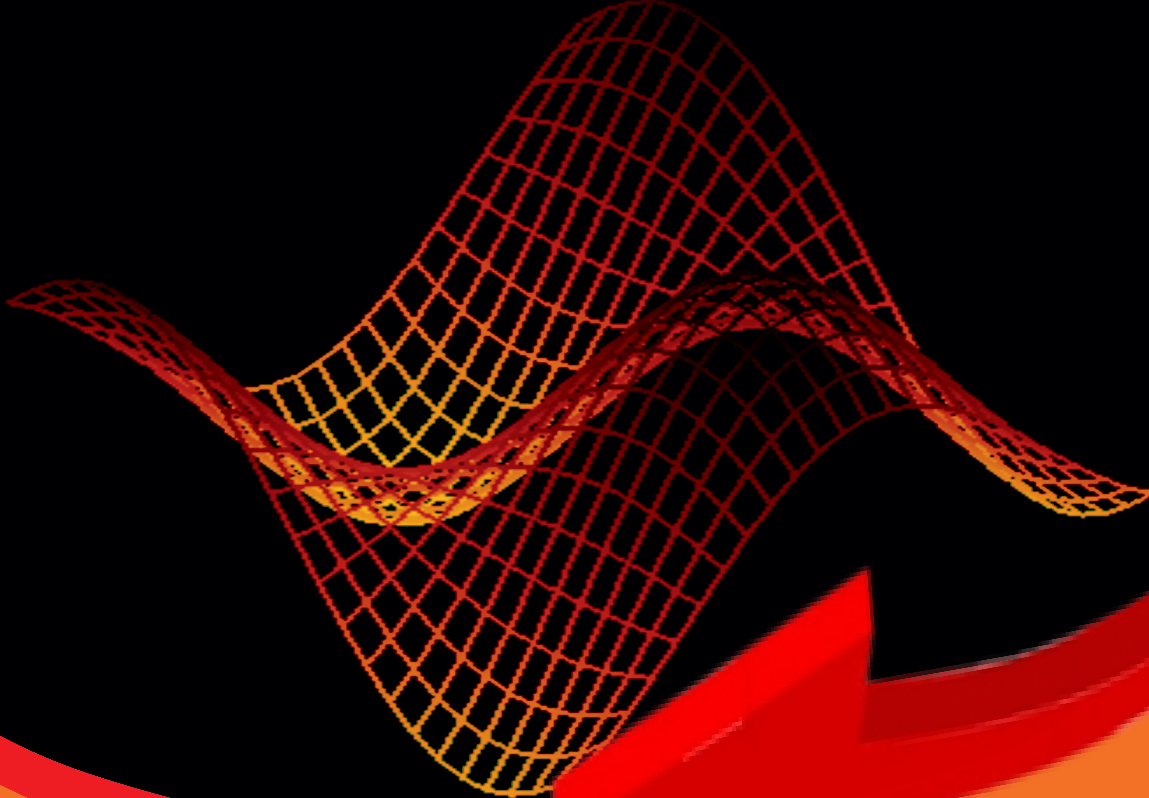

Differential Calculus Booster

with Problems & Solutions

for

JEE

Main and Advanced

McGraw Hill Education

Rejaul Makshud

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About the Author

REJAUL MAKSHUD (RM)
Post Graduated from Calcutta University in PURE MATHEMATICS.
Presently, he trains IIT Aspirants at **RACE IIT Academy**, Jamshedpur.

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Dedicated to
My Beloved Mom and Dad

Preface

DIFFERENTIAL CALCULUS BOOSTER with Problems & Solutions for JEE Main and Advanced is meant for aspirants preparing for the entrance examinations of different technical institutions, especially NIT/IIT/BITSAT/IISc. In writing this book, I have drawn heavily from my long teaching experience at National Level Institutes. After many years of teaching I have realised the need of designing a book that will help the readers to build their base, improve their level of mathematical concepts and enjoy the subject.

This book is designed keeping in view the new pattern of questions asked in JEE Main and Advanced Exams. It has nine chapters. Each chapter has the concept booster followed by a large number of exercises with the exact solutions to the problems as given below:

Level - I	: Problems based on Fundamentals
Level - II	: Mixed Problems (Objective Type Questions)
Level - III	: Problems for JEE Advanced
Level - IV	: Tougher Problems for JEE Advanced
(0.......9)	: Integer Type Questions
Passages	: Comprehensive Link Passages
Matching	: Matrix Match
Reasoning	: Assertion and Reason
Previous years papers	: Questions asked in Previous Years' IIT-JEE Exams

Remember friends, no problem in mathematics is difficult. Once you understand the concept, they will become easy. So please don't jump to exercise problems before you go through the Concept Booster and the objectives. Once you are confident in the theory part, attempt the exercises. The exercise problems are arranged in a manner that they gradually require advanced thinking.

I hope this book will help you to build your base, enjoy the subject and improve your confidence to tackle any type of problem easily and skillfully.

My special thanks goes to Mr. M.P. Singh (IISc. Bangalore), Mr. Yogesh Sindhwani (Head of School, Lancers International School, Gurugram), Mr. Manoj Kumar (IIT, Delhi), Mr. Nazre Hussain (B.Tech.), Dr. Syed Kashan Ali (MBBS) and Mr. Shahid Iqbal, who have helped, inspired and motivated me to accomplish this task. As a matter of fact, teaching being the best learning process, I must thank all my students who inspired me most for writing this book.

I would like to convey my affectionate thanks to my wife, who helped me immensely and my children who bore with patience my neglect during the period I remained devoted to this book.

I also convey my sincere thanks to Mr Biswajit Das of McGraw Hill Education for publishing this book in such a beautiful format.

I owe a special debt of gratitude to my father and elder brother, who taught me the first lesson of Mathematics and to all my learned teachers—Mr. Swapan Halder, Mr. Jadunandan Mishra, Mr. Mahadev Roy and Mr. Dilip Bhattacharya, who instilled the value of quality teaching in me.

I have tried my best to keep this book error-free. I shall be grateful to the readers for their constructive suggestions toward the improvement of the book.

Rejaul Makshud
M. Sc. (Calcutta University, Kolkata)

Contents

CHAPTER

1 Real Function

CONCEPT BOOSTER

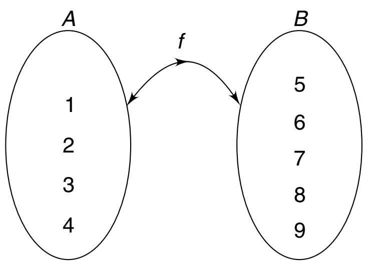
1. Basic Concepts of Real Functions

1.1 Definition

Let A and B be two non-empty sets. A function 'f' is a rule between two sets A and B in such a way that for every element in A there exists a unique element in B. It is denoted as $f: A \to B$ and it is read as f is a function from A to B or f maps from A to B.

Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$

and $f(x) = x + 4$.



> **Note:**
> 1. Every point in A is related to some point in B
> 2. A point in A cannot be related to two or more points in B.
> 3. Two or more points in A can be related to a single point in B.
> 4. There some point in B which are not related to any point in A.
> 5. Every function is a relation but every relation is not a function.
> 6. If the sets A and B consists of m and n elements respectively, then the total number of function between A to B is n^m and the total number of relations between A to B is 2^{nm}.

1.2 Domain

The first set is called the domain of a function. It is denoted as D_f. In $f(x) = x + 4$ above, $D_f = A$

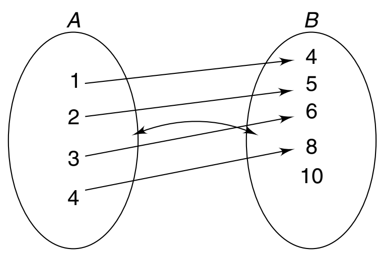
1.3 Co-domain

The second set is called the co-domain of a function. It is denoted as C_f. In $f(x) = x + 4$ above, $C_f = B$

Example-1. Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 8, 10\}$ and f is a relation from A to B such that $f = \{(1, 4), (2, 5), (3, 6), (4, 8)\}$ is f a function?

If so, then find its domain.

Sol.



Yes, it is a function. Domain = $\{1, 2, 3, 4\}$

1.4 Image

If $f(1) = 5$, then 5 is called in image of 1 under the function f and 1 is called an inverse element of 5 or pre-image of 5.

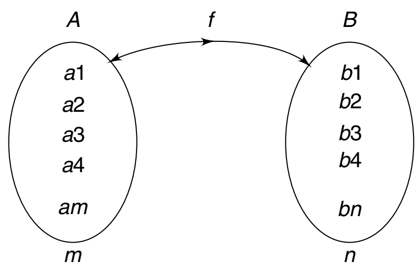
1.5 Range

The set of all images is called the range of a function. It is denoted as R_f Here, $R_f = \{5, 6, 7, 8\} \subseteq B$. In another way, we can define, the range of a function is a subset of co-domain.

1.6 Real Function

If the domain and co-domain are the subsets of a real number, then it is called a real valued function or simply a real function. It is generally denoted as $f: R \to R$.

1.7 Number of real functions between two sets *A* and *B*



If the number of elements of a set A contains are m and the set B are n, then the number of real functions between two sets A and B are n^m.

Let $A = \{1, 2, 3\}$ and $B = \{2, 6\}$. Then the number of functions between two sets A and B are $2^3 = 8$

2. Algebraic Operation on Domain of a Function

(i) Dom $(f \pm g)$ = Dom $f \cap$ Dom g

(ii) Dom $(f \pm g \pm h)$ = Dom $f \cap$ Dom g Dom h

(iii) Dom $(f \cdot g)$ = Dom $f \cap$ Dom g

(iv) Dom $(f \cdot g \cdot h)$ = Dom $f \cap$ Dom $g \cap$ Dom h

(v) Dom $\left(\frac{f}{g}\right)$ = Dom $f \cap$ Dom $g - \{x : g(x) = 0\}$

(vi) Dom $(\sqrt{f(x)}) = \{x : f(x) \geq 0\}$

(vii) Dom $(\log_a (f(x))) = \{x : f(x) > 0\}$

(viii) Dom $(a^{f(x)})$ = Dom f provided $a > 0$

3. Range of a Function

There is no specific method to find out the range of a function. But the following points should be kept in mind in finding the range of a function.

(i) First we find the domain of a function $f(x)$.

(ii) If D_f lie finite number of points, then the R_f is the set of corresponding values of $\{f(x)\}$

(iii) If $D_f = R$, R – {some finite points}, then we express x in terms of y and define x.

(iv) If D_f is a finite interval, say, $[a, b]$, then R_f is the greatest and the least values of $y = f(x)$ i.e. R_f = [Least Value, Greatest Value]

> ***Note:*** We should note that determining range of a function is comparatively more difficult proposition than determining domain of a function.

Q. Find the domains and ranges of each of the following functions

1. $y = x^2 + 2$
2. $y = \sqrt{x - 2}$
3. $y = \sqrt{9 - x^2}$
4. $y = \sqrt{x^2 - 4}$
5. $y = \sqrt{x^2 - 3x + 2}$

Ans.

1. $D_f = R$, $R_f = [2, \infty)$
2. $D_f = [2, \infty)$, $R_f = [0, \infty)$
3. $D_f = [-3, 3]$, $R_f = [0, 3]$
4. $D_f = (-\infty, 2] \cup [2, \infty)$ and $R_f = [0, \infty)$
5. $D_f = (-\infty, 1] \cup [2, \infty)$ and $R_f = [0, \infty)$

4. Types of Functions

Basic functions can be divide into two categories

(i) Algebraic Function

(ii) Transcendental Function

4.1 Algebraic Functions

(A) Polynomial Functions

(i) Constant Function

(ii) Identity Function

(iii) Parabolic Function

(iv) Even Power Function

(v) Cubical Function

(vi) Odd Power Function

(B) Rational Functions

(i) Reciprocal Function

(ii) Even power reciprocal Function

(iii) Odd power reciprocal Function

(C) Irrational Functions

(i) Square root Function

(ii) Even root Function

(iii) Odd root Function

(D) Piece-wise Defined Functions

(i) Modulus Function

(ii) Signum Function/Sign Function

(iii) Greatest Integer Function

(iv) Least Integer Function

(v) Fractional part Function

4.2 Transcendental Functions (Non-algebraic Functions)

(i) Trigonometric Functions

(ii) Inverse Trigonometric Functions

(iii) Exponential Functions

(iv) Logarithmic Functions

(A) Polynomial Function

A function $f : R \to R$ is defined as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \text{ where } n \in W$$

Degree of a Polynomial

The highest index power of x having non-zero co-efficient is called the degree of the polynomial. The degree of the above polynomial is n, when $a_n \neq 0$.

For examples,

The degree of the polynomials

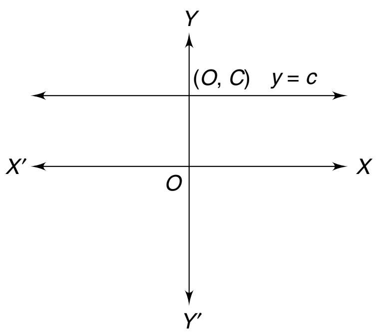
1. $f(x) = 3$ is 0
2. $f(x) = 2x + 4$ is 1
3. $f(x) = x^2 + 3x + 2$ is 2
4. $f(x) = x^3 + 3x^2 + 5x + 6$ is 3
5. $f(x) = x^{2014} + 10x^{2000} + 5x^{199} + 6$ is 2014
6. $f(x) = 0$ is undefined.

Note:

1. The domains and ranges of a polynomial depends on the degree of a polynomial.
2. If the degree of a polynomial is odd, then its domains and ranges are R.
3. If the degree of a polynomial is even, then its will not be all real number.

(i) **Constant Function**

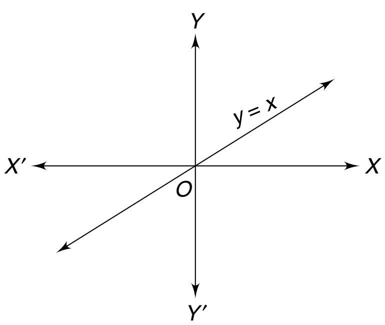
A function $f: R \to R$ is defined as $f(x)$ = constant $= c$



$$D_f = R,\ R_f = \{c\}$$

(ii) **Identity Function**

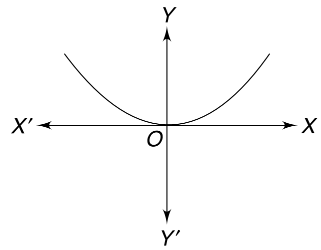
A function $f: R \to R$ is defined as $f(x) = x$



$$D_f = R,\ R_f = R$$

(iii) **Parabolic Function**

A function $f: R \to R$ is defined as $f(x) = x^2$



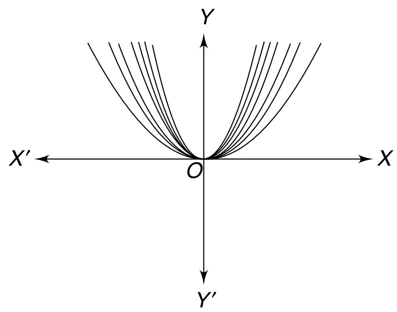
$$D_f = R \text{ and } R_f = [0, \infty)$$

(iv) **Even power Function**

A function $f: R \to R$ is defined as

$$f(x) = x^{2n},\ n \in N$$

i.e. $f(x) = x^2, x^4, x^6, x^8, x^{10}, \ldots\ldots$



Domains and ranges will remain same

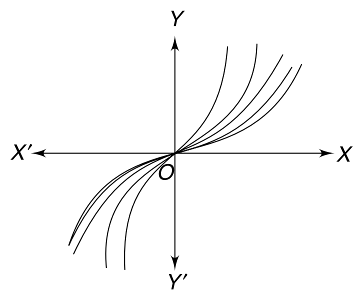
i.e $D_f = R$ and $R_f = [0, \infty)$

(v) **Odd Power Function**

A function $f: R \to R$ is defined as

$$f(x) = x^{2n+1},\ n \in N$$

i.e. $f(x) = x^3, x^5, x^7, x^9, x^{11}, \ldots\ldots$



$$D_f = R \text{ and } R_f = R$$

(B) Rational Functions

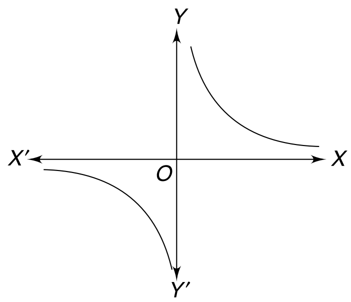
A function $f: R \to R$ is defined as

$f(x) = \dfrac{g(x)}{h(x)}$, $h(x) \neq 0$, $g(x)$ and $h(x)$ are two polynomials functions.

(i) Reciprocal Function

A function $f: R \to R$ is defined as

$$f(x) = \frac{1}{x}$$



$$D_f = R - \{0\} = R_f$$

Asymptote

It is a straight line which touches the curve at infinity.

There are three types of asymptotes

(i) Vertical Asymptote
(ii) Horizontal Asymptote
(iii) Oblique Asymptote.

(i) **Vertical Asymptotes** A line $x = a$ is said to be a vertical asymptotes of the graph $y = f(x)$ if

$$\lim_{x \to a} f(x) = \pm \infty$$

Suppose $f(x) = \frac{1}{x - 2}$

Then the vertical asymptote is $x - 2 = 0$

$\Rightarrow \quad x = 2$

Also, if $f(x) = \frac{1}{x^2 - 9}$, then its vertical asymptotes are $x^2 - 9 = 0 \Rightarrow x = \pm 3$

(ii) **Horizontal Asymptotes** A line $y = b$ is said to be a horizontal asymptote of the graph $y = f(x)$ if

$$\lim_{x \to \infty} f(x) = b = \lim_{x \to -\infty} f(x)$$

Suppose $f(x) = \frac{1}{x - 1}$

Then the horizontal asymptote is

$$y = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{1}{1 - x}\right) = 0$$

Also, let $f(x) = \frac{x - 2}{x + 2}$

Then its horizontal asymptote is

$$y = \lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{x + 2}{x - 2}\right) = 1$$

(iii) **Oblique Asymptotes**

A line $y = mx + c$ is said to be an oblique asymptote of the graph $y = f(x)$ if

$$\lim_{x \to \pm\infty} \left(\frac{y}{x}\right) = \lim_{x \to \pm\infty} \left(\frac{f(x)}{x}\right) = m$$

and $\lim_{x \to \pm\infty} (y - mx) = \lim_{x \to \pm\infty} (f(x) - mx) = c.$

Suppose $f(x) = x + \frac{1}{x}$

Now $\lim_{x \to \infty} \left(\frac{y}{x}\right) = \lim_{x \to \infty} \left(\frac{f(x)}{x}\right) = \lim_{x \to \infty} \left(\frac{x + \frac{1}{x}}{x}\right) = 1$

and $\lim_{x \to \infty} (y - x) = \lim_{x \to \infty} (y - x) = \lim_{x \to \infty} \left(x + \frac{1}{x} - x\right)$
$= 0$

$\Rightarrow \quad y = x$ is an oblique asymptote

Note: 1. Let $f(x) = \frac{1}{x - 2}$

Vertical Asymptote: $Dr = 0$

$\Rightarrow \quad x - 2 = 0$

$\Rightarrow \quad x = 2$

Horizontal Asymptote: $y = \lim_{x \to \infty} f(x)$

$\Rightarrow \quad y = \lim_{x \to \infty} \left(\frac{1}{x - 2}\right) = 0$

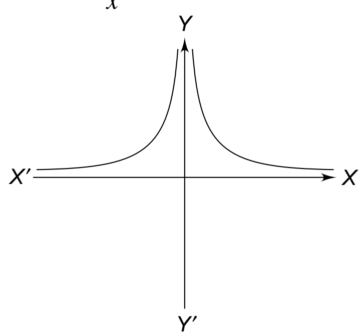
Then $D_f = R - \text{V.A} = R - \{2\}$

and $R_f = R - \text{H.A} = R - \{0\}$.

(ii) Even Power Reciprocal Function

A function $f: R \to R$ is defined as

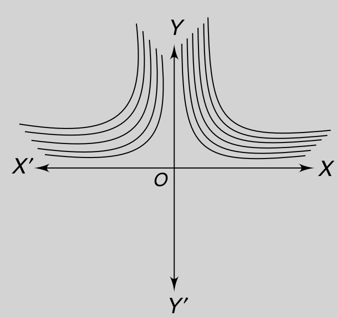
$$f(x) = \frac{1}{x^2}$$



$$D_f = R - \{0\} \text{ and } R_f = R^+$$

Note: The graph of $f(x) = \frac{1}{x^{2n}}, n \in N - \{1\}$

i.e. $\quad f(x) = \frac{1}{x^4}, \frac{1}{x^6}, \frac{1}{x^8}, \frac{1}{x^{10}}, ...$



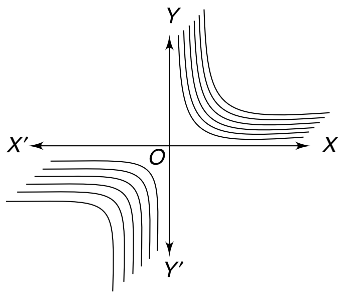
(iii) Odd Power Reciprocal Functions

A function $f: R \to R$ is defined as

$$f(x) = \frac{1}{x^{2n+1}},\ n \in N$$

i.e. $$f(x) = \frac{1}{x^3}, \frac{1}{x^5}, \frac{1}{x^7}, \frac{1}{x^9}, \ldots$$

$$D_f = R - \{0\} = R_f$$



(C) Irrational Functions

The algebraic function containing one or more terms having non-integral rational powers of x are called irrational functions.

A function $f: R \to R$ is defined as

$$f(x) = (g(x))^{\frac{2p+1}{m}},\ p,\ m, \in N$$

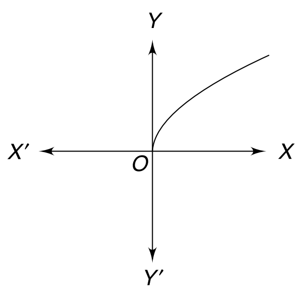
It is undefined for $g(x) < 0$

More over these functions are also not defined when denominator is zero.

(i) **Square root Function**

A function $f: R \to R$ is defined as

$$f(x) = \sqrt{x}$$

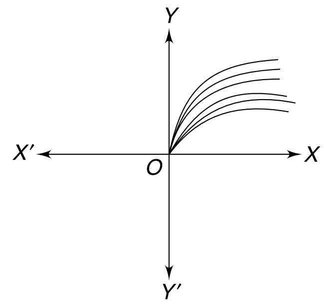


$$D_f = [0, \infty) \text{ and } R_f = R_f = [0, \infty)$$

(ii) **Even root Function**

A function $f: R \to R$ is defined as

$$f(x) = \sqrt[2n]{x},\ n \in N$$



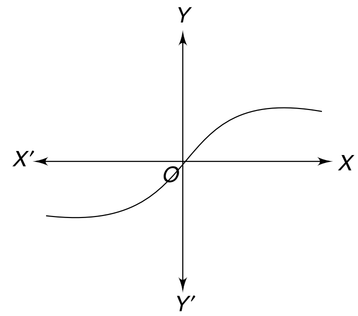
i.e. $$f(x) = x^{1/2}, x^{1/4}, x^{1/6}, x^{1/8}, \ldots$$

$$D_f = [0, \infty) \text{ and } R_f = [0, \infty)$$

(iii) **Cube root Function**

A function $f: R \to R$ is defined as

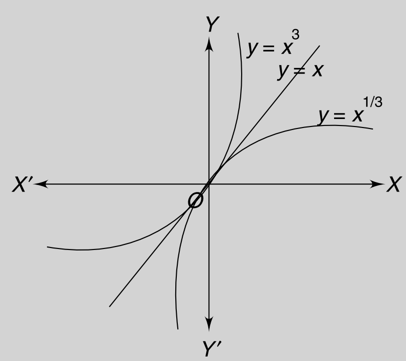
$$f(x) = \sqrt[3]{x}$$



$$D_f = R = R_f$$

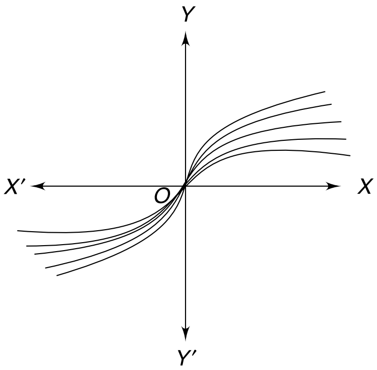
Note:

1. The graph of $y = f(x) = \sqrt[3]{x}$ is the image of the graph of $y = f(x) = x^3$ with respect to the line $y = x$.



Note: The graph of $y = f(x) = \sqrt[2n+1]{x},\ n \in N$

i.e $$y = x^{1/3}, x^{1/5}, x^{1/7}, \ldots$$



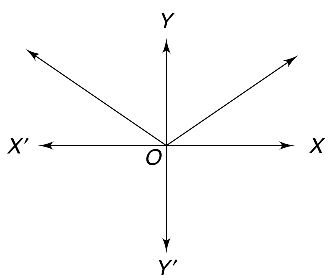
$D_f = R = R_f$

(D) Piece-wise Defined Functions

(i) **Modulus Function**

A function $f: R \to R$ is defined as

$$f(x) = |x| = \begin{cases} x & : \ x \geq 0 \\ -x & : \ x < 0 \end{cases}$$



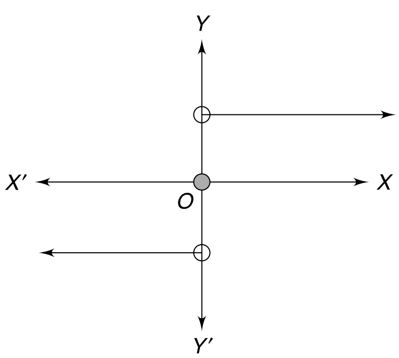
$D_f = R$ and $R_f = [0, \infty)$

(ii) **Signum Function (Sign function/sgn function)**

A function $f: R \to R$ is defined as $f(x) = \text{sgn}(x)$

$$= \begin{cases} \dfrac{|x|}{x} & : \ x \neq 0 \\ 0 & : \ x = 0 \end{cases} \begin{cases} \dfrac{x}{|x|} & : \ x \neq 0 \\ 0 & : \ x = 0 \end{cases}$$

$$= \begin{cases} 1 & : x > 0 \\ 0 & : x = 0. \\ -1 & : x < 0 \end{cases}$$



$D_f = R$ and $R_f = \{-1, 0, 1\}$

(iii) **Greatest Integer Function (G.I.F)**

A function $f: R \to R$ is defined as

$f(x) = [x] \leq x$

Greatest integer of x means, we shall consider of all those integers which are less than or equal to x.

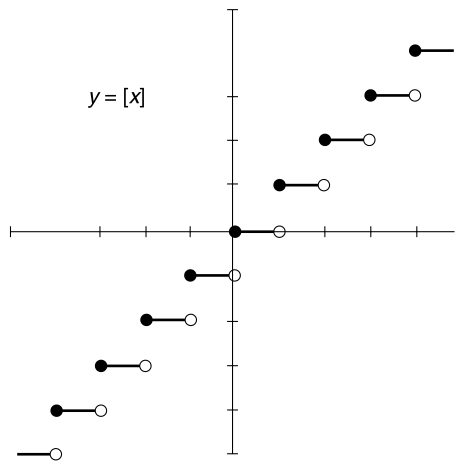
i.e. any Real Number = Integral + Fractional Part

$\Rightarrow$ Greatest integers of x = Integral part of x.

Thus, $[2.1] = 2$, $[3.5] = 3$, $[0.8] = 0$, $[1.9] = 1$, $[-0.7] = -1$, $[-1.7] = -2$, $[-2.8] = -3$ etc.

Therefore, $[x] = n$, if $n \leq x < n + 1$.

Thus $$y = f(x) = [x] = \begin{cases} -2 & : \ -2 \leq x < -1 \\ -1 & : \ -1 \leq x < 0 \\ 0 & : \ 0 \leq x < 1 \\ 1 & : \ 1 \leq x < 2 \\ 2 & : \ 2 \leq x < 3 \end{cases}$$



$D_f = R$ and $R_f = I$

Properties of Greatest Integer Function

(i) $[x] = x$, if $x \in I$

(ii) $[-x] = -1 - [x]$, if $x \notin I$

(iii) $[x] + [-x] = \begin{cases} 0 & : \ x \in I \\ -1 & : \ x \notin I \end{cases}$

(iv) $[x + m] = [x] + m$, $m \in I$

(v) $[x + y] = \begin{cases} x + y & : \ x, y \in I \\ [x] + [y] & : \ 0 \leq f_x + f_y < 1 \\ [x] + [y] + 1 & : \ 1 \leq f_x + f_y < 2 \end{cases}$

(vi) If $[x] = n$ if $n \leq x < + 1$

(vii) $[x] \geq n \Rightarrow x \geq n$, $n \in I$

(viii) $[x] > n \Rightarrow x \geq n + 1$, $n \in I$

(ix) $[x] \leq n \Rightarrow x < n + 1$, $n \in I$

(x) $[x] < n \Rightarrow x < n,\ n \in I$

(xi) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx],\ n \in N$

(xii) $\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \left[\frac{n+8}{16}\right] + \ldots = n,\ n \in N$

(iv) Least Integer Function

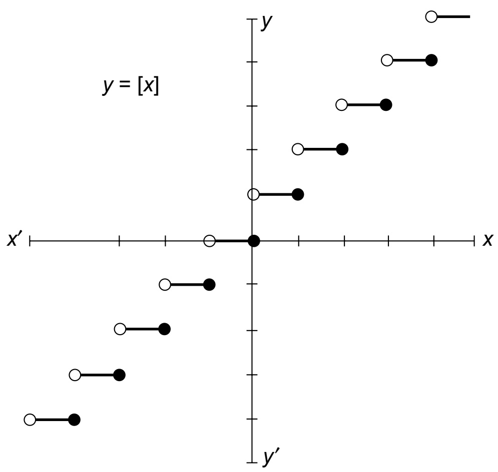
A function $f: R \to R$ is defined as $f(x) = (x) \geq x$

The least integer of x means, we shal considere of all those integers, which are more than and equal to x.

For examples, (2.5) = 3, (3.1) = 4, (1.7) = 2, (0.5) = 1 (–1.4) = –1 etc.

$$y = f(x) = \begin{cases} -2 & : -3 < x \leq -2 \\ -1 & : -2 < x \leq -1 \\ 0 & : -1 < x \leq 0 \\ 1 & : 0 < x \leq 1 \\ 2 & : 1 < x \leq 2 \\ 3 & : 2 < x \leq 3 \end{cases}$$

Thus, $y = (x) = n,\ n < x \leq n + 1$



$D_f = R$ and $R_f = I$

Properties of Least Integer Function

(i) $(x) = x,\ x \in I$

(ii) $(x) = [x],\ x \in I$

(iii) $(x) = [x] + 1,\ x \notin I$

In fact, $(x) = -[-x]$

(iv) $(x + n) = (x) + n,\ n \in I$

(v) $(-x) = -x,\ x \in I$

(vi) $(-x) = 1 - (x),\ x \notin I$

(v) Fractional Part Function

A function $f: R \to R$ is defined as

$$f(x) = \{x\} = x - [x]$$

For examples,

(i) If $x = 3.45$, $[x] = 3$

Thus, $\{x\} = x - [x] = 3.45 - 3 = 0.45$

(ii) If $x = 5$, then $[x] = 5$

Thus, $\{x\} = x - [x] = 5 - 5 = 0$

(iii) If $x = -2.75$, $[x] = -3$

Thus, $\{x\} = x - [x] = -2.75 + 3 = 0.25$

As we know that, $x - 1 < [x] \leq x$

$\Rightarrow \quad -x \leq -[x] < 1 - x$

$\Rightarrow \quad -x + x \leq x - [x] < 1 - x + x$

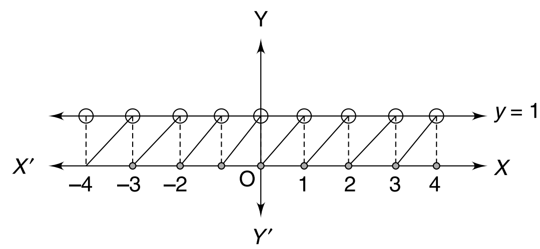
$\Rightarrow \quad 0 \leq x - [x] < 1.$

$\Rightarrow \quad 0 \leq \{x\} < 1.$

Though '0' is not a function, but fractional part function evaluates to zero for integral values. We should keep this exception in mind, while working with fractional part function.

$$y = f(x) = \begin{cases} x+3 & : -3 \leq x < -2 \\ x+2 & : -2 \leq x < -1 \\ x+1 & : -1 \leq x < 0 \\ x & : 0 \leq x < 1 \\ x-1 & : 1 \leq x < 2 \\ x-2 & : 2 \leq x < 3 \end{cases}$$

Thus, $\quad y = \{x\} = x - n,\ n \leq x < n + 1$



$D_f = R$ and $R_f = [0, 1)$.

Properties of Fractional part Function

(i) $0 \leq \{x\} < 1$

(ii) $\{-x\} = 1 - \{x\},\ x \notin I$

(iii) $\{-x\} = 0,\ x \in I$

(iv) $\{x + m\} = \{x\},\ m \in I$

(v) If $\{x\} = f,\ 0 < f < 1$

then $x = n + f$, when $n \in I$

For examples, if $\{x\} = \frac{1}{2}$

then $x = n + \frac{1}{2},\ n \in I$

(vi) If $\{x\} < f,\ 0 < f < 1$

then $n \leq x < n + f,\ n \in I$

For examples, if $\{x\} < \frac{1}{3}$

then $n \leq x < n + \frac{1}{3},\ n \in I$

(vii) If $\{x\} > f,\ 0 < f < 1$
then $n + f < x < n + 1,\ n \in I$
For examples, if $\{x\} > \frac{1}{4}$
then $n + \frac{1}{4} < x < n + 1,\ n \in I$

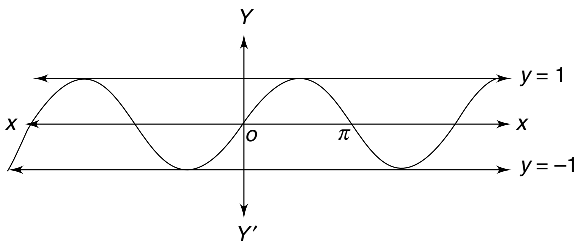
4.2 Transcendental Functions

All functions which are not algebraic are called transcendental functions.

(i) Trigonometric Functions

(a) **Sine function:** *A* function $f: R \to R$ is defined as $f(x) = \sin x$

Graph of $f(x) = \sin x$

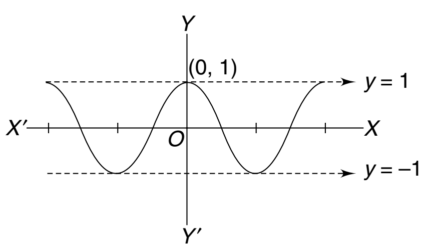


Characteristics of sine function

1. $D_f = R$
2. $R_f = [-1,1]$
3. It is an odd function.
4. It is a periodic function.
5. It is non-monotonic function.
6. If $\sin x = 1 \Rightarrow x = (4n + 1)\ \frac{\pi}{4},\ n \in I$
7. $\sin x = -1 \Rightarrow x = (4n - 1)\ \frac{\pi}{4},\ n \in I$
8. $\sin x = 0 \Rightarrow x = n\pi,\ n \in I$
9. If $\sin x > 0 \Rightarrow x \in (2n\pi, (2n + 1)\pi),\ n \in I$
10. If $\sin x < 0 \Rightarrow x \in ((2n - 1)\pi, 2n\pi),\ n \in I.$
11. If $x > y \Rightarrow \sin x > \sin y,\ \forall\ x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(b) **Cosine function:** *A* function $f: R \to R$ is defined as $f(x) = \cos x$

Graph of $f(x) = \cos x$

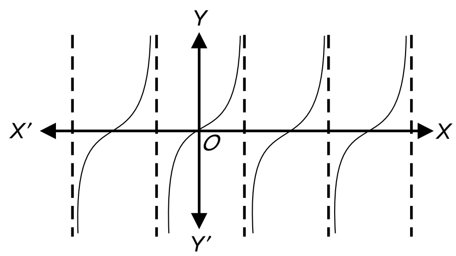


Characteristics of cosine function

1. $D_f = R$
2. $R_f = [-1, 1]$
3. It is an even function.
4. It is a periodic function.
5. It is non-monotonic function.
6. If $\cos x = 1 \Rightarrow x = 2n\pi,\ n \in I$
7. $\cos x = -1 \Rightarrow x = (2n + 1)\ \pi,\ n \in I$
8. $\cos x = 0 \Rightarrow x = (2n + 1)\ \frac{\pi}{2},\ n \in I$
9. If $\cos x > 0$
   $\Rightarrow \quad x \in \left((2n - 1)\frac{\pi}{2}, (2n + 1)\frac{\pi}{2}\right),\ n \in I$
10. If $\cos x < 0$
    $\Rightarrow \quad x \in \left((2n + 1)\frac{\pi}{2}, (2n + 3)\frac{\pi}{2}\right),\ n \in I$
11. If $x > y \Rightarrow \cos x < \cos y,\ \forall\ x, y \in (0, \pi)$

(c) **Tangent function:** A function $f: R \to R$ is defined as $f(x) = \tan x$.

Graph of $f(x) = \tan x$

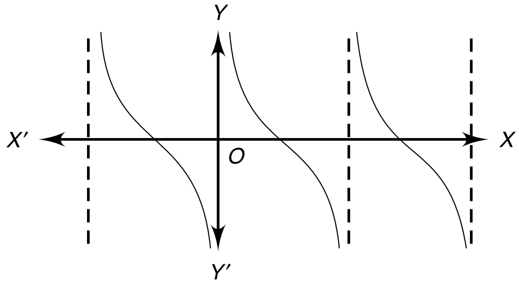


Characteristics of tangent function

1. $D_f = R - (2n + 1)\frac{\pi}{2},\ n \in I$
2. $R_f = R$
3. It is an odd function.
4. It is a periodic function.
5. It is monotonic function.
6. If $\tan x = 1 \Rightarrow x = (4n + 1)\frac{\pi}{4},\ n \in I$
7. If $\tan x = -1 \Rightarrow x = (4n - 1)\frac{\pi}{4},\ n \in I$
8. If $\tan x = 0 \Rightarrow x = n\pi,\ n \in I$
9. If $\tan x > 0$
   $\Rightarrow \quad x \in \left(n\pi, (2n + 1)\frac{\pi}{2}\right),\ n \in I$
10. If $\tan x < 0$
    $\Rightarrow \quad x \in \left((2n - 1)\frac{\pi}{2}, n\pi\right),\ n \in I$
11. If $x > y \Rightarrow \tan x > \tan y\ \forall\ x, y \in R - n\pi,\ n \in I$

(d) **Co-tangent function:** *A* function $f: R \to R$ is defined as $f(x) = \cot x$

Graph of $f(x)$ = cot x

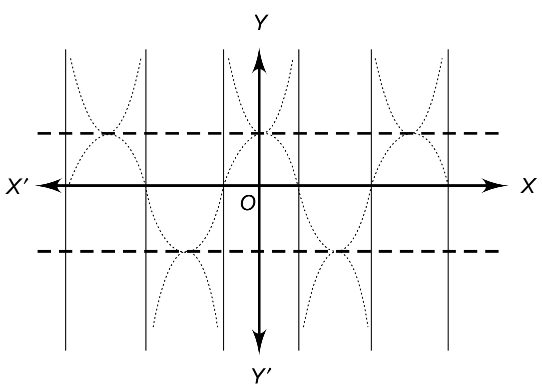


Characteristics of co-tangent function

1. $D_f = R - n\pi,\ n \in I$
2. $R_f = R$
3. It is an odd function.
4. It is a periodic function.
5. It is monotonic function.
6. If cot $x = 1 \Rightarrow x = (4n + 1)\dfrac{\pi}{4},\ n \in I$
7. If cot $x = -1 \Rightarrow x = (4n - 1)\dfrac{\pi}{4},\ n \in I$
8. If cot $x = 0 \Rightarrow x = (2n + 1)\ \dfrac{\pi}{2},\ n \in I$
9. If cot $x > 0$
   $\Rightarrow \quad x \in \left(n\pi,\ (2n + 1)\dfrac{\pi}{2}\right),\ n \in I$
10. If cot $x < 0$
    $\Rightarrow \quad x \in \left((2n - 1)\dfrac{\pi}{2},\ n\pi\right),\ n \in I$

(e) Co-secant function: *A* function $f: R \to R$ is defined as $f(x)$ = cosec x

Graph of $f(x)$ = cosec x

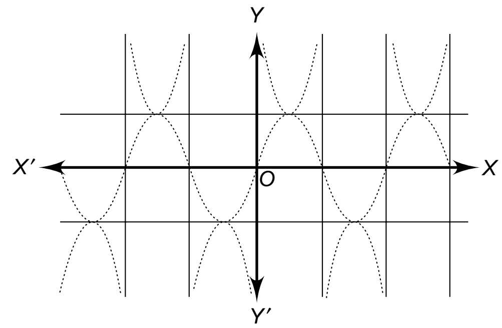


Characteristics of co-secant function

1. $D_f = R - n\pi,\ n \in I$
2. $R_f = (-\infty, -1] \cup [1, \infty)$
3. It is an odd function.
4. It is a periodic function.
5. It is non-monotonic function.
6. If cosec $x = 1 \Rightarrow x = (4n + 1)\dfrac{\pi}{4},\ n \in I$
7. If cosec $x = -1 \Rightarrow x = (4n - 1)\dfrac{\pi}{4},\ n \in I$
8. cosec x can never be zero.
9. If cosec $x > 0 \Rightarrow x \in (2n\pi,\ (2n + 1)\pi),\ n \in I$
10. If cosec $x < 0 \Rightarrow x \in ((2n - 1)\pi,\ 2n\pi),\ n \in I.$

(f) **Secant function:** *A* function $f: R \to R$ is defined as $f(x)$ = sec x

Graph of $f(x)$ = sec x



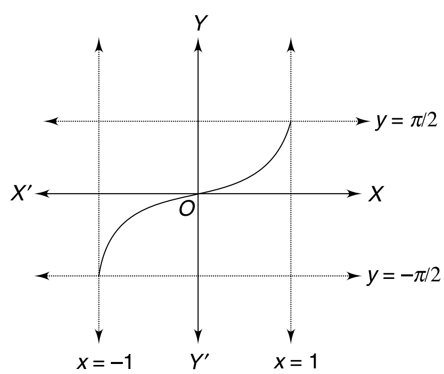
Characteristics of secant function

1. $D_f = R - (2n + 1)\ \dfrac{\pi}{2},\ n \in I$
2. $R_f = (-\infty, -1] \cup [1, \infty)$
3. It is an even function.
4. It is a periodic function.
5. It is non-monotonic function.
6. If sec $x = 1 \Rightarrow = 2n\pi,\ n \in I$
7. If sec $x = -1 \Rightarrow x = (2n + 1)\ \pi,\ n \in I$
8. sec x can never be zero.
9. If sec $x > 0$
   $\Rightarrow \quad x \in \left((4n - 1)\dfrac{\pi}{2},\ (4n + 1)\dfrac{\pi}{2}\right),\ n \in I$
10. If sec $x < 0$
    $\Rightarrow \quad x \in \left((4n + 1)\dfrac{\pi}{2},\ (4n + 3)\dfrac{\pi}{2}\right),\ n \in I$

(ii) **Inverse Trigonometric Functions**

(a) $\sin^{-1}x$: A function $f: [-1, 1] \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ is defined as $f(x) = \sin^{-1}x$ = arc sin x

Graph of $f(x) = \sin^{-1} x$.

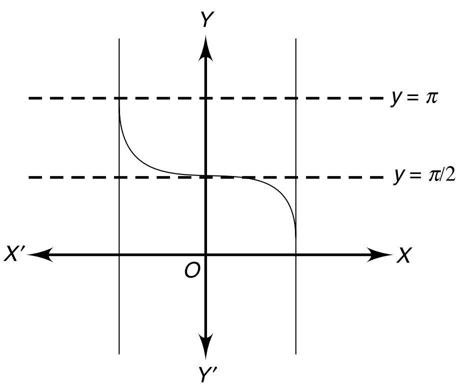


Characteristics of arc sine function

1. $D_f = [-1, 1]$
2. $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
3. It is not a periodic function.
4. It is an odd function.
   since, $\sin^{-1}(-x) = -\sin^{-1}x$
5. It is strictly increasing function.
6. It is one-one function.
7. For $0 < x < \frac{\pi}{2}$, $\sin x < x < \sin^{-1}$.

(b) $\cos^{-1}x$: A function $f: [-1, 1] \to [0, \pi]$ is defined as $f(x) = \cos^{-1}x = \text{arc}\cos x$

Graph of $f(x) = \cos^{-1}x$

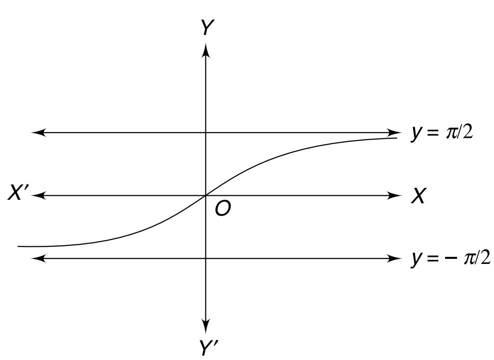


Characteristics of arc cosine function

1. $D_f = [-1, 1]$
2. $[0, \pi]$
3. It is not a periodic function.
4. It is neither even nor odd function
   since, $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$
5. It is strictly decreasing function.
6. It is one-one function.
7. For $0 < x < \frac{\pi}{2}$, $\cos^{-1}x < x < \cos x$

(c) $\tan^{-1}x$: A function $f: R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is defined as $f(x) = \tan^{-1} x$.

Graph of $f(x) = \tan^{-1}x$

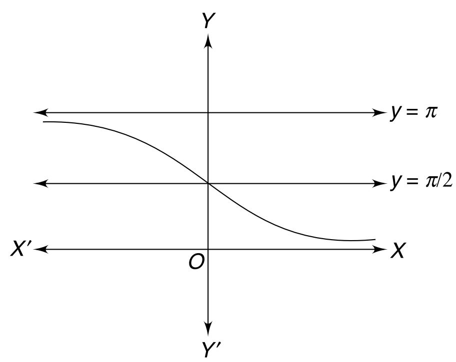


Characteristics of arc tangent function

1. $D_f = R$
2. $R_f = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
3. It is not a periodic function.
4. It is an odd function since, $\tan^{-1}(-x) = -\tan^{-1}x$
5. It is strictly increasing function.
6. It is one-one function
7. For $0 < x < \frac{\pi}{2}$, $\tan^{-1}x < x < \tan x$.

(d) $\cot^{-1} x$: A function $f: R \to (0, \pi)$ is defined as $f(x) = \cot^{-1}x$.

Graph of $f(x) = \cot^{-1}x$

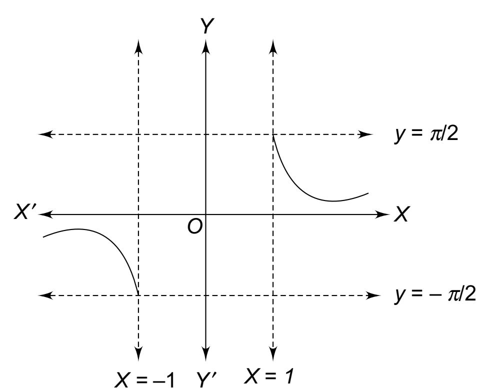


Characteristics of arc co-tangent function

1. $D_f = R$
2. $R_f = (0, \pi)$
3. It is not a periodic function.
4. It is neither even nor odd function
   since, $\cot^{-1}(-x) = \pi - \cot^{-1} x$
5. It is strictly decreasing function.
6. It is one-one function
7. For $0 < x < \frac{\pi}{2}$, $\cot x < x < \cot^{-1}x$

(e) cosec^{-1}: A function $f: (-\infty, -1] \cup [1, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is defined as $f(x) = \text{cosec}^{-1}x$.

Graph of $f(x) = \text{cosec}^{-1}x$.

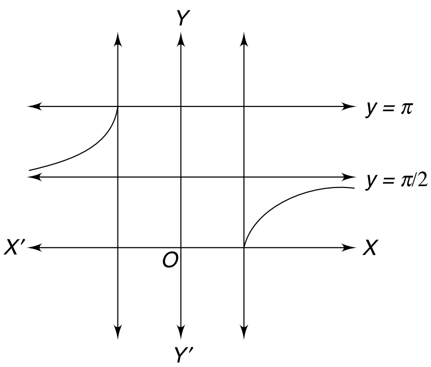


Characteristics of arc co-secant function

1. $D_f = (-\infty, -1] \cup [1, \infty)$
2. $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
3. It is an odd function, since
   $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$
4. It is non-periodic function.
5. It is one-one function.
6. It is strictly decreasing function with respect to its domain.
7. For $0 < x < \frac{\pi}{2}$
   $\text{cosec}^{-1}x < \text{cosec } x$

(v) $\sec^{-1}x$: A function $f: (-\infty, -1] \cup [1, \infty) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$ is defined as $f(x) = \sec^{-1} x$.

Graph of $f(x) = \sec^{-1} x$.



Characteristics of arc secant function

1. $D_f = (-\infty, -1] \cup [1, \infty)$
2. $R_f = [0, \pi] - \left\{\frac{\pi}{2}\right\}$
3. It is neither an even function nor an odd function, since $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$
4. It is non-periodic function.
5. It is one-one function.
6. It is strictly decreasing function with respect to its domain.
7. For $0 < x < \frac{\pi}{2}$
   $\sec^{-1}x < x < \sec x$.

(ii) The maximum and minimum values of

$f(x) = a \cos x + b \sin x + c$

We have $f(x) = a \cos x + b \sin x + c$

Let $\quad a = r \sin \theta$ and $b = r \cos \theta$

Then $\quad r = \sqrt{a^2 + b^2}$ and $\tan(\theta) = \frac{a}{b}$

Now, $\quad f(x) = a \cos x + b \sin x + c$

$= r (\sin \theta \cos x + \cos \theta \sin x)$

$= r = \sin (\theta + x)$

As we know that, $-1 \le \sin (\theta + x) \le 1$

$\Rightarrow \quad -r + c \le r \sin (\theta + x) + c \le r + c$

$\Rightarrow \quad -r + c \le f(x) \le r + c$

$-\sqrt{a^2 + b^2} + c \le f(x) \le \sqrt{a^2 + b^2} + c$

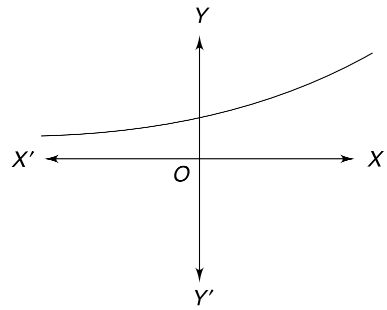
Thus, the maximum value of $f(x)$ is $\sqrt{a^2 + b^2} + c$ and the minimum values of $f(x)$ is $-\sqrt{a^2 + b^2} + c$

(iii) Exponential Function

A function $f: R \to R$ is defined as

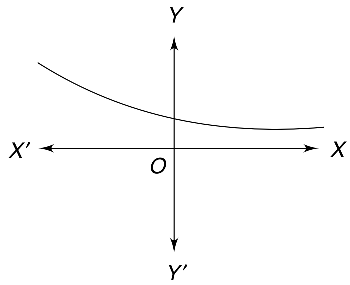
$f(x) = a^x, a \ne 1, a > 0, x \in \text{R}$

Case I: When $a > 1$



$D_f = R$ and $R_f = R^+$

Case II: When $0 < a < 1$

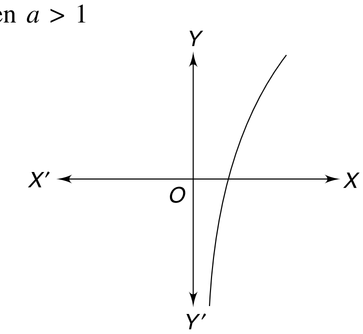


$D_f = R$ and $R_f = R^+$

(iv) Logarithmic Function

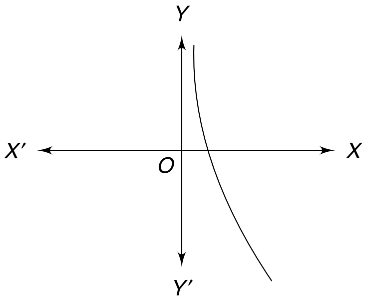
A function $f: R \to R$ is defined as $f(x) = \log_a x$, $a \ne 1$, $a > 0$, $x > 0$

Case I: When $a > 1$



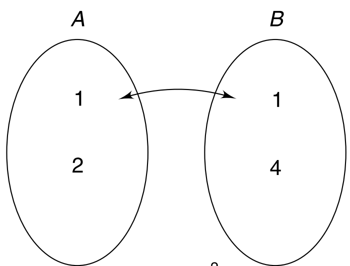
$D_f = R^+$ and $R_f = R$

Case II: When $0 < a < 1$

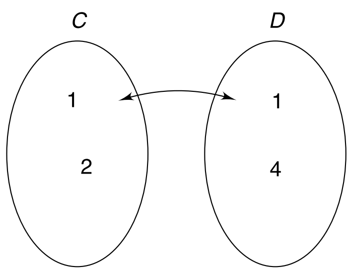


$D_f = R^+$ and $R_f = R$

(v) Equal Functions



$f(x) = x^2$



$g(x) = 3x - 2$

Two functions f and g are said to be equal if

(i) $D_f = D_g$

(ii) $R_f = R_g$

(iii) $f(x) = g(x), \forall x \in D$

5. Classification of Functions with Respect to its Images

There are 5 types of functions w.r.t its images such as

(i) One-One Function

(ii) Many-One Function

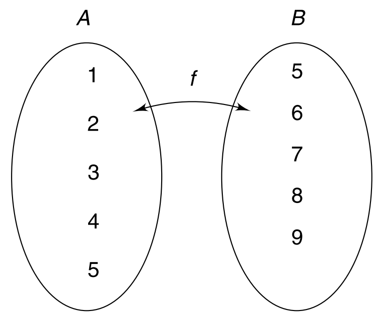
(iii) Onto Function

(iv) Into Function

(v) One-one-onto Function

5.1 One-One function

(a) **Definition:** If different elements of the first set provide us different images in the second set, then it is known as one-one function or injective function.



Let $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$ and $f(x) = x + 4$.

Thus, $f(1) = 5, f(2) = 7, ..., f(4) = 8$. Clearly f is a one-one function.

(b) **Checking algebraically:**

(i) If $x \neq y \Rightarrow f(x) \neq f(y)$, then it is one-one function.

(ii) If $x \neq y \Rightarrow f(x) = f(y)$, then it is not a one-one function.

(c) **Checking geometrically:** Draw a system of lines parallel to x-axis on the given curve. If those lines intersects the curve in only one point, then it is one-one function otherwise many-one function.

(d) **Checking by Calculus:** If a function is either strictly increasing or strictly decreasing function, then it is one-one function.

i.e. either $f'(x) > 0$ or $f'(x) < 0$, then it is one-one function.

Example-1. Let $y = f(x) = 3x + 4$

Then $f'(x) = 3 > 0$

$\Rightarrow f(x)$ is strictly increasing function.

$\Rightarrow f(x)$ is one-one function.

Example-2. Let $f(x) = \tan^{-1}(3 \log (x) + 5)$

Then $f'(x) = \dfrac{1}{1 + (3 \log x + 5)^2} \times \dfrac{1}{x} > 0$ for all positive x

$\Rightarrow f(x)$ is strictly increasing function.

$\Rightarrow f(x)$ is one-one function.

Example-3. Let $f(x) = 2 \tan x + 3 \sin x + 4 \cos x + 10$

Then $f'(x) = 2 \sec 2x + 3 \cos x - 4 \sin x$.

$\Rightarrow f'(x) > 0$ for all x in R.

$\Rightarrow f(x)$ is strictly increasing function.

$\Rightarrow f(x)$ is one-one function.

Example-4. Let $f(x) = \dfrac{x}{1 + |x|}$

Case I: when $x \geq 0$

Then $f(x) = \dfrac{x}{1 + x}$

$\Rightarrow \quad f(x) = \dfrac{(1 + x) - 1}{1 + x} = 1 - \dfrac{1}{1 + x}$

$\Rightarrow \qquad f'(x) = \dfrac{1}{(1+x)^2} > 0$

Case II: when $x < 0$

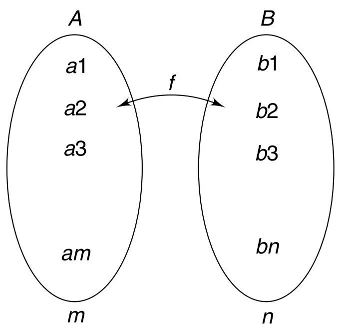
Then $f(x) = \dfrac{x}{1-x}$

$$f(x) = -\frac{x-1+1}{x-1} = -1 - \frac{1}{x-1}$$

$$f'(x) = \frac{1}{(x-1)^2} > 0 \text{ for all } x < 0$$

Thus f is strictly increasing function. Hence, f is one-one function.

(e) **Number of one-one function between two sets A and B.**



If A and B having m and n elements respectively, then the total number of one-one functions between two sets A and B is $= \begin{cases} 0 & : \ n < m \\ {}^{n}P_{m} & : \ n \geq m \end{cases}$

5.2 Many-One Function

If a function is not one-one function, then it is many-one function.

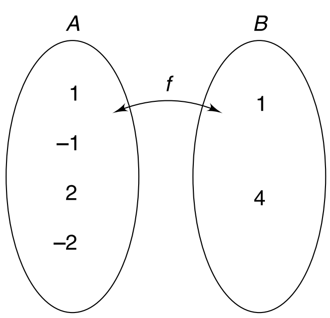
Number of many one function

Number of many one function = Number of total function – Number of one-one function.

5.3 Onto Function

(a) **Definition:** If the range of a function is equal to its co-domain, then it is called Onto function or surjective function

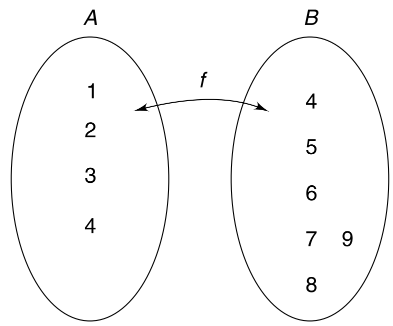
i.e. $\qquad R_f$ = Co-domain.



(b) **Number of onto function between two sets A and B:** If two set A and B having m and n elements respectively, where $m \geq n$, then the number of onto function between two sets A and B = Number of distribution of m balls into n boxes where no box is remain empty.

5.4 Into Function

If a function is not onto function, then it is into function. In another way we can say, if the range of a function is a proper subset of a co-domain then it is into function.



The number of into functions

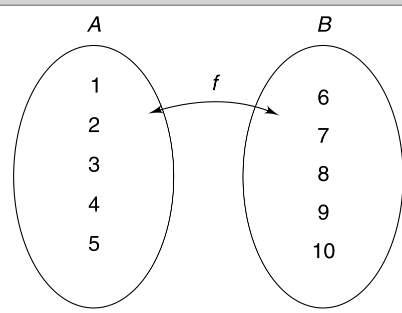
= The total number of functions

– The number of onto functions.

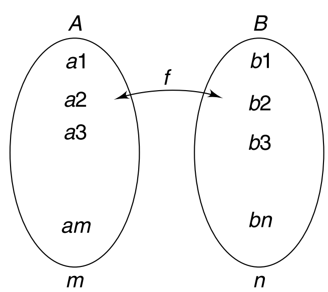
5.5 One-one onto Function

(a) **Definition:** If a function is one-one as well as onto function, then it is one-one-onto function or bijective function.

> ***Note:*** One-one-onto function exists only when the number of elements of both the sets are same.



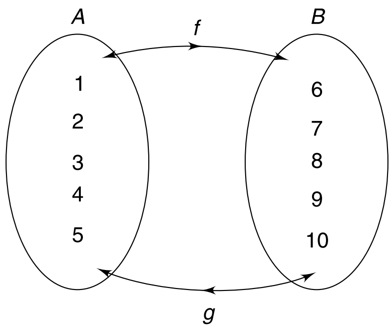
(b) **Number of one-one onto functons between two sets A and B.**



Let two sets A and B having m and n elements respectiovely.

Then the number of one-one onto functions between two sets A and B = $\begin{cases} 0 & : m \neq n \\ {}^nP_m & : m = n \end{cases}$

6. Inverse Function



Let $f: X \to Y$ be a bijective function. If we can make a function g from Y to X, then we shall say that g is the inverse of f.

i.e. $g = f^{-1} \neq \frac{1}{f}$.

Thus, $f^{-1}(f(x)) = x$

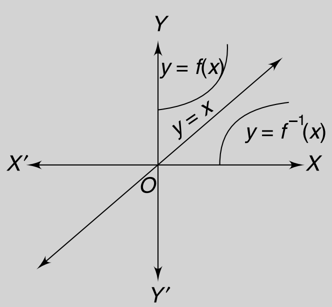
Note:

(i) The inverse of a function exists only when the function f is bijective.

(ii) If the inverse of a function is exists, then it is called an invertible function.

(iii) The inverse of a bijective function is unique.

(iv) Geometrically $f^{-1}(x)$ is the image of $f(x)$ with respect to the line $y = x$.



(v) In another can say that $f^{-1}(x)$ is the symmetrical with respect to the line $y = x$.

(vi) A function $f(x)$ is said to be involution if for all x for which $f(x)$ and $f(f(x))$ are defined such that $f(f(x)) = x$.

(vii) If f is an invertible function, then $(f^{-1}) = f$.

(viii) If $f: A \to B$ be a one one function, then $f^{-1}of = I_A$ and $fof^{-1} = I_B$, where I_A and I_B are the identity functions of the sets A and B respectively.

(ix) Let $f: A \to B$, $g: B \to C$ be two invertible functions, then gof is also invertible with $(gof)^{-1} = (f^{-1}og^{-1})$.

Rule to find out the Inverse of a Function

(i) First, we check the given function is bijective or not.

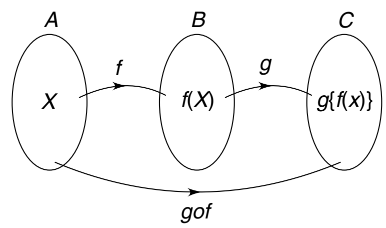
(ii) If the function is bijective, then inverse exists, otherwise not.

(iii) Find x in terms of y and replace y by x, then we get inverse of f. i.e. $f^{-1}(x)$.

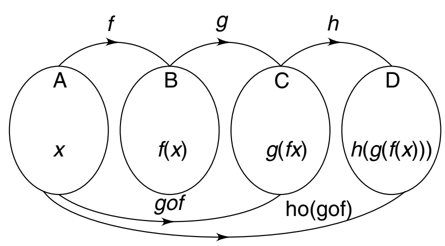
7. Compostion of Functions

Let f and g be two real functions such that $f: A \to B$ and $g: B \to C$.

Here the set B is common to the two functions. Thus we can define a function $h: A \to C$ for which $h(x) = g(f(x))$, where for all $x \in A$, $f(x) \in B$, $g(f(x)) \in C$. Then $h = g_o f$ is the composition of f and g. We read as h as g of f or "g composed with f"



Similarly we can define a function $u: A \to D$ such that $u(x) = (h_o\,(g_o f))(x) = h(g(f(x)))$



Properties of Composition of Functions

(i) Let $f: A \to B$ and $g: B \to C$ be two real functions. Then $g_o f$ is defined only when if $R_f \subseteq D_g$.

(ii) Composition of functions is not commutative i.e. if f and g are two real functions such that $f_o g$ & $g_o f$, then $f_o g \neq g_o f$.

(iii) Composition of functions is associative. i.e. if f, g, h are three real functions such that $(f_o(g_o)_o h))$ are exist, then $(f_o(g_o h)) = ((f_o g)_o h)$

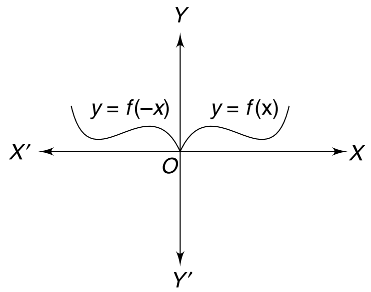
(iv) The composition of two bijections is bijection i.e f and g are two bijections, then $g_o f$ is also bijection

(v) The composition of any function with the identity function is the function itself.
i.e. $f_o I = I_o f = f$

8. Even and Odd Functions

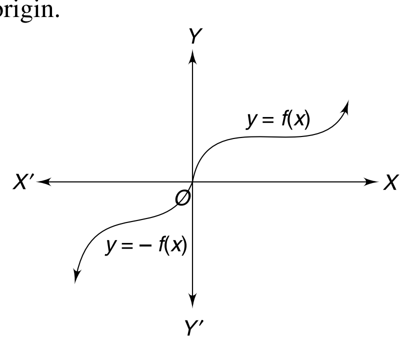
8.1 Even Function

A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x in D_f. Geometrically, an even function is symmetrical about y-axis.



8.2 Odd Function

A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$, $\forall\ x \in D_f$. Geometrically, an odd function is symmetrical about the origin.



9. Algebra of Even and Odd Functions

Let the even function, odd function and neither even nor odd functions are denoted as E, O and $\to$ ~~EO~~

1. Addition
   (i) $E + E = E$
   (ii) $O + O = O$
   (iii) $E + O =$ ~~EO~~
   (iv) $O + E =$ ~~EO~~
2. Negativity
   (i) $-E = E$
   (ii) $-O = O$
3. Difference
   (i) $E - E = E$
   (ii) $O - O = O$
   (iii) $E - O =$ ~~EO~~
   (iv) $O - E =$ ~~EO~~
4. Product
   (i) $E \times E = E$
   (ii) $O \times O = E$
   (iii) $E \times O = O$
   (iv) $O \times E = O$
5. Reciprocality
   (i) $\frac{1}{E} = E$
   (ii) $\frac{1}{O} = O$
6. Quotient
   (i) $\frac{E}{E} = E \times \frac{1}{E} = E \times E = E$
   (ii) $\frac{O}{O} = O \times \frac{1}{O} = O \times O = E$
   (iii) $\frac{O}{E} = O \times \frac{1}{E} = O \times E = O$
   (iv) $\frac{E}{O} = E \times \frac{1}{O} = E \times O = O$
7. Composition
   (i) $E(E) = E$
   (ii) $E(O) = E$
   (iii) $O(E) = E$
   (iv) $O(O) = O$

Note: Composition of functions will be odd only when all are odd.

Properties of even and odd Functions

1. A function which is both even as well as odd is a zero function.
   i.e. $f(x) = 0$, $\forall\ x \in D_f$
2. A non-zero constant function is an even function.
   i.e. $f(x) = c$, $c \neq 0$ is an even function.
3. Odd function always pass through the origin.
4. Even function is many-one function.
5. The derivative of an odd function is even and an even function is odd.
6. The integral of an odd function is even and even function is odd.
7. Inverse of an even function is not defined.
8. Every function can be expressed as a sum of an even and an odd function.

$$\text{Let } f(x) = \frac{1}{2} \times (2f(x))$$
$$= \frac{1}{2} \times ((f(x) + f(-x)) + (f(x) - f(-x)))$$
$$= \frac{1}{2}(f(x) + f(x)) + \frac{1}{2}(f(x) - f(-x))$$
$$= g(x) + h(x), \text{ (say)}$$
$$= E + O$$

To prove $g(-x) = g(x)$ & $h(-x)$

Now, $g(x) = \frac{1}{2}(f(x) + f(-x))$

$\Rightarrow \quad g(-x) = \frac{1}{2}(f(-x) + f(x))$

$\Rightarrow \quad g(-x) = \frac{1}{2}(f(x) + f(-x)) = g(x)$

$\Rightarrow \quad g(x)$ is an even function.

Also, $\quad h(x) = \frac{1}{2}(f(x) - f(-x))$

$\Rightarrow \quad h(-x) = \frac{1}{2}(f(-x) - f(x))$

$\Rightarrow \quad h(-x) = -\frac{1}{2}(f(x) - f(-x)) = -h(x)$

$\Rightarrow \quad h(x)$ is an odd function.

Hence, the result.

10. Even and Odd Extensions of a Function

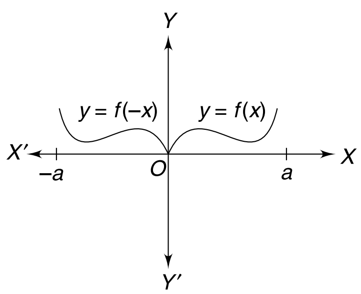
10.1 Even Extension

Let $f(x)$ be defined on $[0, a]$

Then the even extension of f is defined as

$$g(x) = \begin{cases} f(x) & : \ 0 \le x \le a \\ f(-x) & : \ -a \le x < 0 \end{cases}$$

Geometrically, an even extension of a function is the mirror image of the graph of $f(x)$ with respect to y-axis.



Let $f(x)$ be defined on $[a, b]$. Then its even extension can be written as

$$g(x) = \begin{cases} f(x) & : \ a \le x \le b \\ f(-x) & : \ -b \le x < -a \end{cases}$$

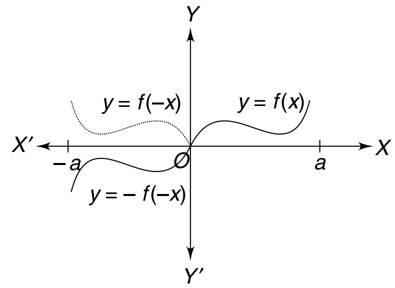
10.2 Odd Extension

Let $f(x)$ be defined on $[0, a]$

Then its odd extension is defined as

$$g(x) = \begin{cases} f(x) & : \ 0 \le x \le a \\ -f(-x) & : \ -a \le x < -0 \end{cases}$$

Geometrically, odd extension of a function is symmetrical about the origin.



Let $f(x)$ be defined on $[a, b]$

Then its odd extension can be written as

$$g(x) = \begin{cases} f(x) & : \ a \le x \le b \\ -f(-x) & : \ -b \le x < -a \end{cases}$$

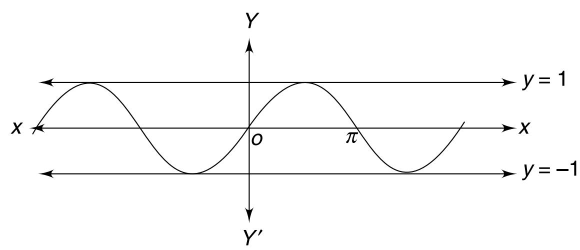
11. Periodic Function

A function $f : R \to R$ is said to a periodic function if there exists a positive real number T such that $f(x + T) = f(x), \forall x \in R$

The least positive value of T is called the fundamental period or simply the period of a function.

Geometrically, if the graph of $y = f(x)$ repeat itself after a fixed interval, then the width of the interval is called the period of the function $y = f(x)$.

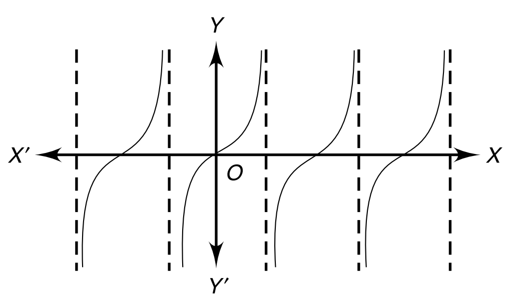
Example-1. Let $y = \sin x$



Since the graph of $y = \sin x$ repeats after a fixed interval 2π

Thus, the period of $\sin x$ is 2π

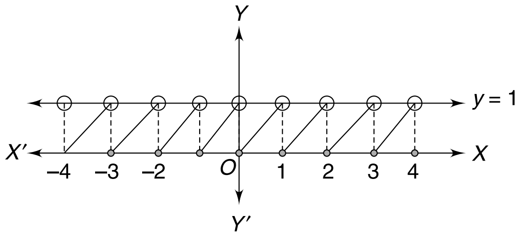
Example-2. Let $y = \tan x$



Since the graph of $y = \tan x$ repeats after a fixed interval π

Thus, the period of $y = \tan x$ is π

Example-3. Let $y = \{x\} = x - [x]$



Since the graph of $y = x - [x]$ repeats after a fixed interval 1.

So the period of $y = x - [x]$ is 1.

Basic Concepts of L.C.M

1. L.C.M of $\left\{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, ...\right\} = \frac{\text{L.C.M. of } \{a, c, e, ...\}}{\text{H.C.F. of } \{b, d, f, ...\}}$
2. L.C.M of rational with rational is possible.
3. L.C.M of irrational with irrational is also possible.
4. But L.C.M of rational with irrational is not possible.
5. Two important irrational number are π, e.

Example-4. The L.C.M of

$$\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\} = \frac{\text{L.C.M. of } \{1, 1, 1\}}{\text{H.C.F. of } \{2, 3, 4\}}$$

$$= \frac{1}{1} = 1$$

Properties of Periodic Functions

1. If $f(x)$ is a periodic function with period T then the period of

(i) $f(a\,x)$

(ii) $f(ax + b)$

(iii) $f(ax + b) + c$

(iv) $df(ax + b) + c$ is $\frac{T}{|a|}$

Example-5. The period of

(i) $\sin 2x$

(ii) $\sin (2x + 3)$

(iii) $\sin (2x + 3) + 5$

(iv) $7 \sin (2x + 3) + 5$ is $\frac{2\pi}{2} = \pi$

2. If $f(x)$ and $g(x)$ are two periodic functions with periods T_1 and T_2 respectively, then the period of

(i) $f(x) + g(x)$

(ii) $f(x) - g(x)$

(iii) $f(x).\,g(x)$

(iv) $\frac{f(x)}{g(x)}$, $g(x) \neq 0$ is the L.C.M of $\{T_1, T_2\}$

Example-6. Find the period of $f(x) = \sin x + \sin 2x$

Sol. Here, the period of $\sin x = 2\pi$ and the period of $\sin 2x = \frac{2\pi}{2} = \pi$

Thus, the period of $f(x)$ = L.C.M of $\{2\pi, \pi\} = 2\pi$

Example-7. Find the period of $f(x) = \sin 2x + \cos 3x$

Sol. Here, the period of $\sin 2x = \frac{2\pi}{2} = \pi$ and

the period of $\cos 3x = \frac{2\pi}{3}$

Thus, the period of $f(x)$ = L.C.M of $\left\{\pi, \frac{2\pi}{3}\right\}$

$$= \frac{\text{L.C.M. of } \{\pi, 2\pi\}}{\text{H.C.F. of } \{1,3\}}$$

$$= \frac{2\pi}{1} = 2\pi$$

3. If $f(x)$ and $g(x)$ both are periodic with periods T_1 and T_2 respectively, and both are even & co-functions and pairwise comparable funtions, then the period of

(i) $f(x) + g(x)$

(ii) $f(x) - g(x)$

(iii) $f(x).g(x)$

(iv) $\frac{f(x)}{g(x)}$, $g(x) \neq 0$ is the L.C. M of $\frac{1}{2} \times \{T_1, T_2\}$

Example-8. Find the period of $f(x) = \sin^4 x + \cos^4 x$

Sol. Here, the period of $\sin^4 x = \pi$ and the period of $\cos^4 x = \pi$

Since both the functions are periodic, even, co-function as well as both are pairwise comparable functions, so the period of $f(x)$ is

$$= \frac{1}{2} \times \text{L.C.M. of } \{\pi, \pi\}$$

$$= \frac{\pi}{2}$$

Example-9. Find the period of $f(x) = \cos(\cos x) + \cos(\sin x)$

Sol. Here, the period of $\cos(\cos x)$ is π and the period of $\cos(\sin x)$ is also π

Since both the functions are periodic, even, co-function as well as both are pairwise comparable functions, so the period of $f(x)$ is

$$= \frac{1}{2} \times \text{L.C.M. of } \{\pi, \pi\}$$

$$= \frac{\pi}{2}$$

4. If $f(x)$ is a periodic functions with period T, then the period of

(i) $\frac{1}{f(x)}$

(ii) $k^{f(x)}$, $k \in R^+ - \{1\}$

(iii) $(f(x))^{1/n}$ is also T.

Example-10. Find the period of $f(x) = \sqrt{\sin x}$

Sol. Since the period of sin x is 2π, so the period of $f(x)$ is also 2π

Example-11. Find the period of $f(x) = \sqrt[2015]{\cos x}$

Sol. Since the period of cos x is 2π, so the period of $f(x)$ is also 2π

Example-12. Find the period of $f(x) = \frac{1}{\sin x}$

Sol. Since the period of sin x is 2π, so the period of $f(x)$ is also 2π

5. If $f(x)$ is a periodic function with period T and $g(x)$ be a non-periodic function, then $g(f(x))$ is periodic with period T and $f(g(x))$ is non periodic.

Example-13. Find the period of $f(x) = \sin^{-1}(\sin x)$

Sol. Let $h(x) = \sin x$ and $g(x) = \sin^{-1}x$

Thus, the period of $h(x)$ is 2π and $g(x)$ is non-periodic. Therefore, the period of $f(x)$ is $= 2\pi$

> ***Note:*** 1. The function $f(x) = \sin(\sin^{-1}x)$ is non-periodic, whereas sin x is periodic but $\sin^{-1}x$ is non-periodic.

6. If $f(x)$ is a periodic function with period T then $f'(x)$ is also periodic with period T

Example-14. Find the period of $f(x) = \sin x$

Sol. We have $f'(x) = \cos x$

Thus, the period of cos x is 2π, whereas the period of sin x is also 2π

7. Constant function is periodic having no fundamental period.

8. Algebraic function is non periodic.

> ***Note:***
> 1. The functions $\sin^n x$, $\cos^n x$, $\text{cosec}^n x$, $\sec^n x$ are periodic with period 2π when n is odd and π when n is even.
> 2. The functions $\tan^n x$, $\cot^n x$ are periodic with periods π, whatever n may be.
> 3. The functions $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$ $|\text{cosec } x|$, $|\sec x|$ are periodic with period π.
> 4. The functions $|\sin x| + |\cos x|$, $|\tan x| + |\cot x|$, $|\text{cosec } x| + |\sec x|$ are periodic with period $\frac{\pi}{2}$
> 5. The functions $|\sin x| - |\cos x|$, $|\tan x| - |\cot x|$, $|\text{cosec } x| - |\sec x|$ are periodic with period π.
> 6. The functions $\sqrt{\sin x}$, $\sqrt{\cos x}$, $\sqrt{\text{cosec } x}$ and $\sqrt{\sec x}$ are periodic with period 2π.
> 7. The functions $\sin^{-1}(\sin x)$, $\cos^{-1}(\cos x)$ $\sec^{-1}(\sec x)$, $\text{cosec}^{-1}(\text{cosec } x)$ are periodic with period 2π.
> 8. The functions $\tan^{-1}(\tan x)$, $\cot^{-1}(\cot x)$ are periodic with period π.
> 9. The functions $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$ are non-periodic.
> 10. The function $x - [x]$ is periodic with period 1.

12. Functional Equation

It is an equation where the unknown is a function. A functional equation asks for a formula satisfying certain conditions. On solving such type of equation we can get one or more functions as solutions.

Exercises

Level I

(Problems Based on Fundamentals)

Basics Concepts of Real Functions

1. If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x - 1)$.
2. If $f(x) = \frac{x}{x+1}$, prove that $(f(x))^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$
3. If $f(x) = \log\left(\frac{1-x}{1+x}\right)$, show that,
$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$$
4. If $f(x) = x^3 - \frac{1}{x^3}$, show that, $f(x) + f\left(\frac{1}{x}\right) = 0$.
5. If $f(x) = \frac{2x}{1+x^2}$, show that, $f(\tan\theta) = \sin 2\theta$
6. If $f(x) = \log\left(\frac{x}{x-1}\right)$, show that,
$$f(x+1) + f(x) = \log\left(\frac{x+1}{x-1}\right)$$
7. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, show that, $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$
8. If $f(x) = \cos(\log_e x)$, find the value of
$$f(x)f(y) - \frac{1}{2}(f(x) + f(y))$$
9. If $y = f(x) = \frac{ax-b}{bx-a}$, show that $x = f(y)$

10. If for non zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \neq b$, find $f(x)$.

11. If $f(x) = \frac{9^x}{9^x + 3}$, then prove that $f(x) + f(1 - x) = 1$.

12. If $f\left(\frac{2\tan x}{1 + \tan^2 x}\right) = \frac{1}{2}(1 + \cos 2x)(\sec^2 x + 2\tan x)$ then find $f(x)$.

ABC of a Real Functions

13. Which one of the followings are functions? If so then find its Domains and Ranges. If not, then find when it can be a function and also find their Domains and Ranges..
    (i) $y = mx + c$
    (ii) $y = 0$
    (iii) $x = 0$
    (iv) $y = 5$
    (v) $x = 3$
    (vi) $x^2 + y^2 = 9$
    (vii) $y^2 = 4x$
    (viii) $x^2 = 4y$
    (ix) $\frac{x^2}{9} + \frac{y^2}{4} = 1$
    (x) $\frac{x^2}{16} - \frac{y^2}{9} = 1$
    (xi) $x^2 - y^2 = 4$
    (xii) $y = (x - 1)(x - 2)(x - 3)$

Domain of Real Functions

14 Find the domain of each of the following functions.
    (i) $f(x) = \frac{x - 2}{x + 3}$
    (ii) $f(x) = \frac{x + 4}{x - 3}$
    (iii) $f(x) = \frac{1}{x} - 1$
    (iv) $f(x) = \frac{1}{x} + 1$
    (v) $f(x) = \frac{x - 1}{x^2 - 3x + 2}$
    (vi) $f(x) = \frac{x^2 - 5x + 4}{x^2 - 3x + 2}$
    (vii) $f(x) = \frac{x^2 + 4}{x^2 + 2}$
    (viii) $f(x) = \frac{x^2 + 9}{x^2 - 4}$
    (ix) $f(x) = \frac{x^2 + x + 1}{x^2 - x + 1}$
    (x) $f(x) = \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$

15. Find the domain of each of the following functions
    (i) $f(x) = \sqrt{x - 2}$
    (ii) $f(x) = \sqrt{x + 5}$
    (iii) $f(x) = \sqrt{4 - x}$
    (iv) $f(x) = \sqrt{x - 2} + \sqrt{4 - x}$
    (v) $f(x) = \sqrt{x - 3} - \sqrt{7 - x}$
    (vi) $f(x) = \sqrt{\frac{x - 2}{5 - x}}$
    (vii) $f(x) = \sqrt{\frac{x}{x - 1}}$
    (viii) $f(x) = \sqrt{1 - \frac{1}{x}}$
    (ix) $f(x) = \sqrt{\frac{1}{x} - 1}$
    (x) $f(x) = \sqrt{\frac{x - 2}{x}}$

16 Find the domain of each of the following functions
    (i) $f(x) = \sqrt{|x| - 2}$
    (ii) $f(x) = \sqrt{4 - |x|}$
    (iii) $f(x) = \sqrt{|x| - 1} + \sqrt{4 - |x|}$
    (iv) $f(x) = \sqrt{\frac{|x| - 1}{3 - |x|}}$
    (v) $f(x) = \sqrt{|x| - x}$
    (vi) $f(x) = \frac{1}{\sqrt{|x| - x}}$
    (vii) $f(x) = \frac{1}{\sqrt{x - |x|}}$
    (viii) $f(x) = \sqrt{[x] - x}$
    (ix) $f(x) = \frac{1}{\sqrt{[x] - x}}$
    (x) $f(x) = \frac{1}{\sqrt{x - [x]}}$

Algebraical Functions

17. Find the domain of $f(x) = \frac{1}{[x - 2]}$

18. Find the domain of $f(x) = \frac{x + 3}{[x + 1]}$

19. Find the domain of $f(x) = \frac{x^2 + x + 1}{[x] - 4}$

20. Find the domain of $f(x) = \frac{1}{\sqrt{x - [x]}}$

21. Find the domain of $f(x) = \frac{1}{\sqrt{[x] - x}}$

22. Let $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$

    Then find $f\left(\frac{\pi}{2}\right), f(\pi)$

23. Find the exact value of
    $[\log_e 1] + [\log_e 2] + [\log_e 3] + \ldots + [\log_e 66]$
24. Solve for $[x]^2 - 3[x] + 2 = 0$
25. Solve for $[x]^2 - 2[x] - 8 = 0$
26. Find the number of solution of $x^2 - 4 - [x] = 0$
27. Find the value of
    $\left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{100}\right] + \left[\frac{1}{4} + \frac{2}{100}\right] + \ldots + \left[\frac{1}{4} + \frac{99}{100}\right]$
    where [,] = G.I.F.
28. Find the value of
    $[1007] + \left[503\frac{3}{4}\right] + \left[252\frac{1}{8}\right] + \left[126\frac{5}{16}\right] + \ldots$
29. Solve the equation $[\sin x] = 0$
30. If $y = 2[x] + 3 = 3[x - 2] + 5$, then find the value of $[x + y]$
31. Find the doamin of $f(x) = \sqrt{[x] - 2}$
32. Find the domain of $f(x) = \sqrt{4 - [x]}$.
33. Find the domain of $f(x) = \sqrt{[x] - 1} + \sqrt{3 - [x]}$
34. Find the domain of $f(x) = \sqrt{\frac{[x] - 2}{5 - [x]}}$
35. Find the value of
    (i) $\left[\frac{\sin^2 x}{2}\right]$
    (ii) $\left[\sin^2\left(\frac{\pi}{2}\right)x\right]$
    (iii) $\left[\cos^2\left(\frac{\pi}{2}\right)x\right]$
    (iv) $\left[\frac{\cos^2 x}{2014}\right]$.
36. Find the set of values of x satisfying
    $[\sin x] + [\cos x] = 1, \forall x \in [0, 2\pi]$
37. If x satisfies the equation
    $[x + 0.19] + [x + 0.20] + [x + 0.21] + \ldots + [x + 0.22] + \ldots + [x + 0.91] = 542$
    Then find the value of $[100x]$
38. Find the value of
    $\left[\frac{2015}{2}\right] + \left[\frac{2016}{4}\right] + \left[\frac{2018}{8}\right] + \left[\frac{2022}{16}\right] + \ldots$
39. Prove that $[2x] = [x] + \left[x + \frac{1}{2}\right]$
40. Prove that $[3x] = [x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right]$
41. Find the number of integral values of $\left[\frac{2x^2}{x^2 + 1}\right]$
42. Solve for $x: \{x\} + \{\sin x\} = 2$
43. If $\{x\}$ and $[x]$ represent fractional and integral part of x respectively, then find the value of
    $[x] + \sum_{r=1}^{2014}\left(\frac{\{x + r\}}{2014}\right)$.
44. Solve for $x: 4\{x\} = x + [x]$, where $\{x\}$ and $[x]$ are the fractional and integral part of x respectively.
45. Solve for $x: 2x + 3\{x\} = 4[x] - 2$
46. Solve for $x: x^2 - 4x + [x] + 3 = 0$
47. Find the number of values of x satisfying $\{x^2\} + [x^4] = 1$
48. Find the number of solutions of
    $\{x\} + \{\tan \pi x\} = 0$
49. Find the number of integral values of
    $y = \left\{\frac{x^2 + 1}{2}\right\}, 0 \le x \le 2$
50. Find the values of x satisfying the equation
    $\left\{\frac{\cos^2 x - 2}{2}\right\} = \frac{1}{4}$
51. Find the domain of the function
    $f(x) = \sqrt{2\{x\}^2 - 3\{x\} + 1}, x > 0$

Trigonometric Functions

52. Find the max and min values of
    $f(x) = 3\sin x + 4\cos x + 10$
53. Find the range of $f(x) = \sin x + \cos x + 3$
54. Let $A = \sin^4\theta + \cos^4\theta$, then find A
55. Find the max and min values of
    $$f(\theta) = \sin^6\theta + \cos^6\theta$$
56. If $A = \cos^2\theta + \sin^4\theta$ and $B = \cos^4\theta + \sin^2\theta$ such that m_1 = Max of A and m_2 = Min of B then find the value of $m_1^2 + m_2^2 + m_1 m_2$
57. Find the max and min values of
    $f(\theta) = \sin^2(\sin\theta) + \cos^2(\cos\theta)$
58. Find the minimum value of
    $f(\theta) = (3\sin(\theta) - 4\cos(\theta) - 10)$
    $(3\sin(\theta) + 4\cos(\theta) - 10)$

59. Find the range of $A = \sin^{2010}\theta + \cos^{2014}\theta$

60. Find the minimum value of $f(x) = \dfrac{x^2\sin^2 x + 4}{x \sin x}$, where $x \in \left(0, \dfrac{\pi}{2}\right)$

Exponential Functions

61. Find the domain of $f(x) = e^{\frac{x}{2}-1}$
62. Find the domain of $f(x) = |e^x - 2|$
63. Find the domain of $f(x) = |e^{|x|} - 2|$
64. Find the domain of $f(x) = \left|e^{-|x|} - \dfrac{1}{2}\right|$
65. Find the domain of $f(x) = \sqrt{e^{-|x|} - \dfrac{1}{2}}$
66. Find the domain of $f(x) = (e^{2x} + e^x + 1)$
67. Find the domain of $f(x) = (e^x + 1)^2 + 3$
68. Find the domain of $f(x)$ satisfying $e^x + e^{f(x)} = e$
69. Find the number of solutions of $2^x + 3^x + 4^x - 5^x = 0$.
70. Find the number of solutions of $1 + 3^{x/2} = 2^x$

Logarithmic Functions

71. Find the domain of the function

$f(x) = \log_2 (x^2 - 4x + 3)$

72. Find the domain of the function

$f(x) = e^{\frac{x}{2}-1} + \log(1 - x) + x^{1001}$

73. Find the domain of the function

$f(x) = \log\left(\dfrac{x}{x-2}\right)$

74. Find the domain of the function

$f(x) = \dfrac{1}{\log(x-2)}$

75. Find the domain of the functions

$f(x) = \dfrac{\log(1-x)}{x^2 - 3x + 2}$

76. Find the domain of the function

$f(x) = \dfrac{\log(3-x)}{x^2 - 5x + 4}$

77. Find the domain of the function

$f(x) = \dfrac{1}{\log(9 - x^2)}$

78. Find the domain of the function

$f(x) = \log_{1/2} (\sqrt{x-1} + \sqrt{3-x})$

79. Find the domain of the function

$f(x) = \log_{1/2}\left(\dfrac{2-x}{x-4}\right)$

80. Find the domain of the function

$f(x) = \log_{10}(1 - \log_{10}(x^2 - 5x + 16))$

81. Find the domain of the function

$f(x) = \log_{10} ((\log_{10} x^2) - 5\log_{10}x + 6)$

82. Find the domain of the function

$f(x) = \log\left(\dfrac{1}{x - [x]}\right)$.

83. Find the number of real solutions of $2 - x - \log_e x = 0$

Equal Functions

84. Let $f: \{1, 2\} \to \{1, 4\}$ and $g: \{1, 2\} \to \{1, 4\}$ such that $f(x) = x^2$ & $g(x) = 3x - 2$ Is $f = g$?
85. Let $f(x) = x$ and $g(x) = \dfrac{x^2}{x}$ Is $f = g$?
86. Let $f(x) = x$ & $g(x) = \sqrt{x^2}$ Is $f = g$?
87. Let $f(x) = 2\log x$ & $g(x) = \log(x^2)$ Is $f = g$?
88. Let $f(x) = \log\left(\dfrac{x-1}{x-2}\right)$ and

$g(x) = \log(x-1) - \log(x-2)$. Is $f = g$?

89. Let $f(x) = \sin x$ & $g(x) = \sqrt{\dfrac{1-\cos 2x}{2}}$ Is $f = g$?

Classification of Functions

90. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$
Then find the total number of one-one function between two sets A and B.
91. Let $A = \{1, 2, 3\}$ and $B = \{p, q, r\}$
Then find the total number of function between A and B.
92. Let $A = \{1, 2, 3, 4\}$ and B $\{a, b, c, d, e\}$
Then find the number of one-one function between A and B.
93. Let a function $f: A \to B$, where $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$ such that $f(1) = 3$. Then find the number of one-one function between A to B.
94. Let a function $f: A \to B$, where $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ such that $f(1) \neq 4$. Then find the number of one-one function between A to B.
95. Let a function $f: A \to B$, where
$A = \{1, 2, 3\}$ and $B = \{6, 7, 8\}$
such that $a < b \Rightarrow f(a) < f(b)$,
where $a \in A$ & $b \in B$.

Then find the number of one-one function between A to B.

96. Find the number of many-one function between two sets $A = \{3, 4, 5\}$ and $B = \{2, 3, 4, 5\}$.

97. Find the number of many-one function between two sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$.

98. Find the number of many-one function between two sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ such that $f(1) \neq 4$

99. A function $f: R - \{-1\} \to R - \{1\}$ is defined as $f(x) = \dfrac{x}{x+1}$. prove that f is onto function.

100. Let a function $f: R \to A$ is defined as $f(x) = \dfrac{1}{x^2+1}$. If f is onto function, then find the set of values of A.

101. Let a function $f: A \to B$ is defined as $f(x) = \dfrac{x^2}{x^2+1}$. If f is onto function, then find the set of values of $A \cap B$.

102. Find the number of onto function between two sets $A = \{1, 2, 3, 4\}$ and $B = \{4, 5\}$

103. Find the number of onto function between two sets $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6\}$

104. Find the number of onto function between two sets A and B where $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8\}$.

105. Find the number of into function between two sets A and B, where $A = \{1, 2, 3\}$ and $B = \{4, 5\}$.

106. Find the number of into function between two sets A and B where $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7\}$.

107. Find the number of into functions between two sets A and B where $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8\}$.

108. A function $f: R \to R$ is defined as $f(x) = 3x + 5$. Prove that f is a bijective function.

109. A function $f: R^+ \to (1, \infty)$ is defined as $f(x) = x^2 + 1$. Prove that the function is bijective.

110. A function $f: R^+ \to (0, 1)$ is defined as $f(x) = \dfrac{1}{x^2+1}$. Prove that f is a bijective function.

111. A function $f: R \to \left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ is defined as $f(x) = \dfrac{x}{x^2+1}$, $\forall\ x \in (-1, 1)$. Prove that $f(x)$ is a bijective function.

112. If $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4\}$ then find the number of one-one onto functions between A and B.

113. Find the number of bijective functions between two sets A and B, where $A = \{a, b, c, d, e\}$ and $B = \{p, q, r, s, t\}$

114. Find the number of bijective functions between two sets $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$ such that $f(1) = 5$.

Inverse of a Function

115. A function $f: R \to R$ is defined as $f(x) = 3x + 5$. Find $f^{-1}(x)$.

116. A function $f: (0, \infty) \to (2, \infty)$ is defined as $f(x) = x^2 + 2$. Find $f^{-1}(x)$.

117. A function $f: R^+ \to [0, 1]$ is defined as $f(x) = \dfrac{x^2}{x^2+1}$. Find $f^{-1}(x)$.

118. A function $f: [1, \infty) \to [1, \infty)$ is defined as $f(x) = 2^{x(x-1)}$. Find $f^{-1}(x)$.

119. If a function f is bijective such that $f(x) = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$. Find $f^{-1}(x)$

120. A function $f: R \to R$ is defined as $f(x) = x + \sin x$. Find $f^{-1}(x)$

121. A function $f: [2, \infty) \to [5, \infty)$ is defined as $f(x) = x^2 - 4x + 9$. Find its inverse.

122. Find all the real solutions to the equation
$$x^2 - \frac{1}{4} = \sqrt{x + \frac{1}{4}}.$$

123. For what values of m is $f(x) = (m+2)x^3 - 3mx^2 + 9mx - 1$ is invertible?

124. Let $f(x) = \sqrt[3]{a - x^3 + 3bx^2 - 3bx + b^3} + b$. Find b, if $f(x)$ is inverse of itself.

Inverse Trigonometric Functions

125. Find the domain of $f(x) = \sin^{-1}(3x + 4)$

126. Find the domain of $f(x) = \cos^{-1}(4x + 5)$.

127. Find the domains of
$$f(x) = \cos^{-1}\left(\frac{x}{2} - 1\right) + e^x + \frac{1}{|x - 1|}$$

128. Find the domains of
$$f(x) = \cos^{-1}\left(\frac{|x| - 3}{5}\right) + \frac{1}{e^x + 1}$$

129. Find the domains of
$$f(x) = \sin^{-1}\left(\frac{1 - |x|}{2}\right) + \left(\frac{e^x - 1}{e^x + 1}\right) + 2015$$

130. Find the domain of $f(x) = \cos^{-1}\left(\frac{3}{4 + 2\sin x}\right)$

131. Find the domain of

$$f(x) = \sin^{-1}\left(\frac{1 + x^2}{2x}\right) + \cos(\sin x).$$

132. Find the domains of $f(x) = \sin^{-1}(\log_2 x)$.

133. Find the domains of $f(x) = \sin^{-1}(\log_4 x^2)$.

134. Find the domains of

$$f(x) = \sin^{-1}\left(\frac{1}{|x^2 - 1|}\right)$$

Composition of Functions

135. Let two real functions f and g are defined as $f: R \to R, f(x) = x^2 + 1$ and $g: R \to R, f(x) = x - 1$. Determine $f_o g$ & $g_o f$.

136. Find $f_o g$ and $g_o f$ for the functions $f(x) = \sin x$ and $g(x) = \sqrt{x - 2}$

137. Find the domain of the function $f_o g$, where $f(x) = \sqrt{x - 3}$ and $g(x) = x^2 + 1$.

138. Find $f_o g$, where $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$.

139. A function $f(x)$ is defined as $f(x) = (a - x^n)^{1/n}$, $x > 0$, $n \in I^+$

Then find $f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right)$.

