectangle of area 96 cm whose perimeter is the least, find also its perimeter

(a) $6\sqrt{6}$ (b) $16\sqrt{6}$

(c) $\sqrt{6}$ (d) 16

2. The radius of the cylinder of maximum volume which can be inscribed in a sphere of radius R is ***[AMU-1999]***

(a) $\frac{2}{3}R$ (b) $\sqrt{\frac{2}{3}}R$

(c) $\frac{3}{4}R$ (d) $\sqrt{\frac{3}{4}}R$

3. x^x has a stationary point at ***[Karnatka CET-1993]***

(a) $x = e$ (b) $x = 1/e$

(c) $x = 1$ (d) $x = \sqrt{e}$

4. The adjacent sides of a rectangle with given perimeter as 100 cm and enclosing maximum area are ***[MPPET-1993]***

(a) 10 cm and 40 cm (b) 20 cm and 30 cm

(c) 25 cm and 25 cm (d) 15 cm and 35 cm

5. If two sides of a triangle be given, then the area of the triangle will be maximum if the angle between the given sides be

(a) $\pi/3$ (b) $\pi/4$

(c) $\pi/6$ (d) $\pi/2$

6. The point on the curve $y^2 = 4x$ which is at minimum distance from the point (2, 1) will be ***[KCET-2000]***

(a) $(1, -2)$ (b) $(-2, 1)$

(c) $(1, 2\sqrt{2})$ (d) $(1, 2)$

7. The function $f(x) = x + \sin x$ has ***[AMU-2000]***

(a) A minimum but not maximum

(b) A maximum but no minimum

(c) Niether maximum nor minimum

(d) Both maximum and minimum

8. The maximum value of $f(x) = \dfrac{x}{4 + x + x^2}$ on $[-1, 1]$ is ***[MPPET-2000]***

(a) $-1/4$ (b) $-1/3$

(c) $1/6$ (d) $1/5$

9. The minimum value of exp $(2 + \sqrt{3}\cos x + \sin x)$ is ***[AMU-1999]***

(a) exp (2) (b) exp $(2 - \sqrt{3})$

(c) exp (4) (d) 1

WORKSHEET: TO CHECK THE PREPARATION LEVEL

Important Instructions

1. The answer sheet is immediately below the worksheet.
2. The test is of 15 minutes.
3. The worksheet consists of 15 questions. The maximum marks are 45.
4. Use Blue/Black Ball point pen only for writing particulars/marking responses. Use of pencil is strictly prohibited.

1. A fucntion $f(x)$ has a minimum value at $x = c$ if $f'(c) = 0$ and
(a) $f'(x)$ changes sign from +ve to negative as x passes through c
(b) $f'(x)$ changes sign from negative to positive as x passes through c
(c) $f'(x)$ does not change sign as x-passes through $x = c$
(d) None of these

2. The maximum value of xy when $x + 2y = 8$ is:
(a) 20 (b) 16
(c) 24 (d) 8

3. The function $f(x) = x^2 (x - 3)^2$
(a) increases on $(0, 3/2) \cup (3, \infty)$
(b) decreases on $(-\infty, 0) \cup (3/2, 3)$
(c) has minimum value 0
(d) has maximum value at $x = 3/2$

4. Let $f(x) = \begin{cases} x+2, & -1 \le x < 0 \\ 1, & x = 0 \\ x/2, & 0 < x \le 1 \end{cases}$. Then on $[-1, 1]$, this function has:
(a) a minimum
(b) a maximum
(c) neither a maximum nor a minimum
(d) $f'(0)$ does not exist.

5. $f(x) = \sin^6 x + \cos^6 x$ attains:
(a) a maximum value of 1
(b) a minimum value of 1
(c) a maximum value of 7/4
(d) a minimum value of 1/4

6. Let $f(x) = (x - 1)^2 (x - 2)^3 e^x$. Then $f(x)$ has:
(a) local maximum at $x = 1$
(b) point of inflexion at $x = 1$
(c) local minimum at $x = 2$
(d) point of inflexion at $x = 2$

7. A stone thrown vertically upwards satisfyies the equation $S = 80t - 16t^2$. The time required to reach the maximum height is:
[Kerala PET-2001]
(a) 2 (b) 4
(c) 3 (d) 3.5

8. The function $f(x) = 2x^3 - 15x^2 + 36x + 4$ is maximum at:
(a) $x = 3$ (b) $x = 0$
(c) $x = 4$ (d) $x = 2$

9. The value of k in order that $f(x) = \sin x - \cos x - kx + b$ decreases for all real values is given by:
[Him. CET-2002]
(a) $k < 1$ (b) $k \ge 1$
(c) $k \ge \sqrt{2}$ (d) $k < \sqrt{2}$

10. The minimum value of $x \log_e x$ is equal to:
[Kerala PET-2001]
(a) e (b) $1/e$
(c) $-1/e$ (d) $2/e$

11. If $x - 2y = 4$ then least value of xy is
[UPSEAT-2003]
(a) 2 (b) –2
(c) 0 (d) –3

12. Which point on the line $3x - 4y = 25$ is nearest to the origin?
[NDA-2004; VIT-2004]
(a) (–1, –7) (b) (3, –4)
(c) (–5, –8) (d) (3, 4)

13. What is the area of the largest rectangular field which can be enclosed with 200 metre of fencing?
[NDA-2008]
(a) 1600 m^2 (b) 2100 m^2
(c) 2400 m^2 (d) 2500m^2

14. What is the maximum value of $x \cdot y$ subject to the condition $x + y = 8$?
[NDA-2008]
(a) 8 (b) 16
(c) 24 (d) 32

15. The perimeter of a sector is p. The area of the sector is maximum when its radius is: ***[Karnataka-CET-2002]***

(a) $1/\sqrt{p}$ (b) $p/2$
(c) $p/4$ (d) $\sqrt{p}$

ANSWER SHEET

1. ⓐ ⓑ ⓒ ⓓ	6. ⓐ ⓑ ⓒ ⓓ	11. ⓐ ⓑ ⓒ ⓓ
2. ⓐ ⓑ ⓒ ⓓ	7. ⓐ ⓑ ⓒ ⓓ	12. ⓐ ⓑ ⓒ ⓓ
3. ⓐ ⓑ ⓒ ⓓ	8. ⓐ ⓑ ⓒ ⓓ	13. ⓐ ⓑ ⓒ ⓓ
4. ⓐ ⓑ ⓒ ⓓ	9. ⓐ ⓑ ⓒ ⓓ	14. ⓐ ⓑ ⓒ ⓓ
5. ⓐ ⓑ ⓒ ⓓ	10. ⓐ ⓑ ⓒ ⓓ	15. ⓐ ⓑ ⓒ ⓓ