140. Find the number of distinct real solutions of the equation $f(f(f(x))) = 0$, where $f(x) = x^2 - 1$

141. If $f(x) = \begin{cases} x^2 & : \; x \ge 0 \\ x & : \; x < 0 \end{cases}$ and $g(x) = -|x|, x \in R$

Then find $f_o g$ & $g_o f$

142. A function f is defined as

$$f(x) = \begin{cases} 1 + x: & x \ge 0 \\ x - x: & x < 0 \end{cases}.$$ Then find $f_o f$

143. Let $f: R^+ \to R$ be defined as $f(x) = x^2 - x + 2$ and $g: [1, 2] \to [1, 2]$ be defined as $g(x) = \{x\} + 1$, where $\{,\}$ = Fractional part function. If the domain and range of $f(g(x))$ are $[a, b]$ and $[c, d)$, then find the value of $\frac{b}{a} + \frac{d}{c}$.

144. If g is the inverse of f and $f'(x) = \sin x$, find $g'(x)$.

145. If $g(x) = \frac{1}{1-x}$ and $g_2(x) = g(g(x))$ and $g_3(x) = g(g(g(x)))$, then find the range of $g_{2016}(x)$.

146. If the roots of $(c - 1)(x^2 + x + 1)^2 - (c + 1)(x^4 + x^2 + 1) = 0$ are real and distinct and $f(x) = \frac{1 - x}{1 + x}$, then find the value of $f(f(x)) + f\left(f\left(\frac{1}{2}\right)\right)$.

147. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then find $(g_o f)(x)$

148. Let $f(x) = 1 + x^2$. Find a function $g(x)$ such that $f(g(x)) = 1 + x^2 - 2x^3 + x^4$

149. Let $g(x) = 1 + x - [x]$ and

$$f(x) = \begin{cases} -1 & : \; x < 0 \\ 0 & : \; x = 0 \\ 1 & : \; x > 0 \end{cases}$$ then for all x, find $f(g(x))$

150 Let $f(x) = \frac{\alpha x}{x + 1}$, $x \ne -1$ then for what value of α is $f(f(x)) = x$?

Even and Odd Functions

Determine the nature of each of the following functions.

151. $f(x) = x\left(\frac{3^x - 1}{3^x + 1}\right)$

152. $f(x) = x\sin(x^2 + 1)$

153. $f(x) = \tan^{-1}\left(\sin(\cos^{-1}(x))\right)$

154. $f(x) = \sin x + \cos x$

155. $f(x) = \left(\frac{x}{e^x - 1} + \frac{x}{2} + 1\right)$

156. $f(x) = \log\left(\frac{3 - x}{3 + x}\right)$

157. $f(x) = \log\left(x + \sqrt{x^2 + 1}\right)$

158. $f(x) = \sqrt{x^2 - x + 1} - \sqrt{x^2 + x + 1}$

159. Determine the nature of the function

$$f(x) = x\sin^2 x + \tan(x^5) + \log\left(\frac{2 - x}{2 + x}\right).$$

160. Determine the nature of the function

$$f(x) = (\tan(x^5))e^{x^2\, sgn(x^7)}$$

161. Determine the nature of the function

$$f(x) = \frac{x^2(\log(x + \sqrt{x^2 + 1}))}{(x\cos^2 x + \tan x + x^7)}$$

162 Determine the nature of the function

$$f(x) = \frac{\sin\left(\tan\left(\log\left(x + \sqrt{x^2 + 1}\right)\right)\right)}{\sqrt{x^2 + 1} + \sin(\cos x) + \cos(\sin x)}$$

163. Express the function $f(x) = 4^{\sin x}$ as a sum of an even and an odd function.

164. Express the function $f(x) = (1 + x)^{2015}$ as a sum of an even and an odd function.

165. Let $f(x) = \begin{cases} x + x^2 & : \quad 0 \le x < 3 \\ x + x & : \quad 3 < x \le 5 \end{cases}$

Then find its even and odd extension.

166. Let $f(x) = x + e^x + \sin x$ be defined on [0, 2]. Then find its even and odd extension.

167. Let $f(x) = x^2 + x + \sin x - \cos x + \log_e (1 + x)$ be defined on [0, 1]. Find its even and odd extension in the interval [–1, 1]

Periodicity of Functions

168. Find the period of $f(x) = 3 \sin 4x + 4 \cos 3x$

169. Find the period of $f(x) = 3 \sin 4x + 4 |\sin 4x|$

170. Find the period of $f(x) = \sin x . \operatorname{cosec} x$

171. Find the period of $f(x) = \tan x . \cot x$.

172. Find the period of

$f(x) = 5 \sin (2\sqrt{2}x) + 7 \cos (3\sqrt{2}x)$.

173. Find the period of

$f(x) = 3 \sin (2\sqrt{3}x) + 2 \cos (5\sqrt{3}x)$.

174. Find the period of

$f(x) = \sin^2 x + \sin^2 \left(x + \frac{\pi}{3}\right) - \cos x \cos \left(x + \frac{\pi}{3}\right)$

175. Find the period of $f(x) = \frac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

176. Find the period of $f(x) = \frac{|\sin x + \cos x|}{|\sin x| - |\cos x|}$

177. Find the period of $f(x) = \frac{1}{2}\left(\frac{|\sin x|}{\cos x} + \frac{\sin x}{|\cos x|}\right)$

178. Find the period of $f(x) = 3 \sin \{2x\} + 2 \cos \{3x\}$

179. Let $f(x) + f(x + 3) = 5$ for all x in R. Prove that $f(x)$ is periodic with period 6.

180. Find the period of the function 'f' which satisfy the equation $f(x + 4) + f(x - 4) = f(x)$

181. If f is a function satisfying the equation

$f(x - 1) + f(x + 1) = \sqrt{2}\, f(x)$, then prove that $f(x)$ is periodic with period 8.

182. Let $f(x + 1) + f(x + 5) = f(x + 3) + f(x + 7)$ for all x in R. Prove that $f(x)$ is periodic Functional Equation.

183. Let $f : R - \{0\} \to R$ be any real function such that $f(x) + 3f\left(\frac{1}{x}\right) = 5x$. Find $f(x)$.

184. Let $2f(\sin x) + 3f(\cos x) = 5, \forall x \in R$. Find $f(x)$.

185. Let a function f saisfy $f(x + 1) = f(x) + x, \forall x \in N$ where $f(1) = 0$. Find $f(x)$.

186. Suppose f is real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$

Then find $f(5)$.

187. Find all polynomial $P(x)$ which satisfy the relation $P(x + 1) = P(x) + 2x + 1$, where $P(0) = 0$

188. Find the natural number 'a' for which

$\sum_{k=1}^{n} f(a + k) = 16 (2^n - 1)$, where f satisfies the relation $f(x + y)$

$= f(x). f(y)$ for all x, y are in N and $f(1) = 2$.

189. Let a polynomail function f satisfies the relation

$f(x) + f\left(\frac{1}{x}\right) = f(x). f\left(\frac{1}{x}\right)$ for all $x \in R - \{0\}$

Then find $f(x)$.

190. If $f(x)$ is a polynomial function of x, satisfying $f(x). f(y) = f(x) + f(y) + f(xy) - 2$ such that $f(1) = 2$ and $f(2) = 5$, then find the value of $f(6) + 2009$.

191. If $P(x)$ be a polynomial satisfying the identity $P(x^2) + 2x^2 + 10x = 2x\, P(x + 1) + 3$, find $P(x)$.

192. If $f(x)$ is a polynomial of degree 2 such that $f(0) = 1$ and $f(x + 2) = f(x) + 4x + 2$, find the polynomial $f(x)$.

Level II

(Mixed Problems)

Choose the most appropriate one

1. If $f(x) = \sqrt{|x - 1|}$ and $g(x) = \sin x$, then $(fog)(x) =$

(a) $\sin \sqrt{|x - 1|}$ (b) $|\sin x/2 - \cos x/2|$

(c) $|\sin x - \cos x|$ (d) None of these

2. If $f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right)$, $x_1 x_2 \in (-1, 1)$, then $f(x) =$

(a) $\log\left(\frac{1 - x}{1 + x}\right)$ (b) $\tan^{-1}\left(\frac{1 - x}{1 + x}\right)$

(c) $\log\left(\frac{2x}{1 + x^2}\right)$ (d) $\tan^{-1}\left(\frac{1 + x}{1 - x}\right)$

3. If $f(x) = \frac{2^x + 2^{-x}}{2}$, then $f(x + y)\, f(x - y) =$

(a) $\frac{1}{2}[f(x + y) + f(x - y)]$

(b) $\frac{1}{2}[f(2x) + f(2y)]$

(c) $\frac{1}{2}[f(x + y). f(x - y)]$

(d) None of these

4. The domain of the function $f(x) = \sqrt{\log\left(\frac{1}{|\sin x|}\right)}$ is

(a) $R-\{-\pi, \pi\}$ (b) $R-\{n\pi/n\in Z\}$

(c) $R-\{2n\pi/n \in Z\}$ (d) $(-\infty, \infty)$

5. The inverse of the function $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is

(a) $\log_{10}(2 - x)$ (b) $\frac{1}{2}\log_{10}\left(\frac{1 + x}{1 - x}\right)$

(c) $\frac{1}{2}\log_{10}(2x - 1)$ (d) $\frac{1}{4}\log_{10}\left(\frac{2x}{2 - x}\right)$

6. Domain of the function

$f(x) = \sqrt{\log_{0.5}(3x - 8) - \log_{0.5}(x^2 + 4)}$ is

(a) $\left(\frac{8}{3}, \infty\right)$ (b) $\left(-\infty, \frac{8}{3}\right)$

(c) $(-\infty, \infty)$ (d) $(0, \infty)$

7. The domain of the function

$f(x) = {}^{16-x}C_{2x-1} + {}^{20-3x}P_{4x-5}$

(a) {2, 3} (b) {2, 3, 4}

(c) {1, 2, 3, 4} (d) (1, 2, 3, 4, 5}

8. Range of the function $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$

(a) R (b) $[3, \infty)$

(c) $\left[\frac{1}{3}, 3\right]$ (d) None of these

9. Function $f: R \to R$, $f(x) = x^3 + 7$ is

(a) One-one onto (b) One-one into

(c) Many-one onto (d) Many one into

10. Let $f: (-1, 1) \to B$ be a function defined by

$f(x) = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$ then f is both one-one and onto when B is in the interval

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left[0, \frac{\pi}{2}\right)$

11. If $A = \{x| -1 \le x \le 1\} = B$ and $f: A \to B$ $f(x) = \sin(\pi x)$, then f is

(a) One-one (b) Onto

(c) One-one onto (d) Many-one into

12. If $f(x)$ and $g(x)$ are two functions of x such that $f(x) + g(x) = e^x$ and $f(x) - g(x) = e^{-x}$ then

(a) $f(x)$ is an odd function

(b) $g(x)$ is an odd function

(c) $f(x)$ is an even function

(d) $g(x)$ is an even function

13. The period of the function $\sin\left(\frac{\pi x}{2}\right) + \cos\left(\frac{\pi x}{2}\right)$ is

(a) 4 (b) 6

(c) 12 (d) 24

14. The domain of definition of the function $y(x)$ is given by the equation $2^x + 2^y = 2$ is

(a) $0 < x$ £ 1 (b) 0 £ x £ 1

(c) $-\infty < x$ £ 0 (d) $-\infty < x < 1$

15. Suppose $f(x) = (x + 1)^2$ for $x \ge -1$. If $g(x)$ is the function whose graph is reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals:

(a) $-\sqrt{x} - 1, x \ge 0$ (b) $\frac{1}{(x + 1)^2}, x > -1$

(c) $\sqrt{x} + 1, x \ge -1$ (d) $\sqrt{x} -1, x \ge 0$

16. If $f: [0, \infty) \to [0, \infty)$ and $f(x) = \frac{x}{1+x}$, then f is

(a) One-one and onto

(b) One-one but not onto

(c) Onto but not one-one

(d) Neither one-one nor onto

17. Let $f(x) = |x - 1|$. Then

(a) $f(x^2) = (f(x))^2$ (b) $f(x + y) = f(x) + f(y)$

(c) $f(|x|) = |f(x)|$ (d) None of these

18. Let function $f: R \to R$ be defined by

$f(x) = 2x + \sin x$ for $x\in$ R. Then f is:

(a) One-one and onto

(b) One-one but not onto

(c) Onto but not one-one

(d) Neither one-one nor onto

19. Let $f(\theta) = \sin\theta\,(\sin\theta + \sin 3\theta)$. Then $f(\theta)$

(a) ≥ 0 only when $q \ge 0$ (b) £ 0 for all real q

(c) ≥ 0 for all real q (d) £ 0 only when q £ 0

20. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$ then $f^{-1}(x) =$

(a) $\frac{x+\sqrt{x^2 - 4}}{2}$ (b) $\frac{x}{1 + x^2}$

(c) $\frac{x-\sqrt{x^2-4}}{2}$ (d) $1 + \sqrt{x^2 - 4}$

21. If $f(x) = \cos(\log x)$, then

$f(x)\, f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value

(a) -1 (b) $\frac{1}{2}$

(c) -2 (d) None of these

22. The domain of definition of $f(x) = \frac{\log_2(x + 3)}{x^2 + 3x + 2}$ is

(a) $R/\{-1, -2\}$ (b) $(-2, \infty)$

(c) $R/\{-1, -2, -3\}$ (d) $(-3, \infty)/\{-1, 2\}$

23. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is

(a) $[0, 1]$ (b) $\left[0, \frac{1}{2}\right]$

(c) $\left[\frac{1}{2}, 1\right]$ (d) $(0, 1]$

24. The domain of definition of the function

$y = \frac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$ is:

(a) $(-3, -2)$ excluding -2.5

(b) $[0, 1]$ excluding 0.5

(c) $(-2, 1)$ excluding 0

(d) None of these.

25. Let $g(x) = 1 + x - [x]$ and

$f(x) = \begin{cases} -1, x < 0 \\ 0, x = 0 \\ 1, x > 0 \end{cases}$, then for all x, $f(g(x)) =$

(a) x (b) 1

(c) $f(x)$ (d) $g(x)$

26. If $f(x) = 3x - 5$, then $f^{-1}(x)$

(a) Is given by $\frac{1}{3x - 5}$

(b) Is given by $\frac{x + 5}{3}$

(c) Does not exist because f is not one-one

(d) Does not exist because f is not onto

27. Range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}$; $x \in R$ is

(a) $(1, \infty)$ (b) $(1, 11/7)$

(c) $(1, 7/3]$ (d) $(1, 7/5)$

28. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is:

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1 + \sqrt{1 + 4 \log_2 x})$

(c) $\frac{1}{2}(1 - \sqrt{1 + 4 \log_2 x})$ (d) Not defined

29. Let $f(x) = \frac{\alpha x}{x + 1}$, $x \neq -1$. Then, for what value of a is $f(f(x))$?

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) 1 (d) -1

30. Which of the following function is periodic

(a) $f(x) = x - [x]$ where $[x]$ denotes the greatest integer less than or equal to the real number x

(b) $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$

(c) $f(x) = x \cos x$

(d) None of these

31. Let $f: (0, \infty) \to R$ and $F(x) = \int_0^x f(t)\, dt$.

If $F(x^2) = x^2(1 + x)$, then $f(4)$ equals:

(a) 5/4 (b) 7

(c) 4 (d) 2

32. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin \sqrt{x})^2$, then:

(a) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$

(b) $f(x) = \sin x$, $g(x) = |x|$

(c) $f(x) = x^2$, $g(x) = \sin \sqrt{x}$

(d) f and g can not be determined

33. Let $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then the number of onto functions from E to F is:

(a) 14 (b) 16

(c) 12 (d) 8

34. If $f(x) = \cos [p^2]\, x + \cos [-p^2]\, x$, where $[x]$ stands for the greatest integer function, then:

(a) $f\left(\frac{\pi}{2}\right) = -1$ (b) $f(p) = 1$

(c) $f(-p) = 0$ (d) $f\left(\frac{\pi}{4}\right) = 1$

35. If S is the set of all real x such that $\frac{2x - 1}{2x^3 + 3x^2 + x}$ is positive, then S contains:

(a) $\left(-\infty, -\frac{3}{2}\right)$ (b) $\left(-\frac{3}{2}, -\frac{1}{4}\right)$

(c) $\left(-\frac{1}{4}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, 3\right)$

36. Domain of definition of the function

$f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ for real valued x, is

(a) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(c) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (d) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

37. For a positive integer n, let

$f_n(\theta) = \left\{\left(\tan\frac{\theta}{2}\right)(1 + \sec\theta)(1 + \sec 2\theta)\right.$ $\left.(1 + \sec 4\theta) \ldots (1 + \sec 2^n\theta)\right\}$ then:

(a) $f_2\left(\frac{\pi}{16}\right) = 1$ (b) $f_3\left(\frac{\pi}{32}\right) = 1$

(c) $f_4\left(\frac{\pi}{64}\right) = 1$ (d) $f_5\left(\frac{\pi}{128}\right) = 1$

38. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $g(f(x))$ is invertible in the domain:

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[0, p]$

39. The range of the function

$f: [0, 1] \to R$, $f(x) = x^3 - x^2 + 4x + 2\sin^{-1}x$ is

(a) $[-\pi - 2, 0]$ (b) $[2, 3]$

(c) $[0, 4 + \pi]$ (d) $(0, 2 + \pi]$

40. The range of the function

$f(x) = \sqrt{\sin(\cos x)} + \sqrt{\cos(\sin x)}$ is

(a) $\lfloor 1, 1 + \sqrt{\cos(1)} \rfloor$

(b) $\lfloor \sqrt{\cos(1)}, 1 + \sqrt{\cos(1)} \rfloor$

(c) $\lfloor \sqrt{\cos(1)}, 1 + \sqrt{\sin(1)} \rfloor$

(d) $\lfloor 1, 1 + \sqrt{\sin(1)} \rfloor$

41. If $f(x) = \dfrac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$, then the period of $f(x)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) π

42. If the period of $f(x) = \cos(x + 4x + 9x + \ldots + n^2x)$ is $\frac{\pi}{7}$ then the value of n, $n \in N$, is

(a) 2 (b) 3

(c) 4 (d) 5

43. Let $f(x) = [\sin 3x] + |\cos 6x|$, where [,] = G.I.F. Then the period of $f(x)$ is

(a) $\pi/3$ (b) $2\pi/3$

(c) $\pi/6$ (c) $\pi/12$

44. Let $f(x) = \dfrac{2x(\sin x + \tan x)}{2\left[\frac{x + 21\pi}{\pi}\right] - 41}$. Then the function $f(x)$ is

(a) odd (b) even

(c) periodic (d) None.

45. Let f: $[-2, 2] \to R$, where

$f(x) = x^3 + \sin x + \left[\frac{x^2 + 1}{a}\right]$ be an odd function. Then the set of values of 'a' is

(a) R (b) $(-\infty, 5)$

(c) $(5, \infty)$ (d) $[-5, 5]$

46. Let $A = \{a_1, \ldots\ldots, a_5\}$ and $B = \{y_1, \ldots\ldots, y_5\}$. Then the number of one-one functions of

$f: A \to B$ such that $f(x_i) \neq y_i$, $i = 1, 2, 3, 4, 5$ is

(a) 40 (b) 44

(c) 24 (d) 60

47. The number of bijective functions of $f: A \to A$, where $A = \{1, 2, 3\}$ such that $f(1) \neq 3$, $f(2) \neq 1$, $f(3) \neq 2$ is

(a) 1 (b) 2

(c) 9 (d) None

48. The number of onto functions from A to B where $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 5\}$ is

(a) 100 (b) 120

(c) 140 (d) 150

49. The domain of the function $f(x) = \dfrac{1}{\sqrt[x-1]{\sqrt{81} - 3}}$ is

(a) $\{2, 3\}$ (b) $\{3, 4\}$

(c) $(2, 3)$ (d) $(3, 4)$

50. The domain of the function

$f(x) = \text{logs}_{(4-x)}(x - 1) - \sin^{-1}[2x - 3]$ is

(a) $(1, 2)$ (b) $(1, 2.5)$

(c) $(1, 1.5)$ (d) $(3, 4)$.

Level III

(Problems for JEE-Advanced)

1. If $2f(x) + 3f\left(\frac{1}{x}\right) = x - 3$, $x \neq 0$, find $f(x)$

2. If $f(x) + 2f(1 - x) = x^2 + 2$, $\forall x \in R$, find $f(x)$.

3. If $f(2x + 3y, 2x - 7y) = 20x$, find $f(x, y)$.

4. If $f(x)$ is a polynomial function satisfying $f(x) + f\left(\frac{1}{x}\right) = f(x).\, f\left(\frac{1}{x}\right)$ and $f(3) = 28$, find $f(4)$.

5. If $f(x + y, x - y) = xy$, find the arithmetic mean of $f(x, y)$ and $f(y, x)$.

6. If $F(n + 1) = \frac{2F(n) + 1}{2}$, $n \in N$ and $F(1) = 2$, find the value of $F(2015)$

7. If $f(x) = \cos(\log_e x)$, find the value of $f(x).\, f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(x\, y)\right\}$

8. If $f(x + 1) + f(x - 1) = 2f(x)$ and $f(0) = 0$ then find $f(n)$, $n \in N$.

9. Let $f(x) = \frac{4^x}{4^x + 2}$, find the value of $f\left(\frac{1}{2015}\right) + f\left(\frac{2}{2015}\right) + f\left(\frac{3}{2015}\right) + ... + f\left(\frac{2014}{2015}\right)$.

10. Let $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, $x \neq 0$, find $f(x)$

11. Let $f(1) = 1$ and $f(n) = 2\sum_{r=1}^{n-1} f(r)$, find $\sum_{r=1}^{m} f(n)$

12. Consider the function $f(x)$ and $g(x)$ are defined as $f(x) = (x + 1)(x^2 + 1)(x^4 + 1)...(x^{2^{2007}} + 1)$ and $g(x)(x^{2^{2008}} - 1) = (f(x) - 1)$, find $g(2)$.

13. If $f(x - 1) = x^2 - 1$, find $f(x)$.

14. Let f be a function such that $f\left(\frac{8}{\sqrt{1} + \sqrt{x}}\right) = x$, for all $x \geq 0$, find the value of $f(4)$.

15. Find the natural domain of the function

$f(x) = \sqrt{1 - x} + \frac{1}{\sqrt{1 + x}}$

16. Find the domain of the function

$f(x) = \sqrt{x - 2} + \sqrt{5 - x}$

17. Find the domain of the function

$f(x) = \log_2(\sqrt{x - 2} + \sqrt{4 - x})$

18. Find the domain of the function

$f(x) = \frac{\log(3 - x)}{x^2 - 3x + 2}$.

19. Find the domian of $f(x) = \sqrt{|x| - 1} + \sqrt{3 - |x|}$

20. Find the domain of $f(x) = \frac{\sqrt{|x| - 1}}{\sqrt{4 - |x|}}$

21. Find the domain of

$f(x) = \sqrt{\frac{[x] - 1}{4 - [x]}}$, where [,] = G.I.F

22. Find the domain of $f(x) = \sin^{-1}(\log_2 x)$

23. Find the domain of $f(x) = \sin^{-1}\left(\frac{1 + x^2}{2x}\right)$

24. Find the domain of $f(x) = \cos^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$

25. Find the domain of

$f(x) = \log_4(\log_5(\log_3(18x - x^2 - 77)))$

26. Find the domain of $f(x) = \sqrt{1 - \log_x \log_2(4^x - 12)}$

27. Find the domain of the function $y = f(x)$ where $3^y + 2^{x^4} = 2^{4x^2 - 3}$

28. Find the domian of definitions of

$f(x) = \sin\left(\log_e\left(\frac{\sqrt{4 - x^2}}{1 - x}\right)\right)$

29. Find the domain of the function

$f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-1/2}$

30. Find the range of the function $f(x) = x^2 + x + 1$

31. Find the range of the function $f(x) = \frac{x^2}{x^2 + 1}$

32. Find the range of the function $f(x) = \frac{x^2}{x^4 + 1}$

33. Find the range of the function

$f(x) = \frac{x^2 + x + 1}{x^4 + x^2 + 1}$

34. Find the range of the function $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$

35. Find the range of the function

$f(x) = \log_2(\sqrt{x - 2} + \sqrt{4 - x})$

36. Find the range of the function $f(x) = x^2 + \frac{1}{x^2 + 1}$

37. Find the range of the function

$f(x) = x^4 + 4x^2 + \frac{1}{x^4 + 4x^2 + 9} + 10$

38. Find the range of the function

$f(x) = \log_2\left(\log_{1/2}(x^2 + 4x + 4)\right)$

39. Find the range of the function

$f(x) = \sin^{-1}\left(\frac{\cos^2 x + 1}{2}\right)$

40 Find the range of the function

$f(x) = 2^x + 3^x + 4^x + 2^{-x} + 3^{-x} + 4^{-x} + 10$

41. Find the range of the function

$f(x) = 3\tan^2 x + 12\cot^2 x + 5$

42. Find the range of the function

$f(x) = \sqrt{3}\sin x + \cos x + 4$

43. Find the range of the function

$f(x) = \frac{1}{2\cos^2 x + 4\sin x\cos x + 4}$

44. Find the range of the function $f(x) = \cos^{-1}\left(\frac{x^2}{1 + x^2}\right)$

45. Find the range of the function

$f(x) = \cos^{-1}\left(\frac{\sqrt{2x^2 + 1}}{x^2 + 1}\right)$

46. Find the range of the function

$f(x) = \sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$ where [,] = G.I.F

47. Find the range of the function

$f(x) = \frac{x^2 + 2x + 3}{x}, x > 0$

48. Find the range of the function

$f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}$ where $x \in R$

49. Find the range of the function $f(x) = \frac{e^x}{1 + [x]}$

50. Find the range of the function $f(x) = \frac{\{x\}}{1 + \{x\}}$, where {,} = F.PF

51. Find the range of the function

$f(x) = 2\cos x + \sec^2 x,\ x \in \left(0, \frac{\pi}{2}\right)$

52. Find the range of the function

$f(x) = 3\tan x + \cot^3 x,\ x\left(0, \frac{\pi}{2}\right)$

53. Determine the nature of the function

$f(x) = \frac{\sin^4 x e^{x^2}}{\left[\frac{x + 3\pi}{\pi}\right] - \frac{5}{2}}$, where [,] = G.I.F

54. Determine the nature of the function

$f(x) = \left(\frac{2x(\sin^{17} x + \tan^{99} x)}{2\left[\frac{x + 21\pi}{\pi}\right] - 41}\right), [x] \le x$

55. Find the inverse of the functions

$f(x) = \sqrt{x + \sqrt{x}}$, where $f: [0, \infty) \to [0, \infty)$

56. Find the inverse of the functions

$f(x) = 4x^3 - 3x$, where $f: \left[\frac{1}{2}, 1\right] \to [-1, 1]$

57. Let $f(x) = \sqrt[3]{a - x^3 + 3x^2 - 3bx + b^3 + b}$.

Find b if $f(x)$ is the inverse of itself.

58. Find the value of a so that $f(x) = \frac{ax + 1}{x + 3}$ is identical with $f^{-1}(x)$

59. Determine the nature of the function $f(x)$, where $f(x)$ satisfying the relation $f(x + y) + f(x - y) = 2f(x).f(y)$

60. If f is a polynomial function satisfying $2 + f(x).f(y) = f(x) + f(y) + f(xy)$ for all x, y in R and if $f(2) = 5$, then find $f(f(2))$

61. If $f(x)$ is symmetrical about $x = a$ and $x = b$, then find the period of $f(x)$.

62. Find the number of integral values of a for which the function $f(x) = \log(\log_{1/3}(\log_7(\sin x + a)))$ be defined for every real value of x.

63. Find the domain of the function

$f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{x^{1/4}}\right) - 1\right)$

64. Find the domain of the function

$f(x) = \log_2(\log_3(\log_4(\tan^{-1}x)^2))$

65. Find the domain of the function

$f(x) = \sin^{-1}\left(\frac{1}{|x^2 - 1|}\right) + \frac{1}{\sqrt{\sin^2 x + \sin x + 1}}$

66. Find the domain of the function

$f(x) = \frac{1}{\sqrt{4x - |x^2 - 10x + 9|}}$

67. Find the domain of the function $f(x)$

$= \sqrt{\sin^{-1}|\sin x| - \cos^{-1}|\cos x|}$ for all x $[0, 2\pi]$

68. Find the domain of the function

$f(x) = \sqrt{\log_{0.5}\left(\frac{3x - x^2}{x - 1}\right)}$

69. Find the domain of the function

$f(x) = \log_7\left(\log_5\left(\log_3\left(\log_2(2x^3 + 5x^2 - 14x)\right)\right)\right)$

70. Find the domain of the function

$f(x) = \sqrt{3^{x-1} + 5^{x-1} + 7^{x-1} - 83}$

71. Find the domain of the function

$f(x) = \sqrt{3 - 2^x - 2^{1-x}}$

72. Find the domain of the function

$\log_{2015}\left(1 - \log_7(x^2 - 5x + 13)\right)$

73. Find the domain of the function

$f(x)\ \sqrt{24 - [x^2]} + \sqrt{|x| - 4}$

74. Find the range of the function $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$

75. Find the range of the function $f(x) = \cos^{-1}\left(\frac{x^2}{1+x^2}\right)$

76. Find the range of the function $f(x) = \sin^{-1}\left(\frac{x^2+1}{x^2+2}\right)$

77. Find the range of the function $f(x) = \tan\left(\sqrt{\frac{\pi^2}{9} - x}\right)$

78. Find the range of the function $f(x) = \sin\left(\sqrt{\frac{\pi^2}{16} - x}\right)$

79. Find the range of the function $f(x) = \log_2(4x - x^2)$

80. Find the range of the function $f(x) = \log\left(\frac{x^2 + e}{x^2 + 1}\right)$

81. Find the range of the function

$f(x) = \sin\left(\log\left(\sqrt{\frac{4 - x^2}{1 - x}}\right)\right)$

82. Find the range of the function

$f(x) = \log\left(\sin^{-1}(\sqrt{x^2 + x + 1})\right)$

83. Find the range of the function

$f(x) = \sin\left(\log(5x^2 - 8x + 4)\right)$

84. Find the range of the function

$f(x) = \cot^{-1}\left(\log_{0.5}(x^4 - 2x^2 + 3)\right)$

85. Find the range of the function

$f(x) = \sqrt{1 - \cos x\sqrt{1 - \cos x\sqrt{1 - \cos x\sqrt{\ldots\infty}}}}$

86. If $f(x) = ax + b$ and $f(f(f(x))) = 27x + 26$ where $a, b \in R$, find the value of $a^2 + b^2 + 2$

87. If $f(x) = \frac{x}{\sqrt{1 + x^2}}$, prove that $f(f(f(x))) = \frac{x}{\sqrt{1 + 3x^2}}$

88. Find the domain of the function

$f(x) = \sqrt{\log_{10}(\log_{10}x) - \log_{10}(4 - \log_{10}3) - \log_{10}3}$

89. Find the domain of the function

$f(x) = \sqrt{\log_{1/2}\log_2[x^2 + 4x + 5]},\ [x] \le x$

90. Let $f(x) = 1 + x^2$. Find a function $g(x)$ such that $f(g(x)) = 1 + x^2 - 2x^3 + x^4$.

91. Let $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x.\cos\left(x + \frac{\pi}{3}\right)$

and $g(x) = \begin{cases} 2x & : 0 \le x < 1 \\ x + \frac{1}{4} & : 1 \le x < 2 \end{cases}$ then find $g(f(x))$

92. Let $f(x) = \frac{ax^2 + 2x + 1}{2x^2 - 2x + 1}$. Find the values of a if

$f: R \to [-1, 2]$ is onto.

93. Find x, if $\left|\left||x^2 - x + 4| - 2\right| - 3\right| = x^2 + x - 12$.

94. Find the set of values of a for which the function $f: R \to R$ is given by $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$ is one-one.

95. Let $g(x) = f(x) - 1$. If $f(x) + f(1 - x) = 2$ for all x in R, then find the line about which $g(x)$ is symmetrical.

96. If $f(2 + x) = f(2 - x)$ and $f(7 - x) = f(7 + x)$ and $f(0) = 0$, find the minimum number of roots of $f(x) = 0$, where $20 \le x \le 20$.

97. If $a, b \in R^+$ and for all x in R, $f(a + x) = b + [b^3 + 1 - 3b^2 f(x) + 3b\{f(x)\}^2 - \{f(x)\}^3]^{1/3}$. Prove that $f(x)$ is periodic.

98. If $4^x - 2^{x+2} + 5 + ||b - 1| - 3| = |\sin y|,\ \forall\ x, y \in R$, find the possible value of b.

Level III

(Tougher Problems for JEE-Advanced)

1. Find the domain of the function

$f(x) = \log_2\left(\log_3\left(\log_{4/\pi}(\tan^{-1} x)^{-1}\right)\right)$

2. Find the domain of the function

$f(x) = \sin^{-1}\left(\frac{1}{|x^2 - 1|}\right) + \frac{1}{\sqrt{\sin^2 x + \sin x + 1}}$

3. Find the domain of the function

$f(x) = \dfrac{1}{\sqrt{4x - |x^2 - 10x + 9|}}$.

4. Find the domain of the function

$f(x) = \sqrt{\sin^{-1}|\sin x| - \cos^{-1}(\cos x)}$ in $[0, 2\pi]$

5. Find the domain of the function

$f(x) = \sqrt{2\{x\}^2 - 3\{x\} + 1}$, $x \in [-1, 1]$,

where $\{,\}$ = F.P.F

6. Find the domain of the function

$f(x) = \sqrt{\log_{0.3}\left(\dfrac{3x - x^2}{x - 1}\right)}$.

7. Find the domain of the function

$f(x) = \log_{10}\left(1 - \log_{10}(x^2 - 5x + 16)\right)$

8. Find the domain of the function

$f(x) = \dfrac{\sqrt{-\log_{0.3}(x - 1)}}{\sqrt{-x^2 + 2x + 8}}$

9. Find the domain of $f(x) = \sqrt{\dfrac{x^{-1}}{x - 2\{x\}}}$

10. Find the domain of $f(x) = \log_{[x^2]}(4 - |x|)$

11. Find the domain of

$f(x) = \cos^{-1}\left(\dfrac{[x^2 - 3]}{5}\right) + \log_2(|x| - 1)$

12. Find the domain of $f(x) = \sqrt{[x] - 1 + x^2}$

13. Find the domain of $f(x) = \sqrt{\sin x - (\cos x)}$, where $(x) \geq x$.

14. Find the domain of definition of the function

$f(x) = \dfrac{1}{[x]} + \log_{(2\{x\}-5)}(x^2 - 3x + 10) + \dfrac{1}{\sqrt{1 - |x|}}$

15. Find the domain of definition of the function

$f(x) = \sqrt{\sin(\cos x)} + \log(-2\cos^2 x + 3\cos x - 1)$

$+ e^{\cos^{-1}\left(\frac{2\sin x + 1}{2\sqrt{\sin x}}\right)}$

16 Find the domain of the definition of the function

$f(x) = \sqrt{1 - \log_x \log_2(4^x - 12)}$

17. Find the domain of the definition of the function

$f(x) = \sqrt{\log_{1/2}\log_2[x^2 + 4x + 5]}$

18 Find the range of the function

$f(x) = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$

19. Find the range of the function

$f(x) = (\tan^{-1}x)^2 + \dfrac{2}{\sqrt{1 + x^2}}$

20. Find the range of the function

$f(x) = \sqrt{a^2\cos^2 x + b^2\sin^2 x} + \sqrt{b^2\cos^2 x + a^2\sin^2 x}$, $b > a$

21. Find the range of the function

$f(x) = \log\left((\cos x)^{\cos x} + 1\right)$, $x \in \left(0, \dfrac{\pi}{2}\right)$

22. Find the range of the function

$f(x) = \cos^{-1}\left(\dfrac{\sqrt{2x^2 + 1}}{x^2 + 1}\right)$.

23. Find the range of the function

$f(x) = \log_2\left(2 - \log_{\sqrt{2}}(16\sin^2 x + 1)\right)$

24. Find the range of the function

$f(x) = \tan^{-1}\left(\log_{5/4}(5x^2 - 8x + 4)\right)$

25. Find the range of the function

$f(x) = 6^x + 3^{-x} + 6^{-x} + 3^x + 2$.

26. Find the Range of

$f(x) = \cos^{-1}\left(\dfrac{1 + x^2}{2x}\right) + \sqrt{2 - x^2}$

27. Find the range of

$f(x) = \dfrac{1}{\pi}(\tan^{-1}x + \sin^{-1}x) + \dfrac{x + 1}{x^2 + 2x + 5}$

28. Find the range of

$f(x) = \sin^{-1}\left[x^2 + \dfrac{1}{2}\right] + \cos^{-1}\left[x^2 + \dfrac{1}{2}\right]$, where $[,]$ = G.I.F

29. Find the range of $f(x) = \log\left\{\cos\sqrt{\dfrac{\pi^2}{9} - x^2}\right\}$

30. Find the range of $f(x) = [1 + \sin x] + [\cos x - 1] + [\tan^{-1} x]$ where $[,]$ = G.I.F.

31. Find the range of the function

$f(x) = \log_{10}\left\{\sin^{-1}(\sqrt{x - 5}) + \dfrac{3\pi}{2}\right\}$

32. Find the range of the function

$f(x) = \log_e(2\sin x + \tan x - 3x + 1)$,

where $\dfrac{\pi}{6} \leq x \leq \dfrac{\pi}{3}$

33. Find the range of the function

$f(x) = \log_{\sqrt{5}}\left\{\sqrt{2}(\sin x - \cos x) + 3\right\}$

34. Solve for $x: \sqrt{3 - 4\cos^2 x} > 2\sin x + 1$

35. Let $m = \left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{200}\right] + \left[\frac{1}{4} + \frac{2}{200}\right] + ... +$

$\left[\frac{1}{4} + \frac{199}{200}\right]$. Find the value of $m + 50$.

36. Solve for $x: \{x\} + \sin\{x\} = 2$, where $\{,\}$ = F.I.F

37. Solve for $x: x^2 - 4x + [x] + 3 = 0$

38. Let $F(x)$ be a function defined by $F(x) = x - [x]$, $x \in R - \{0\}$, [,] = G.I.F, find the number of solutions of $F(x) + F\left(\frac{1}{x}\right) = 1$

39. Find the number of values of x satisfying $\{x^2\} + [x^4] = 1$

40. Find the least value of the function $f(x) = |x-a| + |x-b| + |x-c| + |x-d|$ where $a < b < c < d$ and x takes any arbitrary real number.

41. Find the value of $\sum_{n=1}^{89}\left(\frac{1}{1+(\tan n^{\circ})^2}\right)$

42. Find the set of values of x satisfying the equality $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$, where [,] = G..I.F belongs to the interval $\left(a, \frac{b}{c}\right]$, where a, b, c are natural number and $\frac{b}{c}$ is in its lowest forms. Find the value of $a + b + c + abc + 30$.

43. If the equation $\left(\frac{1}{4}\right)^x + \left(\frac{1}{2}\right)^{x-1} + b = 0$ has a positive solution, find the value of b.

44. If $f(x) = px + q$ and $f(f(f(x))) = 8x + 21$ where p and q are real numbers, find the value of $p^2 + q^2 + p + q$.

45. Find the set of values of p for which the function $f(x) = x^3 - 2x^2 - px + 1$ is one-one.

46. A function $f: R \to R$ is given by $f(x) = \frac{ax^2 + 6a - 8}{a + 6x - 8x^2}$.

Find the values of 'a' for which the function 'f' is onto.

47. Prove that $f(x) = \frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[\frac{x + 2\pi}{\pi}\right] - 3}$ is an odd function, where [,] = G.I.F

48. Find the solution of the equation

$$x^2 - x + 1 = \frac{1}{2} + \sqrt{\left(x - \frac{3}{4}\right)}$$

49. Let $f(x) = \sin^{2016} x - x - \cos^{2016} x$ and, $g(x) = \sin x + \cos x$, then find the general solution of $f(x) = \left[g\left(\frac{\pi}{10}\right)\right]$ where [,] = G.I.F.

50. Find the maximum and minimum values of $f(x, y) = 7x^2 + 4xy + 3y^2$, where subject to the condition $x^2 + y^2 = 1$.

Integer Type Questions

1. If the fundamental period of the function $f(x) = 4\cos^4\left(\frac{x - \pi}{4\pi^2}\right) - 2\cos\left(\frac{x - \pi}{2\pi^2}\right)$ is $m(\pi)^n$, where m and n are positive integers, then $(m + n)$ is
2. If the domain of the function $f(x) = \log_2(\sqrt{x - 2} + \sqrt{4 - x})$ is $[a, b]$ then find the value of $(2b - 3a)$ is
3. If the natural domain of the function $f(x) = \sqrt{\log_2\left(\frac{x - 2}{3 - x}\right)}$ is $[a, b)$, then find the value of $(2a - b)$ is
4. If the equation $|x - 2| - |x + 1| = p$ has exactly one solution, then find the number of integral values of p.
5. If the range of the function $f(x) = \frac{x^2 + x + a}{x^2 + 2x + a}$, $x \in R$ is $\left[\frac{5}{6}, \frac{3}{2}\right]$ then find the value of $\left(\frac{a^2 + a}{5}\right)$
6. The number of integral values of x satisfying the equation $\text{sgn}\left(\left[\frac{15}{x^2 + 1}\right]\right) = [1 + \{2x\}]$ is
7. Find the number of integral values of x for which satisfying the equation $\sqrt[3]{x + 1} = \sqrt{x - 3}$
8. Find the number of solutions of $\text{sgn}(x + 1) = 2x^2 - x$
9. Find the number of integers in the do main of the function $f(x) = \frac{1}{\sqrt[x-1]{81} - 3}$
10. If the range of the function $f(x) = \frac{x^2}{x^4 + 1}$ is $\left[a, \frac{b}{c}\right]$, where $a, b, c \in I$

then find the value of $(a + b + c + 2)$

11. If the range of the function
$f(x) = \dfrac{e^x - 1}{e^x + 1}$ is (a, b), then find the value of
$(a^2 + b^2 + 2)$

12. If $f(x) = ax + b$ and $f(f(x)) = 8x + 21$ where $a, b \in R$, then find the value of $(a + b + 3)$.

Comprehensive Link Passages

Passage I

If the function $f: R \to R$ and $g: R \to R$ be defined as $f(x) = ex$ and $g(x) = 3x - 2$. On the basis of the above information answers the following questions.

(i) The function $f_o g$ is
(a) e^{3x-2} (b) e^{3x+2}
(c) $3e^x - 2$ (d) $3e^x + 2$

(ii) The function goo is
(a) e^{3x-2} (b) e^{3x+2}
(c) $3e^x - 2$ (d) $3e^x + 2$

(iii) Domain of $(f_o g)$ –1 (x) is
(a) $(-\infty, \infty)$ (b) $(-\infty, 0)$
(c) $(0, \infty)$ (d) $[1, 3]$

(iv) Domain of (goo) –1 (x) is
(a) $(-\infty, 2]$ (b) $[-2, 2]$
(c) $(-\infty, \infty)$ (d) $(-2, \infty)$

Passage II

Let $f(x)$ be a function satisfying the functional equation $f(x) + f(1 - x) = k$, for every $x \in Q$, where k is a constant quantity.

Let m be a positive integer.

Put $x = \dfrac{r}{m + 1}$ in the given equation we get

$$f\left(\frac{r}{m+1}\right) + f\left(\frac{m+1-r}{m+1}\right) = k$$

$$\Rightarrow \sum_{r=1}^{m} f\left(\frac{r}{m+1}\right) + \sum_{r=1}^{m} f\left(\frac{m+1-r}{m+1}\right) = mk$$

$$\Rightarrow \sum_{r=1}^{m} f\left(\frac{r}{m+1}\right) + \sum_{r=1}^{m} f\left(\frac{t}{m+1}\right) = mk$$

where $t = m + 1 - r$.

$$\Rightarrow 2\sum_{r=1}^{m} f\left(\frac{r}{m+1}\right) = mk$$

$$\Rightarrow \sum_{r=1}^{m} f\left(\frac{r}{m+1}\right) = \frac{mk}{2}$$

(i) If $f(x) = \dfrac{4^x}{4^x+2}$ where $x \in Q$, then $\sum_{r=1}^{2008} f\left(\dfrac{r}{2009}\right)$ is equal to
(a) 1004 (b) 2006
(c) 2007 (d) None

(ii) If $f(x) = \dfrac{3^{x-3}}{3^{1-x} + 3^x}$, $x \in Q$, then the value of $\sum_{r=1}^{54} f\left(\dfrac{r}{55}\right)$ is
(a) 54 (b) 27
(c) 1 (d) 55

(iii) If $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$ $(a > 0)$, then the value of $\sum_{r=1}^{2n-1} 2f\left(\dfrac{r}{2^n}\right)$
(a) 1 (b) $2n$
(c) $2n–1$ (d) $(2n-1)\dfrac{a}{2}$.

Passage III

Let [,] = G.I.F, then $\left[x + \frac{1}{2}\right] = [2x] - [x]$ for every $x \in R$.

Any real x can be taken as $k + f$ or $k + \frac{1}{2} + f$, where k is an integer and. $0 \le f < \frac{1}{2}$

If $x = k + f$, then $[x] = k$, $\left[x + \frac{1}{2}\right] = k$ and $[2x] = [2k + 2f] = 2k$.

Thus $\left[x + \frac{1}{2}\right] = [2x] - [x]$.

Again, if $x = k + \frac{1}{2} + f$, then $[x] = k$, $\left[x + \frac{1}{2}\right] = k + 1$ and $[2x] = [2k + 2f + 1] = 2k + 1$.

Thus, $\left[x + \frac{1}{2}\right] = [2x] - [x]$.

(i) If $x < 2k$, x and k being positive integers then the value of $\left[\dfrac{n + 2^k}{2^{k+1}}\right]$ is
(a) n (b) $n + 1$
(c) 0 (d) None

(ii) For every real x, $\left[x + \frac{1}{2}\right] + \left[x + \frac{1}{2}\right] + [x]$ is equal to
(a) $[3x]$ (b) $[2x] + 1$
(c) $[2x] + 3$ (d) None

(iii) For every positive integers n, the sum of
$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{2^2}\right] + \left[\frac{n+2^2}{2^3}\right] + \dots + \left[\frac{n+2k}{2^{k+1}}\right] \text{ is}$$

(a) 0 (b) n
(c) $2^k\, n$ (d) infinite.

Passage IV

Let two sets A and B consists of m and n elements respectively. Then the number of onto functions between two sets A and B is the number of distributions of 'm' balls into 'n' boxes, where no box is remain empty.

(i) The number of onto functions between two sets $A = \{1, 2, 3, 4\}$ and $B = \{5, 6\}$ is
(a) 16 (b) 14
(c) 20 (d) 18

(ii) The number of onto functions between two sets $A = \{a, b, c, d\}$ and $\{1, 2, 3, 4, 5\}$ is
(a) 0 (b) 10
(c) 20 (d) 625

(iii) The number of onto functions between two sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{5, 10, 15\}$ is
(a) 100 (b) 150
(c) 200 (d) 125

Passage V

Let two sets A and B consists of m and n elements respectively. Then the number of one-one function between two sets is $= \begin{cases} {}^nP_m & : n > m \\ 0 & : n < m \end{cases}$

(i) The number of one-one functions between two sets $A = \{1, 2, 3\}$ and $B = \{3, 4\}$ is
(a) 0 (b) 3
(c) 2 (d) 1

(ii) The number of one-one functions between two sets $A = \{2, 4, 6\}$ and $B = \{4, 6, 8\}$ is
(a) 4 (b) 6
(c) 8 (d) 10

(iii) The number of one-one function between two sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$ is
(a) 4 (b) 5
(c) 6 (d) 8

(iv) The number of one-one functions between tswo sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ such that $f(1) = 4$ is
(a) 3 (b) 2
(c) 1 (d) 6.

(v) The number of one-one functions between two sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ such that $f(1) \neq 4$ is
(a) 3 (b) 2
(c) 4 (d) 6

Matrix Match
(For JEE-Advanced Examination only)

1. **Match the statement in Column-I with the statement in Column-II**

Column I		Column II	
(A)	Range of $f(x) = \cos^{-1}(\lvert 1 - x^2 \rvert)$	(P)	$\left(0, \frac{\pi}{4}\right]$
(B)	Range of $f(x) = \cot^{-1}(x^2 - 4x + 5)$	(Q)	$\left[0, \frac{\pi}{2}\right]$
(C)	Range of $f(x) = \log \lvert \log x \rvert$	(R)	$\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$
(D)	Range of $f(x) = \dfrac{\log_{2x} 3}{\cos^{-1}(2x-1)}$	(S)	$(0, 1) \cup (1, \infty)$

2. **Match the statement in Column-I with the statement in Column-II**

Column I		Column II	
(A)	Period of $f(x) = \frac{1}{3}\{\sin(3x) + \lvert \sin 3x \rvert + [\sin 3x]\}$	(P)	1
(B)	Period of $f(x) = \cos(\tan x + \cot x) + \cos(\tan x - \cot x)$	(Q)	$\frac{2\pi}{3}$
(C)	Period of $f(x) = e^{\cos^4(\pi x) + x - [x]} + \cos^2(\pi x)$	(R)	10π
(D)	Period of $f(x) = \left\lvert \sin^3 \frac{x}{2} \right\rvert + \left\lvert \cos^5 \frac{x}{5} \right\rvert$	(S)	$\frac{\pi}{2}$

3. **Match the statement in Column-I with the statement in Column-II**

Column I (Functions)		Column II (Ranges)	
(A)	$f(x) = \sin(\pi x)$	(P)	$\left[0, \frac{1}{2}\right)$
(B)	$f(x) = \dfrac{x - [x]}{1 - [x] + x}$	(Q)	$[1, \infty)$
(C)	$f(x) = \dfrac{e^x}{1 + [x]}$	(R)	$\left[\frac{\pi}{4}, \pi\right)$
(D)	$f(x) = \cot^{-1}(2x - x^2)$	(S)	$[-1, 1]$

4. Match the statement in Column-I with the statement in Column-II

Column I		Column II	
(A)	$f(x) = 2x+(1-x)\ \text{sgn}(x)$	(P)	one-one into
(B)	$f(x) = 2x+(x-1)\ \text{sgn}(x)$	(Q)	Many-one onto
(C)	$f(x) = x^3-x+1$	(R)	Onto but not one-one.
(D)	$f(x) = \dfrac{2x+1}{3x+4}$	(S)	neither one-one nor onto.

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.

(B) Both A and R are true but R is not the correct explanation of A.

(C) A is true and R is false.

(D) A is false and R is true.

1. **Assertion (A):** The number of one-one function between two sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ is 6.

   Reason (R): The number of onto function between two sets $A = \{1, 2, 3, 4\}$ and $B = \{5, 6\}$ is 14.

2. **Aseertion (A):** The range of the function

   $f(x) = 2^x + 3^x + 4^x + 2^{-x} + 3^{-x} + 4^{-x} + 10$ is 16

   Reason (R): For any two positive numbers a, b, $\dfrac{a+b}{2} \geq \sqrt{ab}$

3. **Assertion (A):** The solution set of $\left|\dfrac{x}{x-1}\right| + |x| = \dfrac{x^2}{|x-1|}$ is $(1, \infty) \cup \{0\}$

   Reason (R): If $|f(x) + g(x)| = |f(x)| + |g(x)|$, then $f(x)\, g(x) \geq 0$

4. **Assertion (A):** The number of solutions of $|x| + |x^2-1| = \dfrac{5}{6}$ is 6

   Reason (R): The number of solutions of $\sin x = x^2 + x + 1$ is 0

5. **Assetion (A):** The domain of the function

   $f(x) = \sqrt{x-2} + \sqrt{4-x}$ is [2, 4]

   Reason (R): The range of the function $f(x)$ is $\lfloor\sqrt{2}, 2\rfloor$

6. **Assetion (A):** The period of the function $f(x) = \sin^{2n+2} x + \cos^{2n+2} x$ is $\dfrac{\pi}{2}$, $n \in N$

   Reason (R): If $f(x)$ and $g(x)$ be two periodic functions with periods T_1, T_2 respectively, as well as both are even, co-functions and comparable functions, then the period of $f(x) + g(x)$ is $\dfrac{1}{2} \times$ L.C.M of $\{T_1, T_2\}$.

7. **Assetion (A):** The range of the function

   $f(x) = \sin^{-1}\left[x^2 + \dfrac{1}{2}\right] + \cos^{-1}\left[x^2 - \dfrac{1}{2}\right]$ is π

   Reason (R): The range of the above function is a singletone set.

8. **Assetion (A):** A function $f: R \to R$ is defined as $f(x + y) = f(x) + f(y)$, then it is an odd function.

   Reason (R): A function $f(x) = 0$ is the only one function, which is even as well as odd function.

9. **Assetion (A):** Let $f(x) = \left|\left|\dfrac{1}{x}\right| - 2\right|$ Then the number of solutions of $f(x) = 1$ is 4.

   Reason (R): Inverse of an even function is defined.

10. **Assetion (A):** Every periodic function is an even function.

    Reason (R): The domain of $f(x) = \sin^{-1}\left[\log_3\left(\dfrac{x}{3}\right)\right]$ is {1, 9]

Questions asked in Previous Years' JEE Main Exams (2002 to 2014)

1. Which one is not periodic?

   (a) $|\sin 3x| + \sin^2 x$ (b) $\cos \sqrt{x} + \cos^2 x$

   (c) $\cos 4x + \tan^2 x$ (d) $\cos 2x + \sin x$

   [JEE-Main, 2002]

2. The domain of $\sin^{-1}\left(\log_3\left(\dfrac{x}{3}\right)\right)$ is

   (a) [1, 9] (b) [–1, 9]

   (c) [–9, 1] (d) [–9, –1]

   [JEE-Main, 2002]

3. Domain of definition of the function

   $f(x) = \dfrac{3}{4-x^2} + \log_{10}(x^3 - x)$ is

   (a) $(-1, 0) \cup (1, 2)$

   (b) $(1, 2) \cup (2, \infty)$

   (c) $(-1, 0) \cup (1, 2) \cup (2, \infty)$

   (d) $(1, 2)$ **[JEE-Main, 2003]**

4. If $R \to R$ satisfies $f(x + y) = f(x) + f(y)$ for all x, y in R such that $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is

(a) $\frac{7(n+1)}{2}$ (b) $7n(n+1)$

(c) $\frac{7n(n+1)}{2}$ (d) $\frac{7n}{2}$

[JEE-Main, 2003]

5. The range of the function $f(x) = {}^{7-x}P_{x-3}$ is

(a) {1, 2, 3, 4} (b) {1, 2, 3, 4, 5, 6}

(c) {1, 2, 3} (d) {1, 2, 3, 4,5}

[JEE-Main, 2004]

6. The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(a) [1, 2] (b) [2, 3)

(c) [2, 3] (d) [1, 2)

[JEE-Main, 2004]

7. If $R \to S$ defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$ is onto, then S is

(a) [0, 1] (b) [−1, 1]

(c) [0, 3] (d) [−1, 3]

[JEE-Main, 2004]

8. The graph of the function is symmetrical about the line $x = 2$, then

(a) $f(x) = f(-x)$ (b) $f(2 + x) = f(2 - x)$

(c) $f(x + 2) = f(x - 2)$ (d) $f(x) = -f(-x)$

[JEE-Main, 2004]

9. Let $f:(-1, 1) \to B$ be a function is defined by

$f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then f is one-one and onto, when B is the interval

(a) $\left[0, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{2}\right)$

(c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[JEE-Main, 2005]

10. A real valued function $f(x)$ satisfies the functional equation $f(x - y) = f(x) f(y) - f(a - x) f(a + y)$ where a is a given constant and $f(0) = 1$, then $f(2a - x)$ is

(a) $f(x)$ (b) $-f(x)$

(c) $f(-x)$ (d) $f(a) + f(a - x)$

[JEE-Main, 2005]

11. If x is real, the max value of $\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is

(a) 1/4 (b) 41

(c) 1 (d) 17/7

[JEE-Main, 2006]

12. The largest interval lying in

$f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$ is defined is

(a) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left[0, \frac{\pi}{2}\right)$

(c) $[0, \pi]$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

[JEE-Main, 2007]

13. Let $f: N \to Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N: y = 4x + 3 \text{ for some } x \in N\}$ and f is invertible, then $f^{-1}(x)$ is

(a) $\frac{x-3}{4}$ (b) $\frac{3x+4}{3}$

(c) $4 + \frac{x+3}{4}$ (d) $\frac{x+3}{4}$

[JEE-Main, 2008]

14. Let $f(x) = (x + 1)^2 - 1, x \geq -1$

Statement-I: The set $\{x: f(x) = f^{-1}(x)\} = \{0, 1\}$
Statement-II: f is a bijection

[JEE-Main, 2009]

15. No question asked in 2010.

16. The domain of the definition $f(x) = \frac{1}{\sqrt{|x| - x}}$ is

(a) $(-\infty, 0)$ (b) $(-\infty, \infty) - \{0\}$

(c) $(-\infty, \infty)$ (d) $(0, \infty)$

[JEE-Main, 2011]

17. No question asked in 2012.

18. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is

(1) 220 (2) 219

(3) 211 (4) 256

[JEE-Main, 2013]

19. If $X = \{4^n - 3n - 1: n \in N\}$ and $Y = \{9(n - 1): n \in N\}$ where N is the set of natural numbers, then $X \cup Y$ is equal to

(1) N (2) $Y - X$

(3) X (4) Y

[JEE-Main, 2014]

20. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is

(1) 256 (2) 275

(3) 510 (4) 219

[JEE-Main, 2015]

Question asked in Previous Years' JEE-Advanced Examinations

1. Given $A = \left\{x: \frac{\pi}{6} \le x \le \frac{\pi}{3}\right\}$ and $f(x) = \cos x - x(x + 1)$, find $f(A)$

[IIT-JEE, 1980]

2. Let f be a function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statement is true and the remaining two are false: $f(x) = 1$, $f(y) \ne 1$, $f(z) \ne 2$ Determine $f^{-1}(1)$

[IIT-J22, 1982]

3. The values of $f(x) = 3\sin\left(\sqrt{\frac{\pi^2}{16} - x^2}\right)$ lies in the interval .... **[IIT-JEE, 1983]**

4. If $f(x) = (a - x^n)^{1/n}$, where $a > 0$ and n is a positive integer, then $f(f(x)) = x$ Is it true? **[IIT-JEE, 1983]**

5. The fucntion $f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$ is not one to one. Is it true? **[IIT-JEE, 1983]**

6. If $f(x) = |x - 1|$. Then

(a) $f(x^2) = (f(x))^2$ (b) $f(x + y) = f(x) + f(y)$
(c) $f(|x|) = |f(x)|$ (d) None of these.

[IIT-JEE, 1983]

7. If $f(x) = \cos(\log(x))$, then $f(x)\,f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value

(a) -1 (b) $1/2$
(c) -2 (d) None.

[IIT-JEE, 1983]

8. The domain of the function

$f(x) = \frac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$ is

(a) $(-3, -2) - \{-2.5\}$ (b) $(0, 1) - \{0. 5\}$
(c) $(-2, 1) - \{0\}$ (d) None of these.

[IIT-JEE, 1983]

9. Which of the following functions are periodic?

(a) $f(x) = x - [x]$, [.] = G.I.F

(b) $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & : x \ne 0 \\ 0 & : x = 0 \end{cases}$

(c) $f(x) = x\cos x$

(d) None of these. **[IIT-JEE, 1983]**

10. The domain of the function $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$ is ..... **[IIT-JEE, 1984]**

11. For real x, the function $f(x) = \frac{(x - a)(x - b)}{(x - c)}$ will assume all real value provided

(a) $a > b > c$ (b) $a < b < c$
(c) $a > c < b$ (d) $a \le c \le b$

[IIT-JEE, 1984]

12. If $f(x) = \sin\left(\ln \frac{\sqrt{4 - x^2}}{1 - x}\right)$, then the domain of $f(x)$ is .... **[IIT-JEE, 1985]**

13. No questions asked in 1986.

14. Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$ for all x. If $f(0)$ exists, then find its value

[IIT-JEE, 1987]

15. If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively, then $(f_1(x) + f_2(x))$ is defined on $D_1 \cup D_2$. Is it true? **[IIT-JEE, 1988]**

16. There are exactly two distinct linear functions .... and .... which map $[-1, 1]$ onto $[0, 2]$.

[IIT-JEE, 1989

17. If the function $f(x) = \frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)$, $-1 < x < 1$ and $g(x) = \sqrt{3 + 4x - 4x^2}$, then the domain of $(g_o f)$ is

(a) $(-1, 1)$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(c) $\left[-1, \frac{1}{2}\right]$ (d) $\left[\frac{1}{2}, 1\right]$

[IIT-JEE, 1990]

18. No questions asked in 1991.

19. The values of b and c for which the identity $f(x + 1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$, are

(a) $b = 2, c = 1$ (b) $b = 4, c = -1$
(c) $b = -1, c = 4$ (d) None

[IIT-JEE, 1992]

20. The value of the parameter α, for which the function $f(x) = 1 + \alpha x$, $\alpha \ne 0$ is the inverse of itself, if

(a) -2 (b) -1
(c) 1 (d) 2

[IIT-JEE, 1992]

21. No questions asked in 1993.

22. Let $f(x) = \sin x$ and $g(x) = \ell n|x|$. If the ranges of the composite functions $(f_o g)$ and $(g_o f)$ are R_1 and R_2 respectively then

(a) $R_1 = \{u: -1 < u < 1\}$, $R_2 = \{v: -\infty < v < 0\}$
(b) $R_1 = \{u: -\infty < u \le 1\}$, $R_2 = \{v: -1 \le v \le 0\}$

(c) $R_1 = \{u: -1 < u < 1\}$, $R_2 = \{v: -\infty < v < 0\}$
(d) $R_1 = \{u: -1 \le u \le 1\}$, $R_2 = \{v: -\infty < v \le 0\}$

[IIT-JEE, 1994]

23. Let $f(x) = (x + 1)^2 - 1$, $x \ge -1$. Then the set $\{x : f(x) = f^{-1}(x)\}$ is
(a) $\left\{0, -1, \frac{-3 + i\sqrt{3}}{2}, \frac{-3 - i\sqrt{3}}{2}\right\}$
(b) $\{-1, 0, 1\}$
(c) $\{-1, 0\}$
(d) Empty **[IIT-JEE, 1995]**

24. If f is an even function defined on the interval $(-5, 5)$, then the real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are ..... **[IIT-JEE, 1996]**

25. If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right), = 1$, then $(g_o f)(x)$ is
(a) -2 (b) -1
(c) 2 (d) 1

[IIT-JEE, 1996]

26. A function $f: R \to R$, where R is the set of real-numbers, is defined by $f(x) = \frac{\alpha x^2 + 5x - 8}{\alpha + 5x - 8x^2}$. Find the interval of values of a for which is onto. Is the function one to one for $\alpha = 3$? Justify your answer. **[IIT-JEE, 1996]**

27. No questions asked in 1997.

28. If $g(f(x)) = |\sin(x)|$ and $f(g(x)) = (\sin(\sqrt{x}))^2$ then
(a) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$
(b) $f(x) = \sin x$, $g(x) = |x|$
(c) $f(x) = x^2$, $g(x) = \sin(\sqrt{x})$
(d) f and g can not be determined.

[IIT-JEE, 1998]

29. If $f(x) = 3x - 5$, then $f^{-1}(x)$ is
(a) $\frac{1}{3x - 5}$
(b) $\left(x + \frac{5}{3}\right)$
(c) does not exist because f is not one-one
(d) does not exist because f is not onto.

[IIT-JEE, 1999]

30. If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is
(a) $\left(\frac{1}{2}\right)^{x(x-2)}$ (b) $\frac{1}{2}(1 + \sqrt{1 + 4\log_2 x})$
(c) $\frac{1}{2}(1 + 4\log_2 x)$ (d) Not defined

[IIT-JEE, 1999]

31. The domain of definition of the function $y(x)$ is given by the equation $2^x + 2^y = 2$ is
(a) $0 < x \le 1$ (b) $0 \le x \le 1$
(c) $-\infty < x \le 0$ (d) $-\infty < x < 1$

[IIT-JEE, 2000]

32. The domain of definition of the function $f(x) = \frac{\log_2(x + 3)}{x^2 + 3x + 2}$ is
(a) $R - \{-1, -2\}$ (b) $(-2, \infty)$
(c) $R - \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

[IIT-JEE, 2001]

33. Let $g(x) = 1 - x + [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$, then for all x is
(a) x (b) 1
(c) $f(x)$ (d) $g(x)$

[IIT-JEE, 2001]

34. If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals
(a) $\frac{x + \sqrt{x^2 - 4}}{2}$ (b) $\frac{x + \sqrt{x^2 - 4}}{4}$
(c) $\frac{x - \sqrt{x^2 - 4}}{2}$ (d) $1 + \sqrt{x^2 - 4}$

[IIT-JEE, 2001]

35. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is
(a) $[0, 1]$ (b) $\left[0, \frac{1}{2}\right]$
(c) $\left[\frac{1}{2}, 1\right]$ (d) $(0, 1]$

[IIT-JEE, 2001]

36. Let e $\{1, 2, 3, 4\}$ and $F = \{1, 2\}$. Then the number of onto functions from E to F is
(a) 14 (b) 16
(c) 12 (d) 8

[IIT-JEE, 2001]

37. Let $f(x) = \frac{\alpha x}{x + 1}$, $x \ne -1$. Then for what value of α is $f(f(x)) = x$?
(a) $\sqrt{2}$ (b) $-\sqrt{2}$
(c) 1 (d) -1

[IIT-JEE, 2001]

38. Suppose $f(x) = (x + 1)^2$, $x \geq -1$. If $g(x)$ is the inverse whose graph is reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals

(a) $-\sqrt{x} - 1,\ x \geq 0$ (b) $\dfrac{1}{(x+1)^2},\ x > -1$

(c) $\sqrt{x+1},\ x \geq -1$ (d) $\sqrt{x} - 1,\ x \geq 0$

[IIT-JEE, 2002]

39. Let function $f: R \to R$ be defined by $f(x) = 2x + \sin x$ for all x in R. Then 'f' is
(a) one-to-one and onto
(b) one-to-one but not onto.
(c) onto but not one-to-one.
(d) neither one-to-one nor onto.

[IIT-JEE, 2002]

40. If $f: [0, \infty) \to [0, \infty)$ and $f(x) = \dfrac{x}{x+1}$, then f is
(a) one-to-one and onto
(b) one-to-one but not onto.
(c) onto but not one-to-one.
(d) neither one-to-one nor onto.

[IIT-JEE, 2003]

41. Range of the function $f(x) = \dfrac{x^2 + x + 2}{x^2 + x + 1}$ for all x in R, is

(a) $(1, \infty)$ (b) $\left(1, \dfrac{11}{7}\right)$

(c) $\left(1, \dfrac{7}{3}\right]$ (d) $\left(1, \dfrac{7}{5}\right)$

[IIT-JEE, 2003]

42. The domain of definition of the function

$f(x) = \sqrt{\sin^{-1}(2x) + \dfrac{\pi}{6}}$ for real valued x is

(a) $\left[-\dfrac{1}{4}, \dfrac{1}{2}\right]$ (b) $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(c) $\left(\dfrac{-1}{2}, \dfrac{1}{9}\right)$ (d) $\left[-\dfrac{1}{4}, \dfrac{1}{4}\right]$

[IIT-JEE, 2003]

43. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $g(f(x))$ is invertible in the domain

(a) $\left[0, \dfrac{\pi}{2}\right]$ (b) $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

(c) $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ (d) $[0, \pi]$

[IIT-JEE, 2004]

44. Let $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

and $g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$

Then $f - g$ is
(a) one-one and into
(b) neither one-one nor onto.
(c) many one and onto
(d) one-one and onto.

[IIT-JEE, 2005]

45. X and Y are two sets and $f: X \to Y$ If $\{f(c) = y;\ c \subset X,\ y \subset Y\}$ and $\{f^{-1}(d) = x;\ d \subset Y,\ x \subset X\}$ then the true statement is

(a) $f(f^{-1}(b)) = b$ (b) $f^{-1}(f(a)) = a$

(c) $f(f^{-1}(b)) = b,\ b \subset Y$ (d) $f^{-1}(f(a)) = a,\ a \subset X$

[IIT-JEE, 2005]

46. No question asked in between 2006 to 2010.

47. Find the domain of the function

$f(x) = \sin^{-1}\left(\dfrac{8.3^{x-1}}{1 - 3^{2(x-1)}}\right)$ **[IIT-JEE, 2011]**

48. The function $f: [0, 3] \to [1, 29]$ is defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is
(a) one-one and onto
(b) onto but not one-one
(c) one-one but not onto
(d) neither one-one not onto **[IIT-JEE, 2012]**

49. Let $f: (-1, 1) \to R$ be such that for $f(\cos 4\theta) = \dfrac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. Then find the value of $f\left(\dfrac{1}{3}\right)$ is/are

(a) $\left(1 - \sqrt{\dfrac{3}{2}}\right)$ (b) $\left(1 + \sqrt{\dfrac{3}{2}}\right)$

(c) $\left(1 - \sqrt{\dfrac{2}{3}}\right)$ (d) $\left(1 + \sqrt{\dfrac{2}{3}}\right)$

[IIT-JEE, 2012]

50. No questions asked in 2013.

51. Let $f: \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$ Then
(a) $f(x)$ is an odd function
(b) $f(x)$ is a one-one function
(c) $f(x)$ is an onto function
(d) $f(x)$ is an even function.

[IIT-JEE, 2014]

52. No questions asked in 2015, 2016.

ANSWERS

LEVEL II

1. (b)	2. (a)	3. (b)	4. (b)	5. (b)
6. (a)	7. (a)	8. (c)	9. (a)	10. (b)
11. (b)	12. (b,c)	13. (a)	14. (d)	15. (d)
16. (b)	17. (d)	18. (a)	19. (c)	20. (a)
21. (d)	22. (d)	23. (d)	24. (c)	25. (b)
26. (b)	27. (c)	28. (b)	29. (d)	30. (a)
31. (c)	32. (a)	33. (a)	34. (a,c)	35. (a,d)
36. (a)	37. (a,b,c,d)		38. (b).	39. (c)
40. (c)	41. (c)	42. (b)	43. (b)	44. (a)
45. (c)	46. (b)	47. (b)	48. (c)	49. (b)
50. (b)				

INTEGER TYPE QUESTIONS

1. (5)	2. (2)	3. (2)	4. (5)	5. (4)
6. (7)	7. (1)	8. (2)	9. (2)	10. (5)
11. (4)	12. (8)			

MATRIX MATCH

1. (A) → (Q), (B) → (P), (C) → (S), (D) → (R)
2. (A) → (Q), (B) → (S), (C) → (P), (D) → (R)
3. (A) → (S), (B) → (P), (C) → (Q), (D) → (R)
4. (A) → (S), (B) → (R), (C) → (Q), (D) → (P).

COMPREHENSIVE LINK PASSAGES

P-I : (i) (a), (ii) (c), (iii) (c), (iv) (d)
P-II : (i) (a), (ii) (c), (iii) (c)
P-III : (i) (c), (ii) (a), (iii) (b)
P-IV : (i) (b), (ii) (a), (iii) (b)
P-V : (i) (a), (ii) (b), (iii) (a) (iv) (b), (v) (c)

ASSERTION AND REASON

1. (B)	2. (A)	3. (A)	4. (B)	5. (B)
6. (A)	7. (B)	8. (B)		

HINTS AND SOLUTIONS

Level I

1. Given $f(x) = 3x^4 - 5x^2 + 9$

Thus, $f(x - 1)$

$= 3(x - 1)^4 - 5(x - 1)^2 + 9$

$= 3(x^4 - 4x^3 + 6x^2 - 4x + 1) - 5(x^2 - 2x + 1) + 9$

$= 3x^4 - 12x^3 + 13x^2 - 2x + 7$

2. If $f(x) = x + \frac{1}{x}$, prove that $(f(x))^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$

2. Given $f(x) = x + \frac{1}{x}$

We have $f(x^3) + 3f\left(\frac{1}{x}\right)$

$$= x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right)$$

$$= \left(x + \frac{1}{x}\right)^3$$

$$= (f(x))^3$$

Hence, the result.

3. We have $f(x) + f(y)$

$$= \log\left(\frac{1-x}{1+x}\right) + \log\left(\frac{1-y}{1+y}\right)$$

$$= \log\left(\frac{1-x}{1+x} \times \frac{1-y}{1+y}\right)$$

$$= \log\left(\frac{1 - x - y + xy}{1 + x + y + xy}\right)$$

$$= \log\left(\frac{1 + xy - x - y}{1 + xy + x + y}\right)$$

$$= \log\left(\frac{1 - \frac{x+y}{1+xy}}{1 + \frac{x+y}{1+xy}}\right)$$

$$= f\left(\frac{x+y}{1+xy}\right)$$

Hence, the result.

4. Given $f(x) = x^3 - \frac{1}{x^3}$

Now, $f(x) + f\left(\frac{1}{x}\right)$

$$= x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3$$

$$= 0$$

Hence, the result.

5. Given $f(x) = \frac{2x}{1 + x^2}$

Now, $f(\tan\theta)$

$$= \frac{2\tan\theta}{1 + \tan^2\theta}$$

$$= \sin(2\theta)$$

Hence, the result.

6. Given $f(x) = \log\left(\frac{x}{x-1}\right)$

We have $f(x+1) + f(x)$

$$= \log\left(\frac{x+1}{x}\right) + \log\left(\frac{x}{x-1}\right)$$

$$= \log\left(\frac{x+1}{x} \times \frac{x}{x-1}\right)$$

$$= \log\left(\frac{x+1}{x-1}\right)$$

Hence, the result.

7. Given $f(x) = \log\left(\frac{1+x}{1-x}\right)$

Now, $f\left(\frac{2x}{1+x^2}\right)$

$$= \log\left(\frac{1+\frac{2x}{1+x^3}}{1-\frac{2x}{1+x^2}}\right)$$

$$= \log\left(\frac{x^2+2x+1}{x^2-2x+1}\right)$$

$$= \log\left\{\left(\frac{x+1}{x-1}\right)^2\right\}$$

$$= 2\log\left(\frac{x+1}{x-1}\right)$$

$$= 2f(x)$$

Hence, the result.

9. If $f(x) = \cos(\log_e x)$, then find the value of

$f(x)f(y) - \frac{1}{2}\left(f(xy) + f\left(\frac{x}{y}\right)\right)$

9. Given $f(x) = \cos(\log_e x)$

We have $f(x)f(y) - \frac{1}{2}\left(f(xy) + f\left(\frac{x}{y}\right)\right)$

$$= \frac{1}{2}\left\{2f(x)f(y) - \left(f(xy) + f\left(\frac{x}{y}\right)\right)\right\}$$

$$= \frac{1}{2}\left\{\begin{array}{c} 2\cos(\log x)\cos(\log y) - \\ (\cos(\log x) + \cos(\log y)) \end{array}\right\}$$

$$= \frac{1}{2}\left\{\begin{array}{c} \cos(\log x + \log y) + \cos(\log x - \log y) \\ -\left(\cos(\log xy) + \cos\left(\log \frac{x}{y}\right)\right) \end{array}\right\}$$

$$= \frac{1}{2}\left\{\begin{array}{c} \cos(\log(xy)) + \cos\left(\log\left(\frac{x}{y}\right)\right) \\ -\left(\cos(\log xy) + \cos\left(\log\left(\frac{x}{y}\right)\right)\right) \end{array}\right\}$$

$$= 0$$

9. We have $y = f(x) = \frac{ax-b}{bx-a}$

$$\Rightarrow \quad (bx-a)y = ax-b$$

$$\Rightarrow \quad bxy - ay = ax - b$$

$$\Rightarrow \quad x(by-a) = ay-b$$

$$\Rightarrow \quad x = \frac{ay-b}{by-a}$$

$$\Rightarrow \quad x = f(y)$$

Hence, the result.

10. We have

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad \text{...(i)}$$

Replace x by $\frac{1}{x}$, we get

$$af\left(\frac{1}{x}\right) + bf(x) = x - 5 \quad \text{...(ii)}$$

Multiplying (i) by a (ii) by b and subtracting, we get

$$(a^2 - b^2)f(x) = a\left(\frac{1}{x} - 5\right) - b(x-5)$$

$$f(x) = \frac{a\left(\frac{1}{x} - 5\right) - b(x-5)}{(a^2 - b^2)}$$

11. Given $f(x) = \frac{9^x}{9^x + 3}$

We have $f(x) + f(1-x)$

$$= \frac{9^x}{9^x+3} + \frac{9^{1-x}}{9^{1-x}+3}$$

$$= \frac{9^x}{9^x+3} + \frac{9}{9+3\cdot 9^x}$$

$$= \frac{9^x}{9^x+3} + \frac{3}{3+9^x}$$

$$= \frac{9^x}{9^x+3} + \frac{3}{9^x+3}$$

$$= \frac{9^x+3}{9^x+3}$$

$$= 1.$$

Hence, the result.

12. We have $f\left(\frac{2\tan x}{1+\tan^2 x}\right)$

$$= \frac{1}{2}(1 + \cos 2x)(\sec^2 x + \tan x)$$

$$= \frac{1}{2} \times 2\cos^2 x \times (1 + \tan^2 x + 2\tan x)$$

$$= \cos^2 x \times (1 + \tan x)^2$$

$$= \{\cos x \times (1 + \tan x)\}^2$$

$$= (\cos x + \sin x)^2$$

$$= 1 + \sin(2x)$$

Thus, $f(\sin 2x) = 1 + \sin(2x)$

$$\Rightarrow \quad f(x) = 1 + x$$

13. Do Yourself

14. (i) $D_f = R - \{-3\}$
(ii) $D_f = R - \{3\}$
(iii) $D_f = R - \{0\}$
(iv) $D_f = R - \{0\}$
(v) $D_f = R - \{1, 2\}$
(vi) $D_f = R - \{1, 2\}$
(vii) $D_f = R$
(viii) $D_f = R - \{-2, 2\}$
(ix) $D_f = R$
(x) $D_f = R$

15. (i) $D_f = [2, \infty)$
(ii) $D_f = [-5, \infty)$
(iii) $D_f = (-\infty, 4]$
(iv) $D_f = [2, 4]$
(v) $D_f = [3, 7]$
(vi) $D_f = [2, 5]$
(vii) $D_f = [-\infty, 0] \cup [1, \infty)$
(viii) $D_f = (-\infty, 0) \cup [1, \infty)$
(ix) $D_f = [0, 1]$
(x) $D_f = (-\infty, 1] \cup [2, \infty)$

16. (i) Given $f(x) = \sqrt{|x| - 2}$
Thus, $|x| - 2 \geq 0$
$\Rightarrow |x| \geq 2$
$\Rightarrow x \geq 2, x \leq -2$
Therefore, $D_f = (-\infty, -2] \cup [2, \infty)$

(ii) Given $f(x) = \sqrt{4 - |x|}$
Thus, $4 - |x| \geq 2$
$\Rightarrow |x| \leq 4$
$\Rightarrow -4 \leq x \leq 4$
Therefore, $D_f = [-4, 4]$

(iii) Given $f(x) = \sqrt{|x| - 1} + \sqrt{4 - |x|}$
We have $|x| - 1 \geq 0$ and $4 - |x| \geq 0$
$\Rightarrow |x| \geq 1$ and $|x| \leq 4$
$\Rightarrow x \geq 1, x \leq -1$ and $-4 \leq x \leq 4$
$\Rightarrow x \in [-4, -1] \cup [1, 4]$
Therefore, $D_f = [-4, -1] \cup [1, 4]$

(iv) Given $f(x) = \sqrt{\dfrac{|x| - 1}{3 - |x|}}$
We have $\dfrac{|x| - 1}{3 - |x|} \geq 0$
$\Rightarrow \dfrac{|x| - 1}{|x| - 3} \leq 0$
$\Rightarrow 1 \leq |x| \leq 3$
$\Rightarrow x \in [-3, -1] \cup [1, 3]$
Therefore, $D_f = [-3, -1] \cup [1, 3]$

(v) $D_f = [0, \infty)$
(vi) $D_f = (0, \infty)$
(vii) $D_f = \phi$
(viii) $D_f = \phi$
(ix) $D_f = \phi$
(x) $D_f = R - I$

17. Dr = 0 gives $[x - 2] = 0$
$\Rightarrow 0 \leq (x - 2) < 1$
$\Rightarrow 2 \leq x < 3$
Thus, $D_f = R - [2, 3)$

18. Dr = 0 gives $[x + 1] = 0$
$\Rightarrow 0 \leq x + 1 < 1$
$\Rightarrow -1 \leq x < 0$
Thus, $D_f = R - [-1, 0)$

19. Dr = 0 gives $[x] - 4 = 0$
$\Rightarrow [x] = 4$
$\Rightarrow 4 \leq x < 5$
Thus, $D_f = R - [4, 5)$

20. Dr = 0 gives $x - [x] = 0$
$\Rightarrow [x] = x$
$\Rightarrow x \in I$
Also, $x - [x] > 0$
$\Rightarrow x > [x]$
$\Rightarrow x \in f, 0 < f < 1$
Thus, $D_f = R - I$

21. Dr = 0 gives $[x] - x = 0$
$\Rightarrow [x] = x$
$\Rightarrow x \in I$
Also, $[x] - x > 0$
$\Rightarrow [x] > x$
$\Rightarrow x = \phi$
Thus, $D_f = \phi$

22. We have $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$
$\Rightarrow f(x) = \cos 9x + \cos(-10)x$
$\Rightarrow f(x) = \cos 9x + \cos(10)x$
Now, $f\left(\dfrac{\pi}{2}\right)$
$= \cos\left(\dfrac{9\pi}{2}\right) + \cos(5\pi)$
$= 0 - 1$
$= -1$
Also, $f(\pi)$
$= \cos(9\pi) + \cos(10\pi)$
$= -1 + 1$
$= 0$

23. We have

$[\log_2 1] + [\log_2 2] + [\log_2 3] + ... + [\log_2 64]$

$= [\log_2 1] + [\log_2 2] + ... + [\log_2 4]$

$+ [\log_2 5] + [\log_2 6] + ... + [\log_2 8]$

$+ [\log_2 9] + [\log_2 10] + ... + [\log_2 16]$

$+ [\log_2 17] + [\log_2 18] + ... + [\log_2 32]$

$+ [\log_2 33] + [\log_2 34] + ... [\log_2 64]$

$+ [\log_2 65] + [\log_2 + 66]$

$= 2.3 + 3.4 + 4.16 + 5.32 + 6.2$

$= 6 + 12 + 64 + 160 + 12$

$= 254.$

24. We have $[x]^2 - 3[x] + 2 = 0$

$\Rightarrow \quad a^2 - 3a + 2 = 0, a = [x]$

$\Rightarrow \quad (a - 1)(a - 2) = 0$

$\Rightarrow \quad a = 1, 2$

when $a = 1 \Rightarrow [x] = 1$

$\Rightarrow 1 \le x < 2$

$\Rightarrow x \in [1, 2)$

When $a = 2 \Rightarrow [x] = 2$

$\Rightarrow 2 \le x < 3$

$\Rightarrow x \in [2, 3)$

Hence, the solution set is $x \in [1, 3)$

25. We have $[x]^2 - 2[x] - 8 = 0$

$\Rightarrow \quad b^2 - 2b - 8 = 0, b = [x]$

$\Rightarrow \quad (b - 4)(b + 2) = 0$

$\Rightarrow \quad b = -2, 4$

when $b = -2 \Rightarrow [x] = -2$

$\Rightarrow -2 \le x < 1$

$\Rightarrow x \in [-2, -1)$

when $b = 4 \Rightarrow [x] = 4$

$\Rightarrow 4 \le x < 5$

$\Rightarrow x \in [4, 5)$

Hence, the solution set is

$x \in [-2, -1) \cup [4, 5).$

26. We have $x^2 - 4 [x] = 0$

Case-I: When $x \in I$

$\Rightarrow \quad x^2 - 4 - x = 0$

$\Rightarrow \quad x^2 - x - 4 = 0$

$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 + 16}}{2} = \frac{1 \pm \sqrt{17}}{2}$

Case-II: When $x \notin I$

$\Rightarrow \quad x^2 - 4 - (x - \{x\}) = 0$

$\Rightarrow \quad x^2 - 4 - x + \{x\} = 0$

$\Rightarrow \quad \{x\} = x + 4 - x^2$

$\Rightarrow \quad 0 \le x + 4 - x^2 < 1$

Now, $\quad x + 4 - x^2 \ge 0$

$\Rightarrow \quad x^2 - x - 4 \le 0$

$\Rightarrow \quad x \in \left[\frac{1 - \sqrt{17}}{2}, \frac{1 + \sqrt{17}}{2}\right]$

Also, $\quad x + 4 - x^2 < 1$

$\Rightarrow \quad x + 3 - x^2 < 0$

$\Rightarrow \quad x^2 - x - 3 > 0$

$\Rightarrow \quad x \in \left(-\infty \frac{1 - \sqrt{13}}{2}\right) \cup \left(\infty, \frac{1 + \sqrt{13}}{2}\right)$

Thus, the number of solution is infinite.

27. We have

$$\left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{100}\right] + \left[\frac{1}{4} + \frac{2}{100}\right] + ... + \left[\frac{1}{4} + \frac{99}{100}\right]$$

$$= \left[\frac{1}{4} \times 100\right] = 25$$

28. We have

$$[1007] + \left[503\frac{3}{4}\right] + \left[252\frac{1}{8}\right] + \left[126\frac{5}{16}\right] + ...$$

$$= \left[\frac{2013 + 1}{2}\right] + \left[\frac{2013 + 2}{4}\right] + \left[\frac{2013 + 4}{8}\right]$$

$$+ \left[\frac{2013 + 8}{16}\right] + ...$$

$= 2013$

29. We have $[\sin x] = 0$

$\Rightarrow \quad 0 \le \sin x < 1$

$$\Rightarrow \quad \begin{cases} \sin x \ge 0 \\ \sin x < 1 \end{cases}$$

$$\Rightarrow \quad \begin{cases} 2n\pi \le x \le (2n+1)\pi \\ 2n\pi + \frac{\pi}{2} \le x < 2n\pi - \frac{\pi}{2} \end{cases} : n \in I$$

30. We have $2[x] + 3 = 3[x - 2] + 5$

$\Rightarrow \quad 2[x] + 3 = 3[x] - 6 + 5$

$\Rightarrow \quad [x] = 4$

$\Rightarrow \quad 4 \le x < 5$

Also, $y = 2[x] + 3 = 2.4 + 3 = 11$

Now, $[x + y] = [x + 11] = [x] + 11 = 4 + 11 = 15$

31. We have $f(x) = [x] - 2$

$\Rightarrow \quad [x] - 2 \ge 0$

$\Rightarrow \quad [x] \ge 2$

$\Rightarrow \quad x \geq 2$

$\Rightarrow \quad x \in [2, \infty)$

Thus, $\quad D_f = [2, \infty)$

32. We have $4 - [x] \geq 0$

$\Rightarrow \quad [x] \leq 4$

$\Rightarrow \quad x < 4 + 1 = 5$

$\Rightarrow \quad x \in (-\infty, 5)$

Thus, $\quad D_f = (-\infty, 5)$

33. Let $D_1 : [x] - 1 \geq 0$

$\Rightarrow \quad [x] \geq 1$

$\Rightarrow \quad x \geq 1$

Also, let $D_2 : 3 - [x] \geq 0$

$\Rightarrow \quad [x] \leq 3$

$\Rightarrow \quad x < 3 + 1 = 4$

Thus, $D_f = [1, 4)$

34. We have $\dfrac{[x] - 2}{5 - [x]} \geq 0,\ [x] \neq 5$

$\Rightarrow \quad \dfrac{[x] - 2}{5 - [x]} \leq 0,\ [x] \neq [5, 6)$

$\Rightarrow \quad 2 \leq [x] \leq 5,\ x \neq [5, 6)$

$\Rightarrow \quad 2 \leq x \leq 4$

$D_f = [2, 4]$

35. As we know that, $0 \leq \sin^2 x, \cos^2 x \leq 1$

Thus, the values of

$$\left[\frac{\sin^2 x}{2}\right] = 0 = \left[\sin^2\left(\frac{\pi}{2}\right)x\right] = \left[\cos^2\left(\frac{\pi}{2}\right)x\right]$$

$$= \left[\frac{\cos^2 x}{2014}\right]$$

36. When $[\sin x] = 0$, then $[\cos x] = 1$

$\Rightarrow \quad x = 0, \pi, 2\pi$ and $x = 0, 2\pi$

$\Rightarrow \quad x = 0, 2\pi$

Also, when $[\cos x] = 0$, then $= [\sin x] = 1$

$\Rightarrow \quad x = \dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$

$\Rightarrow \quad x = \dfrac{\pi}{2}$

Thus, the solution set is $\left\{0, \dfrac{\pi}{2}, 2\pi\right\}$

37. We have

$[x + 0.19] + [x + 0.20] + [x + 0.21] + \ldots$

$+ \ldots + [x + 0.22] + \ldots + [x + 0.91] = 542$

$$\Rightarrow \left[x + \frac{19}{100}\right] + \left[x + \frac{20}{100}\right] + \ldots + \left[x + \frac{91}{100}\right] = 542$$

$$\Rightarrow [100x] - \left([x] + \left[x + \frac{1}{100}\right] + \ldots + \left[x + \frac{18}{100}\right]\right)$$

$$- \left(\left[x + \frac{92}{100}\right] + \ldots + \left[x + \frac{99}{100}\right]\right) = 542$$

$\Rightarrow [100x] - 7 \times 19 - 8 \times 8 = 542$

$[\because 7.42 \leq x < 8]$

$\Rightarrow [100x] = 542 + 133 + 64 = 739$

Henec, the result.

38. We have

$$\left[\frac{2015}{2}\right] + \left[\frac{2016}{4}\right] + \left[\frac{2018}{8}\right] + \left[\frac{2022}{16}\right] + \ldots$$

$$= \left[\frac{2014 + 1}{2}\right] + \left[\frac{2014 + 2}{4}\right] + \left[\frac{2014 + 4}{8}\right]$$

$$+ \left[\frac{2014 + 8}{16}\right] + \ldots$$

$= 2014.$

39. **Case-I:** When x is integer

i.e $x = n$

L.H.S $= [2n] = 2n$

R.H.S $= [n] + \left[n + \dfrac{1}{2}\right] = n + n = 2n$

Case-II: When x is not an integer.

i.e. $x = n + f$, where $0 \leq f < 1$

L.H.S $= [2(n + f)] = 2n + [2f]$

$$= \begin{cases} 2n & : 0 \leq f < 0.5 \\ 2n + 1 & : 0.5 \leq f < 1 \end{cases}$$

R.H.S $= [n + f] + \left[n + f + \dfrac{1}{2}\right]$

$= n + [f] + n + \left[f + \dfrac{1}{2}\right]$

$= 2n + [f] + \left[f + \dfrac{1}{2}\right]$

$$= \begin{cases} 2n & : 0 \leq f < 0.5 \\ 2n + 1 & : 0.5 \leq f < 1 \end{cases}$$

Hence, the result.

40. **Case-I:** When x is an integer.

i.e $x = n$

L.H.S $= [3n] = 3n$

R.H.S $= [n] + \left[n + \dfrac{1}{3}\right] + \left[n + \dfrac{2}{3}\right]$

$= n + n + n = 3n$

Case-II: When x is not an integer

i.e. $x = n + f$

L.H.S $= [3[n + f] = [3n + 3f] = 3n + [3f]$

$$= \begin{cases} 3n & : 0 \le f < \frac{1}{3} \\ 3n+1 : \frac{1}{3} \le f < \frac{2}{3} \\ 3n+2 : \frac{2}{3} \le f < 1 \end{cases}$$

R.H.S $= [n + f] + \left[n + f + \frac{1}{3}\right] + \left[n + f + \frac{2}{3}\right]$

$$= 3n + [f] + \left[f + \frac{1}{3}\right] + \left[f + \frac{2}{3}\right]$$

$$= \begin{cases} 3n & : 0 \le f < \frac{1}{3} \\ 3n+1 : \frac{1}{3} \le f < \frac{2}{3} \\ 3n+2 : \frac{2}{3} \le f < 1 \end{cases}$$

Hence, the result.

41. We have $\left[\frac{2x^2}{x^2 + 1}\right]$

$$= \left[2 - \frac{2}{x^2 + 1}\right]$$

$$= 2 + \left[- \frac{2}{x^2 + 1}\right]$$

$= 2 + (-2),\ 2 + (-1)\ \}$ $\left(\because -2 \le \left(-\frac{2}{x^2 + 1}\right) < 0\right)$

$= 0, 1$

Hence, the number of integral values is 2.

42. We have $\{x\} + \{\sin x\} = 2$

since $\{x\} < 1$ and $\{\sin x\} < 1$

so $\{x\} + \{\sin x\} < 2$

Thus, the given equation has no solution.

Therefore, $x = \varphi$

43. As we know that $\{x + r\} = \{x\}$ as $x \in I$

we have $[x] + \sum_{r=1}^{2014} \left(\frac{\{x + r\}}{2014}\right)$

$$= [x] + \sum_{r=1}^{2014} \left(\frac{\{x + r\}}{2014}\right)$$

$$= [x] + \left(\left(\frac{\{x\}}{2014}\right) + \left(\frac{\{x\}}{2014}\right) + \ldots \text{ 2014 times}\right)$$

$$= [x] + \left(2014 \times \left(\frac{\{x\}}{2014}\right)\right)$$

$$= [x] + \{x\}$$

$$= x$$

Thus, $[x] + \sum_{r=1}^{2014} \left(\frac{\{x + r\}}{2014}\right) = x$

44. We have $4\{x\} = x + [x]$

$\Rightarrow \quad 4\{x\} = [x] + \{x\} + [x]$

$\Rightarrow \quad 3\{x\} = 2[x]$

$\Rightarrow \quad \{x\} = \frac{2}{3}[x]$

As we know that, $0 \le \{x\} < 1$

$\Rightarrow \quad 0 \le \frac{2}{3}[x] < 1$

$\Rightarrow \quad 0 \le [x] < 2$

$\Rightarrow \quad [x] = 0, 1$

when $[x] = 0 \Rightarrow \{x\} = 0$

when $[x] = 1 \Rightarrow \{x\} = \frac{2}{3}$

Thus $x = [x] + \{x\} = 0, 1 + \frac{2}{3} = 0, \frac{5}{3}$

Hence, the solutions are $\left\{0, \frac{5}{3}\right\}$

45. We have $2x + 3\{x\} = 4[x] - 2$

$\Rightarrow \quad 2([x] + \{x\}) + 3\{x\} = 4[x] - 2$

$\Rightarrow \quad 5\{x\} = 2[x] - 2$

$\Rightarrow \quad \{x\} = \left(\frac{2[x] - 2}{5}\right)$

As we know that, $0 \le \{x\} < 1$

$\Rightarrow \quad 0 \le \left\{\frac{2[x] - 2}{5}\right\} < 1$

$\Rightarrow \quad 0 \le 2[x] - 2 < 5$

$\Rightarrow \quad 2 \le [x] < \frac{7}{2}$

$\Rightarrow \quad [x] = 2, 3$

when $[x] = 2 \Rightarrow \{x\} = \frac{2}{5}$

Thus. $x = [x] + \{x\} = 2 + \frac{2}{5} = \frac{12}{5}$

Also, $[x] = 3 \Rightarrow \{x\} = \frac{4}{5}$

Thus, $x = [x] + \{x\} = 3 + \frac{4}{5} = \frac{19}{5}$

Hence, the solutions are $\left\{\frac{12}{5}, \frac{19}{5}\right\}$.

46. We have $x^2 - 4x + [x] + 3 = 0$

$\Rightarrow \quad x^2 - 4x + x - \{x\} + 3 = 0$

$\Rightarrow \quad x^2 - 3x + 3 = \{x\}$

As we know that $0 \le \{x\} < 1$

$\Rightarrow \quad 0 \le x^2 - 3x + 3 < 1$

when $\quad x^2 - 3x + 3 < 1$

$\Rightarrow \quad x^2 - 3x + 2 < 0$

$\Rightarrow \quad (x-1)(x-2) < 0$

$\Rightarrow \quad 1 < x < 2$

when $\quad x^2 - 3x + 3 \ge 0$

$\Rightarrow \quad x \in R$

Thus the solution set is $1 < x < 2$ which does not satisfy the given equation.

Hence, the equation does not have any solution.

47. It has two solutions only when at $x = 1$ and $x = -1$.

48. We have $0 \le \{x\} < 1$ and $0 \le \{\tan(\pi x)\} < 1$

Now, $\{x\} + \{\tan(\pi x)\} \ge 0$

Thus, the equation holds only when

$\{x\} = 0 \Rightarrow x \in 1$

and $\quad \{\tan(\pi x)\} = 0 \Rightarrow x \in I$

Thus, the equation have infinitely many solution.

49. Given, $y = \left\{\dfrac{x^2+1}{2}\right\}$

$\Rightarrow \quad y = \left(\dfrac{x^2+1}{2}\right) - \left[\dfrac{x^2+1}{2}\right]$

when $x \in [0, 1)$

$\Rightarrow \quad y \in (0, 1)$

when $x \in [1, 2)$

$\Rightarrow \quad y \in [0, 1)$

when $x = 2$

$\Rightarrow \quad y \in (0, 1)$

Thus the number of integral values of y satisfy the given expression is 1.

i.e. $y = 0$ only.

50. We have $\left(\dfrac{\cos^2 x - 2}{2}\right) - \left[\dfrac{\cos^2 x - 2}{2}\right] = \dfrac{1}{4}$ ...(i)

Also, $\quad 0 \le \cos^2 x \le 1$

$\Rightarrow \quad 0 - 2 \le \cos^2 x - 2 \le 1 - 2$

$\Rightarrow \quad -2 \le (\cos^2 x - 2) \le -1$

$\Rightarrow \quad \dfrac{-2}{2} \le \left(\dfrac{\cos^2 x - 2}{2}\right) \le \dfrac{-1}{2}$

$\Rightarrow \quad -1 \le \left(\dfrac{\cos^2 x - 2}{2}\right) \le -\dfrac{1}{2}$

From (i), we get, $\left(\dfrac{\cos^2 x - 2}{2}\right) - (-1) = \dfrac{1}{4}$

$\Rightarrow \quad \left(\dfrac{\cos^2 x - 2}{2}\right) = -\dfrac{3}{4}$

$\Rightarrow \quad (\cos^2 x - 2) = -\dfrac{3}{2}$

$\Rightarrow \quad \cos^2 x = 2 - \dfrac{3}{2} = \dfrac{1}{2}$

$\Rightarrow \quad \cos^2 x = \left(\dfrac{1}{\sqrt{2}}\right)^2 = \cos^2\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \quad x = 2n\pi \pm \dfrac{\pi}{4}, n \in I$

51. $F(x)$ is defined only when $2\{x\}^2 - 3\{x\} + 1 \ge 0$

$\Rightarrow \quad 2\{x\}^2 - 2\{x\} - \{x\} + 1 \ge 0$

$\Rightarrow \quad (2\{x\} - 1)(\{x\} - 1) \ge 0$

$\Rightarrow \quad \{x\} \le \dfrac{1}{2}, \{x\} \ge 1$

But $\{x\} \ge 1$ is not possible.

Thus, $\quad \{x\} \le \dfrac{1}{2}$

$\Rightarrow \quad D_f = \left[-1, -\dfrac{1}{2}\right] \cup \left[0, \dfrac{1}{2}\right] \cup \{1\}$

52. Here, $a = 3$, $b = 4$ and $c = 10$

Thus, the minimum values of $f(x)$

$= -\sqrt{a^2 + b^2} + c = -5 + 10 = 5$

and the maximum values of $f(x)$

$= \sqrt{a^2 + b^2} + c = 5 + 10 = 15.$

53. $R_f = [\min f(x), \max f(x)] = \lfloor -\sqrt{2} + 3, \sqrt{2} + 3 \rfloor$

54. We have, $A = \sin^4\theta + \cos^4\theta$

$= (\sin^2\theta)^2 + (\cos^2\theta)^2$

$= 1 - 2\sin^2\theta . \cos^2\theta$

$= 1 - \dfrac{(\sin 2\theta)^2}{2}$

$= 1 + \dfrac{\{-\sin^2(2\theta)\}}{2}$

As we know that, $-1 \le \{-\sin^2(2\theta)\} \le 0$

$\Rightarrow \quad -\dfrac{1}{2} \le \dfrac{\{-\sin^2(2\theta)\}}{2} \le 0 + 1$

$\Rightarrow \quad -\dfrac{1}{2} + 1 \le \dfrac{\{-\sin^2(2\theta)\}}{2} \le 0 + 1$

$\Rightarrow \quad \dfrac{1}{2} \le A \le 1$

55. We have, $f(\theta) = \sin^6\theta + \cos^6\theta$

$= (\sin^2\theta)^3 + (\cos^2\theta)^3$

$= (\sin^2\theta + \cos^2\theta)^2$

$\quad - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$

$= 1 - 3\sin^2\theta\cos^2\theta$

$= 1 - \frac{3}{4}(4\sin^2\theta\cos^2\theta)$

$= 1 - \frac{3}{4}(\sin^2 2\theta)$

$= 1 + \frac{3}{4}(-\sin^2 2\theta)$

As we know that, $-1 \le (-\sin^2 2\theta) \le 0$

$\Rightarrow \quad -\frac{3}{4} \le \frac{3(-\sin^2 2\theta)}{4} \le 0$

$\Rightarrow \quad 1-\frac{3}{4} \le 1 + \frac{3(-\sin^2 2\theta)}{4} \le 1$

$\Rightarrow \quad \frac{1}{4} \le f(\theta) \le 1$

Hence, the maximum value = 1 and the minimum value = 1/4

56. We have

$A = \cos^2\theta + \sin^4\theta$

$= \frac{1}{2}(2\cos^2\theta) + \frac{1}{4}(2\sin^2\theta)^2$

$= \frac{1}{2}(1 + \cos(2\theta)) + \frac{1}{4}(1 - 2\cos(2\theta))^2$

$= \frac{1}{2}(1 + \cos(2\theta)) + \frac{1}{4}(1 - 2\cos(2\theta) + \cos^2(2\theta))$

$= \frac{1}{2} + \frac{1}{2}\cos(2\theta) + \frac{1}{4} - \frac{1}{2}\cos(2\theta) + \frac{1}{4}(\cos^2(2\theta))$

$= \frac{3}{4} + \frac{1}{4}(\cos^2(2\theta))$

Max value of $A = m_1 = \frac{3}{4} + \frac{1}{4}.\ 1 = 1$

Also,

$B = \sin^2\theta + \cos^4\theta$

$= \frac{1}{2}(2\sin^2\theta) + \frac{1}{4}(4\cos^4\theta)$

$= \frac{1}{2}(2\sin^2\theta) + \frac{1}{4}(2\cos^2\theta)^2$

$= \frac{1}{2}(1 - \cos(2\theta)) + \frac{1}{4}(1 + \cos(2\theta))^2$

$= \frac{1}{2}(1 - \cos(2\theta)) + \frac{1}{4}(1 + 2\cos(2\theta) + \cos^2(2\theta))$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\cos^2(2\theta)$

$= \frac{3}{4} + \frac{1}{4}\cos^2(2\theta)$

Thus, the minimum value of

$B = m_2 = \frac{3}{4} + \frac{1}{4}.\ 0 = \frac{3}{4}$

Now, the value of $m_1^2 + m_2^2 + m_1m_2$

$= 1 + \frac{9}{16} + \frac{3}{4} = \frac{37}{16}$

57. We have, $\sin^2(\sin\theta) + \cos^2(\cos\theta)$

$= \sin^2(\cos\theta) + \cos^2(\cos\theta) + \sin^2(\sin\theta) - \sin^2(\cos\theta)$

$= (\sin^2(\cos\theta) + \cos^2(\cos\theta)) + \sin^2(\sin\theta) - \sin^2(\cos\theta)$

$= 1 + (\sin^2(\sin\theta) - \sin^2(\cos\theta))$

Max value of $f(\theta)$

$= 1 + \left(\sin^2\left(\sin\left(\frac{\pi}{2}\right)\right) - \sin^2\left(\cos\left(\frac{\pi}{2}\right)\right)\right)$

$= 1 + \sin^2(1)$

Min value of $f(\theta)$

$= 1 + (\sin^2(\sin(0)) - \sin^2(\cos(0)))$

$= 1 - \sin^2(1)$

58. We have, $f(\theta) = (3\sin(\theta) - 4\cos(\theta) - 10)$
$(3\sin(\theta) + 4\cos(\theta) - 10)$

$= (9\sin^2(\theta) - 16\cos^2(\theta))$

$\quad -10(3\sin\theta + 4\cos\theta) - 10(3\sin\theta - 4\cos\theta)$

$= (9\sin^2(\theta) - 16\cos^2(\theta)) - 10(3\sin\theta + 4\cos\theta$

$\quad + 3\sin\theta - 4\cos\theta)$

$= (9\sin^2(\theta) - 16\cos^2(\theta)) - 60\sin(\theta)$

$= 25\sin^2\theta - 60\sin(\theta) - 16$

$= (5\sin\theta - 6)^2 - 36 - 16$

$= (5\sin\theta - 6)^2 - 52$

Hence, the minimum value of $f(\theta) = 121 - 52 = 69$

59. Now, $0 < \sin^{2010}\theta \le \sin^2\theta$ ....(i)

and $\quad 0 < \cos^{2014}\theta \le \cos^2\theta$ ....(ii)

Adding (i) and (ii), we get,

$0 < \sin^{2012}\theta + \cos^{2014}\theta \le \sin^2\theta + \cos^2\theta$

$\Rightarrow \quad 0 < A \le 1$

Thus, the range of $A = (0, 1]$

60. We have, $f(x) = \frac{x^2\sin^2 x + 4}{x\sin x}$

$= x\sin x + \frac{4}{x\sin x} \le 4$

Hence, the minimum values of $f(x)$ is 4

61. Here, the function f is defined for all x in R

Thus, $\quad D_f = R$

62. Here, the function f is defined for all x in R.

Thus, $\quad D_f = R$

63. Here, the function f is defined for all x in R.

Thus, $\quad D_f = R$

64. Here, the function f is defined for all x in R.

Thus, $D_f = R$

65. Here, the function f is defined for

$$e^{-|x|} - \frac{1}{2} \geq 0$$

$$\Rightarrow \quad e^{-|x|} \geq \frac{1}{2}$$

$$\Rightarrow \quad -|x| \geq \log_e \left(\frac{1}{2}\right)$$

$$\Rightarrow \quad -|x| \geq \log_e 1 - \log_e 2 = -\log_e 2$$

$$\Rightarrow \quad |x| \leq \log_e 2$$

$$\Rightarrow \quad -\log_e 2 \leq x \leq \log_e 2$$

$$\Rightarrow \quad \log_e \left(\frac{1}{2}\right) \leq x \leq \log_e 2$$

$$\Rightarrow \quad \left[x \in \log_e \left(\frac{1}{2}\right), \log_e 2\right]$$

Thus, $D_f = \left[\log_e \left(\frac{1}{2}\right), \log_e 2\right]$

66. Here, the function f is defined for all x in R

Thus, $D_f = R$

67. Here, the function f is defined for all x in R

Thus, $D_f = R$

68. We have $e^x + e^{f(x)} = e$

$$\Rightarrow \quad e^{f(x)} = e - e^x$$

$$\Rightarrow \quad f(x) = \log_e (e - e^x)$$

It is defined for $e - e^x > 0$

$$\Rightarrow \quad e > e^x$$

$$\Rightarrow \quad e^x < e$$

$$\Rightarrow \quad x < 1$$

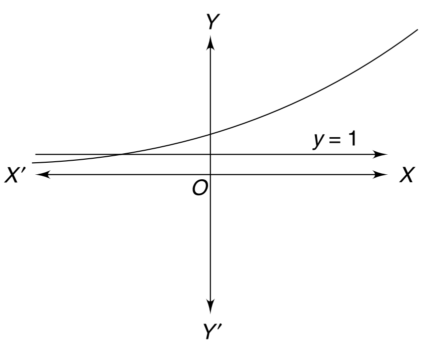
$$\Rightarrow \quad x \in (-\infty, 1)$$

Thus, $D_f = (-\infty, 1)$

69. We have $2^x + 3^x + 4^x - 5^x = 0$

$$\Rightarrow \quad 2^x + 3^x + 4^x = 5^x$$

$$\Rightarrow \quad \left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x = 1$$



Clearly it has only 1 solution.

70. We have $1 + 3^{x/2} = 2^x$

The given equation holds good only for $x = 2$

Thus, the number of solution is 1.

71. f is defined if $(x^2 - 4x + 3)$

$$\Rightarrow \quad (x - 1)(x - 3) > 0$$

$$\Rightarrow \quad x < 1, x > 3$$

Thus, $D_f = (-\infty, 1) \cup (3, \infty)$

72. Here, $e^{\frac{x-1}{2}}$ is defined for all x in R $\log(1 - x)$ is defined for $1 - x > 0$

$$\Rightarrow \quad x < 1$$

$$\Rightarrow \quad x \in (-\infty, 1)$$

Also, x^{1001} is defined for all x in R.

Hence, the domain of the given function

$$= R \cap (-\infty, 0) \cap R$$

$$= (-\infty, 1)$$

73. Here, the function f is defined for $\frac{x}{x-2} > 0$

$$\Rightarrow \quad x < 0 \text{ and } x > 2$$

$$\Rightarrow \quad x \in (-\infty, 0) \cup (2, \infty)$$

Thus $D_f = (-\infty, 0) \cup (2, \infty)$

74. Here, the function f is defined for $(x - 2) > 0$

$$\Rightarrow \quad x > 2$$

$$\Rightarrow \quad x \in (2, \infty)$$

Thus, $D_f = (2, \infty)$

75. Here, $\log(1 - x)$ is defined for $1 - x > 0$

$$\Rightarrow \quad x < 1$$

$$\Rightarrow \quad x \in (-\infty, 1)$$

Also, $x^2 - 3x + 2 = 0$ gives $x = 1, 2$

Thus, $D_f = (-\infty, 1)$

76. Here, the function $\log(3 - x)$ is defined for

$3 - x > 0 \Rightarrow x < 3$ $x \in (-\infty, 3)$

Also, $x^2 - 5x + 4 = 0$ gives $x = 1, 4$

Thus, $D_f = (-\infty, 3) - \{1\} = (-\infty, 1) \cup (1, 3)$

77. Here, the function f is defined for

$$\Rightarrow \quad (9 - x^2) > 0, (9 - x^2) \neq 1, (9 - x^2) \neq 0$$

$$\Rightarrow \quad (x^2 - 9) < 0, x^2 \neq 8, x^2 \neq 9$$

$$\Rightarrow \quad (x + 3)(x - 3) < 0, x \neq \pm 2\sqrt{2}, x \neq \pm 3$$

$$\Rightarrow \quad -3 < x < 3, x \neq \pm 2\sqrt{2}, x \neq \pm 3$$

Thus, $D_f = (-3, -2\sqrt{2}) \cup (2\sqrt{2}, 3)$

78. Here, the function f is defined for

$$(x - 1) \geq 0 \ \& \ (3 - x) \leq 0$$

$$\Rightarrow \quad x \geq 1 \ \& \ x \leq 3$$

$$\Rightarrow \quad x \in [1, 3]$$

$$D_f = [1, 3]$$

79. Here, the function f is defined for $\left(\frac{2-x}{x-4}\right) > 0$

$\Rightarrow \quad \left(\frac{2-x}{x-4}\right) < 0$

$\Rightarrow \quad 2 < x < 4$

Thus, $D_f = (2, 4)$

80. Here, the function f is defined for

$(1 - \log_{10}(x^2 - 5x + 16)) > 0$

$\Rightarrow \quad \log_{10}(x^2 - 5x + 16) < 1$

$\Rightarrow \quad (x^2 - 5x + 16) < 10$

$\Rightarrow \quad (x^2 - 5x + 6) < 0$

$\Rightarrow \quad (x - 2)(x - 3) < 0$

$\Rightarrow \quad 2 < x < 3$

Thus, $D_f = (2, 3)$

81. Here the function f is defined for

$(\log_{10}x)^2 - 5(\log_{10}x) + 6 > 0$

$\Rightarrow \quad a^2 - 5a + 6 > 0 \; a = \log_{10}x$

$\Rightarrow \quad (a - 2)(a - 3) > 0, \; a = \log_{10}x$

$\Rightarrow \quad a < 2 \;\&\; a > 3, \; a = \log_{10}x$

$\Rightarrow \quad \log_{10}x < 2 \;\&\; \log_{10}x > 3$

$\Rightarrow \quad x < 10^2 \;\&\; x > 10^3$ and $x > 0$

$\Rightarrow \quad x \in (0, 10^2) \cup (10^3, \infty)$

Thus, $D_f = (0, 10^2) \cup (10^3, \infty)$

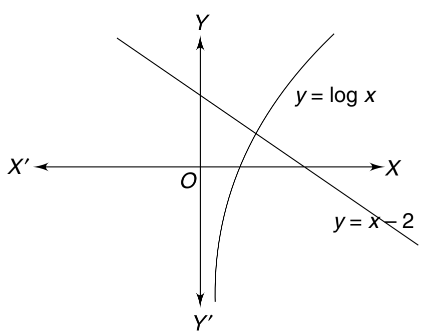
82. Here, the function f is defined for

$x - [x] > 0, \; x \notin I$

$\Rightarrow \quad x > [x], \; x \notin I$

Thus, $D_f = R^+ - I$

83. Given equation can be written as $2 - x = \log_e x$
From the graph it is clear that it has only one solution.



84. Here, $D_f = \{1, 2\} = D_g$ and $R_f = \{1, 4\} = R_g$
Also $f(1) = 1, f(2) = 4$ and $g(1) = 1, g(2) = 4$
Thus all the conditions of equal functions are satisfied.
Hence, $f = g$

85. Here $D_f = R \;\&\; D_g = R - \{0\}$
Since $D_f \neq D_g$, so $f \neq g$

86. Here, $f(x) = x \;\&\; g(x) = \sqrt{x^2} = |x|$
Thus, $R_f = R \;\&\; R_g = [0, \infty)$
Since $R_f \neq R_g$, so $f \neq g$

87. Here, $D_f = R^+ \;\&\; D_g = R - \{0\}$
Since, $D_f \neq D_g$, so $f \neq g$

88. Here, f is defined for $\left(\frac{x-1}{x-2}\right) > 0$

$\Rightarrow \quad x < 1$ and $x > 2$

$\Rightarrow \quad x \in (-\infty, 1) \cup (2, \infty)$

Thus, $D_f = (-\infty, 1) \cup (2, \infty)$

Also, the function g is defined for

$x > 1$ and $x > 2$

$\Rightarrow \quad x > 2$

Thus, $D_g = (2, \infty)$

Since $D_f \neq D_g$, so $f \neq g$

89. Given $f(x) = \sin x$ and

$g(x) = \sqrt{\frac{1 - \cos 2x}{2}} = \sqrt{\sin^2 x} = |\sin x|$

Clearly, $R_f = [-1, 1] \;\&\; R_g = [0, 1]$

Since $R_f \neq R_g$, so $f \neq g$

90. 0
91. $^3P_3 = 3! = 6$
92. $^5P_4 = 5! = 120.$
93. $^3P_3 = 3! = 6$
94. 4
95. is 3 since
(i) $1 < 2 \Rightarrow f(1) = 6 < f(2) = 7$
(ii) $1 < 3 \Rightarrow f(1) = 6 < f(3) = 8$
(iii) $2 < 3 \Rightarrow f(2) = 7 < f(3) = 8.$

96. Number of many one function = Number of total function – Number of one-one function.
$= 4^3 - {}^4P_3 = 64 - 24 = 40.$

97. Number of many one function = Number of total function – Number of one-one function.
$= 3^4 - 0 = 81$

98. Number of many one function = Number of total function – Number of one-one function.
$= 3^3 - 2 = 27 - 2 = 25$

99. Now, let $y = \frac{x}{x+1}$

$\Rightarrow \quad yx + y = x$

$\Rightarrow \quad x(y-1) = -y$

$\Rightarrow \quad x = \frac{-y}{(y-1)}$

Then the range of a function is $R - \{1\}$

Thus, $R_f = R - \{1\}$ = Co-domain

Hence, f is onto function.

100. Clearly, the range of a function is = (0, 1] i.e. $R_f = (0, 1]$. Since f is onto function, so R_f = Co-domain

$\Rightarrow \quad A = R_f = (0, 1]$

Hence, the set A is (0, 1].

101. Clearly, domain of a function is R.

i.e. $\quad D_f = A = R$.

Now, range of the function is [0, 1)

Thus, $R_f = [0, 1)$.

Since f is onto function, so $R_f = B$.

Thus, $B = [0, 1)$

Hence, $A \cap B = [0, 1)$.

102. Number of onto function = Number of distribution of 4 balls into 2 boxes, where no box is remain empty.

$$= \frac{4!}{3! \times 1!} \times 2! + \frac{4!}{2! \times 2!} \times \frac{2!}{2!}$$

$= 8 + 6 = 14$.

103. Number of onto function = Number of distribution of 4 balls into 3 boxes, where no box is remain empty.

$$= \frac{4!}{3! \times 1!} \times 2!$$

$= 8$

104. Number of onto function = Number of distribution of 5 balls into 3 boxes, where no box is remain empty.

$$= \frac{5!}{3! \times 1! \times 1!} \times \frac{3!}{2!} + \frac{5!}{2! \times 2! \times 1!} \times \frac{3!}{2!}$$

$= 60 + 90$

$= 150$.

105. The number of into function = Total number of functions – the number of onto functions

$= 2^3 - 6$

$= 8 - 6$

$= 2$.

106. The number of into functions = Total number of functions – the number of onto functions

$$= 3^4 - \left(\frac{4!}{2! \times 1! \times 1!} \times \frac{3!}{2!}\right)$$

$= 81 - 36$

$= 45$.

107. The number of into functions = The total number of functions – the number of onto functions.

$$= 3^5 - \left(\frac{5!}{3! \times 1! \times 1!} \times \frac{3!}{2!} + \frac{5!}{2! \times 2! \times 1!} \times \frac{3!}{2!}\right)$$

$= 243 - 150$

$= 93$.

108. Given $f(x) = 3x + 5$.

$\Rightarrow \quad f'(x) = 3 > 0$

$\Rightarrow \quad f$ is strictly increasing function.

$\Rightarrow \quad f$ is one-one function.

Also, the range of a function is R

$\Rightarrow \quad f$ is onto function.

Hence, f is a bijective function.

109. Given $f(x) = x^2 + 1$

$\Rightarrow \quad f'(x) = 2x > 0 \ \forall \ x \in R^+$

$\Rightarrow \quad f$ is strictly increasing function.

$\Rightarrow \quad f$ is one-one function.

Also, the range of a function is $(1, \infty)$

$\Rightarrow \quad R_f = (1, \infty)$ = Co-domain

$\Rightarrow \quad f$ is onto function.

Hence, the function f is bijective.

110. Given $f(x) = \frac{1}{x^2 + 1}$

$\Rightarrow \quad f'(x) = \frac{2x}{(x^2+1)^2} > 0, \ \forall \ x \in R^+$

$\Rightarrow \quad f$ is strictly increasing function.

$\Rightarrow \quad f$ is one-one function.

Also, the range of a function is (0, 1)

$\Rightarrow \quad R_f = (0, 1)$ = co-domain

$\Rightarrow \quad f$ is onto function

Thus, f is a bijective function.

111. Given $\quad f(x) = \frac{x}{x^2+1}, \ \forall \ x \in (-1, 1)$

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2} > 0 \ \forall \ x \in (-1, 1)$$

$\Rightarrow \quad f$ is strictly increasing function

$\Rightarrow \quad f$ is one-one function.

Also, the range of a function is $\left(-\frac{1}{2}, \frac{1}{2}\right)$

$\Rightarrow \quad R_f = \left(-\frac{1}{2}, \frac{1}{2}\right)$ = Co-domain

$\Rightarrow$ f is onto function.

Thus f is a bijective function.

112. Since the number of elements of A and B are not same, so the number of one-one onto function is 0.

113. Since the number of elements of both the sets are same, so the number of bijective function is ${}^5P_5 = 5! = 120$.

114. Hence, the number of bijective functions $= {}^3P_3 = 3! = 6$.

115. Given $f(x) = 3x + 5$

$\Rightarrow$ $f'(x) = 3 > 0$

$\Rightarrow$ f is strictly increasing function.

$\Rightarrow$ f is one-one function

Also, $R_f = R$ = Co-domain

$\Rightarrow$ f is onto function.

Thus, f is a bijective function.

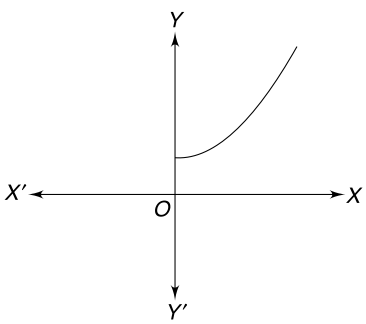
Hence, f^{-1} is exists.

Let $y = 3x + 5$

$\Rightarrow$ $x = \frac{y-5}{3}$

Thus, $f^{-1}(x) = \frac{x-5}{3}$.

116. Given $f(x) = x^2 + 2$



$\Rightarrow$ $f'(x) = 2x > 0$ for every $x > 0$

$\Rightarrow$ f is strictly increasing function.

$\Rightarrow$ f is one-one function.

Also, $R_f = (2, \infty)$ = Co-domain

$\Rightarrow$ f is onto function.

Thus f is a bijective function.

Therefore, the inverse of the given function is exists.

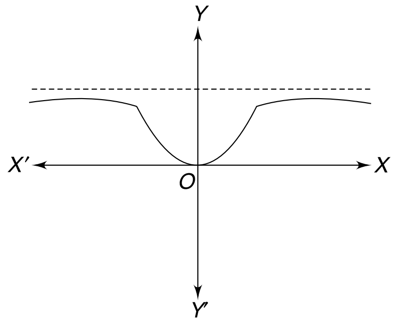
Let $y = x^2 + 2$

$\Rightarrow$ $x^2 = y - 2$

$\Rightarrow$ $x = \sqrt{y-2}$

Hence, $f^{-1}(x) = \sqrt{x-2}$

117. Given $f(x) = \frac{x^2}{x^2+1}$



$f(x) = 1 - \frac{1}{x^2+1}$

$f'(x) = \frac{2x}{(x^2+1)^2} > 0, \forall x \in R^+$

$\Rightarrow$ f is strictly increasing function.

$\Rightarrow$ f is one-one function.

Also, let $y = \frac{x^2}{x^2+1}$

$\Rightarrow$ $y.x^2 + y = x^2$

$\Rightarrow$ $x^2(y-1) = -y$

$\Rightarrow$ $x^2 = \frac{-y}{(y-1)} = \frac{y}{(1-y)}$

$\Rightarrow$ $x = \sqrt{\frac{y}{(1-y)}}$

$\Rightarrow$ $R_f = (0, 1)$ = Co-domain

$\Rightarrow$ f is onto function.

Thus, f is a bijective function.

$\Rightarrow$ $f^{-1}(x)$ is exists.

Hence, $f^{-1}(x) = \sqrt{\frac{x}{1-x}}$.

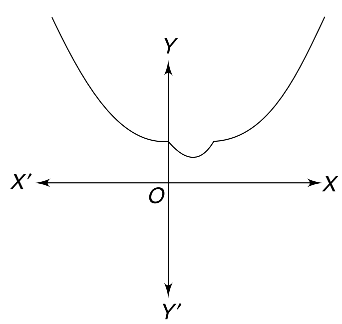
118. Given $f(x) = 2^{x(x-1)}$.

$\Rightarrow$ $f'(x) = 2^{x(x-1)} \times (2x-1) \times \log_e 2 > 0$

for all x in $[1, \infty)$

$\Rightarrow$ f is strictly increasing function.

$\Rightarrow$ f is one-one function.



Also, $R_f = [1, \infty)$

$\Rightarrow$ $R_f = [1, \infty)$ = Co-domain

$\Rightarrow$ f is onto function.

Thus, f is a bijective function.

So its inverse is exists.

Let $y = 2^{x(x-1)}$

$\Rightarrow$ $y = 2^{x^2 - x}$

$\Rightarrow$ $x^2 - x = \log_2(y)$

$\Rightarrow$ $x^2 - x - \log_2(y) = 0$

$\Rightarrow$ $x = \dfrac{1 \pm \sqrt{1 + 4\log_2(y)}}{2}$

$\Rightarrow$ $x = \dfrac{1 + \sqrt{1 + 4\log_2(y)}}{2}$

Thus, $f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4\log_2(y)}}{2}$

119. Since f is a bijective function, so its inverse is exists.

Let $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}} = \dfrac{10^{2x} - 1}{10^{2x} + 1}$.

$\Rightarrow$ $y.10^{2x} + y = 10^{2x} - 1$

$\Rightarrow$ $10^{2x}(y - 1) = -y - 1$

$\Rightarrow$ $10^{2x} = -\dfrac{y+1}{y-1} = \dfrac{y+1}{1-y}$

$\Rightarrow$ $2x = \log_{10}\left(\dfrac{y+1}{1-y}\right)$

$\Rightarrow$ $x = \dfrac{1}{2}\log_{10}\left(\dfrac{1+y}{1-y}\right)$

Thus, $f^{-1}(x) = \dfrac{1}{2}\log_{10}\left(\dfrac{1+x}{1-x}\right)$.

120. Given $f(x) = x + \sin x$

$\Rightarrow$ $f'(x) = 1 + \cos x \geq 0$ for all x in R.

$\Rightarrow$ f is strictly increasing function

$\Rightarrow$ f is one-one function.

Also, the range of a function is R.

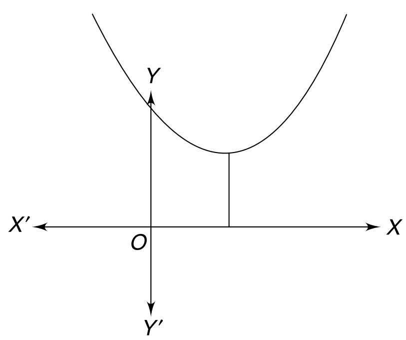
$\Rightarrow$ f is a onto function.

Thus, f is a bijective function.

Hence, f^{-1} is exists.

Therefore, $f^{-1}(x) = x - \sin x$

121. Given $f(x) = x^2 - 4x + 9$



$\Rightarrow$ $f'(x) = 2x - 4 \geq 0$ for all x in D_f

$\Rightarrow$ f is strictly increasing function.

$\Rightarrow$ f is one-one function.

Also, $R_f = [5, \infty)$ = Co-domain

$\Rightarrow$ f is onto function.

Therefore, f is a bijective function.

Hence, its inverse is exists.

Let $y = x^2 - 4x + 5$

$\Rightarrow$ $x^2 - 4x + (5 - y) = 0$

$\Rightarrow$ $x = \dfrac{4 \pm \sqrt{16 - 4(5-y)}}{2}$

$\Rightarrow$ $x = \dfrac{4 \pm \sqrt{4y + 16 - 20}}{2}$

$\Rightarrow$ $x = \dfrac{4 \pm \sqrt{4(y-1)}}{2} = 2 \pm \sqrt{(y-1)}$

$\Rightarrow$ $x = 2 + \sqrt{(y-1)}$, since $x \geq 2$

$\Rightarrow$ $f^{-1}(x) = 2 + \sqrt{(x-1)}$

122. Consider the function $f: [0, \infty) \to \left(-\dfrac{1}{4}, \infty\right)$, where.

$f(x) = x^2 - \dfrac{1}{4}$

Clearly, f is one-one and onto function. So its inverse is exists.

Let its inverse is $f^{-1}(x) : \left[-\dfrac{1}{4}, \infty\right) \to [0, \infty)$.

$\Rightarrow$ $f^{-1}(x) = \sqrt{x + \dfrac{1}{4}}$.

Consequently, we can say that, the two sides of the given equation are inverse to each other.

Thus, the intersection point is the solution of the given equation. $f(x) = x$.

$\Rightarrow$ $x^2 - \dfrac{1}{4} = x$

$\Rightarrow$ $x^2 - x = \dfrac{1}{4}$

$\Rightarrow$ $\left(x - \dfrac{1}{2}\right)^2 = \dfrac{1}{2}$

$\Rightarrow$ $\left(x - \dfrac{1}{2}\right) = \pm\dfrac{1}{\sqrt{2}}$

$\Rightarrow$ $x = \dfrac{1}{2} \pm \dfrac{1}{\sqrt{2}}$

Hence, the solutions are $\left\{\dfrac{1}{2} + \dfrac{1}{\sqrt{2}}, \dfrac{1}{2} - \dfrac{1}{\sqrt{2}}\right\}$.

123. Since the f is invertible, so f is one-one function.

Given $f(x) = (m + 2)x^2 - 3mx^2 + 9mx - 1$

$\Rightarrow$ $f'(x) = 3(m + 2)x^2 - 6mx + 9m$

$\Rightarrow$ $f'(x) = 3(m + 2)x^2 - 6mx + 9m > 0$

Thus, $(m + 2) > 0$ and $36m^2 - 108m(m - 2) < 0$

$\Rightarrow \quad m > -2$ & $m^2 - 3m(m + 2) < 0$

$\Rightarrow \quad m > -2$ & $m^2 + 3m > 0$

$\Rightarrow \quad m > -2$ & $m \in (-\infty, -3) \cup (0, \infty)$

$\Rightarrow \quad m \in (-2, -3) \cup (0, \infty)$

124. Given $f(x) = f^{-1}(x)$

Now, $y = \sqrt[3]{a - x^3 + 3bx^2 - 3bx + b^3} + b$

$\Rightarrow \quad (y - b)^3 = a - (x^3 - 3bx^2 + 3bx - b^3)$

$\Rightarrow \quad (y - b)^3 = a - (x - b)^3$

$\Rightarrow \quad (x - b)^3 = a - (y - b)^3$

$\Rightarrow \quad (x - b) = \sqrt[3]{a - (y - b)^3}$

$\Rightarrow \quad x = b + \sqrt[3]{a - (y - b)^3}$

Thus, $\quad f^{-1}(x) = b + \sqrt[3]{a - (x - b)^3}$

125. We have $f(x) = \sin^{-1}(3x + 4)$

$\Rightarrow \quad -1 \le 3x + 4 \le 1$

$\Rightarrow \quad -5 \le 3x \le -3$

$\Rightarrow \quad \frac{5}{3} \le x \le -1$

$\Rightarrow \quad x \in \left[-\frac{5}{3}, -1\right]$

Thus, $D_f = \left[-\frac{5}{3}, -1\right]$

126. Given $f(x) = \cos^{-1}(4x + 5)$

$\Rightarrow \quad -1 \le 4x + 5 \le 1$

$\Rightarrow \quad -6 \le 4x \le -4$

$\Rightarrow \quad -\frac{3}{2} \le x \le -1$

$\Rightarrow \quad x \in \left[-\frac{3}{2}, -1\right]$

Thus, $D_f = \left[-\frac{3}{2}, -1\right]$

127. Given $f(x) = \cos^{-1}\left(\frac{x}{2} - 1\right) + e^x + \frac{1}{|x - 1|}$

Here, $\cos^{-1}\left(\frac{x}{2} - 1\right)$ is defined for

$\Rightarrow \quad -1 \le \left(\frac{x}{2} - 1\right) \le 1$

$\Rightarrow \quad 0 \le \frac{x}{2} \le 2$

$\Rightarrow \quad 0 \le x \le 4$

$\Rightarrow \quad x \in [0, 4]$

e^x is defined for all x in R.

Also, $\frac{1}{|x - 1|}$ is defined for $x \in R - \{1\}$

Thus, $D_f = [0, 4] \cup R \cup R - \{1\}$

$= [0, 4] - \{1\}$

$= [0, 1) \cup (1, 4]$

128. Here, $\cos^{-1}\left(\frac{|x| - 3}{3}\right)$ is defined for

$\Rightarrow \quad -1 \le \left(\frac{|x| - 3}{3}\right) \le 1$

$\Rightarrow \quad -3 \le |x| - 3 \le 3$

$\Rightarrow \quad 0 \le |x| \le 6$

$\Rightarrow \quad -6 \le x \le 6$

$x \in [-6, 6]$

Also, $\frac{1}{e_x + 1}$ is defined for all x in R

Thus, $D_f = [-6, 6] \cap R = [-6, 6]$

129. Here, $\sin^{-1}\left(\frac{1 - |x|}{2}\right)$ is defined for

$\Rightarrow \quad -1 \le \left(\frac{1 - |x|}{2}\right) \le 1$

$\Rightarrow \quad -2 \le (1 - |x|) \le 2$

$\Rightarrow \quad -3 \le (-|x|) \le 1$

$\Rightarrow \quad -1 \le |x| \le 3$

$\Rightarrow \quad -3 \le x \le 3$

Also, $\frac{e^x - 1}{e^x + 1}$ is defined for all x in R.

Thus, $D_f = [-3, 3] \cap R = [-3, 3]$

130. Here, the given function is defined for

$-1 \le \left(\frac{3}{4 + 2\sin x}\right) \le 1$

$\Rightarrow \quad -1 \le \left(\frac{4 + 2\sin x}{3}\right) \le 1$

$\Rightarrow \quad -3 \le (4 + 2\sin x) \le 3$

$\Rightarrow \quad -7 \le (2\sin x) \le -1$

$\Rightarrow \quad -\frac{7}{2} \le (\sin x) \le -\frac{1}{2}$

$\Rightarrow \quad -1 \le (\sin x) \le -\frac{1}{2}$

$\Rightarrow \quad \frac{-\pi}{2} \le x \le -\frac{\pi}{6}$

Thus, $D_f = \left[-\frac{\pi}{2}, -\frac{\pi}{6}\right]$

131. Here, $\sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is defined for

$$-1 \le \left(\frac{1+x^2}{2x}\right) \le 1$$

$\Rightarrow \quad \left|\frac{1+x^2}{2x}\right| \le 1$

$\Rightarrow \quad \frac{|1+x^2|}{2|x|} \le 1$

$\Rightarrow \quad 1 + x^2 \le 2|x|$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \le 0$

$\Rightarrow \quad (|x| - 1)^2 \le 0$

$\Rightarrow \quad (|x| - 1)^2 = 0$

$\Rightarrow \quad (|x| - 1) = 0$

$\Rightarrow \quad |x| = 1$

$\Rightarrow \quad x = -1, 1$

Also, the function $\cos(\sin x)$ is defined for all x in R.

Thus, $D_f = \{-1, 1\} \cap R \{-1, 1\}$

132. Here, the given function is defined for

$$-1 \le \log_2(x) \le 1$$

$\Rightarrow \quad 2^{-1} \le (x) \le 2^1$

$\Rightarrow \quad \frac{1}{2} \le x \le 2$

Thus, $\quad D_f = \left[\frac{1}{2}, 2\right]$

133. Here, the given function is defined for

$$-1 \le \log_4(x^2) \le 1$$

$\Rightarrow \quad 4^{-1} \le (x^2) \le 4$

$\Rightarrow \quad \frac{1}{4} \le x^2 \le 4$

$\Rightarrow \quad \sqrt{\frac{1}{4}} \le \sqrt{x^2} \le \sqrt{4}$

$\Rightarrow \quad \frac{1}{2} \le |x| \le 2$

When $\quad |x| \le 2 \Rightarrow -2 \le x \le 2$

When $\quad |x| \ge \frac{1}{2} \Rightarrow x \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$

Thus, $\quad D_f = \left[-\frac{1}{2}, -2\right] \cup \left[\frac{1}{2}, 2\right]$.

134. Here, the function f is defined for

$$-1 \le \left(\frac{1}{|x^2 - 1|}\right) \le 1$$

$\Rightarrow \quad -1 \le (|x^2 - 1|) \le 1$

$\Rightarrow \quad -1 \le (x^2 - 1) \le 1$

$\Rightarrow \quad 0 \le (x^2) \le 2$

$\Rightarrow \quad 0 \le |x| \le \sqrt{2}$

$\Rightarrow \quad -\sqrt{2} \le x \le \sqrt{2}$

$\Rightarrow \quad x \in \lfloor -\sqrt{2}, \sqrt{2} \rfloor$

Thus, $\quad D_f = \lfloor -\sqrt{2}, \sqrt{2} \rfloor$

135. We have $(f_o g)(x) = f(g(x))$

$= f(x - 1)$

$= (x - 1)^2 + 1$

$= x^2 - 2x + 2$

Also, $(g_o f)(x) = g(f(x))$

$= g(x^2 + 1)$

$= g(b),\ b = x^2 + 1$

$= b - 1$

$= (x^2+1) - 1$

$= x^2$

Note: Clearly, $f_o g \ne g_o f$, so composition of functions is not commutative.

136. We have $(f_o g)(x) = f(g(x))$

$= f(\sqrt{x-2})$

$= \sin(\sqrt{x-2})$

Also, $(g_o f)(x) = g(f(x))$

$= g(\sin x)$

$= \sqrt{\sin x - 2}$

137. We have $(f_o g)(x)$

$= f(g(x))$

$= f(x^2 + 1)$

$= \sqrt{(x^2 + 1) - 3}$

$= \sqrt{x^2 - 2}$

Thus, the domain of the function $f_o g$

$= (-\infty, -\sqrt{2}\rfloor \cup \lfloor\sqrt{2}, \infty)$

138. Here, $D_f = [0, \infty)$ and $R_f = [0, \infty)$

Also, $D_g = R$ and $R_g = [-1, \infty)$

As we know that $f_o g$ is defined only when

if $R_g \subseteq D_f$

But $[-1, \infty) \not\subseteq [0, \infty)$

Therefore, $f_o g$ is not defined.

139. Now, $f(f(x)) = f\left((a - x^n)^{1/n}\right)$

$= f(b)$, where $b = (a - x^n)^{1/n}$

$= (a - b^n)^{1/n}$

$= \left(a - (a - x^n)^{\frac{1}{n}\times n}\right)^{1/n}$

$= (x^n)^{1/n}$

$= x$

Also, $f\left(f\left(\frac{1}{x}\right)\right) = f\left(\left(a - \left(\left(a - \frac{1}{x^n}\right)^{1/n}\right)^n\right)^n\right)$

$= f\left(a - \left(a - \frac{1}{x^n}\right)\right)^{1/n}$

$= \left(\frac{1}{x^n}\right)^{1/n} = \frac{1}{x}$

Therefore, $f(f(x)) + f\left(f\left(\frac{1}{x}\right)\right) = x + \frac{1}{x}$.

140. We have $f(f(f(x))) = 0$

$\Rightarrow \quad f\left(f(x^2 - 1)\right) = 0$

$\Rightarrow \quad f\left((x^2 - 1)^2 - 1\right) = 0$

$\Rightarrow \quad f\left((x^4 - 2x^2 + 1 - 1)\right) = 0$

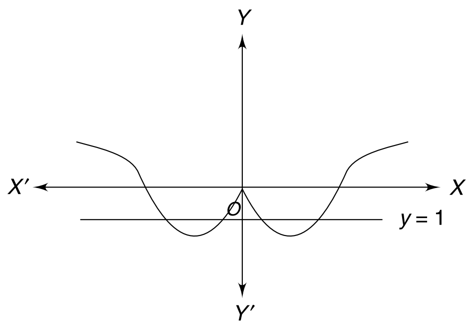
$\Rightarrow \quad f(x^4 - 2x^2) = 0$

$\Rightarrow \quad (x^4 - 2x^2)^2 - 1 = 0$

$\Rightarrow \quad (x^4 - 2x^2)^2 = 1$

$\Rightarrow \quad (x^4 - 2x^2) = \pm 1$

When $(x^4 - 2x^2) = -1 \Rightarrow x = \pm 1$



From the graph it is clear that, it has two solutions Thus, the number of distinct real solutions of the given equation is 4.

141. We have $f(x) = \begin{cases} x^2 : x \geq 0 \\ x : x < 0 \end{cases}$

and $\quad g(x) = \begin{cases} -x, x \geq 0 \\ x, x < 0 \end{cases}$

Now, $f(g(x)) = \begin{cases} f(-x) : x \geq 0 \\ f(x) : x < 0 \end{cases}$

$= \begin{cases} (-x)^2 : & x \geq 0 \\ (-x) \ : x \geq 0 \text{ and } & x < 0 \\ x^2 \ : x \geq 0 \text{ and } & x < 0 \\ x \ : & x < 0 \end{cases}$

$= \begin{cases} x^2 : x \geq 0 \\ -x : x = 0 \\ x^2 : x = 0 \\ x : x < 0 \end{cases}$

142. We have $(f_o f)(x) = f(f(x))$

$= \begin{cases} 1 + (1 + x) : 1 + x \geq 0 \text{ and } x \geq 0 \\ 1 - (1 + x) : 1 + x < 0 \text{ and } x \geq 0 \\ 1 + (1 - x) : 1 - x \geq 0 \text{ and } x \leq 0 \\ 1 - (1 - x) : 1 - x < 0 \text{ and } x < 0 \end{cases}$

$= \begin{cases} 2 + x : x \geq 0 \\ 1 - (1 + x) : \text{no common value} \\ 1 + (1 - x) : \text{no common value} \\ 1 - (1 - x) : x < 0 \end{cases}$

$= \begin{cases} 2 + x : x \geq 0 \\ x \ : x < 0 \end{cases}$

143. We have $f(g(x))$

$= f(\{x\} + 1) : 1 \leq x \leq 2$

$= (\{x\} + 1)^2 - (\{x\} + 1) + 2 : x \in R^+$ and $1 \leq x \leq 2$

$= \{x\}^2 + \{x\} + 2 \ 1 \leq x \leq 2$

Thus, domain of $f(g(x)) = [1, 2]$

and the range of $f(g(x)) = [2, 4)$

Thus, the value of $\frac{b}{a} + \frac{d}{c}$

$= \frac{2}{1} + \frac{4}{2} = 2 + 2 = 4$

144. Given g is the inverse of f

i.e. $\quad g(x) = f^{-1}(x)$

$\Rightarrow \quad f(g(x)) = x$

$\Rightarrow \quad f'(g(x)) \cdot g'(x) = 1$

$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$

$\Rightarrow \quad g'(x) = \frac{1}{\sin\,(g(x))} = \text{cosec}\,(g(x))$

145. Now, $g_2(x) = g(g(x))$

$= g\left(\frac{1}{1 - x}\right)$

$$= \frac{1}{1 - \frac{1}{1 - x}} = \frac{1 - x}{1 - x - 1} = \frac{x - 1}{x}$$

Also, $g_3(x) = g(g(g(x)))$

$$= \frac{1}{1 - \frac{x-1}{x}} = \frac{x}{x - x + 1} = x$$

Again, $g_4(x) = g(g(g(g(x))))$

$$= \frac{1}{1 - x}$$

Thus, the period of $g(x)$ is 3.

Now, $g_{2016}(x) = g_3(x) = x$

Thus, the range of $g_{2016}(x)$ is R.

146. We have

$$(c - 1)(x^2 + x + 1)^2 - (c + 1)(x^4 + x^2 + 1) = 0$$

$$\Rightarrow (x^2 + x + 1)[(c - 1)(x^2 + x + 1) - (c + 1)(x^2 - x + 1)] = 0$$

$$\Rightarrow (c - 1)(x^2 + x + 1) - (c + 1)(x^2 - x + 1) = 0$$

$$\Rightarrow x^2(c - 1 - c - 1) + x(c - 1 + c + 1) + (c - 1 + c + 1) = 0$$

$$\Rightarrow -2x^2 + 2cx - 2 = 0$$

$$\Rightarrow x^2 - cx + 1 = 0$$

$$\Rightarrow x^2 + 1 = cx$$

Now, $f(f(x)) = f\left(\frac{1 - x}{1 + x}\right)$

$$= \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \frac{1 + x - 1 + x}{1 + x + 1 - x} = \frac{2x}{2} = x$$

Also, $f\left(f\left(\frac{1}{x}\right)\right) = f\left(\frac{x - 1}{x + 1}\right)$

$$= \frac{1 - \frac{x-1}{x+1}}{1 + \frac{x-1}{x+1}} = \frac{x + 1 - x + 1}{x + 1 + x - 1} = \frac{2}{2x} = \frac{1}{x}$$

Therefore, $f(x(x)) + f\left(f\left(\frac{1}{x}\right)\right)$

$$= x + \frac{1}{x}$$

$$= \frac{x^2 + 1}{x} = \frac{cx}{x} = c$$

147. We have $f(x)$

$$= \frac{1}{2}\left[2\sin^2 x + 2\sin^2\left(x + \frac{\pi}{3}\right) + 2\cos x \cos\left(x + \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[(1 - \cos 2x) + \left(1 - \cos\left(2x + \frac{2\pi}{3}\right)\right) + \cos\left(2x + \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[1 + 1 + \frac{1}{2} - \cos 2x + \cos\left(2x + \frac{\pi}{3}\right) - \cos\left(2x + \frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} - \cos 2x + 2\sin\left(2x + \frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} - \cos 2x + 2\cos 2x \times \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} - \cos 2x + \cos 2x\right]$$

$$= \frac{5}{4}$$

Also $g\left(\frac{5}{4}\right) = 1$

Thus, $(g_o f)(x) = g(f(x)) = g\left(\frac{5}{4}\right) = 1$

Hence, the result.

148. We have $f(g(x)) = 1 + x^2 - 2x^3 + x^4$

$$= 1 + (x^4 - 2x^3 + x^2)$$

$$= 1 + x^2(x^2 - 2x + 1)$$

$$= 1 + x^2(x - 1)^2$$

$$= 1 + (x(x - 1))^2$$

Thus, $g(x) = \pm x(x - 1)$

149. We have $f(g(x))$

$$= f(1 + x - [x])$$

$= 1$, since the $R_g = [1, 2)$

150. Given $f(f(x)) = x$

$$\Rightarrow f\left(\frac{ax}{x + 1}\right) = x$$

$$\Rightarrow \frac{a \cdot \left(\frac{ax}{x + 1}\right)}{\left(\frac{ax}{x + 1}\right) + 1} = x$$

$$\Rightarrow \frac{a^2 x}{ax + x + 1} = x$$

$$\Rightarrow a^2 x = (a + 1)x^2 + x$$

Comparing the co-efficients of x and x^2

We get, $a^2 = 1$ & $a + 1 = 0$

$\Rightarrow a^2 = 1$ & $a = -1$

$\Rightarrow a = \pm 1$ & $a = -1$

$\Rightarrow a = 1$

151. Given $f(x) = x\left(\dfrac{3^x - 1}{3^x + 1}\right)$

$\Rightarrow \quad f(-x) = -x\left(\dfrac{3^{-x} - 1}{3^{-x} + 1}\right) = -x\left(\dfrac{1 - 3^x}{1 + 3^x}\right)$

$\Rightarrow \quad f(-x) = x\left(\dfrac{3^x - 1}{3^x + 1}\right) = f(x)$

$\Rightarrow \quad f(x)$ is an even function.

152. Given $f(x) = x \sin(x^2 + 1)$

$\Rightarrow \quad f(-x) = -x \sin((-x)^2 + 1)$

$\Rightarrow \quad f(-x) = -x \sin(x^2 + 1) = -f(x)$

$\Rightarrow \quad f(x)$ is an odd function.

153. Given $f(x) = \tan^{-1}(\sin(\cos^{-1}(x)))$

$\Rightarrow \quad f(-x) = \tan^{-1}(\sin(\cos^{-1}(-x)))$

$\Rightarrow \quad f(-x) = \tan^{-1}(\sin(\pi - \cos^{-1} x))$

$\Rightarrow \quad f(-x) = \tan^{-1}(\sin(\cos^{-1} x))$

$\Rightarrow \quad f(x)$ is an even function.

154. Given $f(x) = \sin x + \cos x$

$\Rightarrow \quad f(-x) = \sin(-x) + \cos(-x)$

$\Rightarrow \quad f(-x) = -\sin x + \cos x$

$\Rightarrow \quad \neq f(x), -f(x)$

$\Rightarrow \quad f(x)$ is neither even nor odd function.

155. Given $f(x) = \left(\dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1\right)$

$$\Rightarrow \quad f(-x) = \frac{-x}{e^{-x} - 1} - \frac{x}{2} + 1$$

$$= \frac{x}{1 - e^{-x}} - \frac{x}{2} + 1$$

$$= \frac{xe^x}{e^x - 1} - \frac{x}{2} + 1$$

$$= \left(\frac{xe^x}{e^x - 1} - x\right) + \frac{x}{2} + 1$$

$$= \left(\frac{xe^x - xe^x + x}{e^x - 1}\right) + \frac{x}{2} + 1$$

$$= \left(\frac{x}{e^x - 1}\right) + \frac{x}{2} + 1$$

$$= f(x)$$

$\Rightarrow \quad f(x)$ is an even function.

156. Given $f(x) = \log\left(\dfrac{3 - x}{3 + x}\right)$

$\Rightarrow \quad f(-x) = \log\left(\dfrac{3 + x}{3 - x}\right)$

$\Rightarrow \quad f(x) + f(-x) = \log\left(\dfrac{3 - x}{3 + x}\right) + \log\left(\dfrac{3 + x}{3 - x}\right)$

$\Rightarrow \quad f(x) + f(-x) = \log\left(\dfrac{3 - x}{3 + x} \times \dfrac{3 + x}{3 - x}\right) = \log 1$

$= 0$

$\Rightarrow \quad f(-x) = -f(x)$

$\Rightarrow \quad f(x)$ is an odd function.

157. Given $f(x) = \log\left(x + \sqrt{x^2 + 1}\right)$

$\Rightarrow \quad f(-x) = \log\left(-x + \sqrt{x^2 + 1}\right)$

$\Rightarrow \quad f(-x) = \log\left(\sqrt{x^2 + 1} - x\right)$

$\Rightarrow \quad f(x) + f(-x) = \log\left(x + \sqrt{x^2 + 1}\right) + \log\left(\sqrt{x^2 + 1} - x\right)$

$\Rightarrow \quad f(x) + f(-x) = \log\left\{\left(x + \sqrt{x^2 + 1}\right)\left(\sqrt{x^2 + 1} - x\right)\right\}$

$\Rightarrow \quad f(x) + f(-x) = \log(x^2 - x^2 + 1) = 0$

$\Rightarrow \quad f(-x) = -f(x)$

$\Rightarrow \quad f(x)$ is an odd function.

158. Given $f(x) = \sqrt{x^2 - x + 1} - \sqrt{x^2 + x + 1}$

$\Rightarrow \quad f(-x) = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1} = -f(x)$

$\Rightarrow \quad f(x)$ is an odd function.

159. We have $f(x)$

$= O + O + O$

$=$ Odd function

160. We have $f(x)$

$$= \begin{array}{cccc} (\tan(x^5)) & e^{x^3} & \text{sgn} & x^7 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ O & O & O & O \end{array}$$

$=$ even function

161. We have

Numerator $= E \times O = O$

Denominator $= O + O + O = O$

Thus, $f(x) = \dfrac{O}{O} = E =$ even function.

162. We have

Numerator $= O(O(O)) = O$

Denominator $= E + E + E = E$

Thus, $f(x) = \dfrac{O}{E} = O =$ Odd function.

163. We have $f(x)$

$$= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$$

$= \frac{1}{2}(4^{\sin x} + 4^{-\sin x}) + \frac{1}{2}\left(4^{\sin x} - 4^{-\sin x}\right)$

= (Even Function) + (Odd Function)

Hence, the result.

164. We have $f(x)$

$= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x)) - f(-x))$

$= \frac{1}{2}\left((1 + x)^{2015} + (1 - x)^{2015}\right)$

$+ \frac{1}{2}\left((1 + x)^{2015} - (1 - x)^{2015}\right)$

= (Even function) + (Odd function)

Hence, the result.

165. Even extension of $f(x)$ is

$$f(x) = \begin{cases} x + x^2 : 0 \le x < 3 \\ x + 1 \ \ : 3 < x \le 5 \\ -x + x^2 : -3 < x \le 0 \\ -x + 1 \ \ : -5 \le x < -3 \end{cases}$$

and odd extension of $f(x)$ is

$$f(x) = \begin{cases} x + x^2 : 0 \le x < 3 \\ x + 1 \ \ : 3 < x \le 5 \\ x - x^2 : -3 < x \le 0 \\ x - 1 \ \ : -5 \le x < -3 \end{cases}$$

Hence, the result.

166. Even extension

$$= \begin{cases} f(x) \ \ : 0 \le x \le 2 \\ f(-x) : -2 \le x < 0 \end{cases}$$

$$= \begin{cases} x + e^x + \sin x : 0 \le x \le 2 \\ -x + e^{-x} - \sin x : -2 \le x < 0 \end{cases}$$

We can not find the odd extension of $f(x)$

Since $f(0) = 1$.

In case of odd function, the value of $f(0)$ must be zero.

167. Even extension

$$f(x) = \begin{cases} f(x) : 0 \le x \le 1 \\ f(-x) : -1 \le x < 0 \end{cases}$$

$$f(x) = \begin{cases} x^2 + x + \sin x - \cos x + \log(1 + x) : 0 \le x \le 1 \\ x^2 - x - \sin x - \cos x + \log(1 - x) : -1 \le x < 0 \end{cases}$$

We can not find the odd extension, since $f(0) = -1$.

168. Here, the period of 3 sin $4x = \frac{2\pi}{4} = \frac{\pi}{2}$ and the period of 4 $\cos 3x = \frac{2\pi}{3}$.

Thus, the period of $f(x)$ is

$= \text{L.C.M of } \left\{\frac{\pi}{2}, \frac{2\pi}{3}\right\}$

$= \frac{\text{L.C.M of } \{\pi, 2\pi\}}{\text{H.C.F of } \{2, 3\}}$

$= \frac{2\pi}{1} = 2\pi$

169. Here, the period of 3 sin $4x$ is $= \pi/2$ and the period of $4|\sin 4x|$ is $= \frac{\pi}{4}$

Thus, the period of $f(x)$ is

$= \text{L.C.M of } \left\{\frac{\pi}{2}, \frac{\pi}{4}\right\}$

$= \frac{\text{L.C.M of } \{\pi, \pi\}}{\text{H.C.F of } \{2, 4\}}$

$= \frac{\pi}{2}$

170. We have $f(x) = \sin x \cdot \operatorname{cosec} x$

$= 1$

Thus, the function is periodic but it has no fundamental period.

171. We have $f(x) = \tan x \cdot \cot x$

$= 1$

Thus, the function is periodic but it has no fundamental period.

172. Here, the period of $5\sin(2\sqrt{2}x)$ is

$= \frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$

and the period of 7 $\cos(3\sqrt{2}x)$ is

$= \frac{2\pi}{3\sqrt{2}}$

Thus, the period of $f(x)$ is

$= \text{L.C.M of } \left\{\frac{\pi}{\sqrt{2}}, \frac{2\pi}{3\sqrt{2}}\right\}$

$= \frac{\text{L.C.M of } \{\pi, 2\pi\}}{\text{H.C.F of } \{\sqrt{2}, 3\sqrt{2}\}}$

$= \frac{2\pi}{\sqrt{2}} = \pi/\sqrt{2}$

173. Here, the period of 3 $\sin(2\sqrt{3}x)$ is $\frac{2\pi}{2\sqrt{3}} = \frac{\pi}{\sqrt{3}}$

and the period of 2 $\cos(5\sqrt{3}x)$ is $\frac{2\pi}{5\sqrt{3}}$

Thus, the period of $f(x)$ is

$= \text{L.C.M of } \left\{\frac{\pi}{\sqrt{3}}, \frac{2\pi}{5\sqrt{3}}\right\}$

Since, two irrational roots are not of the same kind, so we can not find its L.C.M

So, the period of the given function is not exist.

174. Here, the period of $\sin^2 x$ is π, the period of

$$\sin^2\left(x+\frac{\pi}{3}\right) \text{ is also } \pi.$$

Finally, the period of $\cos x \cos\left(x+\frac{\pi}{3}\right)$

$$= \frac{1}{2}\times\left(2\cos x\cos\left(x+\frac{\pi}{3}\right)\right)$$

$$= \frac{1}{2}\times\left(\cos\left(2x+\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{2\pi}{2} = \pi$$

Thus, the period of $f(x)$ is

$$= \text{L.C.M of } \{\pi, \pi, \pi\}$$

$$= \pi$$

175. Here, the period of $|\sin x + \cos x|$ is

$$= \sqrt{2}\left|\sin\left(x+\frac{\pi}{4}\right)\right|$$

$$= 2\pi$$

and the period of $|\sin x| + |\cos x|$ is $\frac{\pi}{2}$

Hence, the period of $f(x)$ is

$$= \text{L.C.M of } \left\{2\pi, \frac{\pi}{2}\right\}$$

$$= \frac{\text{L.C.M of } \{\pi, 2\pi\}}{\text{H.C.F of } \{1, 2\}}$$

$$= \frac{2\pi}{1} = 2\pi$$

176. Here, the period of $|\sin x + \cos x|$ is

$$= \sqrt{2}\left|\sin\left(x+\frac{\pi}{4}\right)\right|$$

$$= 2\pi$$

and the period of $|\sin x| - |\cos x|$ is π

Hence, the period of $f(x)$ is

$$= \text{L.C.M of } \{2\pi, \pi\}$$

$$= 2\pi$$

177. Here, the period of $\frac{1}{2}\left(\frac{|\sin x|}{\cos x}\right)$ is

$= \text{L.C.M of } \{\pi, 2\pi\} = 2\pi$ and the period of $\frac{1}{2}\left(\frac{\sin x}{|\cos x|}\right)$ is $= \{2\pi, \pi\} = 2\pi$

Thus, the period of $f(x)$ is

$$= \text{L.C.M of } \{2\pi, 2\pi\}$$

$$= 2\pi$$

178. Here, the period of $3\sin\{2x\}$ is

$= \frac{1}{2}$ and the period of $2\cos\{3x\}$ is $= \frac{1}{3}$

Hence, the period of $f(x)$ is

$$= \text{L.C.M of } \left\{\frac{1}{2}, \frac{1}{3}\right\}$$

$$= \frac{\text{L.C.M of } \{1, 1\}}{\text{H.C.F of } \{2, 3\}}$$

$$= \frac{1}{1} = 1$$

179. We have, $f(x) + f(x+3) = 5$ ...(i)

Replacing x by $x + 3$, we get,

$$f(x+3) + f(x+6) = 5 \quad \text{...(ii)}$$

Subtracting (ii) from (i), we get,

$\Rightarrow \quad f(x) + f(x+3) - f(x+3) - f(x+6) = 0$

$\Rightarrow \quad f(x) - f(x+6) = 0$

$\Rightarrow \quad f(x+6) = f(x)$

Thus, $f(x)$ is periodic with period 6.

180. We have $f(x+4) + f(x-4) = f(x)$ ...(i)

Replacing x by $x + 4$, we get,

$$f(x+8) + f(x) = f(x+4) \quad \text{...(ii)}$$

From (i) and (ii), we get,

$$f(x+8) + f(x+4) + f(x-4) = f(x+4)$$

$\Rightarrow \quad f(x+8) + f(x-4) = 0$

Replacing x by $x + 4$, we get,

$$f(x+12) + f(x) = 0 \quad \text{...(iii)}$$

Again replacing x by $x + 12$, we get,

$$f(x+24) + f(x+12) = 0 \quad \text{...(iv)}$$

Subtracting (iii) from (iv), we get,

$$f(x+24) + f(x+12) - f(x+12) - f(x) = 0$$

$\Rightarrow \quad f(x+24) - f(x) = 0$

$\Rightarrow \quad f(x+24) = f(x)$

Thus $f(x)$ is periodic with period 24.

181. We have $f(x-1) + f(x+1) = \sqrt{2}\, f(x)$ ...(i)

Replacing x by $x + 1$ and x by $x - 1$, we get,

$$f(x) + f(x+2) = \sqrt{2}\, f(x+1) \quad \text{...(ii)}$$

and $\quad f(x-2) + f(x) = \sqrt{2}\, f(x-1)$ ...(iii)

Adding (ii) and (iii), we get,

$$f(x+2) + f(x-2) + 2f(x)$$

$$= \sqrt{2}\,(f(x-1) + f(x+1))$$

$$= \sqrt{2}\left(\sqrt{2}\, f(x)\right)$$

$$= 2\, f(x)$$

$$f(x+2)+f(x-2)=0 \quad \text{...(iv)}$$

Replacing x by $x + 2$, we get,

$$f(x+4)+f(x)=0 \quad \text{...(v)}$$

Again replacing x by $x + 4$, we get,

$$f(x+8)+f(x+4)=0 \quad \text{...(vi)}$$

Subtracting (v) from (vi), we get,

$$f(x+8)+f(x+4)-f(x+4)-f(x)=0$$

$$\Rightarrow \quad f(x+8)-f(x)=0$$

$$\Rightarrow \quad f(x+8)=f(x)=0$$

Hence, the function $f(x)$ is periodic with period 8.

182. We have $f(x+1)+f(x+5)=f(x+3)+f(x+7)$ ...(i)

Replacing x by $x + 2$, we get,

$$f(x+3)+f(x+7)=f(x+5)+f(x+9) \quad \text{...(ii)}$$

Adding (i) and (ii), we get,

$$f(x+1)+f(x+5)+f(x+3)+f(x+7)$$

$$=f(x+3)+f(x+7)+f(x+5)+f(x+9)$$

$$\Rightarrow \quad f(x+1)=f(x+9) \quad \text{...(iii)}$$

Finally replacing x by $(x - 1)$ in (iii), we get,

$$f(x)=f(x+8)$$

Hence, the period of $f(x)$ is 8.

183. Given $\quad f(x)+3f\left(\frac{1}{x}\right)=5x$ ...(i)

Replacing x by $1/x$ in (i), we get,

$$f\left(\frac{1}{x}\right)+3f(x)=5\frac{1}{x} \quad \text{...(ii)}$$

Multiplying (ii) by '3' and subtracting from (i) we get,

$$f(x)+3f\left(\frac{1}{x}\right)-3f\left(\frac{1}{x}\right)-15f(x)=\frac{15}{x}$$

$$\Rightarrow \quad -14f(x)=\frac{15}{x}$$

$$\Rightarrow \quad f(x)=-\frac{15}{14x}$$

184. Given $\quad 2f(\sin x)+3f(\cos x)=5$ ...(i)

Replacing x by $\left(\frac{\pi}{2}-x\right)$ in (i), we get,

$$2f\left(\sin\left(\frac{\pi}{2}-x\right)\right)+3f\left(\cos\left(\frac{\pi}{2}-x\right)\right)=5$$

$$\Rightarrow \quad 2f(\cos x)+3f(\sin x)=5 \quad \text{...(ii)}$$

Multiplying (i) by 2 and (ii) by 3, we get,

$$4f(\sin x)+6f(\cos x)=10 \quad \text{...(iii)}$$

and $9f(\sin x)+6f(\cos x)=15$ ...(iv)

Subtracting (iii) from (iv), we get,

$$5f(\sin x)=5$$

$$\Rightarrow \quad f(\sin x)=1$$

$$\Rightarrow \quad f(x)=1,\ \forall\ x\in(0,1)$$

185. Given $\quad f(1)=0$

$$f(2)=f(1)+1=0+1=1$$

$$f(3)=f(2)+1=1+1=2$$

$$f(4)=f(3)+1=2+1=3,\ \ldots$$

Now, $\quad f(2)-f(1)=1$

$$f(3)-f(2)=2$$

$$f(4)-f(3)=3$$

... ...

... ...

... ...

$$f(x+1)-f(x)=x$$

On addition, we get,

$$f(x+1)-f(1)=1+2+3+\ldots+x$$

$$\Rightarrow \quad f(x+1)=f(1)+1+2+3+\ldots+x$$

$$\Rightarrow \quad f(x+1)=0+1+2+3+\ldots+x=\frac{x(x-1)}{2}$$

Replacing x by $x - 1$, we get,

$$f(x)=\frac{(x-1)(x-2)}{2} \text{ for all } x \text{ in } N.$$

186. Given $\quad f(x+f(x))=4f(x)$

Put $\quad x=1$, we get, $f(1+f(1))=4f(1)$

$$\Rightarrow \quad f(1+4)=4\times 4$$

$$\Rightarrow \quad f(5)=16$$

187. Given $\quad P(x+1)-P(x)=2x+1$

Put $\quad x=1,2,3,\ldots,n$, we get,

$$P(2)-P(1)=3$$

$$P(3)-P(2)=5$$

$$P(4)-P(3)=7$$

... ...

... ...

... ...

$$P(n+1)-P(n)=2n+1$$

On addition, we get,

$$P(n+1) - P(1) = 3 + 5 + 7 + ... + (2n+1)$$

$\Rightarrow$ $P(n+1) = P(1) = 3 + 5 + 7 + ... + (2n+1)$

$\Rightarrow$ $P(n+1) = 1 + 3 + ... + (2n+1) = n(n+1)$

$\Rightarrow$ $P(n+1) = n(n+1)$

$\Rightarrow$ $P(n) = n(n-1)$

Hence, the polynomial $P(x)$ is $P(x) = x(x-1)$

188. Given $f(x+y) = f(x).\,f(y)$

Put $x = 1 = y, f(2) = f(1)\cdot f(1) = 4 = 2^2$

Put $x = 2,\ y = 1, f(3) = f(2)\cdot f(1) = 2^3$

Put $x = 2 = y, f(4) = f(2)\cdot f(2) = 2^4$

Now, $\sum_{k=1}^{n} f(a+k)$

$= f(a+1) + f(a+2) + f(a+3) + ... + f(a+n)$

$= f(a)\cdot f(1) + f(a)\cdot f(2) + ... + f(a)\cdot f(3)$

$= f(a)(f(1) + f(2) + f(3) + ... + f(n))$

$= f(a)(2 + 2^2 + 2^3 + 2^4 + ... + 2^n)$

Also, $f(a)(2 + 2^2 + 2^3 + ... + 2^n) = 16(2^n - 1)$

$\Rightarrow$ $f(a)\cdot 2\left(\frac{2^n - 1}{2 - 1}\right) = 16(2^n - 1)$

$\Rightarrow$ $f(a) = 8$

$\Rightarrow$ $2^a = 8 = 2^3$

$\Rightarrow$ $a = 3$

189. Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

Given relation is $f(x) + f\left(\frac{1}{x}\right) = f(x)\cdot f\left(\frac{1}{x}\right)$

$$\left(a_ox^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n\right)$$

$$+ \left(\frac{a_o}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + ... + a_n\right)$$

$$= \left(a_ox^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n\right)$$

$$.\left(\frac{a_o}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + ... + a_n\right)$$

$\Rightarrow$ $\left(a_ox^{2n} + a_1x^{2n-1} + a_2x^{2n-2} + ... + a^nx^n\right)$

$$+ \left(a_o + a_1x + a_2x^2 + ... + a_nx^n\right)$$

$$= \left(a_ox^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n\right)$$

$$.\left(a_o + a_1x + a_2x^2 + ... + a_nx^n\right)$$

Since, the equation is valid for all x except 0, we compare the co-efficients of various powers of x on both sides

$x^{2n} : a_o = a_n\, a_o \Rightarrow a_n = 1$, where $a_o \neq 0$

$x^{2n-1} : a_1 = a_oa_{n-1} + a_1a_n \Rightarrow a_{n-1} = 0$

$x^{2n-2} : a_{n-2} = 0$

$x^{2n-3} : a_{n-3} = 0$

... ...

... ...

... ...

$x^2 : a_2 = 0$

$x : a_1 = 0$

Now, comparing the co-efficients of x^n, we get,

$$2a_n = a_0^2 + a_n^2$$

$\Rightarrow$ $2 = a_0^2 + 1$

$\Rightarrow$ $a_0 = \pm 1$

Thus, $f(x) = \pm x^n + 1$

190. Given $f(x)\,f(y) = f(x) + f(y) + f(xy) - 2$

Put $y = \frac{1}{x}$

$\Rightarrow$ $f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + f(1) - 2$

$\Rightarrow$ $f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) + 2 - 2$

$\Rightarrow$ $f(x)\cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$

$\Rightarrow$ $f(x) = x^n + 1$

Also, it is given that, $f(2) = 5$

$\Rightarrow$ $2^n + 1 = 5$

$\Rightarrow$ $2^n = 4$

$\Rightarrow$ $n = 2$

Thus, $f(6) + 2009$

$= 36 + 2009$

$= 2045.$

191. Given $P(x^2) + 2x^2 + 10x = 2x\,P(x+1) + 3$

Put $x = 0$, we get, $P(0) = 3$.

Put $x = -1$, we get,

$P(1) + 2 - 10 = -2P(0) + 3 = -6 + 3 = -3$

$P(1) = -3 - 2 + 10 = 5$

Let $P(x) = ax + b$

Then $P(0) = b \Rightarrow b = 3$

Also, $P(1) = a + b \Rightarrow a + b = 5$

$\Rightarrow$ $a + 3 = 5 \Rightarrow a = 2$

Thus, $P(x) = 2x + 3$

192. Let $f(x) = ax^2 + bx + c$

Put $x = 0 \Rightarrow c = f(0) = 1.$

Also, $f(x + 2) - f(x) = 4x + 2$

$\Rightarrow$ $a(x + 2)^2 + b(x + 2) + c - ax^2 - bx - c = 4x + 2$

$\Rightarrow$ $a(x^2 + 4x + 4) + b(x + 2) + c - ax^2 - bx - c = 4x + 2$

$\Rightarrow$ $a(4x + 4) + 2b = 4x + 2$

$\Rightarrow$ $4a = 4$ and $b = -2$

$\Rightarrow$ $a = 1$ and $b = -1$

Thus, $f(x) = x^2 - x + 1$

Level III

(Problems for JEE-Advanced)

1. Given $2f(x) + 3f\left(\frac{1}{x}\right) = x - 3,\ x \neq 0$ ...(i)

Replacing x by $1/x$ in (i), we get,

$$2f\left(\frac{1}{x}\right) + 3f(x) = \frac{1}{x} - 3 \quad \text{...(ii)}$$

Multiplying (i) by 2 and (ii) by 3 and subtracting (i) from (ii), we get,

$$9f(x) - 6f(x) = \frac{3}{x} - 9 - 2x + 6$$

$\Rightarrow$ $3f(x) = \frac{3}{x} - 2x - 3$

$$f(x) = \left(\frac{1}{x} - \frac{2}{3}x - 1\right)$$

2. Given $f(x) + 2f(1 - x) = x^2 + 2$ ...(i)

Replacing x by $1 - x$, we get,

$$f(1 - x) + 2f(x) = (1 - x)^2 + 2 \quad \text{...(ii)}$$

Multiplying (ii) by 2 and (i) by 1 and subtracting (i) from (ii), we get,

$\Rightarrow$ $3f(x) = 2(1 - x)^2 + 4 - x^2 - 2$

$\Rightarrow$ $3f(x) = x^2 - 4x + 4 = (x - 2)^2$

$\Rightarrow$ $f(x) = \frac{1}{3}(x - 2)^2$

3. Let $2x + 3y = u$ and $2x - 7y = v$

Then $x = \frac{9u + v}{20}$ & $y = \frac{u - v}{10}$

Now, $f(u, v) = 20 \times \left(\frac{9u + v}{20}\right) = 9u + v$

$\Rightarrow$ $f(x, y) = 9x + y$

4. As we know that, if a polynomial satisfying

$f(x) + f\left(\frac{1}{x}\right) = f(x)\cdot f\left(\frac{1}{x}\right)$, then the polynomial

$f(x) = x^n + 1$

Also, $f(3) = 28$

$\Rightarrow$ $3^n + 1 = 28$

$n = 3$

Thus, $f(4) = 4^3 + 1 = 65$

5. Given $f(x + y, x - y) = xy$

Let $x + y = u$ and $x - y = v$

Then $x = \frac{1}{2}(u+v)$ & $y = \frac{1}{2}(u - v)$

Therefore, $f(u, v) = \frac{1}{4}(u^2 - v^2)$

$$f(x,y) = \frac{1}{4}(x^2 - y^2)$$

Also, $f(y, x) = \frac{1}{4}(y^2 - x^2)$

Now, $\frac{f(x, y) + f(y, x)}{2}$

$$\frac{1}{8}(x^2 - y^2 + y^2 - x^2) = 0$$

6. Given $F(n + 1) = \frac{2F(n) + 1}{2},\ n \in N$

$\Rightarrow$ $F(n + 1) - F(n) = \frac{1}{2}$ ...(i)

Put $n = 1, 2, 3, ..., 2014$ in (i). we get,

$F(2) - F(1) = 1/2$

$F(3) - F(2) = 1/2$

$F(4) - F(3) = 1/2$

... ...

... ...

$F(2015) - F(2014) = 1/2$

On addition, we get,

$$F(2015) - F(1) = \frac{1}{2} \times 2014 = 1007$$

$\Rightarrow$ $F(2015) = F(1) + 1007 = 1009$

7. Now, $f(x)\cdot f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(x\,y)\right\}$

$$= \frac{1}{2}\left(2f(x)f(y) - f\left(\frac{x}{y}\right) - f(x\,y)\right)$$

$$= \frac{1}{2}\left[2\cos(\log x).\cos(\log y) - \cos\left(\log\left(\frac{x}{y}\right)\right) - \cos(\log(xy))\right]$$

$$= \frac{1}{2}\left[\cos(\log(xy)) + \cos\left(\log\left(\frac{x}{y}\right)\right) - \cos\left(\log\left(\frac{x}{y}\right)\right) - \cos(\log(xy))\right]$$

$$= 0$$

8. Given $f(x + 1) + f(x - 1) = 2f(x)$

Put $x = 1, f(2) + f(0) = 2 \cdot f(1) \Rightarrow f(2) = 2f(1)$

Put $x = 2, f(3) + f(1) = 2 \cdot f(2)$

$\Rightarrow f(3) = 2f(2) - f(1) = 4f(1) - f(1) = 3f(1)$

Put $x = 3, f(4) + f(2) = 2 \cdot f(3)$

$\Rightarrow f(4) = 6\,f(1) - 2\,f(1) = 4\,f(1)$

Therefore by similarity, we can say that,

$f(n) = nf(1), n \in N$

9. Given $f(x) = \dfrac{4^x}{4^x + 2}$

Also, $f(1 - x) = \dfrac{4^{1-x}}{4^{1-x} + 2}$

Now, $f(x) + f(1 - x)$

$$= \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2}$$

$$= \frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2} \times \frac{4^x}{4^x}$$

$$= \frac{4^x}{4^x + 2} + \frac{4}{4 + 2.4^x}$$

$$= \frac{4^x}{4x + 2} + \frac{2}{2 + 4^x}$$

$$= \frac{4^x + 2}{4^x + 2}$$

$$= 1$$

Now, $f\left(\frac{1}{2015}\right) + f\left(\frac{2}{2015}\right) + f\left(\frac{3}{2015}\right) + \ldots + f\left(\frac{2014}{2015}\right)$

$$= \left(f\left(\frac{1}{2015}\right) + f\left(\frac{2014}{2015}\right)\right) + \left(f\left(\frac{2}{2015}\right) + f\left(\frac{2013}{2014}\right)\right) + \left(f\left(\frac{3}{2015}\right) + f\left(\frac{2012}{2015}\right)\right)$$

$$+ \ldots + \left(f\left(\frac{1007}{2015}\right) + f\left(\frac{1008}{2015}\right)\right) + f\left(\frac{1008}{2015}\right)$$

$$= 1 + 1 + 1 + \ldots + 1\,(1007 \text{ times}) + f\left(\frac{1008}{2015}\right)$$

$$= 1007 + f\left(\frac{1008}{2015}\right)$$

10. Given $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}, x \neq 0$

$f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right)^2 - 2$

Replacing $\left(x + \frac{1}{x}\right)$ by y, we get,

$f(y) = y^2 - 2$

$\Rightarrow f(x) = x^2 - 2$

11. *

12. 2

13. Given $f(x - 1) = x^2 - 1$

Replacing $x - 1$ by x, we get,

$f(x) = (x - 1)^2 - 1 = x^2 - 2x$

14. Given $f\left(\dfrac{8}{\sqrt{1} + \sqrt{x}}\right) = x$, for all $x \geq 0$

Put $x = 9$, we get

$f\left(\dfrac{8}{\sqrt{1} + \sqrt{9}}\right) = 9$

$\Rightarrow f\left(\dfrac{8}{\sqrt{1 + 3}}\right) = 9$

$\Rightarrow f(4) = 9$

15. The function $\sqrt{1 - x}$ is defined for $(1 - x) \geq 0$

$\Rightarrow x \leq 1$

$\Rightarrow x \in (-\infty, 1]$

Also, the function $\dfrac{1}{\sqrt{x + 1}}$ is defined for

$x + 1 > 0$

$\Rightarrow x > -1$

$\Rightarrow x \in (-1, \infty)$

Thus, domain of the function f

$= (-1, 1]$

16. Given $f(x) = \sqrt{x - 2} + \sqrt{5 - x}$

Now, $\sqrt{x - 2}$ is defined for $x \geq 2$

and $\sqrt{5 - x}$ is defined for $x \leq 5$

Thus, $D_f = [2, 5]$

17. Here, $\sqrt{x-2}$ is defined for $x \geq 2$

Also, $\sqrt{4-x}$ is defined for $x \leq 4$

Thus, $D_f = [2, 4]$

18. Here, $x^2 - 3x + 2 = 0$ gives $x = 1, 2$

Also $\log(3-x)$ is defined for $3 - x > 0$

$\Rightarrow x < 3$

$\Rightarrow x \in (-\infty, 3)$

Thus, $D_f = (-\infty, 3) - \{1, 2\}$

19. Here, $\sqrt{|x|-1}$ is defined for $|x| - 1 \geq 0$

$\Rightarrow |x| \geq 1$

$\Rightarrow x \in (-\infty, 1] \cup [1, \infty)$

Also, $\sqrt{3-|x|}$ is defined for $3 - |x| \geq 0$

$\Rightarrow |x| \leq 3$

$\Rightarrow -3 \leq x \leq 3$

Thus, $D_f = [-3, -1] \cup [1, 3]$

20. The function f is defined for

$$\frac{|x|-1}{4-|x|} \geq 0 \ |x| \neq 4$$

$$\Rightarrow \frac{|x|-1}{|x|-4} \leq 0, |x| \neq 4$$

$\Rightarrow 1 \leq |x| \leq 4, |x| \neq 4$

$\Rightarrow x \in (-4, -1] \cup [1, 4)$

Thus, $D_f = (-4, -1] \cup [1, 4)$

21. The function f is defined for

$$\frac{[x]-1}{4-[x]} \geq 0, [x] \neq 4$$

$$\Rightarrow \frac{[x]-1}{[x]-4} \leq 0, [x] \neq 4$$

$\Rightarrow 1 \leq [x] \leq 4, [x] \neq 4$

$\Rightarrow 1 \leq x < 5, x \notin [4, 5)$

$\Rightarrow 1 \leq x < 4$

$\Rightarrow x \in [1, 4)$

Thus, $D_f = [1, 4)$

22. The function f is defined for

$$-1 \leq \log_2(x) \leq 1$$

$\Rightarrow 2^{-1} \leq x \leq 2$

$\Rightarrow \frac{1}{2} \leq x \leq 2$

Thus, $D_f = \left[\frac{1}{2}, 2\right]$

23. The function f is defined for

$$-1 \leq \left(\frac{1+x^2}{2x}\right) \leq 1$$

$$\Rightarrow \left|\frac{1+x^2}{2x}\right| \leq 1$$

$$\Rightarrow \frac{|1+x^2|}{|2x|} \leq 1$$

$\Rightarrow |1 + x^2| \leq |2x|$

$\Rightarrow 1 + |x^2| \leq 2|x|$

$\Rightarrow |x|^2 - 2|x| + 1 \leq 0$

$\Rightarrow (|x| - 1)^2 \leq 0$

$\Rightarrow (|x| - 1)^2 = 0$

$\Rightarrow |x| - 1 = 0$

$\Rightarrow x = \pm 1$

Thus, $D_f = \{-1, 1\}$

24. The function f is defined for

$$-1 \leq \log_2\left(\frac{x^2}{2}\right) \leq 1$$

$$\Rightarrow 2^{-1} \leq \left(\frac{x^2}{2}\right) \leq 2$$

$$\Rightarrow \frac{1}{2} \leq \left(\frac{x^2}{2}\right) \leq 2$$

$\Rightarrow 1 \leq x^2 \leq 4$

$\Rightarrow \sqrt{1} \leq \sqrt{x^2} \leq \sqrt{4}$

$\Rightarrow 1 \leq |x| \leq 2$

$\Rightarrow \in [-2, -1] \cup [1, 2]$

Thus, $D_f = [-2, -1] \cup [1, 2]$

25. The function f is defined for

$$\log_5(\log_3(18x - x^2 - 77)) > 0$$

$\Rightarrow (\log_3(18x - x^2 - 77)) > 1$

$\Rightarrow (18x - x^2 - 77) > 3$

$\Rightarrow (18x - x^2 - 80) > 0$

$\Rightarrow x^2 - 18x + 80 < 0$

$(x-8)(x-10) < 0$

$\Rightarrow 8 < x < 10$

Thus, $D_f = (8, 10)$

26. The function f is defined for

$1 - \log_x \log_2 (4^x - 12) \geq 0$

$\Rightarrow \log_x \log_2 (4^x - 12) \leq 1$

$\Rightarrow \log_2 (4^x - 12) \leq x$

$\Rightarrow (4^x - 12) \leq 2^x$

$\Rightarrow ((2^x)^2 - 2^x - 12) \leq 0$

$\Rightarrow (2^x - 4)(2^x + 3) \leq 0$

$\Rightarrow -3 \leq 2^x \leq 4$

$\Rightarrow 2^x \leq 4 = 2^2$

$\Rightarrow x \leq 2$

Also, $\log_2 (4^x - 12) > 0$

$\Rightarrow (4^x - 12) > 2^0 = 1$

$\Rightarrow 4^x > 13$

$\Rightarrow x > \log_4(13)$

Thus, $D_f = (\log_4 (13), 2]$

27. Given $3^y + 2^{x^4} = 2^{4x^2-3}$

$\Rightarrow y = \log_3 \left(2^{4x^2-3} - 2^{x^4}\right)$

It is defined for $\left(2^{4x^2-3} - 2^{x^4}\right) > 0$

$\Rightarrow 2^{x^4} < 2^{4x^2-3}$

$\Rightarrow x^4 < 4x^2 - 3$

$\Rightarrow x^4 - 4x^2 + 3 < 0$

$\Rightarrow (x^2 - 1)(x^2 - 3) < 0$

$\Rightarrow (x + 1)(x - 1)(x + \sqrt{3})(x - \sqrt{3}) < 0$

$\Rightarrow x \in (-\sqrt{3}, -1) \cup (1, \sqrt{3})$

Thus, $D_f = (-\sqrt{3}, -1) \cup (1, \sqrt{3})$

28. Given $f(x) = \sin\left(\log_e\left(\frac{\sqrt{4 - x^2}}{1 - x}\right)\right)$

Now, $\sqrt{4 - x^2}$ is defined for $x^2 < 4$

$\Rightarrow x^2 - 4 < 0$

$\Rightarrow (x + 2)(x - 2) < 0$

$\Rightarrow -2 < x < 2$

Also, $1 - x > 0$

$\Rightarrow x < 1$

Thus, $D_f = (-2, 1)$.

29. Given $f(x) = (x^{12} - x^9 + x^4 - x + 1)^{-1/2}$

Case-I: When $x \geq 1$

Also, $x^a > x^b \Rightarrow a > b$

Thus, $(x^{12} - x^9) + (x^4 - x) + 1 > 0$

Case-II: When $0 \leq x < 1$

Also, $x^a < x^b \Rightarrow a > b$

Thus, $(x^9 - x^4) + (x - 1) < 0$

So, $x^{12} - (x^9 - x^4) + (x - 1) > 0$

Case-III: When $x < 0$

Also, $a > b \Rightarrow x^a < x^b$

Thus, $(x^9 - x^4) + (x - 1) < 0$

Therefore, $x^{12} - (x^9 - x^4) + (x - 1) > 0$

Thus, the domain of the function is

$= D_f = R$

30. Given $y = f(x) = x^2 + x + 1$

$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

Thus, $R_f = \left[\frac{3}{4}, \infty\right)$

31. Given $y = f(x) = \frac{x^2}{x^2 + 1}$

$= \left(\frac{x^2 + 1 - 1}{x^2 + 1}\right)$

$= \left(1 - \frac{1}{x^2 + 1}\right)$

Here, $y \to 1, x \to \infty$

Thus, $R_f = [0, 1)$

32. Given $y = f(x) = \frac{x^2}{x^4 + 1} = \frac{1}{x^2 + \frac{1}{x^2}}$

$\Rightarrow y = f(x) = \frac{1}{x^2 + \frac{1}{x^2}} \leq \frac{1}{2}$

Thus, $R_f = \left[0, \frac{1}{2}\right]$

33. Given $f(x) = \frac{x^2 + x + 1}{x^4 + x^2 + 1}$

$= \frac{(x^2 + x + 1)}{(x^2 + x + 1)(x^2 - x + 1)}$

$= \frac{1}{(x^2 - x + 1)}$

Let $g(x) = (x^2 - x + 1) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

Thus, $R_f = \left[1, \frac{4}{3}\right]$

34. Given $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

$= \dfrac{(x^2 + x + 1) - 2x}{(x^2 + x + 1)}$

$= 1 - \dfrac{2x}{(x^2 + x + 1)}$

$= 1 - \dfrac{2}{\left(x + \frac{1}{x} + 1\right)}$

Let $g(x) = \left(x + \frac{1}{x} + 1\right)$

$g'(x) = \left(1 - \frac{1}{x^2}\right)$

Now, $g'(x) = 0$ gives $x = \pm 1$

At the nbd of $x = -1$, $g'(x)$ changes from positive to negative, so $g(x)$ will provide us the max value and the max value of $g(x)$ is -1.

At the nbd of $x = 1$, g'(x) changes from negetive to positive, so $g(x)$ will provide us the min value and the min value of $g(x)$ is 3.

Thus, the range of the function is $= \left[\frac{1}{3}, 3\right]$

35. Given $y = f(x) = \log_2\left(\sqrt{x - 2} + \sqrt{4 - x}\right)$

Clearly, domain of the function = [2, 4]

At $x = 2$, $y = \log_2(\sqrt{2}) = \frac{1}{2}(\log_2 2) = \frac{1}{2}$

At $x = 4$, $y = \log_2(\sqrt{2}) = \frac{1}{2}(\log_2 2) = \frac{1}{2}$

At $x = 3$, $\log_2(2) = 1$

Thus, $R_f = \left[\frac{1}{2}, 1\right]$

36. Given $y = f(x) = x^2 + \dfrac{1}{x^2 + 1}$

$= (x^2 + 1) + \dfrac{1}{(x^2 + 1)} - 1$

Applying, A.M ≥ G.M, we get,

$y \geq 2 - 1 = 1$

Thus, $R_f = [1, \infty)$

37. Given $y = f(x) = x^4 + 4x^2 + \dfrac{1}{x^4 + 4x^2 + 9} + 10$

$= (x^4 + 4x^2 + 9) + \dfrac{1}{(x^4 + 4x^2 + 9)} + 10$

Applying, A.M ≥ G.M, we get,

$y \geq 2 + 10 = 12$

Thus, $R_f = [12, \infty)$

38. Let $g(x) = (x^2 + 4x + 4) = (x + 2)^2$

Thus $D_g = R$ & $R_g = [0, \infty)$

So, $-\infty < \log_{1/2}(x^2 + 4x + 4) < \infty$

$\Rightarrow$ $0 < \log_{1/2}(x^2 + 4x + 4) < \infty$

$\Rightarrow$ $-\infty < \log_2(\log_{1/2}(x^2 + 4x + 4)) < \infty$

Thus, $R_f = (-\infty, \infty)$

39. Let $g(x) = \left(\dfrac{\cos^2 x + 1}{2}\right)$

Thus, $R_g = \left[\frac{1}{2}, 1\right]$

Therefore,

$R_f = \left[\sin^{-1}\left(\frac{1}{2}\right), \sin^{-1}(1)\right] = \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

40. We have

$f(x) = 2^x + 3^x + 4^x + 2^{-x} + 3^{-x} + 4^{-x} + 10$

$= (2^x + 2^{-x}) + (3^x + 3^{-x}) + (4^x + 4^{-x}) + 10$

$\geq 2 + 2 + 2 + 10 = 16$, since A.M ≥ G.M

Thus, $R_f = [16, \infty)$

41. Given $f(x) = 3\tan^2 x + 12\cot^2 x + 5$

$= (3\tan^2 x + 12\cot^2 x) + 5$

$= (3\tan^2 x + 12\cot^2 x) + 5$

$\geq 12 + 5 = 17$, since A.M ≥ G.M

Thus, $R_f = [17, \infty)$

42. Given $f(x) = \sqrt{3}\sin x + \cos x + 4$

Max value of $f(x) = \sqrt{3 + 1} + 4 = 2 + 4 = 6$

Min value of $f(x) = -\sqrt{3 + 1} + 4 = -2 + 4 = 2$

Thus, $R_f = [2, 6]$

43. Given $f(x) = \dfrac{1}{2\cos^2 x + 4\sin x\cos x + 4}$

Let $g(x) = 2\cos^2 x + 4\sin x\cos x + 4$

$= 1 + \cos 2x + 2\sin 2x + 4$

$= \cos 2x + 2\sin 2x + 5$

Thus, $R_g = \lfloor 5 + \sqrt{5}, 5 - \sqrt{5}\rfloor$

Therefore, $R_f = \left[\dfrac{1}{(5 - \sqrt{5})}, \dfrac{1}{(5 + \sqrt{5})}\right]$

44. Given $f(x) = \cos^{-1}\left(\dfrac{x^2}{1+x^2}\right)$

Let $g(x) = \dfrac{x^2}{x^2+1} = \left(1 - \dfrac{1}{x^2+1}\right)$

Thus, $R_g = [0, 1)$

Now, $\cos^{-1}(0) = \cos^{-1}\left(\cos\left(\dfrac{\pi}{2}\right)\right) = \dfrac{\pi}{2}$

and $\cos^{-1}(1) = \cos^{-1}(\cos(0)) = 0$

Thus, $R_f = 0, \dfrac{\pi}{2}$

45. Given $f(x) = \cos^{-1}\left(\dfrac{\sqrt{2x^2+1}}{x^2+1}\right)$

$$= \sin^{-1}\left(\sqrt{1 - \frac{2x^2+1}{(x^2+1)^2}}\right)$$

$$= \sin^{-1}\left(\sqrt{\frac{(x^2+1)^2 - 2x^2 - 1}{(x^2+1)^2}}\right)$$

$$= \sin^{-1}\left(\sqrt{\frac{x^4 + 2x^2 + 1 - 2x^2 - 1}{(x^2+1)^2}}\right)$$

$$= \sin^{-1}\left(\sqrt{\frac{x^4}{(x^2+1)^2}}\right)$$

$$= \sin^{-1}\left(\frac{x^2}{(x^2+1)}\right)$$

Let $g(x) = \dfrac{x^2}{x^2+1}$

Thus, $R_g = [0, 1)$

Therefore, $R_f = [\sin^{-1}(0), \sin^{-1}(1)) = \left[0, \dfrac{\pi}{2}\right)$

46. Given $f(x) = \sin^{-1}\left[x^2 + \dfrac{1}{2}\right] + \cos^{-1}\left[x^2 - \dfrac{1}{2}\right]$

Now, $\left[x^2 + \dfrac{1}{2}\right] = \left[x^2 - \dfrac{1}{2} + 1\right] = \left[x^2 - \dfrac{1}{2}\right] + 1$

Thus, $f(x) = \sin^{-1}\left[x^2 + \dfrac{1}{2}\right] + \cos^{-1}\left[x^2 - \dfrac{1}{2}\right]$

$$= \sin^{-1}\left[\left(x^2 - \frac{1}{2}\right) + 1\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$$

It is defined for $\left[x^2 - \dfrac{1}{2}\right] = 0, -1$

Therefore,

$$f(x) = \sin^{-1}(1) + \cos^{-1}(0) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Also, $f(x) = \sin^{-1}(0) + \cos^{-1}(-1) = 0 + \pi = \pi$

Thus, $R_f = \{\pi\}$

47. Given $f(x) = \dfrac{x^2 + 2x + 3}{x}, x > 0$

$$= \left(x + \frac{3}{x} + 2\right)$$

$$\geq 2\sqrt{3} + 2, \text{ since A.M} \geq \text{G.M}$$

Thus, $R_f = \lfloor 2 + 2\sqrt{3}, \infty)$

48. Given $f(x) = \dfrac{x^2 + x + 2}{x^2 + x + 1}$

$$= \frac{(x^2 + x + 1) + 1}{(x^2 + x + 1)}$$

$$= 1 + \frac{1}{(x^2 + x + 1)}$$

Let $g(x) = (x^2 + x + 1)$

$$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

Thus, max value of $f(x) = 1 + \dfrac{4}{3} = \dfrac{7}{3}$

Therefore, $R_f = \left[2, \dfrac{7}{3}\right]$

49. Given $f(x) = \dfrac{e^x}{1 + [x]}$

Clearly, $R_f = R$

50. Given $y = f(x) = \dfrac{\{x\}}{1 + \{x\}}$

$\Rightarrow$ $y = 1 - \dfrac{1}{1 + \{x\}}$

Also, $0 \leq \{x\} < 1$

$\Rightarrow$ $1 \leq 1 + \{x\} < 2$

$\Rightarrow$ $\dfrac{1}{2} < \dfrac{1}{1 + \{x\}} \leq 1$

$\Rightarrow$ $-1 \leq -\dfrac{1}{1 + \{x\}} < -\dfrac{1}{2}$

$\Rightarrow$ $1 - 1 \leq 1 - \dfrac{1}{1 + \{x\}} < 1 - \dfrac{1}{2}$

$\Rightarrow$ $0 \leq \left(1 - \dfrac{1}{1 + \{x\}}\right) < \dfrac{1}{2}$

$\Rightarrow$ $0 \leq y < \dfrac{1}{2}$

Thus, $R_f = \left[0, \frac{1}{2}\right)$

51. Given $f(x) = 2\cos x + \sec^2 x,\ x \in \left(0, \frac{\pi}{2}\right)$

$$= \cos x + \cos x + \sec^2 x$$

Applying, A.M ≥ G.M, we get,

$$\frac{\cos x + \cos x + \sec^2 x}{3} \ge \sqrt[3]{\cos x.\cos x \sec^2 x} = 1$$

$$\Rightarrow \quad \frac{\cos x + \cos x + \sec^2}{3} \ge 1$$

$$\Rightarrow \quad 2\cos x + \sec^2 x \ge 3$$

Thus, $R_f = [3, \infty)$

52. Given $f(x) = 3\tan x + \cot^3 x,\ x \in \left(0, \frac{\pi}{2}\right)$

$$= \tan x + \tan x + \tan x + \cot^3 x$$

Applying, A.M ≥ G.M, we get,

$$\Rightarrow \quad \frac{\tan x + \tan x + \tan x + \cot^3 x}{4}$$

$$\ge \sqrt[4]{\tan x \cdot \tan x \cdot \tan x \cdot \cot^3 x} = 1$$

$$\Rightarrow \quad \frac{\tan x + \tan x + \tan x + \cot^3 x}{4} \ge 1$$

$$\Rightarrow \quad 3\tan x + \cot^3 x \ge 4$$

Thus, $R_f = [4, \infty)$

53. Given $f(x) = \dfrac{\sin^4 x e^{x^2}}{\left[\frac{x + 3\pi}{\pi}\right] - \frac{5}{2}}$

$$= \frac{\sin^4 x \cdot e^{x^2}}{\left[\frac{x}{\pi} + 3\right] - \frac{5}{2}}$$

$$= \frac{\sin^4 x \cdot e^{x^2}}{\left[\frac{x}{\pi}\right] + 3 - \frac{5}{2}}$$

$$= \frac{\sin^4 x . e^{x^2}}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$$

Now, $f(-x) = \dfrac{\sin^4(-x) \cdot e^{(-x)^2}}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}}$

$$= \frac{\sin^4(x) \cdot e^{(x)^2}}{-\left[\frac{x}{\pi}\right] - 1 + \frac{1}{2}}$$

$$= \frac{\sin^4(x) \cdot e^{(x)^2}}{-\left[\frac{x}{\pi}\right] - \frac{1}{2}}$$

$$= \frac{\sin^4(x) . e^{(x)^2}}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$$

$$= -f(x)$$

Thus, $f(x)$ is an odd function.