HINTS AND EXPLANATIONS

2. (b) Let equation of parabola be
$f(x) = ax^2 + bx + c$

Line $y = x$ touches curve at (1, 1); $f(1) = 1$

$\therefore \quad 1 = a + b + c \qquad$ (i)

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \text{slope of tangent} = 1$$

$f'(1) = 1 \Rightarrow 2a + b = 1 \qquad$ (ii)

Also $f(0) = c;\ f'(0) = b$

Now, $2(i) - (ii)$ given

$b + 2c = 1$

$f'(0) + 2f(0) = 1$

3. $f(x) = \begin{cases} k - 2x\,; & \text{if } x \le -1 \\ 2x + 3\,; & \text{if } x > -1 \end{cases}$

If $x = -1$ is a local minima

then $f(-1-h) \ge f(-1) \le f(-1+h)$

$\therefore$ consider $f(-1) \le f(-1+h)$

$k + 2 \le 2(-1) + 3$

$k \le -1$

$\therefore$ from options $k = -1$ is only possible.

14. (b) Step 1: $x(8 - x);\ x + y = 8$

and for maximum P $\dfrac{dp}{dx} = 0, \dfrac{d^2p}{dx^2} < 0$

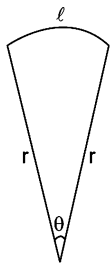
Step 2: $\dfrac{dp}{dx} = 8 - 2x = 0 \Rightarrow x = 4$

Hence $y = 4$

15. (c) Step 1: Radius of circle $= r$, angle of sector $= \theta$

where $l = r\theta$ and $p = r\theta + 2r$ (given)

$$\theta = \frac{p - 2r}{r} = \frac{p}{r} - 2$$



Step 2: Area of sector A

$$= \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{p}{r} - 2\right) = \frac{p}{2}r - r^2$$

For maximum area $\dfrac{dA}{dr} = 0, \dfrac{d^2A}{dr^2} < 0$

$$\therefore \frac{dA}{dr} = \frac{P}{2} - 2r = 0 \Rightarrow r = \frac{P}{4}$$

LECTURE

6

Test Your Skills

ASSERTION/REASONING

ASSERTION AND REASONING TYPE QUESTIONS

Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct.

(a) **Assertion** is True, **Reason** is True and **Reason** is a correct explanation for **Assertion.**
(b) **Assertion** is True, **Reason** is True and **Reason** is not a correct explanation for **Assertion.**
(c) **Assertion** is True and **Reason** is False.
(d) **Assertion** is False and **Reason** is True.

1. **Assertion:** The points of contact of the vertical tangents to $x = 2 - 3 \sin \theta, y = 3 + 2 \cos \theta$ are $(-1, 3)$ and $(5, 3)$.

   Reason: For vertical tangent, $\frac{dx}{d\theta} = 0$.

2. **Assertion:** If $y^2 = 3 + 2x - x^2$ then, at $(3, 0)$ and $(-1, 0)$ tangent is perpendicular to x-axis.

   Reason: At $(3, 0)$ and $(-1, 0)$, $\frac{dy}{dx} = \infty$.

3. **Assertion:** the points on the curve $y^2 = x + \sin x$ at which the tangent is parallel to x-axis lie on a straight line.
   Reason: Tangent is parallel to x-axis, then $\frac{dy}{dx} = 0$ or $\frac{dx}{dy} = \infty$.

4. **Assertion:** Equation of tangents to the curve $f(x) = x^2$ at the point where slope of tangent is equal to functional value of the curve is $4x - y - 4 = 0, y = 0$.
   Reason: $f'(x) = f(x)$.

5. **Assertion:** If S be the area of a circle having radius x and A is the area of an equilateral triangle having side πx at any instant, then $\frac{dA}{dt} > \frac{ds}{dt}$.
   Reason: $A > S$.

6. **Assertion:** The ordinate of a point describing the circle $x^2 + y^2 = 25$ decreases at the rate of 1.5 cm/s. The rate of change of the abscissa of the point when ordinate equals 4 cm is 2 cm/s.
   Reason: $x\, dx + y\, dy = 0$.

7. **Assertion:** The possible percentage error in computing the parallel resistance R of three resistances R_1, R_2, R_3 from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$, if R_1, R_2, R_3 are each in error by plus 1.2% is 1.2%.

   Reason: $\frac{\Delta R}{R^2} = \frac{\Delta R_1}{R_1^2} + \frac{\Delta R_2}{R_2^2} + \frac{\Delta R_3}{R_3^2}$.

8. **Assertion:** Two cyclists move on two seperate roads angled at 60° to each other at the rate of 5 m/s and 10 m/s respectively. The rate which they are seperating from each othen is $5\sqrt{3}$ m/s.

   Reason: Using sine Rule.

9. **Assertion:** $f(x) = \dfrac{1}{(x-5)}$ is decreasing in $x \in (-\infty, 5) \cup (5, \infty)$.

Reason: $f'(x) < 0$ for all $x \neq 5$.

10. **Assertion:** If Rolle's theorem be applied in $f(x)$, then LMVT is also applied in $f(x)$.

Reason: Both Rolle's theorem and LMVT cannot be applied in $f(x) = |\sin|x||$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$.

11. **Assertion:** If Rolle's theorem is applicable in $f(x)$ then $f(x)$ is many one function.

Reason: If LMVT is applicable in $f(x)$, then $f(x)$ is one-one function.

12. **Assertion:** The function $f(x) = \dfrac{ae^x + be^{-x}}{ce^x + de^{-x}}$ is increasing function of x, then $bc > ad$.

Reason: $f'(x) > 0$ for all x.

13. **Assertion:** Let $f: R \to R$ be a fucntion such that $f(x) = x^3 = x^2 + 3x + \sin x$. Then, f is one-one.

Reason: $f(x)$ is decreasing fucntion.

14. **Assertion:** If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists at least one $c \in (a, b)$, then $\dfrac{f(b) - f(a)}{b^3 - a^3} = \dfrac{f'(c)}{3c^2}$.

Reason: $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, $c \in (a, b)$

15. **Assertion:** If $f(x) = x(x + 3)e^{-x/2}$, then Rolle's theorem applies for $f(x)$ in $[-3, 0]$.

Reason: LMVT is applied in $f(x) = x(x + 3)e^{-x/2}$ in any interval.

16. **Assertion:** A tangent parallel to x-axis can be drawn for $f(x) = (x - 1)(x - 2)(x - 3)$ in the interval $[1, 3]$.

Reason: A horizontal tangent can be drawn in Rolle's theorem.

17. **Assertion:** If $f(x) = \max\{|6 - x^2|, |x|\}$ the minimum value of $f(x)$ in the interval $[-3, 3]$ is 2.

Reason: The minimum value of $f(x)$ attains only at $x = 2$.

18. **Assertion:** If $f(x) = (x - 3)^3$, then $f(x)$ has neither maximum nor minimum at $x = 3$.

Reason: $f'(x) = 0, f''(x) = 0$ at $x = 3$.

19. **Assertion:** If $f'(x) = (x - 1)^3(x - 2)^8$, then $f(x)$ has neither maximum nor minimum at $x = 2$.

Reason: $f'(x)$ changes sign from negative to positive at $x = 2$.

20. **Assertion:** If $f(x) = \max\{x^2 - 2x + 2, |x - 1|\}$ the greatest value of $f(x)$ on the interval $[0, 3]$ is 5.

Reason: Greatest value $f(3) = \max(5, 2) = 5$.

21. **Assertion:** The graph $y = x^3 + ax^2 + bx + c$ has no extremum, if $a^2 < 3b$.

Reason: y is either increasing or decreasing $\forall x \in R$.

22. **Assertion:** The least value of the function $f(x) = -x^2 + 4x + 1 + \sin^{-1}\left(\dfrac{x}{2}\right)$ on the interval $[-1, 1]$ is $-4 - \pi/6$.

Reason: The least value of $f(x)$ in $[-1, 1] = \min\{f(-1), f(1)\} = \min\left\{-4 - \dfrac{\pi}{6}, 4 + \dfrac{\pi}{6}\right\} = -4 - \dfrac{\pi}{6}$

23. **Assertion:** The greatest value of abc for positive values of a, b, c subject to the condition $ab + bc + ca = 12$ is 8.

Reason: $ab = bc = ca$.

24. **Assertion:** The function $f(x) = x^2 + \dfrac{16}{x}$ has a minimum value 12 at $x = 2$.

Reason: As x increases through 2, $f'(x)$ changes sign from positive to negative.

ASSERTION/REASONING: SOLUTIONS

1. (a) For vertical tangent $\frac{dx}{d\theta} = 0$

$\therefore$ $-3\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \pi/2, 3\pi/2$

at $\theta = \pi/2; x = 2 - 3 = -1$

and $y = 3 + 0 = 3$ i.e., $(-1, 3)$.

and at $\theta = 3\pi/2; x = 2 + 3 = 5$

and $y = 3 + 0 = 3$ i.e., $(5, 3)$.

2. (a) $2y\frac{dy}{dx} = (2-2x) \Rightarrow \frac{dy}{dx} = \left(\frac{1-x}{y}\right)$

$$\frac{dy}{dx}\Big|_{(3,0)} = \infty \text{ and } \frac{dy}{dx}\Big|_{(-1,0)} = \infty$$

i.e., tangents makes an angle $\frac{\pi}{2}$ with x-axis.

3. (d) $\because y^2 = x + \sin x$

$\therefore$ $2y\frac{dy}{dx} = 1 + \cos x = 0 \quad \left(\because \frac{dy}{dx} = 0\right)$

$\therefore$ $\cos x = -1$, then $\sin x = 0$. From equation (i), $y^2 = x$ (Parabola)

4. (a) Given $f'(x) = f(x)$

$\Rightarrow$ $2x = x^2$

$\Rightarrow$ $x = 0, 2$

at $x = 0, y = 0$ and at $x = 2, y = 4$

So we have to find equation of tangents at (0, 0) and (2, 4) at (0, 0), $f'(0) = 0$ and at (2, 4), $f'(2) = 4$

$\therefore$ Tangents are $y - 0 = 0\,(x - 0)$ and $y - 4 = 4\,(x - 2)$

i.e., $y = 0$ and $4x - y - 4 = 0$.

5. (b) $\because S = \pi x^2 \Rightarrow \frac{dS}{dt} = 2\pi x\frac{dx}{dt}$ and

$$A = \frac{\sqrt{3}}{4}\pi^2 x^2$$

$$\Rightarrow \frac{dA}{dt} = 2\pi^2\frac{\sqrt{3}}{4}x\frac{dx}{dt} \text{ then } \frac{dS}{dA} = \frac{ds/dt}{dA/dt}$$

$$= \frac{2\pi x\frac{dx}{dt}}{\pi^2\frac{\sqrt{3}x}{2}\frac{dx}{dt}} = \frac{4}{\pi\sqrt{3}} < 1$$

$$\therefore \frac{dS}{dt} < \frac{dA}{dt} \text{ or } \frac{dA}{dt} > \frac{dS}{dt}$$

But $\frac{A}{S} = \frac{\frac{\sqrt{3}}{4}\pi^2 x^2}{\pi x^2} = \frac{\sqrt{3}}{4}\pi > 1$

$\therefore$ $A > S$.

6. (a) $\because x^2 + y^2 = 25$

$\Rightarrow$ $2x\,dx + 2y\,dy = 0$

or $\frac{dy}{dx} = -\frac{x}{y} \Rightarrow \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{x}{y}$

$$\Rightarrow \frac{-1.5}{dx/dt} = -\frac{3}{4}$$

or $\frac{dx}{dt} = \frac{1.5\times 4}{3} = 2$ cm/s.

7. (a) $\because \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$

$$\Rightarrow -\frac{1}{R^2}\Delta R = -\frac{1}{R_1^2}\Delta R_1 - \frac{1}{R_2^2}\Delta R_2 - \frac{1}{R_3^2}\Delta R_3$$

$$\Rightarrow \frac{1}{R}\left(\frac{\Delta R}{R}\right) = \frac{1}{R_1}\left(\frac{\Delta R_1}{R_1}\right) + \frac{1}{R_2}\left(\frac{\Delta R_2}{R_2}\right) + \frac{1}{R_3}\left(\frac{\Delta R_3}{R_3}\right)$$

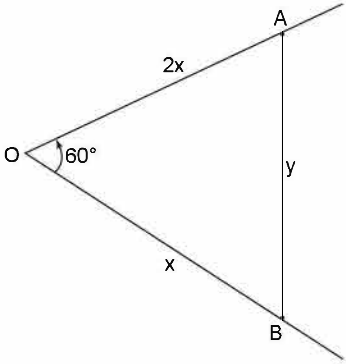
$$\Rightarrow \frac{1}{R}\left(\frac{\Delta R}{R}\times 100\right) = \frac{1}{R_1}\left(\frac{\Delta R_1}{R_1}\times 100\right) + \frac{1}{R_2}\left(\frac{\Delta R_2}{R_2}\times 100\right) + \frac{1}{R_3}\left(\frac{\Delta R_3}{R_3}\times 100\right)$$

$$\Rightarrow \frac{1}{R}\left(\frac{\Delta R}{R}\times 100\right) = (1.2)$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = (1.2)\left(\frac{1}{R}\right)$$

$$\therefore \frac{\Delta R}{R}\times 100 = 1.2\%$$

8. (c) $\therefore\ \cos 60° = \dfrac{x^2 + (2x)^2 - y^2}{2 \cdot x \cdot 2x}$



$\Rightarrow\quad 2x^2 = 5x^2 - y^2$

$\therefore\quad y = x\sqrt{3}$

$\therefore\quad \dfrac{dy}{dt} = \sqrt{3}\dfrac{dx}{dt} = 5\sqrt{3}$ m/s.