54. Given $f(x) = \left(\dfrac{2x(\sin^{17} x + \tan^{99} x)}{2\left[\frac{x + 21\pi}{\pi}\right] - 41}\right)$

$$= \left(\frac{2x(\sin^{17} x + \tan^{99} x)}{2\left[\frac{x}{\pi} + 21\right] - 41}\right)$$

$$= \left(\frac{2x(\sin^{17} x + \tan^{99} x)}{2\left[\frac{x}{\pi}\right] + 42 - 41}\right)$$

$$= \left(\frac{2x(\sin^{17} x + \tan^{99} x)}{2\left[\frac{x}{\pi}\right] + 1}\right)$$

Nowt, $f(-x)$

$$= \left(\frac{2(-x)(\sin^{17}(-x) + \tan^{99}(-x))}{2\left[-\frac{x}{\pi}\right] + 1}\right)$$

$$= \left(\frac{2(x)(\sin^{17}(x) + \tan^{99}(x))}{-2\left[\frac{x}{\pi}\right] - 2 + 1}\right)$$

$$= \left(\frac{2(x)(\sin^{17}(x) + \tan^{99}(x))}{-\left(2\left[\frac{x}{\pi}\right] + 1\right)}\right)$$

$$= -\left(\frac{2(x)(\sin^{17}(x) + \tan^{99}(x))}{\left(2\left[\frac{x}{\pi}\right] + 1\right)}\right)$$

Thus, $f(x)$ is an odd function.

55. Given $f(x) = \sqrt{x + \sqrt{x}}$

$$f'(x) = \frac{1}{2\sqrt{x + \sqrt{x}}} \times \left(1 + \frac{1}{2\sqrt{x}}\right)$$

$$> 0$$

Thus $f(x)$ is strictly increasing function.

Thus f is one-one function.

Also, let $y = \sqrt{x + \sqrt{x}}$

$$\Rightarrow \quad y^2 = x + \sqrt{x}$$

$$\Rightarrow \quad (\sqrt{x})^2 + \sqrt{x} - y^2 = 0$$

$$\Rightarrow \quad (\sqrt{x}) = \frac{-1 \pm \sqrt{1 + 4y^2}}{2}$$

$\Rightarrow \quad (\sqrt{x}) = \dfrac{-1 + \sqrt{1 + 4y^2}}{2}$

$\Rightarrow \quad x = \dfrac{\left(-1 + \sqrt{1 + 4y^2}\right)^2}{4}$

Clearly, x is defined for $[0, \infty)$

So it is onto function

Thus, f is a bijective function.

So, its inverse is exist.

Hence, $f^{-1}(x) = \dfrac{\left(1 - \sqrt{1 + 4x^2}\right)^2}{4}$

$= \dfrac{1 - 2\sqrt{1 + 4x^2} + 1 + 4x^2}{4}$

$= \dfrac{2 - 2\sqrt{1 + 4x^2} + 4x^2}{4}$

$= \left(\dfrac{1 - \sqrt{1 + 4x^2}}{2} + x^2\right)$

56. Given $y = f(x) = 4x^3 - 3x$

$\Rightarrow \quad y = f(\cos\theta) = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \quad y = f(\cos\theta) = \cos 3\theta$

Clearly f is one-one and onto

So, its inverse exist.

Thus, $f^{-1}(x) = \cos\left(\dfrac{1}{3}\cos^{-1} x\right)$

57. Here $y = f(x) \Rightarrow x = f^{-1}(x)$

Now, $y = \sqrt[3]{a - x^3 + 3x^2 - 3bx + b^3} + b$

$\Rightarrow \quad (y - b)^3 = a - (x - b)^3$

$\Rightarrow \quad (x - b)^3 = a - (y - b)^3$

$\Rightarrow \quad (x - b) = \sqrt[3]{a - (y - b)^3}$

$\Rightarrow \quad x = \sqrt[3]{a - (y - b)^3} + b$

Thus, $f^{-1}(x) = \sqrt[3]{a - (x - b)^3} + b$

Therefore, $f(x) = f^{-1}(x)$

So, $f(f(x)) = x$

$\Rightarrow \quad b \in R$

58. Given $f(x) = f^{-1}(x)$

$\Rightarrow \quad f(f(x)) = x$

$\Rightarrow \quad \dfrac{a\left(\dfrac{ax + 1}{x + 3}\right) + 1}{\left(\dfrac{ax + 1}{x + 3}\right) + 3} = x$

$\Rightarrow \quad a^2x + a + x + 3 = x\,(ax + 1 + 3x + a)$

$\Rightarrow \quad (a^2 + 1)x + a + 3 = (a + 3)x^2 + 10x$

$\Rightarrow \quad a + 3 = 0$ and $a^2 + 1 = 10$

$\Rightarrow \quad a = -3$ and $a = \pm 3$

$a = -3$

59. Given $f(x + y) + f(x - y) = 2f(x)\cdot f(y)$ ...(i)

Putting $y = x$, we get,

$f(2x) + f(0) = 2(f(x))^2$ ...(ii)

Putting $y = -x$ in (i), we get,

$f(0) + f(2x) = 2\,f(x)\,f(-x)$ ...(iii)

Subtracting (iii) from (ii), we get,

$2(f(x))^2 = 2f(x)\cdot f(-x)$

$\Rightarrow \quad f(x)(f(x) - f(-x)) = 0$

$\Rightarrow \quad f(x) = 0$ or $f(-x) = f(x)$

Thus, in both the cases, function is an even function.

60. Given $2 + f(x)\cdot f(y) = f(x) + f(y) + f(xy)$ ...(i)

Put $y = \dfrac{1}{x}$, we get,

$2 + f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right) + f(1)$ ...(ii)

Again put $x = 1$ in (ii), we get,

$2 + f(1)\cdot f(1) = f(1) + f(1) + f(1)$

$\Rightarrow \quad (f(1))^2 - 3f(1) + 2 = 0$

$\Rightarrow \quad (f(1) - 1)(f(1) - 2) = 0$

$\Rightarrow \quad f(1) = 1$ or 2

But $f(1) \neq 1$, thus, $f(1) = 2$

Thus, $f(x)\cdot f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right)$

$\Rightarrow \quad f(x) = x^n + 1$

Also, $f(2) = 5 \Rightarrow 2^n + 1 = 5$

$\Rightarrow \quad n = 2$

Therefore, $f(f(2)) = f(5) = 5^2 + 1 = 26$

61. Since $f(x)$ is symmetrical about the lines

$x = a$ and $x = b$

Thus, $f(a - x) = f(a + x)$

and $f(b - x) = f(b + x)$

Now, $f(x) = f(a + (x - a))$

$= f(a - (x - a))$

$= f(2a - x)$

$= f(b + (2a - x - b))$

$= f(b - (2a - x - b))$

$= f(2b - 2a + x)$

$= f(x + (2b - 2a))$

Thus, $f(x)$ is a periodic with period $(2b - 2a)$

62. The function f is defined for

$\log_{1/3} (\log_7 (\sin x + a)) > 0$

$\Rightarrow \quad 0 < (\log_7(\sin x + a)) < 1$

$\Rightarrow \quad 1 < (\sin x + a) < 7$

$\Rightarrow \quad 1 - \sin x < a < 7 - \sin x$

$\Rightarrow \quad 2 < a < 6$

63. Given $\quad f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{x^{1/4}}\right) - 1\right)$

It is defined for $\left(-\log_{1/2}\left(1 + \frac{1}{x^{1/4}}\right) - 1\right) > 0$

$\Rightarrow \quad \left(\log_{1/2}\left(1 + \frac{1}{x^{1/4}}\right) + 1\right) < 0$

$\Rightarrow \quad \left(\log_{1/2}\left(1 + \frac{1}{x^{1/4}}\right)\right) < -1$

$\Rightarrow \quad \left(1 + \frac{1}{x^{1/4}}\right) > \left(\frac{1}{2}\right)^{-1}$

$\Rightarrow \quad \left(1 + \frac{1}{x^{1/4}}\right) > 2$

$\Rightarrow \quad \frac{1}{x^{1/4}} > 1$

$\Rightarrow \quad 0 < x < 1$

Thus, $\quad D_f = (0, 1)$

64. The function f is defined for

$\log_3(\log_4(\tan^{-1} x)^2) > 0$

$\Rightarrow \quad (\log_4(\tan^{-1} x)^2) > 1$

$\Rightarrow \quad (\tan^{-1} x)^2 > 4$

$\Rightarrow \quad \left((\tan^{-1} x) + 2\right) \left((\tan^{-1} x) - 2\right) > 0$

$\Rightarrow \quad x < -\tan 2 \text{ and } x > \tan 2 \quad$ ...(i)

Also, $\quad \log_4(\tan^{-1} x)^2 > 0$

$\Rightarrow \quad (\tan^{-1} x)^2 > 1$

$\Rightarrow \quad ((\tan^{-1} x) - 1)((\tan^{-1} x) - 1) > 0$

$\Rightarrow \quad x < -\tan 1 \text{ and } x > \tan 1 \quad$ ...(ii)

Again, $\quad (\tan^{-1} x)^2 > 0$

$\Rightarrow \quad (\tan^{-1} x) > 0$

$\Rightarrow \quad x > 0 \quad$ ...(iii)

From (i), (ii) and (iii), we get,

$x > \tan 2$

Thus, $\quad D_f = (\tan 2, \infty)$

65. Here, $\sin^2 x + \sin x + 1 > 0$, which is true for all x in R.

Also, $\quad \sin^{-1}\left(\frac{1}{|x^2 - 1|}\right)$ is defined for

$\Rightarrow \quad -1 \le \frac{1}{|x^2 - 1|} \le 1$

$\Rightarrow \quad 0 < \frac{1}{|x^2 - 1|} \le 1$

$\Rightarrow \quad |x^2 - 1| \ge 1 \text{ and } x^2 - 1 \le -1$

$\Rightarrow \quad (x^2 - 1) \ge 1 \text{ and } x^2 \le 0$

$\Rightarrow \quad x^2 \ge 2 \text{ and } x = 0$

$\Rightarrow \quad |x| \ge \sqrt{2} \text{ and } x = 0$

$\Rightarrow \quad x \in \left(-\infty, -\sqrt{2}\right] \cup \left[\sqrt{2}, \infty\right) \cup \{0\}$

Thus, $\quad D_f = \left(-\infty, -\sqrt{2}\right] \cup \left[\sqrt{2}, \infty\right) \cup \{0\}$

66. The function f is defined for

$4x - |x^2 - 10x + 9| > 0$

$\Rightarrow \quad |x^2 - 10x + 9| < 4x$

$\Rightarrow \quad -4x < (x^2 - 10x + 9) < 4x$

$\Rightarrow \quad (x^2 - 10x + 9) < 4x$

and $\quad -4x < (x^2 - 10x + 9)$

$\Rightarrow \quad x^2 - 14x + 9 < 0 \text{ and } x^2 - 6x + 9 > 0$

$\Rightarrow \quad x - (7 + \sqrt{40})\, x - (7 - \sqrt{40}) < 0$

and $\quad (x - 3)^2 > 0$

$\Rightarrow \quad x \in (7 - \sqrt{40}, 7 + \sqrt{40}) - \{3\}$

Thus, $\quad D_f = (7 - \sqrt{40}, 7 + \sqrt{40}) - \{3\}$

67. The function f is defined for

$\sin^{-1} |\sin x| \ge \cos^{-1} (\cos x)$

Now, $\sin^{-1}|\sin x| = \begin{cases} x & : 0 \le x \le \frac{\pi}{2} \\ \pi - x & : \frac{\pi}{2} \le x \le \pi \\ x - \pi & : \pi \le x \le \frac{3\pi}{2} \\ 2\pi - x & : \frac{3\pi}{2} \le x \le 2\pi \end{cases}$

Also, $\cos^{-1}(\cos x) = \begin{cases} x & : 0 \le x \le \pi \\ 2\pi - x & : \pi \le x \le 2\pi \end{cases}$

From the above relation, it is clear that

$x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

Thus, $D_f = \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

68. The function f is defined for

$\Rightarrow \quad \log_{0.5}\left(\frac{3x - x^2}{x - 1}\right) \ge 0$ and $\left(\frac{3x - x^2}{x - 1}\right) > 0$

$\Rightarrow \quad \left(\frac{3x - x^2}{x - 1}\right) \le 1$ and $\left(\frac{x^2 - 3x}{x - 1}\right) < 0$

$\Rightarrow \quad \frac{x^2 - 2x - 1}{x - 1} \ge 0$ and $\frac{x(x - 3)}{(x - 1)} < 0$

$\Rightarrow \quad x \in \lfloor 1 - \sqrt{2}, 1) \cup \lfloor 1 + \sqrt{2}, \infty)$

and $\quad x \in (-\infty, 0) \cup (1, 3)$

Thus, $D_f = \lfloor 1 - \sqrt{2}, 0) \cup \lfloor 1 + \sqrt{2}, 3)$

69. The function f is defined for

$\log_5(\log_3(\log_2(2x^3 + 5x^2 - 14x))) > 0$

$\Rightarrow \quad (\log_3(\log_2(2x^3 + 5x^2 - 14x))) > 1$

$\Rightarrow \quad (\log_2(2x^3 + 5x^2 - 14x)) > 3$

$\Rightarrow \quad (2x^3 + 5x^2 - 14x) > 8$

$\Rightarrow \quad (2x^3 + 5x^2 - 14x - 8) > 0$

$\Rightarrow \quad (x - 2)(x + 4)(2x + 1) > 0$

$\Rightarrow \quad x \in \left(-4, -\frac{1}{2}\right) \cup (2, \infty)$

Thus, $D_f = \left(-4, -\frac{1}{2}\right) \cup (2, \infty)$

70. The function f is defined for

$\Rightarrow \quad 3^{x-1} + 5^{x-1} + 7^{x-1} - 83 \ge 0$

$\Rightarrow \quad 3^{x-1} + 5^{x-1} + 7^{x-1} \ge 83$

$\Rightarrow \quad (3^{x-1} + 5^{x-1} + 7^{x-1}) \ge (3^2 + 5^2 + 7^2)$

$\Rightarrow \quad x - 1 \ge 2$

$\Rightarrow \quad x \ge 3$

Thus, $D_f = [3, \infty)$

71. The function f is defined for

$3 - 2^x - 2^{1-x} \ge 0$

$\Rightarrow \quad 3.2^x - (2^x)^2 - 2 \ge 0$

$\Rightarrow \quad (2^x)^2 - 3.2^x + 2 \le 0$

$\Rightarrow \quad (2^x - 1)(2^x - 2) \le 0$

$\Rightarrow \quad 1 \le 2^x \le 2$

$\Rightarrow \quad 2^0 \le 2^x \le 2^1$

$\Rightarrow \quad 0 \le x \le 1$

Thus, $D_f = [0, 1]$

72. The function f is defined for

$(1 - \log_7(x^2 - 5x + 13)) > 0$

$\Rightarrow \quad \log_7(x^2 - 5x + 13) < 1$

$\Rightarrow \quad (x^2 - 5x + 13) < 7$

$\Rightarrow \quad (x^2 - 5x + 6) < 0$

$\Rightarrow \quad (x - 2)(x - 3) < 0$

$\Rightarrow \quad 2 < x < 3$

Thus, $D_f = (2, 3)$

73. The function f is defined for

$24 - [x^2] \ge 0$ and $|x| - 4 \ge 0$

$\Rightarrow \quad [x^2] \le 24$ and $|x| \ge 4$

$\Rightarrow \quad 24 \le x^2 < 25$ and $|x| \ge 4$

$\Rightarrow \quad \sqrt{24} \le \sqrt{x^2} < \sqrt{25}$ and $|x| \ge 4$

$\Rightarrow \quad 2\sqrt{6} \le |x| < 5$ and $x \in (-\infty, -4] \cup [4, \infty)$

$\Rightarrow \quad x \in (-5, -2\sqrt{6}\rfloor \cup \lfloor 2\sqrt{6}, 5)$

and $\quad x \in (-\infty, -4] \cup [4, \infty)$

Thus, $D_f = (-5, -4] \cup [4, 5)$

74. Clearly, $D_f = \{-1, 1\}$

Thus, $R_f = \left\{\frac{\pi}{2}\right\}$

75. Given $y = f(x) = \cos^{-1}\left(\frac{x^2}{1 + x^2}\right)$

Clearly, range of the function $\left(\frac{x^2}{x^2 + 1}\right)$ is $[0, 1)$

$\Rightarrow \quad y \in \left(0, \frac{\pi}{2}\right]$ ...(i)

Also, $0 \le y \le \pi$ ...(ii)

From (i) and (ii), we get,

$$y \in \left(0, \frac{\pi}{2}\right]$$

Thus, $R_f = \left(0, \frac{\pi}{2}\right]$

76. Given $y = f(x) = \sin^{-1}\left(\frac{x^2+1}{x^2+2}\right)$

Clearly, the range of the function $\left(\frac{x^2+1}{x^2+2}\right)$ is $\left[\frac{1}{2}, 1\right)$

Thus, $R_f = \left[\sin^{-1}\left(\frac{1}{2}\right), \sin^{-1}(1)\right)$

$$= \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$$

77. Clearly, $D_f = \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

Thus, $R_f = \left[\tan\left(-\frac{\pi}{3}\right), \tan\left(\frac{\pi}{3}\right)\right]$

$$= \lfloor -\sqrt{3}, \sqrt{3} \rfloor$$

78. Clearly, $D_f = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Thus, $R_f = \left[\sin\left(-\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right]$

$$= \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$

79. The function f is defined for

$(4x - x^2) > 0$

$\Rightarrow (x^2 - 4x) < 0$

$\Rightarrow x(x - 4) < 0$

$\Rightarrow 0 < x < 4$

Thus, $D_f = (0, 4)$

Therefore, $R_f = (\log_2(0), \log_2 4) = (-\infty, 2)$.

80. Given $y = f(x) = \log\left(\frac{x^2+e}{x^2+1}\right)$

$$= \log\left(\frac{x^2+1+e-1}{x^2+1}\right)$$

$$= \log\left(1 + \frac{e-1}{x^2+1}\right)$$

Thus, $R_f = (0, 1]$

81. Since the range of $\left(\frac{\sqrt{4-x^2}}{1-\text{x}}\right)$ is R

Thus, $R_f = [-1, 1]$

82. We have, $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

Also, the function is defined for

$\Rightarrow x^2 + x + 1 \le 1$

Therefore, $\frac{3}{4} \le x^2 + x + 1 \le 1$

$\Rightarrow \sqrt{\frac{3}{4}} \le \sqrt{x^2+x+1} \le \sqrt{1}$

$\Rightarrow \sqrt{\frac{3}{2}} \le \sqrt{x^2+x+1} \le 1$

$\Rightarrow \sin^{-1}\left(\sqrt{\frac{3}{2}}\right) \le \sin^{-1}\left(\sqrt{x^2+x+1}\right) \le \sin^{-1}(1)$

$\Rightarrow \frac{\pi}{3} \le \sin^{-1}\left(\sqrt{x^2+x+1}\right) \le \frac{\pi}{2}$

$\Rightarrow \log\left(\frac{\pi}{3}\right) \le \ln\left(\sin^{-1}\left(\sqrt{x^2+x+1}\right)\right) \le \log\left(\frac{\pi}{2}\right)$

Thus, $R_f = \left[\log\left(\frac{\pi}{3}\right), \log\left(\frac{\pi}{3}\right)\right]$

83. Clearly, $5x^2 - 8x + 4 > 0, \forall x \in R$

Also, $5x^2 - 8x + 4 = 5\left(x - \frac{4}{5}\right)^2 + \frac{4}{5}$

Thus, the range of $\log(5x^2 - 8x + 4)$ is

$$= \left(\log\left(\frac{4}{5}\right), \infty\right)$$

Hence, $R_f = [-1, 1]$

84. We have $(x^4 - 2x^2 + 3) = (x^2 - 1)^2 + 2 \ge 2$

Let $g(x) = (x^4 - 2x^3 + 3)$

and $h(x) = \log(x^4 - 2x^2 + 3)$

Now, $R_g = [2, \infty)$

and $R_h = (\log_{0.5}(\infty), \log_{0.5}(2)] = (-\infty, -1]$

Also, Range of $\cot^{-1} x$ is $(0, \pi)$

and $\cot^{-1} x$ is a decreasing function.

Thus, $R_f = [\cot^{-1}(-1), \pi) = \left[3\frac{\pi}{4}, \pi\right)$

85. Given

$$y = f(x) = \sqrt{1 - \cos x\sqrt{1 - \cos x\sqrt{1 - \cos x\sqrt{\ldots\infty}}}}$$

$$= (1 - \cos x)^{1/2} \cdot (1 - \cos x)^{1/4} \cdot (1 - \cos x)^{1/8} \ldots$$

$= (1 - \cos x)^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\ldots \text{ to } \infty}$

$= (1 - \cos x)^{\frac{\frac{1}{2}}{1-\frac{1}{2}}}$

$= (1 - \cos x)$

$= 2\sin^2\left(\frac{x}{2}\right)$

Thus, $R_f = [0, 2]$

86. Given $f(x) = ax + b$

Now, $f(f(f(x)))$

$= f(f(ax + b))$

$= f(a(ax + b)) + b)$

$= f((a^2x + ab + b))$

$= a(a^2x + ab + b) + b$

$= (a^3x + a^2b + ab + b)$

Thus, $(a^3x + a^2b + ab + b) = 27x + 26$

$\Rightarrow$ $a^3 = 27,\ a^2b + ab + b = 26$

$\Rightarrow$ $a = 3,\ 9\ b + 3\ b + b = 26$

$\Rightarrow$ $a = 3,\ 13\ b = 26$

$\Rightarrow$ $a = 3,\ b = 2$

Therefore, $a^2 + b^2 + 2$

$= 9 + 4 + 2$

$= 15$

87. Given $f(x) = \frac{x}{\sqrt{1 + x^2}}$

Now, $f(f(f(x))) = f\left(f\left(\frac{x}{\sqrt{1 + x^2}}\right)\right)$

$= f(f(a)),\ a = \frac{2}{\sqrt{1 + x^2}}$

$= f\left(\frac{a}{\sqrt{1 + a^2}}\right)$

$= f\left(\frac{\frac{x}{\sqrt{1 + x^2}}}{\sqrt{1 + \frac{x^2}{1 + x^2}}}\right)$

$= f\left(\frac{x}{\sqrt{1 + 2x^2}}\right)$

$= f(b),\ b = \frac{x}{\sqrt{1 + 2x^2}}$

$= \frac{b}{\sqrt{1 + b^2}}$

$= \frac{\frac{x}{\sqrt{1 + 2x^2}}}{\sqrt{1 + \frac{x^2}{1 + 2x^2}}}$

$= \frac{x}{\sqrt{1 + 3x^2}}$

Hence, the result.

88. The function f is defined for

$\log_{10}a - \log_{10}(4 - a) - \log_{10}3 \geq 0$

$\Rightarrow$ $\log_{10}a - \log_{10}\{3(4 - a)\} \geq 0$

$\Rightarrow$ $\log_{10}\left(\frac{a}{3\ (4 - a)}\right) \geq 0$

$\Rightarrow$ $\left(\frac{a}{3(4 - a)}\right) \geq 1$

$\Rightarrow$ $\left(\frac{a}{12 - 3a} - 1\right) \geq 0$

$\Rightarrow$ $\left(\frac{a - 12 + 3a}{12 - 3a}\right) \geq 0$

$\Rightarrow$ $\left(\frac{4a - 12}{3a - 12}\right) \leq 0$

$\Rightarrow$ $\left(\frac{a - 3}{a - 4}\right) \leq 0$

$\Rightarrow$ $3 \leq a < 4$

$\Rightarrow$ $3 \leq \log_{10}x < 4$

$\Rightarrow$ $10^3 \leq x < 10^4$

Thus, $D_f = [10^3,\ 10^4)$

89. The function f is defined for

$\log_{1/2}(\log_2[x^2 + 4x + 5]) \geq 0$

$\Rightarrow$ $(\log_2[x^2 + 4x + 5]) \leq 1$

$\Rightarrow$ $([x^2 + 4x + 5]) \leq 2$

$\Rightarrow$ $2 \leq (x^2 + 4x + 5) < 3$

when $2 \leq (x^2 + 4x + 5)$

$\Rightarrow$ $(x^2 + 4x + 5) \geq 0$

$\Rightarrow$ $(x + 1)(x + 3) \geq 0$

$\Rightarrow$ $\in (-\infty, -3] \cup [-1, \infty)$

when $(x^2 + 4x + 5) < 3$

$\Rightarrow$ $(x^2 + 4x + 2) < 0$

$\Rightarrow \quad (x + 2 + \sqrt{2})(x + 2 - \sqrt{2}) < 0$

$\Rightarrow \quad -(2 + \sqrt{2}) < x < (-2 + \sqrt{2})$

Thus, D_f

$= (-(2 + \sqrt{2}), -3\rfloor \cup \lfloor -1, -2 + \sqrt{2})$

90. Given $f(g(x)) = 1 + x^2 - 2x^3 + x^4$

$= 1 + (x^4 - 2x^3 + x^2)$

$= 1 + x^2(x^2 - 2x + 1)$

$= 1 + x^2(x - 1)^2$

$= 1 + (x(x - 1))^2$

$= g(x) = \pm(x(1 - x))$

91. Given $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right)$

$+ \cos x \cdot \cos\left(x + \frac{\pi}{3}\right)$

$= \frac{1}{2}\left[2\sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + 2\cos x . \cos\left(x + \frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left[1 - \cos 2x + 1 - \cos 2\left(x + \frac{\pi}{3}\right)\right.$

$\left. + \cos\left(2x + \frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left(\frac{5}{2} - \cos 2x - \cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(2x + \frac{\pi}{3}\right)\right)$

$= \frac{1}{2}\left(\frac{5}{2} - \cos 2x + \cos\left(2x + \frac{\pi}{3}\right) - \cos\left(2x + \frac{2\pi}{3}\right)\right)$

$= \frac{1}{2}\left(\frac{5}{2} - \cos 2x + 2\sin\left(2x + \frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)\right)$

$= \frac{1}{2}\left(\frac{5}{2} - \cos 2x + 2\sin\left(2x + \frac{\pi}{2}\right) \times \frac{1}{2}\right)$

$= \frac{1}{2}\left(\frac{5}{2} - \cos 2x + \sin\left(2x + \frac{\pi}{2}\right)\right)$

$= \frac{1}{2}\left(\frac{5}{2} - \cos 2x + \cos 2x\right)$

$= \frac{5}{4}$

Now, $g(f(x)) = g\left(\frac{5}{4}\right) = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}$

92. Let $y = \dfrac{ax^2 + 2x + 1}{2x^2 - 2x + 1}$

$\Rightarrow \quad 2x^2 y - 2xy + y - ax^2 - 2x - 1 = 0$

$\Rightarrow \quad (2x^2 - a)y - (2y + 1)x + (y - 1) = 0$

since $x \in R$, so $D \geq 0$

$\Rightarrow \quad 4(y + 1)^2 - 4(2y - a)(y - 1) \geq 0$

$\Rightarrow \quad y^2 - (a + 4)y + (a - 1) \leq 0$

Since the function is onto,

so, $R_f = C_f = [-1, 2]$

when $y = 2,\ 4 - 2(a + 4) + (a - 1) \leq 0$

$\Rightarrow \quad 4 - 2a - 8 + (a - 1) \leq 0$

$\Rightarrow \quad -a - 5 \leq 0$

$\Rightarrow \quad a \geq 5$

when $y = -1,\ 1 + (a + 4) + (a - 1) \leq 0$

$\Rightarrow \quad 2a + 4 \leq 0$

$\Rightarrow \quad a \leq -2$

Thus, $\in [-5, -2]$

93. Given $|||x^2 - x + 4| - 2| - 3| = x^2 + x - 12$

$\Rightarrow \quad |||x^2 - x + 2| - 3|| = x^2 + x - 12$

$\Rightarrow \quad |x^2 - x - 1| = x^2 + x - 12$

$\Rightarrow \quad x^2 - x - 1 = x^2 + x - 12$

$\Rightarrow \quad 2x = 11$

$\Rightarrow \quad x = 11/2$

Hence, the value of x is 11/2.

94. Given $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$

Since the function is one-one, so f is monotonic.

Now, $f'(x) = 3x^2 + 2(a + 2)x + 3a > 0$

$\Rightarrow \quad 4(a + 2)^2 - 36 < 0$

$\Rightarrow \quad (a + 2)^2 - 9 < 0$

$\Rightarrow \quad (a + 2)^2 - 3^2 < 0$

$\Rightarrow \quad (a + 2 + 3)(a + 2 - 3) < 0$

$\Rightarrow \quad (a + 5)(a - 1) < 0$

$\Rightarrow \quad -5 < a < 1$

95. Given $f(x) = g(x) + 1$

Also, $f(x) + f(1 - x) = 2$

$\Rightarrow \quad g(x) + 1 + g(1 - x) + 1 = 2$

$\Rightarrow \quad g(x) + g(1 - x) = 0$ ...(i)

Replacing x by $(1 - x)$ in (i), we get,

$g(1 - x) + g(-x) = 0$ ...(ii)

From (i) and (ii), we get,

$g(x) - g(-x) = 0$

$\Rightarrow \quad g(x) = g(-x)$

Thus, $g(x)$ is symmetrical about the line $x = 0$.

96. Given $f(2 + x) = f(2 - x)$

and $f(7 - x) = f(7 + x)$

Now, $f(x) = f(2 + (x-2))$

$= f(2-(x-2))$

$= f(4-x)$

$= f(7+(4 - x - 7))$

$= f(7-(4 - x - 7))$

$= f(10 + x)$

Thus, $f(x)$ is periodic with period 10.

Therefore, the number roots of $f(x) = 0$ is 5.

97. We have $f(a + x) - b$

$= \{1+(b-f(x))^3\}^{1/3}$

$\Rightarrow (f(a + x)- b)^3 = 1 + (b - f(x))^3$

$\Rightarrow (f(a + x) - b)^3 + (f(x) - b)^3 = 1$ ...(i)

Replacing x by $x + a$, we get,

$(f(x + 2a) - b)^3 + (f(x + a)- b)^3 = 1$ ...(ii)

From (i) and (ii), we get,

$\Rightarrow (f(x + 2a) - b)^3 - (f(x) - b)^3 = 0$

$\Rightarrow (f(x + 2a)- b)^3 = (f(x) - b)^3$

$\Rightarrow (f(x + 2a)- b) = (f(x)- b)$

$\Rightarrow f(x) = f(x + 2a)$

Thus $f(x)$ is periodic with period $2a$.

98. We have $4^x - 2^{x+2} + 5 +||b-1| -3| = |\sin y|$

$\Rightarrow (4^x - 4.2^x + 4) + 1 + ||b-1|-3| = |\sin y|$

$\Rightarrow (2^x - 2)^2 + 1 + ||b-1|-3| = |\sin y|$

L.H.S ≥ 1 & R.H.S ≤ 1

Thus, $(2^x - 2) = 0$, $|b - 1| -3 = 0$

$\Rightarrow |b - 1| -3 = 0$

$\Rightarrow |b - 1| = 3$

$\Rightarrow (b - 1) = \pm 3$

$\Rightarrow b = -2, 4$

Hence, the values of b are $-2, 4$

Hints and Solutions

Level I

(Tougher Problems for JEE-Advanced)

1. We have

$\log_3(\log_{4/\pi}(\tan^{-1} x)^{-1}) > 0$

$\Rightarrow (\log_{4/\pi}(\tan^{-1} x)^{-1}) > 0$

$\Rightarrow (\tan^{-1} x)^{-1} > \left(\frac{4}{\pi}\right)$

$\Rightarrow (\tan^{-1} x) < \left(\frac{\pi}{4}\right)$

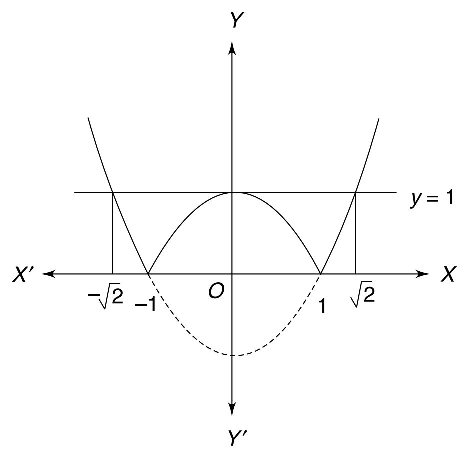
$x < 1$

Also, $f(x)$ is defined only when $x > 0$

Thus, $0 < x < 1$

$x \in (0, 1)$

2. f is defined for $-1 \leq \frac{1}{|x^2 - 1|} \leq 1$ and $\sin^2 x + \sin x + 1 > 0$ is true for every x in R.



Thus, $0, \frac{1}{|x^2 - 1|} \leq 1$

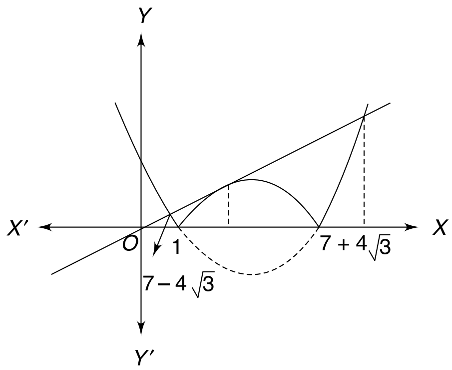
$\Rightarrow |x^2 - 1| \geq 1$

$\Rightarrow x \in \left(-\infty, -\sqrt{2}\right] \cup \left[\sqrt{2}, \infty\right) \cup \{0\}$

3. f is defined for

$4x - |x^2 - 10x + 9| > 0$

$\Rightarrow |x^2 - 10x + 9| < 4x$ ...(i)



We draw $y = x^2 - 10x + 9$ and $y = 4x$

and solving $x^2 - 10x + 9 = \pm 4x$

we get, $x = 7 \pm 4\sqrt{3}, 3$

since $|x^2 - 10x + 9| < 4x$

so $x \in (7 - 4\sqrt{3}, 7 + 4\sqrt{3}) - \{3\}$

Thus, $D_f = (7 - 4\sqrt{3}, 7 + 4\sqrt{3}) - \{3\}$

4. f is defined for

$$\sin^{-1}(|\sin x|) - \cos^{-1}(|\cos x|) \ge 0$$

$$\Rightarrow \sin^{-1}(|\sin x|) \ge \cos^{-1}(|\cos x|)$$

It is possible only when

$$x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$$

Thus, $D_f = \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

5. f is defined for

$$2\{x\}^2 - 3\{x\} + 1 \ge 0$$

$$(2\{x\} - 1)(\{x\} - 1) \ge 0$$

$$(2\{x\} - 1) \le 0, (\{x\} - 1) \ge 0$$

$$\{x\} \le \frac{1}{2}, \{x\} \ge 1$$

$\{x\} \le \frac{1}{2}$, since $\{x\} \ge 1$ is not possible

Thus, $x \in \left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right] \cup \{1\}$

$$D_f = \left[-1, \frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right] \cup \{1\}$$

6. f is defined for

$$\log_{0.3}\left(\frac{3x - x^2}{x - 1}\right) \ge 0 \text{ and } \left(\frac{3x - x^2}{x - 1}\right) > 0$$

$$\left(\frac{3x - x^2}{x - 1}\right) \le 1 \text{ and } \left(\frac{3x - x^2}{x - 1}\right) > 0$$

Now, $\left(\frac{3x - x^2}{x - 1}\right) \le 1$

$$\Rightarrow \left(\frac{3x - x^2}{x - 1} - 1\right) \le 0$$

$$\Rightarrow \left(\frac{3x - x^2 - x + 1}{x - 1}\right) \le 0$$

$$\Rightarrow \left(\frac{2x - x^2 + 1}{x - 1}\right) \le 0$$

$$\Rightarrow \left(\frac{x^2 - 2x - 1}{x - 1}\right) \le 0$$

$$\Rightarrow x \in \lfloor 1 - \sqrt{2}, 1) \cup (1 + \sqrt{2}, 1\rfloor \quad \text{...(i)}$$

Again $\left(\frac{3x - x^2}{x - 1}\right) > 0$

$$\Rightarrow \left(\frac{x(x - 3)}{x - 1}\right) < 0$$

$$x \in (-\infty, 0) \cup (1, 3) \quad \text{...(ii)}$$

From (i) and (ii), we get,

$$x \in \lfloor 1 - \sqrt{2}, 0) \cup (\lfloor 1 + \sqrt{2}, 3)$$

$$D_f = \lfloor 1 - \sqrt{2}, 0) \cup \lfloor 1 + \sqrt{2}, 3)$$

7. f is defined for

$$(1 - \log_{10}(x^2 - 5x + 16)) > 0$$

$$\Rightarrow \log_{10}(x^2 - 5x + 16) < 1$$

$$\Rightarrow (x^2 - 5x + 16) < 10$$

$$\Rightarrow (x^2 - 5x + 6) < 0$$

$$\Rightarrow (x - 2)(x - 3) < 0$$

$$\Rightarrow 2 < x < 3$$

Thus, $D_f = (2, 3)$

8. f is defined for

$$-\log_{0.3}(x - 1) \ge 0, (x - 4)(x + 2) < 0$$

$$(x - 1) > 0 \text{ and } x \ne -2, 4$$

$$\Rightarrow x \ge 2, -2 < x < 4, x > 1, x \ne -2, 4$$

$$\Rightarrow x \in [2, 4)$$

$$\Rightarrow D_f = [2, 4)$$

9. Given $f(x) = \sqrt{\frac{x - 1}{x - 2\{x\}}}$

f is defined for $\frac{x - 1}{x - 2\{x\}} \ge 0$

Case-I: When $(x - 1) \ge 0$ and $(x - \{2x\}) > 0$

$x \ge 1$ and $x > 2\{x\}$

$x \ge 2$

Case-II: $(x - 1) \le 0$ and $(x - 2\{x\}) < 0$

$x \le 1$ and $x < 2\{x\}$

$x \le 1$ and $x \ne 0$

$x \in (-\infty, 0) \cup (0, 1]$

Hence, the domain of the given function

$= (-\infty, 0) \cup (0, 1] \cup [2, \infty)$

10. Given $f(x) = \log_{[x^2]}(4 - |x|)$

f is defined for

$(4 - |x|) > 0$, $[x^2] > 0$ and $[x^2] \ne 1$

$-4 < x < 4$, $[x^2] > 0$ and $[x^2] \ne 1$

$-4 < x < 4$, $[x^2] = 2, 3, 4, \ldots$

$-4 < x < 4$, $[x^2] \ge 2$

$-4 < x < 4$, $x^2 \ge 2$

$-4 < x < 4$, $x \ge \sqrt{2}$ and $x \le -\sqrt{2}$

Thus, $D_f = (-4, -\sqrt{2}\rfloor \cup \lfloor\sqrt{2}, 4)$

11. Given $f(x) = \cos^{-1}\left(\frac{[x^2 - 3]}{5}\right) + \log_2(|x| - 1)$

f is defined for

$-1 \le \left(\frac{[x^2 - 3]}{5}\right) \le 1$, $(|x| - 1) > 0$

$\Rightarrow$ $-5 \le [x^2 - 3] \le 5$, $|x| > 1$

$\Rightarrow$ $-5 \le [x^2] - 3 \le 5$, $|x| > 1$

$\Rightarrow$ $-2 \le [x^2] \le 8$, $|x| > 1$

$\Rightarrow$ $-2 \le (x^2) < 9$, $|x| > 1$

$\Rightarrow$ $-3 < x < 3$, $x < -1$, $x > 1$

Thus, $D_f = (-3, -1) \cup (1, 3)$

12. Given $f(x) = \sqrt{[x] - 1 + x^2}$ f is defined for

$[x] - 1 + x^2 \ge 0$

$\Rightarrow$ $[x] \ge 1 - x^2$

$\Rightarrow$ $1 - x^2 \le [x]$

It is possible only when

$x \ge 1$ and $x \le -\sqrt{3}$

Thus, $D_f = (-\infty, -\sqrt{3}\rfloor \cup [1, \infty)$

13. Given $f(x) = \sqrt{(\sin x) - (\cos x)}$

f is defined for $(\sin x) - (\cos x) \ge 0$

$(\sin x) \ge (\cos x)$

It is true for $x \in \left(0, \frac{3\pi}{2}\right)$

Thus, $D_f = \left(2n\pi, 2n\pi + \frac{3\pi}{2}\right), n \in I$

14. Given $f(x) = \frac{1}{[x]} + \log_{(2\{x\}-5)}(x^2 - 3x + 10) + \frac{1}{\sqrt{1 - |x|}}$

f is defined for

$R - [0, 1)$, $|x| < 1$, $2\{x\} - 5 > 0, \ne 1$

$R - [0, 1)$, $-1 < x < 1$, $\{x\} > \frac{5}{2}, \ne 3$

Thus, $D_f = \varphi$.

15. f is defined for

(i) $\sin(\cos x) \ge 0$

(ii) $-2\cos^2 x + 3\cos x - 1 > 0$

(iii) $-1 \le \frac{2\sin x + 1}{2\sqrt{2\sin x}} \le 1$

Now, from (i), we get,

$0 \le \cos x \le \pi$

as $\pi > 1$

$0 \le \cos x \le 1$

$2n\pi \le x \le \left(2n\pi + \frac{\pi}{2}\right), n \in I$ ...(1)

Also, from (ii), we get,

$-2\cos^2 x + 3\cos x - 1 > 0$

$2\cos^2 x - 3\cos x + 1 < 0$

$\left(\cos x - \frac{1}{2}\right)(\cos x - 1) < 0$

$\frac{1}{2} < \cos x < 1$

$2n\pi - \frac{\pi}{3} < x < 2n\pi + \frac{\pi}{3}, n \in I$ ...(2)

Again from (iii), we get,

$2\sin x + 2\sqrt{2\sin x} + 1 \ge 0$

and $2\sin x - 2\sqrt{2\sin x} + 1 \le 0$

$(\sqrt{2\sin x} + 1)^2 \ge 0$ and $(\sqrt{2\sin x} - 1)^2 \le 0$

It is possible only when $\sin x = \frac{1}{2}$

$\cos x = \frac{\sqrt{3}}{2}$

$x = 2n\pi \pm \frac{\pi}{6}, n \in I$ ...(3)

From (1), (2) and (3), we get,

$x = 2n\pi + \frac{\pi}{6}, n \in I$

Thus, $D_f = 2n\pi + \frac{\pi}{6}, n \in I$

16. Given $f(x) = \sqrt{1 - \log_x \log_2(4^x - 12)}$

f is defined for

(i) $(4^x - 12) > 0$

(ii) $\log_2(4^x - 12) > 0$

(iii) $\log x(\log_2(4^x - 12)) \le 1$

From (i), we get,

$$4^x > 12$$

$$x > \log_4(12) \quad ...(1)$$

Also, from (ii), we get,

$$4^x - 12 > 1$$

$$4^x > 13$$

$$x > \log_4(13) \quad ...(2)$$

Again, from (iii), we get,

$$\log_2(4^x - 12) \le x$$

$$(4^x - 12) \le 2^x$$

$$(4^x - 2^x - 12) \le 0$$

$$((2^x)^2 - 2^x - 12) \le 0$$

$$(2^x - 4)(2^x - 3) \le 0$$

$$3 \le 2^x \le 4$$

$$\log_2(3) \le x \le \log_2(4) \quad ...(3)$$

From (1), (2) and (3), we get,

$$x \in [\log_4(13), 2]$$

Thus, $D_f = [\log_4(13), 2]$

17. Given $f(x) = \sqrt{\log_{1/2} \log_2[x^2 + 4x + 5]}$

f is defined for

(i) $\log_{1/2}(\log_2[x^2 + 4x + 5]) \ge 0$

(ii) $\log_2[x^2 + 4x + 5] > 0$

(iii) $[x^2 + 4x + 5] > 0$

Now from (i), we get,

$$(\log_2[x^2 + 4x + 5]) \le 1$$

$$[x^2 + 4x + 5] \le 2$$

$$x^2 + 4x + 5 < 3$$

$$x^2 + 4x + 2 < 0$$

$$x \in (-2 - \sqrt{2}, -2 + \sqrt{2}) \quad ...(1)$$

Also, from (ii), we get,

$$\log_2[x^2 + 4x + 5] > 0$$

$$[x^2 + 4x + 5] > 1$$

$$x^2 + 4x + 5 \ge 2$$

$$x^2 + 4x + 3 \ge 0$$

$$(x + 1)(x + 3) \ge 0$$

$$x \le -3, x \ge -1 \quad ...(2)$$

Again, from (iii), we get,

$$[x^2 + 4x + 5] > 0$$

$$x^2 + 4x + 5 \ge 1$$

$$x^2 + 4x + 4 \ge 0$$

$$(x + 2)^2 \ge 0$$

which is true for all x in R. ...(3)

From (1), (2) and (3), we get,

$$x \in \left(-2 - \sqrt{2}, -3\right] \cup \left[-1, -2 + \sqrt{2}\right)$$

Thus, $D_f = \left(-2 - \sqrt{2}, -3\right] \cup \left[-1, -2 + \sqrt{2}\right)$

18. Let $y = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$

$$\Rightarrow \quad y = \frac{(x - 1)(x - 2)}{(x + 3)(x - 2)} = \frac{(x - 1)}{(x + 3)}, x \ne 2$$

$$x = \frac{3y + 1}{1 - y}$$

Also, when $x = 2$, then $y = \dfrac{1}{5}$

Hence, the range is $R - \left\{1, \dfrac{1}{5}\right\}$.

19. Clearly, the domain of the given function is $[0, \infty)$

Thus, $R_f = \left[f(0), \lim_{x \to \infty} f(x)\right)$

$$= \left[2, \frac{\pi^2}{4}\right)$$

20. Let $y = \sqrt{a^2 \cos^2 x + b^2 \sin^2 x} + \sqrt{a^2 \sin^2 x + b^2 \cos^2 x}$

$$\Rightarrow \quad y = \sqrt{\lambda} + \sqrt{(a^2 + b^2) - \lambda}, \text{ where}$$

$$\lambda = a^2 \cos^2 x + b^2 \sin^2 x$$

$$\Rightarrow \quad y^2 = \lambda + (a^2 + b^2) - \lambda + 2\sqrt{\lambda(a^2 + b^2) - \lambda^2}$$

$$\Rightarrow \quad y^2 = (a^2 + b^2) + 2\sqrt{\lambda(a^2 + b^2) - \lambda^2}$$

Max value of

$$y^2 = 2(a^2 + b^2), \text{ when } \lambda = \left(\frac{a^2 + b^2}{2}\right)$$

Min value of

$$y^2 = \sqrt{a^2 + b^2 + 2ab}, \text{ when } \lambda = b^2$$

Thus, $R_f = \left[(a + b), 2\sqrt{a^2 + b^2}\right]$

21. As $0 < x < \dfrac{\pi}{2}$, so $0 < x < 1$

Now, range of

$$y = \ln\{(\cos x)^{\cos x} + 1\} \text{ in } \left(0, \frac{\pi}{2}\right)$$

$R_y = \ln(x^x + 1)$ in (0, 1)

Let $g(x) = x^x + 1$

$\Rightarrow \quad g'(x) = x^x(1 + \ln x)$

For max/min, $g'(x) = 0$ gives

$\Rightarrow \quad x^x(1 + \ln x) = 0$

$\Rightarrow \quad (1 + \ln x) = 0$

$\Rightarrow \quad \ln x = -1$

$\Rightarrow \quad x = e^{-1} = \frac{1}{e}$

Now, $g(x)$ is minimum at $x = \frac{1}{e}$

Minimum value of $g(x)$ is

$$= g\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} + 1$$

Also,
$$g(0) = \lim_{x\to 0}(x^x + 1)$$
$$= \lim_{x\to 0}\left(e^{x\log x} + 1\right)$$
$$= e^{\lim_{x\to 0}\left(\frac{\log x}{1/x}\right)} + 1$$
$$= e^{\lim_{x\to 0}\left(\frac{\frac{1}{x}}{\frac{-1}{x^2}}\right)} + 1$$
$$= e^{\lim_{x\to 0}(-x)} + 1$$
$$= 1 + 1$$
$$= 2.$$

Thus, $1 + \left(\frac{1}{e}\right)^{1/e} < x < 2$

$$\Rightarrow \quad \ln\left(1 + \left(\frac{1}{e}\right)^{1/e}\right) < \ln(x^x + 1) < \ln(2)$$

$$\Rightarrow \quad \ln\left(1 + \left(\frac{1}{e}\right)^{1/e}\right) < \ln\left((\cos x)^{\cos x} + 1\right) < \ln(2)$$

$$\Rightarrow \quad R_f = \left(\ln\left(1 + \left(\frac{1}{e}\right)^{1/e}\right), \ln(2)\right)$$

22. Given
$$f(x) = \cos^{-1}\left(\frac{\sqrt{2x^2 + 1}}{x^2 + 1}\right)$$
$$= \sin^{-1}\left(\sqrt{1 - \frac{2x^2 + 1}{(x^2 + 1)^2}}\right)$$
$$= \sin^{-1}\left(\sqrt{\frac{x^4}{(x^2 + 1)^2}}\right)$$
$$= \sin^{-1}\left(\frac{x^2}{(x^2 + 1)}\right)$$

As we know that

$$0 \le \frac{x^2}{(x^2 + 1)} < 1$$

$$\Rightarrow \quad \sin^{-1}(0) \le \sin^{-1}\left(\frac{x^2}{(x^2 + 1)}\right) < \sin^{-1}(1)$$

$$\Rightarrow \quad 0 \le \sin^{-1}\left(\frac{x^2}{(x^2 + 1)}\right) < \frac{\pi}{2}$$

Thus, the range of a function is

$$R_f = \left[0, \frac{\pi}{2}\right)$$

23. Given $f(x) = \log_2\left(2 - \log_{\sqrt{2}}\left(16\sin^2 x + 1\right)\right)$

f is defined for

(i) $2 - \log_{\sqrt{2}}\left(16\sin^2 x + 1\right) > 0$

(ii) $\left(16\sin^2 x + 1\right) > 0$

which is true for all x in R

Now, from (i), we get,

$(16\sin^2 x + 1) < 2$

$0 \le (16\sin^2 x) < 1$

$1 \le (16\sin^2 x + 1) < 2$

$\log_{\sqrt{2}}(1) \le \log_{\sqrt{2}}(16\sin^2 x + 1) < \log_{\sqrt{2}}(2)$

$0 \le \log_{\sqrt{2}}(16\sin^2 x + 1) < 2$

$-2 < -\log_{\sqrt{2}}(16\sin^2 x + 1) \le 0$

$2 - 2 < 2 - \log_{\sqrt{2}}(16\sin^2 x + 1) \le 2$

$0 < 2 - \log_{\sqrt{2}}(16\sin^2 x + 1) \le 2$

$\log_2(0) < \log_2\left(2 - \log_{\sqrt{2}}\left(16\sin^2 x + 1\right)\right) \le \log_2 2$

$-\infty < \log_2\left(2 - \log_{\sqrt{2}}\left(16\sin^2 x + 1\right)\right) \le 1$

Thus, $R_f = (-\infty, 1]$

24. Let
$$g(x) = (5x^2 - 8x + 4)$$
$$= 5\left(x^2 - \frac{8}{5}x + \frac{4}{5}\right)$$
$$= \left(5\left(x - \frac{4}{5}\right)^2 + \frac{4}{5}\right)$$

Thus, $R_g = \left[\frac{4}{5}, \infty\right)$

$$\frac{4}{5} \le g(x) < \infty$$

$$\log_{5/4}\left(\frac{4}{5}\right) \le \log_{5/4}(g(x)) < \log_{5/4}(\infty)$$

$$-1 \le \log_{5/4}(g(x)) < \infty$$

$$\tan^{-1}(-1) \le \tan^{-1}\left(\log_{5/4}(g(x))\right) < \tan^{-1}(\infty)$$

$$-\frac{\pi}{4} \le \tan^{-1}\left(\log_{5/4}((g(x))\right) < \frac{\pi}{2}$$

$$-\frac{\pi}{4} \le f(x) < \frac{\pi}{2}$$

Thus, $R_f = \left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$

25. Given $f(x) = 6^x + 3^{-x} + 6^{-x} + 3^x + 2.$

$$= (6^x + 6^{-x}) + (3^x + 3^{-x}) + 2$$

$$\geq 2 + 2 + 2 = 6$$

Thus, $R_f = [6, \infty)$

26. Given $f(x) = \cos^{-1}\left(\dfrac{1 + x^2}{2x}\right) + \sqrt{2 - x^2}$

f is defined for

(i) $-1 \leq \left(\dfrac{1 + x^2}{2x}\right) \leq 1$

(ii) $2 - x^2 \geq 0$

Now, from (i), we get,

$$\left|\frac{1 + x^2}{2x}\right| \leq 1$$

$$|1 + x^2| \leq 2\,|x|$$

$$(1 + x^2) \leq 2\,|x|$$

$$|x|^2 - 2\,|x| + 1 \leq 0$$

$$(|x| - 1)^2 \leq 0$$

$$(|x| - 1) = 0$$

$$x = \pm 1 \quad \text{...(1)}$$

Also, from (ii), we get,

$$2 - x^2 \geq 0$$

$$x^2 - 2 \leq 0$$

$$x^2 - (\sqrt{2})^2 \leq 0$$

$$(x + \sqrt{2})(x - \sqrt{2}) \leq 0$$

$$-\sqrt{2} \leq x \leq \sqrt{2} \quad \text{...(2)}$$

From (1) and (2), we get,

$$x = \{-1, 1\}$$

Thus, $D_f = \{-1, 1\}$

$$R_f = \{f(1), f(-1)\}$$

$$= \{1, 1 + \pi\}$$

27. Given $f(x) = \dfrac{1}{\pi}(\tan^{-1}x + \sin^{-1}x) + \dfrac{x + 1}{x^2 + 2x + 5}$

f is defined for

(i) $-1 \leq x \leq 1$

(ii) $x^2 + 2x + 5 \neq 0$

which is true for all x in R

From (i) and (ii), we get

$$x \in [-1, 1]$$

$$D_f = [-1, 1]$$

Thus, $R_f = [f(-1), f(1)]$

$$= \left[-\frac{3}{4}, 1\right]$$

28. Given $f(x) = \sin^{-1}\left[x^2 + \dfrac{1}{2}\right] + \cos^{-1}\left[x^2 - \dfrac{1}{2}\right]$

Now, $\left[x^2 + \dfrac{1}{2}\right] = \left[x^2 - \dfrac{1}{2} + 1\right] = \left[x^2 - \dfrac{1}{2}\right] + 1$

Thus, $f(x) = \sin^{-1}\left[x^2 + \dfrac{1}{2}\right] + \cos^{-1}\left[x^2 - \dfrac{1}{2}\right]$

$$= \sin^{-1}\left[\left(x^2 - \frac{1}{2}\right) + 1\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$$

It is defined for $\left[x^2 - \dfrac{1}{2}\right] = 0, -1$

Therefore,

$$f(x) = \sin^{-1}(1) + \cos^{-1}(0) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Also, $f(x) = \sin^{-1}(0) + \cos^{-1}(-1) = 0 = \pi = \pi$

Thus, $R_f = \{\pi\}$

29. Given $f(x) = \log\left\{\cos\sqrt{\dfrac{\pi^2}{9} - x^2}\right\}$

f is defined for

(i) $\cos\left(\sqrt{\dfrac{\pi^2}{9} - x^2}\right) > 0$

(ii) $\left(\dfrac{\pi^2}{9} - x^2\right) \geq 0$

Now, from (ii), we get,

$$\left(x^2 - \frac{\pi^2}{9}\right) \leq 0$$

$$\left(x + \frac{\pi}{3}\right)\left(\frac{x - \pi}{3}\right) \leq 0$$

$$-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$$

Thus, $D_f = \left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$

Also, $0 \leq \sqrt{\dfrac{\pi^2}{9} - x^2} \leq \dfrac{\pi}{3}$

$$\cos\left(\frac{\pi}{3}\right) \leq \cos\left(\sqrt{\frac{\pi^2}{9} - x^2}\right) \leq \cos(0)$$

$$\frac{1}{2} \leq \cos\left(\sqrt{\frac{\pi^2}{9} - x^2}\right) \leq 1$$

$$\log\left(\frac{1}{2}\right) \le \log\left(\cos\left(\sqrt{\frac{\pi^2}{9} - x^2}\right)\right) \le \log(1)$$

$$-\log(2) \le \log\left(\cos\left(\sqrt{\frac{\pi^2}{9} - x^2}\right)\right) \le 0$$

Thus, $R_f = [-\log(2), 0]$

30. Given $f(x) = [1 + \sin x] + [\cos x - 1] + [\tan^{-1} x]$

$= 1 + [\sin x] + [\cos x] - 1 + [\tan^{-1} x]$

$= [\sin x] + [\cos x] + [\tan^{-1} x]$

$$= \begin{cases} 1 : x = 0 \\ 0 : 0 < x < \tan(1) \\ 1 : \tan(1) \le x < \dfrac{\pi}{2} \\ 2 : x = \dfrac{\pi}{2} \\ 0 : \dfrac{\pi}{2} < x < \pi \\ -1 : \pi \le x \le \dfrac{3\pi}{2} \\ 0 : \dfrac{3\pi}{2} \le x \le 2\pi \\ 2 : x = 2\pi \end{cases}$$

Hence, the range of the function is

$= \{-1, 0, 1, 2\}$

31. Given $f(x) = \log_{10}\left\{\sin^{-1}(\sqrt{x-5}) + \dfrac{3\pi}{2}\right\}$

f is defined for

(i) $(x - 5) \ge 0$

$x \ge 5$

(ii) $-1 \le \sqrt{x-5} \le 1$

$0 \le (x - 5) \le 1$

$5 \le x \le 6$

Thus, $D_f = [5, 6]$

Now, $0 \le \sin^{-1}\sqrt{x-5} \le \dfrac{\pi}{2}$

$$\frac{3\pi}{2} \le \sin^{-1}(\sqrt{x-5}) + \frac{3\pi}{2} \le \frac{\pi}{2} + \frac{3\pi}{2}$$

$$\frac{3\pi}{2} \le \sin^{-1}(\sqrt{x-5}) + \frac{3\pi}{2} \le 2\pi$$

$$\log\left(\frac{3\pi}{2}\right) \le \log\left(\sin^{-1}(\sqrt{x-5}) + \frac{3\pi}{2}\right) \le \log(2\pi)$$

$$\log\left(\frac{3\pi}{2}\right) \le f(x) \le \log(2\pi)$$

Thus, $R_f = \left[\log\left(\dfrac{3\pi}{2}\right), \log(2\pi)\right]$

32. Find the range of the function

$f(x) = \log_e(2\sin x + \tan x - 3x + 1)$,

where $\dfrac{\pi}{6} \le x \le \dfrac{\pi}{3}$

33. Given $f(x) = \log_{\sqrt{5}}\{\sqrt{2}(\sin x - \cos x) + 3\}$

Now, $\{\sqrt{2}(\sin x - \cos x) + 3\}$

$= \left\{2\sin\left(x - \dfrac{\pi}{4}\right) + 3\right\}$

$-1 \le \sin\left(x - \dfrac{\pi}{4}\right) \le 1$

$-2 \le \sin\left(x - \dfrac{\pi}{4}\right) \le 2$

$-2 + 3 \le \sin\left(x - \dfrac{\pi}{4}\right) + 3 \le 2 + 3$

$1 \le \sin\left(x - \dfrac{\pi}{4}\right) + 3 \le 5$

$\log_{\sqrt{5}}(1) \le \log_{\sqrt{5}}\left\{\sin\left(x - \dfrac{\pi}{4}\right) + 3\right\} \le \log_{\sqrt{5}}(5)$

$0 \le \log_{\sqrt{5}}\left\{\sin\left(x - \dfrac{\pi}{4}\right) + 3\right\} \le 2$

$0 \le f(x) \le 2$

$R_f = [0\ 2]$

34. Given equation is

$$\sqrt{3 - 4\cos^2 x} > 2\sin x + 1$$

$$\sqrt{4\sin^2 x - 1} > 2\sin x + 1$$

Let $\sin x = t$

So, the given equation becomes

$$\sqrt{4t^2 - 1} > 2t + 1$$

It is defined for $4t^2 - 1 \ge 0$

$\Rightarrow$ $(2t + 1)(2t - 1) \ge 0$

$\Rightarrow$ $t \ge \dfrac{1}{2}$ and $t \le -\dfrac{1}{2}$

Case-I: When $(2t + 1) \ge 0$

The given equation becomes

$\Rightarrow$ $(4t^2 - 1) > (2t + 1)^2$

$\Rightarrow$ $(4t^2 - 1) > 4t^2 + 4t + 1$

$\Rightarrow$ $4t < -2$

$\Rightarrow$ $t < -\dfrac{1}{2}$

So, it has no solution

Case-II: When $2t + 1 < 0$

$\Rightarrow \quad t < -\frac{1}{2}$

$\Rightarrow \quad \sin x < -\frac{1}{2}$

$\Rightarrow \quad 2nx - \frac{5\pi}{6} < x < 2n\pi - \frac{\pi}{6}, n \in I$

35. Given $m = \left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{200}\right] + \left[\frac{1}{4} + \frac{2}{200}\right] + \ldots + \left[\frac{1}{4} + \frac{199}{200}\right]$

$= \left[\frac{1}{4} \times 200\right] = 50$

Hence, the value of $m + 50$.

$= 50 + 50$

$= 100$

36. Given equation is $\{x\} + \sin\{x\} = 2$

Now, $\{x\} < 1$ and $\{\sin x\} < 1$

$\Rightarrow \quad \{x\} + \{\sin x\} < 2$

Hence, it has no solution.

37. The given equation is

$x^2 - 4x + [x] + 3 = 0 \quad$ ...(i)

$\Rightarrow \quad x^2 - 4x + x - \{x\} + 3 = 0$

$\Rightarrow \quad x^2 - 3x + 3 = \{x\}$

$\Rightarrow \quad 0 \le x^2 - 3x + 3 < 1$

Now, $x^2 - 3x + 3 \ge 0$

It is true for all x, since $D < 0$

Also, $x^2 - 3x + 3 < 1$

$\Rightarrow \quad x^2 - 3x + 2 < 0$

$\Rightarrow \quad (x - 1)(x - 2) < 0$

$\Rightarrow \quad 1 < x < 2$

$\Rightarrow \quad [x] = 1$

Now, from (i), we get,

$x^2 - 4x + 4 = 0$

$\Rightarrow \quad (x - 2)^2 = 0$

$\Rightarrow \quad x = 2$

which does not satisfy $1 < x < 2$

Thus, the given equation has no solution.

38. Given $F(x) = x - [x]$

Now, $F(x) + \frac{1}{F(x)} = 1$

$\Rightarrow \quad (x - [x]) + \frac{1}{x} - \frac{1}{[x]} = 1$

$\Rightarrow \quad x + \frac{1}{x} = [x] + \frac{1}{[x]} + 1$

$\Rightarrow \quad \left(x + \frac{1}{x}\right) - \left([x] + \frac{1}{[x]}\right) = 1 \quad$ ...(i)

Since R.H.S is an integer, so L.H.S is also an integer.

Let $\left([x] + \frac{1}{[x]} + 1\right) = t$ (Integer)

$\Rightarrow \quad \left(x + \frac{1}{x}\right) = t$, from (i)

$\Rightarrow \quad x^2 - tx + 1 = 0$

$\Rightarrow \quad x = \frac{t \pm \sqrt{t^2 - 4}}{2}$

For real x, $t^2 - 4 \ge 0$

$\Rightarrow \quad (t + 2)(t - 2) \ge 0$

$\Rightarrow \quad t \le -2$ and $t \ge 2$

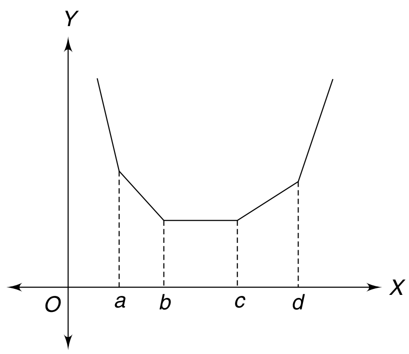
$\Rightarrow \quad t = 2$ and -2 does not satisfy the equation (i)

So, the equation (i) has infinitely many solutions.

39. Given equation is $\{x^2\} + [x^4] = 1$

It can be solved graphically, it has only 2 real solutions.

40. Given $f(x) = |x - a| + |x - b| + |x - c| + |x - d|$



Hence, the least value of the function is

$= (c + d - a - b)$ at $x = b$ or c.

41. Let $S = \sum_{n=1}^{89} \left(\frac{1}{1 + (\tan n°)^2}\right)$

$= \frac{1}{1 + \tan^2(1°)} + \frac{1}{1 = \tan^2(2°)} + \frac{1}{1 + \tan^2(3°)}$

$\ldots + \frac{1}{1 + \tan^2(87°)} + \frac{1}{1 = \tan^2(88°)} + \frac{1}{1 + \tan^2(89°)}$

...(i)

$$= \frac{1}{1+\cot^2(89°)} + \frac{1}{1+\cot^2(88°)} + \frac{1}{1+\cot^2(87°)}$$

$$\ldots + \frac{1}{1+\cot^2(3°)} + \frac{1}{1+\cot^2(2°)} + \frac{1}{1+\cot^2(1°)} \quad \ldots\text{(ii)}$$

Adding (i) and (ii), we get,

$$\Rightarrow \quad 2S = \sum_{n=1}^{89} \frac{1}{1+\tan^2(n°)} + \frac{2}{1+\cot^2(n°)}$$

$$\Rightarrow \quad 2S = \sum_{n=1}^{89} \frac{1}{1+\tan^2(n°)} + \frac{\tan^2(n°)}{1+\tan^2(n°)}$$

$$\Rightarrow \quad 2S = \sum_{n=1}^{89} = \left(\frac{1 + \tan^2(n°)}{1 + \tan^2(n°)}\right)$$

$$\Rightarrow \quad 2S = \sum_{n=1}^{89} 1 = 1 + 1 + 1 + \ldots + 1\,(89 \text{ times})$$

$$\Rightarrow \quad 2S = 89$$

$$\Rightarrow \quad S = 44\frac{1}{4}$$

42. Given equation is $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$

when x is negative, the value of

$\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right]$ can never be equal to 5

when x is positive, there are 3 possibilities

(i) $\left[\frac{3}{x}\right] = 0$ and $\left[\frac{4}{x}\right] = 5$

(ii) $\left[\frac{3}{x}\right] = 1$ and $\left[\frac{4}{x}\right] = 4$

(iii) $\left[\frac{3}{x}\right] = 2$ and $\left[\frac{4}{x}\right] = 3$

From (i) $\left[\frac{3}{x}\right] = 0$, then $0 \le \frac{3}{x} < 1$

$\Rightarrow \quad x > 3$

and $\quad \left[\frac{4}{x}\right] = 5$

$\Rightarrow \quad 5 \le \frac{4}{x} < 6$

$\Rightarrow \quad \frac{1}{6} < \frac{x}{4} \le \frac{1}{5}$

$\Rightarrow \quad \frac{2}{3} < x \le \frac{4}{5}$

This is not possible simultaneously,
So, it has no solution.

From (ii), $\left[\frac{3}{x}\right] = 1$, then $1 \le \frac{3}{x} < 2$

$\Rightarrow \quad \frac{1}{2} < \frac{x}{3} \le 1$

$\Rightarrow \quad \frac{3}{2} < x \le 3$

and $\quad \left[\frac{4}{x}\right] = 4$

$\Rightarrow \quad 4 \le \frac{4}{x} < 5$

$\Rightarrow \quad \frac{4}{5} < x \le 1$

This is not possible simultaneously,
So, it has no solution.
From (iii), we get,

$$\left[\frac{3}{x}\right] = 2 \text{ and } \left[\frac{4}{x}\right] = 3$$

$\Rightarrow \quad 2 \le \frac{3}{x} < 3$ and $3 \le \frac{4}{x} < 4$

$\Rightarrow \quad \frac{1}{3} < \frac{x}{3} \le \frac{1}{2}$ and $\frac{1}{4} < \frac{x}{4} \le \frac{1}{3}$

$\Rightarrow \quad 1 < x \le \frac{3}{2}$ and $1 < x \le \frac{4}{3}$

$\Rightarrow \quad 1 < x \le \frac{4}{3}$

$\Rightarrow \quad x \in \left(1, \frac{4}{3}\right]$

Thus, $\quad a = 1, b = 4, c = 3$

Hence, the value of $a + b + c + abc + 30$

$= 1 + 4 + 3 + 12 + 30$

$= 50.$

43. Given equation is

$$\left(\frac{1}{4}\right)^x + \left(\frac{1}{2}\right)^{x-1} + b = 0$$

$$\Rightarrow \quad \left(\left(\frac{1}{2}\right)^x\right)^2 + \left(\frac{1}{2}\right)^{x-1} + \text{b} = 0$$

$$\Rightarrow \quad t^2 + 2t + b = 0,\ t = \left(\frac{1}{2}\right)^x$$

where $\quad 0 < t < 1$

Sum of the roots $= -2 < 0$

So, there is a + ve root in (0, 1)

Thus, $\quad f(0), f(1) < 0$

$\Rightarrow \quad b\,(b + 3) < 0$

$\Rightarrow \quad -3 < b < 0$

$\Rightarrow \quad b \in (-3, 0)$

44. Given $\quad f(x) = px + q$

Now, $\quad f(f(f(x)))$

$= f(f(px + q))$

$= f(f(a)), \ a = px + q$

$= f(pa + q)$

$= f(p(px + q) + q)$

$= f(p^2x + pq + q)$

$= f(c), \ c = (p^2x + pq + q)$

$= pc + q$

$= p(p^2x + pq + q) + q$

$= p^3x + p^2q + pq + q$

Given $f(f(f(x))) = 8x + 21$

Thus, $p^3 = 8, \ p^2q + pq + q = 21$

$\Rightarrow \quad p = 2, \ q = 3$

Hence, the value of $p^2 + q^2 + p + q$

$= 4 + 9 + 2 + 3$

$= 18.$

45. Given $f(x) = x^3 - 2x^2 - px + 1$

$f'(x) = 3x^2 - 4x - p$

Since f is one-one function

So, f is either strictly increasing or decreasing function

Thus, $f'(x) > 0$ (since, $a = 3 > 0$)

$\Rightarrow \quad D < 0$

$\Rightarrow \quad 16 + 12p < 0$

$\Rightarrow \quad 4 + 3p < 0$

$\Rightarrow \quad p < -\dfrac{4}{3}$

46. Since the given function is onto, so the range of a function is equal to its co-domain

Thus, $R_f = R$

Let $y = \dfrac{ax^2 + 6a - 8}{a + 6x - 8x^2}$

$\Rightarrow \quad (a + 8y)x^2 + 6(1 - y)x - (8 + ay) = 0$

since x is real, so $D \geq 0$

$\Rightarrow \quad 36(1 - y)^2 + 4(a + 8y)(8 + ay) \geq 0$

$\Rightarrow \quad 9(1 - y)^2 + (a + 8y)(8 + ay) \geq 0$

$\Rightarrow \quad (9 + 8a)y^2 + (a^2 + 46)y + (9 + 8a) \geq 0$

Since the range of a function is R, so the given equality holds good for all real values of y

Thus, $(9 + 8a) > 0$ and $D < 0$

$\Rightarrow \quad a > -\dfrac{9}{8}$ and $D < 0$

Now, $D < 0$

$\Rightarrow \quad (a^2 + 46)^2 - \{2(9 + 8a)^2\} < 0$

$\Rightarrow \quad (a^2 + 16a + 64)(a^2 - 16a + 28) < 0$

$\Rightarrow \quad (a + 8)^2(a - 2)(a - 14) < 0$

$\Rightarrow \quad (a - 2)(a - 14) < 0$

$\Rightarrow \quad 2 < a < 14$

Hence, the value of a is (2, 14).

47. Given $f(x) = \dfrac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[\dfrac{x + 2\pi}{\pi}\right] - 3}$

$$= \frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[2 + \frac{x}{\pi}\right] - 3}$$

$$= \frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[\frac{x}{\pi}\right] + 1}$$

$$f(-x) = \frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[-\frac{x}{\pi}\right] + 1}$$

$$= \frac{2(e^x - e^{-x})(\sin x + \tan x)}{-2\left[\frac{x}{\pi}\right] - 1}$$

$$= -\frac{2(e^x - e^{-x})(\sin x + \tan x)}{2\left[\frac{x}{\pi}\right] + 1} = -f(x)$$

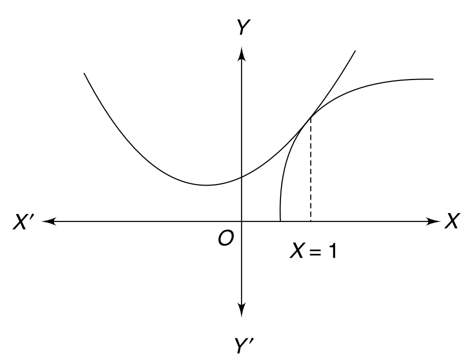
Thus, $f(x)$ is an odd function.

48. Given equation is

$$x^2 - x + 1 = \frac{1}{2} + \sqrt{\left(x - \frac{3}{4}\right)}$$

$$\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{1}{2} + \sqrt{\left(x - \frac{3}{4}\right)}$$

$$\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} = \sqrt{\left(x - \frac{3}{4}\right)}$$



Clearly, it has a solution at $x = 1$.

49. We have $f(x) = \left[g\left(\frac{\pi}{10}\right)\right]$

$= \left[\sin\left(\frac{\pi}{18}\right) + \cos\left(\frac{\pi}{18}\right)\right]$

$= [\sin(18°) + \cos(18°)]$

$= \left[\frac{\sqrt{5}-1}{4} + \frac{\sqrt{10+2\sqrt{5}}}{4}\right]$

$= [1.26]$

$= 1$

$\Rightarrow \quad \sin^{2016}x - \cos^{2016}x = 1$

$\Rightarrow \quad \sin^{2016}x = \cos^{2016}x + 1 \geq 1$

It is possible only when $\cos^{2016}x = 0$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{2}, n \in I$

50. Given $f(x, y) = 7x^2 + 4xy + 3y^2$

Put $x = \cos\theta, y = \sin\theta$

Thus, $y = f(\theta)$

$= 7\cos 2\theta + 4\sin\theta\cos\theta + 3\sin 2\theta$

$= 3(\sin 2\theta + \cos 2\theta) + 4\sin\theta\cos\theta + 4\cos 2\theta$

$= 3 + 2\sin 2\theta + 2(1 + \cos 2\theta)$

$= 5 + 2\sin 2\theta + 2\cos 2\theta$

$= 5 + 2(\sin 2\theta + \cos 2\theta)$

Max value $= (5 + 2\sqrt{2})$

Min Value $= (5 - 2\sqrt{2})$

INTEGER TYPE QUESTIONS

1. We have

$f(x) = 4\cos^4\left(\frac{x-\pi}{4\pi^2}\right) - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$

$= \left\{2\cos^2\left(\frac{x-\pi}{4\pi^2}\right)\right\}^2 - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$

$= \left\{1 + \cos\left(\frac{x-\pi}{2\pi^2}\right)\right\}^2 - 2\cos\left(\frac{x-\pi}{2\pi^2}\right)$

$= 1 + \cos^2\left(\frac{x-\pi}{2\pi^2}\right)$

Thus, the period of $f(x)$ is

$= \frac{\pi}{\frac{1}{2\pi^2}} = 2\pi^3$

Clearly, $m = 2$ and $n = 3$

Hence, the value of $(m + n)$ is 5.

2. Given $f(x) = \log_2(\sqrt{x-2} + \sqrt{4-x})$

Clearly, $D_f = [2, 4]$

Thus, $a = 2, b = 4$

Hence, the value of $(2b - 3a)$

$= 8 - 6 = 2$

3. Given $f(x) = \sqrt{\log_2\left(\frac{x-2}{3-x}\right)}$

Thus, $\left(\frac{x-2}{3-x}\right) > 0$ and $\left(\frac{x-2}{3-x}\right) \geq 1$

$\Rightarrow \quad \left(\frac{x-2}{x-3}\right) < 0$ and $\left(\frac{x-2}{3-x}\right) - 1 \geq 0$

$\Rightarrow \quad \left(\frac{x-2}{x-3}\right) < 0$ and $\left(\frac{2x-5}{x-3}\right) \leq 0$

$\Rightarrow \quad 2 < x < 3$ and $\frac{5}{2} \leq x \leq 3$

$\Rightarrow \quad D_f = \left[\frac{5}{2}, 3\right)$

Thus, $a = 5/2, b = 3$

Therefore, the value of $(2a - b)$

$= 5 - 3 = 2$

4. Let $f(x) = |x - 2| - |x + 1|$

Clearly, Range of the function $f(x)$ is $[-3, 3]$

Thus, $-3 \leq p \leq 3$

Therefore, the number of integral values of p is 7, where $p = -3, -2, -1, 0, 1, 2, 3$.

5. Let $y = \frac{x^2 + x + a}{x^2 + 2x + a}$

$= \frac{x + \frac{a}{x} + 1}{x + \frac{a}{x} + 2}$

Let $g(x) = x + \frac{a}{x}$

$\Rightarrow \quad g'(x) = 1 - \frac{a}{x^2}$

For max or min, $g'(x) = 0$

$\Rightarrow \quad 1 - \frac{a}{x^2} = 0$

$\Rightarrow \quad x = \pm\sqrt{a}$

Minimum value at $x = \sqrt{a}$ is

$= \frac{2\sqrt{a}+1}{2\sqrt{a}+2}$

Maximum value at $x = -\sqrt{a}$ is

$= \frac{2\sqrt{a}-1}{2\sqrt{a}-2}$

Clearly, $\frac{2\sqrt{a}+1}{2\sqrt{a}+2} = \frac{5}{6}$

$\Rightarrow \quad 12\sqrt{a} + 6 = 10\sqrt{a} + 10\sqrt{a} = 2$

$\Rightarrow \quad 2\sqrt{a} = 4$

$\Rightarrow \quad \sqrt{a} = 2$

$\Rightarrow \quad a = 4$

Hence, the value of $\left(\frac{a^2 + a}{5}\right)$ is 4

6. Given equation is

$$\text{sgn}\left(\left[\frac{15}{x^2 + 1}\right]\right) = [1 + \{2x\}]$$

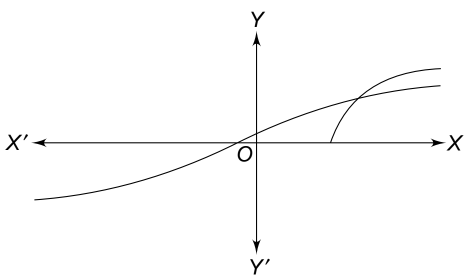
$$\Rightarrow \quad \text{sgn}\left(\left[\frac{15}{x^2 + 1}\right]\right) = 1$$

Thus, $\quad 1 + x^2 \le 15$

$\Rightarrow \quad x^2 \le 14$

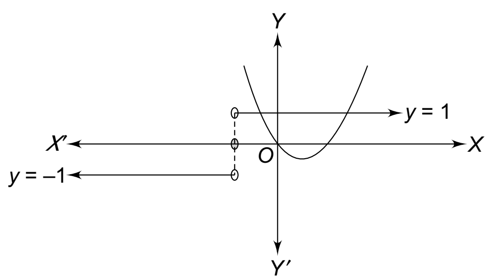
Hence, the number of integral values of x are 7 where $x = -3, -2, -1, 0, 1, 2, 3$.

7. Given equation is $\sqrt[3]{x + 1} = \sqrt{x - 3}$



From the graph, it is clear that, the number of solutions is 1.

8. Given equation is $\quad \text{sgn}(x + 1) = 2x^2 - x$



Clearly, the number of solutions is 2.

9. Given function is $f(x) = \frac{1}{\sqrt{\sqrt[x-1]{81}}} - 3$

It is defined for

$$x - 1 \ge 2, \; x - 1 \in N \text{ and } \sqrt[x-1]{81} > 3$$

$\Rightarrow \quad x \ge 3, \; x \in N$ and $81 > 3^{x-1}$

$\Rightarrow \quad x \ge 3, \; x \in N$ and $3^4 > 3^{x-1}$

$\Rightarrow \quad x \ge 3, \; x \in N$ and $x < 5$

$\Rightarrow \quad x = \{3, 4\}$

Thus, the integral values of x is 2

10. Clearly, $f(x) \ge 0$ for any x in R

When x is non zero,

$$f(x) = \frac{1}{x^2 + \frac{1}{x^2}} \le \frac{1}{2}$$

But at $x = 0$, $f(0) = 0$

Thus, the range of the given function is $\left[0, \frac{1}{2}\right]$

Clearly, $a = 0$, $b = 1$ and $c = 2$

Hence, the value of $(a + b + c + 2)$ is 5

11. Let $\quad y = \frac{e^x - 1}{e^x + 1}$

$\Rightarrow \quad e^x = \frac{1 + y}{1 - y}$

$\Rightarrow \quad \frac{1 + y}{1 - y} = e^x > 0$

$\Rightarrow \quad \frac{y + 1}{y - 1} < 0$

$\Rightarrow \quad -1 < y < 1$

Thus, the range is $(-1, 1)$

Clearly, $a = -1$, $b = 1$

Hence, the value of $(a^2 + b^2 + 2)$ is 4.

12. Given $f(x) = ax + b$

We have $f(f(x))$

$= f(ax + b)$

$= a(ax + b) + b$

$= a^2x + (ab + b)$

Also, $f(f(f(x)))$

$= f(a^2x + (ab + b))$

$= a(a^2x + (ab + b)) + b$

$= a^3x + a(ab + b) + b$

$= a^3x + a^2b + ab + b$

$= 8x + 21$

Clearly, $a = 2$, $b = 3$

Hence, the value of $(a + b + 3)$ is 8.

Questions asked in Past JEE Main Exams (2002 to 2014)

1. Clearly the option is (*b*)

Let $\quad f(x) = \cos x$ and $g(x) = \sqrt{x}$

Thus, $\quad f(g(x)) = f(\sqrt{x}) \cos\sqrt{x}$ is non-periodic.

2. We have $-1 \le \left(\log_3\left(\frac{x}{3}\right)\right) \le 1$

$\Rightarrow \quad 3^{-1} \le \left(\frac{x}{3}\right) \le 3$

$\Rightarrow \quad \frac{1}{3} \le \left(\frac{x}{3}\right) \le 3$

$\Rightarrow \quad 1 \le x \le 9$

$\Rightarrow \quad x \in [1, 9]$

Thus, $\quad D_f = [1, 9]$

Ans. (a)

3. The expression $\log_{10}(x^3 - x)$ is defined for

$(x^3 - x) > 0$

$x(x^2 - 1) > 0$

$x(x + 1)(x - 1) > 0$

$x \in (-1, 0) \cup (1, \infty)$

Also, the function $\frac{3}{4 - x^2}$ is defined only when

$x \ne \pm 2$

Thus, $D_f = (-1, 0) \cup (1, \infty) - \{2\}$

$= (-1, 0) \cup (1, 2) \cup (2, \infty)$

Ans. (c)

4. Given $\quad f(x + y) = f(x) + f(y)$

$f(2) = 7 + 7 = 14$

$f(3) = 14 + 7 = 21$

$f(4) = 21 + 7 = 28$

... ...

... ...

... ...

$f(n) = 7 \cdot n$

Thus, $\quad \sum_{r=1}^{n} f(r)$

$= (7 + 14 + 21 + 28 + ... + 7 \cdot n)$

$= 7(1 + 2 + 3 + ... + n)$

$= \frac{7n(n + 1)}{2}$

Ans. (c)

5. The function ${}^{7-x}P_{x-3}$ is defined for

$x - 3 \ge 0,\ 7 - x > 0\ \&\ x - 3 \le 7 - x$

$\Rightarrow \quad x \ge 3,\ x < 7\ \&\ x \le 5$

$\Rightarrow \quad x = 3, 4, 5$

Thus, $\quad D_f = \{3, 4, 5\}$

when $\quad x = 3,\ y = {}^4P_0 = 1$

when $\quad x = 4,\ y = {}^3P_1 = 3$

when $\quad x = 5,\ y = {}^2P_1 = 2$

Thus, $\quad R_f = \{1, 2, 3\}$

Ans. (c)

6. The function $\sin^{-1}(x - 3)$ is defined for

$-1 \le (x - 3) \le 1$

$\Rightarrow \quad 2 \le x \le 4 \quad$ ...(i)

Also, the function $\sqrt{9 - x^2}$ is defined for

$9 - x^2 > 0$

$\Rightarrow \quad x^2 - 9 < 0$

$\Rightarrow \quad (x + 3)(x - 3) < 0$

$\Rightarrow \quad -3 < x < 3 \quad$ ...(ii)

From (i) and (ii), we get, $x \in [2, 3)$

Thus, $D_f = [2, 3)$

7. Since the function is onto, so $R_f = C_f = S$

Max value of $f = \sqrt{1 + 3} + 1 = 2 + 1 = 3$

Min value of $f = -\sqrt{1 + 3} + 1 = -2 + 1 = -1$

Thus, $\quad S = R_f = [-1, 3]$

Ans. (d)

8. Since $f(x)$ is symmetrical about the line $x = 2$

then $\quad f(2 + x) = f(2 - x)$

Ans. (b)

9. The range of the function f is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Since the function f is onto, so $R_f = C_f = B$

Thus, $\quad B = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

10. Given $\quad f(x - y) = f(x)f(y) - f(a - x)f(a + y)$

Put $\quad x = 0 = y$, we get,

$f(0) = f(0)f(0) - (f(a))^2$

$1 = 1 - (f(a))^2$

$f(a) = 0$

Also, $\quad f(2a - x) = f(a - (x - a))$

$= f(a)f(x - a) - f(a - a)f(a + x - a)$

$= f(a)f(x - a) - f(0)f(x)$

$= 0 \cdot f(x - a) - 1 \cdot f(x)$

$= -f(x)$

Ans. (b)

11. The given expression

$$= \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$$

$$= \frac{(3x^2 + 9x + 7) + 10}{(3x^2 + 9x + 7)}$$

$$= 1 + \frac{10}{(3x^3 + 9x + 7)}$$

Let $g(x) = (3x^2 + 9x + 7)$

Min value of $g(x) = \frac{D}{4a}$

$$= -\frac{81 - 84}{12} = \frac{1}{4}$$

Thus, the max value of the given expression

$$= 1 + 10 \times 4 = 41$$

Ans. (b)

12. Here, 4^{-x^2} is defined for all R.

$\cos^{-1}\left(\frac{x}{2} - 1\right)$ is defined for $-1 \le \left(\frac{x}{2} - 1\right) \le 1$

$\Rightarrow \quad 0 \le \left(\frac{x}{2}\right) \le 2$

$\Rightarrow \quad 0 \le x \le 4$

Also, log (cos$x$) is defined for $\cos x > 0$

$\Rightarrow \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

Thus, the function f is defined for

$$x \in \left[0, \frac{\pi}{2}\right)$$

$\Rightarrow \quad D_f = \left[0, \frac{\pi}{2}\right)$

Ans. (b)

13. Since f is invertivle, so its inverse is exist.

Let $y = 4x + 3$

$\Rightarrow \quad x = \frac{y - 3}{4}$

Thus, $f^{-1}(x) = \frac{x - 3}{4}$

Ans. (a)

14. Since f is a bijection, so its inverse is exist.

Given $y = f(x) = (x - 1)^2 - 1$

$\Rightarrow \quad (x - 1)^2 = y + 1$

$\Rightarrow \quad (x - 1) = \sqrt{y + 1}$

$\Rightarrow \quad x = \sqrt{y + 1} - 1$

Thus, $f^{-1}(x) = \sqrt{x + 1} - 1$

It is also given that, $f^{-1}(x) = f(x)$

$\Rightarrow \quad \sqrt{x + 1} - 1 = (x + 1)^2 - 1$

$\Rightarrow \quad \sqrt{x + 1} = (x + 1)^2$

$\Rightarrow \quad (x + 1)^2 - \sqrt{x + 1} = 0$

$\Rightarrow \quad \sqrt{x + 1}\left((x + 1)^{3/2} - 1\right) = 0$

$\Rightarrow \quad \sqrt{x + 1} = 0, \left((x + 1)^{3/2-1}\right) = 0$

$\Rightarrow \quad x = -1, 0$

Thus, the solution set is $\{-1, 0\}$

Hence, the statement-II is a correct explanation for statement-I

Ans. (a)

16. The function f is defined for $|x| - x > 0$

$\Rightarrow \quad |x| > x$

$\Rightarrow \quad x \in (-\infty, 0)$

Thus, $D_f = (-\infty, 0)$

Ans. (a)

17. No questions asked in 2012.

18. *Ans.* (2)

Here $A \times B$ has 8 elements respectively

Now, total subsets of $A \times B$ is 2^8

Thus, the number of subsets of 2^8 having 3 or more elements

$$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2$$
$$= 256 - (1 + 8 + 28)$$
$$= 256 - 37$$
$$= 219$$

19. *Ans.* (3)

We have

$X = \{4n - 3n - 1 : n \in N\}$

$= \{(1 + 3)^n - 3n - 1\}$

$= \{(1 + {}^nC_1 \cdot 3 + {}^nC_2 \cdot 3^2 + {}^nC_3 \cdot 3^3 + \ldots + {}^nC_n \cdot 3^n) - 3n - 1\}$

$= ({}^nC_2 \cdot 3^2 + {}^nC_3 \cdot 3^3 + \ldots {}^nC_n \cdot 3^n)$

$= 9({}^nC_2 + {}^nC_3 \cdot 3 + \ldots {}^nC_n \cdot 3^{n-2})$

and

$Y = \{9(n - 1) : n \in N\}$

$= \{0, 9, 18, 27, 36, \ldots\}$

$\Rightarrow \quad Y \subset X$

$\Rightarrow \quad X \cup Y = X$

20. *Ans.* (4)

Here $A \times B$ has 8 elements respectively

Now, total subsets of $A \times B$ is 2^8

Thus, the number of subsets of 2^8 having 3 or more elements

$$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2$$

$= 256 - (1 + 8 + 28)$
$= 256 - 37$
$= 219$

Questions asked in Past IIT-JEE Examinations

1. Given $f(x) = \cos x - x(x + 1)$

$f'(x) = -\sin x - 2x - 1$
$= -(\sin x + 2x + 1)$
$< 0, \forall x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

Clearly, f is a decreasing function

Thus, the range of the function $f = f(A)$

$= \left[f\left(\frac{\pi}{3}\right), f\left(\frac{\pi}{6}\right)\right]$

$= \left[\frac{1}{2} - \frac{\pi^2}{9} - \frac{\pi}{3}, \frac{\sqrt{3}}{2}, \frac{\pi^2}{36} - \frac{\pi}{6}\right]$

2. **Case-I:** Let $f(x) = 1$ is true

Then $f(y) = 1$ and $f(z) = 2$

So, f is not one-one, since $f(x) = 1 = f(y)$

Case-II: Let $f(y) \neq 1$ is true

Then $f(x) \neq 1$ and $f(z) = 2$

So, f is not one, since $f(x) = 2 = f(y)$

or $f(x) = 2 = f(z), f(y) = 3$

Case-III: Let $f(z) \neq 2$ is true

Then $f(x) \neq 1, f(y) = 1$

Thus, f is one-one,

since $f(x) = 2, f(y) = 1, f(z) = 3$

Therefore, $f^{-1}(y) = 1$

3. Clearly, the domain of the function is $\left[0, \frac{\pi}{4}\right]$ and it is a decreasing function.

Hence, the range of the function is

$= \left[f\left(\frac{\pi}{4}\right), f(0)\right]$

$= \left[0, \frac{3}{\sqrt{2}}\right]$

4. Given $f(x) = (a - x^n)^{1/n}$

$f(f(x)) = f(a - x^n)^{1/n}$
$= f(b), b = (a - x^n)^{1/n}$
$= (a - b^n)^{1/n}$
$= \left(a - \left\{(a - a^n)^{1/n}\right\}^n\right)^{1/n}$
$= (a - (a - x^n))^{1/n}$
$= (x^n)^{1/n}$
$= x$

So, it is true.

5. Here, we can consider any value of $f(x)$ other than 1

Let $f(x) = 2$

Then $\frac{x^2 + 4x + 30}{x^2 - 8x + 18} = 2$

$\Rightarrow 2x^2 - 16x + 36 = x^2 + 4x + 30$

$\Rightarrow x^2 - 20x + 6 = 0$

$\Rightarrow x = \frac{20 \pm \sqrt{400 - 24}}{2}$

$\Rightarrow x = 10 \pm \sqrt{94}$

Thus, for $x = 10 + \sqrt{94}$ and $10 - \sqrt{94}$

the value of $f(x) = 2$

So, f is not one-one function. It is false.

6. *Ans.* (d)

Now, $f(x^2) = |x^2 - 1| \neq |x - 1|^2$

$f(x + y) = |x + y - 1| \neq |x - 1| + |y - 1|$

$f(|x|) = ||x| - 1| \neq ||x - 1||$

7. *Ans.* (d)

Given $f(x) = \cos(\log(x))$

Now, $f(x)\,f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$

$= \frac{1}{2}\left[2f(x)\,f(y) - f\left(\frac{x}{y}\right) - f(xy)\right]$

$= \frac{1}{2}\left[2\cos(\log x)\cos(\log y) - \cos\left(\log\left(\frac{x}{y}\right)\right) - \cos(\log(xy))\right]$

$= \frac{1}{2}\left[\cos(\log x + \log y) + \cos(\log x - \log y) - \cos\left(\log\left(\frac{x}{y}\right)\right) - \cos(\log(xy))\right]$

$= \frac{1}{2}\left[\cos(\log(xy)) + \cos\left(\log\left(\frac{x}{y}\right)\right) - \cos\left(\log\left(\frac{x}{y}\right)\right) - \cos(\log(xy))\right]$

$= 0$

8. *Ans.* (c)

Given $f(x) = \frac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$

f is defined for $(1 - x) > 0$, $(1 - x) \neq 0, 1$; $x \geq -2$

$\Rightarrow \quad x < 1, x \neq 1, 0; x \geq -2$

$\Rightarrow \quad [-2, 1) - \{0\}$

9. Ans. (a)

Here, $\quad f(x) = x - [x], [.] = \text{G.I.F}$

is a periodic function with period 1.

10. We have

$$f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$$

f is defined for $-1 \leq \log_2\left(\frac{x^2}{2}\right) \leq 1$

$\Rightarrow \quad 2^{-1} \leq \left(\frac{x^2}{2}\right) \leq 2^1$

$\Rightarrow \quad \frac{1}{2} \leq \left(\frac{x^2}{2}\right) \leq 2$

$\Rightarrow \quad 1 \leq x^2 \leq 4$

$\Rightarrow \quad 1 \leq \sqrt{x^2} \leq 2$

$\Rightarrow \quad 1 \leq |x| \leq 2$

$\Rightarrow \quad |x| \geq 1, |x| \leq 2$

$\Rightarrow \quad x \geq 1, x \leq -1; -2 \leq x \leq 2$

$\Rightarrow \quad x \in [-2, 1] \cup [1, 2]$

12. Given $f(x) = \sin\left(\ln\left(\frac{\sqrt{4 - x^2}}{1 - x}\right)\right)$

f is defined for

$$\left(\frac{\sqrt{4 - x^2}}{(1 - x)}\right) > 0$$

$(4 - x^2) \geq 0, 1 - x > 0$

$(x^2 - 4) \geq 0, x < 1$

$-2 \leq x \leq 2, x < 1$

Thus, $\quad D_f = [-2, 1)$

Clearly, the range of the function f is

$= [-1, 1]$

13. No questions asked in 1986.

14. Clearly, $\quad f(x) = 0$

which is an even function as well as it has a value at $x = 0$

15. The domain of $(f_1(x) + f_2(x))$ is $D_1 \cap D_2$ not $D_1 \cup D_2$. It is false.

16. Let $f(x) = ax + b$, where $f: [-1, 1] \to [0, 2]$

Clearly, $\quad f(-1) = 0, f(1) = 2$

or $\quad f(-1) = 2, f(1) = 0$

$\Rightarrow \quad \begin{cases} -a + b = 0 \\ a + b = 2 \end{cases}$ or $\begin{cases} -a + b = 2 \\ a + b = 0 \end{cases}$

$\Rightarrow \quad \begin{cases} a = 1, b = 1 \\ a = -1, b = 1 \end{cases}$

$\Rightarrow \quad f(x) = x + 1, 1 - x$

17. *Ans.* (b)

Now, $\quad (g_0 f)(x)$

$= g(f(x))$

$= g\left(\frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)\right)$

$$= \sqrt{3 + 4\left(\frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)\right) - 4\left(\frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)\right)^2}$$

$$= \sqrt{4 - \left(2\left(\frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)\right) - 1\right)^2}$$

$$= \sqrt{4 + 4\tan^2\left(\frac{\pi x}{2}\right)}$$

$$= \sqrt{4\left(1 + \tan^2\left(\frac{\pi x}{2}\right)\right)}$$

$$= \sqrt{4\sec^2\left(\frac{\pi x}{2}\right)}$$

$$= 2\left|\sec\left(\frac{\pi x}{2}\right)\right|$$

It is defined for, $x \in R - (2n + 1), n \in I$

Also, $\quad 3 + 4x - 4x^2 \geq 0$

$\Rightarrow \quad 4x^2 - 4x - 3 \leq 0$

$\Rightarrow \quad (2x - 3)(2x + 1) \leq 0$

$\Rightarrow \quad -\frac{1}{2} \leq x \leq \frac{3}{2}$

Thus,

$-\frac{1}{2} \leq x \leq \frac{3}{2}; -1 < x < 1; x \in R - (2n + 1), n \in I$

Therefore, the domain of $(g_0 f)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

18. No questions asked in 1991.
19. Ans. (b)

Given $f(x) = bx^2 + cx + d$

and $f(x+1) - f(x) = 8x + 3$

$b(x+1)^2 + c(x+1) + d - (bx^2 + cx + d)$

$= 8x + 3$

$b((x+1)^2 - x^2) + c(x + 1 - x) = 8x + 3$

$b(2x+1) + c = 8x + 3$

$2bx + (b + c) = 8x + 3$

$b = 4$ and $c = -1$

20. *Ans.* (b)

Let $y = 1 + ax$

$\Rightarrow \quad x = \dfrac{y-1}{a}$

$\Rightarrow \quad f^{-1}(x) = \dfrac{x-1}{a}$

It is given that, $f(x) = f^{-1}(x)$

$\Rightarrow \quad 1 + \alpha x = \dfrac{x-1}{\alpha}$

Comparing the co-efficients, we get,

$\alpha^2 = 1$ and $\alpha = -1$

$\Rightarrow \quad \alpha = -1$

21. No questions asked in 1993.
22. *Ans.* (d)

We have $(f_o g)(x) = f(g(x))$

$= f(\ln|x|) = \sin(\ln|x|)$

Thus, $R_1 = \{u : -1 \le u \le 1\}$

Also, $(g_o f)(x)$

$= g(f(x)) = g(\sin x) = \ln|\sin(x)|$

Since $0 \le |\sin(x)|, \le 1$ we get,

$R_2 = \{v : -\infty < v \le 0\}$

23. Ans. (a)

Let $y = (x+1)^2 - 1$

$\Rightarrow \quad (x+1)^2 = y + 1$

$\Rightarrow \quad (x+1) = \sqrt{y+1}$

$\Rightarrow \quad x = \sqrt{y+1} - 1$

Thus, $f^{-1}(x) = \sqrt{x+1} - 1$

Now, $f(x) = f^{-1}(x)$

$\Rightarrow \quad (x+1)^2 - 1 = \sqrt{x+1} - 1$

$\Rightarrow \quad (x+1)^2 = \sqrt{x+1}$

$\Rightarrow \quad (x+1)^4 = (x+1)$

$\Rightarrow \quad (x+1)((x+1)^3 - 1) = 0$

$\Rightarrow \quad (x+1)x((x+1)^2 + (x+1) + 1) = 0$

$\Rightarrow \quad x(x+1)(x^2 + 3x + 3) = 0$

$\Rightarrow \quad x = 0, -1, \dfrac{-3 \pm i\sqrt{3}}{2}$

Hence, the solutions are

$$\left\{-1, 0, \frac{-3 + i\sqrt{3}}{2}, \frac{-3 - i\sqrt{3}}{2}\right\}$$

24. Given $f(x) = f\left(\dfrac{x+1}{x+2}\right)$

$\Rightarrow \quad x = \pm\left(\dfrac{x+1}{x+2}\right)$

$\Rightarrow \quad x = \left(\dfrac{x+1}{x+2}\right)$ and $x = -\left(\dfrac{x+1}{x+2}\right)$

$\Rightarrow \quad x^2 + x - 1 = 0$ and $x^2 + 3x + 1 = 0$

$\Rightarrow \quad x = \dfrac{-1 \pm \sqrt{5}}{2}$ and $x = \dfrac{-3 \pm \sqrt{5}}{2}$

25. *Ans.* (d)

We have

$f(x) = \sin^2 x + \sin^2\left(x + \dfrac{\pi}{3}\right) + \cos x \cos\left(x + \dfrac{\pi}{3}\right)$

$= \dfrac{1}{2}\left[2\sin^2 x + 2\sin^2\left(x + \dfrac{\pi}{3}\right) + 2\cos x \cos\left(x + \dfrac{\pi}{3}\right)\right]$

$= \dfrac{1}{2}\left[1 - \cos 2x + 1 - \cos\left(2x + \dfrac{2\pi}{3}\right)\right.$

$\left. + \cos\left(2x + \dfrac{\pi}{3}\right) + \cos\left(\dfrac{\pi}{3}\right)\right]$

$= \left[\dfrac{1}{2}\ 1 + 1 + \dfrac{1}{2} - \cos 2x + \cos\left(2x + \dfrac{\pi}{3}\right)\right.$

$\left. - \cos\left(2x + \dfrac{2\pi}{3}\right)\right]$

$= \dfrac{1}{2}\left[\dfrac{5}{2} - \cos 2x + 2\cos 2x \times \dfrac{1}{2}\right]$

$= \dfrac{1}{2}\left[\dfrac{5}{2} - \cos 2x + \cos 2x\right]$

$= \dfrac{5}{4}$

Now, $(g_0 f)(x)$

$= g(f(x))$

$= g\left(\dfrac{5}{4}\right)$

$= 1$

26. Let $y = \dfrac{\alpha x^2 + 5x - 8}{\alpha + 5x - 8x^2}$

$\Rightarrow \quad y(\alpha + 5x - 8x^2) = \alpha x^2 + 5x - 8$

$(\alpha + 8y)x^2 + 5(1 - y)x - (8 + y\alpha) = 0$

Since for all x in R, so $D \geq 0$

$\Rightarrow 25(1 - y)^2 + 4(\alpha + 8y)(8 + y\alpha) \geq 0$

$\Rightarrow 25(y^2 - 2y + 1) + 4(8\alpha + 64y\ \alpha^2 y + 8\alpha y^2) \geq 0$

$\Rightarrow (25 + 32\alpha)y^2 + (4\alpha^2 + 256 - 50)y + (25 + 32\alpha) \geq 0$

$\Rightarrow (25 + 32\alpha)y^2 + (4\alpha^2 + 206)y + (25 + 32\alpha) \geq 0$

Since the range of a function is R, so

$\Rightarrow (25 + 256\alpha) > 0$ and

$\Rightarrow (4\alpha^2 + 206)^2 - 4(25 + 32\alpha)(25 + 32\alpha) < 0$

$\Rightarrow (4\alpha^2 + 206 - 50 - 64\alpha)(4\alpha^2 + 206 + 50 + 64\alpha) < 0$

$\Rightarrow (4\alpha^2 - 64\alpha + 156)(5\alpha^2 + 64\alpha + 256) < 0$

$\Rightarrow (\alpha^2 - 16\alpha + 39)(\alpha^2 + 16\alpha + 64) < 0$

$\Rightarrow (\alpha - 3)(\alpha - 13)(\alpha + 8)^2 < 0$

$\Rightarrow (\alpha - 3)(\alpha - 13) < 0$

$\Rightarrow 3 < \alpha < 13$

Hence, the value of is α (3, 13).

when $\alpha = 3$, $f(x) = \dfrac{3x^2 + 5x - 8}{3 + 5x - 8x^2}$

$$= -\frac{3x^2 + 5x - 8}{8x^2 - 5x - 3}$$

$$= -\frac{(3x + 8)(x - 1)}{(8x + 3)(x - 1)}$$

$$= -\frac{(3x + 8)}{(8x + 3)},\ x \neq 1$$

$$f'(x) = -\frac{55}{(3x + 8)^2}$$

Thus, $f'(x)$ is not defined at $x = -\dfrac{8}{3}$

So f is neither increasing nor decreasing function

Therefore, f is not one-one function.

27. No questions asked in 1997.

28. *Ans.* (a)

Given $\quad g(f(x)) = |\sin(x)|$

and $\quad f(g(x)) = (\sin(\sqrt{x}))^2$

$\Rightarrow \quad f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

29. *Ans.* (b)

Let $\quad y = 3x - 5$

$\Rightarrow \quad x = \dfrac{y + 5}{3}$

$\Rightarrow \quad f^{-1}(x) = \dfrac{x + 5}{3}$

30. *Ans.* (b)

Let $\quad y = f(x) = 2^{x(x-1)}$

$\Rightarrow \quad y = 2^{x^2 - x}$

$\Rightarrow \quad x^2 - x = \log_2(y)$

$\Rightarrow \quad x^2 - x - \log_2(y) = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1 + 4\log_2(y)}}{2}\ f(x)$

since f is defined for $[1, \infty)$ to $[1, \infty)$

so, $\quad x = \dfrac{1 + \sqrt{1 + 4\log_2(y)}}{2}$

Thus, $\quad f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4\log_2(x)}}{2}$

31. Given equation is $2^x + 2^y = 2$

$\Rightarrow \quad 2^y = 2 - 2^x$

$\Rightarrow \quad y = \log_2(2 - 2^x)$

It is defined for $(2 - 2^x) > 0$

$\Rightarrow \quad 2^x < 2 = 2^1$

$\Rightarrow \quad x < 1$

Thus, $\quad D_f = (-\infty, 1)$

Ans. (d)

32. *Ans.* (d)

Here, $\quad x^2 + 3x + 2 = 0$ gives $x = -1, -2$

The function $\log_2(x + 3)$ is defined for $x + 3 > 0$

$\Rightarrow \quad x > -3$

$\Rightarrow \quad x \in (-3, \infty)$

Thus domian of the function f is

$= (-3, \infty) - (-1, -2)$

33. Here, $\quad g(x) = 1 + x - [x] = 1 + \{x\} \geq 1$

Thus, $\quad f(g(x)) = 1$

Ans. (b)

34. Clearly f is bijective.

So, its inverse exists.

Let $y = x + \frac{1}{x}$

$\Rightarrow$ $x^2 - xy + 1 = 0$

$\Rightarrow$ $x = \frac{y \pm \sqrt{y^2 - 4}}{2}$

$\Rightarrow$ $x = \frac{y + \sqrt{y^2 - 4}}{2}, \frac{y - \sqrt{y^2 - 4}}{2}$

Since the values of x and y are positive,

so $x = \frac{y + \sqrt{y^2 - 4}}{2}$

$\Rightarrow$ $f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$

Ans. (a)

35. Given $f(x) = (1 + b^2)x^2 + 2bx + 1$

Now, $m(b)$ = Min value of $f(x)$

$= -\frac{D}{4a}$

$= -\frac{(4b^2 - 4(1 + b^2))}{4(b^2 + 1)}$

$= \frac{1}{(b^2 + 1)}$

Thus, the range of $m(b) = (0, 1]$

Ans. (d)

36. The number of onto function

$= 2^4 - 2$

$= 14$

Ans. (a)

37. Given $f(x) = \frac{\alpha x}{x + 1}, x \neq -1$

Now, $f(f(x)) = x$

$\Rightarrow$ $f\left(\frac{\alpha x}{x + 1}\right) = x$

$\Rightarrow$ $\frac{\alpha\left(\frac{\alpha x}{x + 1}\right)}{\frac{\alpha x}{x + 1} + 1} = x$

$\Rightarrow$ $\frac{a^2 x}{\alpha x + x + 1} = x$

$\Rightarrow$ $(\alpha + 1)x^2 + (1 - \alpha^2)x = 0$

$\Rightarrow$ $(\alpha + 1) = 0$ & $(1 - \alpha^2) = 0$

$\Rightarrow$ $\alpha = -1$ & $\alpha = \pm 1$

$\Rightarrow$ $\alpha = -1$

Ans. (d)

38. Clearly $g(x)$ is the inverse of $f(x)$

Let $y = (x + 1)^2$

$(x + 1) = \sqrt{y}$

$x = \sqrt{y} - 1$

Thus, $f^{-1}(x) = g(x) = \sqrt{x} - 1, x \geq 0$

39. Given $f(x) = 2x + \sin x$

$\Rightarrow$ $f'(x) = 2 + \cos x > 0$

$\Rightarrow$ f is monotonic function

$\Rightarrow$ f is one-one function

Also, $R_f = R$ = Co-domain

Thus, f is one-one and onto function.

Ans. (a)

40. Given $f(x) = \frac{x}{x + 1} = 1 - \frac{1}{x + 1}$

$f'(x) = \frac{1}{(x + 1)^2} > 0$

f is monotonic function

f is one-one function

Also, let $y = \frac{x}{x + 1}$

$xy + y = x$

$(y - 1)x = -y$

$x = \frac{-y}{(y - 1)} = \frac{y}{(1 - y)}$

Thus, $R_f = R - \{1\} \neq$ Co-domain

Therefore f is not onto.

Ans. (b)

41. Given $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}$

$= \frac{(x^2 + x + 1) + 1}{(x^2 + x + 1)}$

$= 1 + \frac{1}{(x^2 + x + 1)}$

Let $g(x) = (x^2 + x + 1)$

Thus, the min value of $g(x)$

$= -\frac{D}{4a} = -\frac{(1-4)}{4} = \frac{3}{4}$

Thus, max value of $f(x)$ is $= 1 + \frac{4}{3} = \frac{7}{3}$

Also, $f(x) \to 1$ as $x \to \infty$

Thus, $R_f = \left(1, \frac{7}{3}\right]$

Ans. (c)

42. Given $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$

It is defined for $\sin^{-1}(2x) + \frac{\pi}{6} \geq 0$

$\Rightarrow \quad \sin^{-1}(2x) \geq -\frac{\pi}{6}$

$2x \geq \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

$\Rightarrow \quad x \geq -\frac{1}{4}$

Also, $-1 \leq (2x) \leq 1$

$-\frac{1}{2} \leq x \leq \frac{1}{2}$

Thus, $D_f = \left[-\frac{1}{4}, \frac{1}{2}\right]$

Ans. (a)

43. Given $f(x) = \sin x + \cos x,\ g(x) = x^2 - 1$

Now, $g(f(x)) = (\sin x + \cos x)^2 - 1$

$= \sin 2x$

Clearly $g(f(x))$ is invertible in the domain of

$-\frac{\pi}{2} \leq 2x \leq \frac{\pi}{2}$

$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

Thus, $D_{g(f(x))} = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Ans. (b)

44. Given $f(x) = \begin{cases} x : \text{If } x \text{ is rational} \\ x : \text{If } x \text{ is irrational} \end{cases}$

where $f : R \to R$

and $g(x) = \begin{cases} 0, \text{ if } x \text{ is rational} \\ x, \text{ if } x \text{ is irrational} \end{cases}$

where $g : R \to R$

Now, $f(x) - g(x)$

$= \begin{cases} -x : x \in \text{Rational} \\ x : x \in \text{Irrational} \end{cases}$

where $(f - g) : R \to R$

Clearly, $(f - g)$ is one-one and onto.

Ans. (a)

45. Given that X and Y are two sets and $f : X \to Y$,

$\{f(c) = y;\ c \subset X,\ y \subset Y\}$ and

$\{f^{-1}(d) = x\ ;\ d \subset Y,\ x \subset X\}$

since $f^{-1}(d) = x \Rightarrow f(x) = d$

Now, if $a \subset X \Rightarrow f(a) \subset f(x) = d$

$\Rightarrow \quad f^{-1}(f(a)) = a$

Ans. (d)

47. The function f is defined for

$$-1 \leq \frac{8.3^x}{1 - 3^{2(x-1)}} \leq 1$$

$$\Rightarrow \quad -1 \leq \frac{3^x - 3^{x-2}}{1 - 3^{2(x-1)}} \leq 1$$

Case-I: When $\frac{3^x - 3^{x-2}}{1 - 3^{2(x-1)}} \leq 1$

$$\Rightarrow \quad \frac{3^x - 3^{x-2} - 1 + 3^{2x-2}}{1 - 3^{2x-2}} \leq 0$$

$$\Rightarrow \quad \frac{(3^x - 1)(3^{x-2} - 1)}{(3^{2x-2} - 1)} \leq 0$$

$\Rightarrow \quad x \in (-\infty, 0] \cup (1, \infty)$

Case-II: When $\frac{3^x - 3^{x-2}}{1 - 3^{2(x-1)}} \geq -1$

$$\Rightarrow \quad \frac{3^x - 3^{x-2}}{1 - 3^{2x-2}} + 1 \geq 0$$

$$\Rightarrow \quad \frac{3^x - 3^{x-2} + 1 - 3^{2x-2}}{1 - 3^{2x-2}} \geq 0$$

$$\Rightarrow \quad \frac{(3^{x-2} - 1)(3^x + 1)}{(3^x . 3^{x-2} - 1)} \geq 0$$

$\Rightarrow \quad x \in (-\infty, 1) \cup [2, \infty)$

Thus, $D_f = (-\infty, 0] \cup [2, \infty)$

48. Given $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\Rightarrow \quad f'(x) = 6x^2 - 30x + 36$

$\Rightarrow \quad f'(x) = 6(x^2 - 5x + 6)$

$= 6(x - 2)(x - 3)$

Thus, $f'(x)$ is positive at $x < 2$

$\Rightarrow \quad f'(x)$ is negative at $2 < x < 3$

$\Rightarrow \quad f'(x)$ is positive at $x > 3$

Therefore $f(x)$ is many-one function.

Also, $R_f = [1, 29]$ = Co-domain

Thus f is many-one onto function.

Ans. (b)

49. Given $f(\cos 4\theta) = \frac{2}{2 - 2\sec^2\theta}$ for

$$\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

Let $\cos(4\theta) = \frac{1}{3}$

$\Rightarrow \quad 2\cos^2(2\theta) - 1 = \frac{1}{3}$

$\Rightarrow \quad 2\cos^2(2\theta) = \frac{4}{3}$

$\Rightarrow \quad \cos^2(2\theta) = \frac{2}{3}$

$\cos(2\theta) = \pm\sqrt{\frac{2}{3}}$

Now, $f\left(\frac{1}{3}\right) = \frac{2}{2 - \sec^2\theta}$

$= \frac{2\cos^2\theta}{2\cos^2\theta - 1}$

$= \frac{\cos 2\theta + 1}{\cos 2\theta}$

$= 1 + \frac{1}{\cos 2\theta}$

$= \left(1 \pm \sqrt{\frac{3}{2}}\right)$

Ans. (a, b)

50. No question asked in 2013.

51. We have $f(x) = (\log(\sec x + \tan x))^3$

$f(-x) = \{\log(\sec(-x) + \tan(-x))\}^3$

$= \{\log(\sec(x) - \tan(x))\}^3$

$= \left\{\log\left(\frac{\sec(x) - \tan(x))(\sec(x) + \tan(x))}{(\sec(x) + \tan(x))}\right)\right\}^3$

$= \left\{\log\left(\frac{1}{(\sec(x) + \tan(x))}\right)\right\}^3$

$= \{-\log(\sec(x) + \tan(x))\}^3$

$= -\{\log(\sec(x) + \tan(x))\}^3$

$= -f(x)$

Thus, $f(x)$ is an odd function.

52. No questions asked in 2015, 2016.

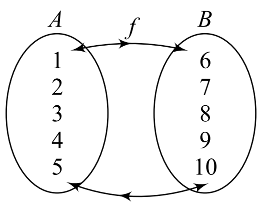
CHAPTER 2

Inverse Trigonometric Functions

INVERSE FUNCTION

CONCEPT BOOSTER

2.1 INTRODUCTION TO INVERSE FUNCTION



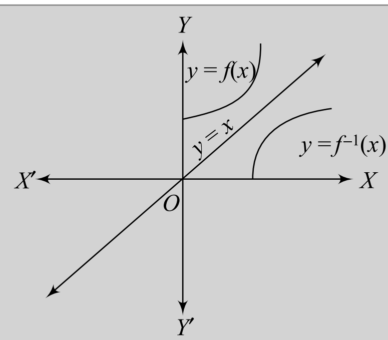
Let $f : X \to Y$ be a bijective function.

If we can make another function g from Y to X, then we shall say that g is the inverse of f.

i.e., $g = f^{-1} \neq \frac{1}{f}$

Thus, $f^{-1}(f(x)) = x$

Note:
- (i) The inverse of a function exists only when the function f is bijective.
- (ii) If the inverse of a function is exists, then it is called an invertible function.
- (iii) The inverse of a bijective function is unique.
- (iv) Geometrically $f^{-1}(x)$ is the image of $f(x)$ with respect to the line $y = x$.
- (v) Another way also we can say that $f^{-1}(x)$ is the symmetrical with respect to the line $y = x$.
- (vi) A function $f(x)$ is said to be involution if for all x for which $f(x)$ and $f(f(x))$ are defined such that $f(f(x)) = x$.



- (vii) If f is an invertible function, then $(f^{-1})^{-1} = f$.
- (viii) If $f : A \to B$ be a one one function, then $f^{-1}of = I_A$ and $fof^{-1} = I_B$, where I_A and I_B are the identity functions of the sets A and B respectively.
- (ix) Let $f : A \to B$, $g : B \to C$ be two invertible functions, then gof is also invertible with $(gof)^{-1} = (f^{-1}og^{-1})$.

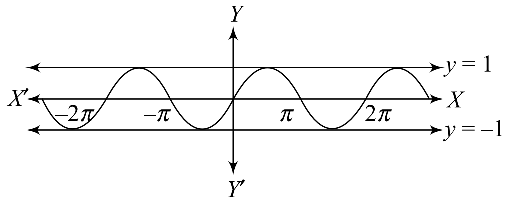
Rule to Find out the Inverse of a Function

- (i) First, we check the given function is bijective or not.
- (ii) If the function is bijective, then inverse exists, otherwise not.
- (iii) Find x in terms of y
- (iv) And then replace y by x, then we get inverse of f. i.e., $f^{-1}(x)$.

2.2 INVERSE TRIGONOMETRIC FUNCTIONS

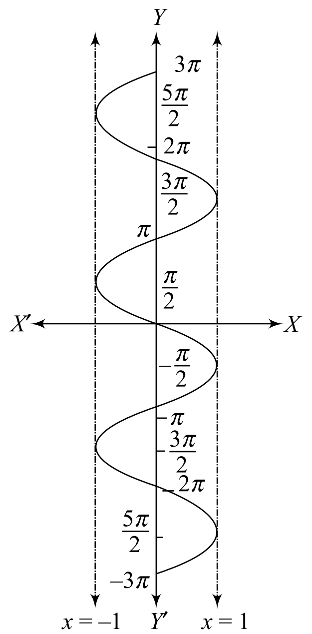
We know that sine function is defined only for every real number and the range of sine function is [–1, 1]. Thus, the graph of $f(x) = \sin(x)$ is as follows

Graph of $f(x) = \sin(x)$:



From the graph, we can say that, it will be one one and onto only when we considered it in some particular intervals like $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, and so on. If we consider the whole function, then it is not one one as well as onto.

Also, when we think the inverse function, then domain and range are interchanged. So the graph of this function is as follows.



As a whole, inverse of this function does not exist. Its inverse exists only when, we restrict its range.

So the intervals are $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$, and so on.

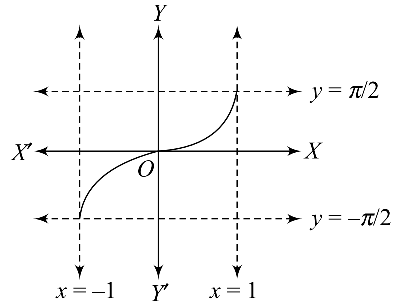
In the conventional mathematics, we consider it in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Thus, sin inverse function is defined as

$$\sin^{-1}:[-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Therefore, a function $f:[-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as $f(x) = \sin^{-1}x$.

So the graph of $f(x) = \sin^{-1}x$ is



Thus, $D_f = [-1, 1]$ and $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

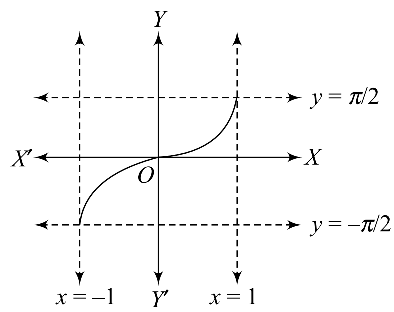
Now, we shall discuss the graphs of other inverse trigonometric functions and their characteristics.

2.3 Graphs of Inverse Trigonometric Functions

(i) $\sin^{-1}x$:

A function $f:[-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined as $f(x) = \sin^{-1}x = \text{arc} \sin x$

Graph of $f(x) = \sin^{-1}x$.



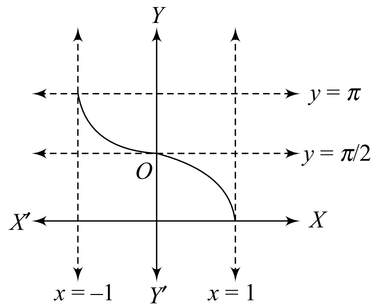
Characteristics of ARC Sine Function

1. $D_f = [-1, 1]$
2. $R_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
3. It is not a periodic function.
4. It is an odd function.
   since, $\sin^{-1}(-x) = -\sin^{-1}x$
5. It is a strictly increasing function.
6. It is a one one function.
7. For $0 < x < \frac{\pi}{2}$, $\sin x < x < \sin^{-1}x$.

(ii) $\cos^{-1}x$:

A function $f: [-1, 1] \rightarrow [0, \pi]$
is defined as $f(x) = \cos^{-1}x = \text{arc} \cos x$

Graph of $f(x) = \cos^{-1}x$

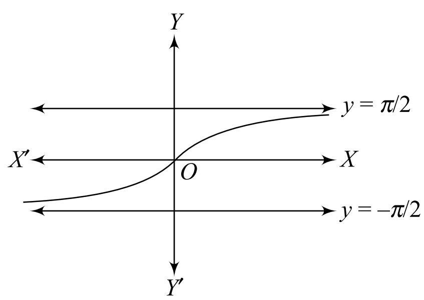


Characteristics of ARC Cosine Function

1. $D_f = [-1, 1]$
2. $[0, \pi]$
3. It is not a periodic function.
4. It is neither even nor odd function since, $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$
5. It is a strictly decreasing function.
6. It is a one one function.
7 For $0 < x < \dfrac{\pi}{2}$,
   $\cos^{-1}x < x < \cos x$

(iii) $\tan^{-1}x$:
   A function $f: R \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is defined as $f(x) = \tan^{-1}x$.

Graph of $f(x) = \tan^{-1}x$:

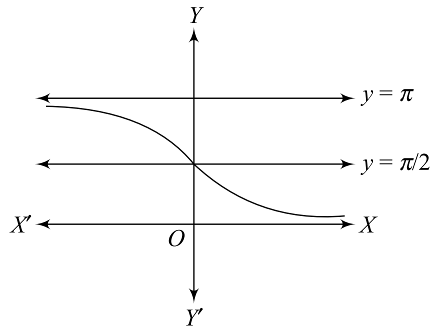


Characteristics of ARC Tangent Function

1. $D_f = R$
2. $R_f = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
3. It is not a periodic function
4. It is an odd function.
   Since, $\tan^{-1}(-x) = -\tan^{-1}x$
5. It is a strictly increasing function.
6. It is a one one function.
7. For $0 < x < \dfrac{\pi}{2}$, $\tan^{-1}x < x < \tan x$.

(iv) $\cot^{-1}x$:
   A function $f: R \to (0, \pi)$ is defined as $f(x) = \cot^{-1}x$.

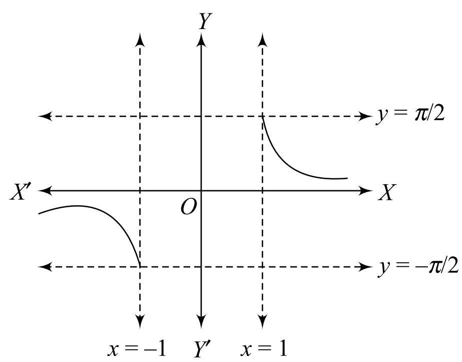
Graph of $f(x) = \cot^{-1}x$



Characteristics of ARC Co-tangent Function

1. $D_f = R$
2. $R_f = (0, \pi)$
3. It is not a periodic function.
4. It is neither even nor odd function since, $\cot^{-1}(-x) = \pi - \cot^{-1}x$
5. It is a strictly decreasing function.
6. It is a one one function.
7. For $0 < x < \dfrac{\pi}{2}$,
   $\cot x < x < \cot^{-1}x$

(v) $\text{cosec}^{-1}x$:
   A function
   $$f: (-\infty, -1] \cup [1, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$
   is defined as $f(x) = \text{cosec}^{-1}x$.
   Graph of $f(x) = \text{cosec}^{-1}x$.



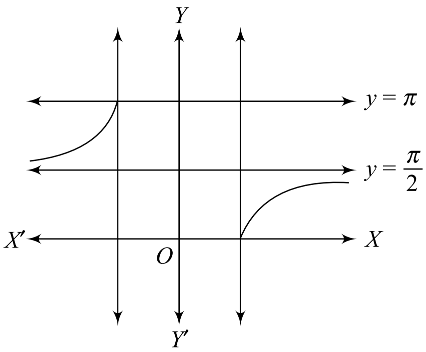
Characteristics of ARC Co-secant Function

1. $D_f = (-\infty, -1] \cup [1, \infty)$
2. $R_f = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$
3. It is an odd function, since
   $$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$$
4. It is a non periodic function.
5. It is a one one function.
6. It is a strictly decreasing function with respect to its domain.

7. For $0 < x < \frac{\pi}{2}$, $\text{cosec}^{-1} x < x < \text{cosec } x$

(v) $\sec^{-1}x$: A function $f: (-\infty, -1] \cup [1, \infty) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$ is defined as $f(x) = \sec^{-1} x$

Graph of $f(x) = \sec^{-1} x$



Characteristics of ARC Secant Function

1. $D_f = (-\infty, -1] \cup [1, \infty]$
2. $R_f = [0, \pi] - \left\{\frac{\pi}{2}\right\}$
3. It is neither an even function nor an odd odd function, since $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$
4. It is a non periodic function.
5. It is a one one function.
6. It is strictly decreasing function with respect to its domain.
7. For $0 < x < \frac{\pi}{2}$, $\sec^{-1} x < x < \sec x$

2.4 Constant Property

(i) $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}, \forall x \in [-1, 1]$

(ii) $\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}, \forall x \in R$

(iii) $\text{cosec}^{-1}(x) + \sec^{-1}(x) = \frac{\pi}{2}, \forall x \in R - (-1, 1)$

2.5 Conversion of Inverse Trigonometric Functions

Case I: When $x > 0$

Functions	Principal values
1. $\sin^{-1}x$	$\left[0, \frac{\pi}{2}\right]$
2. $\cos^{-1}x$	$\left[0, \frac{\pi}{2}\right]$
3. $\tan^{-1}x$	$\left(0, \frac{\pi}{2}\right)$
4. $\cot^{-1}x$	$\left(0, \frac{\pi}{2}\right]$
5. $\text{cosec}^{-1}x$	$\left(0, \frac{\pi}{2}\right]$
6. $\sec^{-1}x$	$\left[0, \frac{\pi}{2}\right)$

Case II: When $x < 0$

Functions	Principal values
1. $\sin^{-1}x$	$\left[-\frac{\pi}{2}, 0\right]$
2. $\text{cosec}^{-1}x$	$\left[-\frac{\pi}{2}, 0\right)$
3. $\tan^{-1}x$	$\left(-\frac{\pi}{2}, 0\right)$
4. $\cos^{-1}x$	$\left[\frac{\pi}{2}, \pi\right]$
5. $\sec^{-1}x$	$\left(\frac{\pi}{2}, \pi\right]$
6. $\cot^{-1}x$	$\left[\frac{\pi}{2}, \pi\right)$

Here, we shall discuss, how any inverse trigonometric function can be expressed in terms of any other inverse trigonometric functions.

Step I:

(i) $\sin^{-1}(x) = \text{cosec}^{-1}\left(\frac{1}{x}\right)$, $x \in [-1, 1] - \{0\}$

(ii) $\text{cosec}^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$

(iii) $\cos^{-1}(x) = \sec^{-1}\left(\frac{1}{x}\right)$, $x \in [-1, 1] - \{0\}$

(iv) $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$

(v) $\tan^{-1}(x) = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right) : x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right) : x < 0 \end{cases}$

(vi) $\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) \quad : x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right) : x < 0 \end{cases}$

Step II:

(i) $\sin^{-1}x = \begin{cases} \cos^{-1}\left(\sqrt{1-x^2}\right) & : \quad 0 \le x \le 1 \\ -\cos^{-1}(\sqrt{1-x^2}) & : -1 \le x < 0 \end{cases}$

(ii) $\sin^{-1}x = \begin{cases} \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & : \quad 0 \le x \le 1 \\ -\sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & : -1 \le x < 0 \end{cases}$

(iii) $\cos^{-1}x = \begin{cases} \sin^{-1}(\sqrt{1-x^2}) & : \quad 0 \le x \le 1 \\ \pi - \sin^{-1}(\sqrt{1-x^2}) & : -1 \le x < 0 \end{cases}$

(iv) $\cos^{-1}x = \begin{cases} \text{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & : \quad 0 < x \le 1 \\ \pi - \text{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & : -1 \le x < 0 \end{cases}$

(v) $\sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) : -1 < x < 1$

(vi) $\sin^{-1}x = \begin{cases} \cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) & : 0 < x \le 1 \\ -\pi + \cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) & : -1 \le x < 0 \end{cases}$

2.6 Composition of Trigonometric Functions and its Inverse

Let $y = \sin^{-1}x$

$\Rightarrow \quad x = \sin y$

$\Rightarrow \quad x = \sin(\sin^{-1}x)$

$\Rightarrow \quad \sin(\sin^{-1}x) = x$

Therefore, $\sin(\sin^{-1}x)$ provide us a real value lies in $[-1, 1]$

Hence,

(i) $\sin(\sin^{-1}x) = x, |x| \le 1$

(ii) $\cos(\cos^{-1}x) = x, |x| \le 1$

(iii) $\tan(\tan^{-1}x) = x, x \in R$

(iv) $\cot(\cot^{-1}x) = x, x \in R$

(v) $\text{cosec}(\text{cosec}^{-1}x) = x, |x| \ge 1$

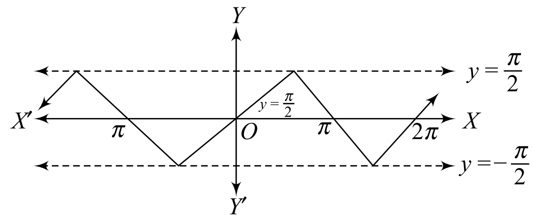
(vi) $\sec(\sec^{-1}x) = x, |x| \ge 1$

2.7 Composition of Inverse Trigonometric Functions and Trigonometric Functions

(i) A function $f: R \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ is

defined as $f(x) = \sin^{-1}(\sin x)$

Graph of $f(x) = \sin^{-1}(\sin x)$

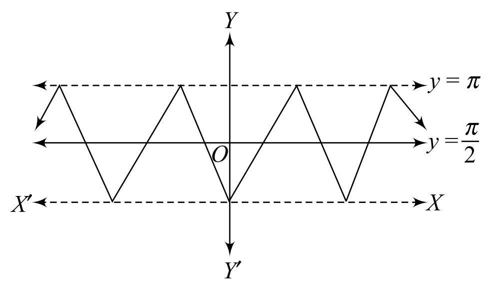


1. $D_f = R$
2. $R_f = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
3. It is an odd function.
4. It is a periodic function with period 2π
5. $\sin^{-1}(\sin x) = \begin{cases} x & : -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x & : \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \\ x - 2\pi & : \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \\ -\pi - x & : -\dfrac{3\pi}{2} \le x \le -\dfrac{\pi}{2} \end{cases}$

(ii) $\cos^{-1}(\cos x)$:

A function $f: R \to [0, \pi]$ is defined as $f(x) = \cos^{-1}(\cos x)$

Graph of $f(x) = \cos^{-1}(\cos x)$:



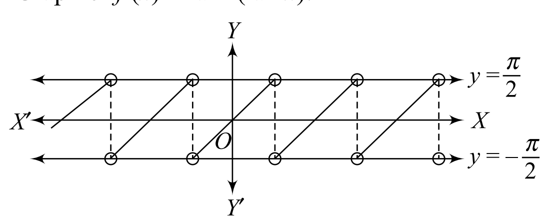
1. $D_f = R$
2. $R_f = [0, \pi]$
3. It is neither an odd nor an even function.
4. It is a periodic function with period 2π
5. $\cos^{-1}(\cos x) = \begin{cases} x & : 0 \le x \le \pi \\ 2\pi - x & : \pi \le x \le 2\pi \\ x - 2\pi & : 2\pi \le x \le 3\pi \\ -x & : -\pi \le x \le 0 \end{cases}$

(iii) $\tan^{-1}(\tan x)$:

A function $f: R - (2n+1)\dfrac{\pi}{2} \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

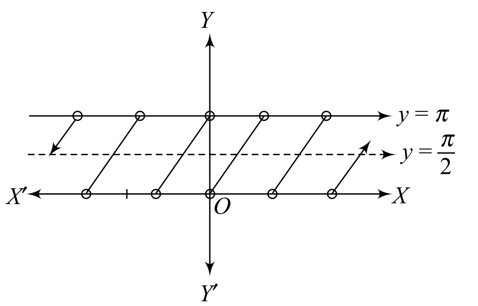
is defined as $f(x) = \tan^{-1}(\tan x)$

Graph of $f(x) = \tan^{-1}(\tan x)$:



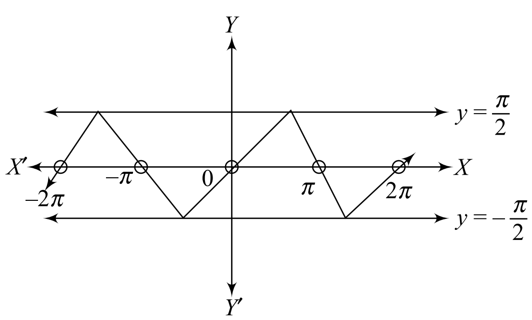
1. $D_f = R-(2n+1)\dfrac{\pi}{2}, n \in I$
2. $R_f = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
3. It is an odd function.
4. It is a periodic function with period π
5. $\tan^{-1}(\tan x) = \begin{cases} x & : \dfrac{-\pi}{2} < x < \dfrac{\pi}{2} \\ x-\pi & : \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x-2\pi & : \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x+\pi & : \dfrac{-3\pi}{2} < x < \dfrac{-\pi}{2} \end{cases}$

(iv) $\cot^{-1}(\cot x)$:
A function $f : R - (n\pi) \to (0, \pi)$ is defined as $f(x) = \cot^{-1}(\cot x)$
Graph of $f(x) = \cot^{-1}(\cot x)$:



1. $D_f = R - n\pi, n \in I$
2. $R_f = (0, \pi)$
3. It is neither an even nor an odd function.
4. It is a periodic function with period π
5. $\cot^{-1}(\cot x) = \begin{cases} x & : 0 < x < \pi \\ x-\pi & : \pi < x < 2\pi \\ x-2\pi & : 2\pi < x < 3\pi \\ \pi + x & : -\pi < x < 0 \end{cases}$

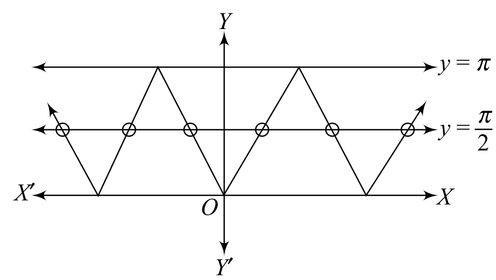
(v) $\text{cosec}^{-1}(\text{cosec } x)$: A function
$f : R-(n\pi) \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$ is defined
as $f(x) = \text{cosec}^{-1}(\text{cosec } x)$
Graph of $f(x) = \text{cosec}^{-1}(\text{cosec } x)$



1. $D_f = R - n\pi, n \in I$
2. $R_f = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$
3. It is an odd function.
4. It is a periodic function with period 2π
5. $\text{cosec}^{-1}(\text{cosec } x)$

$$= \begin{cases} x & : \dfrac{-\pi}{2} \le x \le \dfrac{\pi}{2} \\ \pi - x & : \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2} \\ x - 2\pi & : \dfrac{3\pi}{2} \le x \le \dfrac{5\pi}{2} \\ -x-\pi & : \dfrac{-3\pi}{2} \le x \le \dfrac{-\pi}{2} \end{cases}$$

(vi) $\sec^{-1}(\sec x)$: A function
$f : R-(2n+1)\dfrac{\pi}{2} \to [0, \pi] - \left\{\dfrac{\pi}{2}\right\}$ is defined as
$f(x) = \sec^{-1}(\sec x)$
Graph of $f(x) = \sec^{-1}(\sec x)$



1. $D_f = R - (2n+1)\dfrac{\pi}{2}, n \in I$
2. $R_f = [0, \pi] - \left\{\dfrac{\pi}{2}\right\}$
3. It is neither an even nor an odd function.
4. It is a periodic function with period 2π.
5. $\sec^{-1}(\sec x) = \begin{cases} x & : 0 \le x \le \pi \\ 2\pi - x & : \pi \le x \le 2\pi \\ x - 2\pi & : 2\pi \le x \le 3\pi \\ -x & : -\pi \le x \le 0 \end{cases}$

2.8 Sum of Angles

(i) $\sin^{-1}x + \sin^{-1}y$

$$= \begin{cases} \alpha : & x^2+y^2 \le 1 \\ \pi - \alpha : & x>0, y>0, x^2+y^2>1 \\ \alpha : & xy<0, x^2+y^2>1 \\ -\pi-\alpha : & x<0, y>0, x^2+y^2>1 \end{cases}$$

where $\alpha = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

(ii) $\sin^{-1}x - \sin^{-1}y$

$$= \begin{cases} \alpha : & x^2 + y^2 \le 1 \\ \pi - \alpha : & x>0, y<0, x^2 + y^2 > 1 \\ \alpha : & x\,y>0,\ x^2 + y^2 > 1 \\ -\pi - \alpha : & x<0, y>0, x^2 + y^2 > 1 \end{cases}$$

Where $\alpha = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$

(iii) $\cos^{-1}x + \cos^{-1}y$

$$= \begin{cases} \alpha & : x + y \ge 0 \\ 2\pi - \alpha & : x + y < 0 \end{cases}$$

where $\alpha = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$

(iv) $\cos^{-1}x - \cos^{-1}y$

$$= \begin{cases} \alpha & : \quad x \le y \\ -\alpha & : \quad x > y \end{cases}$$

where $\alpha = \cos^{-1}\ (xy + \sqrt{1-x^2}\sqrt{1-y^2})$

(v) $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \alpha : & xy < 1 \\ \pi + \alpha & : x > 0,\ y > 0,\ xy > 1 \\ -\pi + \alpha & : x < 0, y < 0,\ xy > 1 \\ \dfrac{\pi}{2} & : x > 0,\ y > 0,\ xy = 1 \\ \dfrac{\pi}{2} & : x < 0,\ y < 0,\ xy = 1 \end{cases}$$

where $\alpha = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

(vi) $\tan^{-1} - \tan^{-1}y$

$$= \begin{cases} \alpha & : \quad x\,y > -1 \\ \pi + \alpha & : xy < -1, x > 0,\ y < 0 \\ -\pi + \alpha & : xy < -1, x < 0, y > 0 \\ \dfrac{\pi}{2} & : xy = -1, x > 0,\ y < 0 \\ -\dfrac{\pi}{2} & : xy = -1,\ x < 0, y > 0 \end{cases}$$

where $\alpha = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

2.9 Multiple Angles

(i) $\sin^{-1}(2x\sqrt{1-x^2})$

$$= \begin{cases} 2\ \sin^{-1}x & : \ -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & : \quad \dfrac{1}{\sqrt{2}} < x \le 1 \\ -\pi - 2\sin^{-1}x & : \ -1 \le x < -\dfrac{1}{\sqrt{2}} \end{cases}$$

(ii) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & : 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & : -1 \le x < 0 \end{cases}$

(iii) $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \begin{cases} \alpha & : -1 < x < \\ -\pi + \alpha & : x > 1 \\ \pi + \alpha & : x < -1 \end{cases}$

where $\alpha = 2\ \tan^{-1}(x)$.

(iv) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \begin{cases} \alpha & : -1 \le x \le 1 \\ \pi - \alpha & : x > 1 \\ -\pi - \alpha & : x < -1 \end{cases}$

where $\alpha = 2\ \tan^{-1}(x)$

(v) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\ \tan^{-1}(x) & : x \ge 0 \\ -2\ \tan^{-1}(x) & : x \le 0 \end{cases}$

2.10 More Multiple Angles

(i) $\sin^{-1}(3x - 4x^3)$

$$= \begin{cases} 3\ \sin^{-1}x & : -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & : \dfrac{1}{2} < x \le 1 \\ -\pi - 3\sin^{-1}x & : -1 \le x < -\dfrac{1}{2} \end{cases}$$

(ii) $\cos^{-1}(4x^3 - 3x)$

$$= \begin{cases} 3\cos^{-1}x & : \dfrac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1}x & : -\dfrac{1}{2} \le x < \dfrac{1}{2} \\ -2\pi + 3\cos^{-1}x & : -1 \le x < -\dfrac{1}{2} \end{cases}$$

(iii) $\tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right)$

$$= \begin{cases} 3\tan^{-1}x & : \ -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + 3\ \tan^{-1}x & : \ -\infty < x < -\dfrac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & : \quad \dfrac{1}{\sqrt{3}} < x < \infty \end{cases}$$

EXERCISES

Level I

(Problems Based on Fundamentals)

ABC of Inverse Function

1. A function $f: R \to R$ is defined as $f(x) = 3x + 5$. Find $f^{-1}(x)$.
2. A function $f: (0, \infty) \to (2, \infty)$ is defined as $f(x) = x^2 + 2$. Then find $f^{-1}(x)$.
3. A function $f: R^+ \to [0, 1)$ is defined as $f(x) = \frac{x^2}{x^2+1}$. Then find $f^{-1}(x)$.
4. A function $f: [1, \infty) \to [1, \infty)$ is defined as $f.(x) = 2^{x(x-1)}$. Find $f^{-1}(x)$.
5. If a function f is bijective such that $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$, then find $f^{-1}(x)$
6. A function $f: R \to R$ is defined as $f(x) = x + \sin x$. Find $f^{-1}(x)$.
7. A function f: $[2, \infty) \to [5, \infty)$ is defined as $f(x) = x^2 - 4x + 9$. Find its inverse.
8. Find all the real solutions to the equation $$x^2 - \frac{1}{4} = \sqrt{x + \frac{1}{4}}$$
9. A function f is defined as $f(x) = 3z + 5$ where $f: R \to R$, then find $f^{-1}(x)$
10. A function f is defined as $f(x) = \frac{x}{x-1}$ where $f: R - \{1\} \to R - \{1\}$, then find $f^{-1}(x)$
11 A function f is defined as $f(x) = \frac{1}{x^2+1}$ where f: $R^+ \cup \{0\} \to (0, 1]$, find $f^{-1}(x)$
12 A function f is bijective such that $f(x) = \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$, then find $f^{-1}(x)$.
13. A function $f: [-1, 1] \to \left[-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right]$ is defined as $f(x) = \frac{x}{x^2+1}$, then find $f^{-1}(x)$.

ABC of inverse trigonometric functions

14. Find the domain of $f(x) = \sin^{-1}(3x + 5)$
15. Find the domain of $f(x) = \sin^{-1}\left(\frac{x}{x+1}\right)$.
16. Find the domain of $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$.
17. Find the domain of $f(x) = \sin^{-1}\left(\frac{|x|-1}{2}\right)$.
18. Find the domain of $f(x) = \sin^{-1}(\log_2 x)$.
19. Find the domain of $f(x) = \sin^{-1}(\log_4 x^2)$.
20. Solve for x and y : $\sin^{-1} x + \sin^{-1} y = \pi$
21. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then find the value of $$x^{2013} + y^{2013} + z^{2013} - \frac{9}{x^{2014} + y^{2014} + z^{2014}}$$
22. Find the range of $f(x) = 2\sin^{-1}(3x+5) + \frac{\pi}{4}$.
23. Solve the inequality: $\sin^{-1} x > \sin^{-1}(3x - 1)$.
24. Find the domain of $f(x) = \cos^{-1}(2x + 4)$.
25. Find the range of $f(x) = 2\cos^{-1}(3x+5) + \frac{\pi}{4}$.
26. Find the range of $f(x) = 3\cos^{-1}(-x^2) - \frac{\pi}{2}$.
27. Solve for x: $\cos^{-1} x + \cos^{-1} x^2 = 0$.
28. Solve for x: $[\sin^{-1} x] + [\cos^{-1} x] = 0$, where x is a non negative real number and [,] denotes the greatest integer function.
29. Find the domain of $f(x) = \cos^{-1}\left(\frac{x^2}{x^2+1}\right)$.
30. Solve for x: $\cos^{-1}(x) > \cos^{-1}(x^2)$.
31. Find the domain of $f(x) = \tan^{-1}(\sqrt{9-x^2})$.
32. Find the range of the function $$f(x) = 2\tan^{-1}(1-x^2) + \frac{\pi}{6}$$
33. Find the range of $f(x) = \cot^{-1}(2x - x^2)$.
34. Solve for x: $[\cot^{-1} x] + [\cos^{-1} x] = 0$,
35. Find the number of solutions of $\sin\{x\} = \cos\{x\}, \forall x \in [0, 2\pi]$

Q. Find the domains of each of the following functions:

36. $f(x) = \sin^{-1}\left(\frac{|x|-2}{3}\right) + \cos^{-1}\left(\frac{1-|x|}{4}\right)$
37. $f(x) = \sin^{-1}(2x^2 - 1)$
38. $f(x) = \sqrt{5\pi \sin^{-1}x - 6(\sin^{-1}x)^2}$
39. $f(x) = \log_2\left(\frac{3\tan^{-1}x + \pi}{\pi - 4\tan^{-1}x}\right)$
40. $f(x) = \cos^{-1}\left(\frac{3}{2+\sin x}\right)$
41. $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$
42. $f(x) = \cos^{-1}\left(\frac{x^2+1}{x^2}\right)$
43. $f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4)$
44. $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$
45. $f(x) = \sin^{-1}[2 - 3x^2]$
46. $f(x) = \frac{1}{x} + 3^{\sin^{-1}x} + \frac{1}{\sqrt{x-2}}$
47. $f(x) = \sin^{-1}(\log_2 x^2)$
48. $f(x) = e^x + \sin^{-1}\left(\frac{x}{2} - 1\right) + \frac{1}{x}$

49. $f(x)=\sqrt{\sin^{-1}(\log_x 2)}$
50. $f(x)=\sqrt{\sin^{-1}(\log_2 x)}$

Q. Find the ranges of each of the following functions:

51. $f(x) = \sin^{-1}(2x-3)$
52. $f(x)=2\sin^{-1}(2x-1)-\dfrac{\pi}{4}$
53. $f(x) = 2\cos^{-1}(-x^2) - \pi$
54. $f(x)=\dfrac{1}{2}\tan^{-1}(1-x^2)-\dfrac{\pi}{4}$
55. $f(x) = \cot^{-1}(2x - x^2)$
56. $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$
57. $f(x) = \sin^{-1} x + \sec^{-1} x + \tan^{-1} x$
58. $f(x)=3\cot^{-1}x+2\tan^{-1}x+\dfrac{\pi}{4}$
59. $f(x) = \operatorname{cosec}^{-1}[1 + \sin^2 x]$
60. $f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4))$

Constant Property

61. Find the range of
$$f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$$
62. Solve for x: $4\sin^{-1}(x-2) + \cos^{-1}(x-2) = \pi$
63. Solve for x:
$$\sin^{-1}(x^2-2x+1)+\cos^{-1}(x^2-x)=\frac{\pi}{2}$$
64. Find the number of real solutions of
$$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}.$$
65. If $\sin^{-1}\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{4}-........\right)$
$$+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-.......\right)=\frac{\pi}{2}\text{, for}$$
$0<|x|<\sqrt{2}$, then find x.
66. Solve for x: $\sin^{-1} x > \cos^{-1} x$

Q. Solve for x:

67. $(\sin^{-1} x)^2 - 3\sin^{-1} x + 2 = 0$
68. $\sin^{-1} x + \sin^{-1} 2y = \pi$
69. $\cos^{-1} x + \cos^{-1} x^2 = 2\pi$
70. $\cos^{-1} x + \cos^{-1} x^2 = 0$
71. $4\sin^{-1}(x-1) + \cos^{-1}(x-1) = \pi$
72. $\cot^{-1}\left(\dfrac{1}{x^2-1}\right)+\tan^{-1}(x^2-1)=\dfrac{\pi}{2}$
73. $\cot^{-1}\left(\dfrac{x^2-1}{2x}\right)+\tan^{-1}\left(\dfrac{2x}{x^2-1}\right)=\dfrac{2\pi}{3}$
74. $4\sin^{-1}x+\cos^{-1}x=\dfrac{3\pi}{4}$
75. $5\tan^{-1}x+3\cot^{-1}x=\dfrac{7\pi}{4}$
76. $5\tan^{-1} x + 4\cot^{-1} x = 2\pi$
77. $\cot^{-1}x-\cot^{-1}(x+1)=\dfrac{\pi}{2}$
78. $[\sin^{-1} x] + [\cos^{-1} x] = 0$
79. $[\tan^{-1} x] + [\cot^{-1} x] = 0$
80. $[\sin^{-1} \cos^{-1} \sin^{-1} \tan^{-1} x] = 0$
81. $[\sin^{-1} \cos^{-1} \sin^{-1} \tan^{-1} x] = 1$
82. $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\dfrac{5\pi^2}{8}$

Conversion of Inverse Trigonometric Functions

83. Find the value of $\cos\left(\dfrac{1}{2}\cos^{-1}\left(\dfrac{3}{5}\right)\right)$.
84. Find the value of $\sin\left(\dfrac{\pi}{4}+\sin^{-1}\left(\dfrac{1}{2}\right)\right)$.
85. If m is a root of $x^2 + 3x + 1 = 0$, then find the value of $\tan^{-1}(m)+\tan^{-1}\left(\dfrac{1}{m}\right)$.
86. Prove that
$$\cos(\tan^{-1}(\sin(\cot^{-1} x))) = \sqrt{\frac{x^2+1}{x^2+2}}$$

Q. Solve for x:

87. $6(\sin^{-1} x)^2 - \pi\sin^{-1} x \le 0$
88. $\dfrac{2\tan^{-1}x+\pi}{4\tan^{-1}x-\pi}\le 0$
89. $\sin^{-1} x < \sin^{-1} x^2$
90. $\cos^{-1} x > \cos^{-1} x^2$
91. $\log^2(\tan^{-1} x) > 1$
92. $(\cot^{-1} x)^2 - 5\cot^{-1} x + 6 > 0$
93. $\sin^{-1} x < \cos^{-1} x$
94. $\sin^{-1} x > \sin^{-1}(1-x)$
95. $\sin^{-1} 2x > \operatorname{cosec}^{-1} x$
96. $\tan^{-1} 3x < \cot^{-1} x$
97. $\cos^{-1} 2x^3 \sin^{-1} x$
98. $x^2 - 2x < \sin^{-1}(\sin 2)$
99. $\sin^{-1}\left(\dfrac{x}{2}\right)<\cos^{-1}(x+1)$
100. $\tan^{-1} 2x > 2\tan^{-1} x$
101. $\tan(\cos^{-1} x)\le\sin\left(\cot^{-1}\left(\dfrac{1}{2}\right)\right)$

Composition of Trigonometric Functions and its Inverse

102. Let $f(x) = \sin^{-1} x + \cos^{-1} x$

Then find the value of:

(i) $f\left(\dfrac{1}{m^2+1}\right)$, $m\in R$

(ii) $f\left(\dfrac{m^2}{m^2+1}\right)$, $m\in R$

(iii) $f\left(\dfrac{m}{m^2+1}\right)$, $m\in R$

(iv) $f(m^2 - 2m + 6)$, $m \in R$

(v) $f(m^2 + 1)$, $m \in R$

103. If $\cos^{-1}x + \cos^{-1}y = \frac{2\pi}{3}$, then find the value of $\sin^{-1} x + \sin^{-1} y$.

104. If m is the root of $x^2 + 3x + 1 = 0$, then find the value of $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$.

105. Solve for x:
$$\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+2}\right)\right) > \sin^{-1}(\sin 3)$$

Composition of Inverse Trigonometric Functions and Trigonometric Functions

106. Find the values of:
(i) $\sin^{-1}(\sin 3)$ (ii) $\sin^{-1}(\sin 53)$
(iii) $\sin^{-1}(\sin 7)$ (iv) $\sin^{-1}(\sin 10)$
(v) $\sin^{-1}(\sin 20)$

107. Find the values of:
(i) $\cos^{-1}(\cos 2)$ (ii) $\cos^{-1}(\cos 3)$
(iii) $\cos^{-1}(\cos 5)$ (iv) $\cos^{-1}(\cos 7)$
(v) $\cos^{-1}(\cos 10)$

108. Find the values of:
(i) $\tan^{-1}(\tan 3)$ (ii) $\tan^{-1}(\tan 5)$
(iii) $\tan^{-1}(\tan 7)$ (iv) $\tan^{-1}(\tan 10)$
(v) $\tan^{-1}(\tan 15)$

109. Find the value of $\cos^{-1}(\sin(-5))$

110. Find $f'(x)$, where $f(x) = \sin^{-1}(\sin x)$ and $-2\pi \le x \le \pi$

111. Find $f'(x)$, where $f(x) = \cos^{-1}(\cos x)$ and $-\pi \le x \le 2\pi$

112. Solve for x: $\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+1}\right)\right) < \pi - 3$

113. Find the integral values of x satisfying the inequality, $x^2 - 3x < \sin^{-1}(\sin 2)$

114. Find the value of
$\sin^{-1}(\text{si } 50) + \cos^{-1}(\cos 50) + \tan^{-1}(\tan 50)$

Q. Find the values of:

115. $\sin^{-1}(\sin 1) + \sin^{-1}(\sin 2) + \sin^{-1}(\sin 3)$

116. $\sin^{-1}(\sin 10) + \sin^{-1}(\sin 20)$
$+ \sin^{-1}(\sin 30) + \sin^{-1}(\sin 40)$

117. $\cos^{-1}(\cos 1) + \cos^{-1}(\cos 2)$
$+ \cos^{-1}(\cos 3) + \cos^{-1}(\cos 4)$

118. $\cos^{-1}(\cos 10) + \cos^{-1}(\cos 20)$
$+ \cos^{-1}(\cos 30) + \cos^{-1}(\cos 40)$

119. $\sin^{-1}(\sin 10) + \cos^{-1}(\cos 10)$

120. $\sin^{-1}(\sin 50) + \cos^{-1}(\cos 50)$

121. $\sin^{-1}(\sin 100) + \cos^{-1}(\cos 100)$

122. $\cos^{-1}(\sin(-5)) + \sin^{-1}(\cos(-5))$

123. Find the number of ordered pairs of (x, y) satisfying the equations $y = |\sin x|$ and $y = \cos^{-1}(\cos x)$, where $x \in [-2\pi, 2\pi]$

124. Let $f(x) = \cos^{-1}(\cos x) - \sin^{-1}(\sin x)$ in $[0, \pi]$. Find the area bounded by $f(x)$ and x-axis.

125. $\tan^{-1}(\tan 1) + \tan^{-1}(\tan 2)$
$+ \tan^{-1}(\tan 3) + \tan^{-1}(\tan 4)$

126. $\tan^{-1}(\tan 20) + \tan^{-1}(\tan 40)$
$+ \tan^{-1}(\tan 60) + \tan^{-1}(\tan 80)$

127. $\sin^{-1}(\sin 15) + \cos^{-1}(\cos 15) + \tan^{-1}(\tan 15)$

128. $\sin^{-1}(\sin 50) + \cos^{-1}(\cos 50) - \tan^{-1}(\tan 50)$

129. $3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$

130. $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$

Sum of Angles

131. Find the value of $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$.

132. Find the value of
$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$

133. Find the value of $\tan^{-1}(9) + \tan^{-1}\left(\frac{5}{4}\right)$.

134. Find the value of
$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

135. Prove that $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

136. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, prove that
$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

137. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$
prove that $x^2 + y^2 + z^2 + 2xyz = 1$

138. If $\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$, prove that
$$9x^2 + 12xy\cos\theta + 4y^2 = 36\sin^2\theta$$

139. Let $m = \frac{\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)}{\cot^{-1}(1) + \cot^{-1}(2) + \cot^{-1}(3)}$
Then find the value of $(m-1)^{2013}$.

140. Solve for x: $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{3\pi}{4}$

141. Solve for x: $\sin^{-1}(x) + \sin^{-1}(2x) = \frac{\pi}{3}$

142. Let $f(x) = \cos^{-1}(x)$
$$+ \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right), \text{ for } \frac{1}{2} \le x \le 1$$
Then find $f(2013)$

143. $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$

144. $x^2 - 4x > \sin^{-1}(\sin(\pi^{3/2})) + \cos^{-1}(\cos[\pi^{3/2}])$

145. $\cos(\tan^{-1} x) = x$

146. $\sin(\tan^{-1} x) = \cos(\cot^{-1}(x+1))$

147. $\sec^{-1}\left(\frac{x}{2}\right) - \sec^{-1}x = \sec^{-1}2$

148. $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x + \frac{3}{2}\right)\right)\right)\right) + \tan(\sec^{-1} x) = 0$

149. Find the smallest +ve integer x so that
$$\tan\left(\tan^{-1}\left(\frac{x}{10}\right) + \tan^{-1}\left(\frac{1}{x+1}\right)\right) = \tan\left(\frac{\pi}{4}\right)$$

150. Find the least integral value of k for which $(k-2)x^2 + 8x + k + 4 > \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$ holds for all x in R.

151 If $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ and $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$.

152. Let $f(x) = \sin^{-1}(\sin x)$, $\forall x \in [-\pi, 2\pi]$. Then find $f'(x)$.

153. Let $f(x) = \cos^{-1}(\cos x)$, $\forall x \in [-2\pi, \pi]$. Then find $f'(x)$.

154. Let, $f(x) = \tan^{-1}(\tan x)$, $\forall x \in \left[-\frac{3\pi}{2}, \frac{5\pi}{2}\right]$. Then find $f'(x)$.

155. Prove that $\sin^{-1}\left(\frac{1}{5}\right) + \cot^{-1}(3) = \frac{\pi}{4}$.

156. Prove that $2\tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{12}{5}\right) = \pi$.

Q. Find the simplest form of:

157. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

158. $\sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})$

159. $\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

160. $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \frac{\pi}{4} < x < \frac{5\pi}{4}$

161. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

162. $\sin^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$

Multiple Angles

163. Find the value of $\sin\left(2\sin^{-1}\left(\frac{1}{4}\right)\right)$

164. Find the value of $\cos\left(2\cos^{-1}\left(\frac{1}{3}\right)\right)$

165. Find the value of $\cos\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)$

166. Find the value of $\sin\left(\frac{1}{2}\cot^{-2}\left(\frac{3}{4}\right)\right)$

167. Find the value of $\tan^{-1}\left(\frac{3\pi}{4} - 2\tan^{-1}\left(\frac{3}{4}\right)\right)$

168. Prove that $\sin\left(2\sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{\sqrt{3}}{2}$

169. Prove that $\sin\left(3\sin^{-1}\left(\frac{1}{3}\right)\right) = \frac{23}{27}$

170. Prove that $\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right) = \frac{3}{4}$

171. Prove that $\cos\left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{10}\right)\right) = \frac{3\sqrt{5}}{10}$

172. Prove that $\sin\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{9}\right)\right) = \frac{2}{3}$

173. Prove that $\sin\left(\frac{1}{4}\tan^{-1}(\sqrt{63})\right) = \frac{1}{2\sqrt{2}}$

174. Prove that $\cos\left(\frac{1}{4}\left(\tan^{-1}\left(\frac{24}{7}\right)\right)\right) = \frac{3}{\sqrt{10}}$

175. Prove that $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right) = \frac{1}{\sqrt{5}}$

176. Prove that $\tan\left(2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right) = -\frac{7}{17}$

177. $\tan\left(\frac{3\pi}{4} - \frac{1}{4}\sin^{-1}\left(-\frac{4}{5}\right)\right) = \frac{1-\sqrt{5}}{2}$

178. Find the integral values of x satisfying the inequation $x^2 - 3x < \sin^{-1}(\sin 2)$

179. Find the value of x satisfying the inequation $3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$

180. For what value of x,

$$f(x) = \cos^{-1}x + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{1-3x^2}}{2}\right\}$$

is a constant function.

More Multiple angles

181. Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + 2\tan^{-1}(x)$, $x > 1$

Then find the value of $f(2013)$

182. Let $f(x) = 2\tan^{-1}\left(\frac{1+x}{1-x}\right) + \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

for $0 \le x < 1$. Then find the value of $f\left(\frac{1}{2014}\right)$

183. Let $f(x) = \sin^{-1}\left(\frac{6x}{x^2+9}\right) + 2\tan^{-1}\left(-\frac{x}{3}\right)$

is independent of x, then find the value of x

184. Find the interval of x for which the function $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}(x)$ is a constant function.

185. Find the interval of x for which the function $f(x) = 3\cos^{-1}(2x^2 - 1) + 2\cos^{-1}(4x^3 - 3x)$ is independent of x

186. If $\tan^{-1} y : \tan^{-1} x = 4 : 1$, then express y as algebraic function of x. Also, prove that $\tan\left(22\frac{1}{2}^\circ\right)$ is a root of $x^4 - 6x^2 + 1 = 0$

Board Special Problems

Q. Prove that:

187. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2\right)$

188. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$

189. $\tan^{-1}\left(\frac{p-q}{1+pq}\right)+\tan^{-1}\left(\frac{q-r}{1+qr}\right)+\tan^{-1}\left(\frac{r-p}{1+pr}\right)=\pi$
where $p > q > 0$ and $pr < -1 < qr$

190. $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)=0$

191. $\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right)$
$=\left(\frac{x+y}{1-xy}\right)$, $xy<1$

192. $\tan^{-1}\left(\frac{1-x}{1+x}\right)-\tan^{-1}\left(\frac{1-y}{1+y}\right)=\sin^{-1}\left(\frac{y-x}{\sqrt{(1+x^2)(1+y^2)}}\right)$

193. $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=0$

194. $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\theta}{2}\right)\right)=\cos^{-1}\left(\frac{b+a\cos\theta}{a+b\cos\theta}\right)$

195. $\tan(2\tan^{-1}a)=2\tan(\tan^{-1}a+\tan^{-1}a^3)$

196. $\cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right)=\frac{\pi}{3}, \frac{1}{2}<x<1$

197. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$,
then prove that
$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$

198. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$,
then prove that
$x^2+y^2+z^2+2xyz=1$

199. If $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$, then prove that
$9x^2-12xy\cos\theta+4y^2=36\sin^2\theta$.

200. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then prove that
$x^2+y^2+z^2-2xyz=1$.

201. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then prove that
$xy+yz+zx=3$.

202. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, then find the value of
$$x^{2012}+y^{2012}+z^{2012}-\frac{9}{x^{2013}+y^{2013}+z^{2013}}$$

203. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then prove that, $xy+yz+zx=3$.

204. If $\text{co}^{-1}x+\cos^{-1}y+\cos^{-1}z=3\pi$, then find the value of
$$\left(\frac{x^{2013}+y^{2013}+z^{2013}+6}{x^{2014}+y^{2014}+z^{2014}}\right)$$

205. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then prove that $xy+yz+zx=1$.

206. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$, then prove that $x+y+xy=1$.

207. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then prove that $x+y+z=xyz$.

208. If $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)=\alpha$, then prove that $x^2=\sin 2\alpha$

209. Let $m=\tan^2(\sec^{-1}2)+\cot^2(\text{cosec}^{-1}3)$. Then find the value of (m^2+m+10).

210. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)=\frac{\pi}{4}$, then find the value of $\tan\theta$.

211. Let $m=\frac{(\tan^{-1}1+\tan^{-1}2+\tan^{-1}3)}{(\cot^{-1}1+\cot^{-1}2+\cot^{-1}3)}$, then prove that $(m+2)^{m+1}=64$.

Q. Solve for *x*:

212. $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{3\pi}{4}$

213. $\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$

214. $\sin^{-1}(2x)+\sin^{-1}(x)=\frac{\pi}{3}$

215. $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$

216. $\sin^{-1}(x)+\sin^{-1}(3x)=\frac{\pi}{3}$

217. $\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{1+4x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$

218. $2\tan^{-1}(2x+1)=\cos^{-1}x$

219. $\cos^{-1}x-\sin^{-1}x=\cos^{-1}(x\sqrt{3})$

220. If $\tan^{-1}y:\tan^{-1}x=4:1$, express y as an algebraic function of x. Hence, prove that $\tan\left(\frac{\pi}{8}\right)$ is a root of $x^4+1=6x^2$

Level II

(Mixed Problems)

1. The set of values of k for which $x^2-kx+\sin^{-1}(\sin 4)>0$ for all real x is
(a) $\{0\}$ (b) $(-2, 2)$
(c) R (d) None of these

2. If $x < 0$ then value of

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) =$$

(a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$
(c) 0 (d) None of these

3. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then $\cos^{-1} x + \cos^{-1} y$ is

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) π

4. Let $f(x) = \sin^{-1} x + \cos^{-1} x$. Then $\frac{\pi}{2}$ is equal to

(a) $f\left(\frac{1}{2}\right)$ (b) $f(k^2 - 2k + 3), k \in R$
(c) $f\left(\frac{1}{1+k^2}\right), k \in R$ (d) $f(-2)$

5. Which one of the following is correct?
(a) $\tan 1 > \tan^{-1} 1$ (b) $\tan 1 < \tan^{-1} 1$
(c) $\tan 1 = \tan^{-1} 1$ (d) None

6. If $a \sin^{-1} x - b \cos^{-1} x = c$, then the value of a $\sin^{-1} x + b \cos^{-1} x$ is

(a) 0 (b) $\frac{\pi ab + c(b-a)}{a+b}$
(c) $\frac{\pi ab - c(b-a)}{a+b}$ (d) $\frac{\pi}{2}$

7. The number of solutions of the equation $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$ is
(a) 0 (b) 1
(c) 2 (d) More than two

8. The smallest and the largest values of $\tan^{-1}\left(\frac{1-x}{1+x}\right), 0 \le x \le 1$ are

(a) $0, \pi$ (b) $0, \frac{\pi}{4}$ (c) $-\frac{\pi}{4}, \frac{\pi}{4}$ (d) $\frac{\pi}{4}, \frac{\pi}{2}$

9. The equation $\sin^{-1} x - \cos^{-1} x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has
(a) No solution
(b) Unique solution
(c) Infinite number of solution
(d) None

10. If $-\pi \le x \le 2\pi$, then $\cos^{-1}(\cos x)$ is
(a) x (b) $\pi - x$ (c) $2\pi + x$ (d) $2\pi - x$

11. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to

(a) 0 (b) $\frac{1}{\sqrt{5}}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{\sqrt{3}}{2}$

12. If $\cos[\tan^{-1}\{\sin(\cot^{-1}\sqrt{3})\}] = y$, then the value of y is

(a) $y = \frac{4}{5}$ (b) $y = \frac{2}{\sqrt{5}}$
(c) $y = -\frac{2}{\sqrt{5}}$ (d) $y = \frac{\sqrt{3}}{2}$

13. If $x = \frac{1}{5}$, then the value of $\cos(\cos^{-1} x + 2\sin^{-1} x)$ is

(a) $\sqrt{\frac{24}{25}}$ (b) $-\sqrt{\frac{24}{25}}$ (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

14. $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$ is equal to

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) None of these

15. $\tan^{-1} a + \tan^{-1} b$, where $a > 0, b > 0, ab > 1$ is equal to

(a) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (b) $\tan^{-1}\left(\frac{a+b}{1-ab}\right) - \pi$
(c) $\pi + \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (d) $\pi - \tan^{-1}\left(\frac{a+b}{1-ab}\right)$

16. A solution to the equation

$$\tan^{-1}(1+x) + \tan^{-1}(1-x) = \frac{\pi}{2}$$ is

(a) $x = 1$ (b) $x = -1$ (c) $x = 0$ (d) $x = \pi$

17. All possible values of p and q for which

$$\cos^{-1}(\sqrt{p}) + \cos^{-1}(\sqrt{1-p}) + \cos^{-1}(\sqrt{1-q}) = \frac{3\pi}{4}$$

holds, is
(a) $p = 1, q = 1/2$ (b) $q > 1, p = 1/2$
(c) $0 < p < 1, q = 1/2$ (d) None

18. $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1} x\right)$, $x \ne 0$, is equal to

(a) x (b) $2x$ (c) $\frac{2}{x}$ (d) $\frac{x}{2}$

19. The value of $\cot^{-1}(3) + \operatorname{cosec}^{-1}(\sqrt{5})$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

20. If $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi$, then $\sum_{i=1}^{2n} x_i$ is

(a) n (b) $2n$
(c) $\frac{n(n+1)}{2}$ (d) $\frac{n(n-1)}{2}$

21. If $u = \cot^{-1}(\sqrt{\tan\alpha}) - \tan^{-1}(\sqrt{\tan\alpha})$, then $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$ is equal to

(a) $\sqrt{\tan\alpha}$ (b) $\sqrt{\cot\alpha}$
(c) $\tan\alpha$ (d) $\cot\alpha$

22. The value of $\tan^{-1}\left(\frac{a}{b+c}\right) + \tan^{-1}\left(\frac{b}{a+c}\right)$, if $\angle C = 90$, in triangle ABC is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

23. If $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$, $n \in N$, then the maximum value of 'n' is
(a) 1 (b) 5
(c) 9 (d) None of these

24. $\sin^{-1} x > \cos^{-1} x$ holds for
(a) all values of x (b) $x \in \left(0, \frac{1}{\sqrt{2}}\right)$
(c) $\left(\frac{1}{\sqrt{2}}, 1\right)$ (d) $x = 0.75$

25. The value of $\cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right)$ is equal to
(a) $\frac{3}{4}$ (b) $-\frac{3}{4}$ (c) $\frac{1}{16}$ (d) 4

26. The values of x satisfying
$\tan(\sec^{-1}x) = \sin\left(\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$ is
(a) $\pm\frac{\sqrt{5}}{3}$ (b) $\pm\frac{3}{\sqrt{5}}$ (c) $\pm\frac{\sqrt{3}}{5}$ (d) $\pm\frac{3}{5}$

27. If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then the value of x is
(a) 0 (b) –1 (c) –2 (d) –3

28. The number of real solutions of $\cos^{-1} x + \cos^{-1} 2x = -\pi$ are
(a) 0 (b) 1
(c) 2 (d) infinitely many

29. Let a, b, c be positive real numbers and
$$\theta = \tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right) + \tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ac}}\right) + \tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ba}}\right),$$
then the value of $\tan\theta$ is
(a) 0 (b) 1 (c) –1 (d) None

30. The set of values of x satisfying the inequation $\tan^2(\sin^{-1} x) > 1$ is
(a) $[-1, 1]$ (b) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$
(c) $(-1,1) - \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (d) $[-1,1] - \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

31. The value of a for which
$ax^2 + \sin^{-1}(x^2 - 2x + 2) + \cos^{-1}(x^2 - 2x + 2) = 0$
has a real solution, is
(a) $\frac{\pi}{2}$ (b) $\frac{-\pi}{2}$ (c) $\frac{2}{\pi}$ (d) $\frac{-2}{\pi}$

32. The value of
$$\sin^{-1}\left[\cot\left(\sin^{-1}\sqrt{\left(\frac{2-\sqrt{3}}{4}\right)} + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}\sqrt{2}\right)\right]$$ is
(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

33. The number of positive integral solutions of $\tan^{-1}x + \cot^{-1}\left(\frac{1}{y}\right) = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$ is
(a) 0 (b) 1 (c) 2 (d) 3

34. The value of
$\cos\left[\frac{1}{2}\cos\left\{\cos\left(\sin^{-1}\left(\frac{\sqrt{63}}{8}\right)\right)\right\}\right]$ is
(a) $\frac{3}{16}$ (b) $\frac{3}{8}$ (c) $\frac{3}{4}$ (d) $\frac{3}{2}$

35. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then the value of $\frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy}$ is
(a) 0 (b) 1 (c) $\frac{1}{xyz}$ (d) xyz

36. If $x < 0$, then $\tan^{-1}\left(\frac{1}{x}\right)$ is
(a) $\cot^{-1}(x)$ (b) $-\cot^{-1}(x)$
(c) $-\pi + \cot^{-1}(x)$ (d) None

37. The number of triplets satisfying
$\sin^{-1} x + \cos^{-1} y + \sin^{-1} z = 2\pi$, is
(a) 0 (b) 2 (c) 1 (d) infinite

38. If $x^2 + y^2 + z^2 = r^2$, then
$\tan^{-1}\left(\frac{xy}{zr}\right) + \tan^{-1}\left(\frac{zy}{xr}\right) + \tan^{-1}\left(\frac{zx}{yr}\right)$ is equal to ....
(a) π (b) $\frac{\pi}{2}$ (c) 0 (d) None

39. If $\tan^{-1} x + \tan^{-1} 2x + \tan^{-1} 3x = \pi$, then the value of x is
(a) 0 (b) –1 (c) 1 (d) ϕ

40. The number of solutions of the equation $1 + x^2 + 2x \sin(\cos^{-1} y) = 0$ is
(a) 1 (b) 2 (c) 3 (d) 4

41. If α is the only real root of the equation $x^3 + bx^2 + cx + 1 = 0$, then the value of $\tan^{-1}\alpha + \tan^{-1}\left(\frac{1}{\alpha}\right)$ is equal to
(a) $\frac{\pi}{2}$ (b) $\frac{-\pi}{2}$ (c) 0 (d) None

42. If α, β, γ are the roots of $x^3 + px^2 + 2x + p = 0$, the the general value of $\tan^{-1}\alpha + \tan^{-1}\beta \tan^{-1}\gamma$ is
(a) $n\pi$ (b) $\frac{n\pi}{2}$
(c) $\frac{(2n+1)\pi}{2}$ (d) depend on p

43. If $[\sin^{-1}(\cos^{-1})(\cos^{-1}(\sin^{-1}(\tan^{-1} x)))] = 1$, where [,] = GIF, the value of x lies in
(a) [tan sin cos 1, tan sin cos sin 1]
(b) (tan sin cos 1, tan sin cos sin 1)
(c) $[-1, 1]$
(d) [sin cos tan 1, sin cos sin tan 1]

Level IIA

(Problems For JEE Main)

1. Find the principal value of $\sin^{-1}(\sin 10)$
2. Find the principal value of $\cos^{-1}(\cos 5)$

3. Find the value of
$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$$
4. Find x if $\sin^{-1} x > \cos^{-1} x$
5. Find x if $\sin^{-1} x < \cos^{-1} x$
6. Find x if $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$
7. Find x if $3 \sin^{-1} x = \pi + \sin^{-1}(3x - 4x^3)$
8. Find x if $2\tan^{-1}x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$
9. Find the value of $\cos\left(\frac{\pi}{6} + \cos^{-1}\left(-\frac{1}{2}\right)\right)$
10. Find the value of $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)))$
11. Find the value of $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$.
12. Find the value of $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$.
13. Find the value of $\sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$
14. Find the value of
$$\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right) + \tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)$$
$$+ \tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right) + ... + \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)$$
$$+ \tan^{-1}\left(\frac{1}{a_n}\right)$$, where $x, y, a_1, a_2, \ldots a_n \in R^+$
15. Find x if $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{\pi^2}{8}$
16. Find the maximum value of $f(x)$, if
$$f(x) = (\sec^{-1} x)^2 + (\operatorname{cosec}^{-1} x)^2$$
17. Find the minimum value of $f(x)$, if
$$f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$$
18. Find x if $[\cot^{-1} x] + [\cos^{-1} x] = 0$
19. Find x if $[\sin^{-1} x] + [\cos^{-1} x] = 0$.
20. Find x if $[\tan^{-1} x] + [\cot^{-1} x] = 2$
21. Find x if $[\sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x)))] = 1$
22. Find the range of
$$f(x) = \sin^{-1} x + \tan^{-1} x + \cot^{-1} x$$
23. Find the range of
$$f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x.$$
24. Find the range of
$$f(x) = \sin^{-1} x + \sec^{-1} x + \tan^{-1} x$$
25. If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, then find x
26 If $\cos^{-1}x = \cot^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$, then find x.
27. If $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$, where $n \in N$, then find the maximum value of n.
28. If $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$, then find x.
29. If $x = \sin^{-1}(b^6 + 1) + \cos^{-1}(b^4 + 1) + \tan^{-1}(a^2 + 1)$ then find the value of
$$\sin\left(x + \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)$$
30. Find the number of integral values of k for which the equation $\sin^{-1} x + \tan^{-1} x = 2k + 1$ has a solution.
31. If $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$, then find the value of $\left(\frac{1+x^4+y^4}{x^2-x^2y^2+y^2}\right)$
32. If $\cos^{-1} x + \cos^{-1}(2x) + \cos^{-1}(3x)$ and x satisfies the equation $ax^3 + bx^2 + cx = 1$ then find the value of
$$a^2 + b^2 + c^2 + 10.$$
33. If $f(x) = \sin^{-1} x + \tan^{-1} x + x^2 + 4x + 5$ such that $R_f = [a, b]$, find the value of $a + b + 5$.
34. If $\cot^{-1}(\sqrt{\cos\alpha}) + \tan^{-1}(\sqrt{\cos\alpha}) = x$, then $\sin x$ is

(a) 1 (b) $\cot^2(\alpha/2)$
(c) $\tan\alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

[JEE Main, 2002]

35. The domain of $\sin^{-1}\left(\log_3\left(\frac{x}{3}\right)\right)$ is

(a) [1, 9] (b) [–1, 9]
(c) [–9, 1] (d) [–9, –1]

[JEE Main, 2002]

36. The trigonometric equation $\sin^{-1} x = 2 \sin^{-1} a$ has a solution for

(a) all real values (b) $|a| < \frac{1}{2}$
(c) $|a| \le \frac{1}{\sqrt{2}}$ (d) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$

[JEE Main, 2003]

37. The domain of the function
$$f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$$ is

(a) [1, 2] (b) [2, 3) (c) [2, 3] (d) [1, 2)

[JEE Main, 2004]

38. Let f: (–1, 1) → B be a function defined as $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then f is both one one and onto, when B lies in

(a) $\left[0, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{2}\right)$
(c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[JEE Main, 2005]

39. If $\cos^{-1}x - \cos^{-1}\left(\frac{y}{2}\right) = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2$ is

(a) 4 (b) $2 \sin \alpha$
(c) $-4 \sin^2 \alpha$ (d) $4 \sin^2 \alpha$

[JEE Main, 2005]

40. If $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then x is

(a) 4 (b) 5 (c) 1 (d) 3

[JEE Main, 2007]

41. Find the value of

$\cot\left(\text{cosec}^{-1}\left(\frac{5}{3}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$ is

(a) $\frac{5}{17}$ (b) $\frac{6}{17}$ (c) $\frac{3}{17}$ (d) $\frac{4}{17}$

[JEE Main, 2008]

Level III

(Problems for JEE Advanced)

1. Find the domain of

$$f(x) = \sin^{-1}\left(\frac{|x| - 2}{3}\right) + \cos^{-1}\left(\frac{1 - |x|}{4}\right)$$

2. Find the domain of

$$f(x) = \sqrt{5\pi \sin^{-1}x - 6(\sin^{-1}x)^2}$$

3. Find the domain of

$$f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4)).$$

4. Solve for x: $\cos^{-1} x + \cos^{-1} x^2 = 2\pi$
5. Solve for x:

$$\cot^{-1}\left(\frac{1}{x^2 - 1}\right) + \tan^{-1}(x^2 - 1) = \frac{\pi}{2}$$

6. Solve for x:

$$\cot^{-1}\left(\frac{x^2 - 1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2 - 1}\right) = \frac{2\pi}{3}$$

7. Solve for x:

$$\sin^{-1}\left(\sin\left(\frac{2x^2 + 4}{x^2 + 1}\right)\right) < \pi - 3$$

8. Solve for x:

$$x^2 - 4x > \sin^{-1}(\sin [\pi^{3/2}]) + \cos^{-1}(\cos [\pi^{3/2}])$$

9. Solve for x:

$$\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x + \frac{3}{2}\right)\right)\right)\right) + \tan(\sec^{-1} x) = 0$$

10. Solve for x:

$$\tan\left(\tan^{-1}\left(\frac{x}{10}\right) + \tan^{-1}\left(\frac{1}{x + 1}\right)\right) = \tan\left(\frac{\pi}{4}\right)$$

11. If $\alpha = 2\tan^{-1}\left(\frac{1 + x}{1 - x}\right)$ and $\beta = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$

12. Find the range of $f(x) = 2 \sin^{-1}(2x - 3)$
13. Find the range of

$$f(x) = 2\sin^{-1}(2x - 1) - \frac{\pi}{4}$$

14. Find the range of

$$f(x) = 2\cos^{-1}(-x)^2 - \pi$$

15. Find the range of

$$f(x) = \frac{1}{2}\tan^{-1}(1 - x^2) - \frac{\pi}{4}$$

16. Find the range of $f(x) = \cot^{-1}(2x - x^2)$.
17. Find the range of

$$f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$$

18. Find the range of

$$f(x) = \sin^{-1} x + \sec^{-1} x + \tan^{-1} x$$

19. Find the range of

$$f(x) = 3\cot^{-1}x + 2\tan^{-1}x + \frac{\pi}{4}$$

20. Prove that $\sin(\cot^{-1}(\tan(\cos^{1} x))) = x, \forall x \in (0, 1]$
21. Prove that $\sin(\text{cosec}^{-1}(\cot(\tan^{-1} x))) = x, \forall x \in (0, 1]$
22. Find the value of $\sin^{-1}(\sin 5) +$ $\cos^{-1}(\cos 10) + \tan^{-1}(\tan(-6)) + \cot^{-1}(\cot(-10))$
23. If $U = \cot^{-1}(\sqrt{\cos 2\theta}) - \tan^{-1}(\sqrt{\cos 2\theta})$, then prove that $\sin U = \tan^2 \theta$
24. Prove that

$$\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$$

25. Prove that

$$\cos^{-1}\left(\frac{\cos x + \cos y}{1 + \cos x \cos y}\right) = 2\tan^{-1}\left(\tan\left(\frac{x}{2}\right)\tan\left(\frac{y}{2}\right)\right)$$

26. Prove that

$$2\tan^{-1}\left(\sqrt{\frac{a - b}{a + b}}\tan\left(\frac{x}{2}\right)\right) = \cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right)$$

27. If $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in AP then prove that $(x + z)$ $y^2 + 2y(1 - xz)$, where $y \in (0, 1)$, $xz < 1$, $x > 0$ and $z > 0$.
28. Prove that

$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right) = \frac{13\pi}{7}$$

29. Solve for x and y:

$$\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}, \cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}.$$

30. If $y = \tan^{-1}\left(\frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}\right)$, then prove that

$$x^2 = \sin(2y).$$

31. Prove that

$$\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right) + \frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right) = (\alpha+\beta)(\alpha^2+\beta^2)$$

32. Find the minimum value of n, if

$$\cot^{-1}\left(\frac{n^2-10n+21.6}{\pi}\right) > \frac{\pi}{6}, n \in N$$

33. Prove that

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}(\sqrt{2})\right\}\right\}=0$$

34. Solve for x:

$$[\sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x)))] = 1$$

where [,] = GIF

35. Find the interval for which

$2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is independent of x.

36. If $x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))))$ and
$y = \sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1} a)))))$,
where $a \in [0, 1]$, then find the relation between x and y.

37. Find the sum of the infinite series.

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\ldots$$

38. Find the sum of

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)$$

$$+\ldots\ldots+\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\times\sqrt{n+1}}\right)+\ldots \text{ to } \infty$$

39. Find the sum of infinite series:

$$\cot^{-1}(2.1^2)+\cot^{-1}(2.2^2)+\cot^{-1}(2.3^2)+\ldots$$

40. If $\cos^{-1}\left(\frac{x}{2}\right)+\cos^{-1}\left(\frac{y}{3}\right)=\theta$, then prove that

$$9x^2 - 2xy\cos\theta + 4y^2 = 36\sin^2\theta$$

[Roorkee, 1984]

41. Evaluate: $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)$ **[Roorkee, 1986]**

Note: No questions asked between 1987 and 1991.

42. Solve for x: $\sin[2\cos^{-1}\{\cot(2\tan^{-1} x)\}]$

[Roorkee, 1992]

43. Find all positive integral solutions of

$$\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$$

[Roorkee, 1993]

44. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then find the value of $x^2 + y^2 + z^2 + 2xyz$.

[Roorkee, 1994]

45. Convert the trigonometric function $\sin[2\cos^{-1}\{\cot(2\tan^{-1} x)\}]$ into an algebraic function $f(x)$. Then from the algebraic function $f(x)$, find all values of x for which $f(x)$ is zero.

Also, express the values of x in the form of $a \pm \sqrt{b}$, where a and b are rational numbers.

[Roorkee, 1995]

Note No questions asked in 1996.

46. If $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$,

then find the general value of θ

[Roorkee, 1997]

Note No questions asked in 1998.

47. Using the principal values, express the following expression as a single angle

$$3\tan^{-1}\left(\frac{1}{2}\right)+2\tan^{-1}\left(\frac{1}{5}\right)+\sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$$

[Roorkee, 1999]

48. Solve for x:

$$\sin^{-1}\left(\frac{ax}{c}\right)+\sin^{-1}\left(\frac{bx}{c}\right)=\sin^{-1}x$$

where $a^2 + b^2 = c^2, c \neq 0$ **[Roorkee, 2000]**

49. Solve for x:

$$\cos^{-1}(x\sqrt{6})+\cos^{-1}(3\sqrt{3}x^2)=\frac{\pi}{2}$$

[Roorkee, 2001]

50. Let x_1, x_2, x_3, x_4 be four non zero numbers satisfying the equation

$$\tan^{-1}\left(\frac{a}{x}\right)+\tan^{-1}\left(\frac{b}{x}\right)+\tan^{-1}\left(\frac{c}{x}\right)+\tan^{-1}\left(\frac{d}{x}\right)=\frac{\pi}{2}$$

then prove that

(i) $\sum_{i=1}^{4} x_i = 0$ (ii) $\sum_{i=1}^{4}\left(\frac{1}{x_i}\right)=0$

(iii) $\prod_{i=1}^{4}(x_i) = abcd$ (iv) $\Pi(x_1+x_2+x_3) = abcd$

51. Let $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$

If x satisfies the cubic equation

$$ax^3 + bx^2 + cx - 1 = 0,$$

then find the value of $(a + b + c + 2)$.

52. If $x = \sin(2\tan^{-1}2)$, $y = \sin\left(\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right)$ then prove that $y^2 = 1 - x$

Level 10

(Tougher Problems for JEE Advanced)

1. Prove that
   $\sin^{-1}(\cos(\sin^{-1} x)) + \cos^{-1}(\sin(\cos^{-1} x))$
2. Prove that
   $$\tan^{-1}\{\operatorname{cosec}(\tan^{-1}x) - \tan(\cot^{-1}x)\} = \frac{1}{2}\tan^{-1}x$$
   where $x \neq 0$
3. Prove that
   $\tan(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$
   $= \cot(\cot^{-1} x + \cot^{-1} y + \cot^{-1} z)$.
4. Prove that $\sin(\cot^{-1}(\tan(\cos^{-1} x))) = \forall x \in (0, 1]$
5. Prove that $\sin(\operatorname{cosec}^{-1}(\cot(\tan^{-1} x))) = x \; \forall x \in (0, 1]$
6. Find the value of
   $\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}(\tan(-6)) + \cot^{-1}(\cot(1-10))$
7. Find the simplest value of
   $$\cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right), \forall x \in \left(\frac{1}{2}, 1\right)$$
8. Find the value of
   $$\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\sqrt{5-2\sqrt{6}}}{1+\sqrt{6}}\right)$$
9. Let $m = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^2 + 1)$
   Then find the image of the line $x + y = m$ about the y-axis.
10. If $(\sin^{-1}x)^3 + (\sin^{-1}y)^3 + (\sin^{-1}z)^3 = \frac{(3\pi)^3}{8}$
    then find the value of $(3x + 4y - 5z + 2)$
11. Let $S = \sum_{r=1}^{n} \cot^{-1}\left(2^{r+1} + \frac{1}{2^r}\right)$
    Then find $\lim_{n\to\infty}(S)$
12. Find the value of
    $$\lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{4}{4r^2+3}\right)\right)\right)$$
13. Find the number of solution of the equation
    $$2\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi x^3$$
14. If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$, then prove that,
    $$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$$
15. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$,
    then prove that,
    $$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$
16. Find the greatest and least value of the function
    $f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$.
17. Solve for x: $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{2}$
18. Solve for x:
    $$\tan^{-1}\left(\frac{1}{1+2x}\right) + \tan^{-1}\left(\frac{1}{1+4x}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$
19. Solve for x:
    $\tan^{-1}(x - 1) + \tan^{-1}(x) + \tan^{-1}(x + 1) = \tan^{-1}(3x)$
20. Solve for x: $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}x = \frac{\pi}{4}$
21. Solve for x:
    $$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$
22. Solve for x:
    $$2\tan^{-1}x = \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) - \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right),$$
    $a > 0, b > 0$
23. Solve for x:
    $\cot^{-1} x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n - 1)$
24. Solve for x:
    $$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$
25. Solve for x:
    $$\sec^{-1}\left(\frac{x}{a}\right) - \sec^{-1}\left(\frac{x}{b}\right) = \sec^{-1}b - \sec^{-1}a$$
26. Find the sum of
    $$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{8n}{n^4 - 2n^2 + 5}\right)$$
27. Find the number of real solutions of the equation
    $$\sin^{-1}(e^x) + \cos^{-1}(x^2) = \frac{\pi}{2}$$
28. Find the number of real roots of
    $\sqrt{\sin(x)} = \cos^{-1}(\cos x)$ in $(0, 2\pi)$
29. If $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$
    where $n \in N$, then find n
30. If α is the real root of $x^3 + bx^2 + cx + 1 = 0$ where $b < c$, then find the value of
    $$\tan^{-1}(\alpha) + \tan^{-1}\left(\frac{1}{\alpha}\right)$$
31. If the equation $x^3 + bx^2 + cx + 1 = 0$ has only one root α, then find the value of
    $2\tan^{-1}(\operatorname{cosec}\alpha) + \tan^{-1}(2\sin\alpha\sec^2\alpha)$

Q. Solve the following inequalities:

32. $\sin^{-1} x > \cos^{-1} x$
33. $\cos^{-1} x > \sin^{-1} x$

34. $(\cos^{-1} x)^2 - 5(\cot^{-1} x) + 6 > 0$
35. $\tan^2(\sin^{-1} x) > 1$
36. $4(\tan^{-1} x)^2 - 8(\tan^{-1} x) + 3 < 0$
37. $4 \cot^{-1} x - (\cot^{-1} x)^2 - 3 \geq 0$
38. $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 2$
39. Find the maximum value of
$$f(x) = \{\sin^{-1}(\sin x)\}^2 - \sin^{-1}(\sin x)$$
40. Find the minimum value of
$$f(x) = 8^{\sin^{-1}x} + 8^{\cos^{-1}x}$$
41. Find the set of values of k for which $x^2 - kx + \sin^{-1}(\sin 4) > 0$, for all real x
42. If $A = 2\tan^{-1}(2\sqrt{2}-1)$ and
$$= 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right),$$
then prove that $A > B$
43. Prove that
$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right) + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}(\sqrt{2})\right\}\right\}$$
$= 0$
44. Find the domain of the function
$$f(x) = \sin^{-1}(\cos^{-1} x + \tan^{-1} x + \cot^{-1} x)$$
45. If $\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sin^{-1}\left(\sqrt{1-\frac{y}{4}}\right) + \tan^{-1} y = \frac{2\pi}{3}$
then find the maximum value of $(x^2 + y^2 + 1)$
46. Find the number of integral ordered pairs (x, y) satisfying the equation
$$\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{10}\right)$$
47. Let $\left[\cot\left(\sum_{k=1}^{10} \cot^{-1}(k^2+k+1)\right)\right] = \frac{a}{b}$
where a and b are co-prime, then find the value of $(a + b + 10)$.
48. If $p > q > 0$, $pr < -1 < qr$, then prove that
$$\tan^{-1}\left(\frac{p-q}{1+pq}\right) + \tan^{-1}\left(\frac{q-r}{1+qr}\right) + \tan^{-1}\left(\frac{r-p}{1+rp}\right) = \pi$$
49. Consider the equation
$$(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = a\pi^3$$
find the values of 'a' so that the given equation has a solution.
50. If the range of the function $f(x) = \cot^{-1}\left(\frac{x^2}{x^2+1}\right)$ is (a, b), find the value of $\left(\frac{b}{a}+2\right)$
51. If $\tan^{-1} y = 4 \tan^{-1} x$, $\left(|x| < \tan\left(\frac{\pi}{8}\right)\right)$, find y as an algebraic function of x and hence prove that $\tan\left(\frac{\pi}{8}\right)$ is a root of the equation $x^4 - 6x^2 + 1 = 0$
52. Prove that
$$\tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right) + \tan^{-1}\left(\sqrt{\frac{b(a+b+c)}{ac}}\right)$$
$$+ \tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right) = \pi, \text{ where } a, b, c > 0$$
53. Solve
$$\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5+4\cos 2\theta}\right)$$
54. Simplify
$$\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right)$$
55. Solve
$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \sin^{-1}\left(\frac{2x}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$
56. Prove that
$$\tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{xz}{yr}\right) + \tan^{-1}\left(\frac{xy}{zr}\right) = \frac{\pi}{2}.$$
where $x^2 + y^2 + z^2 = r^2$.
57. If $\sum_{r=1}^{10} \tan^{-1}\left(\frac{3}{9r^2+3r-1}\right) = \cot^{-1}\left(\frac{m}{n}\right)$
where m and n are co-prime, find the value of $(2m + n + 4)$
58. If the sum $\sum_{b=1}^{10}\sum_{a=1}^{10} \tan^{-1}\left(\frac{a}{b}\right) = m\pi$, then find the value of $(m + 4)$
59. Let $f(x) = \frac{1}{\pi}(\sin^{-1}x + \cos^{-1}x + \tan^{-1}x) + \frac{(x+1)}{x^2+2x+10}$
such that the maximum value of $f(x)$ is m, then find the value of $(104m - 90)$.
60. Let m be the number of solutions of
$$\sin(2x) + \cos(2x) + \cos x + 1 = 0 \text{ in}$$
$0 < x < \frac{\pi}{2}$ and
$$n = \sin\left[\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right) + \cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)\right]$$
then find the value of $(m^2 + n^2 + m + n + 4)$
61. Let $f(n) = \sum_{k=-n}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right) - \tan^{-1}(k)\right)$
such that $\sum_{n=2}^{10}(f(n) + f(n-1)) = a\pi$
then find the value of $(a = 1)$

Integer Type Questions

1. If the solution set of
$$\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3 \text{ is}$$
(a, b), where $a, b \in I$, then find $(b - a + 5)$

2. If $a \sin^{-1} x - b \cos^{-1} x = c$, such that the value of $a \sin^{-1} x + b \cos^{-1} x$ is $\frac{m\pi ab + c(a-b)}{a+b}, m \in N$, then find the value of $(m^2 + m + 2)$

3 If m is a root of $x^2 + 3x + 1 = 0$, such that the value of $\tan^{-1}(m)+\tan^{-1}\left(\frac{1}{m}\right)$ is $\frac{k\pi}{2}, k \in I$, then find the value of $(k + 4)$

4. Find the number of real solutions of
$$\sin^{-1}(x^2-2x+1)+\cos^{-1}(x^2-x)=\frac{\pi}{2}$$

5. Let $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$
If x satisfies the cubic equation
$ax^3 + bx^2 + cx + d = 0$, then find the value of $(b + c) - (a + d)$

6. Consider α, β, γ are the roots of $x^3 - x^2 - 3x + 4 = 0$ such that $\tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma = \theta$
If the positive value of tan (θ) is $\frac{p}{q}$, where p and q are natural numbers, then find the value of $(p + q)$

7. If M is the number of real solution of $\cos^{-1} x + \cos^{-1}(2x) + \pi = 0$ and N is the number of values of x satisfying the equation $\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$, then find the value of $M + N + 4$

8. Find the value of
$$4\cos\left[\cos^{-1}\left(\frac{1}{4}(\sqrt{6}-\sqrt{2})\right)-\cos^{-1}\left(\frac{1}{4}(\sqrt{6}+\sqrt{2})\right)\right]$$

9. Find the value of
$$5\cot\left(\sum_{k=1}^{5}\cot^{-1}(k^2+k+1)\right)$$

10. Let $3\sin^{-1}(\log_2 x)+\cos^{-1}(\log_2 y)=\frac{\pi}{2}$
and $\sin^{-1}(\log_2 x)+2\cos^{-1}(\log_2 y)=\frac{11\pi}{6}$
then find the value of $\left(\frac{1}{x^2}+\frac{1}{y^2}+2\right)$

11. If α and β are the roots of $x^2 + 5x - 44 = 0$, then find the value of cot $(\cot^{-1} \alpha + \cot^{-1} \beta)$

12. If x and y are positive integers satisfying $\tan^{-1}\left(\frac{1}{x}\right)+\tan^{-1}\left(\frac{1}{y}\right)=\tan^{-1}\left(\frac{1}{7}\right)$, then find the number of ordered pairs of (x, y)

Comprehensive Link Passages

In these questions, a passage (paragraph) has been given followed by questions based on each of the passage. You have to answer the questions based on the passage given.