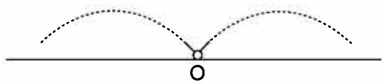
9. (a) $\because\ f(x) = \dfrac{1}{(x-5)}$

$\therefore\quad f'(x) = -\dfrac{1}{(x-5)^2} < 0\ \forall\ x \in R \sim \{5\}$

10. (b) For Rolle's theorem and LMVT, $f(x)$ must be continuous in $[a, b]$ and differentiable in (a, b).

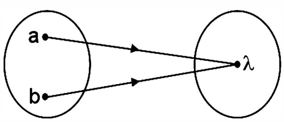
$\therefore$ If Rolle's theorem be applied in $f(x)$, then LMVT is also applied in $f(x)$.

$\because\ f(x) = |\sin|x||$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ is non differentiable at $x = 0$



$\therefore$ Rolle's theorem and LMVT cannot be applicable in $f(x) = |\sin|x||\ \forall x \in \left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$.

11. (c) Suppose, Rolle's theorem is applicable in $[a, b]$, then $f(a) = f(b) = \lambda$ (say) $f(x)$ is many one.



Also, if LMVT is applicable in $[a, b]$, then

$$f'(c) = \frac{f(b) - f(a)}{b - a}\quad \forall\ c \in (a, b)$$

or we can say that Rolle's theorem is a special case of LMVT.

Since, $f(a) = f(b)$ if $f(a) \neq f(b)$

Then, $f'(c) \neq 0\ \therefore f(x)$ is one-one.

12. (d) $\because\ f'(x) = \dfrac{2(ad - bc)}{(ce^x + de^{-x})^2}$

$\because\quad f(x)$ is increasing function $\therefore f'(x) > 0$

$\Rightarrow\quad \dfrac{2(ad - bc)}{(ce^x + de^{-x})^2} > 0\ \therefore 2\,(ad - bc) > 0$

or $\quad ad > bc$ or $bc < ad$

13. (c) $\because f'(x) = 3x^2 + 2x + 3 + \cos x$

$$= 3\left\{x^2 + \frac{2x}{3} + 1\right\} + \cos x$$

$$= 3\left\{\left(x + \frac{1}{3}\right)^2 - \frac{1}{9} + 1\right\} + \cos x$$

$$= 3\left(x + \frac{1}{3}\right)^2 + \frac{8}{3} + \cos x > 0$$

$$\left(\because 3\left(x + \frac{1}{3}\right)^2 \geq 0, -1 \leq \cos x \leq 1\right)$$

$\therefore\quad f(x)$ is an increasing function.

$\Rightarrow\quad f(x)$ is one-one.

14. (b) Let $h(x) = f(x) - f(a) + \lambda\,(x^3 - a^3)$ where, λ is selected in such a way $h(b) = f(b) - f(a) + \lambda(b^3 - a^3) = 0$ ....(i)

but $h(a) = 0$

Hence, $h(x)$ satisfies all conditions of Rolle's theorem.

$\because\ c \in (a, b)\ \therefore\ h'(c) = 0$

$\Rightarrow f'(c) - 0 + 3\lambda c^2 = 0$

$\therefore\quad \lambda = -\dfrac{f'(c)}{3c^2}$

From equation (i), $\lambda = -\dfrac{f(b) - f(a)}{(b^3 - a^3)}$

$\therefore\quad \dfrac{f(b) - f(a)}{b^3 - a^3} = \dfrac{f'(c)}{3c^2}$

15. (b) $\because\ x\,(x + 3)$ and $e^{-x/2}$ are continuous and differentiable everywhere

$\therefore$ $x\,(x + 3)e^{-x/2}$ are continuous and differentiable and $f(-3) = f(0) = 0$ and $f'(x)$

$= (x^2 + 3x) - e^{-x/2}\left(-\frac{1}{2}\right) + e^{-x/2}.(2x + 3)$

$= -\frac{1}{2}e^{-x/2}\,(x^2 + 3x - 4x - 6)$

$= -\frac{1}{2}e^{-x/2}(x^2 - x - 6)$

$= -\frac{1}{2}e^{-x/2}\,(x - 3)(x + 2)$

$\therefore$ $f'(x) = 0 \Rightarrow x = 3, -2$

$3 \notin [-3, 0]$ $\therefore$ $x = -2 \in [-3, 0]$

$\therefore$ Rolle's theorem is verified.

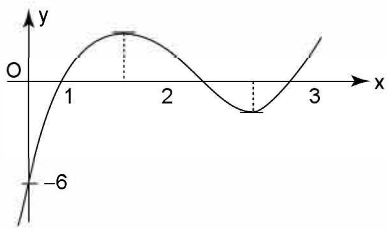
LMVT is also applied.

$\because$ $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

i.e., Rolle's theorem is a special case of LMVT.

Since, $f(a) = f(b) \Rightarrow f'(c) = 0$.

16. (b) $\because$ $f(x) = (x - 1)(x - 2)(x - 3)$
$= x^3 - 6x^2 + 11x - 6$



$\therefore$ $f'(x) = 3x^2 - 12x + 11$ $\because$ $f'(x) = 0$

$\therefore$ $3x^2 - 12x + 11 = 0$

$\Rightarrow$ $x = \dfrac{12 \pm \sqrt{144 - 132}}{6} = 2 \pm \dfrac{1}{\sqrt{3}}$

$\therefore$ $1 \le 2 \pm \dfrac{1}{\sqrt{3}} \le 3$

17. (c) Solve, $|6 - x^2| = |x|$

$\Rightarrow$ $6 - x^2 = \pm x$

$\Rightarrow$ $x^2 \pm x - 6 = 0$

$\Rightarrow$ $x = \pm 2, \pm 3$

In the interval $[-3, 3]$

$$f(x) = \begin{cases} |x| & , \ -3 \le x < -2 \\ |6 - x^2| & , \ -2 \le x \le 2 \\ |x| & , \ 2 < x \le 3 \end{cases}$$

Clearly, minimum value of $f(x)$ obtains at $x = -2, 2$ and is equal to 2.

18. (b) $\because$ $f'(x) = 3(x - 3)^2$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow$ $x = 3$

$\therefore$ $f''(x) = 6(x - 3)$ $\Rightarrow f''(3) = 0$

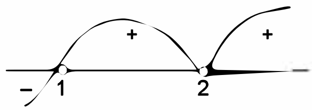
$f'''(x) = 6$

Since, $f'''(x) \neq 0$

Hence, $f(x)$ neither maximum nor minimum at $x = 3$.

19. (c) It is clear from figure $f'(x)$, sign no change at $x = 2$

Hence, $f(x)$ neither maximum nor minimum at $x = 2$



20. (b) $\because$ $f(x) = \max\{(x - 1)^2 + 1, |x - 1|\}$
$= (x - 1)^2 + 1$

$\therefore$ $f'(x) = 2(x - 1) = 0$

$\therefore$ $x = 1 \in [0, 3]$

Greatest value of $f(x) = \max\{f(0), f(1), f(3)\} = \max\{2, 1, 5\} = 5$.

21. (a) For no extremum $\dfrac{dy}{dx} > 0$ or $\dfrac{dy}{dx} < 0$ for all $x \in R$

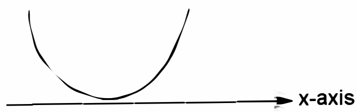
$\therefore$ $\dfrac{dy}{dx} = 3x^2 + 2ax + b > 0$

$\Rightarrow$ $3x^2 + 2ax + b > 0$

$\therefore$ $D < 0$

$\Rightarrow$ $4a^2 - 4.3\,.\,b < 0$

$\Rightarrow$ $a^2 < 3b$



22. (b) $\because$ $f'(x) = -2x + 4 + \dfrac{1}{\sqrt{1 - \left(\dfrac{x}{2}\right)^2}} \cdot \dfrac{1}{2}$

$= 2(2 - x) + \dfrac{1}{\sqrt{(4 - x^2)}} > 0$ in $x \in [-1, 1]$

$\Rightarrow$ $f(x)$ is increasing function in $[-1, 1]$

$\therefore$ Least value of $f(x)$ is $f(-1) = -4 - \pi/6$.

23. (a) Given $ab + bc + ca = 12$ (constant) the value of $(ab)(bc)(ca)$ is greatest when $ab = bc = ca = \lambda$ (say)

$\therefore\quad 3\lambda = 12 \quad \Rightarrow \lambda = 4$

$\Rightarrow\quad (ab)(bc)(ca) = 4^3$

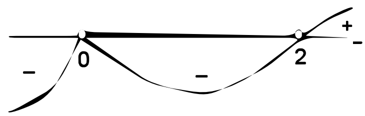
$\Rightarrow\quad abc = 8.$

24. (c) $\because\ f(x) = x^2 + \dfrac{16}{x}$

$$\therefore\quad f'(x) = 2x - \frac{16}{x^2} = \frac{2(x^3-8)}{x^2}$$

$$= \frac{2(x-2)(x^2+2x+4)}{x^2}$$

$$= \frac{2(x-2)((x+2)^2+3)}{x^2}$$



$\Rightarrow\quad f'(x) > 0$ for $x > 2$ and $f'(x)$ change sign From negative to positive

$\therefore\quad$ Minimum value of $f(x)$ is

$$f(2) = 4 + \frac{16}{2} = 12.$$

MENTAL PREPARATION TEST

1. The radius of circular plate is increasing at the rate of 0.2 cm/sec. Find the rate of change of the area of plate when its radius is 10 cm. ***[MP-99]***

2. Using differentials find the approximate value of $(29)^{1/3}$. ***[CBSE-97]***

3. Verify Rolle's theorem for the function $f(x) = x^2 - 4x + 3$ on [1, 3] ***[CBSE-95, 98, 2002C]***

4. Verify Lagrange's Mean value theorem for the function $f(x) = x^2 + x - 1$ on [0, 4]. ***[CBSE-2002]***

5. Prove that the function $4x^3 - 6x^2 + 3x + 12$ is increasing on R. ***[CBSE-92]***

6. Verify Rolle's theorem for the function $f(x) = \cos x$ on $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$. ***[CBSE-96]***

7. Verify Lagrange's mean value theorem for the function $f(x) = x + \dfrac{1}{x}$ on [1, 3] ***[CBSE-87, 94C, 2000]***

8. The radius of a Soap bubble is increasing at the rate of 0.2 cm/s. Find the rate of increase of its volume when the radius is 5 cm. ***[CBSE-97 C]***

9. A particle moves along the curve $y = \left(\dfrac{2}{3}\right)x^3 + 1$. Find the points on the curve at which the y-coordinate is changing twice as fast as the x-coordinate. ***[CBSE-96C]***

10. If $y = x^4 - 10$ and if x changes from 2 to 1.97 what is the approximate change in y? ***[CBSE-99C]***

11. Find the point on the curve $x^2 = 8y$ which is nearrest to the point (2, 4). ***[CBSE-98]***

12. Divide a number 15 into two parts such that the square of first multiplied with the cube of the second is a maximum. ***[MP-99; CBSE-83]***

13. Verify Lagrange's Mean Value Theorem for the function $f(x) = (x-1)(x-2)(x-3)$ on [1, 4]. ***[CBSE-94]***

14. Find the intervals in which the function $f(x) = x^3 - 12x^2 + 36x + 17$ is increasing or decreasing. ***[CBSE-98, 2001, 2006]***

15. A man 2 metres high, walks at a uniform speed of 6 metres per minute a way from a lamp post, 5 metres high. Find the rate at which the length of his shadow increases. ***[CBSE-91, 94]***

16. Show that the height of the cylinder of greater volume which can be inscribed in a right circular cone of height H and having semi vertical angle 30° is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{81}\pi H^2$

[CBSE-2001 C]

17. A radius of circle is increasing at the rate of 0.5 cm./sec. At what rate the circumference is increasing when the radius is 2 cm.