Passage 1

Function	Domain	Co-domain
$\sin^{-1}x$	$[-1, 1]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\cos^{-1}x$	$[-1, 1]$	$[\pi, 2\pi]$
$\cot^{-1}x$	R	$[\pi, 2\pi]$

1. $\sin^{-1}(-x)$ is
   (a) $-\sin^{-1} x$
   (b) $p + \sin^{-1} x$
   (c) $2\pi - \sin^{-1} x$
   (d) $2\pi-\cos^{-1}\sqrt{1-x^2}, x>0$
2. If $f(x) = 3 \sin^{-1} x - 2 \cos^{-1} x$, then $f(x)$ is
   (a) even function
   (b) odd function
   (c) neither even nor odd
   (d) even as well as odd function.
3. The minimum value of $(\sin^{-1} x)^3 - (\cos^{-1} x)^3$ is
   (a) $-\frac{63\pi^3}{8}$ (b) $\frac{63\pi^3}{8}$
   (c) $\frac{125\pi^3}{32}$ (d) $-\frac{125\pi^3}{32}$
4. The value of $\sin^{-1} x + \cos^{-1} x$ is
   (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{2}$ (d) $\frac{7\pi}{2}$
5. If the co-domain of $\sin^{-1} x$ is $\left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right]$ such that $\sin^{-1}x+\cos^{-1}x=\frac{5\pi}{2}$, then the co-domain of $\cos^{-1} x$ is
   (a) $[4\pi, 5\pi]$ (b) $[3\pi, 4\pi]$
   (c) $[6\pi, 7\pi]$ (d) $[5\pi, 6\pi]$

Passage II

We know that corresponding to every bijection function

$f: A \to B$, there exist a bijection.
$g: B \to A$ defined by $g(y) = x$ if and only if $f(x) = y$

The function $g: B \to A$ is called the inverse of function $f: A \to B$ and is denoted by f^{-1}.

Thus, we have $f(x) = y \Rightarrow f^{-1}(y) = x$

We know that trigonometric functions are periodic functions and hence, in general all trigonometric functions are not bijectives.

Consequently, their inverse do not exist.

However, if we restrict their domains and co-domains, they we can make the bijectives and also we can find their inverse. Now, answer the following questions.

1. $\sin^{-1}(\sin\theta) = \theta$, for all θ belonging to

(a) $[0, \pi]$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(c) $\left[-\frac{\pi}{2}, 0\right]$ (d) None of these

2. $\cos^{-1}(\cos\theta) = \theta$, for all θ belonging to

(a) $[0, \pi]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) None of these

3. $\tan^{-1}(\tan\theta) = \theta$, for all θ belonging to

(a) $[0, \pi]$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (d) None of these

4. $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, for all θ belonging to

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$

(c) $[0, \pi]$ (d) $(0, \pi)$

5. $\sec^{-1}(\sec\theta) = \theta$, for all θ belonging to

(a) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (b) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$

(c) $(0, \pi)$ (d) None of these

6. $\sin^{-1}(\sin x) = x$, for all x belonging to

(a) R (b) φ

(c) $[-1,1]$ (d) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

7. The value of $\sin^{-1}(\sin 2) + \cos^{-1}(\cos 2)$ is

(a) 0 (b) $\frac{\pi}{2}$

(c) $-\frac{\pi}{2}$ (d) None of these

Passage III

Let $f(x) = \sin\{\cot^{-1}(x + 1)\} - \cos(\tan^{-1} x)$ and $a = \cos(\tan^{-1}(\sin(\cot^{-1} x)))$ and $b = \cos(2\cos^{-1} x + \sin^{-1} x)$

1. The value of x for which $f(x) = 0$ is

(a) –1/2 (b) 0 (c) 1/2 (d) 1

2. If $f(x) = 0$, then a^2 is equal to

(a) 1/2 (b) 2/3 (c) 5/9 (d) 9/5

3. If $a^2 = \frac{26}{51}$, then b^2 is equal to

(a) 1/25 (b) 24/25 (c) 25/26 (d) 50/51

Passage IV

Every bijective (one-one onto function)

$f: A \to B$ there exists a bijection

$g: B \to A$ is defined by $g(y) = x$

if and only if $f(x) = y$.

The function $g : B \to A$ is called the inverse of function $f : A \to B$ and is denoted by f^{-1}.

If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of that function.

1. The value of $\cos\{\tan^{-1}(\tan 2)\}$ is

(a) $1/\sqrt{5}$ (b) $-1/\sqrt{5}$ (c) cos 2 (d) –cos 2

2. If x takes negative permissible value then $\sin^{-1} x$ is

(a) $\cos^{-1}(\sqrt{1-x^2})$ (b) $-\cos^{-1}(\sqrt{1-x^2})$

(c) $\cos^{-1}(\sqrt{x^2-1})$ (d) $\pi - \cos^{-1}(\sqrt{1-x^2})$.

3. If $x + \frac{1}{x} = 2$, then the value of $\sin^{-1} x$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{2}$

Passage V

Let $\cos^{-1}x + (\sin^{-1}y)^2 = \frac{a\pi^2}{4}$ …(i)

and $\cos^{-1}x \cdot (\sin^{-1}y)^2 = \frac{\pi^2}{16}$ …(ii)

where $-1 \le x, y \le 1$. Then

1. The set of values of 'a' for which the equation (i) holds good is

(a) $\left(0, 2 + \frac{4}{\pi}\right)$ (b) $\left[0, 1 + \frac{4}{\pi}\right)$

(c) R (d) $\left[0, -1 + \frac{4}{\pi}\right)$

2. The set of values of 'a' for which equations (i) and (ii) posses solutions

(a) $(-\infty, 2] \cup [2, \infty)$ (b) $(-2, 2)$

(c) $\left[2, 1 + \frac{4}{\pi}\right]$ (d) R

3. The values of x and y, the system of equations (i) and (ii) posses solutions for integral values of 'a'

(a) $\left\{\cos\left(\frac{\pi^2}{4}\right), 1\right\}$ (b) $\left\{\cos\left(\frac{\pi^2}{4}\right), -1\right\}$

(c) $\left\{\cos\left(\frac{\pi^2}{4}\right), \pm 1\right\}$ (d) $\{(x, y): x \in R, y \in R\}$

Matrix Match
(For JEE-Advanced Examination Only)

Given below are matching type questions, with two columns (each having some items) each.

Each item of column I has to be matched with the items of column II, by encircling the correct match(es).

Note: An item of Column I can be matched with more than one items of Column II. All the items of Column II have to be matched.

1. **Match the following columns:**

Column I		Column II	
(A)	The principal value of $\sin^{-1}(\sin 20)$ is	(P)	$(20 - 6\pi)$
(B)	The principal value of $\sin^{-1}(\sin 10)$ is	(Q)	$(3\pi - 10)$
(C)	The principal value of $\cos^{-1}(\cos 10)$ is	(R)	$(4\pi - 10)$
(D)	The principal value of $\cos^{-1}(\cos 20)$ is	(S)	$(5\pi - 20)$

2. **Match the following columns:**

Column I		Column II	
(A)	The range of $f(x) = 3\sin^{-1} x + 2\cos^{-1} x$ is	(P)	$(0, \pi)$
(B)	The range of $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$ is	(Q)	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
(C)	The range of $f(x) = \sqrt{\sin^{-1}x + \pi}$ is	(R)	$\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$
(D)	The range of $f(x) = 2\tan^{-1}x + \sin^{-1}x + \sec^{-1}\left(\frac{1}{x}\right)$ is	(S)	$[0, \pi]$

3. **Match the following columns:**

Column I		Column II	
(A)	$\sin(\sin^{-1} x) = \sin^{-1}(\sin x)$, if	(P)	$-1 \le x \le 1$
(B)	$\cos(\cos^{-1} x) = \cos^{-1}(\cos x)$, if	(Q)	$0 \le x \le 1$
(C)	$\tan(\tan^{-1} x) = \tan^{-1}(\tan x)$, if	(R)	$0 < x < \pi$
(D)	$\cot(\cot^{-1} x) = \cot^{-1}(\cot x)$, if	(R)	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

4. **Match the following columns:**

Column I		Column II	
The value of			
(A)	$\sin^{-1}\left(\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$	(P)	$\frac{7\pi}{6}$
(B)	$\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(\frac{1}{2}\right)$	(Q)	$\frac{5\pi}{6}$
(C)	$\tan^{-1}(\sqrt{3}) + \cot^{-1}(-\sqrt{3})$	(R)	$\frac{\pi}{6}$
(D)	$\sin^{-1}\left(\frac{1}{2013}\right) + \cos^{-1}\left(\frac{1}{2013}\right)$	(S)	$\frac{\pi}{2}$

5. **Match the following columns:**

Column I		Column II	
(A)	The value of $\tan^{-1} + \tan^{-1} 2 + \tan^{-1} 3$ is	(P)	$\frac{3\pi}{4}$
(B)	The value of $\tan^{-1}1 + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$ is	(Q)	$\frac{\pi}{2}$
(C)	The value of $\tan^{-1}(9) + \tan^{-1}\left(\frac{5}{4}\right)$ is	(R)	π
(D)	The value of $2\tan^{-1}x - \tan^{-1}\left(\frac{2x}{1-x^2}\right), x > 1$ is	(S)	$-\frac{\pi}{2}$

6. **Match the following columns:**

Column I		Column II	
(A)	$(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ is maximum at	(P)	$x = \frac{1}{\sqrt{2}}$
(B)	$(\sin^{-1} x)^2 + (\cos^{-1} x)^2$ is minimum at	(Q)	$x = 1$
(C)	$(\sin^{-1} x) - (\cos^{-1} x)$ is minimum at	(R)	$x = -1$
(D)	$(\tan^{-1} x)^2 + (\cot^{-1} x)^2$ is minimum at	(S)	$x = 0$

Assertion and Reason

Codes:

(A) Both A and R are individually true and R is the correct explanation of A

(B) Both A and R are individually true and R is not the correct explanation of A.

(C) A is true but R is false.

(D) A is false but R is true.

1. *Assertion (A):* If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then the value of

$$(x^{2013} + y^{2013} + z^{2013}) - \frac{9}{(x^{2014} + y^{2014} + z^{2014})}$$ is zero.

Reason (R): Maximum value of $\sin^{-1} x$ is $\frac{\pi}{2}$

(a) A (b) B (c) C (d) D

2. *Assertion (A):* The value of $2\tan^{-1}x - \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ is π

Reason (R): $x > 1$

(a) A (b) B (c) C (d) D

3. *Assertion (A):* The value of

$$\tan^{-1}(p) + \tan^{-1}\left(\frac{1}{p}\right) \text{ is } -\frac{\pi}{2}$$

Reason (R): P is the root of $x^2 + 2013x + 2014 = 0$.

(a) A (b) B (c) C (d) D

4. *Assertion (A):*

If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$, then the value of $\cos^{-1} x + \cos^{-1} y$ is $\frac{\pi}{3}$

Reason (R): $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, when $x \in [-1, 1]$

(a) A (b) B (c) C (d) D

5. *Assertion (A):*

The value of $\cos^{-1} (\cos 10)$ is $(2\pi - 5)$

Reason (R):

The range of $\cos^{-1} x$ is $[0, \pi]$

(a) A (b) B (c) C (d) D

6. *Assertion (A):*

If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then $x^2 + y^2 + z^2 + 2xyz = 1$

Reason(R): For $-1 \le x, y, z \le 1$

(a) A (b) B (c) C (d) D

7. *Assertion (A):*

If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, then x is $\frac{1}{6}$

Reason (R): For $0 < 2x, 3x < 1$

(a) A (b) B (c) C (d) D

8. *Assertion (A):* $\cos\left(\frac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x$

Reason (R): for $x \ge 0$

(a) A (b) B (c) C (d) D

9. *Assertion (A):* $\sin^{-1} (3x - 4x^3) = \pi - 3\sin^{-1} x$

Reason (R): for $\frac{1}{2} < x \le 1$

(a) A (b) B (c) C (d) D

10. *Assertion (A):* $\cos^{-1} (4x^3 - 3x) = 2\pi - 3\cos^{-1} (x)$

Reason (R): For $-\frac{1}{2} \le x < \frac{1}{2}$

(a) A (b) B (c) C (d) D

11. *Assertion (A):* $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right), x > 0$

Reason (R): $\cot^{-1}(x) = \pi + \tan^{-1}\left(\frac{1}{x}\right), x < 0$

(a) A (b) B (c) C (d) D

12. *Assertion (A):*

If $\alpha. \beta$ are the roots of $x^2 - 3x + 2 = 0$, then $\sin^{-1} \alpha$ exists but not $\sin^{-1} \beta$, where $\alpha > \beta$

Reason (R): Domain of $\sin^{-1} x$ is $[-1, 1]$

(a) A (b) B (c) C (d) D

Questions asked In Previous Years' JEE-Advanced Examinations

1. Let a, b, c be positive real numbers such that

$$\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

Then $\tan\theta$ is equal to? **[IIT-JEE, 1981]**

2. The numerical value of

$$\tan^{-1}\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\} \text{ is ?}$$

[IIT-JEE, 1981]

3. Find the value of $\cos (2\cos^{-1} x + \sin^{-1} x)$ at $x = 1/5$, where $0 \le \cos^{-1} x \le \pi$ and $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

[IIT-JEE, 1981]

4. The value of $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(a) 6/17 (b) 17/6 (c) –17/6 (d) –6/17

[IIT-JEE, 1983]

5. No questions asked between 1984 and 1985.

6. The principal value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{3}$ (d) $\frac{5\pi}{3}$

[IIT-JEE, 1986]

7. No questions asked between 1987 and 1988.

8. The greater of the two angles $A = 2\tan^{-1}(2\sqrt{2} - 1)$ and $B = 3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$ is.......

[IIT-JEE, 1989]

9. No questions asked between 1990 and 1998.

10. The number of real solutions of

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2} \text{ is}$$

(a) 0 (b) 1 (c) 2 (d) ∞

[IIT-JEE, 1999]

11. No questions asked in 2000.

12. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - ....\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....\right) = \frac{\pi}{2}$,

for $0 < |x| < \sqrt{2}$, then x is

(a) 1/2 (b) 1 (c) –1/2 (d) –1

[IIT-JEE, 2001]

13. Prove that $\cos (\tan^{-1}(\sin (\cot^{-1} x))) = \sqrt{\frac{x^2+1}{x^2+2}}$

[IIT-JEE, 2002]

14. The domain of $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ is

(a) $\left(-\frac{1}{2}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{4}, \frac{3}{4}\right)$

(c) $\left[-\frac{1}{4}, \frac{1}{4}\right]$ (d) $\left[-\frac{1}{4}, \frac{1}{2}\right]$

[IIT-JEE, 2003]

15. If $\sin(\cot^{-1}(x+1)) = \cos(\tan^{-1} x)$, then the value of x is

(a) –1/2 (b) 1/2 (c) 0 (d) 9/4

[IIT-JEE, 2004]

16. No questions asked in between 2005 to 2006.

17. Match the following columns:

Let (x, y) be such that

$$\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$$

	Column I		Column II
(A)	If $a = 1$ and $b = 0$, then (x, y)	(P)	lies on the circle $x^2 + y^2 = 1$
(B)	If a = 1 and b = 1, then (x, y)	(Q)	lies on $(x^2 - 1)(y^2 - 1) = 0$
(C)	If $a = 1$ and $b = 2$, then (x, y)	(R)	lies on the line $y = x$
(D)	If $a = 2$ and $b = 2$, then (x, y)	(S)	lies on $(4x^2 - 1)(y^2 - 1) = 0$

[IIT-JEE, 2007]

18. If $0 < x < 1$, then

$\sqrt{1+x^2} \times [\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2 - 1]^{1/2}$ equals

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x

(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

[IIT-JEE, 2008]

19. No questions asked between 2009 to 2010.

20. Let $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is.......

[IIT-JEE, 2011]

21 No questions asked in 2012.

22 The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is

(a) 23/25 (b) 25/23 (c) 23/24 (d) 24/23

[IIT-JEE, 2013]

23. The value of

$$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1}y)+y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y)+\tan(\sin^{-1}y)}\right)^2+y^4\right)^{1/2}$$

is.........

[IIT-JEE, 2013]

24 If $\cot(\sin^{-1}\sqrt{1-x^2})=\sin(\tan^{-1}(x\sqrt{6})), x\neq 0$ then the value of x is.......

[IIT-JEE, 2013]

25. The number of positive solutions satisfying the equation

$$\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

is....

[IIT-JEE, 2014]

26. No questions asked in 2015.

Answers

Level II

1. (d)	2. (b)	3. (b)	4. (b)	5. (d)
6. (b)	7. (b)	8. (d)	9. (a)	10. (c)
11. (c)	12. (c)	13. (a, c)	14. (a)	15. (d)
16. (b)	17. (b)	18. (c)	19. (c)	20. (b)
21. (a)	22. (a)	23. (b)	24. (c, d)	25. (a)
26. (b)	27. (b)	28. (a)	29. (a)	30. (c)
31. (b)	32. (a)	33. (a)	34. (c)	35. (b)
36. (c)	37. (c)	38. (b)	39. (c)	40. (a)
41. (b)	42. (a)	43. (a)		

Level III

6. $(8\pi - 21)$

7. $\frac{\pi}{3}$

16. Maximum Value $\frac{7\pi^3}{8}$, when $x = -1$

and minimum Value = $\frac{\pi^3}{32}$, when $x = \frac{1}{\sqrt{2}}$

17 $x=\frac{1}{2}\sqrt{\frac{3}{7}}$

18. $x = 3$

19. $x=0, \frac{1}{2}, \frac{-1}{2}$

20. $x=\frac{3}{\sqrt{10}}$

21. $x = 2-\sqrt{3}, \sqrt{3}$

22. $x=\frac{a-b}{1+ab}$

23. $x = n^2 - n + 1, n$

24. $x=\frac{4}{3}$

25. $x = ab$

26. $x = \frac{1}{2}, y = 1$
27. $x = 1, y = 2$; $x = 2, y = 7$
29. $-\pi$
32. $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$
33. $x \in \left(-1, \frac{1}{\sqrt{2}}\right]$
34. $x \in (-\infty, \cot 3) \cup (\cot 2, \infty)$
35. $x \in \left(\frac{1}{\sqrt{2}}, 1\right) \cup \left(-1, -\frac{1}{\sqrt{2}}\right)$
36. $\tan\left(\frac{1}{2}\right) < x < \tan\left(\frac{3}{2}\right)$
37. $\cot(3) \le x \le \cot(1)$
38. $x \in (-1, 1)$
39. $n = 5$
40. $n = 8$
41. $k = \phi$
44. $\tan(\sin(\cos(\sin 1))) \le x < \tan(\sin(1))$.
45. $[1, \infty)$
49. $a \in \left[\frac{1}{32}, \frac{7}{8}\right]$
50. $x = y = \sqrt{a^2 + 1}$
53. $\theta \in n\pi + \tan^{-1}(-2), n \in Z$
54. θ
55. $x = \frac{1}{\sqrt{3}}$,
56. $\frac{\pi}{2}$
57. 36
58. 29
59. 4
60. 6
61. 100

Integer Type Questions

1. 7
2. 4
3. 3
4. $x = 1$
5. 3
6. 9
7. 5, where $M = 0, N = 1$
8. 2
9. 7
10. 8
11, 9
12. 6

Comprehensive Link Passages

Passage I:	1. (c)	2. (b)	3. (a)	4. (c)
	5. (a)			
Passage II:	1. (b)	2. (a)	3. (c)	4. (a)
	5. (a)	6. (c)	7. (a)	
Passage III:	1. (a)	2. (c)	3. (b)	
Passage IV:	1. (d)	2. (b)	3. (b)	
Passage V:	1. (b)	2. (c)	3. (c)	

Matrix Match

1.(A) → (P); (B) → (Q); (C) → (R); (D) → (P)
2.(A) → (S); (B) → (P); (C) → (Q); (D) → (R)
3.(A) → (P); (B) → (Q); (C) → (S); (D) → (R)
4.(A) → (Q); (B) → (R); (C) → (P); (D) → (S)
5.(A) → (R); (B) → (Q); (C) → (P); (D) → (R)
6.(A) → (R); (B) → (P); (C) → (Q); (D) → (Q)

Assertion and Reason

1. (a)	2. (a)	3. (a)	4. (a)	5. (a)
6. (a)	7. (a)	8. (a)	9. (a)	10. (a)
11. (b)	12. (a)			

Hints and Solutions

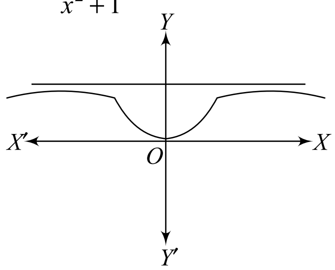
Level I

1. Given, $f(x) = 3x + 5$
$\Rightarrow$ $f'(x) = 3 > 0$
$\Rightarrow$ f is strictly increasing function.
$\Rightarrow$ f is one one function
Also, $R_f = R$ = Co-domain
$\Rightarrow$ f is onto function.
Thus, f is a bijective function.
Hence, f^{-1} is exists.
Let $y = 3x + 5$
$\Rightarrow$ $x = \frac{y-5}{3}$
Thus, $f^{-1}(x) = \frac{x-5}{3}$.

2. Given, $f(x) = x^2 + 2$
$\Rightarrow$ $f'(x) = 2x > 0$ for every $x > 0$
$\Rightarrow$ f is strictly increasing function.
$\Rightarrow$ f is one one function.
Also, $R_f = (2, \infty)$ = Co-domain
$\Rightarrow$ f is ont function.
Thus, f is a bijective function.
Therefore, the inverse of the given function exists.
Let $y = x^2 + 2$
$\Rightarrow$ $x^2 = y - 2$
$\Rightarrow$ $x = \sqrt{y-2}$

Hence, $f^{-1}(x) = \sqrt{x-2}$

3. Given, $f(x) = \dfrac{x^2}{x^2+1}$



$\Rightarrow \quad f(x) = 1 - \dfrac{1}{x^2+1}$

$\Rightarrow \quad f'(x) = \dfrac{2x}{(x^2+1)^2} > 0, \forall\ x \in R^+$

$\Rightarrow$ f is strictly increasing function.

$\Rightarrow$ f is a one one function.

Also, let $y = \dfrac{x^2}{x^2+1}$

$\Rightarrow \quad y \cdot x^2 + y = x^2$

$\Rightarrow \quad x^2(y-1) = -y$

$\Rightarrow \quad x^2 = -\dfrac{y}{(y-1)} = \dfrac{y}{(1-y)}$

$\Rightarrow \quad x = \sqrt{\dfrac{y}{(1-y)}}$

$\Rightarrow$ $R_f = (0, 1) =$ Co-domain

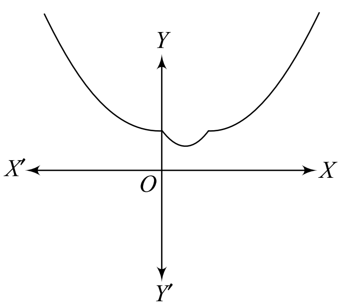
$\Rightarrow$ f is onto function.

Thus, f is a bijective function.

$\Rightarrow$ $f^{-1}(x)$ is exists.

Hence, $f^{-1}(x) = \sqrt{\dfrac{x}{1-x}}$.

4. Given, $f(x) = 2^{x(x-1)}$.



$\Rightarrow \quad f'(x) = 2^{x(x-1)} \times (2x-1) \times \log, 2 > 0$

for all x in $[1, \infty)$

$\Rightarrow$ f is strictly increasing function.

$\Rightarrow$ f is a one one function.

Also, $R_t = [1, \infty)$

$\Rightarrow$ $R_f = [1, \infty)$= Co-domain

$\Rightarrow$ f is onto function.

Thus, f is a bijective function.

So its inverse is exists.

Let $\quad y = 2^{x(x+1)}$

$\Rightarrow \quad y = 2^{x^2-x}$

$\Rightarrow \quad x^2 - x = \log_2(y)$

$\Rightarrow \quad x^2 - x - \log_2(y) = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1 + 4\log_2(y)}}{2}$

$\Rightarrow \quad x = \dfrac{1 + \sqrt{1 + 4\log_2(y)}}{2}$

Thus, $f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4\log_2(x)}}{2}$

5. Since f is a bijective function, so its inverse exists.

Let $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}} = \dfrac{10^{2x} - 1}{10^{2x} + 1}$.

$\Rightarrow \quad y \times 10^{2x} + y = 10^{2x} - 1$

$\Rightarrow \quad 10^{2x}(y-1) = -y - 1$

$\Rightarrow \quad 10^{2x} = -\dfrac{y+1}{y-1} = \dfrac{y+1}{1-y}$

$\Rightarrow \quad 2x = \log_{10}\left(\dfrac{y+1}{1-y}\right)$

$\Rightarrow \quad x = \dfrac{1}{2}\log_{10}\left(\dfrac{1+y}{1-y}\right)$

Thus, $f^{-1}(x) = \dfrac{1}{2}\log_{10}\left(\dfrac{1+x}{1-x}\right)$.

6. Given, $f(x) = x + \sin x$

$\Rightarrow$ $f'(x) = 1 + \cos x \geq 0$ for all x in R.

$\Rightarrow$ f is strictly increasing function

$\Rightarrow$ f is a one one function.

Also, the range of a function is R.

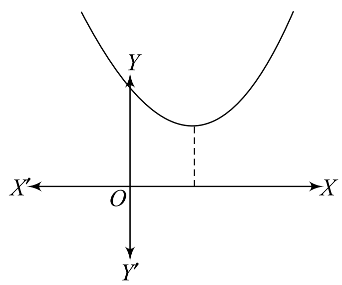
$\Rightarrow$ f is a onto function

Thus, f is a bijective function.

Hence, f^{-1} exists.

Therefore, $f^{-1}(x) = x - \sin x$

7. Given, $f(x) = x^2 - 4x + 9$



$\Rightarrow$ $f'(x) = 2x - 4 \geq 0$ for all x in D_f

$\Rightarrow$ f is strictly increasing function.

$\Rightarrow$ f is a one one function.

Also, $R_f = [5, \infty) =$ Co-domain

$\Rightarrow$ f is onto function.

Therefore, f is a bijective function.
Hence, its inverse is exists.
Let $y = x^2 - 4x + 3$

$\Rightarrow \quad x^2 - 4x + (5 - y) = 9$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{16 - 4(5 - y)}}{2}$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{4y + 16 - 20}}{2}$

$\Rightarrow \quad x = \dfrac{4 \pm \sqrt{4(y-1)}}{2} = 2 \pm \sqrt{(y-1)}$

$\Rightarrow \quad x = 2 + \sqrt{(y-1)}$, since $x \geq 2$

$\Rightarrow \quad f^{-1}(x) = 2 + \sqrt{(x-1)}$

8. Consider the function

$f : [0, \infty) \to \left[-\dfrac{1}{4}, \infty\right)$, where

$f(x) = x^2 - \dfrac{1}{4}$

Clearly, f is a one one and onto function.
So its inverse is exists.

Let its inverse be $f^{-1} : \left[-\dfrac{1}{4}, \infty\right) \to [0, \infty)$.

$\Rightarrow \quad f^{-1}(x) = \sqrt{x + \dfrac{1}{4}}$.

Consequently, we can say that, the two sides of the given equation are inverse to each other.
Thus, the intersection point is the solution of the given equation. $f(x) = x$

$\Rightarrow \quad x^2 - \dfrac{1}{4} = x$

$\Rightarrow \quad x^2 - x = \dfrac{1}{4}$

$\Rightarrow \quad \left(x - \dfrac{1}{2}\right)^2 = \dfrac{1}{2}$

$\Rightarrow \quad \left(x - \dfrac{1}{2}\right) = \pm \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad x = \dfrac{1}{2} \pm \dfrac{1}{\sqrt{2}}$

Hence, the solutions are

$$\left\{\frac{1}{2} + \frac{1}{\sqrt{2}}, \frac{1}{2} - \frac{1}{\sqrt{2}}\right\}$$

9. Clearly, f is bijective
So, its inverse exists
Let $y = 3x + 5$

$\Rightarrow \quad x = \dfrac{y-5}{3}$

Thus, $f^{-1}(x) = \dfrac{x-5}{3}$

10. Clearly, f is bijective
So, its inverse exists

Let $y = \dfrac{x}{x-1}$

$\Rightarrow \quad xy - y = x$

$\Rightarrow \quad x(y-1) = y$

$\Rightarrow \quad x = \dfrac{y}{(y-1)}$

Thus, $f^{-1}(x) = \dfrac{x}{(x-1)}$

11. Clearly, f is bijective
So its inverse exists
Let $y = x^2 + 1$

$\Rightarrow \quad x = \sqrt{y-1}$

Thus, $f^{-1}(x) = \sqrt{x-1}$

12. Since f is bijective, so its inverse exists

Let $y = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$

$\Rightarrow \quad \dfrac{y}{1} = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$

$\Rightarrow \quad \dfrac{y+1}{y-1} = \dfrac{2^x - 2^{-x} + 2^x + 2^{-x}}{2^x - 2^{-x} - 2^x - 2^{-x}}$

$\Rightarrow \quad \dfrac{y+1}{y-1} = -\dfrac{2.2^x}{2.2^{-x}}$

$\Rightarrow \quad 2^{2x} = \dfrac{y+1}{1-y}$

$\Rightarrow \quad 2x = \log_2\left(\dfrac{y+1}{1-\mathrm{y}}\right)$

$\Rightarrow \quad x = \dfrac{1}{2}\log_2\left(\dfrac{y+1}{1-\mathrm{y}}\right)$

Thus, $f^{-1}(x) = \dfrac{1}{2}\log_2\left(\dfrac{x+1}{1-x}\right)$

13. Clearly, f is bijective.
So, its inverse exists

Let $y = \dfrac{x}{x^2+1}$

$\Rightarrow \quad yx^2 - x + y = 0$

$\Rightarrow \quad x = \dfrac{1 \pm \sqrt{1 - 4y^2}}{2y}$

Thus, $f^{-1}(x) = \dfrac{1 + \sqrt{1 - 4x^2}}{2x}$

14. We have

$-1 \leq 3x + 5 \leq 1$

$\Rightarrow \quad -6 \leq 3x \leq -1$

$\Rightarrow \quad -2 \leq x \leq -\dfrac{4}{3}$

$\Rightarrow \quad D_f = x \in \left[-2, -\frac{4}{3}\right]$

15. We have

$$-1 \le \frac{x}{x+1} \le 1$$

Case I: When $\frac{x}{x+1} \le 1$

$\Rightarrow \quad \frac{x}{x+1} - 1 \le 0$

$\Rightarrow \quad \frac{-1}{x+1} \le 0$

$\Rightarrow \quad \frac{1}{x+1} \ge 0$

$\Rightarrow \quad x > -1$

Case II: When $\frac{x}{x+1} \ge -1$

$\Rightarrow \quad \frac{x}{x+1} + 1 \ge 0$

$\Rightarrow \quad \frac{2x+1}{x+1} \ge 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup \left[-\frac{1}{2}, \infty\right)$

Hence, $D_f = \left[-\frac{1}{2}, \infty\right)$.

16. We have

$$-1 \le \frac{x^2+1}{2x} \le 1$$

$\Rightarrow \quad \left|\frac{x^2+1}{2x}\right| \le 1$

$\Rightarrow \quad \frac{|x^2+1|}{|2x|} \le 1$

$\Rightarrow \quad \frac{|x^2+1|}{2|x|} \le 1$

$\Rightarrow \quad x^2 + 1 \le 2|x|$

$\Rightarrow \quad |x|^2 - 2|x| + 1 \le 0$

$\Rightarrow \quad (|x| - 1)^2 \le 0$

$\Rightarrow \quad (|x| - 1)^2 = 0$

$\Rightarrow \quad (|x| - 1) = 0$

$\Rightarrow \quad |x| = 1$

$\Rightarrow \quad x = \pm 1$

Hence, $D_f = \{-1, 1\}$

17. We have $-1 \le \frac{|x|-1}{2} \le 1$

$\Rightarrow \quad -2 \le |x| - 1 \le 2$

$\Rightarrow \quad -1 \le |x| \le 3$

$\Rightarrow \quad |x| \le 3$ ({ $|x|$ ³ –1 is rejected)

$\Rightarrow \quad -3 \le x \le 3$

Hence, D_f [–3, 3]

18. We have $-1 \le (\log_2 x) \le 1$

$\Rightarrow \quad 2^{-1} \le x \le 2^1$

$\Rightarrow \quad \frac{1}{2} \le x \le 2$

Hence, $D_f = \left[\frac{1}{2}, 2\right]$

19. We have

$$-1 \le \log_3 x^2 \le 1$$

$\Rightarrow \quad 4^{-1} \le x^2 \le 4^1$

$\Rightarrow \quad \frac{1}{4} \le x^2 \le 4$

$\Rightarrow \quad \frac{1}{2} \le |x| \le 2$

$\Rightarrow \quad |x| \le 2$ and $|x| \ge \frac{1}{2}$

$\Rightarrow \quad -2 \le x \le 2$ and $x \ge \frac{1}{2}$ and $x \le -\frac{1}{2}$

$\Rightarrow \quad x \in \left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$

Hence, $D_f = \left[-2, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 2\right]$

20. Given, $\sin^{-1} x + \sin^{-1} y = \pi$

It is possible only when each term of the given equation provides the maximum value.

Thus, $\sin^{-1} x = \frac{\pi}{2}$ and $\sin^{-1} y = \frac{\pi}{2}$

$\Rightarrow \quad x = \sin\left(\frac{\pi}{2}\right) = 1$ and $y = \sin\left(\frac{\pi}{2}\right) = 1$

Hence, the solutions are $x = 1$ and $y = 1$.

21. Given $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value.

Thus, $\sin^{-1} x = \frac{\pi}{2}, \sin^{-1} y = \frac{\pi}{2}$

and $\quad \sin^{-1} z = \frac{\pi}{2}$

$\Rightarrow \quad x = 1, y = 1$ and $z = 1$

Hence, the value of

$$x^{2013} + y^{2013} + z^{2013} - \frac{9}{x^{2014} + y^{2014} + z^{2014}}$$

$= 1 + 1 + 1 - \frac{9}{1+1+1}$

$= 3 - 3$

$= 0$

22. We have $-\frac{\pi}{2} \le \sin^{-1}(3x+5) \le \frac{\pi}{2}$

$\Rightarrow \quad -\pi \le 2\sin^{-1}(3x+5) \le \pi$

$\Rightarrow \quad -\pi + \frac{\pi}{4} \le 2\sin^{-1}(3x+5) + \frac{\pi}{4} \le \pi + \frac{\pi}{4}$

$\Rightarrow \quad -\frac{3\pi}{4} \le f(x) \le \frac{5\pi}{4}$

Hence, $R_f = \left[-\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

23. We have $\sin^{-1} x > \sin^{-1} (3x - 1)$

$\Rightarrow \quad x > (3x - 1)$

$\Rightarrow \quad 2x - 1 < 0$

$\Rightarrow \quad x < \frac{1}{2}$

$\Rightarrow \quad x \in \left[-1, \frac{1}{2}\right)$

24. We have $-1 \le 2x + 4 \le 1$

$\Rightarrow \quad -5 \le 2x \le -3$

$\Rightarrow \quad -\frac{5}{2} \le x \le -\frac{3}{2}$

Hence, $D_f = \left[-\frac{5}{2}, -\frac{3}{2}\right]$

25. We have $0 \le \cos^{-1} (3x + 5) \le \pi$

$\Rightarrow \quad 0 \le 2 \cos^{-1} (3x + 5) \le 2\pi$

$\Rightarrow \quad \frac{\pi}{4} \le 2 \cos^{-1}(3x + 5) + \frac{\pi}{4} \le 2\pi + \frac{\pi}{4}$

Hence, $R_f = \left[\frac{\pi}{4}, \frac{9\pi}{4}\right]$

26. We have $\frac{\pi}{2} \le \cos^{-1}(-x^2) \le \pi$

$\Rightarrow \quad \frac{3\pi}{2} \le 3\cos^{-1}(-x^2) \le 3\pi$

$\Rightarrow \quad \frac{3\pi}{2} - \frac{\pi}{2} \le 3 \cos^{-1}(-x^2) - \frac{\pi}{2} \le 3\pi - \frac{\pi}{2}$

$\Rightarrow \quad \pi \le f(x) \le \frac{5\pi}{2}$

Hence, $R_f = \left[\pi, \frac{5\pi}{2}\right]$.

27. Given, $\cos^{-1} x + \cos^{-1} x^2 = 0$

It is possible only when each term will provide us the minimum value.

So, $\cos^{-1} x + \cos^{-1} x^2 = 0$

$\Rightarrow \quad x = 1$ and $x^2 = 1$

$\Rightarrow \quad x = 1$ and $x = \pm 1$

Hence, the solution is $x = 1$.

28. Given, $[\sin^{-1} x] + [\cos^{-1} x]$ and $x \ge 0$

$\Rightarrow \quad [\sin^{-1} x] = 0$ and $[\cos^{-1} x] = 0$

$\Rightarrow \quad x \in [0, \sin 1)\ \&\ x \in (\cos 1, 1]$

$\Rightarrow \quad x \in (\cos 1, \sin 1)$

29. We have $-1 \le \frac{x^2}{x^2 + 1} \le 1$

$\Rightarrow \quad \left|\frac{x^2}{x^2 + 1}\right| \le 1$

$\Rightarrow \quad \frac{|x^2|}{|x^2 + 1|} \le 1$

$\Rightarrow \quad \frac{x^2}{x^2 + 1} \le 1$

$\Rightarrow \quad x^2 + 1 \ge x^2$

$\Rightarrow \quad 1 > 0$

Hence, $x \in R$

30. We have $\cos^{-1} (x) > \cos^{-1} (x^2)$

$\Rightarrow \quad x < x^2$

$\Rightarrow \quad x^2 - x > 0$

$\Rightarrow \quad x(x - 1) > 0$

$\Rightarrow \quad x \in [-1, 0)$

31. Since $\tan^{-1} x$ is defined for all real values of x, so

$9 - x^2 \le 0$

$\Rightarrow \quad x^2 - 9 \le 0$

$\Rightarrow \quad (x + 3)(x - 3) \le 0$

$\Rightarrow \quad -3 \le x \le 3$

Hence, $D_f = [-3, 3]$

32. We have $-\frac{\pi}{2} \le \tan^{-1}(1 - x^2) \le \frac{\pi}{4}$

$\Rightarrow \quad -\pi \le 2 \tan^{-1}(1 - x^2) \le \frac{\pi}{2}$

$\Rightarrow \quad -\pi + \frac{\pi}{6} \le 2 \tan^{-1}(1 - x^2) + \frac{\pi}{6} \le \frac{\pi}{2} + \frac{\pi}{6}$

$\Rightarrow \quad -\frac{5\pi}{6} \le f(x) \le \frac{2\pi}{3}$

Hence, $R_f = \left[-\frac{5\pi}{6}, \frac{2\pi}{3}\right]$

33. We have $f(x) = \cot^{-1} (2x - x^2)$

$\Rightarrow \quad f(x) = \cot^{-1} (1 - (x^2 - 2x + 1))$

$\Rightarrow \quad f(x) = \cot^{-1} (1 - (x - 1)^2)$

Since $(1 - (x -)^2) \le 1$ and $0 \le \cot^{-1} x \le \pi$ and $\cot^{-1} x$ is strictly decreasing function so,

$\cot^{-1} (1) \le \cot^{-1} (1 - (x - 1)^2) \le \cot^{-1} (0)$

$\Rightarrow \quad \frac{\pi}{4} \le f(x) \le \pi$

Hence, $R_f = \left[\frac{\pi}{4}, \pi\right]$

where [,] = GIF

34. We have $[\cot^{-1} x] + [\cos^{-1} x] = 0$

$\Rightarrow \quad [\cot^{-1} x] = 0\ \&\ [\cos^{-1} x] = 0$

$\Rightarrow \quad 0 \le \cot^{-1} x < 1\ \&\ 0 \le \cos^{-1} x < 1$

$\Rightarrow \quad x \in (\cot 1, \infty)\ \&\ x \in (\cos 1, 1]$

$\Rightarrow \quad x \in (\cot 1, 1]$

35. We have $\sin \{x\} = \cos \{x\}, \forall x \in [0, 2\pi]$

$\Rightarrow \quad \tan \{x\} = 1$

$\Rightarrow \quad \{x\} = \tan^{-1}(1) = \frac{\pi}{4}$

Hence, the number of solutions = 6

(Since $\{x\}$ is a periodic function with period 1, it has one solution between 0 to 1. So, there is six solutions between 0 to 6.28).

36. We have,

$$-1 \le \left(\frac{|x| - 2}{3}\right) \le 1$$

$\Rightarrow\quad -3 \le |x| - 2 \le 3$

$\Rightarrow\quad -1 \le |x| \le 5$

$\Rightarrow\quad -5 \le x \le 5 \quad \ldots\text{(i)}$

Also, $-1 \le \left(\dfrac{1-|x|}{4}\right) \le 1$

$\Rightarrow\quad -4 \le 1 - |x| \le 4$

$\Rightarrow\quad -4 \le |x| - 1 \le 4$

$\Rightarrow\quad -3 \le |x| \le 5$

$\Rightarrow\quad -5 \le x \le 5 \quad \ldots\text{(ii)}$

From (i) and (ii), we get

$-5 \le x \le 5$

Thus, $D_f = [-5, 5]$

37. Given, $f(x) = \sin^{-1}(2x^2 - 1)$

So, $-1 \le (2x^2 - 1) \le 1$

$\Rightarrow\quad 0 \le 2x^2 \le 2$

$\Rightarrow\quad 0 \le x^2 \le 1$

$\Rightarrow\quad 0 \le |x| \le 1$

$\Rightarrow\quad -1 \le x \le 1$

Thus, $D_f = [-1, 1]$

38. Given, $f(x) = \sqrt{5\pi \sin^{-1} x - 6(\sin^{-1} x)^2}$

We have $5\pi \sin^{-1} x - 6(\sin^{-1} x) \ge 0$

$\Rightarrow\quad (5\pi - 6(\sin^{-1} x)) \sin^{-1} x \ge 0$

$\Rightarrow\quad (6(\sin^{-1} x) - 5\pi) \sin^{-1} x \le 0$

$\Rightarrow\quad 0 \le \sin^{-1} x \le \dfrac{5\pi}{6}$

$\Rightarrow\quad 0 \le x \le \sin\left(\dfrac{5\pi}{6}\right)$

$\Rightarrow\quad 0 \le x \le \dfrac{1}{2}$

Also, $-1 \le x \le 1$

Thus, $D_f = \left[0, \dfrac{1}{2}\right]$

39. Given, $f(x) = \log_2\left(\dfrac{3\tan^{-1} x + \pi}{\pi - 4\tan^{-1} x}\right)$

So, $\dfrac{3\tan^{-1} x + \pi}{\pi - 4\tan^{-1} x} > 0$

$\Rightarrow\quad \dfrac{3\tan^{-1} x + \pi}{4\tan^{-1} x - \pi} < 0$

$\Rightarrow\quad -\dfrac{\pi}{3} < \tan^{-1} x < \dfrac{\pi}{4}$

$\Rightarrow\quad \tan\left(-\dfrac{\pi}{3}\right) < x < \tan\left(\dfrac{\pi}{4}\right)$

$\Rightarrow\quad -\sqrt{3} < x < 1$

Hence, $D_f = (-\sqrt{3}, 1)$

40. Given, $f(x) = \cos^{-1}\left(\dfrac{3}{2 + \sin x}\right)$

$\Rightarrow\quad -1 \le \left(\dfrac{3}{2 + \sin x}\right) \le 1$

$\Rightarrow\quad -1 \le \left(\dfrac{2 + \sin x}{3}\right) \le 1$

$\Rightarrow\quad -3 \le (2 + \sin x) \le 3$

$\Rightarrow\quad -5 \le (\sin x) \le 1$

$\Rightarrow\quad -1 \le (\sin x) \le 1$

$\Rightarrow\quad \sin^{-1}(-1) \le x \le \sin^{-1}(1)$

$\Rightarrow\quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

Thus, $D_f = \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

41. Given, $f(x) = \sin^{-1}\left(\dfrac{x^2 + 1}{2x}\right)$

$\Rightarrow\quad -1 \le \left(\dfrac{x^2 + 1}{2x}\right) \le 1$

$\Rightarrow\quad \left|\dfrac{x^2 + 1}{2x}\right| \le 1$

$\Rightarrow\quad \dfrac{x^2 + 1}{2|x|} \le 1$

$\Rightarrow\quad x^2 + 1 \le 2|x|$

$\Rightarrow\quad |x|^2 + 1 \le 2|x|$

$\Rightarrow\quad |x|^2 - 2|x| + 1 \le 0$

$\Rightarrow\quad (|x| - 1) \le 0$

$\Rightarrow\quad (|x| - 1) = 0$

$\Rightarrow\quad x = \pm 1$

Thus, $D_f = \{-1, 1)$

42. Given, $f(x) = \cos^{-1}\left(\dfrac{x^2 + 1}{x^2}\right)$

So, $-1 \le \left(\dfrac{x^2 + 1}{x^2}\right) \le 1$

$\Rightarrow\quad -1 \le \left(1 + \dfrac{1}{x^2}\right) \le 1$

$\Rightarrow\quad -2 \le \left(\dfrac{1}{x^2}\right) \le 0$

which is not true

Hence, $D_f = \varphi$

43. Given, $f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4))$

So, $-1 \le \log_2(x^2 + 3x + 4) \le 1$

$\Rightarrow\quad \dfrac{1}{2} \le (x^2 + 3x + 4) \le 2$

when $(x^2 + 3x + 4) \le 2$

$\Rightarrow\quad (x^2 + 3x + 2) \le 0$

$\Rightarrow\quad (x + 1)(x + 2) \le 0$

$\Rightarrow\quad -2 \le x \le -1$

when $x^2 + 3x + 4 \ge \dfrac{1}{2}$

$\Rightarrow\quad 2x^2 + 6x + 7 \ge 0$

Clearly, $D < 0$

So, it is true for all R

Hence, $D_f = [-2, -1]$

44. Given, $f(x)=\sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$

So, $-1 \le \log_2\left(\frac{x^2}{2}\right) \le 1$

$\Rightarrow \quad \frac{1}{2} \le \left(\frac{x^2}{2}\right) \le 2$

$\Rightarrow \quad 1 \le x^2 \le 4$

$\Rightarrow \quad 1 \le |x| \le 2$ …(i)

when $|x| \le 2$

$\Rightarrow \quad -2 \le x \le 2$

when $|x| \ge 1$

$\Rightarrow \quad x \ge 1$ and $x \le -1$ …(ii)

From (i) and (ii), we get,

$x \in [-2, -1] \cup [1, 2]$

Thus, $D_f = [-2, -1] \cup [1, 2]$

45. Given, $f(x) = \sin^{-1}[2 - 3x^2]$

$\Rightarrow \quad -1 \le [2 - 3x^2] \le 1$

when $[2 - 3x^2] \ge 1$

$\Rightarrow \quad 1 \le 2 - 3x^2 < 2$

$\Rightarrow \quad -2 < 3x^2 - 2 \le 1$

$\Rightarrow \quad 0 < 3x^2 \le 3$

$\Rightarrow \quad 0 < x^2 \le 1$

$\Rightarrow \quad 0 < |x| \le 1$

$\Rightarrow \quad -1 \le x \le 1 - \{0\}$ …(i)

Also, when $[2 - x^2] \ge -1$

$\Rightarrow \quad 2 - x^2 \ge -1$

$\Rightarrow \quad x^2 \le 3$

$\Rightarrow \quad |x| \le \sqrt{3}$

$\Rightarrow \quad -\sqrt{3} \le x \le \sqrt{3}$ …(ii)

Hence, $D_f = [-1, 0) \cup (0, 1]$

46. Given, $f(x)=\frac{1}{x}+3^{\sin^{-1}x}+\frac{1}{\sqrt{x-2}}$

Let $D_1 = R - \{0\}$

$D_2 = [-1, 1]$

and $D_3 = (2, \infty)$

Thus, $D_f = D_1 \cap D_2 \cap D_3 = [-1, 1]$

47. Given, $f(x) = \sin^{-1}(\log_2 x^2)$

So, $-1 \le \log_2(x^2) \le 1$

$\Rightarrow \quad \frac{1}{2} \le (x^2) \le 2$

$\Rightarrow \quad \frac{1}{\sqrt{2}} \le |x| \le \sqrt{2}$

Thus, $x \in \left[-\sqrt{2}, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$

Hence, $D_f = \left[-\sqrt{2}, -\frac{1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$

48. Given, $f(x)=e^x+\sin^{-1}\left(\frac{x}{2}-1\right)+\frac{1}{x}$

Let $D_1 = R$

$D_2 : -1 \le \left(\frac{x}{2}-1\right) \le 1$

$\Rightarrow \quad 0 \le \left(\frac{x}{2}\right) \le 2$

$\Rightarrow \quad 0 \le x \le 4$

and $D_3 = R - \{0\}$

Hence, $D_f = D_1 \cap D_2 \cap D_3 = (0, 4]$

49. Given, $f(x)=\sqrt{\sin^{-1}(\log_x 2)}$

We have $\sin^{-1}(\log_x 2) \ge 0$

$\Rightarrow \quad (\log_x 2) \ge \sin(0) = 0$

$\Rightarrow \quad 2 \ge x^0 = 1$

$\Rightarrow \quad 2 > 1$

which is true for all x in R

Also, $x \ne 1$ and $x > 0$

Furthermore, $-1 \le \log_x 2 \le 1$ which is also true for $x \ne 1$ and $x > 0$

Hence, $D_f = (0, 1) \cup (1, \infty)$

50. Given, $f(x)=\sqrt{\sin^{-1}(\log_2 x)}$

We have $\sin^{-1}(\log_2 x) \ge 0$

$\Rightarrow \quad (\log_2 x) \ge 0$

$\Rightarrow \quad x \ge 1$

Also, $-1 \le \log_2 x \le 1$

$\Rightarrow \quad \frac{1}{2} \le x \le 2$

Hence, $D_f = [1, 2]$

51. Given, $f(x) = \sin^{-1}(2x - 3)$

$$R_f=\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

52. Given, $f(x)=2\sin^{-1}(2x-1)-\frac{\pi}{4}$

$$-\frac{\pi}{2} \le \sin^{-1}(2x-1) \le \frac{\pi}{2}$$

$$-\pi \le 2\sin^{-1}(2x-1) \le \pi$$

$$-\pi-\frac{\pi}{4} \le 2\sin^{-1}(2x-1)-\frac{\pi}{4} \le \pi-\frac{\pi}{4}$$

$$-\frac{5\pi}{4} \le \left(2\sin^{-1}(2x-1)-\frac{\pi}{4}\right) \le \frac{3\pi}{4}$$

$$-\frac{5\pi}{4} \le f(x) \le \frac{3\pi}{4}$$

So, $R_f=\left[-\frac{5\pi}{4},\frac{3\pi}{4}\right]$

53. Given, $f(x) = 2\cos^{-1}(-x^2) - \pi$

$= 2(\pi - \cos^{-1}(x^2)) - \pi$

$= \pi - 2\cos^{-1}(x^2)$

54. Given, $f(x)=\frac{1}{2}\tan^{-1}(1-x^2)-\frac{\pi}{4}$

Now, $-\infty < (1 - x^2) \le 1$

$\Rightarrow \quad \tan^{-1}(-\infty) < \tan^{-1}(1 - x^2) \le \tan^{-1}(1)$

$\Rightarrow \quad -\frac{\pi}{2} < \tan^{-1}(1-x^2) \le \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) \le \frac{\pi}{8}$

$\Rightarrow \quad -\frac{\pi}{4} - \frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4} \le \frac{\pi}{8} - \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{2} < f(x) \le -\frac{\pi}{8}$

So, $R_f = \left(-\frac{\pi}{2}, -\frac{\pi}{8}\right]$

55. Given, $f(x) = \cot^{-1}(2x - x^2)$
$= \cot^{-1}(1 - (x-1)^2)$
Clearly, $-\infty < (1 - (x-1)^2) \le 1$
$\Rightarrow \quad \cot^{-1}(1) \le \cot^{-1}((1-(x-1)^2)) \le \cot^{-1}(-\infty)$
$\Rightarrow \quad \frac{\pi}{4} \le \cot^{-1}((1-(x-1)^2)) \le \pi$
$\Rightarrow \quad \frac{\pi}{4} \le f(x) \le \pi$
So, $R_f = \left[\frac{\pi}{4}, \pi\right]$

56. Given, $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$
$D_f = [-1, 1]$
So, $R_f = [f(-1), f(1)]$
$= \left[\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}\right]$
$= \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

57. Given, $f(x) = \sin^{-1} x + \sec^{-1} x + \tan^{-1} x$
Thus, $D_f = \{-1, 1\}$
So, $R_f = \{f(-1), f(1)\}$
$= \left\{-\frac{\pi}{2} + 0 - \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}\right\}$
$= \left\{-\frac{3\pi}{4}, \frac{3\pi}{4}\right\}$

58. Given, $f(x) = 3\cot^{-1}x + 2\tan^{-1}x + \frac{\pi}{4}$
$= 2(\tan^{-1}x + \cot^{-1}x) + \cot^{-1}x + \frac{\pi}{4}$
$= 2 \times \frac{\pi}{2} + \cot^{-1}x + \frac{\pi}{4}$
$= \cot^{-1}x + \frac{5\pi}{4}$
Thus, $0 \le \cot^{-1} x \le \pi$
$\Rightarrow \quad 0 + \frac{5\pi}{4} \le \cot^{-1}x + \frac{5\pi}{4} \le \pi + \frac{5\pi}{4}$
$\Rightarrow \quad \frac{5\pi}{4} \le f(x) \le \frac{9\pi}{4}$
So, $R_f = \left[\frac{5\pi}{4}, \frac{9\pi}{4}\right]$

59. Given, $f(x) = \text{cosec}^{-1}[1 + \sin^2 x]$.
Clearly,
$1 \le (1 + \sin^2 x) \le 2$
$R_f = [\text{cosec}^{-1}(2), \text{cosec}^{-1}(1)]$
$= \left[\sin^{-1}\left(\frac{1}{2}\right), \sin^{-1}(1)\right]$
$= \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

60. Given, $f(x) = \sin^{-1}(\log_2(x^2 + 3x + 4))$
Clearly, $D_f = [-2, -1]$
Thus, $R_f = [f(-2), f(-1)]$
$= \left[\frac{\pi}{2}, \frac{\pi}{2}\right] = \left\{\frac{\pi}{2}\right\}$

61. We have, $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$ is defined only when $-1 \le x \le 1$
Now, $f(1) = \sin^{-1}(1) + \cos^{-1}(1) + \tan^{-1}(1)$
$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$
and $f(-1) = \sin^{-1}(-1) + \cos^{-1}(-1) + \tan^{-1}(-1)$
$= -\frac{\pi}{2} + \pi - \frac{\pi}{4} = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$
Thus, $R_f = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

62. We have $4\sin^{-1}(x-1) + \cos^{-1}(x-2) = \pi$
$\Rightarrow \quad 3\sin^{-1}(x-2) + \frac{\pi}{2} = \pi$
$\Rightarrow \quad 3\sin^{-1}(x-2) = \frac{\pi}{2}$
$\Rightarrow \quad \sin^{-1}(x-2) = \frac{\pi}{6}$
$\Rightarrow \quad (x-2) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
$\Rightarrow \quad x = 2 + \frac{1}{2} = \frac{5}{2}$
Hence, the solution is $x = \frac{5}{2}$

63. As we know that, if $\sin^{-1}(f(x)) + \cos^{-1}(g(x)) = \frac{\pi}{2}$, then
$f(x) = g(x)$
$\Rightarrow \quad (x^2 - 2x + 1) = (x^2 - x)$
$\Rightarrow \quad 2x - x = 1$
$\Rightarrow \quad x = 1$
Hence, the solution is $x = 1$

64. We have
$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$
$\Rightarrow \quad \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right) + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$
$\Rightarrow \quad \left(\frac{1}{\sqrt{x^2+x+1}}\right) = \sqrt{x^2+x+1}$
$\Rightarrow \quad x^2 + x + 1 = 1$
$\Rightarrow \quad x^2 + x = 0$
$\Rightarrow \quad x(x+1) = 0$

$\Rightarrow \quad x = 0$ and -1

Hence, the number of solutions is 2

65. As we know that, if $\sin^{-1}(f(x)) + \cos^{-1}(g(x)) = \frac{\pi}{2}$, then

$f(x) = g(x)$

$\Rightarrow \quad \left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\right) = \left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right)$

$\Rightarrow \quad x\left(1 - \frac{x}{2} + \frac{x^2}{4} - \ldots\right) = x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{4} - \ldots\right)$

$\Rightarrow \quad x\left(\frac{1}{1 + \frac{x}{2}}\right) = x^2\left(\frac{1}{1 + \frac{x^2}{2}}\right)$

$\Rightarrow \quad \left(\frac{2x}{x+2}\right) = \left(\frac{2x^2}{x^2+2}\right)$

$\Rightarrow \quad x\left\{\left(\frac{1}{x+2}\right) - \left(\frac{x}{x^2+2}\right)\right\} = 0$

$\Rightarrow \quad x = 0$ and $\left(\frac{1}{x+2}\right) = \left(\frac{x}{x^2+2}\right)$

$\Rightarrow \quad x =$ and $x = 1$

66. We have $\sin^{-1} x > \cos^{-1} x$

$\Rightarrow \quad 2\sin^{-1} x > \sin^{-1} x + \cos^{-1} x$

$\Rightarrow \quad 2\sin^{-1} x > \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1} x > \frac{\pi}{4}$

$\Rightarrow \quad x > \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad x > \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

67. $(\sin^{-1} x)^2 - 3\sin^{-1} x + 2 = 0$

$\Rightarrow \quad (\sin^{-1} x - 1)(\sin^{-1} x - 2) = 0$

$\Rightarrow \quad (\sin^{-1} x - 1) = 0, (\sin^{-1} x - 2) = 0$

$\Rightarrow \quad \sin^{-1} x = 1, 2$

$\Rightarrow \quad \sin^{-1} x = 1$

$\Rightarrow \quad x \sin(1)$

68. Given equation is $\sin^{-1}x + \sin^{-1}2y = \pi$. It is possible only when

$\Rightarrow \quad \sin^{-1}x = \frac{\pi}{2}, \sin^{-1}(2y) = \frac{\pi}{2}$

$\Rightarrow \quad x = 1, 2y = 1$

$\Rightarrow \quad x = 1, y = \frac{1}{2}$

69. Given equation is $\cos^{-1} x + \cos^{-1} x^2 = 2\pi$. It is possible only when

$\Rightarrow \quad \cos^{21} x = \pi, \cos^{-1}(x^2) = \pi$

$\Rightarrow \quad x = -1, x^2 = -1$

$\Rightarrow \quad x = \varphi$

70. Given equation is $\cos^{-1} x + \cos^{-1} x^2 = 0$

It is possible only when

$\Rightarrow \quad \cos^{-1} x = 0, \cos^{-1}(x^2) = 0$

$\Rightarrow \quad x = 1$ and $x^2 = 1$

$\Rightarrow \quad x = 1$

71. Given equation is

$4\sin^{-1}(x - 1) + \cos^{-1}(x - 1) = \pi$

$\Rightarrow \quad 3\sin^{-1}(x-1) + \frac{\pi}{2} = \pi$

$\Rightarrow \quad 3\sin^{-1}(x-1) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(x-1) = \frac{\pi}{6}$

$\Rightarrow \quad (x-1) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$\Rightarrow \quad x = \frac{3}{2}$

Hence, the solution is $x = \frac{3}{2}$

72. Given equation is

$$\cot^{-1}\left(\frac{1}{x^2-1}\right) + \tan^{-1}(x^2-1) = \frac{\pi}{2}$$

It is possible only when

$\Rightarrow \quad \frac{1}{x^2-1} = x^2 - 1$

$\Rightarrow \quad (x^2 - 1)^2 = 1$

$\Rightarrow \quad (x^2 - 1) = \pm 1$

$\Rightarrow \quad x^2 = 1 \pm 1 = 2, 0$

$\Rightarrow \quad x = \{-\sqrt{2}, 0, \sqrt{2}\}$

73. Given equation is

$$\cot^{-1}\left(\frac{x^2-1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{x^2-1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) = -\frac{\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}x = -\frac{\pi}{3}$

$\Rightarrow \quad \tan^{-1}x = -\frac{\pi}{6}$

$\Rightarrow \quad x = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

Hence, the solution is $x = -\frac{1}{\sqrt{3}}$.

74. Given equation is

$$4\sin^{-1}x + \cos^{-1}x = \frac{3\pi}{4}$$

$\Rightarrow$ $3\sin^{-1}x + \frac{\pi}{2} = \frac{3\pi}{4}$

$\Rightarrow$ $3\sin^{-1}x = \frac{\pi}{4}$

$\Rightarrow$ $\sin^{-1}x = \frac{\pi}{12}$

$\Rightarrow$ $x = \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

75. Given equation is

$$5\tan^{-1}x + 3\cot^{-1}x = \frac{7\pi}{4}$$

$\Rightarrow$ $2\tan^{-1}x + \frac{3\pi}{2} = \frac{7\pi}{4}$

$\Rightarrow$ $2\tan^{-1}x = \frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}x = \frac{\pi}{8}$

$\Rightarrow$ $x = \tan\left(\frac{\pi}{8}\right) = (\sqrt{2}-1)$

76. Given equation is

$5\tan^{-1}x + 4\cot^{-1}x = 2\pi$

$\Rightarrow$ $\tan^{-1}x + 2\pi = 2\pi$

$\Rightarrow$ $\tan^{-1}x = 0$

$\Rightarrow$ $x = \tan(0) = 0$

Hence, the solution is $x = 0$.

77. Given equation is

$$\cot^{-1}x - \cot^{-1}(x+1) = \frac{\pi}{2}$$

$\Rightarrow$ $\tan^{-1}\left(\frac{1}{x}\right) - \tan^{-1}\left(\frac{1}{x+1}\right) = \frac{\pi}{2}$

$\Rightarrow$ $\tan^{-1}\left(\dfrac{\frac{1}{x} - \frac{1}{x+1}}{1 + \frac{1}{x} \times \frac{1}{x+1}}\right) = \frac{\pi}{2}$

$\Rightarrow$ $\tan^{-1}\left(\frac{1}{x^2+x+1}\right) = \frac{\pi}{2}$

$\Rightarrow$ $\left(\frac{1}{x^2+x+1}\right) = \tan\left(\frac{\pi}{2}\right) = \infty$

$\Rightarrow$ $x^2 + x + 1 = \frac{1}{\infty} = 0$

$\Rightarrow$ $x^2 + x = 1 = 0$

So, no real values of x satisfies the above equation.

Hence, the solution is $x = \varphi$

78. Given equation is

$[\sin^{-1}x] + [\cos^{-1}x] = 0$

It is possible only when

$[\sin^{-1}x] = 0, [\cos^{-1}x] = 0$

$\Rightarrow$ $0 \le \sin^{-1}x \le 1$ and $0 \le \cos^{-1}x \le 1$

$\Rightarrow$ $0 \le x \le \sin(1)$ and $\cos(1) \le x \le 1$

$\Rightarrow$ $x \in [\cos(1), \sin(1)]$

79. Given equation is

$[\tan^{-1}x] + [\cot^{-1}x] = 0$

It is possible only when

$[\tan^{-1}x] = 0$ and $[\cot^{-1}x] = 0$

$\Rightarrow$ $0 \le \tan^{-1}x \le 1$ and $0 \le \cot^{-1}x \le 1$

$\Rightarrow$ $\cot(1) \le x \le \tan(1)$

Hence, $x \in [\cot(1), \tan(1)]$

80. Given equation is

$[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 0$

$\Rightarrow$ $0 \le \sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1}x))) < 1$

$\Rightarrow$ $0 \le (\cos^{-1}(\sin^{-1}(\tan^{-1}x))) < \sin(1)$

$\Rightarrow$ $\cos(\sin(1)) < (\sin^{-1}(\tan^{-1}x)) \le 1$

$\Rightarrow$ $\sin(\cos(\sin(1))) < (\tan^{-1}x) \le \sin(1)$

$\Rightarrow$ $\tan(\sin(\cos(\sin(1)))) < x \le \tan(\sin(1))$

81. Do yourself.

82. Given equation is

$$(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$$

$\Rightarrow$ $(\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x \cdot \cot^{-1}x = \frac{5\pi^2}{8}$

$\Rightarrow$ $\frac{\pi^2}{4} - 2a\left(\frac{\pi}{2} - a\right) = \frac{5\pi^2}{8}, a = \tan^{-1}x$

$\Rightarrow$ $2a\left(\frac{\pi}{2} - a\right) + \frac{3\pi^2}{8} = 0$

$\Rightarrow$ $a\pi - 2a^2 + \frac{3\pi^2}{8} = 0$

$\Rightarrow$ $8a\pi - 16a^2 + 3\pi^2 = 0$

$\Rightarrow$ $16a^2 - 8a\pi - 3\pi^2 = 0$

$\Rightarrow$ $16a^2 - 12a\pi + 4a\pi - 3\pi^2 = 0$

$\Rightarrow$ $4a(4a - 3\pi) + \pi(4a - 3\pi) = 0$

$\Rightarrow$ $(4a + \pi)(4a - 3\pi) = 0$

$\Rightarrow$ $a = \frac{3\pi}{4}, -\frac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}x = \frac{3\pi}{4}, -\frac{\pi}{4}$

$\Rightarrow$ $x = \tan\left(\frac{3\pi}{4}\right), \tan\left(-\frac{\pi}{4}\right)$

$x = -1$

83. Let $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right) = \theta$

$\Rightarrow$ $\cos^{-1}\left(\frac{3}{5}\right) = 2\theta$

$\Rightarrow$ $\cos(2\theta) = \frac{3}{5}$

$\Rightarrow$ $2\cos^2\theta - 1 = \frac{3}{5}$

$\Rightarrow \quad 2\cos^2\theta = 1 + \frac{3}{5} = \frac{8}{5}$

$\Rightarrow \quad \cos^2\theta = \frac{4}{5}$

$\Rightarrow \quad \cos\theta = \frac{2}{\sqrt{5}}$

84. We have

$\sin\left(\frac{\pi}{4} + \sin^{-1}\left(\frac{1}{2}\right)\right)$

$= \sin\left(\frac{\pi}{4} + \theta\right), \theta = \sin^{-1}\left(\frac{1}{2}\right)$

$= \sin\left(\frac{\pi}{4} + \theta\right), \sin\theta = \frac{1}{2}$

$= \sin\left(\frac{\pi}{4}\right)\cos(\theta) + \cos\left(\frac{\pi}{4}\right)\sin(\theta)$

$= \frac{1}{\sqrt{2}}\cos(\theta) + \frac{1}{\sqrt{2}}\sin(\theta)$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

85. Let m_1 and m_2 be the roots of

$x^2 + 3x + 1 = 0$

Thus, $m_1 + m_2 = -3 < 0$

and $m_1 \cdot m_2 = 1$

It is possible only when both are negative.

Thus, $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$

$= \tan^{-1}(m) - \pi + \cot^{-1}(m)$

$= \tan^{-1}(m) + \cot^{-1}(m) - \pi$

$= \frac{\pi}{2} - \pi$

$= -\frac{\pi}{2}$

86. We have $\cos(\tan^{-1}(\sin(\cot^{-1} x)))$

$= \cos(\tan^{-1}(\sin\theta)), \cot\theta = x$

$= \cos\left(\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)$

$= \cos\varphi, \tan\varphi = \left(\frac{1}{\sqrt{1+x^2}}\right)$

$= \sqrt{\frac{x^2+1}{x^2+2}}$

87. Given, $6(\sin^{-1} x)^2 - \pi \sin^{-1} x \le 0$

$\Rightarrow \quad \sin^{-1} x\,(6\sin^{-1} x - \pi) \le 0$

$\Rightarrow \quad 0 \le \sin^{-1} x \le \frac{\pi}{6}$

$\Rightarrow \quad 0 \le x \le \frac{1}{2}$

88. Given, in-equation is

$\frac{2\tan^{-1} x + \pi}{4\tan^{-1} x - \pi} \le 0$

$\Rightarrow \quad -\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{4}$

$\Rightarrow \quad -\infty < x < 1$

89. Given inequation is

$\Rightarrow \quad \sin^{-1} x < \sin^{-1} x^2$

$\Rightarrow \quad x^2 > x$

$\Rightarrow \quad x(x-1) > 0$

$\Rightarrow \quad x > 1$ and $x < 0$

$\Rightarrow \quad x \in [-1, 0)$

90. Given in-equation is

$\Rightarrow \quad \cos^{-1} x > \cos^{-1} x^2$

$\Rightarrow \quad x^2 > x$

$\Rightarrow \quad x^2 - x > 0$

$\Rightarrow \quad x(x-1) > 0$

$\Rightarrow \quad -1 \le x < 0$

91. Given in-eqution is

$\log_2(\tan^{-1} x) > 1$

$\Rightarrow \quad \tan^{-1} x > 2$

$\Rightarrow \quad x > \tan(2)$

Hence, the solution is

$(\tan 2, \infty)$

92. Given in-equation is

$(\cot^{-1} x)^2 - 5\cot^{-1} x + 6 > 0$

$\Rightarrow \quad (\cot^{-} x - 2)(\cot^{-1} x - 3) > 0$

$\Rightarrow \quad (\cot^{-1} x - 2) < 0, (\cot^{-1} x - 3) > 0$

$\Rightarrow \quad x > \cot(2), x < \cot(3)$

$\Rightarrow \quad x \in (\cot 2, \cot 3)$

93. Given in-equation is

$\sin^{-1} x < \cos^{-1} x$

$\Rightarrow \quad 2\sin^{-1} x < \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1} x < \frac{\pi}{4}$

$\Rightarrow \quad x < \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x \in \left[-1, \frac{1}{\sqrt{2}}\right)$

94. Given in-equation is

$\sin^{-1} x > \sin^{-1}(1-x)$

$x > (1-x)$

$2x > 1$

$x > \frac{1}{2}$

Hence, the solution is $x \in \left(\frac{1}{2}, 1\right]$

95. Given in-equation is

$\sin^{-1} 2x > \operatorname{cosec}^{-1} x$

$\Rightarrow \quad \sin^{-1}(2x) > \sin^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad 2x > \frac{1}{x}$

$\Rightarrow \quad 2x - \frac{1}{x} > 0$

$\Rightarrow \quad \frac{2x^2 - 1}{x} > 0$

$\Rightarrow \quad \frac{(\sqrt{2}x + 1)(\sqrt{2}x - 1)}{x} > 0$

$\Rightarrow \quad x \in \left(-\frac{1}{\sqrt{2}}, 0\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right]$

96. Given in-equation is

$\tan^{-1} 3x < \cot^{-1} x$

$\Rightarrow \quad \tan^{-1}(3x) < \tan^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \quad (3x) - \left(\frac{1}{x}\right) < 0$

$\Rightarrow \quad \frac{(\sqrt{3}x + 1)(\sqrt{3}x - 1)}{x} < 0$

$\Rightarrow \quad x \in \left(-\frac{1}{\sqrt{3}}, 0\right) \cup \left(0 \frac{1}{\sqrt{3}}\right)$

97. Given in-equation is

$\cos^{-1} 2x \geq \sin^{-1} x$

$\Rightarrow \quad \sin^{-1}\sqrt{1 - 4x^2} > \sin^{-1} x$

$\Rightarrow \quad \sqrt{1 - 4x^2} > x$

$\Rightarrow \quad (1 - 4x^2) > x^2$

$\Rightarrow \quad 5x^2 - 1 < 0$

$\Rightarrow \quad (\sqrt{5}x + 1)(\sqrt{5}x - 1) < 0$

$\Rightarrow \quad -\frac{1}{\sqrt{5}} < x < \frac{1}{\sqrt{5}}$

98. Given in-equation is

$x^2 - 2x < \sin^{-1}(\sin 2)$

$\Rightarrow \quad x^2 - 2x < (\pi - 2)$

$\Rightarrow \quad (x - 1)^2 < (\pi - 1)$

$\Rightarrow \quad |(x - 1)| < \sqrt{(\pi - 1)}$

$\Rightarrow \quad -\sqrt{(\pi - 1)} < (x - 1) < \sqrt{(\pi - 1)}$

$\Rightarrow \quad 1 - \sqrt{(\pi - 1)} < x < 1 + \sqrt{(\pi - 1)}$

99. Given in-equation is

$\sin^{-1}\left(\frac{x}{2}\right) < \cos^{-1}(x + 1)$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{2}\right) < \sin^{-1}(\sqrt{1 - (x + 1)^2})$

$\Rightarrow \quad \left(\frac{x}{2}\right) < (\sqrt{1 - (x + 1)^2})$

$\Rightarrow \quad \left(\frac{x}{2}\right)^2 < 1 - (x + 1)^2$

$\Rightarrow \quad \frac{x^2}{4} < -2x - x^2$

$\Rightarrow \quad \frac{5x^2}{4} + 2x < 0$

$\Rightarrow \quad 5x^2 + 8x < 0$

$\Rightarrow \quad x(5x + 8) < 0$

$\Rightarrow \quad -\frac{8}{5} < x < 0$

100. Given in-equation is

$\tan^{-1} 2x > 2\tan^{-1} x$

$\Rightarrow \quad \tan^{-1}(2x) > \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$

$\Rightarrow \quad (2x) > \left(\frac{2x}{1 - x^2}\right)$

$\Rightarrow \quad (2x)\left(1 - \frac{1}{1 - x^2}\right) > 0$

$\Rightarrow \quad x\left(\frac{1 - x^2 - 1}{1 - x^2}\right) > 0$

$\Rightarrow \quad x\left(\frac{x^2}{x^2 - 1}\right) > 0$

$\Rightarrow \quad \left(\frac{x^3}{(x - 1)(x + 1)}\right) > 0$

$\Rightarrow \quad x \in (-1, 0) \cup (1, \infty)$

101. Given in-equation is

$\tan(\cos^{-1} x) \leq \sin\left(\cot^{-1}\left(\frac{1}{2}\right)\right)$

$\Rightarrow \quad \frac{\sqrt{1 - x^2}}{x} \leq \frac{1}{\sqrt{5}}$

$\Rightarrow \quad \left(\frac{1 - x^2}{x^2}\right) \leq \frac{1}{5}$

$\Rightarrow \quad \left(\frac{1 - x^2}{x^2} - \frac{1}{5}\right) \leq 0$

$\Rightarrow \quad \frac{5 - 5x^2 - x^2}{5x^2} \leq 0$

$\Rightarrow \quad \frac{6x^2 - 5}{x^2} \leq 0$

$\Rightarrow \quad \frac{6x^2 - 5}{x^2} \geq 0$

$\Rightarrow \quad x \in R - \left(-\sqrt{\frac{5}{6}}, \sqrt{\frac{5}{6}}\right)$

102. As we know that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for every x in $[-1, 1]$

(i) Since $0 < \frac{1}{m^2 + 1} \leq 1$

so, $f\left(\frac{1}{m^2 + 1}\right) = \frac{\pi}{2}$

(ii) Since, $0 \le \frac{m^2}{m^2+1} < 1$,

so $f\left(\frac{m^2}{m^2+1}\right) = \frac{\pi}{2}$.

(iii) Since, $-\frac{1}{2} \le \frac{m}{m^2+1} \le \frac{1}{2}$

so, $f\left(\frac{m}{m^2+1}\right) = \frac{\pi}{2}$

(iv) Since $m^2 - 2m + 6 = (m-1)^2 + 5$
Thus, $4 \le (m-1)^2 + 5 < \infty$
Hence, $f((m-1)^2 + 5)$ is not defined.

(v) Also, $1 \le (m-1)^2 + 5 < \infty$
So, $f(m^2+1)$ is not defined.

103. Given, $\cos^{-1}x + \cos^{-1}y = \frac{2\pi}{3}$

Now, $\sin^{-1} x + \sin^{-1} y$

$= \frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y$

$= \pi - (\cos^{-1} x + \cos^{-1} y)$

$= \pi - \frac{2\pi}{3}$

$= \frac{\pi}{3}$

104. Let m_1 and m_2 be the two roots of the given equation.
Now, $m_1 + m_2 = -3$ and $m_1 \cdot m_2 = 1$
$\Rightarrow$ m_1 and m_2 are two negative roots.

Now, $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$

$= \tan^{-1}(m) - \pi + \cot^{-1}(m)$

$= -\pi + \tan^{-1}(m) + \cot^{-1}(m)$

$= -\pi + \frac{\pi}{2}$

$= -\frac{\pi}{2}$

105. Let $m = \frac{2x^2+5}{x^2+2} = 2 + \frac{1}{x^2+2}$

Thus, $m \in \left[2, \frac{5}{2}\right]$

now, $\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+2}\right)\right) > \sin^{-1}(\sin 3)$

$\Rightarrow \sin^{-1}\left(\sin\left(\pi - \frac{2x^2+5}{x^2+2}\right)\right)$

$< \sin^{-1}(\sin(\pi - 3))$

$\Rightarrow \pi - \left(\frac{2x^2+5}{x^2+2}\right) > \pi - 3$

$\Rightarrow \left(\frac{2x^2+5}{x^2+2}\right) < 3$

$\Rightarrow \left(\frac{2x^2+5}{x^2+2} - 3\right) < 0$

$\Rightarrow \left(\frac{-x^2-5}{x^2+2}\right) < 0$

$\Rightarrow \left(\frac{x^2+5}{x^2+2}\right) > 0$

$\Rightarrow x \in R$

106. (i) $\sin^{-1}(\sin 3)$
$= \sin^{-1}(\sin(\pi - 3))$
$= (\pi - 3)$

(ii) $\sin^{-1}(\sin 5)$
$= \sin^{-1}(\sin(5 - 2\pi))$
$= (5 - 2\pi)$

(iii) $\sin^{-1}(\sin 7)$
$= \sin^{-1}(\sin(7 - 2\pi))$
$= (7 - 2\pi)$

(iv) $\sin^{-1}(\sin 10)$
$= \sin^{-1}(\sin(3\pi - 10))$
$= (3\pi - 10)$

(v) $\sin^{-1}(\sin 20)$
$= \sin^{-1}(\sin(20 - 6\pi))$
$= (20 - 6\pi)$

107. (i) $\cos^{-1}(\cos 2) = 2$

(ii) $\cos^{-1}(\cos 3) = 2$

(iii) $\cos^{-1}(\cos 5)$
$= \cos^{-1}(\cos(2\pi - 5))$
$= (2\pi - 5)$

(iv) $\cos^{-1}(\cos 7)$
$= \cos^{-1}(\cos(7 - 2\pi))$
$= (7 - 2\pi)$

(v) $\cos^{-1}(\cos 10)$
$= \cos^{-1}(\cos(4\pi - 10))$
$= (4\pi - 10)$

108. (i) $\tan^{-1}(\tan 3)$
$= \tan^{-1}(\tan(3 - \pi))$
$= (3 - \pi)$

(ii) $\tan^{-1}(\tan 5)$
$= \tan^{-1}(\tan(5 - 2\pi))$
$= (5 - 2\pi)$

(iii) $\tan^{-1}(\tan 7)$
$= \tan^{-1}(\tan(7 - 2\pi))$
$= (7 - 2\pi)$

(iv) $\tan^{-1}(\tan 10)$
$= \tan^{-1}(\tan(10 - 3\pi))$
$= (10 - 3\pi)$

(v) $\tan^{-1}(\tan 15)$
$= \tan^{-1}(\tan(15 - 5\pi))$
$= (15 - 5\pi)$

109. We have
$\cos^{-1}(\sin(-5))$
$= \cos^{-1}(-\sin 5)$
$= \pi - \cos^{-1}(\sin 5)$

$$= \pi - \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 5\right)\right)$$
$$= \pi - \left(\frac{\pi}{2} - 5\right)$$
$$= \frac{\pi}{2} + 5$$

110. We have
$$f(x) = \sin^{-1}(\sin x)$$
$$= x + 2\pi - \pi - x + x + \pi - x$$
$$= 2\pi$$
$$\Rightarrow \quad f(x) = 0$$

111. We have
$$f(x) = \cos^{-1}(\cos x)$$
$$= -x + x + 2\pi = x$$
$$= 2\pi - x$$
$$\Rightarrow \quad f(x) = -1$$

112. We have
$$\sin^{-1}\left(\sin\left(\frac{2x^2+5}{x^2+1}\right)\right) < \pi - 3$$
$$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi - \left(\frac{2x^2+5}{x^2+1}\right)\right)\right) > \pi - 3$$
$$\Rightarrow \quad \left(\pi - \left(\frac{2x^2+5}{x^2+1}\right)\right) < \pi - 3$$
$$\Rightarrow \quad -\left(\frac{2x^2+5}{x^2+1}\right) < -3$$
$$\Rightarrow \quad \left(\frac{2x^2+5}{x^2+1}\right) > 3$$
$$\Rightarrow \quad \left(\frac{2x^2+5}{x^2+1} - 3\right) > 0$$
$$\Rightarrow \quad \left(\frac{2x^2+5-3x^2-3}{x^2+1}\right) > 0$$
$$\Rightarrow \quad x^2 < 2$$
$$\Rightarrow \quad -\sqrt{2} < x < \sqrt{2}$$

113. We have
$$x^2 - 3x < \sin^{-1}(\sin 2)$$
$$\Rightarrow \quad x^2 - 3x < \sin^{-1}(\sin(\pi - 2))$$
$$\Rightarrow \quad x^2 - 3x < (\pi - 2)$$
$$\Rightarrow \quad x^2 - 3x + (2 - \pi) < 0$$
$$\Rightarrow \quad \left(x - \frac{3+\sqrt{1+4\pi}}{2}\right)\left(x - \frac{3-\sqrt{1+4\pi}}{2}\right) < 0$$
$$\Rightarrow \quad \frac{3-\sqrt{1+4\pi}}{2} < x < \frac{3+\sqrt{1+4\pi}}{2}$$

114. We have
$$= \sin^{-1}(\sin 50) + \cos^{-1}(\cos 50) + \tan^{-1}(\tan 50)$$
$$= \sin^{-1}(\sin(50 - 16\pi)) + \cos^{-1}(\cos(16\pi - 50) + \tan^{-1}(\tan(50 - 16\pi)$$
$$= (50 - 16\pi) + (16\pi - 50) + (50 - 16\pi)$$
$$= (50 - 16\pi)$$

115. $\sin^{-1}(\sin 1) + \sin^{-1}(\sin 2) + \sin^{-1}(\sin 3)$
$$= 1 + (\pi - 2) + (\pi - 3)$$
$$= (2\pi - 4)$$

116. $\sin^{-1}(\sin 10) + \sin^{-1}(\sin 20) + \sin^{-1}(\sin 30) + \sin^{-1}(\sin 40)$
$$= (3\pi - 10) + (20 - 6\pi) + (30 - 10\pi) + (13\pi - 40)$$
$$= 0$$

117. $\cos^{-1}(\cos 1) + \cos^{-1}(\cos 2) + \cos^{-1}(\cos 3) + \cos^{-1}(\cos 4)$
$$= 1 + 2 + 3 + (2\pi - 4)$$
$$= 2(\pi + 1)$$

118. $\cos^{-1}(\cos 10) + \cos^{-1}(\cos 20) + \cos^{-1}(\cos 30) + \cos^{-1}(\cos 40)$
$$= (4\pi + 10) + (20 + 6\pi) + (10\pi + 30) + (40 - 12\pi)$$
$$= (20 - 4\pi)$$

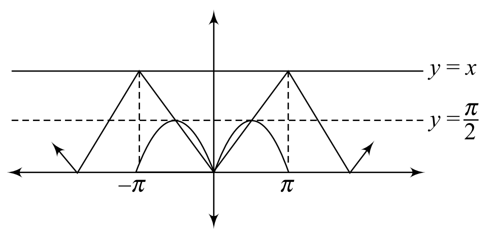
119. $\sin^{-1}(\sin 10) + \cos^{-1}(\cos 10)$
$$= (3\pi - 10) + (4\pi - 10)$$
$$= (7\pi - 20)$$

120. $\sin^{-1}(\sin 50) + \cos^{-1}(\cos 50)$
$$= (50 - 16\pi) + (16\pi - 50)$$
$$= 0$$

121. $\sin^{-1}(\sin 100) + \cos^{-1}(\cos 100)$
$$= (100 - 32\pi) + (32\pi - 100)$$
$$= 0$$

122. $\cos^{-1}(\sin(-5)) + \sin^{-1}(\cos(-5))$
$$= \pi - \cos^{-1}(\sin 5) + \sin^{-1}(\cos 5)$$
$$= \pi - \cos^{-1}\left(\cos\left(5 - \frac{3\pi}{2}\right)\right) + \sin^{-1}\left(\sin\left(5 - \frac{3\pi}{2}\right)\right)$$
$$= \pi - \left(5 - \frac{3\pi}{2}\right) + \left(5 - \frac{3\pi}{2}\right)$$
$$= \pi$$

123.



Hence, the number of solutions is 3.
Thus, the ordered pairs are
$$\left(-\frac{\pi}{2}, 1\right), (0, 0), \left(\frac{\pi}{2}, 1\right)$$

124. Given,
$$f(x) = \cos^{-1}(\cos x) - \sin^{-1}(\sin x)$$
$$= x - (x + \pi - x)$$
$$= x - \pi$$
Hence, the required area
$$= \frac{1}{2} \times \pi \times \pi = \frac{\pi^2}{2}$$

125. $\tan^{-1}(\tan 1) + \tan^{-1}(\tan 2) + \tan^{-1}(\tan 3) + \tan^{-1}(\tan 4)$
$$= 1 + (2 - \pi) + (3 - \pi) + (4 - \pi)$$
$$= (10 - 3\pi)$$

126. $\tan^{-1}(\tan 20) + \tan^{-1}(\tan 40)$
$+ \tan^{-1}(\tan 60) + \tan^{-1}(\tan 80)$
$= (20 - 6\pi) + (40 - 13\pi) + (60 - 21\pi) + (80 - 26\pi)$
$= (200 - 66\pi)$

127. $\sin^{-1}(\sin 15) + \cos^{-1}(\cos 15) + \tan^{-1}(\tan 15)$
$= (5\pi - 15) + (15 - 4\pi) + (15 - 5\pi)$
$= (15 - 4\pi)$

128. $\sin^{-1}(\sin 50) + \cos^{-1}(\cos 50) - \tan^{-1}(\tan 50)$
$= (50 - 16\pi) + (16\pi - 50) - (50 - 16\pi)$
$= (16\pi - 50)$

129. $3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$
$\Rightarrow 3x^2 + 2x < 2(\pi - 4) - (2\pi - 4)$
$\Rightarrow 3x^2 + 2x < -4$
$\Rightarrow 3x^2 + 8x + 4 < 0$
$\Rightarrow 3x^2 + 6x + 2x + 4 < 0$
$\Rightarrow 3x(x + 2) + 2(x + 2) < 0$
$\Rightarrow (3x + 2)(x + 2) < 0$
$\Rightarrow -2 < x < -\frac{2}{3}$

130. We have

$$\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$$

$$\Rightarrow \sin^{-1}\left(\sin\left(\pi - \left(\frac{2x^2+4}{x^2+1}\right)\right)\right) < \pi - 3$$

$$\Rightarrow \left(\pi - \left(\frac{2x^2+4}{x^2+1}\right)\right) < \pi - 3$$

$$\Rightarrow -\left(\frac{2x^2+4}{x^2+1}\right) < -3$$

$$\Rightarrow \left(\frac{2x^2+4}{x^2+1}\right) > 3$$

$$\Rightarrow \left(\frac{2x^2+4}{x^2+1} - 3\right) > 0$$

$$\Rightarrow \left(\frac{2x^2+4-3x^2-3}{x^2+1}\right) > 0$$

$\Rightarrow x^2 < 1$
$\Rightarrow -1 < x < 1$

131. We have $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$$

$$= \tan^{-1}\left(\frac{5/6}{1-1/6}\right)$$

$$= \tan^{-1}\left(\frac{5/6}{5/6}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

132. We have
$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$

$$= \frac{\pi}{4} + \pi + \tan^{-1}\left(\frac{2+3}{1-2.3}\right)$$

$$= \frac{\pi}{4} + \pi + \tan^{-1}(-1)$$

$$= \frac{\pi}{4} + \pi - \frac{\pi}{4}$$

$= \pi$

133. We have

$$\tan^{-1}(9) + \tan^{-1}\left(\frac{5}{4}\right)$$

$$= \pi + \tan^{-1}\left(\frac{9+\frac{5}{4}}{1-9\cdot\frac{5}{4}}\right)$$

$$= \pi + \tan^{-1}\left(\frac{\frac{41}{4}}{-\frac{41}{4}}\right)$$

$$= \pi + \tan^{-1}(-1)$$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

134. We have

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\cdot\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\cdot\sqrt{1-\left(\frac{4}{5}\right)^2}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\cdot\frac{12}{13} + \frac{5}{13}\cdot\frac{3}{13}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{48}{65} + \frac{15}{65}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$$= \sin^{-1}\left(\frac{63}{65}\right) - \sin^{-1}\left(\frac{63}{65}\right)$$

$= 0$

135. We have

$$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{2\cdot\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{25}{25}\right)$$

$$= \frac{\pi}{4}$$

136. Let $\sin^{-1} x = A$, $\sin^{-1} y = B$, $\sin^{-1} z = C$

Then $x = \sin A$, $y = \sin B$, $z = \sin C$

we have, $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$

$= \sin A \cdot \cos A + \sin B \cdot \cos B + \sin C \cdot \cos C$

$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$

$= \frac{1}{2}(4\sin A \cdot \sin B \cdot \sin C)$

$= 2 \sin A \cdot \sin B \cdot \sin C$

$= 2xyz$

137. We have

$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$\Rightarrow \cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$

$\Rightarrow \cos^{-1} x + \cos^{-1} y = \cos^{-1}(-z)$

$\Rightarrow \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos(-z)$

$\Rightarrow (xy + z)^2 = (1 - x^2)(1 - y^2)$

$\Rightarrow x^2y^2 + 2xyz + z^2 = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

138. Given, $\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$

$\Rightarrow \cos^{-1}\left(\frac{x}{2}\cdot\frac{y}{3} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \theta$

$\Rightarrow \left(\frac{xy}{6} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \cos\theta$

$\Rightarrow \left(\frac{xy}{6} - \cos\theta\right)^2 = \left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{9}\right)$

$\frac{x^2y^2}{36} - \frac{xy}{3}\cos\theta + \cos^2\theta$

$\Rightarrow = 1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36}$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} - \frac{xy}{3}\cos\theta = 1 - \cos^2\theta$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} - \frac{xy}{3}\cos\theta = \sin^2\theta$

$\Rightarrow 9x^2 + 4y^2 - 12\,xy\cos\theta = 36\sin^2\theta$

139. We have

$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$

and $\cot^{-1}(1) + \cot^{-1}(2) + \cot^{-1}(3)$

$= \tan^{-1}(1) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$

Hence, $m = \frac{\pi}{\pi/2} = 2$

140. **Case I:** When $x \le 0$

Then, $\tan^{-1}(2x) \le 0$, $\tan^{-1}(3x) \le 0$,

$\Rightarrow x \le 0$

So, it has no solution.

Case II: When $x > 0$, $2x \cdot 3x = 6x^2 < 1$

$\Rightarrow x < \frac{1}{\sqrt{6}}$

Then $\frac{3\pi}{4} = \tan^{-1}(2x) + \tan^{-1}(3x) < \frac{\pi}{2}$

So, it is not possible.

Case III: When $x > 0$, $2x \cdot 3x > 1$

$\Rightarrow x > \frac{1}{\sqrt{6}}$

Then $\frac{3\pi}{4} = \tan^{-1}(2x) + \tan^{-1}(3x)$

$\Rightarrow \pi + \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{3\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{3\pi}{4} - \pi = -\frac{\pi}{4}$

$\Rightarrow \left(\frac{5x}{1-6x^2}\right) = -1$

$\Rightarrow 6x^2 - 5x - 1 = 0$

$\Rightarrow x = 1, -1/6$

Thus, $x = 1$ is a solution.

141. We have

$\sin^{-1}(x) + \sin^{-1}(2x) = \frac{\pi}{3}$

$\Rightarrow \sin^{-1}x + \sin^{-1}(2x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \sin^{-1}x - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\sin^{-1}(2x)$

$\Rightarrow \sin^{-1}\left(\frac{x}{2} - \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right) = \sin^{-1}(-2x)$

$\Rightarrow \left(\frac{x}{2} - \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right) = -2x$

$\Rightarrow 5x = \sqrt{3}\sqrt{1-x^2}$

$\Rightarrow 25\,x^3 = 3(1 - x^2)$

$\Rightarrow 28x^2 = 3$

$\Rightarrow x = \pm\frac{\sqrt{3}}{2\sqrt{7}}$

$\Rightarrow \quad x = \frac{\sqrt{3}}{2\sqrt{7}}$, negative value of x does not satisfy the given equation.