[MP-2005]

18. Using differentials find the approximate value of $\sqrt{50}$. ***[CBSE-2000]***

19. Verify Rolle's Theorem for the functions $f(x) = x^2 - 5x + 6$ on [2, 3]

[CBSE-96, 98, 2002C]

20. Verify Lagrange's Mean Value Theorem for the function $f(x) = x^2 - 2x + 4$ on [1, 5]

[CBSE-04]

21. Prove that the function $y = \sin x$ is maximum at $x = \frac{\pi}{2}$. ***[MP-98]***

22. Verify lagrange's mean value Theorem for the function $f(x) = \log_e x$ on [1, 2].

[PB-94C, 98; HPSB-99S; NCERT BOOK]

23. The radius of a spherical soap bubble is increasing at the rate of 0.2 cm/sec. Find the rate of increases of its surface area when the radius is 4 cm. ***[CBSE-92 C]***

24. Find the interval in which the function $f(x) = 5x^2 + 7x - 13$ is increasing or decreasing.

[CBSE-82]

25. For the function $y = x^3 + 21$, find the value of x when y increases 75 times as fast as x.

[CBSE-99]

26. Find the intervals in which the function $f(x) = 8 + 36x + 3x^2 - 2x^3$ is increasing or decreasing. ***[CBSE-95 C]***

27. Find the equation of the tangent to the curve $x^2 + 3y - 3 = 0$ which is parallel to the line $y = 4x - 5$. ***[CBSE-2001 C, 2005]***

28. A man 2 metres high walks at a uniform speed of 5 km/hr. away from a lamp-post 6 metres high. Find the rate at which the length of his shadow increases.

[CBSE-94C; HSB-94C; PSB-91C, 92]

29. The radius of a circle is increasing uniformly at the rate of 2 cm per second. At what rate is the area increasing when the radius is 8 cm?

[MP-2003]

30. Using differentials find the approximate value of $\sqrt{0.48}$. ***[CBSE-2002C]***

31. Find the point on the curve $y = x^2 - 4x - 32$ at which tangent is parallel to x-axis.

[CBSE-95]

32. Verify Rolle's Theorem for the function $f(x) = \cos x - 1$ on $[0, 2\pi]$

[HPB-97; NCERT BOOK]

33. At what points of the ellipse $16x^2 + 9y^2 = 400$ does the ordinate decreases at the same rate which the abscissa increases?

[CBSE-99]

34. If $y = x^4 + 4$ and x changes from 2 to 2.1, find the approximate change in y.

[CBSE-99 C]

35. Find two positive numbers whose sum is 14 and the sum of whose square is minimum.

[HB-91]

36. Show that the curves $2x = y^2$ and $2xy = k$ cut at right angles if $k^2 = 8$.

[CBSE-96]

37. Show that the height of the cylinder of maximum volume that can be inscribed in a right circular cone of height 9 m is 3 m.

[HB-2002]

38. A wire of length 36 m is to be cut into two pieces. One of the pieces is to made into a square and the other into a circle. What should be the length of the two pieces, so that the combined area of the square and the circle is minimum? ***[CBSE-96]***

39. A particle is moving in a straight line. The distance x (in metres) travelled by it in time t is given by the relation $x = 4t + 2t^2$. Find the velocity and acceleration of the particle after 4 seconds. ***[MP-2004]***

40. The distance S travelled by a particle in t seconds is given by $S = 7t^2 - 4t + 1$ find the velocity and acceleration at $t = 3/2$ sec.

[MP-2003]

41. A rectangular window is surmounted by an equilateral triangle. Given that the perimeter is P, find the dimensions of the window so that maximum amount of light may enter.

[CBSE-(SP)-2006]

42. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

[CBSE-2008]

43. Show that the volume of the greatest cylinder that can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$.

[CBSE-2008]

44. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is ***[CBSE-2008]***

(i) increasing and

(ii) decreasing.

45. At what points will the tangent to the curve $y = 2x^3 - 15x^2 + 36x - 21$ be parallel to x-axis? Also, find the equations of tangents to the curve at those points. ***[CBSE-2008]***

LECTUREWISE WARMUP TEST

1. If the sum of the squares of the intercepts made by the tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$ $(a > 0)$ at point $(a/8, a/8)$ on coordinate axes is 2, then the value of a is:

(a) 1 (b) 2

(c) 4 (d) 8

2. The function $f(x) = 2x^3 - 3x^2 + 90x + 174$ is increasing in the interval:

(a) $\frac{1}{2} < x < 1$ (b) $\frac{1}{2} < x < 2$

(c) $3 < x < \frac{59}{4}$ (d) $-\infty < x < \infty$

3. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is:

(a) $3y = 9x + 2$ (b) $y = 2x + 1$

(c) $2y = x + 8$ (d) $y = x + 2$

4. Let the function $f : R \to R$ be defined by $f(x) = 2x + \cos x$. Then f:

[Karnataka CET-2002]

(a) has maximum at $x = 0$

(b) has minimum at $x = \pi$

(c) is an increasing function

(d) is a decreasing function

5. The rate of change of the surface area of a sphere of radius r when the radius is increasing at the rate of 2 cm/sec. is proportional to:

[Karnataka CET-2003]

(a) $1/r^2$ (b) $1/r$

(c) r^2 (d) r

6. Gas is being pumped into a spherical balloon at the rate of 30 ft³/min. Then the rate at which the radius increases when it reaches the value 15 ft. is: ***[EAMCET-2003]***

(a) $\frac{1}{30\pi}$ ft/min. (b) $\frac{1}{15\pi}$ ft./min

(c) $\frac{1}{20}$ ft./min (d) $\frac{1}{25}$ ft/min

7. Find which function does not obey, mean value theorem in [0, 1]:

[IIT (Screening)-2003]

(a) $f(x) = \begin{cases} \frac{1}{2} - x; & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2: & x \ge \frac{1}{2} \end{cases}$

(b) $f(x) = \begin{cases} \frac{\sin x}{x}; & x \ne 0 \\ 1; & x = 0 \end{cases}$

(c) $f(x) = x\,|x|$

(d) $f(x) = |x|$

8. The necessary and sufficient conditions that continuous and differentiable function $f(x)$ has a maximum at $x = a$ are:

(a) $f'(a) = 0$ and $f''(a) > 0$
(b) $f'(a) = 0$ and $f''(a) < 0$
(c) $f'(a) < 0$ and $f''(a) = 0$
(d) $f'(a) > 0$ and $f''(a) > 0$

9. Let $f(x)$ be an even function in R. If $f(x)$ is monotonic increasing in [2, 6], then:
(a) $f(3) > f(-5)$ (b) $f(-2) < f(2)$
(c) $f(-2) > f(2)$ (d) $f(-3) < f(5)$