142. We have

$$f(x) = \cos^{-1}(x) + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right)$$

$$= \cos^{-1}(x) + \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(x)$$

$$= \frac{\pi}{3}$$

Now, $f(2013) = \frac{\pi}{3}$

143. $\sin^{-1} x + \sin^{-1}(1 - x) = \cos^{-1} x$

$\Rightarrow \quad \sin^{-1}(1-x) = \frac{\pi}{2} - 2\sin^{-1}x$

$\Rightarrow \quad (1-x) = \sin\left(\frac{\pi}{2} - 2\sin^{-1}x\right)$

$\Rightarrow \quad (1 - x) = \cos(2\sin^{-1} x)$

$\Rightarrow \quad (1 - x) = 1 - 2x^2$

$\Rightarrow \quad x(2x - 1) = 0$

$\Rightarrow \quad x = 0, (2x - 1) = 0$

$\Rightarrow \quad x = 0, \frac{1}{2}$

144. $x^2 - 4x > \sin^{-1}(\sin[\pi^{3/2}]) + \cos^{-1}(\cos[\pi^{3/2}])$

$\Rightarrow \quad x^2 - 4x > \sin^{-1}(\sin 5.5) + \cos^{-1}(\cos 5.5)$

$\Rightarrow \quad x^2 - 4x > (5.5 - 2\pi) + (2\pi - 5.5)$

$\Rightarrow \quad x^2 - 4x > 0$

$\Rightarrow \quad x(x - 4) > 0$

$\Rightarrow \quad x < 0$ and $x > 4$

$\Rightarrow \quad x \in (-\infty, 0) \cup (4, \infty)$

145. $\cos(\tan^{-1} x) = x$

$\Rightarrow \quad \frac{1}{\sqrt{x^2+1}} = x$

$\Rightarrow \quad x^2(x^2 + 1) = 1$

$\Rightarrow \quad x^4 + x^2 - 1 = 0$

$\Rightarrow \quad x^2 = \frac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad x^2 = \frac{\sqrt{5}-1}{2}$

$\Rightarrow \quad x = \pm\sqrt{\frac{\sqrt{5}-1}{2}}$

146. $\sin(\tan^{-1} x) = \cos(\cot^{-1}(x + 1))$

$\Rightarrow \quad \frac{x}{\sqrt{x^2+1}} = \frac{x+1}{\sqrt{1+(x+1)^2}}$

$\Rightarrow \quad \frac{x^2}{x^2+1} = \frac{(x+1)^2}{1+(x+1)^2}$

$\Rightarrow \quad \frac{x^2}{x^2+1} - 1 = \frac{(x+1)^2}{1+(x+1)^2} - 1$

$\Rightarrow \quad \frac{-1}{x^2+1} = \frac{-1}{1+(x+1)^2}$

$\Rightarrow \quad x^2 + 1 = 1 + (x + 1)^2$

$\Rightarrow \quad x^2 = (x + 1)^2$

$\Rightarrow \quad 2x = -1$

$\Rightarrow \quad x = -\frac{1}{2}$

147. $\sec^{-1}\left(\frac{x}{2}\right) - \sec^{-1}x = \sec^{-1}2$

$\Rightarrow \quad \cos^{-1}\left(\frac{x}{2}\right) = \cos^{-1}\left(\frac{1}{x}\right) + \cos^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \quad \cos^{-1}\left(\frac{x}{2}\right) = \cos^{-1}\left(\frac{1}{2}\cdot\frac{1}{\text{x}} - \sqrt{1-\frac{1}{x^2}}\cdot\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad \left(\frac{x}{2}\right) = \left(\frac{1}{2}\cdot\frac{1}{\text{x}} - \sqrt{1-\frac{1}{x^2}}\cdot\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad x - \frac{1}{x} = -\sqrt{3}\sqrt{1-\frac{1}{x^2}}$

$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = 3\left(1 - \frac{1}{x^2}\right)$

$\Rightarrow \quad x^2 + \frac{1}{x^2} - 2 = 3 - \frac{3}{x^2}$

$\Rightarrow \quad x^2 + \frac{4}{x^2} - 5 = 0$

$\Rightarrow \quad x^4 - 5x^2 + 4 = 0$

$\Rightarrow \quad (x^2 - 1)(x^2 - 4) = 0$

$\Rightarrow \quad x = \pm 2, \pm 1$

148. $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(\text{x} + \frac{3}{2}\right)\right)\right)\right) + \tan(\sec^{-1}x) = 0$

$\Rightarrow \quad \left(\frac{2x+3}{2}\right) = \sqrt{x^2-1}$

$\Rightarrow \quad (2x + 3)^2 = 4(x^2 - 1)$

$\Rightarrow \quad 12x + 9 = -4$

$\Rightarrow \quad x = -\frac{13}{12}$

149. Given equation is

$$\tan\left(\tan^{-1}\left(\frac{x}{10}\right) + \tan^{-1}\left(\frac{1}{x+1}\right)\right) = \tan\left(\frac{\pi}{4}\right)$$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x}{12} + \frac{1}{x+1}}{1 - \frac{x}{12}\cdot\frac{1}{x+1}}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \frac{x^2 + x + 12}{11x + 12} = 1$

$\Rightarrow \quad x^2 + x + 12 = 11x + 12$

$\Rightarrow \quad x^2 - 10x = 0$

$\Rightarrow \quad x(x - 10) = 0$

$\Rightarrow \quad x = 0, 10$

150. $(x-2)x^2 + 8x + k + 4$

$> \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$

$\Rightarrow (k-2)x^2 + 8x + k + 4$

$> (12 - 4\pi) + (4\pi - 12)$

$\Rightarrow (k-2)x^2 + 8x + (k+4) > 0$

For all x in R, $D \geq 0$

$\Rightarrow 64 - 4(k-2)(k+4) \geq 0$

$\Rightarrow 16 - (k-2)(k+4) \geq 0$

$\Rightarrow (k-2)(k+4) - 16 \leq 0$

$\Rightarrow k^2 + 2k - 24 \leq 0$

$\Rightarrow (k+6)(k-4) \leq 0$

$\Rightarrow -6 \leq k \leq 4$

Thus, the least integral value of k is -6

151. Do yourself.

152. Given,

$f(x) = \sin^{-1}(\sin x), \forall x \in [-\pi, 2\pi]$.

$= (-\pi - x) - x + (\pi - x) + (x - 2\pi)$

$= -2\pi - 2x$

Thus, $f'(x) = -2$

153. Given,

$f(x) = \cos^{-1}(\cos x), \forall x \in [-2\pi, \pi]$

$= (x + 2\pi) - x + x$

$= (x + 2\pi)$

Thus, $f'(x) = 1$

154. Given,

$f(x) = \tan^{-1}(\tan x), \forall x \in \left[-\frac{3\pi}{2}, \frac{5\pi}{2}\right]$

$= (x + \pi) + x + (x - \pi) + (x - 2\pi)$

$= (4x - 2\pi)$

Thus, $f'(x) = 4$

155. We have

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cot^{-1}(3)$$

$$= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

156. We have $2\tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{12}{5}\right)$

$$= \tan^{-1}\left(\frac{2\cdot\frac{3}{2}}{1-\frac{9}{4}}\right) + \tan^{-1}\left(\frac{12}{5}\right)$$

$$= \tan^{-1}\left(-\frac{12}{5}\right) + \tan^{-1}\left(\frac{12}{5}\right)$$

$$= \pi - \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{12}{5}\right)$$

$$= \pi$$

157. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$$= \cot^{-1}\left(\frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}\right)$$

$$= \cot^{-1}\left(\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)$$

$$= \cot^{-1}\left(\cot\left(\frac{x}{2}\right)\right) = \frac{x}{2}$$

158. $\sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})$

$$= \sin^{-1}(x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2})$$

$$= \sin^{-1}(x) - \sin^{-1}(\sqrt{x})$$

159. $\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$$= \sin^{-1}\left(\sin\left(x + \frac{\pi}{4}\right)\right)$$

$$= \left(x + \frac{\pi}{4}\right)$$

160. $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right), \frac{\pi}{4} < x < \frac{5\pi}{4}$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \cos^{-1}\left(\cos\left(x - \frac{\pi}{4}\right)\right)$$

$$= \left(x - \frac{\pi}{4}\right)$$

161. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Put $x^2 = \cos 2\pi$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \theta\right)\right)$$

$$= \left(\frac{\pi}{4} + \theta\right)$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}(x^2)\right)$$

162. $\sin^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$

$= \sin^{-1}(\sin \alpha \cos x + \cos \alpha \sin x)$

$= \sin^{-1}(\sin (x + \alpha))$

$= (x + \alpha)$

$= x + \tan^{-1}\left(\frac{3}{4}\right)$

163. Let $\sin^{-1}\left(\frac{1}{4}\right) = \theta$

$\Rightarrow \quad \sin \theta = \frac{1}{4}$

Now,

$\sin (2\theta) = 2 \sin \theta \cdot \cos \theta$

$= 2 \times \frac{1}{4} \times \sqrt{1 - \frac{1}{16}}$

$= \frac{\sqrt{15}}{8}$

164. Let $\cos^{-1}\left(\frac{1}{3}\right) = \theta$

$\Rightarrow \quad \cos \theta = \frac{1}{3}$

Now,

$\cos (2\theta) = 2 \cos^2 \theta - 1$

$= \frac{2}{9} - 1$

$= -\frac{7}{9}$

165. Let $\tan^{-1}\left(\frac{1}{3}\right) = \theta$

$\Rightarrow \quad \tan \theta = \frac{1}{3}$

Now,

$$\cos (2\theta) = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$$= \frac{1 - \left(\frac{1}{3}\right)^2}{1 + \left(\frac{1}{3}\right)^2}$$

$$= \frac{9 - 1}{9 + 1}$$

$$= \frac{8}{10} = \frac{4}{5}$$

166. Let $\cot^{-1}\left(\frac{3}{4}\right) = \theta$

Now, $\sin\left(\frac{\theta}{2}\right)$

$$= \sqrt{\frac{1 - \cos \theta}{2}}$$

$$= \sqrt{\frac{1 - \frac{\cos \theta}{\sin \theta \cdot \operatorname{cosec} \theta}}{2}}$$

$$= \sqrt{\frac{1 - \frac{\cot \theta}{\operatorname{cosec} \theta}}{2}}$$

$$= \sqrt{\frac{1 - \frac{\cot \theta}{\sqrt{1 + \cot^2\theta}}}{2}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1 - \frac{3/4}{\sqrt{1 + 9/16}}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1 - \frac{3}{5}}$$

$$= \frac{2}{\sqrt{5}}$$

167. Let $\tan^{-1}\left(\frac{3}{4}\right) = \theta$

$\Rightarrow \quad \tan \theta = \frac{3}{4}$

We have

$$\tan\left(\frac{3\pi}{4} - 2\theta\right) = \frac{\tan\left(\frac{3\pi}{4}\right) - \tan (2\theta)}{1 + \tan\left(\frac{3\pi}{4}\right) \cdot \tan (2\theta)}$$

$$= \frac{-1 - \tan (2\theta)}{1 - \tan (2\theta)}$$

$$= \frac{-1 - \frac{2 \tan \theta}{1 - \tan^2\theta}}{1 - \frac{2 \tan \theta}{1 - \tan^2\theta}}$$

$$= \frac{\tan^2\theta - 2 \tan \theta - 1}{1 - \tan^2\theta - 2 \tan \theta}$$

$$= \frac{\frac{9}{16} - \frac{6}{4} - 1}{1 - \frac{9}{16} - \frac{6}{4}}$$

$$= \frac{41}{17}$$

168. Let $\sin^{-1}\left(\frac{1}{2}\right) = \theta$

Then $\sin \theta = \frac{1}{2}$

Now, $\sin\left(2 \sin^{-1}\left(\frac{1}{2}\right)\right)$

$= \sin(2\theta)$
$= 2\sin\theta\cos\theta$
$= 2\times\frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

169. Let $\sin^{-1}\left(\frac{1}{3}\right) = \theta$

Then $\sin\theta = \frac{1}{3}$

Now, $\sin\left(3\sin^{-1}\left(\frac{1}{3}\right)\right)$
$= \sin(3\theta)$
$= 3\sin\theta - 4\sin^3\theta$
$= 3\cdot\frac{1}{3} - 4\cdot\left(\frac{1}{3}\right)^3$
$= 1 - \frac{4}{27} = \frac{23}{27}$

170. Let $\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right) = \theta$

Then $\cos^{-1}\left(\frac{1}{8}\right) = 2\theta$

$\Rightarrow \quad \cos(2\theta) = \frac{1}{8}$

$\Rightarrow \quad 2\cos^2\theta - 1 = \frac{1}{8}$

$\Rightarrow \quad 2\cos^2\theta = \frac{9}{8}$

$\Rightarrow \quad \cos^2\theta = \frac{9}{16}$

$\Rightarrow \quad \cos\theta = \frac{3}{4}$

$\Rightarrow \quad \cos\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right) = \frac{3}{4}$

171. Let $\frac{1}{2}\cos^{-1}\left(-\frac{1}{10}\right) = \theta$

Then $\cos^{-1}\left(-\frac{1}{10}\right) = 2\theta$

$\Rightarrow \quad \cos(2\theta) = -\frac{1}{10}$

$\Rightarrow \quad 2\cos^2\theta - 1 = -\frac{1}{10}$

$\Rightarrow \quad 2\cos^2\theta = 1 - \frac{1}{10} = \frac{9}{10}$

$\Rightarrow \quad \cos^2\theta = \frac{9}{20}$

$\Rightarrow \quad \cos\theta = \frac{3}{2\sqrt{5}}$

$\Rightarrow \quad \cos\theta = \frac{3}{2\sqrt{5}} = \frac{3\sqrt{5}}{2\sqrt{5}\times\sqrt{5}} = \frac{3\sqrt{5}}{10}$

$\Rightarrow \quad \cos\left(\frac{1}{2}\cos^{-1}\left(-\frac{1}{10}\right)\right) = \frac{3\sqrt{5}}{10}$

172. Let $\frac{1}{2}\cos^{-1}\left(\frac{1}{9}\right) = \theta$

$\Rightarrow \quad \cos^{-1}\left(\frac{1}{9}\right) = 2\theta$

$\Rightarrow \quad \cos(2\theta) = \frac{1}{9}$

$\Rightarrow \quad 2\cos^2(\theta) - 1 = \frac{1}{9}$

$\Rightarrow \quad 2\cos^2(\theta) = 1 + \frac{1}{9} = \frac{10}{9}$

$\Rightarrow \quad \cos^2(\theta) = \frac{5}{9}$

$\Rightarrow \quad \sin^2(\theta) = 1 - \frac{5}{9} = \frac{4}{9}$

$\Rightarrow \quad \sin(\theta) = \frac{2}{3}$

$\Rightarrow \quad \sin\left(\frac{1}{2}\cos^{-1}\left(\frac{1}{9}\right)\right) = \frac{2}{3}$

173. Let $\frac{1}{4}\tan^{-1}(\sqrt{63}) = \theta$

$\Rightarrow \quad \tan^{-1}(\sqrt{63}) = 4\theta$

$\Rightarrow \quad \tan(4\theta) = \sqrt{63}$

$\Rightarrow \quad \tan(4\theta) = \sqrt{63}$

$\Rightarrow \quad \frac{\sin(4\theta)}{\cos(4\theta)} = \sqrt{63}$

$\Rightarrow \quad \frac{\sin(4\theta)}{\sqrt{63}} = \frac{\cos(4\theta)}{1} = \frac{1}{8}$

Now, $\cos(4\theta) = \frac{1}{8}$

$\Rightarrow \quad 2\cos^2(2\theta) - 1 = \frac{1}{8}$

$\Rightarrow \quad 2\cos^2(2\theta) = 1 + \frac{1}{8} = \frac{9}{8}$

$\Rightarrow \quad \cos^2(2\theta) = \frac{9}{16}$

$\Rightarrow \quad \cos(2\theta) = \frac{3}{4}$ ...(i)

$\Rightarrow \quad 2\cos^2(\theta) - 1 = \frac{3}{4}$

$\Rightarrow \quad 2\cos^2(\theta) = \frac{7}{4}$

$\Rightarrow \quad \cos^2(\theta) = \frac{7}{8}$

$\Rightarrow \quad \cos(\theta) = \frac{\sqrt{7}}{2\sqrt{2}}$ ...(ii)

Also, $\sin(4\theta) = \frac{\sqrt{63}}{8}$

$\Rightarrow \quad 2\sin(2\theta)\cos(2\theta) = \frac{\sqrt{63}}{8}$

$\Rightarrow \quad 2\sin(2\theta) \times \frac{3}{4} = \frac{\sqrt{63}}{8}$, from (i)

$\Rightarrow \quad \sin(2\theta) = \frac{\sqrt{63}}{12}$

$\Rightarrow \quad 2\sin(\theta)\cos(\theta) = \frac{\sqrt{63}}{12}$, from (ii)

$\Rightarrow \quad 2\sin(\theta) \times \frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{63}}{12}$

$\Rightarrow \quad \sin(\theta) \times \frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{63}}{12}$

$\Rightarrow \quad \sin(\theta) = \frac{3\sqrt{2}}{12}$

$\Rightarrow \quad \sin(\theta) = \frac{\sqrt{2}}{4}$

$\Rightarrow \quad \sin(\theta) = \frac{1}{2\sqrt{2}}$

$\Rightarrow \quad \sin\left(\frac{1}{4}\tan^{-1}(\sqrt{63})\right) = \frac{1}{2\sqrt{2}}$

174. Let $\frac{1}{4}\tan^{-1}\left(\frac{24}{7}\right) = \theta$

$\Rightarrow \quad \tan^{-1}\left(\frac{24}{7}\right) = 4\theta$

$\Rightarrow \quad \tan(4\theta) = \frac{24}{7}$

$\Rightarrow \quad \frac{\sin(4\theta)}{24} = \frac{\cos(4\theta)}{7} = \frac{1}{25}$

Now, $\cos(4\theta) = \frac{7}{25}$

$\Rightarrow \quad 2\cos^2(2\theta) - 1 = \frac{7}{25}$

$\Rightarrow \quad 2\cos^2(2\theta) = 1 + \frac{7}{25} = \frac{32}{25}$

$\Rightarrow \quad \cos^2(2\theta) = \frac{32}{50}$

$\Rightarrow \quad \cos(2\theta) = \sqrt{\frac{32}{50}}$

$\Rightarrow \quad 2\cos^2(\theta) - 1 = \sqrt{\frac{32}{50}}$

$\Rightarrow \quad 2\cos^2(\theta) = 1 + \frac{8}{10} = \frac{18}{10}$

$\Rightarrow \quad \cos^2(\theta) = \frac{9}{10}$

$\Rightarrow \quad \cos(\theta) = \frac{3}{\sqrt{10}}$

$\Rightarrow \quad \cos\left(\frac{1}{4}\left(\tan^{-1}\left(\frac{24}{7}\right)\right)\right) = \frac{3}{\sqrt{10}}$

175. Let $\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right) = \theta$

$\Rightarrow \quad \cos^{-1}\left(\frac{2}{3}\right) = 2\theta$

$\Rightarrow \quad \cos(2\theta) = \frac{2}{3}$

$\Rightarrow \quad \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{2}{3}$

$\Rightarrow \quad 2 + 2\tan^2\theta = 3 - 3\tan^2\theta$

$\Rightarrow \quad 5\tan^2\theta = 1$

$\Rightarrow \quad \tan^2\theta = \frac{1}{5}$

$\Rightarrow \quad \tan\theta = \frac{1}{\sqrt{5}}$

$\Rightarrow \quad \tan\left(\frac{1}{2}\cos^{-1}\left(\frac{2}{3}\right)\right) = \frac{1}{\sqrt{5}}$

176. We have $\tan\left(2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right) = -\frac{7}{17}$

Let $2\tan^{-1}\left(\frac{1}{5}\right) = \theta$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{2}{5}}{1 - \frac{1}{25}}\right) = \theta$

$\Rightarrow \quad \tan\theta = \left(\frac{\frac{2}{5}}{1 - \frac{1}{25}}\right) = \frac{10}{24} = \frac{5}{12}$

Now, $\tan\left(\theta - \frac{\pi}{4}\right)$

$= \frac{\tan\theta - 1}{1 + \tan\theta}$

$= \frac{\frac{5}{12} - 1}{\frac{5}{12} + 1} = -\frac{7}{17}$

177. Let $\frac{1}{4}\sin^{-1}\left(-\frac{4}{5}\right) = \theta$

$\Rightarrow \quad \sin^{-1}\left(-\frac{4}{5}\right) = 4\theta$

$\Rightarrow \quad \sin(4\theta) = -\frac{4}{5}$

$\Rightarrow \quad \frac{2\tan(2\theta)}{1+\tan^2(2\theta)} = -\frac{4}{5}$

$\Rightarrow \quad \frac{\tan(2\theta)}{1+\tan^2(2\theta)} = -\frac{2}{5}$

$\Rightarrow \quad 2\tan^2(2\theta) + 5\tan(2\theta) + 2 = 0$

$\Rightarrow \quad \tan(2\theta) = -\frac{1}{2}, -2$

when $\tan(2\theta) = -\frac{1}{2}$

$\Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} = -\frac{1}{2}$

$\Rightarrow \quad \tan^2\theta - 4\tan\theta - 1 = 0$

$\Rightarrow \quad \tan\theta = 2-\sqrt{5}$

Now, $\tan\left(\frac{3\pi}{4} - \frac{1}{4}\sin^{-1}\left(-\frac{4}{5}\right)\right)$

$= \tan\left(\frac{3\pi}{4} - \theta\right)$

$= \tan\left(\pi - \left(\frac{\pi}{4} + \theta\right)\right)$

$= -\tan\left(\frac{\pi}{4} + \theta\right)$

$= -\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$

$= -\left(\frac{1+2-\sqrt{5}}{1-2+\sqrt{5}}\right)$

$= \left(\frac{3-\sqrt{5}}{1-\sqrt{5}}\right)$

$= \left(\frac{(3-\sqrt{5})(1+\sqrt{5})}{-4}\right)$

$= -\frac{1}{4}(3+2\sqrt{5}-5)$

$= -\frac{1}{4}(-2+2\sqrt{5})$

$= \left(\frac{1-\sqrt{5}}{2}\right)$

178. Given equation is

$x^2 - 3x < \sin^{-1}(\sin 2)$

$\Rightarrow \quad x^2 - 3x < (\pi - 2)$

$\Rightarrow \quad \left(x - \frac{3}{2}\right)^2 < \left(\pi - 2 + \frac{9}{4}\right) = \frac{4\pi+1}{4}$

$\Rightarrow \quad \left|\left(x - \frac{3}{2}\right)\right| < \frac{\sqrt{4\pi+1}}{2}$

$\Rightarrow \quad -\frac{\sqrt{4\pi+1}}{2} < \left(x - \frac{3}{2}\right) < \frac{\sqrt{4\pi+1}}{2}$

$\Rightarrow \quad \frac{3}{2} - \frac{\sqrt{4\pi+1}}{2} < x < \frac{\sqrt{4\pi+1}}{2} + \frac{3}{2}$

$\Rightarrow \quad -0.2 < x < 3.3$

Thus, the integral values of x are 0, 1, 2, 3.

179. Given in-equation is

$3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$

$\Rightarrow \quad 3x^2 + 8x < 2(\pi - 4) - (2\pi - 4)$

$\Rightarrow \quad 3x^2 + 8x + 4 < 0$

$\Rightarrow \quad 3x^2 + 6x + 2x + 4 < 0$

$\Rightarrow \quad 3x(x+2) + 2(x+2) < 0$

$\Rightarrow \quad (3x+2)(x+2) < 0$

$\Rightarrow \quad -2 < x < -\frac{2}{3}$

180. We have

$f(x) = \cos^{-1}x + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{1-3x^2}}{2}\right\}$

$= \cos^{-1}(x) + \cos^{-1}\left(\frac{1}{2}x + \sqrt{1-x^2}\sqrt{1-\frac{1}{4}}\right)$

$= \cos^{-1}(x) + \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(x)$

$= \cos^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$

= constant function.

Hence, the result.

181. We have $f(x)$

$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) + 2\tan^{-1}(x)$

$= \pi - 2\tan^{-1}(x) + 2\tan^{-1}(x)$

$= \pi$

Hence, the value of $f(2013) = \pi$

182. We have

$f(x) = 2\tan^{-1}\left(\frac{1+x}{1-x}\right) + \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= 2(\tan^{-1}(1) + \tan^{-1}(x)) + \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= 2\left(\frac{\pi}{4} + \tan^{-1}(x)\right) + \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= \left(\frac{\pi}{2} + 2\tan^{-1}(x)\right) + \left(\frac{\pi}{2} - 2\tan^{-1}(x)\right)$

$= \pi$

Hence, the value of $f\left(\frac{1}{2014}\right) = \pi$

183. We have

$$f(x)=\sin^{-1}\left(\frac{6x}{x^2+9}\right)+2\tan^{-1}\left(-\frac{x}{3}\right)$$

$$=\sin^{-1}\left(\frac{2\cdot\left(\frac{x}{3}\right)}{1+\left(\frac{x}{3}\right)^2}\right)-2\tan^{-1}\left(\frac{x}{3}\right)$$

$$=2\tan^{-1}\left(\frac{x}{3}\right)-2\tan^{-1}\left(\frac{x}{3}\right)$$

$$=0$$

It will happen when $\left|\frac{x}{3}\right|\le 1$

$\Rightarrow\quad |x|\le 3$

$\Rightarrow\quad -3\le x\le 3$

184. We have

$$f(x)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+2\tan^{-1}(x)$$

$$=-2\tan^{-1}(x)+2\tan^{-1}(x),\ x\le 0$$

$$=0$$

It is possible only when $x\le 0$

$\Rightarrow\quad x\in(-\infty, 0]$

185. We have

$$f(x)=3\cos^{-1}(2x^2-1)+2\cos^{-1}(4x^3-3x)$$

$$=3(2\cos^{-1}x)+2(2\pi-3\cos^{-1}x)$$

$$(\text{for } 0\le x<1)\ \left(\text{for } -\frac{1}{2}\le x\le\frac{1}{2}\right)$$

It is possible only when $0\le x\le\frac{1}{2}$

$\Rightarrow\quad x\in\left[0,\frac{1}{2}\right]$

Also,

$$f(x)=3\cos^{-1}(2x^2-1)+2\cos^{-1}(4x^3-3x)$$

$$=3(2\pi-2\cos^{-1}x)+2(-2\pi+3\cos^{-1}x)$$

$$(\text{for } -1\le x\le 0)\ \left(\text{for } -1\le x<-\frac{1}{2}\right)$$

$$=2\pi$$

It is possible only when $x\in\left[-1,-\frac{1}{2}\right)$

Hence, the value of x is

$$\left[-1,-\frac{1}{2}\right)\cup\left[0,\frac{1}{2}\right]$$

186. We have

$$\tan^{-1}y=4\tan^{-1}x$$

$$\Rightarrow\quad \tan^{-1}y=\tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)$$

$$\Rightarrow\quad y=\frac{4x(1-x^2)}{1-6x^2+x^4}$$

Which is a function of x.

Let $\tan^{-1}x=\frac{\pi}{8}$

$$\Rightarrow\quad x=\tan\left(\frac{\pi}{8}\right)$$

$$\Rightarrow\quad \tan^{-1}y=4\tan^{-1}x=\frac{\pi}{2}$$

$$\Rightarrow\quad \frac{4x(1-x^2)}{1-6x^2+x^4}\to\infty$$

$$\Rightarrow\quad 1-6x^2+x^4=0$$

$\Rightarrow\quad x=\tan\left(\frac{\pi}{8}\right)$ is a root of $1+x^4=6x^2$

187. Do yourself.

188. We have

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)$$

$$=\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right),\ \theta=\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)$$

$$=\frac{1+\tan\theta}{1-\tan\theta}+\frac{1-\tan\theta}{1+\tan\theta},\ \cos(2\theta)=\frac{a}{b}$$

$$=\frac{2(1+\tan^2\theta)}{1-\tan^2\theta}$$

$$=\frac{2}{\cos(2\theta)}$$

$$=\frac{2}{\frac{a}{b}}=\frac{2b}{a}$$

189. Do yourself.

190. We have

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

$$=\tan^{-1}\left(\frac{a-b}{1+ab}\right)+\tan^{-1}\left(\frac{b-c}{1+bc}\right)+\tan^{-1}\left(\frac{c-a}{1+ca}\right)$$

$$=(\tan^{-1}a-\tan^{-1}b)+(\tan^{-1}b-\tan^{-1}c)+(\tan^{-1}c-\tan^{-1}a)$$

$$=0$$

191. We have

$$\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right)$$

$$=\tan\left(\frac{1}{2}\cdot 2\tan^{-1}x+\frac{1}{2}\cdot 2\tan^{-1}y\right)$$

$$=\tan(\tan^{-1}x+\tan^{-1}y)$$

$$=\tan\left(\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right)$$

$$=\left(\frac{x+y}{1-xy}\right),\ xy<1$$

192. We have

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)-\tan^{-1}\left(\frac{1-y}{1+y}\right)$$

$$=\{\tan^{-1}(1)-\tan^{-1}(x)\}-\{\tan^{-1}(1)-\tan^{-1}(y)\}$$

$$=\tan^{-1}(y)-\tan^{-1}(x)$$

$$= \tan^{-1}\left(\frac{y-x}{1+xy}\right)$$

$$= \sin^{-1}\left(\frac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}\right)$$

193. We have

$$\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\cot A + \cot^3 A}{1-\cot^4 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(\frac{\tan A}{\tan^2 A - 1}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) + \tan^{-1}\left(-\frac{\tan A}{1-\tan^2 A}\right)$$

$$= \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right) - \tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right)$$

$$= 0$$

194. We have

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{\theta}{2}\right)\right)$$

$$= \cos^{-1}\left(\frac{1-\left(\frac{a-b}{a+\text{b}}\right)\tan^2\left(\frac{\theta}{2}\right)}{1+\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{\theta}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{(a+b)-(a-b)\tan^2\left(\frac{\theta}{2}\right)}{(a+b)+(a-b)\tan^2\left(\frac{\theta}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)+b\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{a\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)+b\left(1-\tan^2\left(\frac{\theta}{2}\right)\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\left(\dfrac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)+b}{a+b\left(\dfrac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\cos\theta + b}{a + b\cos\theta}\right)$$

195. We have

$2\tan(\tan^{-1} a + \tan^{-1} a^3)$

$$= 2\tan\left(\tan^{-1}\left(\frac{a+a^3}{1-a^4}\right)\right)$$

$$= 2\tan\left(\tan^{-1}\left(\frac{a}{1-a^2}\right)\right)$$

$$= \left(\frac{2a}{1-a^2}\right)$$

$$= \tan(2\tan^{-1} a)$$

196. Do yourself

197. Given, $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

Let $A = \sin^{-1} x, B = \sin^{-1} y, C = \sin^{-1} z$

$\Rightarrow \quad x = \sin A, y = \sin B, z = \sin C$

$\Rightarrow \quad \cos A = \sqrt{1-x^2}, \cos B = \sqrt{1-y^2}$

and $\cos C = \sqrt{1-z^2}$

Now, $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$

$= \sin A\cos A + \sin B\cos B + \sin C\cos C$

$$= \frac{1}{2}[2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C]$$

$$= \frac{1}{2}(\sin(2A) + \sin(2B) + \sin(2C))$$

$$= \frac{1}{2}(4\sin A\sin B\sin C)$$

$= (2\sin A\sin B\sin C)$

$= 2xyz$

198. We have

$\cos^{-1} x \cos^{-1} y + \cos^{-1} z = \pi,$

$\Rightarrow \quad \cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$

$\Rightarrow \quad \cos^{-1}(x \cdot y - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos^{-1}(-z)$

$\Rightarrow \quad (xy+z)^2 = (\sqrt{(1-x^2)(1-y^2)})^2$

$\Rightarrow \quad x^2y^2 + 2xyz + z^2 = 1 - x^2 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad 2xyz + z^2 = 1 - x^2 - y^2$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$

199. We have

$$\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{xy}{6} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \theta$$

$$\Rightarrow \quad \left(\frac{xy}{6} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right) = \cos\theta$$

$$\Rightarrow \quad \left(\frac{xy}{6} - \cos\theta\right)^2 = \left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{9}\right)$$

$$\Rightarrow \quad \frac{x^2y^2}{36} - \frac{xy}{3}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36}$$

$$\Rightarrow \quad -\frac{xy}{3}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{4} - \frac{y^2}{9}$$

$\Rightarrow \quad \frac{x^2}{4}+\frac{y^2}{9}-\frac{xy}{3}\cos\theta = 1-\cos^2\theta$

$\Rightarrow \quad \frac{x^2}{4}+\frac{y^2}{9}-\frac{xy}{3}\cos\theta = \sin^2\theta$

$\Rightarrow \quad 9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

Hence, the result.

200. Given, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value

Thus, $\sin^{-1}x=\frac{\pi}{2}=\sin^{-1}y=\sin^{-1}z$

So, $x = 1, y = 1, z = 1$

Hence, the value of

$x^2 + y^2 + z^2 - 2xyz$

$= 1 + 1 + 1 - 2$

$= 1$

201. Given, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value

Thus, $\sin^{-1}x=\frac{\pi}{2}=\sin^{-1}y=\sin^{-1}z$

So, $x = 1, y = 1, z = 1$

Hence, the value of

$xy + yz + zx = 1 + 1 + 1$

$= 3$

202. Given, $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$

It is possible only when each term will provide us the maximum value

Thus, $\sin^{-1}x=\frac{\pi}{2}=\sin^{-1}y=\sin^{-1}z$

So, $x = 1, y = 1, z = 1$

Hence, the value of

$x^{2012}+y^{2012}+z^{2012}-\frac{9}{x^{2013}+y^{2013}+z^{2013}}$

$= 1+1+1-\frac{9}{1+1+1}$

$= 3 - 3$

$= 0$

203. Given, $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

It is possible only when, each term will provide us the maximum value.

Thus, $\cos^{-1}x = \pi$, $\cos^{-1}y = \pi$, $\cos^{-1}z = \pi$

$x = -1, y = -1, z = -1$

Hence, the value of

$xy + yz + zx = 1 + 1 + 1$

$= 3$

204. Given, $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$

It is possible only when, each term will provide us the maximum value.

Thus, $\cos^{-1}x = \pi$, $\cos^{-1}y = \pi$, $\cos^{-1}z = \pi$

$x = -1, y = -1, z = -1$

Hence, the value of

$\left(\frac{x^{2013}+y^{2013}+z^{2013}+6}{x^{2014}+y^{2014}+z^{2014}}\right)$

$=\frac{-1-1-1+6}{1+1+1}$

$=\frac{3}{3}=1$

205. Given, $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y=\frac{\pi}{2}-\tan^{-1}z$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right)=\cot^{-1}(z)$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}\left(\frac{1}{z}\right)$

$\Rightarrow \quad \left(\frac{x+y}{1-xy}\right)=\left(\frac{1}{z}\right)$

$\Rightarrow \quad xz + yz = 1 - xy$

$\Rightarrow \quad xy + yz + zx = 1$

206. Given, $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right)=\frac{\pi}{4}$

$\Rightarrow \quad \left(\frac{x+y}{1-xy}\right)=\tan\left(\frac{\pi}{4}\right)=1$

$\Rightarrow \quad x + y = 1 - xy$

$\Rightarrow \quad x + y + xy = 1$

207. Given, $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)=\pi$

$\Rightarrow \quad \left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)=\tan(\pi)=0$

$\Rightarrow \quad \left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)=0$

$\Rightarrow \quad x + y + z - xyz = 0$

$\Rightarrow \quad x + y + z = xyz$

Hence, the result.

208. Given, $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)=\alpha$

Put $x^2 = \sin(2\theta)$

Now, $\tan^{-1}\left(\frac{\sqrt{1+\sin 2\theta}-\sqrt{1-\sin 2\theta}}{\sqrt{1+\sin 2\theta}+\sqrt{1-\sin 2\theta}}\right)=\alpha$

$\Rightarrow \quad \tan^{-1}\left(\frac{(\cos\theta+\sin\theta)-(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)+(\cos\theta-\sin\theta)}\right)=\alpha$

$\Rightarrow \quad \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)=\alpha$

$\Rightarrow \tan^{-1}(\tan\theta) = \alpha$

$\Rightarrow \theta = \alpha$

$\Rightarrow 2\theta = 2\alpha$

$\Rightarrow \sin(2\theta) = \sin(2\alpha)$

$\Rightarrow x^2 = \sin(2\alpha)$

209. We have

$m = \tan^2(\sec^{-1} 2) + \cot^2(\text{cosec}^{-1} 3)$

$= \tan^2(\theta) + \cot^2(\varphi)$, where

$\sec\theta = 2$ and $\text{cosec}\,\varphi = 3$

$= 1 + \sec^2\theta + 1 + \text{cosec}^2\varphi$

$= 1 + 4 + 1 + 9$

$= 15$

Hence, the value of $(m^2 + m + 10)$

$= 225 + 15 + 10$

$= 250$

210. Given, $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right) = \frac{\pi}{4}$

$\Rightarrow \sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right) = \frac{\pi}{2}$

$\Rightarrow \left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right) = 1$

$\Rightarrow 3\sin 2\theta = 5 + 4\cos 2\theta$

$\Rightarrow \frac{3.2\tan\theta}{1 + \tan^2\theta} = 5 + 4\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)$

$\Rightarrow \frac{6\tan\theta}{1 + \tan^2\theta} = \frac{5 + 5\tan^2\theta + 4 - 4\tan^2\theta}{1 + \tan^2\theta}$

$\Rightarrow 6\tan\theta = \tan^2\theta + 9$

$\Rightarrow \tan^2\theta - 6\tan\theta + 9 = 0$

$\Rightarrow (\tan\theta - 3)^2 = 0$

$\Rightarrow (\tan\theta - 3) = 0$

$\Rightarrow \tan\theta = 3$

211. Given, $m = \frac{(\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3)}{(\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3)}$

$= \frac{\pi}{\frac{\pi}{2}} = 2$

Hence, the value of

$(m + 2)^{m+1} = (2 + 2)^3 = 4^3 = 64$

212. Given equation is

$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{3\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{2x + 3x}{1 - 6x^2}\right) = \frac{\pi}{4}$

$\Rightarrow \left(\frac{5x}{1 - 6x^2}\right) = 1$

$\Rightarrow 5x = 1 - 6x^2$

$\Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow 6x^2 + 6x - x - 1 = 0$

$\Rightarrow 6x(x + 1) - 1(x + 1) = 0$

$\Rightarrow (6x - 1)(x + 1) = 0$

$\Rightarrow x = -1, \frac{1}{6}$

213. $\tan^{-1}\left(\frac{x+1}{x-1}\right) + \tan^{-1}\left(\frac{x-1}{x}\right) = \tan^{-1}(-7)$

$\Rightarrow \tan^{-1}\left(\frac{\frac{x+1}{x-1} + \frac{x-1}{x}}{1 - \frac{x+1}{x-1} \times \frac{x-1}{x}}\right) = \tan^{-1}(-7)$

$\Rightarrow \tan^{-1}\left(\frac{x^2 + x + x^2 - 2x + 1}{x^2 - x - x^2 + 1}\right) = \tan^{-1}(-7)$

$\Rightarrow \tan^{-1}\left(\frac{2x^2 - x + 1}{1 - x}\right) = \tan^{-1}(-7)$

$\Rightarrow \left(\frac{2x^2 - x + 1}{1 - x}\right) = (-7)$

$\Rightarrow 2x^2 - x + 1 = -7 + 7x$

$\Rightarrow 2x^2 - 8x + 8 = 0$

$\Rightarrow x^2 - 4x + 4 = 0$

$\Rightarrow (x - 2)^2 = 0$

$\Rightarrow x = 2$

Hence, the solution is $x = 2$

214. Given equation is

$\sin^{-1}(2x) + \sin^{-1}(x) = \frac{\pi}{3}$

$\Rightarrow \sin^{-1}(2x) = \frac{\pi}{3} - \sin^{-1}x$

$\Rightarrow \sin^{-1}(2x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}x$

$\Rightarrow \sin^{-1}(2x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1 - x^2} - x\sqrt{1 - \frac{3}{4}}\right)$

$\Rightarrow (2x) = \left(\frac{\sqrt{3}}{2}\sqrt{1 - x^2} - x\sqrt{1 - \frac{3}{4}}\right)$

$\Rightarrow (2x) = \left(\frac{\sqrt{3}}{2}\sqrt{1 - x^2} - \frac{x}{2}\right)$

$\Rightarrow \left(2x + \frac{x}{2}\right) = \frac{\sqrt{3}}{2}\sqrt{1 - x^2}$

$\Rightarrow \left(\frac{5x}{2}\right) = \frac{\sqrt{3}}{2}\sqrt{1 - x^2}$

$\Rightarrow \left(\frac{25x^2}{4}\right) = \frac{3(1 - x^2)}{4}$

$\Rightarrow 25x^2 = 3 - 3x^2$

$\Rightarrow 28x^2 = 3$

$\Rightarrow x^2 = \frac{3}{28}$

$\Rightarrow x = \pm\sqrt{\frac{3}{28}} = \pm\frac{\sqrt{3}}{2\sqrt{7}}$

215. Given equation is

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1}x=\frac{\pi}{4}-\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad \cos^{-1}x=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)-\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad \cos^{-1}x=\cos^{-1}\left(\frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{5}}+\sqrt{1-\frac{1}{2}}\sqrt{1-\frac{4}{5}}\right)$$

$$\Rightarrow \quad x=\left(\frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{5}}\right)$$

$$\Rightarrow \quad x=\frac{3}{\sqrt{10}}$$

216. Given equation is

$$\sin^{-1}(x)+\sin^{-1}(3x)=\frac{\pi}{3}$$

$$\Rightarrow \quad \sin^{-1}(3x)=\frac{\pi}{3}-\sin^{-1}(x)$$

$$\Rightarrow \quad \sin^{-1}(3x)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}(x)$$

$$\Rightarrow \quad \sin^{-1}(3x)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1-x^2}-x\sqrt{1-\frac{3}{4}}\right)$$

$$\Rightarrow \quad (3x)=\left(\frac{\sqrt{3}}{2}\sqrt{1-x^2}-x\sqrt{1-\frac{3}{4}}\right)$$

$$\Rightarrow \quad \left(3x+\frac{x}{2}\right)=\frac{\sqrt{3}}{2}\sqrt{1-x^2}$$

$$\Rightarrow \quad \frac{7x}{2}=\frac{\sqrt{3}}{2}\sqrt{1-x^2}$$

$$\Rightarrow \quad 49x^2=3(1-x^2)$$

$$\Rightarrow \quad 52x^2=3$$

$$\Rightarrow \quad x^2=\frac{3}{52}$$

$$\Rightarrow \quad x=\pm\sqrt{\frac{3}{52}}$$

Hence, the solutions are

$$\left\{-\sqrt{\frac{3}{52}},\sqrt{\frac{3}{52}}\right\}$$

217. Given equation is

$$\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{1+4x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\frac{1}{1+2x}\times\frac{1}{1+4x}}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \left(\frac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\frac{1}{1+2x}\times\frac{1}{1+4x}}\right)=\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \left(\frac{1+2x+1+4x}{1+6x+8x^2-1}\right)=\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \left(\frac{2+6x}{6x+8x^2}\right)=\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \frac{1+3x}{3x+4x^2}=\frac{2}{x^2}$$

$$\Rightarrow \quad 3x^3+x^2=6x+8x^2$$

$$\Rightarrow \quad 3x^3-7x^2-6x=0$$

$$\Rightarrow \quad (3x^2-7x-6)\,x=0$$

$$\Rightarrow \quad (3x^2-9x+2x-6)\,x=0$$

$$\Rightarrow \quad x(x-3)(2x+3)=0$$

$$\Rightarrow \quad x=0,3,-\frac{3}{2}$$

Hence, the solutions are

$$\left\{0,3,-\frac{3}{2}\right\}$$

218. Given equation is

$$2\tan^{-1}(2x+1)=\cos^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2(2x+1)}{1-(2x+1)^2}\right)=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$\Rightarrow \quad \frac{2(2x+1)}{1-(2x+1)^2}=\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$\Rightarrow \quad -\frac{2(2x+1)}{(4x^2+4x)}=\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$\Rightarrow \quad -\frac{(2x+1)}{(2x^2+2x)}=\left(\frac{\sqrt{1-x^2}}{x}\right), x=0$$

$$\Rightarrow \quad -\frac{(2x+1)}{(2x+2)}=\sqrt{1-x^2}, x=0$$

$$\Rightarrow \quad \frac{(2x+1)^2}{(2x+2)^2}=(1-x^2), x=0$$

$$\Rightarrow \quad \frac{4x^2+4x+1}{4x^2+8x+4}=1-x^2, x=0$$

$$\Rightarrow \quad 3-4x^4-8x^3-4x^2+4x=0, x=0$$

$$\Rightarrow \quad 4x^4+8x^3+4x^2-4x-3=0, x=0$$

Clearly, it has 3 solutions.

219. Given equation is

$$\cos^{-1}x-\sin^{-1}x=\cos^{-1}(x\sqrt{3})$$

$$\Rightarrow \quad \cos^{-1}x-\frac{\pi}{2}+\cos^{-1}x=\cos^{-1}(x\sqrt{3})$$

$$\Rightarrow \quad 2\cos^{-1}x-\frac{\pi}{2}=\cos^{-1}(x\sqrt{3})$$

$\Rightarrow \quad 2\cos^{-1}x = \frac{\pi}{2} + \cos^{-1}(x\sqrt{3})$

$\Rightarrow \quad \cos^{-1}(2x^2 - 1) = \frac{\pi}{2} + \cos^{-1}(x\sqrt{3})$

$\Rightarrow \quad (2x^2 - 1) = \cos\left(\frac{\pi}{2} + \cos^{-1}(x\sqrt{3})\right)$

$\Rightarrow \quad (2x^2 - 1) = -\sin(\cos^{-1}(x\sqrt{3}))$

$\Rightarrow \quad (2x^2 - 1) = -\sqrt{1 - 3x^2}$

$\Rightarrow \quad (2x^2 - 1)^2 = (1 - 3x^2)$

$\Rightarrow \quad 4x^4 - 4x^2 + 1 = (1 - 3x^2)$

$\Rightarrow \quad 4x^4 - x^2 = 0$

$\Rightarrow \quad x^2(4x^2 - 1) = 0$

$\Rightarrow \quad x^2(2x - 1)(2x + 1) = 0$

$\Rightarrow \quad x = 0, \pm\frac{1}{2}$

Hence, the solutions are

$$\left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$$

220. We have, $\tan^{-1} y = 4\tan^{-1} x$

$\Rightarrow \quad \tan^{-1}y = \tan^{-1}\left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$

$\Rightarrow \quad y = \frac{4x(1 - x^2)}{1 - 6x^2 + x^4}$

Which is a function of x.

Let $\tan^{-1}x = \frac{\pi}{8}$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right)$

$\Rightarrow \quad \tan^{-1}y = 4\tan^{-1}x = \frac{\pi}{2}$

$\Rightarrow \quad \frac{4x(1 - x^2)}{1 - 6x^2 + x^4} \to \infty$

$\Rightarrow \quad 1 - 6x^2 + x^4 = 0$

$\Rightarrow \quad x = \tan\left(\frac{\pi}{8}\right)$ is a root of $1 + x^4 = 6x^2$

Level IIA

1. We have $\sin^{-1}(\sin 10)$
$= \sin^{-1}(\sin(3\pi - 10))$
$= (3\pi - 10)$

2. We have $\cos^{-1}(\cos 5)$
$= \cos^{-1}(\cos(2\pi - 5))$
$= (2\pi - 5)$

3. We have $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3)$
$= \tan^{-1}(1) + \pi + \tan^{-1}\left(\frac{2+3}{1-2.3}\right)$
$= \frac{\pi}{4} + \pi + \tan^{-1}\left(-\frac{5}{5}\right)$
$= \frac{\pi}{4} + \pi + \tan^{-1}(-1)$
$= \frac{\pi}{4} + \pi - \tan^{-1}(1)$
$= \frac{\pi}{4} + \pi - \frac{\pi}{4}$
$= \pi$

4. Given, $\sin^{-1} x > \cos^{-1} x$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}x > \sin^{-1} > \sin^{-1}x + \cos^{-1}x$

$\Rightarrow \quad 2\sin^{-1}x > \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}x > \frac{\pi}{4}$

$\Rightarrow \quad x > \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Also, the domain of $\sin^{-1} x$ is $[-1, 1]$

Thus, the solution set is $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

5. Given, $\sin^{-1} x < \cos^{-1} x$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} x < \sin^{-1} x + \cos^{-1} x$

$\Rightarrow \quad 2\sin^{-1}x < \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}x < \frac{\pi}{4}$

$\Rightarrow \quad x < \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Also, the domain of $\sin^{-1} x$ is $[-1, 1]$

Thus, the solution set is $x \in \left[-1, \frac{1}{\sqrt{2}}\right)$

6. Given, $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1 - x^2})$

Now, the range of $\sin^{-1}(2x\sqrt{1 - x^2})$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Thus, $-\frac{\pi}{2} \le 2\sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1}x \le \frac{\pi}{4}$

$\Rightarrow \quad \sin\left(-\frac{\pi}{4}\right) \le x \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad -\sin\left(\frac{\pi}{4}\right) \le x \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

Hence, the solution set is $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

7. Given, $3\sin^{-1}x = \pi + \sin^{-1}(3x - 4x^3)$

Now, the range of $\pi + \sin^{-1}(3x - 4x^3)$

is $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Thus, $\frac{\pi}{2} \le 3\sin^{-1}x \le \frac{3\pi}{2}$

$\Rightarrow \quad \frac{\pi}{6} \le \sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \quad \sin\left(\frac{\pi}{6}\right) \le x \le \sin\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad \frac{1}{2} \le x \le 1$

Hence, the solution set is $\left[\frac{1}{2}, 1\right]$

8. Given, $2\tan^{-1}x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Now, the range of $\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Thus, $\frac{\pi}{2} \le 2\tan^{-1}x \le \frac{3\pi}{2}$

$\Rightarrow \quad \frac{\pi}{4} \le \tan^{-1}x \le \frac{3\pi}{4}$

$\Rightarrow \quad \tan\left(\frac{\pi}{4}\right) \le x < \tan\left(\frac{\pi}{2}\right)$

and $\quad \tan\left(\frac{\pi}{2}\right) < x \le \tan\left(\frac{3\pi}{4}\right)$

$\Rightarrow \quad 1 \le x < \infty$ and $-\infty < x \le -1$

Therefore, the solution set is

$x \in (-\infty, -1] \cup [1, \infty)$

9. We have

$$\cos\left(\frac{\pi}{6} + \cos^{-1}\left(-\frac{1}{2}\right)\right)$$
$$= \cos\left(\frac{\pi}{6} + \pi - \cos^{-1}\left(\frac{1}{2}\right)\right)$$
$$= \cos\left(\frac{\pi}{6} + \pi - \frac{\pi}{3}\right)$$
$$= \cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$$
$$= \cos\left(\frac{5\pi}{6}\right)$$
$$= \cos\left(\pi - \frac{\pi}{6}\right)$$
$$= -\cos\left(\frac{\pi}{6}\right)$$
$$= -\frac{1}{2}$$

10. We have

$$\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)))$$
$$= \cos^{-1}\left(\cos\left(2\cos^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}}\right)\right)\right)$$
$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2(\sqrt{2}-1)^2}{(4-2\sqrt{2})} - 1\right)\right)\right)$$
$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2(3-2\sqrt{2}) - 4 + 2\sqrt{2}}{(4-2\sqrt{2})}\right)\right)\right)$$
$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2-2\sqrt{2}}{(4-2\sqrt{2})}\right)\right)\right)$$
$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(\frac{2(1-\sqrt{2})}{2\sqrt{2}(\sqrt{2}-1)}\right)\right)\right)$$
$$= \cos^{-1}\left(\cos\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)\right)$$
$$= \cos^{-1}\left(\cos\left(\pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)\right)$$
$$= \cos^{-1}\left(\cos\left(\pi - \frac{\pi}{4}\right)\right)$$
$$= \cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$$
$$= \left(\frac{3\pi}{4}\right)$$

11. We have

$$\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$$
$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r(r+1)}\right)$$
$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{(r+1)-\mathrm{r}}{1+r(r+1)}\right)$$
$$= \sum_{r=0}^{\infty} (\tan^{-1}(r+1) - \tan^{-1}(r))$$
$$= \sum_{r=0}^{n} (\tan^{-1}(r+1) - \tan^{-1}(r)), n \to \infty$$

$= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2))$
$+ (\tan^{-1}(4) - \tan^{-1}(3) + (\tan^{-1}(5) - \tan^{-1}(4))$
$+ \ldots +) \tan^{-1}(n+1) - \tan^{-1}(n))$
$= (\tan^{-1}(n+1) - \tan^{-1}(1)), n \to \infty$

$$= \tan^{-1}\left(\frac{(n+1)-1}{1+(n+1)\cdot 1}\right), n \to \infty$$
$$= \tan^{-1}\left(\frac{n}{n+2}\right), n \to \infty$$

$= \tan^{-1}(1)$, when $n \to \infty$

$= \frac{\pi}{4}$

12. We have

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r-1}}{1+2^r \cdot 2^{r-1}}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^r - 2^{r-1}}{1+2^r \cdot 2^{r-1}}\right)$$

$$= \sum_{r=1}^{n} (\tan^{-1}(2^r) - \tan^{-1}(2^{r-1}))$$

$= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(2^2) - \tan^{-1}(2))$
$+ (\tan^{-1}(2^3) - \tan^{-1}(2^2)) + \ldots$
$+ (\tan^{-1}(2^n) - \tan^{-1}(2^{n-1}))$

$= \tan^{-1}(2^n) - \tan^{-1}(1)$

$= \tan^{-1}(2^n) - \frac{\pi}{4}$

13. We have

$$\sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$$

$$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}\cdot\sqrt{r-1}}\right)$$

$$= \sum_{r=1}^{n} (\tan^{-1}(\sqrt{r}) - \tan^{-1}(\sqrt{r-1}))$$

$= (\tan^{-1}(1) - \tan^{-1}(0) + (\tan^{-1}(2) - \tan^{-1}(1))$
$+ (\tan^{-1}(3) - \tan^{-1}(2)) + (\tan^{-1}(4) - \tan^{-1}(3))$
$+ \ldots + (\tan^{-1}(n) - \tan^{-1}(n-1))$

$= \tan^{-1}(n)$

14. We have

$$\tan^{-1}\left(\frac{a_1 x - y}{a_1 y + x}\right) + \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_3 a_2}\right) + \ldots + \tan^{-1}\left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right)$$

$$= \tan^{-1}\left(\frac{a_1 - \frac{y}{x}}{1 + a_1 \cdot \frac{y}{x}}\right) + \tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_3 a_2}\right) + \ldots + \tan^{-1}\left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right)$$

$$= \tan^{-1}(a_1) - \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1}(a_2) - \tan^{-1}(a_1)$$

$\tan^{-1}(a_3) - \tan^{-1}(a_2) + \ldots + \tan^{-1}(a_n)$
$- \tan^{-1}(a_{n-1}) + \cot^{-1}(a_n)$

$$= \tan^{-1}(a_n) + \cot^{-1}(a_n) - \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \cot^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{x}{y}\right)$$

15. We have $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{\pi^2}{8}$

$\Rightarrow \quad (\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x \cdot \cot^{-1}x = \frac{\pi^2}{8}$

$\Rightarrow \quad \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1}x \cdot \cot^{-1}x = \frac{\pi^2}{8}$

$\Rightarrow \quad \frac{\pi^2}{4} - 2\tan^{-1}x \cdot \cot^{-1}x = \frac{\pi^2}{8}$

$\Rightarrow \quad 2\tan^{-1}x \cdot \cot^{-1}x = \frac{\pi^2}{8}$

$\Rightarrow \quad \tan^{-1}x \cdot \cot^{-1}x = \frac{\pi^2}{16}$

$\Rightarrow \quad a\left(\frac{\pi}{2} - a\right) = \frac{\pi^2}{16}$, where $a = \tan^{-1}x$

$\Rightarrow \quad 16a\left(\frac{\pi}{2} - a\right) = \pi^2$

$\Rightarrow \quad 16a^2 - 8a\pi + \pi^2 = 0$

$\Rightarrow \quad (4a - \pi)^2 = 0$

$\Rightarrow \quad (4a - \pi) = 0$

$\Rightarrow \quad a = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{4}$

$\Rightarrow \quad x = 1$

Hence, the solution is $x = 1$

16. Given,

$f(x) = (\sec^{-1}x)^2 + (\operatorname{cosec}^{-1}x)^2$

$= (\sec^{-1}x + \operatorname{cosec}^{-1}x)^2 - 2\sec^{-1}x \cdot \operatorname{cosec}^{-1}x$

$$= \left(\frac{\pi}{2}\right)^2 - 2\sec^{-1}x\left(\frac{\pi}{2} - \sec^{-1}x\right)$$

$$= \frac{\pi^2}{4} - \pi \cdot \sec^{-1}x + 2(\sec^{-1}x)^2$$

$$= 2\left((\sec^{-1}x)^2 - \frac{\pi}{2}\cdot\sec^{-1}x + \frac{\pi^2}{8}\right)$$

$$= 2\left((\sec^{-1}x)^2 - 2\cdot\sec^{-1}x\cdot\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 + \frac{\pi^2}{8} - \frac{\pi^2}{16}\right)$$

$$= 2\left(\sec^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{8}$$

Maximum value of $f(x)$ is $\frac{\pi^2}{4}$ at $x = 1$.

17. Given,

$$f(x) = (\sin^{-1}x)^3 + (\cos^{-1}x)^3$$
$$= (\sin^{-1}x + \cos^{-1}x)((\sin^{-1}x)^2 + (\cos^{-1}x)^2 - \sin^{-1}x\cos^{-1}x)$$
$$= (\sin^{-1}x + \cos^{-1}x)((\sin^{-1}x + \cos^{-1}x)^2 - 3\sin^{-1}x\cos^{-1}x)$$
$$= \frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)\right)$$
$$= \frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)^2 - 3a\left(\frac{\pi}{2} - a\right)\right)$$
$$= \frac{\pi}{2}\left(\frac{\pi^2}{4} - \frac{3a\pi}{2} + 3a^2\right)$$
$$= \frac{\pi}{2}\left(3\left(a^2 - \frac{a\pi}{2}\right) + \frac{\pi^2}{4}\right)$$
$$= \frac{\pi}{2}\left(3\left(a^2 - 2\cdot\frac{\pi}{4}\cdot a + \left(\frac{\pi}{6}\right)^2\right) + \frac{\pi^2}{4} - \frac{3\pi^2}{16}\right)$$
$$= \frac{\pi}{2}\left(3\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{4} - \frac{3\pi^2}{16}\right)$$
$$= \frac{\pi}{2}\left(3\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right)$$
$$= \left(\frac{3\pi}{2}\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^3}{32}\right)$$
$$= \frac{3\pi}{2}\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^3}{32} \geq \frac{\pi^3}{32}$$

Hence, the minimum value of $f(x)$ is $\frac{\pi^3}{32}$

18. Given,

$$[\cot^{-1}x] + [\cos^{-1}x] = 0$$

It is possible only when

$\Rightarrow$ $\cot^{-1}x = 0$ and $\cos^{-1}x = 0$

$\Rightarrow$ $0 \leq \cot^{-1}x < 1$ and $0 \leq \cos^{-1}x < 1$

$\Rightarrow$ $x \in (\cot 1, \infty)$ and $x \in (\cos 1, 1]$

Thus, the solution is x (cot 1, 1]

19. Given,

$$[\sin^{-1}x] + [\cos^{-1}x] = 0$$

It is possible only when

$\Rightarrow$ $[\sin^{-1}x] = 0$ and $[\cos^{-1}x] = 0$

$\Rightarrow$ $0 \leq \sin^{-1}x < 1$ and $0 \leq \cos^{-1}x < 1$

$\Rightarrow$ $x \in [0, \sin 1)$ and $x \in (\cos 1, 1]$

Thus, the solution is $x \in (\cos 1, \sin 1)$

20. Given,

$$[\tan^{-1}x] + [\cot^{-1}x] = 2$$

The range of $[\tan^{-1}x]$ is $\{-2, -1, 0, 1\}$ and $[\cot^{-1}x]$ is $\{0, 1, 2, 3\}$

Case I: When $[\cot^{-1}x] = 1$ and $[\tan^{-1}x] = 1$

$\Rightarrow$ $1 \leq \cot^{-1}x < 2$ and $1 \leq \tan^{-1}x < 2$

$\Rightarrow$ $x \in (\cot 2, \cot 1]$ and $x \in [\tan 1, \tan 2)$

$\Rightarrow$ $x \in \varphi$ ({ cot 1 < tan 1)

Case II: When $[\cot^{-1}x] = 3$ and $[\tan^{-1}x] = -1$

$\Rightarrow$ $3 \leq \cot^{-1}x < 4$ and $-1 \leq \tan^{-1}x < 0$

$\Rightarrow$ $x \in (\cot 4, \cot 3]$ and $x \in [-\tan 1, 0)$

$\Rightarrow$ $x \in \varphi$, ({ cot 3 < –tan 1)

Case III: When $[\cot^{-1}x] = 2$ and $[\tan^{-1}x] = 0$

$\Rightarrow$ $2 \leq \cot^{-1}x < 3$ and $0 \leq \tan^{-1}x < 1$

$\Rightarrow$ $x \in (3, \cot 2]$ and $x \in [0 \tan 1)$

$\Rightarrow$ $x \in \varphi$, ({ cot 2 < tan 1)

Thus, there is no such value of x, where the equation is valid.

21. Given,

$$[\sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1}x)))] = 1$$

$\Rightarrow$ $0 \leq \sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1}x))) < 1$

$\Rightarrow$ $0 \leq (\cos^{-1}(\sin^{-1}(\tan^{-1}x))) < \sin 1$

$\Rightarrow$ $\cos(\sin 1) < (\sin^{-1}(\tan^{-1}x)) \leq 1$

$\Rightarrow$ $\sin(\cos(\sin 1)) < (\tan^{-1}x) \leq \sin 1$

$\Rightarrow$ $\tan(\sin(\cos(\sin 1))) < x \leq \tan(\sin 1)$

$\Rightarrow$ $x \leq (\tan(\sin(\cos(\sin 1))), \tan(\sin 1)]$

22. Given,

$$f(x) = \sin^{-1}x + \tan^{-1}x + \cot^{-1}x$$

It is defined for $-1 \leq x \leq 1$

Thus,

$$f(-1) = \sin^{-1}(-1) + \tan^{-1}(-1) + \cot^{-1}(-1)$$
$$= -\frac{\pi}{2} - \frac{\pi}{4} + \pi - \frac{\pi}{4}$$
$$= -\pi + \pi = 0$$

and

$$f(1) = \sin^{-1}(1) + \tan^{-1}(1) + \cot^{-1}(1)$$
$$= \frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{4}$$
$$= \pi$$

Thus, range of $f(x)$ is = $[f(-1), f(1)] = [0, \pi]$

23. Given,

$$f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$$
$$= \frac{\pi}{2} + \tan^{-1}x$$

As we know that, range of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, the range of $f(x)$ is $(0, \pi)$

24. Given, $f(x) = \sin^{-1}x + \sec^{-1}x + \tan^{-1}x$

The domain of $f(x)$ is $\{-1, 1\}$

Now, $f(1) = \sin^{-1}(1) + \sec^{-1}(1) + \tan^{-1}(1)$

$$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$$

and $f(-1) = \sin^{-1}(-1) + \sec^{-1}(-1) + \tan^{-1}(-1)$

$$= -\frac{\pi}{2} + \pi - 0 - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Thus, the range of $f(x)$ is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

25. Given, $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x+3x}{1-2x\cdot 3x}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \left(\frac{5x}{1-6x^2}\right) = 1$

$\Rightarrow \quad 6x^2 + 5x - 1 = 0$

$\Rightarrow \quad 6x^2 + 6x - x - 1 = 0$

$\Rightarrow \quad (6x - 1)(x + 1) = 0$

$\Rightarrow \quad x = -1, \frac{1}{6}$

Hence, the solution set is $\left\{-1, \frac{1}{6}\right\}$

26. Given, $\cos^{-1}x = \cot^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \quad \cos^{-1}x = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$\Rightarrow \quad \cos^{-1}x = \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$

$\Rightarrow \quad \cos^{-1}x = \tan^{-1}\left(\frac{25}{25}\right) = \tan^{-1}(1)$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{4}$

$\Rightarrow \quad x = \frac{1}{\sqrt{2}}$

Hence, the solution is $x = \frac{1}{\sqrt{2}}$

27. Given, $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$

$\Rightarrow \quad \frac{n}{\pi} < \cot\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \frac{n}{\pi} < \sqrt{3}$

$\Rightarrow \quad n < \frac{\pi}{\sqrt{3}}$

$\Rightarrow \quad n < \frac{\pi}{\sqrt{3}} = 3.14 \times 1.732 = 5.43848$

Hence, the max. value of n is 5.

28. Given $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$

$$\sin^{-1}\left(\frac{5}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{12}{x}\right)$$

$\Rightarrow \quad \sin^{-1}\left(\frac{5}{x}\right) = \cos^{-1}\left(\frac{12}{x}\right)$

$\Rightarrow \quad \sin^{-1}\left(\frac{5}{x}\right) = \sin^{-1}\sqrt{1-\left(\frac{12}{x}\right)^2}$

$\Rightarrow \quad \left(\frac{5}{x}\right)^2 = 1 - \left(\frac{12}{x}\right)^2$

$\Rightarrow \quad \left(\frac{169}{x^2}\right) = 1$

$\Rightarrow \quad x^2 = 169$

$\Rightarrow \quad x = \pm 13$

29. We have

$$x = \sin^{-1}(b^6 + 1) + \cos^{-1}(b^4 + 1) + \tan^{-1}(a^2 + 1)$$

It is possible only when $a = 0$

Thus, $x = \sin^{-1}(1) + \cos^{-1}(1) + \tan^{-1}(1)$

$$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$$

Therefore, $\sin\left(x + \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)$

$$= \sin\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{4} + \frac{\pi}{4}\right)$$

$$= \sin\pi + \cos\pi$$

$$= 0 - 1 = -1$$

30. Given,

$$\sin^{-1}x + \tan^{-1}x = 2k + 1$$

Let $g(x) = \sin^{-1}x + \tan^{-1}x$

Domain of $g = [-1, 1]$

Now, $g(-1) = \sin^{-1}(-1) + \tan^{-1}(-1)$

$$= -\frac{\pi}{2} - \frac{\pi}{4} = -\frac{3\pi}{4}$$

$$g(1) = \sin^{-1}(1) + \tan^{-1}(1)$$

$$= \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

Range of $g = \left[-\frac{3\pi}{4}, \frac{3\pi}{4}\right]$

Thus, $-\frac{3\pi}{4} \le 2k + 1 \le \frac{3\pi}{4}$

$\Rightarrow \quad -\frac{3\pi}{4} - 1 \le 2k \le \frac{3\pi}{4} - 1$

$\Rightarrow \quad -\frac{3\times 3.14}{4} - 1 \le 2k \le \frac{3\times 3.14}{4} - 1$

$\Rightarrow \quad -2.35 - 1 \le 2k \le 2.35 - 1$

$\Rightarrow \quad -3.35 \le 2k \le 1.35$

$\Rightarrow \quad -\frac{3.35}{2} \le k \le \frac{1.35}{2}$

$\Rightarrow \quad -1.67 \le k \le 6.7$

Thus, the integral values of k are –1 and 0

31. Given,

$$\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$$

$\Rightarrow \quad \sin^{-1}x = \frac{\pi}{2} - \sin^{-1}y$

$\Rightarrow \quad \sin^{-1}x = \cos^{-1}y$

$\Rightarrow \quad \sin^{-1}x = \sin^{-1}\sqrt{1-y^2}$

$\Rightarrow \quad x = \sqrt{1-y^2}$

$\Rightarrow \quad x^2 = 1 - y^2$

$\Rightarrow \quad x^2 = y^2 = 1$

Now,

$$\left(\frac{1+x^4+y^4}{x^2-x^2y^2+y^2}\right) = \left(\frac{1+(x^2+y^2)^2-2x^2y^2}{(x^2+y^2-x^2y^2)}\right)$$
$$= \left(\frac{1+1-2x^2y^2}{(1-x^2y^2)}\right)$$
$$= \frac{2(1-x^2y^2)}{(1-x^2y^2)}$$
$$= 2$$

32. Given, $\cos^{-1}x + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$

$\Rightarrow \quad \cos^{-1}2x + \cos^{-1}3x = \pi - \cos^{-1}x$

$\Rightarrow \quad \cos^{-1}2x + \cos^{-1}3x = \cos^{-1}(-x)$

$\Rightarrow \quad \cos^{-1}(2x\cdot 3x - \sqrt{1-4x^2}\sqrt{1-9x^2}) = \cos^{-1}(-x)$

$\Rightarrow \quad (6x^2 - \sqrt{1-4x^2}\sqrt{1-9x^2}) = -x$

$\Rightarrow \quad (6x^2 + x)^2 = (\sqrt{1-4x^2}\sqrt{1-9x^2})^2$

$\Rightarrow \quad 36x^4 + 12x^3 + x^2 = 1 - 4x^2 - 9x^2 + 36x^4$

$\Rightarrow \quad 12x^3 + 14x^2 = 1$

Also, $ax^3 + bx^2 + cx = 1$

Thus, $a = 12$, $b = 14$ and $c = 0$

Hence, the value of $a^2 + b^2 + c^2 + 10$

$= 144 + 196 + 10$

$= 350$

33. The domain of $\sin^{-1}x + \tan^{-1}x$ is $[-1, 1]$

Now, $f(1) = \sin^{-1}(1)\tan^{-1}(1) + 1 + 4 + 5$

$= \frac{3\pi}{4} + 10$

and $f(-1) = \sin^{-1}(-1) + \tan^{-1}(-1) + 1 - 4 + 5$

$= -\frac{3\pi}{4} + 2$

Therefore, $a + b + 5 = 10 + 2 + 5 = 17$.

34. Clearly, $x = \frac{\pi}{2}$.

Thus, $\sin x = 1$

Ans. (a)

35. As we know that, domain of $\sin^{-1}x$ is $[-1, 1]$

Therefore, $-1 \le \log_3\left(\frac{x}{3}\right) \le 1$

$\Rightarrow \quad 3^{-1} \le \left(\frac{x}{3}\right) \le 3$

$\Rightarrow \quad \frac{1}{3} \le \left(\frac{x}{3}\right) \le 3$

$\Rightarrow \quad 1 \le x \le 9$

Thus the domain of $f(x)$ is $[1, 9]$

36. As we know that, the range of $\sin^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, $-\frac{\pi}{2} \le 2\sin^{-1}a \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1}a \le \frac{\pi}{4}$

$\Rightarrow \quad -\frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}}$

$\Rightarrow \quad |a| \le \frac{1}{\sqrt{2}}$

Ans. (c)

37. Now, $\sin^{-1}(x-3)$ is defined for

$-1 \le (x-3) \le 1$

$\Rightarrow \quad 2 \le x \le 4$

Also, the function $\frac{1}{\sqrt{9-x^2}}$ is defined for

$\Rightarrow \quad 9 - x^2 > 0$

$\Rightarrow \quad x^2 - 9 < 0$

$\Rightarrow \quad (x+3)(x-3) < 0$

$\Rightarrow \quad -3 < x < 3$

Thus, the solution is $x \in [2, 3)$

Hence, the domain is $[2, 3)$

Ans. (b)

38. Since the f is ont, so the range of f is co-domain.

i.e., range $= B$

Clearly, range of f is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Thus, $B = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Ans. (c)

39. Given, $\cos^{-1}x - \cos^{-1}\left(\frac{y}{2}\right) = \alpha$

$\Rightarrow \quad \cos^{-1}\left(x\cdot\frac{y}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$\Rightarrow \quad \left(\frac{xy}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \cos\alpha$

$\Rightarrow \quad \left(\cos\alpha - \frac{xy}{2}\right)^2 = \left(\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right)^2$

$\Rightarrow \quad \left(\cos^2\alpha - xy\cos\alpha + \left(\frac{xy}{2}\right)^2\right)$

$\Rightarrow \quad = 1 - x^2 - \frac{y^2}{4} + \frac{x^2y^2}{4}$

$\Rightarrow \quad x^2 - xy\cos\alpha + \frac{y^2}{4} = 1 - \cos^2\alpha$

$\Rightarrow \quad x^2 - xy\cos\alpha + \frac{y^2}{4} = \sin^2\alpha$

$\Rightarrow \quad 4x^2 - 4xy \cos \alpha + y^2 = 4 \sin^2 \alpha$

Ans. (d)

> ***Note:*** No question asked in 2006.

40. Given, $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \quad x = 3$

Ans. (d)

41. Given, $\cot\left(\text{cosec}^{-1}\left(\frac{5}{3}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}}\right)\right)$

$= \cot\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$

$= \cot\left(\cot^{-1}\left(\frac{6}{17}\right)\right)$

$= \frac{6}{17}$

Ans. (b)

> ***Note:*** No questions asked in between 2009 to 2014.

Level III

1. Let $D_1 : -1 \le \left(\frac{|x| - 2}{3}\right) \le 1$

$\Rightarrow \quad -3 \le (|x| - 2) \le 3$

$\Rightarrow \quad -1 \le |x| \le 5$

$\Rightarrow \quad -5 \le x \le 5$

and D_2: $-1 \le \left(\frac{1 - |x|}{4}\right) \le 1$

$\Rightarrow \quad -4 \le 1 - |x| \le 4$

$\Rightarrow \quad -5 \le -|x| \le 3$

$\Rightarrow \quad -3 \le |x| \le 5$

$\Rightarrow \quad -5 \le x \le 5$

Thus, $D_f = D_1 \cap D_2 = [-5, 5]$

2. The function f is defined for

$\Rightarrow \quad 5\pi \sin^{-1} x - 6 (\sin^{-1} x)^2 \le 0$

$\Rightarrow \quad 6 (\sin^{-1} x)^2 - 5\pi \sin^{-1} x \le 0$

$\Rightarrow \quad (\sin^{-1} x) (6(\sin^{-1} x) - 5\pi) \le 0$

$\Rightarrow \quad 0 \le \sin^{-1} x \le \frac{5\pi}{6}$

$\Rightarrow \quad 0 \le x \le \frac{1}{2}$

Also, $\sin^{-1} x$ is defined for $[-1, 1]$

Thus, $D_f = \left[0, \frac{1}{2}\right]$

3. f is defined for

$-1 \le \log_2(x^2 + 3x + 4) \le 1$

$\Rightarrow \quad 2^{-1} \le (x^2 + 3x + 4) \le 2$

when $(x^2 + 3x + 4) \le 2$

$\Rightarrow \quad (x^2 + 3x + 2) \le 0$

$\Rightarrow \quad (x + 1)(x + 2) \le 0$

$\Rightarrow \quad -2 \le x \le -1$

when $x^2 + 3x + 4 \ge \frac{1}{2}$

$\Rightarrow \quad 2x^2 + 6x + 7 \ge 0$

$\Rightarrow \quad x \in R$

Thus, $D_f = [-2, -1]$

4. Given, $\cos^{-1} x + \cos^{-1} x^2 = 2\pi$

It is possible only when

$\cos^{-1} x = \pi$ and $\cos^{-1} x^2 = \pi$

$x = \cos \pi = -1$ and $x^2 = -1$ and $x^2 = -1$

Thus, no such value of x is exist.

5. It is true only when $\frac{1}{x^2 - 1} = x^2 - 1$

$\Rightarrow \quad (x^2 - 1)^2 = 1$

$\Rightarrow \quad (x^2 - 1) = \pm 1$

$\Rightarrow \quad x^2 = 1 \pm 1 = 2, 0$

$\Rightarrow \quad x = 0, \pm\sqrt{2}$

6. Given, $\cot^{-1}\left(\frac{x^2 - 1}{2x}\right) + \tan^{-1}\left(\frac{2x}{x^2 - 1}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1 - x^2}\right) + \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \left(\frac{2x}{1 - x^2}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\Rightarrow \quad \frac{2x}{\sqrt{3}} = 1 - x^2$

$\Rightarrow \quad x^2 + \frac{2}{\sqrt{3}}x - 1 = 0$

$\Rightarrow \quad \left(x + \frac{1}{\sqrt{3}}\right)^2 = 1 + \frac{1}{3} = \frac{4}{3}$

$\Rightarrow \quad \left(x+\frac{1}{\sqrt{3}}\right)=\pm\frac{2}{\sqrt{3}}$

$\Rightarrow \quad x=-\frac{1}{\sqrt{3}}\pm\frac{2}{\sqrt{3}}$

$\Rightarrow \quad x=\frac{1}{\sqrt{3}},-\sqrt{3}$

Hence, the solution set is $\left\{-\sqrt{3},\frac{1}{\sqrt{3}}\right\}$

7. Given, $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi-\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

$\Rightarrow \quad \left(\pi-\frac{2x^2+4}{x^2+1}\right)<\pi-3$

$\Rightarrow \quad \frac{2x^2+4}{x^2+1}>3$

$\Rightarrow \quad \frac{2x^2+4}{x^2+1}-3>0$

$\Rightarrow \quad \frac{2x^2+4-3x^2-3}{x^2+1}>0$

$\Rightarrow \quad \frac{-x^2+1}{x^2+1}>0$

$\Rightarrow \quad \frac{x^2-1}{x^2+1}<0$

$\Rightarrow \quad \frac{(x-1)(x+1)}{x^2+1}<0$

$\Rightarrow \quad -1<x<1$

8. We have

$\sin^{-1}(\sin[\pi^{3/2}])+\cos^{-1}(\cos[\pi^{3/2}])$

$=\sin^{-1}(\sin[\pi\sqrt{\pi}])+\cos^{-1}(\cos[\pi\sqrt{\pi}])$

$=\sin^{-1}(\sin[5.56])+\cos^{-1}(\cos[5.56])$

$=\sin^{-1}(\sin 5)+\cos^{-1}(\cos 5)$

$=\sin^{-1}(\sin(5-2p))+\cos^{-1}(\cos(2\pi-5))$

$=(5-2\pi)+(2\pi-5)=0$

Thus, the given expression reduces to

$x^2-4x<0$

$x(x-4)<0$

$0<x<4$

9. Now, $\cos\left(\tan^{-1}\left(\cot\left(\sin^{-1}\left(x+\frac{3}{2}\right)\right)\right)\right)$

$=\cos\left(\tan^{-1}\left(\cot\left(\cot^{-1}\left(\frac{\sqrt{1-\left(x+\frac{3}{2}\right)^2}}{\left(x+\frac{3}{2}\right)}\right)\right)\right)\right)$

$=\cos\left(\tan^{-1}\left(\frac{\sqrt{1-\left(x+\frac{3}{2}\right)^2}}{\left(x+\frac{3}{2}\right)}\right)\right)$

$=\cos\left(\cos^{-1}\left(\frac{x+\frac{3}{2}}{1}\right)\right)$

$=\left(x+\frac{3}{2}\right)$

Thus, the given equation reduces to

$\left(x+\frac{3}{2}\right)+\tan(\sec^{-1}x)=0$

$\left(x+\frac{3}{2}\right)+\tan\left(\tan^{-1}\left(\frac{\sqrt{x^2-1}}{1}\right)\right)=0$

$\Rightarrow \quad \left(x+\frac{3}{2}\right)+\left(\frac{\sqrt{x^2-1}}{1}\right)=0$

$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2=(-\sqrt{x^2-1})^2$

$\Rightarrow \quad \left(x+\frac{3}{2}\right)^2=x^2-1$

$\Rightarrow \quad x^2+3x+\frac{9}{4}=x^2-1$

$\Rightarrow \quad 3x+\frac{9}{4}=-1$

$\Rightarrow \quad 3x=-1-\frac{9}{4}=-\frac{13}{4}$

$\Rightarrow \quad x=-\frac{13}{12}$

Hence, the solution is $x=-\frac{13}{12}$

10. We have

$\Rightarrow \quad \tan\left(\tan^{-1}\left(\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}\cdot\frac{1}{x+1}}\right)\right)=1$

$\Rightarrow \quad \left(\frac{\frac{x}{10}+\frac{1}{x+1}}{1-\frac{x}{10}\cdot\frac{1}{x+1}}\right)=1$

$\Rightarrow \quad \left(\frac{x}{10}+\frac{1}{x+1}\right)=\left(1-\frac{x}{10}\cdot\frac{1}{x+1}\right)$

$\Rightarrow \quad x(x+1)+10=10(x+1)-x$

$\Rightarrow \quad x^2+x+10=10x+10-x$

$\Rightarrow \quad x^2-8x=0$

$\Rightarrow \quad x=0, 8$

Hence, the solution set is $\{0, 8\}$.

11. Given

$$\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$$
$$= 2(\tan^{-1} 1 + \tan^{-1} x)$$
$$= 2\left(\frac{\pi}{4} + \tan^{-1}x\right)$$
$$= \frac{\pi}{2} + 2\tan^{-1}x$$

Also, $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$= \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$
$$= \frac{\pi}{2} - 2\tan^{-1}x$$

Thus,

$$\alpha + \beta = \frac{\pi}{2} + 2\tan^{-1}x + \frac{\pi}{2} - 2\tan^{-1}x$$
$$= \pi$$

12. As we know that

$$-\frac{\pi}{2} \le \sin^{-1}(2x-3) \le \frac{\pi}{2}$$
$$\Rightarrow \quad -\frac{2\pi}{2} \le 2\sin^{-1}(2x-3) \le \frac{2\pi}{2}$$
$$\Rightarrow \quad -\pi \le f(x) \le \pi$$

Thus, $R_f = [-\pi, \pi]$

13. We have

$$-\frac{\pi}{2} \le \sin^{-1}(2x-1) \le \frac{\pi}{2}$$
$$\Rightarrow \quad -\pi \le 2\sin^{-1}(2x-1) \le \pi$$
$$\Rightarrow \quad -\pi - \frac{\pi}{4} \le 2\sin^{-1}(2x-1) - \frac{\pi}{4} \le \pi - \frac{\pi}{4}$$
$$\Rightarrow \quad -\frac{5\pi}{4} \le f(x) \le \frac{3\pi}{4}$$

Thus, $R_f = \left[-\frac{5\pi}{4}, \frac{3\pi}{4}\right]$

14. We have

$$f(x) = 2\cos^{-1}(-x^2) - \pi$$
$$= 2(\pi - \cos^{-1}(x^2)) - \pi$$
$$= \pi - 2\cos^{-1}(x^2)$$

As we know that, $0 \le \cos^{-1}(x^2) \le \pi$

$\Rightarrow \quad 0 \le 2\cos^{-1}(x^2) \le 2\pi$

$\Rightarrow \quad -2\pi \le -2\cos^{-1}(x^2) \le 0$

$\Rightarrow \quad -2\pi + \pi \le \pi - 2\cos^{-1}(x^2) \le \pi$

$\Rightarrow \quad -\pi \le f(x) \le \pi$

Thus, $r_f = [-\pi, \pi]$

15. We have

$$-\infty < 1 - x^2 \le 1$$
$$\Rightarrow \quad \tan^{-1}(-\infty) < \tan^{-1}(1-x^2) \le \tan^{-1}(1)$$
$$\Rightarrow \quad -\frac{\pi}{2} < \tan^{-1}(1-x^2) \le \frac{\pi}{4}$$
$$\Rightarrow \quad -\frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) \le \frac{\pi}{8}$$
$$\Rightarrow \quad -\frac{\pi}{4} - \frac{\pi}{4} < \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4} \le \frac{\pi}{8} - \frac{\pi}{4}$$
$$\Rightarrow \quad -\frac{\pi}{2} < \frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4} \le -\frac{\pi}{8}$$

Hence, the range of the function

$$= \left[-\frac{\pi}{2}, -\frac{\pi}{8}\right]$$

16. We have

$$2x - x^2 = -(x^2 - 2x + 1) + 1$$
$$= 1 - (x-1)^2$$

Thus, $-\infty < 1 - (x-1)^2 \le 1$

$\Rightarrow \quad \cot^{-1}(-\infty) < \cot^{-1}(1 - (x-1)^2) \le \cot^{-1}(1)$

$\Rightarrow \quad 0 < \cot^{-1}(1-(x-1)^2) \le \frac{\pi}{4}$

Hence, the range of the function $= \left(0, \frac{\pi}{4}\right]$.

17. The function f is defined for $-1 \le x \le 1$

We have

$$f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$$
$$= \frac{\pi}{2} + \tan^{-1}x$$

Thus, $R_f = [f(-1), f(1)]$

$$= \left[\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} + \frac{\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

18. The function f is defined for $x = \pm 1$

Now, $f(1) = \sin^{-1}(1) + \sec^{-1}(1) + \tan^{-1}(1)$

$$= \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$$

Also, $f(-1) = \sin^{-1}(-1) + \sec^{-1}(-1) + \tan^{-1}(-1)$

$$= -\frac{\pi}{2} + \pi - 0 - \frac{\pi}{4}$$
$$= \frac{3\pi}{4}$$

Thus, $R_f = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

19. We have

$$f(x) = 2(\tan^{-1}x + \cot^{-1}x) + \cot^{-1}x + \frac{\pi}{4}$$
$$= 2\cdot\frac{\pi}{2} + \cot^{-1}x + \frac{\pi}{4}$$
$$= \cot^{-1}x + \frac{5\pi}{4}$$

Also, $0 < \cot^{-1}x < \pi$

$$\Rightarrow \quad \frac{5\pi}{4} < \cot^{-1}x + \frac{5\pi}{4} < \pi + \frac{5\pi}{4}$$

$$\Rightarrow \quad \frac{5\pi}{4} < f(x) < \frac{9\pi}{4}$$

Thus, $R_f = \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right)$

20. We have

$\sin(\cot^{-1}(\tan(\cos^{-1} x)))$

$$= \sin\left(\cot^{-1}\left(\tan\left(\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)\right)\right)$$

$$= \sin\left(\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\frac{x}{1}\right)\right)$$

$$= x$$

21. We have

$\sin(\operatorname{cosec}^{-1}(\cot(\tan^{-1} x)))$

$$= \sin\left(\operatorname{cosec}^{-1}\left(\cot\left(\cot^{-1}\left(\frac{1}{x}\right)\right)\right)\right)$$

$$= \sin\left(\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)\right)$$

$$= \sin(\sin^{-1}(x))$$

$$= x$$

22. We have

$\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10)$

$-\tan^{-1}(\tan 6) + \pi - \cot^{-1}(\cot 10)$

$= \sin^{-1}(\sin(5 - 2\pi)) + \cos^{-1}(\cos(4\pi - 10))$

$+ \pi - \tan^{-1}(\tan(6 - 2\pi)) - \cot^{-1}(\cot(10 - 4\pi))$

$= (5 - 2\pi) + (4\pi - 10) + \pi + (6 - 2\pi) - (10 - 4\pi)$

$= 5\pi - 9$

23. Given, $U = \cot^{-1}(\sqrt{\cos 2\theta}) - \tan^{-1}(\sqrt{\cos 2\theta})$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{1}{\sqrt{\cos 2\theta}}\right) - \tan^{-1}(\sqrt{\cos 2\theta})$$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{\frac{1}{\sqrt{\cos 2\theta}} - \sqrt{\cos 2\theta}}{1 + \frac{1}{\sqrt{\cos 2\theta}} \cdot \sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{1 - \cos 2\theta}{2\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{2\sin^2\theta}{2\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow \quad U = \tan^{-1}\left(\frac{\sin^2\theta}{\sqrt{\cos 2\theta}}\right)$$

$$\Rightarrow \quad U = \sin^{-1}\left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta + 1 - 2\sin^2\theta}}\right)$$

$$\Rightarrow \quad \sin U = \sin\left(\sin^{-1}\left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta + 1 - 2\sin^2\theta}}\right)\right)$$

$$\Rightarrow \quad \sin U = \left(\frac{\sin^2\theta}{\sqrt{\sin^4\theta + 1 - 2\sin^2\theta}}\right)$$

$$\Rightarrow \quad \sin U = \left(\frac{\sin^2\theta}{\sqrt{(\sin^2\theta - 1)^2}}\right) = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$$

Hence, the result.

24. Let $\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right) = \theta$

$$\cos(2\theta) = \frac{a}{b}$$

The given expression reduces to

$$\tan\left(\frac{\pi}{4} + \theta\right) + \tan\left(\frac{\pi}{4} - \theta\right)$$

$$= \frac{1 + \tan\theta}{1 - \tan\theta} + \frac{1 - \tan\theta}{1 + \tan\theta}$$

$$= \frac{(1 + \tan\theta)^2 + (1 - \tan\theta)^2}{(1 - \tan^2\theta)}$$

$$= \frac{2(1 + \tan^2\theta)}{(1 - \tan^2\theta)}$$

$$= \frac{2}{\cos 2\theta}$$

$$= \frac{2b}{a}$$

25. RHS $= 2\tan^{-1}\left(\tan\left(\frac{x}{2}\right)\tan\left(\frac{y}{2}\right)\right)$

$$= \cos^{-1}\left(\frac{1 - \tan^2\left(\frac{x}{2}\right)\tan^2\left(\frac{y}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)\tan^2\left(\frac{y}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{\cos^2\left(\frac{x}{2}\right)\cos^2\left(\frac{y}{2}\right) - \sin^2\left(\frac{x}{2}\right)\sin^2\left(\frac{y}{2}\right)}{\cos^2\left(\frac{x}{2}\right)\cos^2\left(\frac{y}{2}\right) + \sin^2\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{(1 + \cos x)(1 + \cos y) - (1 - \cos x)(1 - \cos y)}{(1 + \cos x)(1 + \cos y) + (1 - \cos x)(1 - \cos y)}\right)$$

$$= \cos^{-1}\left(\frac{(2\cos x + 2\cos y)}{(2 + 2\cos x\cos y)}\right)$$

$$= \cos^{-1}\left(\frac{(\cos x + \cos y)}{(1 + 1\cos x\cos y)}\right)$$

Hence, the result.

26. We have $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}\right)\right)$

$$= \cos^{-1}\left(\frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{x}{2}\right)}{1+\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{x}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{(a+b)-(a-b)\tan^2\left(\frac{x}{2}\right)}{(a+b)+(a-b)\tan^2\left(\frac{x}{2}\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\left(1-\tan^2\left(\frac{x}{2}\right)\right)+b\left(1+\tan^2\left(\frac{x}{2}\right)\right)}{a\left(1+\tan^2\left(\frac{x}{2}\right)\right)+b\left(1-\tan^2\left(\frac{x}{2}\right)\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}\right)+b}{a+b\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}\right)}\right)$$

$$= \cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

27. Given,

$\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$

$\Rightarrow \quad \tan^{-1}x+\tan^{-1}z = 2\tan^{-1}y$

$\Rightarrow \quad \tan^{-1}\left(\frac{x+z}{1-xz}\right) = \tan^{-1}\left(\frac{2y}{1-y^2}\right)$

$\Rightarrow \quad \left(\frac{x+z}{1-xz}\right) = \left(\frac{2y}{1-y^2}\right)$

$\Rightarrow \quad (x+z)(1-y^2) = 2y(1-xz)$

$\Rightarrow \quad y^2(x+z) + 2y(1-xz) = (x+z)$

28. We have

$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)$$
$$+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(4\pi+\frac{5\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(6\pi+\frac{4\pi}{7}\right)\right)$$
$$+\tan^{-1}\left(-\tan\left(2\pi-\frac{3\pi}{8}\right)\right)+\cot^{-1}\left(-\cot\left(2\pi+\frac{3\pi}{8}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{5\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right)$$
$$+\tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right)+\cot^{-1}\left(-\cot\left(\frac{3\pi}{8}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(\pi-\frac{2\pi}{7}\right)\right)+\cos^{-1}\left(\cos\left(\frac{4\pi}{7}\right)\right)$$
$$+\tan^{-1}\left(\tan\left(\frac{3\pi}{8}\right)\right)+\cot^{-1}\left(\cot\left(\frac{5\pi}{8}\right)\right)$$

$$= \frac{2\pi}{7}+\frac{4\pi}{7}+\frac{3\pi}{8}+\frac{5\pi}{8}$$

$$= \frac{6\pi}{7}+\pi$$

$$= \frac{13\pi}{7}$$

29. Given equations are

$$\sin^{-1}x+\sin^{-1}y = \frac{2\pi}{3}, \qquad \ldots\text{(i)}$$

$$\cos^{-1}x-\cos^{-1}y = \frac{\pi}{3} \qquad \ldots\text{(ii)}$$

(i) reduces to

$\Rightarrow \quad \frac{\pi}{2}-\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}y = \frac{2\pi}{3}$

$\pi-(\cos^{-1}x+\cos^{-1}y) = \frac{2\pi}{3}$

$\Rightarrow \quad (\cos^{-1}x+\cos^{-1}y) = \pi-\frac{2\pi}{3} = \frac{\pi}{3} \qquad \ldots\text{(iii)}$

Adding (ii) and (iii), we get

$2\cos^{-1}x = \frac{2\pi}{3}$

$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{3}$

$\Rightarrow \quad x = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

when $x = \frac{1}{2}, y = 0$

Hence, the solutions are $x = 1/2$ and $y = 0$.

30. Given, $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$

Put $x^2 = \cos(2\theta)$

Thus, $y = \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$

$$= \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)$$

$$= \left(\frac{\pi}{4}-\theta\right)$$

$\Rightarrow \quad \theta = \frac{\pi}{4} - y\ y$

$\Rightarrow \quad \frac{1}{2}\cos^{-1}(x^2) = \frac{\pi}{4} - y$

$\Rightarrow \quad \cos^{-1}(x^2) = \frac{\pi}{2} - 2y$

$\Rightarrow \quad (x^2) = \cos\left(\frac{\pi}{2} - 2y\right)$

$\Rightarrow \quad x^2 = \sin(2y)$

31. We have

$$\frac{\beta^3}{2}\text{cosec}^2\left(\frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$= \frac{\beta^3}{2}\text{cosec}^2\left(\frac{\tan^{-1}\left(\frac{\beta}{\alpha}\right)}{2}\right) + \frac{\alpha^3}{2}\sec^2\left(\frac{\tan^{-1}\left(\frac{\alpha}{\beta}\right)}{2}\right)$$

$$= \frac{\beta^3}{2\sin^2\left(\frac{\tan^{-1}\left(\frac{\beta}{\alpha}\right)}{2}\right)} + \frac{\alpha^3}{2\cos^2\left(\frac{\tan^{-1}\left(\frac{\alpha}{\beta}\right)}{2}\right)}$$

$$= \frac{\beta^3}{1-\cos\left(\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)} + \frac{\alpha^3}{1+\cos\left(\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right)}$$

$$= \frac{\beta^3}{1-\cos\left(\cos^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)\right)}$$

$$+ \frac{\alpha^3}{1+\cos\left(\cos^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)\right)}$$

$$= \frac{\beta^3}{1-\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)} + \frac{\alpha^3}{1+\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$= \frac{\beta^3(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}-\alpha} + \frac{\alpha^3(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}+\beta}$$

$$= (\sqrt{\alpha^2+\beta^2})\left(\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}-\alpha} + \frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}+\beta}\right)$$

$$= (\sqrt{\alpha^2+\beta^2})\left(\frac{\beta^3(\sqrt{\alpha^2+\beta^2}+\alpha)}{\beta^2} + \frac{\alpha^3(\sqrt{\alpha^2+\beta^2}-\beta)}{\alpha^2}\right)$$

$$= (\sqrt{\alpha^2+\beta^2})(\beta(\sqrt{\alpha^2+\beta^2}+\alpha) + \alpha(\sqrt{\alpha^2+\beta^2}-\beta))$$

$$= (\sqrt{\alpha^2+\beta^2})\sqrt{\alpha^2+\beta^2}(\alpha+\beta)$$

$$= (\alpha^2+\beta^2)(\alpha+\beta)$$

Hence, the result.

32. Given, $\cot^{-1}\left(\frac{n^2-10n+21.6}{\pi}\right) > \frac{\pi}{6}, n \in N$

$\Rightarrow \quad \left(\frac{n^2-10n+21.6}{\pi}\right) < \cot\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad (n^2 - 10n + 21.6 < \pi\sqrt{3}$

$\Rightarrow \quad (n^2 - 10n) + 21.6 < 5.6$

$\Rightarrow \quad (n^2\ 10n) + 16 < 0$

$\Rightarrow \quad (n-2)(n-8) < 0$

$\Rightarrow \quad 2 < n < 8$

Thus, the minimum value of n is 2.