10. A particle moving on a curve has the position given by $x = f'(t)\sin t + f''(t)\cos t$, $y = f'(t)\cos t - f''(t)\sin t$ at time t where f is a thrice – differentiable function. Then the velocity of the particle at time t is:
(a) $f'''(t)$ (b) $f'(t) + f'''(t)$
(c) $f'(t) + f''(t)$ (d) $f'(t) - f'''(t)$

11. The slope of the tangent to the curve $\tan y = \dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$ at $x = 1/2$ is:
(a) 1/3 (b) $\sqrt{3}$
(c) 1 (d) 1/2

12. The slope of the tangent to the curve $x = \sec y$ at the point $y = \sec^{-1}(-2)$ is:
(a) $\dfrac{1}{2\sqrt{3}}$ (b) $\dfrac{-1}{2\sqrt{3}}$
(c) $\dfrac{1}{2\sqrt{3}}$ or $\dfrac{-1}{2\sqrt{3}}$ (d) None of these

13. If $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$, then slope of the tangent at $x = 0$ is:
(a) 1 (b) –1
(c) 0 (d) 2

14. Which one of the following statements is not correct?
(a) The derivative of $f(x)$ at $x = a$ is the slope of the graph of $f(x)$ at the point $[a, f(a)]$
(b) $f(x)$ has a positive derivative at $x = a$ means $f(x)$ increases as x increases from 'a'
(c) The sum of two differentiable functions is differentiable
(d) If a function is continuous at a point, it is also differentiable at the same point.

15. What is the minimum value of $px + qy$ $(p > 0, q > 0)$ when $xy = r^2$?
(a) $2r\sqrt{pq}$
(b) $2pq\sqrt{r}$
(c) $-2r\sqrt{pq}$
(d) $2\,rpq$

16. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and $m(b)$ is a minimum value of $f(x)$. If b can assume different values, then range of $m(b)$ is equal to:
[IIT (Screening)-2001]
(a) [0, 1] (b) (0, 1/2]
(c) [1/2, 1] (d) (0, 1]

LECTUREWISE WARMUP TEST: SOLUTIONS

1. (c) Equation of tangent (x_1, y_1) is

$$\frac{x}{x_1^{2/3}} + \frac{y}{y_1^{2/3}} = a^{1/3} \qquad \ldots\ldots (1)$$

$\therefore$ Sum of squares of intersects

$$x_1^{4/3}a^{2/3} + y_1^{4/3}a^{2/3}$$

$$= a^{2/3}\left[\left(\frac{a}{8}\right)^{4/3} + \left(\frac{a}{8}\right)^{4/3}\right]$$

$$= a^{2/3}\left(\frac{2\times a^{4/3}}{16}\right) = \frac{a^2}{16} = 1$$

$\Rightarrow$ $a = 4$.

2. (d) $f'(x) = 6(x^2 - x + 15) > 0 \ \forall\ x$

3. (d) Any point on $y^2 = 8x$ is $(2t^2, 4t)$ where the tangent is $yt = x + 2t^2$
Solving it with $xy = -1$, $y(yt - 2t^2) = -1$
or $ty^2 - 2t^2y + 1 = 0$
For common tangent, it should have equal roots .
$\therefore$ $4t^2 - 4t = 0$ $\therefore$ $t = 0, 1$
$\therefore$ The common tangent is $y = x + 2$, (when $t = 0$, it is $x = 0$ which can touch $xy = -1$ at infinity only).

4. (c) $f(x) = 2x + \cos x \Rightarrow f'(x) = 2 - \sin x$
$\because$ $0 \le |\sin x| \le 1$
$\therefore$ $f'(x) > 0$ and thus $f(x)$ is increasing function:

5. (d) Let S be the surface area of sphere

$\therefore S = 4\pi r^2$

$\therefore \quad \frac{ds}{dt} = 4\pi \times 2r\frac{dr}{dt} = 8\pi r \times 2$ cm²/sec. = $16\pi r$ cm²/sec.

Which is proportional to r:

6. (a) Given that $\frac{dv}{dt}$ = 30 ft³/min. and r = 15 ft.

$$v = \frac{4}{3}\pi r^3; \frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}; \frac{dr}{dt}$$

$$= \frac{\frac{dv}{dt}}{4\pi r^2} = \frac{30}{4\times\pi\times 15\times 15} = \frac{1}{30\pi}\text{ft/min.}$$

$\therefore$ Answer is (a)

7. (a) The function defind in option (a) is not differentiable at $x = 1/2$.

8. (b) It is the fundamental concept.

9. (d) As $f(x)$ is given, $f(-2) = f(2)$

As $f(x)$ is monotonic increasing in [2, 6], $f(3) < f(5) = f(-5)$

Also, $f(3) < f(5) \Rightarrow f(-3) < f(5)$.

10. (b) $\frac{dx}{dt} = f''(t)\sin t + f'(t)\cos t + f'''(t)\cos t - f''(t)\sin t$

$= \{f'(t) + f'''(t)\}\cos t$

$\frac{dy}{dt} = f''(t)\cos t - f'(t)\sin t - f'''(t)\sin t - f''(t)\cos t$

$= -\{f'(t) + f'''(t)\}\sin t$

$$\therefore \text{ velocity} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = f'(t) + f'''(t).$$

11. (a) $\tan y = \frac{\left(\sqrt{1+x} - \sqrt{1-x}\right)^2}{1+x-(1-x)}$

$$= \frac{1+x+1-x-2\sqrt{1-x^2}}{2x}$$

$$= \frac{1-\sqrt{1-x^2}}{x}$$

$$\therefore \quad y = \tan^{-1}\frac{1-\sqrt{1-x^2}}{x}$$

Put $x = \sin 2\theta$. Then

$$y = \tan^{-1}\frac{1-\cos 2\theta}{\sin 2\theta} = \tan^{-1}\frac{2\sin^2\theta}{2\sin\theta.\cos\theta}$$

$$= \tan^{-1}\tan\theta = \theta \therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{1}{2\cos 2\theta} = \frac{1}{2\sqrt{1-x^2}}$$

$$\therefore \quad \text{at } x = \frac{1}{2}, \frac{dy}{dx} = \frac{1}{2.\sqrt{1-\frac{1}{4}}} = \frac{1}{\sqrt{3}}$$

12. (a) Here $y = \sec^{-1} x$ So, $\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

$$\therefore \quad \text{at } x = -2, \frac{dy}{dx} = \frac{1}{|-2|\sqrt{(-2)^2-1}} = \frac{1}{2\sqrt{3}}$$

13. (a) We have $y = (1+x)^y + \sin^{-1}(\sin^2 x)$ ... (1)

when $x = 0$, we have $y = 1$, differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = (1+x)^y\left\{\frac{dy}{dx}\log(1+x) + \frac{y}{1+x}\right\}$$

$$+\frac{\sin 2x}{\sqrt{1-\sin^4 x}} \Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = 1$$

14. (d) It is the fundamental concept.

15. (a) $px + qy \geq 2\sqrt{pqxy}$

$\Rightarrow \quad px + qy \geq 2\sqrt{pqr^2}$

$\Rightarrow \quad px + qy \geq 2r\sqrt{pq}$

16. (d) $f'(x) = 2(1+b^2)x + 2b$,

$f''(x) = 2(1+b^2) > 0$

$f'(x) = 0 \Rightarrow x = -\frac{b}{1+b^2}$ where $f'(x) > 0$

$\therefore \quad f(x)$ is minimum at $x = -\frac{b}{1+b^2}$

$$\therefore \quad m(b) = (1+b^2)\frac{b^2}{(1+b^2)^2}$$

$$+ 2b\left(\frac{-b}{1+b^2}\right) + 1 = \frac{1}{1+b^2}$$

$\therefore \quad 0 < m(b) \leq 1.$

ANSWERS

LECTURE 1

Unsolved Objective Problems (Identical Problems for Practice): For Improving Speed with Accuracy

1. (b)	**8.** (c)	**15.** (c)	**22.** (c)
2. (a)	**9.** (c)	**16.** (a)	**23.** (b)
3. (c)	**10.** (b)	**17.** (c)	**24.** (d)
4. (d)	**11.** (b)	**18.** (b)	**25.** (b)
5. (b)	**12.** (c)	**19.** (b)	
6. (d)	**13.** (b)	**20.** (d)	
7. (b)	**14.** (d)	**21.**	

Worksheet: To Check the Preparation Level

1. (b)	**5.** (b)	**9.** (d)	**13.** (b)
2. (c)	**6.** (a)	**10.** (d)	**14.** (a)
3. (b)	**7.** (d)	**11.**	**15.** (b)
4. (a)	**8.** (b)	**12.** (b)	

LECTURE 2

Unsolved Objective Problems (Identical Problems for Practice): For Improving Speed with Accuracy

1. (b)	**8.** (c)	**15.** (b)	**22.** (c)
2. (b)	**9.** (c)	**16.** (a)	**23.** (c)
3. (d)	**10.** (c)	**17.** (c)	**24.** (c)
4. (a)	**11.** (d)	**18.** (b)	**25.** (d)
5. (d)	**12.** (a)	**19.** (b)	
6. (a)	**13.** (a)	**20.** (d)	
7. (a)	**14.** (a)	**21.** (b)	

Worksheet: To Check the Preparation Level

1. (a)	**5.** (c)	**9.** (b)	**13.** (b)
2. (c)	**6.** (d)	**10.** (d)	**14.** (c)
3. (b)	**7.** (c)	**11.** (a)	**15.** (b)
4. (c)	**8.** (a)	**12.** (c)	**16.** (c)

LECTURE 3

Unsolved Objective Problems (Identical Problems for Practice): For Improving Speed with Accuracy

1. (c)	**6.** (d)	**11.** (d)	**16.** (c)
2. (a)	**7.** (a)	**12.** (c)	**17.** (a)
3. (b)	**8.** (c)	**13.** (b)	
4. (d)	**9.** (c)	**14.** (c)	
5. (c)	**10.** (c)	**15.** (a)	

Worksheet: To Check the Preparation Level

1. (b)	**6.** (a)	**11.** (c)	**16.** (d)
2. (b)	**7.** (a)	**12.** (a)	**17.** (a)
3. (a)	**8.** (b)	**13.** (a)	
4. (b)	**9.** (c)	**14.** (d)	
5. (a)	**10.** (c)	**15.** (a)	

LECTURE 4

Unsolved Objective Problems (Identical Problems for Practice): For Improving Speed with Accuracy

1. (a)	**7.** (a)	**13.** (a)	**19.** (d)
2. (d)	**8.** (d)	**14.** (c)	**20.** (d)
3. (b)	**9.** (a)	**15.** (b)	
4. (a)	**10.** (a)	**16.** (c)	
5. (a)	**11.** (c)	**17.** (a)	
6. (a)	**12.** (b)	**18.**	

Worksheet: To Check the Preparation Level

1. (d)	**6.** (c)	**11.** (a)	**16.** (a)
2. (a)	**7.** (d)	**12.** (c)	
3.	**8.** (a)	**13.** (c)	
4. (c)	**9.** (c)	**14.** (d)	
5. (c)	**10.** (c)	**15.** (b)	

LECTURE 5

Unsolved Objective Problems (Identical Problems for Practice): For Improving Speed with Accuracy

1. $16\sqrt{6}$ **4.** (c) **7.** (c)
2. (b) **5.** (d) **8.** (c)
3. (b) **6.** (d) **9.** (d)

Worksheet: To Check the Preparation Level

1. (b) **5.** (a, d) **9.** (c) **13.** (d)
2. (d) **6.** (a, d) **10.** (c) **14.** (b)
3. (a, b, c, d) **7.** (c) **11.** (b) **15.** (c)
4. (c, d) **8.** (d) **12.** (a)

LECTURE 6

Mental Preparation Test

1. $\frac{dA}{dt} = 4\pi \text{ cm}^2/\text{sec}$
2. 3.074
8. 20 π cc/sec
9. (1, 5/3) and (–1, 1/3)
10. y changes from 6 to 5.04
11. (2, 4)
12. 6, 9
14. Increasing $(-\infty, 2) \cup (6, \infty)$, decreasing (2, 6)
15. 4 m/minute
17. $\frac{dC}{dt} = \pi$ cm/sec.
18. 7.07
23. 6. 4 π cm^2/sec.
24. Increasing $(-7/10, \infty)$, Decreasing $(-\infty, -7/10)$
25. $x = \pm 5$
26. Increasing (–2, 3), decreasing $(-\infty, -2) \cup (3, \infty)$
27. $4x - y + 13 = 0$
28. 5/2 km/hr
29. 32 π cm^2/sec.
30. 0.693
31. (2, –36)
33. (3, 16/3) and (–3, –16/3)
34. y changes from 8 to 8.4
35. The required numbers are both equal to 7.
37. $\frac{36\pi}{\pi+4}, \frac{144}{\pi+4}$
39. 20 m/s, 4 m/sec^2
40. 17 cm/sec, 14 cm/sec^2
41. $x = \frac{(6+\sqrt{3})P}{33}$ and $y = \frac{(5-\sqrt{3})P}{22}$