33. We have

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right) + \cos^{-1}\left(\frac{\sqrt{12}}{4}\right) + \sec^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{4-2\sqrt{3}}{8}}\right)\right.\right.$$

$$\left.\left. + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sec^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{(\sqrt{3}-1)^2}{8}}\right) + \frac{\pi}{6} + \frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \frac{\pi}{6} + \frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left(\frac{\pi}{2}\right)\right\}$$

$= \sin^{-1}(0)$

$= 0$

34. Given, $[\sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x)))] = 1$

$\Rightarrow \quad 1 \le \sin^{-1}(\cos^{-1}(\sin^{-1}(\tan^{-1} x))) < 2$

$\Rightarrow \quad \sin(1) \le \cos^{-1}(\sin^{-1}(\tan^{-1} x)) < \sin(2)$

$\Rightarrow \quad \cos(\sin(2)) < \sin^{-1}(\tan^{-1} x) \le \cos(\sin(11)$

$\Rightarrow \quad \sin(\cos(\sin(2))) < (\tan^{-1} x) \le \sin(\cos(\sin(1)))$

$\Rightarrow \quad \tan(\sin(\cos(\sin(2)))) < x \le \tan(\sin(\cos(\sin(1))))$

35. Given, $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$= 2\tan^{-1} x + \pi - 2\tan^{-1} x, x > 1$

$= \pi, x > 1$

Thus, $x \in (1, \infty)$

36. Given, $x = \text{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))))$

$$= \text{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$$

$$= \text{cosec}\left(\tan^{-1}\left(\cos\left(\cos^{-1}\left(\frac{1}{a}\right)\right)\right)\right)$$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\frac{1}{\text{a}}\right)\right)$$

$$= \operatorname{cosec}\,(\operatorname{cosec}^{-1}(\sqrt{a^2+1}))$$

$$= (\sqrt{a^2+1})$$

and $y = \sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1} a)))))$

$$= \sec\left(\cot^{-1}\left(\sin\left(\tan^{-1}\left(\frac{1}{\sqrt{a^2-1}}\right)\right)\right)\right)$$

$$= \sec\left(\cot^{-1}\left(\frac{1}{a}\right)\right)$$

$$= \sec\,(\sec^{-1}(\sqrt{a^2+1}))$$

$$= (\sqrt{a^2+1})$$

Thus $x = y$

37. We have

$$\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{1}{1+2}\right)+\tan^{-1}\left(\frac{1}{1+6}\right)+\tan^{-1}\left(\frac{1}{1+12}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{2-1}{1+2.1}\right)+\tan^{-1}\left(\frac{3-2}{1+3.2}\right)+\tan^{-1}\left(\frac{4-3}{1+4.3}\right)$$

$$+\ldots+\tan^{-1}\left(\frac{n+1-n}{1+(n+1)\cdot n}\right)$$

$$= \tan^{-1}(2)-\tan^{-1}(1)+\tan^{-1}(3)-\tan^{-1}(2)$$
$$+\tan^{-1}(4)-\tan^{-1}(3)+\ldots+\tan^{-1}(n+1)-\tan^{-1}(n)$$

$$= \tan^{-1}(n+1)-\tan^{-1}(1)$$

$$= \tan^{-1}\left(\frac{n+1-1}{1+(n+1)\cdot 1}\right)$$

$$= \tan^{-1}\left(\frac{n}{n+2}\right),\ n\to\infty$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

38. Let I $t_n = \sin^{-1}\left(\dfrac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\times\sqrt{n+1}}\right)$

$$= \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r(r-1)}}\right)$$

$$= \tan^{-1}(\sqrt{r})-\tan^{-1}(\sqrt{r-1})$$

Now, $\displaystyle\sum_{r=1}^{n}\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n}\sqrt{n+1}}\right)$

$$= \sum_{r=1}^{n}\tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}\sqrt{r-1}}\right)$$

$$= \sum_{r=1}^{n}(\tan^{-1}\sqrt{r}-\tan^{-1}\sqrt{r-1})$$

$$= (\tan^{-1}1-\tan^{-1}0)+(\tan^{-1}\sqrt{2}-\tan^{-1}1)$$
$$+(\tan^{-1}\sqrt{3}-\tan^{-1}\sqrt{2})+(\tan^{-1}\sqrt{4}-\tan^{-1}\sqrt{3})$$
$$+\ldots+(\tan^{-1}\sqrt{n}-\tan^{-1}\sqrt{n-1})$$

$$= \tan^{-1}\sqrt{n}-\tan^{-1}0$$

$$= \tan^{-1}\sqrt{n}$$

When $n\to\infty$, the sum is $\tan^{-1}(\infty)=\dfrac{\pi}{2}$.

39. We have

$\cot^{-1}(2.1^2)+\cot^{-1}(2.2^2)+\cot^{-1}(2.3^2)+\ldots$

$$= \tan^{-1}\left(\frac{1}{2.1^2}\right)+\tan^{-1}\left(\frac{1}{2.2^2}\right)+\tan^{-1}\left(\frac{1}{2.3^2}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{2}{4}\right)+\tan^{-1}\left(\frac{2}{16}\right)+\tan^{-1}\left(\frac{2}{36}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{2}{1+3}\right)+\tan^{-1}\left(\frac{2}{1+15}\right)+\tan^{-1}\left(\frac{2}{1+35}\right)+\ldots$$

$$= \tan^{-1}\left(\frac{3-1}{1+3.1}\right)+\tan^{-1}\left(\frac{5-3}{1+5.3}\right)$$
$$+\tan^{-1}\left(\frac{7-5}{1+7.5}\right)+\ldots+\tan^{-1}\left(\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right)$$

$$= (\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}5-\tan^{-1}3)$$
$$+(\tan^{-1}7-\tan^{-1}5)+(\tan^{-1}9-\tan^{-1}7)$$
$$+\ldots+(\tan^{-1}(2n+1)-\tan^{-1}(2n-1))$$

$$= (\tan^{-1}(2n+1)-\tan^{-1}1)$$

$$= \left(\tan^{-1}\frac{(2n+1-1)}{1+(2n+1)\cdot 1}\right)$$

$$= \left(\tan^{-1}\left(\frac{n}{n+1}\right)\right)=\frac{\pi}{4},\ \text{when } n\to\infty$$

40. Given, $\cos^{-1}\left(\dfrac{x}{2}\right)+\cos^{-1}\left(\dfrac{y}{3}\right)=\theta$

$$\Rightarrow\quad \cos^{-1}\left(\frac{x}{2}\cdot\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)=\theta$$

$$\Rightarrow\quad \cos\theta=\left(\frac{x}{2}\cdot\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)$$

$$\Rightarrow\quad \left(\cos\theta-\frac{xy}{6}\right)^2=\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)$$

$$\Rightarrow\quad \left(\cos^2\theta-2\cdot\frac{xy}{6}\cdot\cos\theta+\frac{x^2y^2}{36}\right)$$

$= 1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36}$

$\Rightarrow \cos^2\theta - \frac{xy}{3}\cos\theta = 1 - \frac{x^2}{4} - \frac{y^2}{9}$

$\Rightarrow \frac{x^2}{4} - \frac{xy}{3}\cdot\cos\theta + \frac{y^2}{9} = 1 - \cos^2\theta$

$\Rightarrow 9x^2 - 12xy\cos\theta + 4y^2 = 12\sin^2\theta$

Hence, the result.

Note No questions asked in 1985.

41. Let $\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right) = \theta$

$\Rightarrow \cos^{-1}\left(\frac{\sqrt{5}}{3}\right) = 2\theta$

$\Rightarrow \cos 2\theta = \frac{\sqrt{5}}{3}$

$\Rightarrow \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{\sqrt{5}}{3}$

$\Rightarrow \frac{1-\tan^2\theta+1+\tan^2\theta}{1-\tan^2\theta-1-\tan^2\theta} = \frac{\sqrt{5}+3}{\sqrt{5}-3}$

$\Rightarrow \frac{2}{-2\tan^2\theta} = \frac{\sqrt{5}+3}{\sqrt{5}-3}$

$\Rightarrow \frac{1}{\tan^2\theta} = \frac{\sqrt{5}+3}{3-\sqrt{5}}$

$\Rightarrow \tan^2\theta = \frac{3-\sqrt{5}}{3+\sqrt{5}} = \frac{(3-\sqrt{5})^2}{4}$

$\Rightarrow \tan\theta = \pm\sqrt{\frac{(3-\sqrt{5})^2}{4}} = \pm\frac{(3-\sqrt{5})}{2}$

42. Given, $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}0 = 0$

Let $\tan^{-1}x = \theta \Rightarrow x = \tan\theta$

$\Rightarrow \sin(2\cos^{-1}(\cot(2\theta))) = 0$

$\Rightarrow \sin\left(2\cos^{-1}\left(\frac{1-\tan^2\theta}{2\tan\theta}\right)\right) = 0$

$\Rightarrow \sin\left(2\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right) = 0$

$\Rightarrow \sin\left(\cos^{-1}\left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)\right) = 0$

$\Rightarrow \sin\left(\cos^{-1}\left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)\right) = 0$

$\Rightarrow \sin\left(\sin^{-1}\left(\sqrt{1-\left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)^2}\right)\right) = 0$

$\Rightarrow \sqrt{1-\left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)^2} = 0$

$\Rightarrow 1-\left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)^2 = 0$

$\Rightarrow \left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right)^2 = 1$

$\Rightarrow \left(2\left(\frac{1-x^2}{2x}\right)^2 - 1\right) = \pm 1$

$\Rightarrow 2\left(\frac{1-x^2}{2x}\right)^2 = 1 \pm 1 = 2, 0$

$\Rightarrow 2\left(\frac{1-x^2}{2x}\right)^2 = 2$ and $2\left(\frac{1-x^2}{2x}\right)^2 = 0$

$\Rightarrow \left(\frac{1-x^2}{2x}\right)^2 = 1$ and $1 - x^2 = 0$

$\Rightarrow \left(\frac{1-x^2}{2x}\right) = \pm 1$ and $x = \pm 1$

$\Rightarrow x^2 + 2x - 1 = 0, x^2 - 2x - 1 = 0$ and $x = \pm 1$

$\Rightarrow (x+1)^2 = (\sqrt{2})^2, (x-1)^2 = (\sqrt{2})^2$ and $x = \pm 1$

$\Rightarrow x = -1 \pm \sqrt{2}, 1 \pm \sqrt{2}, \pm 1$

43. Given, $\tan^{-1}x + \cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right) = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$

$\Rightarrow \tan^{-1}x + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}3$

$\Rightarrow \tan^{-1}\left(\frac{x + \frac{1}{y}}{1 - x\cdot\frac{1}{y}}\right) = \tan^{-1}(3)$

$\Rightarrow \tan^{-1}\left(\frac{xy+1}{y-x}\right) = \tan^{-1}(3)$

$\Rightarrow \left(\frac{xy+1}{y-x}\right) = 3$

$\Rightarrow xy + 1 = 3y - 3x$

$\Rightarrow 3x + 1 = y(3 - x)$

$\Rightarrow y = \frac{3x+1}{3-x}$

when $x = 1, y = 2$

Also, when $x = 2, y = 7$

Hence, the positive integral solutions are 2.

44. We have $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \cos^{-1}(-z)$

$\Rightarrow \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos(-z)$

$\Rightarrow \quad (xy + z)^2 = (1 - x^2)(1 - y^2)$

$\Rightarrow \quad x^2y^2 + 2xyz + z^2 = 1 - x^2 - y^2 + x^2y^2$

$\Rightarrow \quad x^2 + y^2 + z^2 + 2xyz = 1$

45. See solutions of Ex-42.

46. Given, $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right)$

Now, $\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right)$

$$= \left(\frac{\frac{6\tan\theta}{1+\tan^2\theta}}{5 + 4\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)}\right)$$

$$= \left(\frac{6\tan\theta}{5(1+\tan^2\theta) + 4(1-\tan^2\theta)}\right)$$

$$= \left(\frac{6\tan\theta}{9+\tan^2\theta}\right)$$

$$= \left(\frac{\frac{2}{3}\tan\theta}{1+\left(\frac{\tan\theta}{3}\right)^2}\right)$$

$$= \left(\frac{2\cdot\frac{\tan\theta}{3}}{1+\left(\frac{\tan\theta}{3}\right)^2}\right)$$

Also, $\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\cdot 2\tan^{-1}\left(\frac{\tan\theta}{3}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{\tan\theta}{3}\right)$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2\tan^2\theta - \frac{\tan\theta}{3}}{1 + 2\tan^2\theta\cdot\frac{\tan\theta}{3}}\right)$

$\Rightarrow \quad \tan\theta = \left(\frac{6\tan^2\theta - \tan\theta}{3 + 2\tan^3\theta}\right)$

$\Rightarrow \quad 3\tan\theta + 2\tan^4\theta - 6\tan^2\theta + \tan\theta = 0$

$\Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 4\tan\theta = 0$

$\Rightarrow \quad \tan^4\theta - 3\tan^2\theta + 2\tan\theta = 0$

$\Rightarrow \quad \tan\theta(\tan 3\theta - 3\tan\theta + 2) = 0$

$\Rightarrow \quad \tan\theta(\tan\theta - 1)^2(\tan\theta + 2) = 0$

$\Rightarrow \quad \tan\theta = 0, 1, -2$

when $\tan\theta = 0 \Rightarrow \theta = n\pi, n \notin I$

when $\tan\theta = 1 \Rightarrow \theta = m\pi + \frac{\pi}{4}, m \in I$

when $\tan q = -2 \Rightarrow q = p\pi + \tan^{-1}(-2), p \in I$

47. We have

$$3\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$$

Now, $3\tan^{-1}\left(\frac{1}{2}\right)$

$$= \tan^{-1}\left(\frac{3\cdot\frac{1}{2} - \left(\frac{1}{2}\right)^3}{1 - 3\left(\frac{1}{2}\right)^2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{2} - \frac{1}{8}}{1 - \frac{3}{4}}\right)$$

$$= \tan^{-1}\left(\frac{12-1}{2}\right)$$

$$= \tan^{-1}\left(\frac{11}{2}\right)$$

Also, $2\tan^{-1}\left(\frac{1}{5}\right)$

$$= \tan^{-1}\left(\frac{2\cdot\frac{1}{5}}{1 - \left(\frac{1}{5}\right)^2}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2}{5}}{\frac{25-1}{25}}\right)$$

$$= \tan^{-1}\left(\frac{2}{5}\times\frac{25}{24}\right)$$

$$= \tan^{-1}\left(\frac{5}{12}\right)$$

Also, $\sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$

$$= \tan^{-1}\left(\frac{142}{31}\right)$$

Therefore,

$$3\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\left(\frac{142}{65\sqrt{5}}\right)$$

$$= \tan^{-1}\left(\frac{11}{2}\right) + \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{142}{31}\right)$$

$$= \tan^{-1}\left(\frac{\frac{11}{2} + \frac{5}{12}}{1 - \frac{11}{2}\cdot\frac{5}{12}}\right) + \tan^{-1}\left(\frac{142}{31}\right)$$

$$= \tan^{-1}\left(\frac{132+10}{24-55}\right) + \tan^{-1}\left(\frac{142}{31}\right)$$

$= \tan^{-1}\left(-\frac{142}{31}\right) + \tan^{-1}\left(\frac{142}{31}\right)$

$= -\tan^{-1}\left(\frac{142}{31}\right) + \tan^{-1}\left(\frac{142}{31}\right)$

$= 0$

48. Given, $\sin^{-1}\left(\frac{ax}{c}\right) + \sin^{-1}\left(\frac{bx}{c}\right) = \sin^{-1}x$

$\Rightarrow \quad \sin^{-1}\left(\frac{ax}{c}\right) = \sin^{-1}x - \sin^{-1}\left(\frac{bx}{c}\right)$

$\Rightarrow \quad \sin^{-1}\left(\frac{ax}{c}\right) = \sin^{-1}\left(x\sqrt{1-\frac{b^2x^2}{c^2}} - \frac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow \quad \left(\frac{ax}{c}\right) = \left(x\sqrt{1-\frac{b^2x^2}{c^2}} - \frac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow \quad \left(\frac{ax}{c}\right) = \left(\frac{x}{c}\sqrt{c^2-b^2x^2} - \frac{bx}{c}\sqrt{1-x^2}\right)$

$\Rightarrow \quad x((\sqrt{c^2-b^2x^2} - b\sqrt{1-x^2}) - a) = 0$

$\Rightarrow \quad x = 0, \sqrt{c^2-b^2x^2} = b\sqrt{1-x^2} + a$

Thus $x = 0$ and

$\sqrt{c^2-b^2x^2} = b\sqrt{1-x^2} + a$

$\Rightarrow \quad c^2 - b^2x^2 = 2ab\sqrt{1-x^2} + b^2(1-x^2) + a^2$

$\Rightarrow \quad c^2 - b^2x^2 = 2ab\sqrt{1-x^2} + b^2 - b^2x^2 + a^2$

$\Rightarrow \quad c^2 = 2ab\sqrt{1-x^2} + b^2 + a^2$

$\Rightarrow \quad c^2 = 2ab\sqrt{1-x^2} + c^2$

$\Rightarrow \quad 2ab\sqrt{1-x^2} = 0$

$\Rightarrow \quad (1-x^2) = 0$

$\Rightarrow \quad x = \pm 1$

Hence, the solution sets is $\{-1, 0, 1\}$

49. Given, $\cos^{-1}(x\sqrt{6}) + \cos^{-1}(3\sqrt{3}\,x^2) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(\sqrt{1-6x^2}) + \cos^{-1}(3\sqrt{3}\,x^2) = \frac{\pi}{2}$

It is possible only when, $\sqrt{1-6x^2} = 3\sqrt{3}x^2$

$\Rightarrow \quad 1 - 6x^2 = 27x^4$

$\Rightarrow \quad 27x^4 + 6x^2 - 1 = 0$

$\Rightarrow \quad 27x^4 - 9x^3 + 9x^3 - 3x^2 + 9x^2 - 1 = 0$

$\Rightarrow \quad 9x^3(3x-1) + 3x^2(3x-1) + (3x-1)(3x+1) = 0$

$\Rightarrow \quad ((3x-1)(9x^3 + 3x^2 + (3x+1)) = 0$

$\Rightarrow \quad (3x-1)(3x^2(3x+1) + (3x+1) = 0$

$\Rightarrow \quad (3x-1)(3x+1)(3x^2+1) = 0$

$\Rightarrow \quad x = -\frac{1}{3}, \frac{1}{3}, \pm\frac{i}{\sqrt{3}}$

Hence, the solutions are $\left\{\pm\frac{1}{3}, \pm\frac{i}{\sqrt{3}}\right\}$

50. Given equation is

$\Rightarrow \tan^{-1}\left(\frac{a}{x}\right) + \tan^{-1}\left(\frac{b}{x}\right) + \tan^{-1}\left(\frac{c}{x}\right) + \tan^{-1}\left(\frac{d}{x}\right) = \frac{\pi}{2}$

$\tan^{-1}\left(\frac{\frac{a}{x}+\frac{b}{x}}{1-\frac{a}{x}\cdot\frac{b}{x}}\right) + \tan^{-1}\left(\frac{\frac{c}{x}+\frac{d}{x}}{1-\frac{c}{x}\frac{d}{x}}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right) + \tan^{-1}\left(\frac{(c+d)x}{x^2-cd}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{(c+d)x}{x^2-cd}\right)$

$\Rightarrow \quad \tan^{-1}\left(\frac{(a+b)x}{x^2-ab}\right) = \cot^{-1}\left(\frac{(c+d)x}{x^2-cd}\right)$

$\Rightarrow \quad \left(\frac{(a+b)x}{x^2-ab}\right) = \left(\frac{x^2-cd}{(c+d)x}\right)$

$\Rightarrow \quad (x^2-ab)(x^2-cd) = (a+b)(c+d)\,x^2$

$\Rightarrow \quad x^4 - (a+b+cd)x^2 + abcd = (a+b)\,(c+d)\,x^2$

$\Rightarrow \quad x^4 - (a+b+cd+(a+b)(c+d))\,^2 + abcd = 0$

since x_1, x_2, x_3, x_4 are the values of the above equation, we have

$x_1 + x_2 + x_3 + x_4 = 0$

$\Sigma x_1x_2 = (ab + cd + (a+b)(c+d))$

$\Sigma x_1x_2x_3 = 0$

$\Sigma x_1x_2x_3x_4 = abcd$

(i) $\sum_{i=1}^{4} x_i = x_1 + x_2 + x_3 + x_4 = 0$

(ii) $\sum_{i=1}^{4}\left(\frac{1}{x_i}\right)$

$= \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$

$= \frac{x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3}{x_1x_2x_3x_4}$

$= \frac{0}{abcd} = 0$

(iii) $\prod_{i=1}^{4}(x_i)$

$= x_1x_2x_3x_4$

$= abcd$

(iv) $\Pi(x_1 + x_2 + x_3)$

$= (x_1 + x_2 + x_3)(x_1 + x_2 + x_4)(x_1 + x_3 + x_4)(x_2 + x_3 + x_4)$

$= (-x_4)(-x_3)(-x_2)(-x_1)$

$= x_1x_2x_3x_4$

$= abcd$

51. We have,

$\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$

$\cos^{-1}(2x) + \cos^{-1}(3x) = \pi - \cos^{-1}(x)$

$\cos^{-1}(2x) + \cos^{-1}(3x) = \cos^{-1}(-x)$

$\cos^{-1}(2x \cdot 3x - \sqrt{(1-4x^2)(1-9x^2)}) = \cos^{-1}(-x)$

$6x^2 - \sqrt{(1-4x^2)(1-9x^2)} = -x$

$(6x^2 + x) = \sqrt{(1-4x^2)(1-9x^2)}$

$(6x^2 + x)^2 = (1 - 4x^2)(1 - 9x^2)$

$36x^4 + 12x + x^2 = 1 - 13x^2 + 36x^4$

$12x^3 + 14x^2 - 1 = 0$

Thus $a = 12$, $b = 14$, $c = 0$

Hence, the value of $(a + b + c + 2)$

$= 28$

52. We have $x = \sin(2\tan^{-1} 2)$

$\Rightarrow \quad x = \sin(2\theta), \tan^{-1} 2 = \theta$

$\Rightarrow \quad x = \dfrac{2\tan\theta}{1+\tan^2\theta}, \tan\theta = 2$

$\Rightarrow \quad x = \dfrac{4}{5}$

Also, $y = \sin\left(\dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right)\right)$

$\Rightarrow \quad y = \sin(\theta), \dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right) = \theta$

$\Rightarrow \quad y = \sin(\theta), \tan(2\theta) = \dfrac{4}{3}$

$\Rightarrow \quad y = \sin(\theta), \tan(\theta) = \dfrac{1}{2}$

$\Rightarrow \quad y = \dfrac{1}{\sqrt{5}}$

Hence, $y^2 = \dfrac{1}{5} = 1 - \dfrac{4}{5} = 1 - x$

Level IV

1. We have

$\sin^{-1}(\cos(\sin^{-1} x)) + \cos^{-1}(\sin(\cos^{-1} x))$

$= \sin^{-1}(\sqrt{1-x^2}) + \cos^{-1}(\sqrt{1-x^2})$

$= \cos^{-1}(x) + \sin^{-1}(x)$

$= \dfrac{\pi}{2}$

2. We have

$\tan^{-1}\{\text{cosec}(\tan^{-1} x) - \tan(\cot^{-1} x)\}$

$= \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

$= \tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right), x = \tan\theta$

$= \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\dfrac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right)$

$= \tan^{-1}\left(\tan\left(\dfrac{\theta}{2}\right)\right)$

$= \dfrac{1}{2}\tan^{-1} x$

3. We have

$(\tan^{-1} x + \tan^{-1} y + \tan^{-1} z)$

$= \tan\left(\tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)\right)$

$= \left(\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right)$

Also, $\cot(\cot^{-1} x \cot^{-1} y + \cot^{-1} z)$

$= \cot\left(\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}\left(\dfrac{1}{y}\right) + \tan^{-1}\left(\dfrac{1}{z}\right)\right)$

$= \cot\left(\tan^{-1}\left(\dfrac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{xyz}}{1-\left(\frac{1}{xy}+\frac{1}{xz}+\frac{1}{yz}\right)}\right)\right)$

$= \cot\left(\tan^{-1}\left(\dfrac{xy+yz+zx-1}{xyz-(x+y+z)}\right)\right)$

$= \cot\left(\cot^{-1}\left(\dfrac{xyz-(x+y+z)}{xy+yz+zx-1}\right)\right)$

$= \cot\left(\cot^{-1}\left(\dfrac{(x+y+z)-xyz}{1-(xy+yz+zx)}\right)\right)$

$= \left(\dfrac{(x+y+z)-xyz}{1-(xy+yz+zx)}\right)$

Hence, the result.

4. Given expression is

$\sin(\cot^{-1}(\tan(\cot^{-1} x)))$

$= \sin(\cot^{-1}(\tan\theta)), \theta = \cos^{-1} x$

$= \sin(\cot^{-1}(\tan\theta)), \cos\theta = x$

$= \sin\left(\cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)\right)$

$= \sin\varphi$, where $\cot\varphi = \dfrac{\sqrt{1-x^2}}{x}$

$= x$

5. Given expression is

$\sin(\text{cosec}^{-1}(\cot(\tan^{-1} x)))$

$= \sin(\text{cosec}^{-1}(\cot\theta))$, where $\tan\theta = x$

$= \sin\left(\text{cosec}^{-1}\left(\dfrac{1}{x}\right)\right)$

$= \sin\varphi$, where $\operatorname{cosec}\varphi = \dfrac{1}{x}$

$= x$

6. Given expression is

$\sin^{-1}(\sin 5) + \cos^{-1}(\cos 10) + \tan^{-1}(\tan(-6)) + \cot^{-1}(\cot(-10))$

$= (5 - 2\pi) + (4\pi - 10)\ (6 - 2\pi) + \pi - (10\ 3\pi)$

$= (5 - 2\pi) + (4\pi - 10) + (2\pi - 6) + \pi + (3\pi - 10)$

$= 8\pi - 21$

7. We have

$$\cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right), \forall\, x \in \left(\frac{1}{2}, 1\right)$$

$$= \cos^{-1}(x) + \cos^{-1}\left(x\cdot\frac{1}{2} + \sqrt{1-\frac{1}{4}}\sqrt{1-x^2}\right)$$

$$= \cos^{-1}(x) + \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(x)$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3}$$

8. We have

$$\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\sqrt{5-2\sqrt{6}}}{1+\sqrt{6}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{\sqrt{(\sqrt{3}-\sqrt{2})^2}}{1+\sqrt{6}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{(\sqrt{3}-\sqrt{2})}{1+\sqrt{3}\sqrt{2}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - (\tan^{-1}(\sqrt{3}) - \tan^{-1}(\sqrt{2}))$$

$$= \cot^{-1}(\sqrt{2}) + \tan^{-1}(\sqrt{2}) - \tan^{-1}(\sqrt{3})$$

$$= \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

9. We have

$m = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^2 + 1)$

since $\sin^{-1}(\ ) + \cos^{-1}(\ )$ is defined for $[-1, 1]$

Put $a = 0$, then

$m = \sin^{-1}(1) + \cos^{-1}(1) - \tan^{-1}(1)$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Hence, the image of the line $x + y = \dfrac{\pi}{4}$ w.r.t. to the y-axis is $x - y + \dfrac{\pi}{4} = 0$

10. Given equation is

$$(\sin^{-1}x)^3 + (\sin^{-1}y)^3 + (\sin^{-1}z)^3 = \frac{(3\pi)^3}{8}$$

It is possible only when

$$\sin^{-1}x = \frac{\pi}{2} = \sin^{-1}y = \sin^{-1}z$$

So, $x = 1, y = 1$ and $z = 1$

Hence, the value of $(3x + 4y - 5z + 2)$

$= 3 + 4 - 5 + 2$

$= 4$

11. We have

$$S = \sum_{r=1}^{n}\cot^{-1}\left(2^{r+1} + \frac{1}{2^r}\right)$$

$$= \sum_{r=1}^{n}\cot^{-1}\left(\frac{2^{2r+1}+1}{2^r}\right)$$

$$= \sum_{r=1}^{n}\tan^{-1}\left(\frac{2^r}{1+2^{2r+1}}\right)$$

$$= \sum_{r=1}^{n}\tan^{-1}\left(\frac{2^{r+1}-2^r}{1+2^{r+1}\cdot 2^r}\right)$$

$$= \sum_{r=1}^{n}[\tan^{-1}(2^{r+1}) - \tan^{-1}(2^r)]$$

$= [\tan^{-1}(2^{n+1}) - \tan^{-1}(2)]$

Thus, $\lim_{n\to\infty}(S)$

$$= \lim_{n\to\infty}[\tan^{-1}(2^{n+1}) - \tan^{-1}(2)]$$

$= [\tan^{-1}(\infty) - \tan^{-1}(2)]$

$$= \frac{\pi}{2} - \tan^{-1}(2)$$

$= \cot^{-1}(2)$

12. We have

$$\lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{4}{4r^2+3}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{r^2+\frac{3}{4}}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{1+\left(r^2-\frac{1}{4}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{\left(r+\frac{1}{2}\right)-\left(r-\frac{1}{2}\right)}{1+\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(r+\frac{1}{2}\right) - \tan^{-1}\left(r-\frac{1}{2}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\tan^{-1}\left(n+\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\tan\left(\tan^{-1}\left(\frac{n+\frac{1}{2}-\frac{1}{2}}{1+\frac{1}{2}\left(n+\frac{1}{2}\right)}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{1+\frac{1}{2}\left(n+\frac{1}{2}\right)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{\frac{1}{n}+\frac{1}{2}\left(1+\frac{1}{2n}\right)}\right)$$

$$= 2$$

13. Given equation is

$$2\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi x^3$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{\pi x^3}{2}$$

$$2\tan^{-1}x = \frac{\pi x^3}{2}$$

$$\tan^{-1}x = \frac{\pi x^3}{4}$$

Clearly, there are 3 solutions at $x = -1, 0, 1$.

14. Given, $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$

$$\Rightarrow \quad \cos^{-1}\left(\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right) = \alpha$$

$$\Rightarrow \quad \left(\frac{xy}{ab} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right) = \cos\alpha$$

$$\Rightarrow \quad \left(\frac{xy}{ab} - \cos\alpha\right)^2 = \left(\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right)^2$$

$$\Rightarrow \quad \left(\frac{xy}{ab}\right)^2 - 2\left(\frac{xy}{ab}\right)\cos\alpha + \cos^2\alpha$$

$$= 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \left(\frac{xy}{ab}\right)^2$$

$$\Rightarrow \quad \frac{x^2}{a^2} - 2\left(\frac{xy}{ab}\right)\cos\alpha + \frac{y^2}{b^2} = 1 - \cos^2\alpha$$

$$\Rightarrow \quad \frac{x^2}{a^2} - 2\left(\frac{xy}{ab}\right)\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$$

15. Put $A = \sin^{-1} x$, $B = \sin^{-1} y$, $C = \sin^{-1} z$

$\Rightarrow x = \sin A, y = \sin B, z = \sin C$

$\Rightarrow \cos A = \sqrt{1-x^2}, \cos B = \sqrt{1-y^2}, \cos C = \sqrt{1-z^2}$

Thus, $A + B + C = \pi$

Now, LHS

$$= x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$$

$$= \sin A\cos A + \sin B\cos B + \sin C\cos C$$

$$= \frac{1}{2}[\sin 2A + \sin 2B + \sin 2C]$$

$$= \frac{1}{2}(4\sin A\sin B\sin C)$$

$$= 2\sin A\sin B\sin C$$

$$= 2xyz$$

Hence, the result.

16. Given,

$$f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$$

$$= (\sin^{-1} x + \cos^{-1} x)\{(\sin^{-1} x)^2 + (\cos^{-1} x)^2 - \sin^{-1} x\cos^{-1} x\}$$

$$= \frac{\pi}{2}\left\{\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}x\cos^{-1}x\right\}$$

$$= \frac{\pi}{2}\left\{\frac{\pi^2}{4} - 3\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)\right\}$$

$$= \frac{\pi}{2}\left\{\frac{\pi^2}{4} - 3a\left(\frac{\pi}{2} - a\right)\right\} \text{ where } a = \sin^{-1}x$$

$$= \frac{\pi}{2}\left\{3a^2 - \frac{3a\pi}{2} + \frac{\pi^2}{4}\right\}$$

$$= \frac{\pi}{8}\{12\alpha^2 - 6\pi a + \pi^2\}$$

$$= \frac{12\pi}{8}\left\{\alpha^2 - \frac{1}{2}\pi a + \frac{\pi^2}{12}\right\}$$

$$= \frac{12\pi}{8}\left\{\left(a - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48}\right\}$$

Minimum value $= \frac{\pi^3}{32}$ at $x = \frac{1}{\sqrt{2}}$

and maximum value $= \frac{7\pi^3}{8}$ at $x = -1$

17. Given equation is

$$\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}(2x) = \frac{\pi}{2} - \sin^{-1}x = \cos^{-1}x$$

$$\Rightarrow \quad \sin^{-1}(2x) = \sin^{-1}(\sqrt{1-\mathrm{x}^2})$$

$$\Rightarrow \quad 2x = \sqrt{1-x^2}$$

$$\Rightarrow \quad 4x^2 = 1 - x^2$$

$$\Rightarrow \quad 5x^2 = 1$$

$$\Rightarrow \quad x^2 = \frac{1}{5}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{1}{5}}$$

18. Given equation is

$$\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{1+4x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\frac{1}{1+2x}\times\frac{1}{1+4x}}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \quad \frac{1+4x+1+2x}{1+6x+8x^2-1}=\frac{2}{x^2}$$

$$\Rightarrow \quad \frac{2+6x}{6x+8x^2}=\frac{2}{x^2}$$

$$\Rightarrow \quad \frac{1+3x}{3x+4x^2}=\frac{2}{x^2}$$

$\Rightarrow \quad 3x^3 + x^2 + 8x^2 + 6x = 0$

$\Rightarrow \quad 3x^3 + 9x^2 + 6x = 0$

$\Rightarrow \quad x^3 + 3x^2 + 2x = 0$

$\Rightarrow \quad x(x^2 + 3x = 2) = 0$

$\Rightarrow \quad x(x + 1)(x + 2) = 0$

$\Rightarrow \quad x = 0, -1, -2$

19. Given equation is

$\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$

$\Rightarrow \quad \tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}(3x) - \tan^{-1}(x)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-1+x+1}{1-(x^2-1)}\right)=\tan^{-1}\left(\frac{3x-x}{1+3x\cdot x}\right)$$

$$\Rightarrow \quad \left(-\frac{2x}{x^2}\right)=\left(\frac{2x}{1+3x^2}\right)$$

$\Rightarrow \quad 2x(1 + 3x^2 + x^2) = 0$

$\Rightarrow \quad 2x = 0, (1 + 4x^2) = 0$

$\Rightarrow \quad x = 0$

Hence, the solution is $x = 0$

20. Given equation is

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1}\left(\sqrt{1-\frac{1}{5}}\right)+\cos^{-1}x=\frac{\pi}{4}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)+\cos^{-1}x=\frac{p}{4}$$

$$\Rightarrow \quad \cos^{-1}x=\frac{\pi}{4}-\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad x=\cos\left(\frac{\pi}{4}-\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

$$\Rightarrow \quad x=\cos\left(\frac{\pi}{4}\right)\cos\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)+\sin\left(\frac{\pi}{4}\right)\sin\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$$

$$\Rightarrow \quad x=\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{5}}=\frac{3}{\sqrt{10}}$$

21. Given equation is

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\tan^{-1}\left(\frac{2x}{x^2-1}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \cos^{-1}\left(-\frac{1-x^2}{x^2+1}\right)+\tan^{-1}\left(-\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-\cos^{-1}\left(\frac{1-x^2}{x^2+1}\right)-\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-2\tan^{-1}(x)-2\tan^{-1}(x)=\frac{2\pi}{3}$$

$$\Rightarrow \quad \pi-4\tan^{-1}(x)=\frac{2\pi}{3}$$

$$\Rightarrow \quad 4\tan^{-1}(x)=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}(x)=\frac{\pi}{12}$$

$$\Rightarrow \quad x=\tan\left(\frac{\pi}{12}\right)=(2-\sqrt{3})$$

22. Given equation is

$$2\tan^{-1}x=\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)$$

$\Rightarrow \quad 2\tan^{-1}x = 2\tan^{-1}(a) - 2\tan^{-1}(b)$

$\Rightarrow \quad \tan^{-1}x = \tan^{-1}(a) - \tan^{-1}(b)$

$$\Rightarrow \quad \tan^{-1}x=\tan^{-1}\left(\frac{a-b}{1+ab}\right)$$

$$\Rightarrow \quad x=\left(\frac{a-b}{1+ab}\right)$$

23. Given equation is

$\cot^{-1}x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n-1)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right)+\tan^{-1}\left(\frac{1}{n^2-x+1}\right)=\tan^{-1}\left(\frac{1}{n-1}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{n^2-x+1}\right)=\tan^{-1}\left(\frac{1}{n-1}\right)-\tan^{-1}\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{n^2-x+1}\right)=\tan^{-1}\left(\frac{\frac{1}{n-1}-\frac{1}{x}}{1+\frac{1}{x(n-1)}}\right)$$

$$\Rightarrow \quad \left(\frac{1}{n^2-x+1}\right)=\frac{x-n+1}{nx-x+1}$$

$\Rightarrow \quad n^2x - x^2 + x - n^3 + nx - n + n^2 - x + 1 = nx - x + 1$

$\Rightarrow \quad n^2x - x^2 + x - n^3 - n + n^2 = 0$

$\Rightarrow \quad (n^2 + 1)x - (n^2 + 1)n - x^2 + n^2 = 0$

$\Rightarrow \quad (n^2 + 1)(x - n) - (x^2 - n^2) = 0$

$\Rightarrow \quad (x - n)(n^2 + 1 - x - n) = 0$

$\Rightarrow \quad (x - n) = 0, (n^2 + 1 - x - n) = 0$

$\Rightarrow \quad x = n, n^2 - n + 1$

Hence, the solutions are

$x = n, n^2 - n + 1$

24. Given equation is

$$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \left(\frac{x-1}{x+1} \cdot \frac{2x-1}{2x+1}\right)}\right) = \tan^{-1}\left(\frac{23}{36}\right)$$

$$\Rightarrow \quad \frac{2x^2 - x - 1 + 2x^2 + x - 1}{2x^2 + 3x + 1 - 2x^2 + 3x - 1} = \frac{23}{26}$$

$$\Rightarrow \quad \frac{4x^2 - 2}{6x} = \frac{23}{36}$$

$$\Rightarrow \quad \frac{2x^2 - 1}{x} = \frac{23}{12}$$

$\Rightarrow \quad 24x^2 - 23x - 12 = 0$

$\Rightarrow \quad 24x^2 - 32x + 9x - 12 = 0$

$\Rightarrow \quad 8x(3x - 4) + 3(3x - 4) = 0$

$\Rightarrow \quad (8x + 3)(3x - 4) = 0$

$$\Rightarrow \quad x = \frac{4}{3}, -\frac{3}{8}$$

Hence, the solutions are $x = \frac{4}{3}, -\frac{3}{8}$

25. Given equation is

$$\sec^{-1}\left(\frac{x}{a}\right) - \sec^{-1}\left(\frac{x}{b}\right) = \sec^{-1}b - \sec^{-1}a$$

$$\Rightarrow \quad \sec^{-1}\left(\frac{x}{a}\right) + \sec^{-1}(a) = \sec^{-1}\left(\frac{x}{b}\right) + \sec^{-1}(b)$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{a}{x}\right) + \cos^{-1}\left(\frac{1}{a}\right) = \cos^{-1}\left(\frac{b}{x}\right) + \cos^{-1}\left(\frac{1}{b}\right)$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{a}{x} \cdot \frac{1}{a} - \sqrt{1 - \left(\frac{a}{x}\right)^2}\sqrt{1 - \frac{1}{a^2}}\right)$$

$$= \cos^{-1}\left(\frac{b}{x} \cdot \frac{1}{b} - \sqrt{1 - \left(\frac{b}{x}\right)^2}\sqrt{1 - \frac{1}{b^2}}\right)$$

$$\Rightarrow \quad \frac{1}{x} - \frac{\sqrt{(x^2 - a^2)(a^2 - 1)}}{ax} = \frac{1}{x} - \frac{\sqrt{(x^2 - b^2)(b^2 - 1)}}{bx}$$

$$\Rightarrow \quad \frac{\sqrt{(x^2 - a^2)(a^2 - 1)}}{a} = \frac{\sqrt{(x^2 - b^2)(b^2 - 1)}}{b}$$

$$\Rightarrow \quad \frac{(x^2 - a^2)(a^2 - 1)}{a^2} = \frac{(x^2 - b^2)(b^2 - 1)}{b^2}$$

$\Rightarrow \quad (x^2 - a^2)(a^2 - 1)\, b^2 = (x^2 - b^2)(b^2 - 1)\, n^2$

$\Rightarrow \quad x^2((a^2 - 1)b^2 - a^2(b^2 - 1)) = a^2b^2(a^2 - 1) - a^2b^2(b^2 - 1)$

$\Rightarrow \quad x^2[a^2 - b^2] = a^2b^2[(a^2 - 1) - (b^2 - 1)]$

$\Rightarrow \quad x^2[(a^2 - b^2)] = a^2b^2[(a^2 - b^2)]$

$\Rightarrow \quad x^2 = a^2b^2$

$\Rightarrow \quad x = ab$

26. We have

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{8n}{n^4 - 2n^2 + 5}\right)$$

$$= \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2\{(n+1)^2 - (n-1)^2\}}{4 + (n-1)^2}\right)$$

$$= \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\left\{\left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2\right\}}{1 + \left(\frac{n+1}{2}\right)^2 \cdot \left(\frac{n-1}{2}\right)^2}\right)$$

$$= \sum_{n=1}^{\infty} \tan^{-1}\left\{\left(\frac{n+1}{2}\right)^2\right\} - \tan^{-1}\left\{\left(\frac{n+1}{2}\right)^2\right\}$$

$$= \left[\tan^{-1}(1^2) - 0 + \tan^{-1}\left(\frac{3}{2}\right)^2 - \tan^{-1}(1^2)\right.$$

$$\left. + \tan^{-1}(2^2) - \tan^{-1}\left(\frac{3}{2}\right)^2 + \ldots\ldots + \right.$$

$$\left. = \tan^{-1}\left(\frac{n+1}{2}\right)^2 - \tan^{-1}\left(\frac{n-1}{2}\right)^2\right], n \to \infty$$

$$= \lim_{n \to \infty}\left\{\tan^{-1}\left(n + \frac{1}{2}\right)^2 - 0\right\}$$

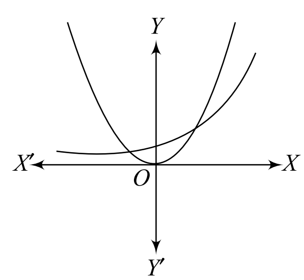
$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

27. Given equation is

$$\sin^{-1}(e^x) + \cos^{-1}(x^2) = \frac{\pi}{2}$$

It solutions exist only when

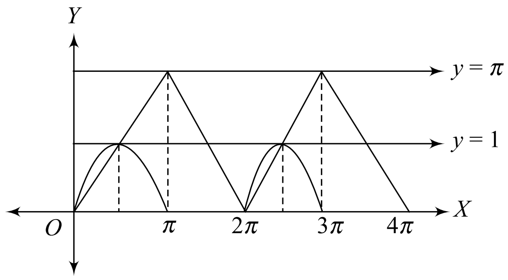
$e^x = x^2$



Hence, the number of solution is 2

28. Given equation is

$$\sqrt{\sin(x)} = \cos^{-1}(\cos x) \text{ in } (0, 2\pi)$$



From the graph, it is clear that, the number of real solution is 1 at $x = \frac{\pi}{2}$

29. Given equation is

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{3}\cdot\frac{1}{4}}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{7}{11}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{7}{11}+\frac{1}{5}}{1-\frac{7}{11}\cdot\frac{1}{5}}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{23}{24}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4} - \tan^{-1}\left(\frac{23}{24}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}(1) - \tan^{-1}\left(\frac{23}{24}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}\left(\frac{1-\frac{23}{24}}{1+\frac{23}{24}}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n}\right) = \tan^{-1}\left(\frac{1}{47}\right)$$

$\Rightarrow n = 47$

30. Given equation is

$$x^3 + bx^2 + cx + 1 = 0$$

Let $f(x) = x^3 + bx^2 + cx + 1$

Now, $f(0) = 1 > 0, f(-1) = b - c < 0$

So, the function $f(x)$ has a root in $-1 < \alpha < 0$

Now, $\tan^{-1}(\alpha) + \tan^{-1}\left(\frac{1}{\alpha}\right)$

$= \tan^{-1}\alpha - \pi + \cot^{-1}\alpha$

$= -\pi + (\tan^{-1}\alpha + \cot^{-1}\alpha)$

$= -\pi + \frac{\pi}{2} = -\frac{\pi}{2}$

31. Given equation is

$$x^3 + bx^2 + cx + 1 = 0$$

Let $f(x) = x^3 + bx^2 + cx + 1$

Now, $f(0) = 1 > 0, f(-1) = b - c < 0$

So, the function $f(x)$ has a root in $-1 < \alpha < 0$

Now, $2\tan^{-1}(\operatorname{cosec}\alpha) + \tan^{-1}(2\sin\alpha\sec^2\alpha)$

$$= 2\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + \tan^{-1}\left(\frac{2\sin\alpha}{\cos^2\alpha}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + \tan^{-1}\left(\frac{2\sin\alpha}{1-\sin^2\alpha}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + 2\tan^{-1}(\sin\alpha)$$

$$= 2\left[\tan^{-1}\left(\frac{1}{\sin\alpha}\right) + \tan^{-1}(\sin\alpha)\right]$$

$$= 2\left(-\frac{\pi}{2}\right), \text{ as } \sin\alpha < 0$$

$$= \pi$$

32. Given, $\sin^{-1}x > \cos^{-1}x$

$\Rightarrow 2\sin^{-1}x > \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}x > \frac{\pi}{4}$

$\Rightarrow x > \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Hence, the solution is $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$

33. Given, $\cos^{-1}x > \sin^{-1}x$

$\Rightarrow \sin^{-1}x + \cos^{-1}x > 2\sin^{-1}x$

$\Rightarrow 2\sin^{-1}x < \frac{\pi}{2}$

$\Rightarrow \sin^{-1}x < \frac{\pi}{4}$

$\Rightarrow x < \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

Hence, the solution set is $x \in \left[-1, \frac{1}{\sqrt{2}}\right)$

34. Let $a = \cot^{-1}x$

The given in-equation reduces to

$a^2 - 5a + 6 > 0$

$\Rightarrow (a-2)(a-3) > 0$

$\Rightarrow a < 2$ and $a > 3$

$\Rightarrow \cot^{-1}x < 2$ and $\cot^{-1}x > 3$

$\Rightarrow x > \cot(2)$ and $x > \cot(3)$

Hence, the solution set is

$(-\infty, \cot(2)) \cup (\cot 3, \infty)$

35. $\tan^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) > 1$

$$\Rightarrow \left\{\tan\left(\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)\right\}^2 > 1$$

$\Rightarrow \quad \frac{x^2}{(1-x^2)} > 1$

$\Rightarrow \quad \frac{x^2}{(1-x^2)} - 1 > 0$

$\Rightarrow \quad \frac{x^2 - 1 + x^2}{(1-x^2)} > 0$

$\Rightarrow \quad \frac{2x^2 - 1}{(x^2 - 1)} < 0$

$\Rightarrow \quad \frac{(\sqrt{2}x+1)(\sqrt{2}x-1)}{(x+1)(x-1)} < 0$

$\Rightarrow \quad x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(1, \frac{1}{\sqrt{2}}\right)$

36. Given, $4(\tan^{-1} x)^2 - 8(\tan^{-1} x) + 3 < 0$

$\Rightarrow \quad 4a^2 - 8a + 3, a = \tan^{-1} x$

$\Rightarrow \quad 4a^2 - 6a - 2a = 3 < 0$

$\Rightarrow \quad 2a(2a-3) - 1(2a-3) < 0$

$\Rightarrow \quad (2a-1)(2a-3) < 0$

$\Rightarrow \quad \frac{1}{2} < a < \frac{3}{2}$

$\Rightarrow \quad \frac{1}{2} < \tan^{-1} x < \frac{3}{2}$

$\Rightarrow \quad \tan\left(\frac{1}{2}\right) < x < \tan\left(\frac{3}{2}\right)$

37. Given, $4 \cot^{-1} x - (\cot^{-1} x)^2 - 3 \geq 0$

$\Rightarrow \quad 4a - a^2 - 3 \geq 0, a = \cot^{-1} x$

$\Rightarrow \quad a^2 - 4a + 3 \leq 0$

$\Rightarrow \quad (a-1)(a-3) \leq 0$

$\Rightarrow \quad 1 \leq a \leq 3$

$\Rightarrow \quad 1 \leq \cot^{-1} \leq x \leq 3$

$\Rightarrow \quad \cot(3) \leq x \leq \cot(1)$

38. Given in equation is

$$\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 2$$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\frac{2(x^2+1)+2}{x^2+1}\right)\right) < \pi - 2$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(2 + \frac{2}{x^2+1}\right)\right) < \pi - 2$

$\Rightarrow \quad \sin^{-1}\left(\sin\left(\pi - \frac{2x^2+4}{x^2+1}\right)\right) < \pi - 2$

$\Rightarrow \quad \left(\pi - \frac{2x^2+4}{x^2+1}\right) < \pi - 2$

$\Rightarrow \quad \left(\frac{2x^2+4}{x^2+1}\right) > 2$

$\Rightarrow \quad \left(\frac{x^2+2}{x^2+1}\right) > 1$

which is a true statement.

Hence, $x \in R$

39. We have

$f(x) = \{\sin^{-1} (\sin x)\}^2 - \sin^{-1} (\sin x)$

$= \left\{\sin^{-1}(\sin x) - \frac{1}{2}\right\}^2 - \frac{1}{4}$

$= \left\{\frac{\pi}{2} + \frac{1}{2}\right\}^2 - \frac{1}{4}$

$= \frac{\pi}{4}(\pi + 2)$), since the maximum value of $\sin^{-1}(\sin x)$ is $-\frac{\pi}{2}$

40. Clearly, both terms are positive.

Applying, AM $\geq$ GM, we get,

$\Rightarrow \quad \frac{8^{\sin^{-1}x} + 8^{\cos^{-1}x}}{2} \geq \sqrt{8^{\sin^{1-}x} \cdot 8^{\cos^{-1}x}}$

$\Rightarrow \quad \frac{f(x)}{2} \geq \sqrt{8^{\sin^{-1}x + \cos^{-1}x}}$

$\Rightarrow \quad \frac{f(x)}{2} \geq \sqrt{8^{\frac{p}{2}}}$

$\Rightarrow \quad f(x) \geq 2\sqrt{8^{\frac{\pi}{2}}} = 2.8^{\frac{\pi}{4}} = 2.2^{\frac{3\pi}{4}} = 2^{1+\frac{3\pi}{4}}$

Hence, the minimum value of $2^{1+\frac{3\pi}{4}}$

41. Given in equation is

$x^2 - kx + \sin^{-1} (\sin 4) > 0$

$\Rightarrow \quad x^2 - kx + \sin^{-1} (\sin (\pi - 4)) > 0$

$\Rightarrow \quad x^2 - kx + (\pi - 4) > 0$

$\Rightarrow \quad$ For, all x in R, $D \geq 0$

$\Rightarrow \quad k^2 - 4(\pi - 4) \geq 0$

$\Rightarrow \quad k^2 \geq 4(\pi - 4)$

So, no real values of k satisfies the above in equation.

Hence, the solution is $k = \varphi$

42. Given,

$A = 2 \tan^{-1}(2\sqrt{2} - 1)$

$\Rightarrow \quad A = 2 \tan^{-1}(2.8 - 1) = 2 \tan^{-1}(1.4)$

$\Rightarrow \quad A > \frac{2\pi}{3}$

and $\quad B = 3 \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$\Rightarrow \quad B = \sin^{-1}\left(\frac{3}{3} - \frac{4}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$\Rightarrow \quad B = \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$\Rightarrow \quad B < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad B < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Hence, $A > B$

43. We have

$$\sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sqrt{\frac{2-\sqrt{3}}{4}}\right)+\cos^{-1}\left(\frac{\sqrt{12}}{4}\right)+\sec^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}(\sqrt{2})\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\sin^{-1}\left(\sin\left(\frac{\pi}{12}\right)\right)+\frac{\pi}{6}+\frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left\{\frac{\pi}{12}+\frac{\pi}{6}+\frac{\pi}{4}\right\}\right\}$$

$$= \sin^{-1}\left\{\cot\left(\frac{6\pi}{12}\right)\right\}$$

$$= \sin^{-1}(0) = 0$$

44. Given,

$f(x) = \sin^{-1}(\cos^{-1} x + \tan^{-1} x + \cot^{-1} x)$

$= \sin^{-1}\left(\frac{\pi}{2} + \cos^{-1}x\right)$

Thus, $-1 \le \left(\frac{\pi}{2} + \cos^{-1}x\right) \le 1$

$\Rightarrow \quad -1 - \frac{\pi}{2} \le \cos^{-1}x \le 1 - \frac{\pi}{2}$

But the ranges of $\cos^{-1} x$ is $[0, \pi]$

So it has no solution

Therefore, $D_f = \varphi$

45. Clearly, $0 \le x \le 4$

We have

$$\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \sin^{-1}\left(\sqrt{1-\frac{x}{4}}\right) + \tan^{-1}y = \frac{2p}{3}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{\sqrt{x}}{2}\right) + \cos^{-1}\left(\frac{\sqrt{x}}{2}\right) + \tan^{-1}(y) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \frac{\pi}{2} + \tan^{-1}(y) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}(y) = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$$

$$\Rightarrow \quad y = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Hence, the maximum value of $(x^2 + y^2 + 1)$ is

$$= \left(16 + \frac{1}{3} + 1\right) = \frac{52}{3}$$

46. Given, $\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{10}\right)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{1}{x}+\frac{1}{y}}{1-\frac{1}{x}\cdot\frac{1}{y}}\right) = \tan^{-1}\left(\frac{1}{10}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{xy-1}\right) = \tan^{-1}\left(\frac{1}{10}\right)$$

$$\Rightarrow \quad \left(\frac{x+y}{xy-1}\right) = \left(\frac{1}{10}\right)$$

$\Rightarrow \quad 10(x + y) = xy - 1$

$\Rightarrow \quad y(10 - x) = -1 - 10x$

$$\Rightarrow \quad y = \left(\frac{1+10x}{10-x}\right)$$

Thus, there are four ordered pairs

(11, 111), (111, 11), (9, –91), (–91, 9)

satisfying the above equations.

47. Given, $\left[\cot\left(\sum_{k=1}^{10}\cot^{-1}(k^2+k+1)\right)\right] = \frac{a}{b}$

$$\Rightarrow \quad \frac{a}{b} = \left[\cot\left(\sum_{k=1}^{10}\cot^{-1}(k^2+k+1)\right)\right]$$

$$= \left[\cot\left(\sum_{k=1}^{10}\tan^{-1}\left(\frac{1}{1+k+k^2}\right)\right)\right]$$

$$= \left[\cot\left(\sum_{k=1}^{10}\tan^{-1}\left(\frac{(k+1)-k}{1+(k+1)k}\right)\right)\right]$$

$$= \left[\cot\left(\sum_{k=1}^{10}[\tan^{-1}(k+1) - \tan^{-1}(k)]\right)\right]$$

$$= [\cot(\tan^{-1}(11) - \tan^{-1}(1))]$$

$$= \left[\cot\left(\tan^{-1}\left(\frac{11-1}{1+11}\right)\right)\right]$$

$$= \left[\cot\left(\tan^{-1}\left(\frac{5}{6}\right)\right)\right]$$

$$= \left[\cot\left(\cot^{-1}\left(\frac{6}{5}\right)\right)\right]$$

$$= \frac{6}{5}$$

Hence, the value of $(a + b + 10)$ is 21.

48. We have

$$\tan^{-1}\left(\frac{p-q}{1+pq}\right) + \tan^{-1}\left(\frac{q-r}{1+qr}\right) + \tan^{-1}\left(\frac{r-p}{1+rp}\right)$$

$= (\tan^{-1} p - \tan^{-1} q) + (\tan^{-1} q - \tan^{-1} r) + p + (\tan^{-1} r - \tan^{-1} p)$

$= \pi$

49. Given equation is

$(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = \pi a^3$

$\Rightarrow \quad \frac{\pi}{2}[(\sin^{-1}x)^2 + (\cos^{-1}x)^2 - 2\sin^{-1}x\cos^{-1}x] = \pi a^3$

$\Rightarrow \quad \left[\frac{\pi^2}{4} - 3\sin^{-1}x\cos^{-1}x\right] = 2a\pi^2$

$\Rightarrow \quad \left[\pi^2 - 12b\left(\frac{\pi}{2} - b\right)\right] = 8a\pi^2$

$\Rightarrow \quad [\pi^2 - 6b\pi + 12b^2] = 8a\pi^2$

$\Rightarrow \quad \left(b^2 - \frac{b\pi}{2} + \frac{\pi^2}{12}\right) = \frac{8a\pi^2}{12} = \frac{2a\pi^2}{3}$

$\Rightarrow \quad \left(b - \frac{\pi}{4}\right)^2 = \frac{p^2}{12}(8a - 1) + \frac{\pi^2}{16}$

$\Rightarrow \quad \left(b - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{48}(32a - 1)$

$\Rightarrow \quad \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{48}(32a - 1)$

As we know that,

$-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2} - \frac{\pi}{4} \le \sin^{-1}x - \frac{\pi}{4} \le \frac{\pi}{2} - \frac{\pi}{4}$

$\Rightarrow \quad -\frac{3\pi}{4} \le \left(\sin^{-1}x - \frac{\pi}{4}\right) \le \frac{\pi}{4}$

$\Rightarrow \quad 0 \le \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \le \frac{9\pi^2}{16}$

$\Rightarrow \quad 0 \le \frac{\pi^2}{48}(32a - 1) \le \frac{9\pi^2}{16}$

$\Rightarrow \quad 0 \le (32a - 1) \le 27$

$\Rightarrow \quad 1 \le 32a \le 28$

$\Rightarrow \quad \frac{1}{32} \le a \le \frac{7}{8}$

50. Let $g(x) = \frac{x^2}{x^2 + 1} = 1 - \frac{1}{1 + x^2}$

Clearly, $R_g = [0, 1)$

Now, $R_f = (f(1), f(0)] = (\cot^{-1}(1), \cot^{-1}(1)]$

$= \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$

Hence, the value of $\left(\frac{b}{a} + 2\right)$

$= \left(\frac{\frac{\pi}{2}}{\frac{\pi}{4}} + 2\right) = 2 + 2 = 4$

51. Given $\tan^{-1} y = 4\tan^{-1} x$

$\Rightarrow \quad \tan^{-1} y = \tan^{-1}\left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$

$\Rightarrow \quad y = \left(\frac{4x - 4x^3}{1 - 6x^2 + x^4}\right)$

$\Rightarrow \quad \frac{1}{y} = \left(\frac{1 - 6x^2 + x^4}{4x - 4x^3}\right)$

$\Rightarrow \quad \frac{1}{y} = \cot(4\theta)$, where, $x = \tan\theta$

$\Rightarrow \quad$ Clearly, $\frac{1}{y} = 0$ is zero only when $\theta = \frac{\pi}{8}$

Hence, $x^4 - 6x^2 + 1 = 0$

52. Let $x = \sqrt{\frac{a(a+b+c)}{bc}}$, $y = \sqrt{\frac{b(a+b+c)}{ac}}$

and $z = \sqrt{\frac{c(a+b+c)}{ab}}$

Now, $x + y + z - xyz$

$= \sqrt{(a+b+c)}\left(\sqrt{\frac{a}{bc}} + \sqrt{\frac{b}{ca}} + \sqrt{\frac{c}{ab}}\right) - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$

$= \sqrt{(a+b+c)}\left(\frac{a+b+c}{\sqrt{abc}}\right) - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$

$= \frac{(a+b+c)^{3/2}}{\sqrt{abc}} - \frac{(a+b+c)^{3/2}}{\sqrt{abc}}$

$= 0$

Now, $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

$= \tan^{-1}\left(\frac{x + y + z - xyz}{1 - xy - yz - zx}\right) = \tan^{-1}(0) = \pi$

Hence, the result.

53. We have

$\theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{6\tan\theta}{9 + \tan^2\theta}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2}\sin^{-1}\left(\frac{2\left(\frac{\tan\theta}{3}\right)}{1 + \frac{1}{3}\tan^2\theta}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \frac{1}{2} \times 2\tan^{-1}\left(\frac{\tan\theta}{3}\right)$

$\Rightarrow \quad \theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{\tan\theta}{3}\right)$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{2\tan^2\theta - \frac{\tan\theta}{3}}{1 + 2\tan^2\theta \cdot \frac{\tan\theta}{3}}\right)$

$\Rightarrow \quad \tan(\theta) = \dfrac{6\tan^2\theta - \tan\theta}{3 + 2\tan^3\theta}$

$\Rightarrow \quad 3\tan(\theta) + 2\tan^4\theta - 6\tan^2\theta = 2\tan\theta = 0$

$\Rightarrow \quad 2\tan^4\theta - 6\tan^2\theta + 5\tan\theta = 0$

$\Rightarrow \quad \tan\theta(2\tan^3\theta - 6\tan\theta + 5) = 0$

$\Rightarrow \quad \tan\theta = 0, (2\tan^2\theta - 6\tan\theta + 5) = 0$

$\Rightarrow \quad \theta = n\pi, n \in I$

54. We have

$$\tan^{-1}\left(\frac{x\cos\theta}{1 - x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x - \sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{x\cos\theta}{1 - x\sin\theta}\right) - \tan^{-1}\left(\frac{x - \sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x\cos\theta}{1 - x\sin\theta} - \frac{x - \sin\theta}{\cos\theta}}{1 + \frac{x\cos\theta}{1 - x\sin\theta} \times \frac{x - \sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{x\cos^2\theta - x + x^2\sin\theta + \sin\theta - x\sin^2\theta}{\cos\theta(1 - x\sin\theta) + x\cos\theta(x - \sin\theta)}\right)$$

$$= \tan^{-1}\left(\frac{(x^2 + 1)\sin\theta - 2x\sin^2\theta}{(x^2 + 1)\cos\theta - x\sin 2\theta}\right)$$

$$= \tan^{-1}\left(\frac{(x^2 - 2x\sin\theta + 1)\sin\theta}{(x^2 - 2x\sin\theta + 1)\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta) = \theta$$

55. Given equation is

$$\cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right) + \sin^{-1}\left(\frac{2x}{x^2 + 1}\right) + \tan^{-1}\left(\frac{2x}{x^2 - 1}\right) = \frac{2\pi}{3}$$

$\Rightarrow \quad \pi - 2\tan^{-1}x + 2\tan^{-1}x - 2\tan^{-1}x = \dfrac{2\pi}{3}$

$\Rightarrow \quad \pi - 2\tan^{-1}x = \dfrac{2\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}x = \dfrac{\pi}{3}$

$\Rightarrow \quad \tan^{-1}x = \dfrac{\pi}{6}$

$\Rightarrow \quad x = \tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$

Hence, the solution is $x = \tan\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$

56. We have

$$\tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{zx}{yr}\right) + \tan^{-1}\left(\frac{xy}{zr}\right)$$

$$= \tan^{-1}\left(\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{xyz}{r^3}}{1 - \left(\frac{z^2}{r^2} + \frac{x^2}{r^2} + \frac{y^2}{r^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{xyz}{r^3}}{1 - \left(\frac{x^2 + y^2 + z^2}{r^2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{yz}{xr} + \frac{zx}{yr} + \frac{xy}{zr} - \frac{xyz}{r^3}}{1 - 1}\right)$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

57. Given, $\displaystyle\sum_{r=1}^{10} \tan^{-1}\left(\frac{3}{9r^2 + 3r - 1}\right)$

$$= \sum_{r=1}^{10} \tan^{-1}\left(\frac{3}{1 + (3r + 2)(3r - 1)}\right)$$

$$= \sum_{r=1}^{10} \tan^{-1}\left(\frac{(3r + 2) - (3r - 1)}{1 + (3r + 2)(3r - 1)}\right)$$

$$= \sum_{r=1}^{10} [\tan^{-1}(3r + 2) - \tan^{-1}(3r - 1)]$$

$$= \tan^{-1}(32) - \tan^{-1}(2)$$

$$= \tan^{-1}\left(\frac{32 - 2}{1 + 32.2}\right)$$

$$= \tan^{-1}\left(\frac{30}{65}\right) = \tan^{-1}\left(\frac{6}{13}\right)$$

$$= \cot^{-1}\left(\frac{13}{6}\right)$$

Hence, the value of $(2m + n + 4)$

$= 26 + 6 + 4$

$= 36$

58. We have

$$S = \sum_{b=1}^{10}\sum_{a=1}^{10} \tan^{-1}\left(\frac{a}{b}\right)$$

$$= \sum_{b=1}^{10}\left[\tan^{-1}\left(\frac{1}{b}\right) + \tan^{-1}\left(\frac{2}{b}\right) + \tan^{-1}\left(\frac{3}{b}\right) + ... + \tan^{-1}\left(\frac{10}{b}\right)\right]$$

$$= \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) + ... + \tan^{-1}\left(\frac{1}{10}\right)$$

$$+ \tan^{-1}\left(\frac{2}{1}\right) + \tan^{-1}\left(\frac{2}{2}\right) + \tan^{-1}\left(\frac{2}{3}\right) + ... + \tan^{-1}\left(\frac{2}{10}\right)$$

$$+ \tan^{-1}\left(\frac{3}{1}\right) + \tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{3}{3}\right) + ... + \tan^{-1}\left(\frac{3}{10}\right)$$

$$+ \tan^{-1}\left(\frac{10}{1}\right) + \tan^{-1}\left(\frac{10}{2}\right) + \tan^{-1}\left(\frac{10}{3}\right) + ... + \tan^{-1}\left(\frac{10}{10}\right)$$

$$= 10 \times \frac{\pi}{4} + 45 \times \frac{\pi}{2}$$

$$= 25\pi$$

Hence, the value of $(m + 4)$ is 29.

59. We have

$$f(x)=\frac{1}{\pi}(\sin^{-1}x+\cos^{-1}x+\tan^{-1}x)+\frac{(x+1)}{x^2+2x+10}$$

It will provide us the max value at $x = 1$

$$f(1)=\frac{1}{\pi}\left(\frac{\pi}{2}+\tan^{-1}(1)\right)+\frac{2}{13}$$

$$=\frac{1}{ð}\times\frac{3\pi}{4}+\frac{2}{13}$$

$$=\frac{3}{4}+\frac{2}{13}=\frac{39+8}{52}=\frac{47}{52}$$

Hence, the value of $(104m - 90)$ is 4.

60. We have

$\sin(2x) + \cos(2x) + \cos x + 1 = 0$

$\sin 2x + (1 + \cos 2x) + \cos x = 0$

Each term of the above equation is positive in $\left(0, \frac{\pi}{2}\right)$.

So it has no solution

Thus, $m = 0$

Also, $$n=\sin\left[\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)+\cos^{-1}\left(\cos\left(\frac{7\pi}{3}\right)\right)\right]$$

$$=\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

Hence, the value of

$(m^2 + n^2 + m + n + 4)$

$= 0 +1 + 0 + 1 + 4$

$= 6$

61. We have

$$f(n)=\sum_{k=-n}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)$$

$$=\sum_{k=-n}^{-1}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)+\sum_{k=1}^{n}\left(\cot^{-1}\left(\frac{1}{k}\right)-\tan^{-1}(k)\right)$$

$$=\sum_{k=-n}^{-1}(\tan^{-1}(k)+\pi-\tan^{-1}(k))+\sum_{k=1}^{n}(\tan^{-1}(k)-\tan^{-1}(k))$$

$$=\sum_{k=-n}^{-1}(\pi)+0$$

$$=n\pi$$

Now, $$\sum_{n=2}^{10}(f(n)+f(n-1))$$

$$=\sum_{n=2}^{10}(n\pi+(n-1)\pi)$$

$$=\sum_{n=2}^{10}((2n-1)\pi)$$

$= (3 + 5 + 7 + 9 + \ldots + 19)\pi$

$= (1 + 3 + 5 + 7 + 9 + \ldots + 19)\pi - \pi$

$= (10^2)\pi - \pi$

$= 99\pi$

Hence, the value of $(a + 1)$ is 100.

Integer Type Questions

1. Given, $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{x^2+1}\right)\right)<\pi-3$

$$\Rightarrow \frac{2x^2+4}{x^2+1}>3$$

$$\Rightarrow \frac{2x^2+4}{x^2+1}-3>0$$

$$\Rightarrow \frac{2x^2+4-3x^2-3}{x^2+1}>0$$

$$\Rightarrow \frac{1-x^2}{x^2+1}>0$$

$$\Rightarrow \frac{x^2-1}{x^2+1}<0$$

$\Rightarrow x \in (-1, 1)$

Hence, the value of $(b - a + 5) = 1 + 1 + 5 = 7$.

2. Given, $a \sin^{-1} x - b \cos^{-1} x = c$

$$\Rightarrow a\sin^{-1}x-b\left(\frac{\pi}{2}-\sin^{-1}x\right)=c$$

$$\Rightarrow (a+b)\sin^{-1}x=c+\frac{b\pi}{2}$$

$$\Rightarrow \sin^{-1}x=\frac{2c+b\pi}{2(a+b)}$$

Now, $$\cos^{-1}x=\frac{\pi}{2}-\frac{2c+b\pi}{2(a+b)}$$

$$\Rightarrow \cos^{-1}x=\frac{(a+b)\pi-2c-b\pi}{2(a+b)}$$

$$\Rightarrow \cos^{-1}x=\frac{a\pi-2c}{2(a+b)}$$

Now, $a \sin^{-1} x - b \cos^{-1} x$

$$=a\left(\frac{2c+b\pi}{2(a+b)}\right)+b\left(\frac{a\pi-2c}{2(a+b)}\right)$$

$$=\left(\frac{2ac+ab\pi}{2(a+b)}\right)+\left(\frac{ab\pi-2bc}{2(a+b)}\right)$$

$$=\left(\frac{2ac+ab\pi+ab\pi-2bc}{2(a+b)}\right)$$

$$=\left(\frac{ab\pi+c(a-b)}{(a+b)}\right)$$

Clearly, $m = 1$

Hence, the value of $(m^2 + m + 2)$ is 4

3. Since sum of the roots is negative and product of the roots is positive.
So, both roots are negative
Thus m is a negative root
Now, $\tan^{-1}(m) + \tan^{-1}\left(\frac{1}{m}\right)$
$= \tan^{-1}(m) - \pi + \cot^{-1}(m)$
$= -\pi + (\tan^{-1}(m) + \cot^{-1}(m))$
$= -\pi + \frac{\pi}{2} = -\frac{\pi}{2}$
Clearly, $k = -1$
Hence, the value of $(k + 4)$ is 3.

4. Given equation is
$$\sin^{-1}(x^2 - 2x + 1) + \cos^{-1}(x^2 - x) = \frac{\pi}{2}$$
It is true only when
$(x^2 - 2x + 1) = x^2 - x$
$\Rightarrow \quad x = 1$
Thus, the number of solutions is 1.

5. Given, $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$
$\Rightarrow \quad \cos^{-1}(3x) + \cos^{-1}(2x) = \pi - \cos^{-1}(x)$
$\Rightarrow \quad \cos^{-1}(3x \cdot 2x - \sqrt{1-9x^2}\sqrt{1-4x^2}) = \cos^{-1}(-x)$
$\Rightarrow \quad (3x \cdot 2x - \sqrt{1-9x^2}\sqrt{1-4x^2}) = (-x)$
$\Rightarrow \quad (6x^2 + x)^2 = (-\sqrt{1-9x^2}\sqrt{1-4x^2})^2$
$\Rightarrow \quad 36x^4 + 12x^3 + x^2 = 1 - 13x^2 + 36x^4$
$\Rightarrow \quad 12x^3 + 14x^2 - 1 = 0$
Thus, $a = 12, b = 14, c = 0, d = -1$
Hence, the value of $(b + c) - (a + d)$
$= 14 - 11 = 3.$

6. Given equation is
$x^3 - x^2 - 3x + 4 = 0$
$\Rightarrow \quad \alpha + \beta + \gamma = 1, \alpha\beta + \beta\gamma + \gamma\alpha = -3, \alpha\beta\gamma = -4$
It is given that,
$\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = 0$
$\Rightarrow \quad \tan^{-1}\left(\frac{\alpha + \beta + \gamma - \alpha\beta\gamma}{1 - (\alpha\beta + \beta\gamma + \gamma\alpha)}\right) = \theta$
$\Rightarrow \quad \tan^{-1}\left(\frac{1+4}{1+3}\right) = \theta$
$\Rightarrow \quad \tan\theta = \frac{5}{4}$
Hence, the value of $(p + q)$ is 9.

7. Given equation is
$\cos^{-1} x + \cos^{-1}(2x) + \pi = 0$
$\Rightarrow \quad \cos^{-1}(2x) + \cos^{-1}(x) = -\pi$
$\Rightarrow \quad \cos^{-1}(2x \cdot x - \sqrt{1-x^2}\sqrt{1-4x^2}) = -\pi$
$\Rightarrow \quad (2x^2 - \sqrt{1-x^2}\sqrt{1-4x^2}) = \cos(-\pi) = -1$
$\Rightarrow \quad (2x^2 + 1)^2 = (1 - x^2)(1 - 4x^2)$
$\Rightarrow \quad 4x^4 + 4x^2 + 1 = 1 - 5x^2 + 4x^4$
$\Rightarrow \quad 9x^2 = 0$
$\Rightarrow \quad x = 0$
But $x = 0$ does not satisfy the equation.
So it has no solution.
Therefore $M = 0$
Again, $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$,
$\Rightarrow \quad \sin^{-1}\left(\frac{5}{x}\right) + \cos^{-1}\left(\sqrt{1 - \frac{144}{x^2}}\right) = \frac{\pi}{2}$
It is true only when,
$\left(\frac{5}{x}\right) = \sqrt{1 - \frac{144}{x^2}}$
$\Rightarrow \quad \frac{25}{x^2} = 1 - \frac{144}{x^2}$
$\Rightarrow \quad \frac{169}{x^2} = 1$
$\Rightarrow \quad x^2 = 169$
$\Rightarrow \quad x = \pm 13$
Clearly, $x = 13$ only satisfies the equation. Thus, $N = 1$.
Hence, the value of $M + N + 4 = 5$.

8. We have
$$4\cos\left[\cos^{-1}\left(\frac{1}{4}(\sqrt{6} - \sqrt{2})\right) - \cos^{-1}\left(\frac{1}{4}(\sqrt{6} + \sqrt{2})\right)\right]$$
$= 4\cos\left[\cos^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) - \cos^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\right]$
$= 4\cos[\cos^{-1}(\cos(75°)) - \cos^{-1}(\cos(15°))]$
$= 4\cos(75° - 15°)$
$= 4\cos(60°)$
$= 2$

9. We have
$$5\cot\left(\sum_{k=1}^{5}\cot^{-1}(k^2 + k + 1)\right)$$
$= 5\cot\left(\sum_{k=1}^{5}\tan^{-1}\left(\frac{1}{1 + k + k^2}\right)\right)$
$= 5\cot\left(\sum_{k=1}^{5}\tan^{-1}\left(\frac{(k+1) - \text{k}}{1 + (k+1)k}\right)\right)$
$= 5\cot\left(\sum_{k=1}^{5}[\tan^{-1}(k+1) - \tan^{-1}k]\right)$
$= 5\cot\left(\tan^{-1}(6) - \tan^{-1}(1)\right)$
$= 5\cot\left(\tan^{-1}\left(\frac{6-1}{1+6.1}\right)\right)$
$= 5\cot\left(\tan^{-1}\left(\frac{5}{7}\right)\right)$

$$= 5\cot\left(\cot^{-1}\left(\frac{7}{5}\right)\right)$$

$$= 7$$

10. $a = \sin^{-1}(\log_2 x)$ and $b = \cos^{-1}(\log_2 x)$

The given equations reduces to

$$\begin{cases} 3a + b = \dfrac{\pi}{2} \\ a + 2b = \dfrac{11\pi}{6} \end{cases}$$

On solving, we get,

$$a = -\frac{\pi}{6} \text{ and } b = \pi$$

$$\Rightarrow \quad \sin^{-1}(\log_2 x) = -\frac{\pi}{6} \text{ and } \cos^{-1}(\log_2 y) = \pi$$

$$\Rightarrow \quad (\log_2 x) = -\frac{1}{2} \text{ and } (\log_2 y) = -1$$

$$\Rightarrow \quad x = 2^{-\frac{1}{2}} \text{ and } y = 2^{-1}$$

$$\Rightarrow \quad \frac{1}{x} = \sqrt{2} \text{ and } \frac{1}{y} = 2$$

$$\Rightarrow \quad \frac{1}{x^2} + \frac{1}{y^2} + 2 = 2 + 4 + 2 = 8$$

11. Here, both roots are negative

Now, $\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$

$$= \cot\left(\pi + \tan^{-1}\left(\frac{1}{\alpha}\right) + \pi + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$

$$= \cot\left(2\pi + \tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{1}{\alpha}\right) + \tan^{-1}\left(\frac{1}{\beta}\right)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{\alpha + \beta}{\alpha\beta - 1}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{\alpha\beta - 1}{\alpha + \beta}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{\alpha\beta - 1}{\alpha + \beta}\right)\right)$$

$$= \left(\frac{44 + 1}{5}\right) = 9$$

12. Given equation is

$$\tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x + y}{xy - 1}\right) = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\Rightarrow \quad \left(\frac{x + y}{xy - 1}\right) = \left(\frac{1}{7}\right)$$

$$\Rightarrow \quad 7x + 7y = xy - 1$$

$$\Rightarrow \quad y = \left(\frac{7x + 1}{x - 7}\right)$$

Hence, the possible ordered pairs are

(8, 57), (9, 32), (12, 17), (17, 12), (32, 9), (57, 8)

Thus, the number of ordered pairs is 6.

Previous Years' JEE-Advanced Examinations

1. Now,

$$\tan^{-1}\left(\sqrt{\frac{a(a+b+c)}{bc}}\right) + \tan^{-1}\left(\sqrt{\frac{b(a+\ +c)}{ca}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\dfrac{a(a+b+c)}{bc}} + \sqrt{\dfrac{b(a+\ +c)}{ca}}}{1 - \sqrt{\dfrac{a(a+b+c)}{bc}} \cdot \sqrt{\dfrac{b(a+\ +c)}{ca}}}\right)$$

$$= \tan^{-1}\left(\frac{\left(\dfrac{(a+b+c)}{\sqrt{c}}\right) + \left(\dfrac{\sqrt{a}}{\sqrt{b}} + \dfrac{\sqrt{b}}{\sqrt{a}}\right)}{1 - \sqrt{\dfrac{(a+b+c)}{c}} \cdot \sqrt{\dfrac{(a+c)}{c}}}\right)$$

$$= \tan^{-1}\left(\frac{\left(\dfrac{(a+b+c)}{\sqrt{c}}\right)\left(\dfrac{a+b}{\sqrt{ab}}\right)}{\left(\dfrac{c-a-b-c}{\sqrt{c}}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\left(\dfrac{(a+b+c}{\sqrt{c}}\right)\left(\dfrac{a+b}{\sqrt{ab}}\right)}{\left(-\dfrac{a+b}{\sqrt{c}}\right)}\right)$$

$$= \tan^{-1}\left(-\sqrt{\frac{c(a+b+c)}{ab}}\right)$$

$$= -\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right)$$

Therefore, θ

$$= -\tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{ab}}\right) + \tan^{-1}\left(\sqrt{\frac{c(a+b+c)}{a}}\right)$$

$$= 0$$

2. Given, $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right\}$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{2}{5}}{1 - \left(\frac{1}{5}\right)^2}\right) - \frac{\pi}{4}\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}(1)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{5}{12} - 1}{1 + \frac{5}{12} \cdot 1}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(-\frac{7}{17}\right)\right)$$

$$= \left(-\frac{7}{17}\right)$$

3. Now, $\cos(2\cos^{-1}x + \sin^{-1}x)$

$$= \cos(\cos^{-1}x + \sin^{-1}x + \cos^{-1}x)$$

$$= \cos\left(\frac{\pi}{2} + \cos^{1}x\right)$$

$$= -\sin(\cos^{-1}x)$$

$$= -\sin(\sin^{-1}\sqrt{1-x^2})$$

$$= -\sqrt{1-x^2}$$

When $x = 1/5$, then the value of the given expression is

$$-\left(\sqrt{1-\frac{1}{25}}\right) = -\frac{2\sqrt{4}}{5}$$

4. Given, $\tan\left[\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right]$

$$= \tan\left(\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right)$$

$$= \left(\frac{9+8}{12-6}\right)$$

$$= -\frac{17}{6}$$

5. No questions asked in between 1984 and 1985

6. Given, $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

$$= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \left(\frac{\pi}{3}\right)$$

7. No questions asked in between 1987 and 1988

8. We have

$$A = 2\tan^{-1}(2\sqrt{2}-1)$$

$$> 2\tan^{-1}(\sqrt{3}) = \frac{2\pi}{3}$$

and $B = 3\tan^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$$= \sin^{-1}\left(3\left(\frac{1}{3}\right) - 4\left(\frac{1}{3}\right)^3\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$< \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\pi}{3}$$

Thus, $A > B$

10. Ans. (c)

Given,

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\Rightarrow \quad \cos^{-1}\left(\frac{1}{\sqrt{x^2+x+1}}\right) + \sin^{-1}(\sqrt{x^2+x+1}) = \frac{\pi}{2}$$

Thus, $\left(\frac{1}{\sqrt{x^2+x+1}}\right) = (\sqrt{x^2+x+1})$

$$\Rightarrow \quad (\sqrt{x^2+x+1})^2 = 1$$

$$\Rightarrow \quad x^2 + x + 1 = 1$$

$$\Rightarrow \quad x^2 + x = 0$$

$$\Rightarrow \quad x(x+1) = 0$$

$$\Rightarrow \quad x = 0, -1$$

11. **No questions asked in 2000.**

12. We have

$$\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\right) = \left(x - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right)$$

$$\Rightarrow \quad x\left(1 - \frac{x}{2} + \frac{x^2}{4} - \ldots\right) = x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{4} - \ldots\right)$$

$$\Rightarrow \quad x\left(\frac{1}{1-\left(-\frac{x}{2}\right)}\right) = x^2\left(\frac{1}{1-\left(-\frac{x^2}{2}\right)}\right)$$

$$\Rightarrow \quad x\left(\frac{1}{1+\left(\frac{x}{2}\right)}\right) = x^2\left(\frac{1}{1+\left(\frac{x^2}{2}\right)}\right)$$

$$\Rightarrow \quad x\left(1+\frac{x^2}{2}\right) = x^2\left(1+\frac{x}{2}\right)$$

$$\Rightarrow \quad x(2+x^2) = x^2(2+x)$$

$$\Rightarrow \quad (2x + x^3) = (2x^2 + x^3)$$

$$\Rightarrow \quad x^2 = x$$

$$\Rightarrow \quad x(x-1) = 0$$

$$\Rightarrow \quad x = 0, 1$$

Hence, the solution set is $\{0, 1\}$

13. We have

$\cos(\tan^{-1}(\sin(\cot^{-1}x)))$

$$= \cos\left(\tan^{-1}\left(\sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)\right)\right)$$

$$= \cos\left(\tan^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)\right)$$

$$= \cos\left(\cos^{-1}\left(\sqrt{\frac{x^2+1}{x^2+2}}\right)\right)$$

$$= \left(\sqrt{\frac{x^2+1}{x^2+2}}\right)$$

14. Ans. (d)

Given, $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$

It is defined for, $\sin^{-1}(2x) + \frac{\pi}{6} \geq 0$

$$\sin^{-1}(2x) \geq -\frac{\pi}{6}$$

$$(2x) \geq \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$x \geq -\frac{1}{4}$$

Also, $-1 \leq 2x \leq 1$

$$-\frac{1}{2} \leq x \leq \frac{1}{2}$$

Thus, the domain of the given function $= \left[-\frac{1}{4}, \frac{1}{2}\right]$

15. Given, $\sin(\cot^{-1}(x+1) = \cos(\tan^{-1} x)$

$\Rightarrow \quad \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+2x+2}}\right)\right)$

$= \cos\left(\cos^{-1}\frac{1}{\sqrt{x^2+1}}\right)$

$\Rightarrow \quad \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{x^2+1}}$

$\Rightarrow \quad \sqrt{x^2+2x+2} = \sqrt{x^2+1}$

$\Rightarrow \quad x^2 + 2x + 2 = x^2 + 1$

$\Rightarrow \quad 2x + 2 = 1$

$\Rightarrow \quad x = -\frac{1}{2}$

16. No questions asked in between 2005 and 2006.

17. Given, $\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$

(A) when $a = 1, b = 0$, then

$\sin^{-1}(x) + \cos^{-1}(y) + \cos^{-1}(0) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(x) + \cos^{-1}(y) + \frac{\pi}{2} = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(x) + \cos^{-1}(y) = 0$

$\Rightarrow \quad \sin^{-1}(x) = -\cos^{-1}(y) = -\sin^{-1}\left(\sqrt{1-y^2}\right)$

$\Rightarrow \quad x = -\sqrt{1-y^2}$

$\Rightarrow \quad x^2\ 1 - y^2$

$\Rightarrow \quad x^{23} + y^2 = 1$

Ans. (P)

(B) When $a = 1, b = 1$, then

$\sin^{-1} x + \cos^{-1} y + \cos^{-1}(xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1} x + \cos^{-1} y + \cos^{-1}(xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1} x = \cos^{-1} y + \cos^{-1}(xy)$

$\Rightarrow \quad \cos^{-1} x - \cos^{-1} y = \cos^{-1}(xy)$

$\Rightarrow \quad \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos^{-1}(xy)$

$\Rightarrow \quad (xy - \sqrt{1-x^2}\sqrt{1-y^2}) = (xy)$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} = 0$

$\Rightarrow \quad (1-x^2)(1-y^2) = 0$

Ans. (Q)

(C) When $a = 1, b = 2$, then

$\Rightarrow \quad \sin^{-1} x + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1} x + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1} x = \cos^{-1} y + \cos^{-1}(2xy)$

$\Rightarrow \quad \mathrm{ccos}^{-1} x - \cos^{-1} y = \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos^{-1}(2xy)$

$\Rightarrow \quad (xy - \sqrt{1-x^2}\sqrt{1-y^2}) = (2xy)$

$\Rightarrow \quad (\sqrt{1-x^2}\sqrt{1-y^2})^2 = (-xy)^2$

$\Rightarrow \quad (1-x^2)(1-y^2) = x^2y^2$

$\Rightarrow \quad 1 - x^2 - y^2 + x^2y^2 = x^2y^2$

$\Rightarrow \quad x^2 + y^2 = 1$

Ans. (P)

(D) When $a = 2, b = 2$, then

$\Rightarrow \quad \sin^{-1}(2x) + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \cos^{-1}(2x) + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}(2x) = \cos^{-1} y + \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}(2x) - \cos^{-1} y = \cos^{-1}(2xy)$

$\Rightarrow \quad \cos^{-1}(2x \cdot y - \sqrt{1-4x^2}\sqrt{1-y^2}) = \cos^{-1}(2xy)$

$\Rightarrow \quad (2x \cdot y - \sqrt{1-4x^2}\sqrt{1-y^2}) = (2xy)$

$\Rightarrow \quad (\sqrt{1-4x^2}\sqrt{1-y^2}) = 0$

$\Rightarrow \quad (1-4x^2)(1-y^2) = 0$

Ans. (S)

18. We have

$\sqrt{1+x^2} \times [\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 - 1]^{1/2}$

$$= \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)\right.\right.$$

$$\left.\left. + \sin\left(\sin^{-1}\left(\frac{1}{\sqrt{x^2+}}\right)\right)\right\}^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}\right)^2 - \right]^{1/2}$$

$$= \sqrt{1+x^2}\left[\left(\frac{x^2+1}{\sqrt{x^2+1}}\right)^2 - 1\right]^{1/2}$$

$$= \sqrt{1+x^2}[(\sqrt{x^2+1})^2 -]^{1/2}$$

$$= \sqrt{1+x^2}[(x^2+1) - 1]^{1/2}$$

$$= x\sqrt{1+x^2}$$

Ans. (c)

19. No questions asked in between 2009 and 2010.

20. Given, $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

$$= \sin\left(\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\sin^2\theta + \cos 2\theta}}\right)\right)$$

$$= \left(\frac{\sin\theta}{\sqrt{\sin^2\theta + \cos^2\theta - \sin^2\theta}}\right)$$

$$= \left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$$

$$= \left(\frac{\sin\theta}{\cos\theta}\right) = \tan\theta$$

Now, $\frac{d}{d(\tan\theta)}(\tan\theta) = 1$

21. No questions asked in 2012

22. $\sum_{k=1}^{n}(2k) = 2(1 + 2 + 3 + \ldots + n)$

$$= 2\left(\frac{n(n+1)}{2}\right)$$

$$= (n^2 + n)$$

Therefore, $\sum_{n=1}^{23}\cot^{-1}(1+n+n^2)$

$$= \sum_{n=1}^{23}\tan^{-1}\left(\frac{1}{1+n+n^2}\right)$$

$$= \sum_{n=1}^{23}\tan^{-1}\left(\frac{1}{1+(n+1)n}\right)$$

$$= \sum_{n=1}^{23}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)$$

$$= \sum_{n=1}^{23}(\tan^{-1}(n+1) - \tan^{-1}(n))$$

$$= (\tan^{-1}(2) - \tan^{-1}(1)) + (\tan^{-1}(3) - \tan^{-1}(2)) + \ldots + (\tan^{-1}(24) - \tan^{-1}(23))$$

$$= \tan^{-1}(24) - \tan^{-1}(1)$$

$$= \tan^{-1}\left(\frac{24-1}{1+24}\right) = \tan^{-1}\left(\frac{23}{25}\right)$$

Thus the given expression reduces to

$$= \cot\left(\tan^{-1}\left(\frac{23}{25}\right)\right)$$

$$= \cot\left(\cot^{-1}\left(\frac{25}{23}\right)\right)$$

$$= \frac{25}{23}$$

23. We have

$$\left(\frac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}\right)$$

$$= \left(\frac{\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{y^2+1}}\right)\right) + y\sin\left(\sin^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)\right)}{\cot\left(\cot^{-1}\left(\frac{\sqrt{1-y^2}}{y}\right)\right) + \tan\left(\tan^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)\right)}\right)$$

$$= \left(\frac{\left(\frac{1}{\sqrt{y^2-1}}\right) + y\left(\frac{y}{\sqrt{1+y^2}}\right)}{\left(\frac{\sqrt{1-y^2}}{y}\right) + \left(\frac{y}{\sqrt{1-y^2}}\right)}\right)$$

$$= \left(\frac{\left(\frac{1+y^2}{\sqrt{y^2+1}}\right)}{\left(\frac{1-y^2+y^2}{y-y^2}\right)}\right)$$

$$= y(\sqrt{1-y^4})$$

Thus, the given expression reduces to

$$= \left(\frac{1}{y^2}(y\sqrt{1-y^4})^2 + y^4\right)^{1/2}$$

$$= \left(\frac{y^2(1-y^4)}{y^2} + y^4\right)^{1/2}$$

$$= ((1-y^4) + y^4)^{1/2}$$

$$= 1$$

24. Given, $\cos(\sin^{-1}\sqrt{1-x^2}) = \sin(\tan^{-1}(x\sqrt{6}))$

$$\Rightarrow \quad \cos\left(\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right) = \sin\left(\sin^{-1}\left(\frac{x\sqrt{6}}{\sqrt{6x^2+1}}\right)\right)$$

$$\Rightarrow \quad \left(\frac{x}{\sqrt{1-x^2}}\right) = \left(\frac{x\sqrt{6}}{\sqrt{6x^2+1}}\right)$$

$$\Rightarrow \quad \left(\frac{1}{\sqrt{1-x^2}}\right) = \left(\frac{\sqrt{6}}{\sqrt{6x^2+1}}\right)$$

$$\Rightarrow \quad 6(1 - x^2) = (6x^2 + 1)$$

$$\Rightarrow \quad 6 - 6x^2 = 6x^2 + 1$$

$$\Rightarrow \quad 12x^2 = 5$$

$$\Rightarrow \quad x^2 = \frac{5}{12}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{12}{5}}$$

CHAPTER

3 The Limit

CONCEPT BOOSTER

1. Meaning of $x \to a$

The symbol $x \to a$ is called as x tends to a or x approaches to a. It implies that x takes values closer and closer to a but not equal to a.

2. Neighbourhood of a Point

Any open interval containing a point a as its mid-point is called a neighbourhood of a. A positive number δ is called a neighbourhood of a, if $a - \delta < x < \delta + a$.

3. Limit of a Function

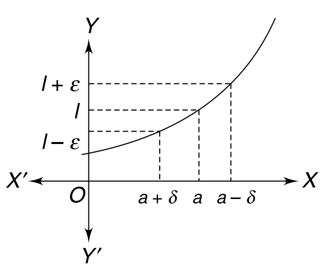
If there is a number l such that x approaches to a, either from the right or from the left, $f(x)$ approaches l, then l is called the limit of $f(x)$ as x approaches to a.

A number l is said to be a limiting value only if it is finite and real, otherwise we say that limit does not exist.

In a piece-wise defined function or more than one function, we shall use the concept of right hand limit as well as left hand limit. In a function(s), if right hand limit as well as left hand limit both are exist and their values are same, then limit exist, otherwise not.

4. Formal Definition of a Limit

A number l is said to be limit of the function $f(x)$ at $x = a$, for any positive number $\in > 0$, there corresponds a positive number δ such that $|f(x) - l| < \in$, $\forall\ x \in D_f$, $x \neq a$, $|x - a| < \delta$.



5. Concept of Infinity

Let us consider that n assumes sucessively the values 1, 2, 3, ... Then n gets larger and larger and there is no limit to the extent of its increase. It is convenient to that n tends to infinity. When we say that n tends to infinity, we simply mean it that n is supposed to assume a series of values which increas beyond limit.

A function $y = f(x)$ which approaches infinity as $x \to a$ does not have a limit in the ordinary sense.

6. Concept of Limit

The concept of limit is used to discuss the behaviour of a function close to a certain point

Let $$f(x) = \frac{x^2 - 1}{x - 1}, x \neq 1$$

Clearly the given function is not defined at $x = 1$.

It is defined only when $x \neq 1$, that means, either $x > 1$ or $x < 1$.

Case I: When $x > 1$ (just slightly more than 1)

$x > 1$	$f(x) = x + 1$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

Thus, $x \to 1^+ \Rightarrow f(x) \to 2^+$

We define it the right hand limit of a function

Case II: When $x < 1$ (just slightly less than 1)

$x < 1$	$f(x) = x + 1$
.9	1.9
.99	1.99
.999	1.999
.9999	1.9999

Thus, $x \to 1^- \Rightarrow f(x) \to 2^-$.

We define it the left hand limit of a function.

Both the concept simultaneously known as limit of a function. i.e. $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right) = 2$

Note:

1. The value of a function at any $x \in D_f$ is the the exact value of a function.
2. But the limiting value of a function is the closer value of the exact value of a function.
3. If the limiting value of a function is a real and finite then limit is exist, otherwise not.
4. We shall use the concepts of R.H.L as well as L.H.L only when, we get a piece-wise defined function or more than one function.

Example 1. $\lim_{x\to 1}[x]$

(i) $\lim_{x\to 1^+}[x] = 1$

(ii) $\lim_{x\to 1^-}[x] = 0$

5. In a function, if the R.H.L and L.H.L both are exist and their values are same, then limit is exist.

Example 2. Let $f(x) = \begin{cases} x^2 & : x > 1 \\ 2 - x & : x < 1 \end{cases}$

Then $\lim_{x\to 1^+} f(x) = 1$

and $\lim_{x\to 1^-} f(x) = 1.$

Thus $\lim_{x\to 1} f(x) = 1.$

6. In a function, if the R.H.L and L.H.L both are exist and their values are not same then limit does not exist.

Example 3. Let $\lim_{x\to 0}\left(\frac{|x|}{x}\right)$.

Then R.H.L $= \lim_{x\to 0^+}\left(\frac{|x|}{x}\right) = \lim_{x\to 0^+}\left(\frac{x}{x}\right) = 1$

and L.H.L $= \lim_{x\to 0^-}\left(\frac{|x|}{x}\right) = \lim_{x\to 0^-}\left(\frac{x}{x}\right) = -1$

Thus, limit does not exist at $x = 0$.

7. Algebra of Limits

Let f and g be two real functions defined in the domain D.

If $\lim_{x\to a} f(x) = m$ and $\lim_{x\to a} g(x) = n$, where m and n are real and finite, then

(i) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = m \pm n$

(ii) $\lim_{x\to a}(f(x) \times g(x)) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x) = m \times n$

(iii) $\lim_{x\to a}(f(x) \div g(x)) = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x) = \frac{m}{n}, n \neq 0$

(iv) $\lim_{x\to a}\left(f(x)^{g(x)}\right) = \lim_{x\to a}(f(x))^{\lim_{x\to a} g(x)} = m^n$

(v) $\lim_{x\to a}(kf(x)) = k\left(\lim_{x\to a}(f(x))\right) = km$

(vi) $\lim_{x\to a}|f(x)| = \left|\lim_{x\to a}(f(x))\right| = |m|$

(vii) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a}(f(x))} = e^m$

(viii) $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a}(g(x))\right) = f(n)$

(ix) $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{m}$

(x) $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n = m^n$

Some Important Expansions to Remember

(i) $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

(iii) $a^x = 1 + (\log_e a)x + \frac{(\log_e a)^2}{2!}x^2 + \frac{(\log_e a)3}{3!}x^3 + \ldots$

(iv) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

(v) $\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

(vi) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

(vii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(viii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

(ix) $(1 + x)^{1/x} = e\left(1 + \frac{x}{2} + \frac{11}{24}x^2 + \ldots\right)$

(x) $f(x + 0) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

which is known as **Maclaurin Series**.

8. Evaluation of Limit

(i) **Algebraic Limit**

(a) Direct Substitution Method (DSM)
(b) Factorisation Method (FM)
(c) Rationalisation Method (RM)
(d) Standard Result Method (SRM)
(e) Infinity Method (IM)

(ii) **Non-algebraic Limit**
 (a) Trigonometric limit
 (b) Inverse trigonometric limit
 (c) Exponential limit
 (d) Logarithmic limit
 (e) Miscellaneous limit.
  (i) L'Hospital Rule
  (ii) Advances Exponential limt (1^∞)
  (iii) Sandwitch Theorem
  (iv) Newton and Leibnitz limit
  (v) Definite integral as the limit of a sum.
  (vi) Limits of the form of (0^0)
  (vii) Limits of the form of (1^∞).

8.1 Algebraic Limit

(i) **Direct substitution method**

In this method, we can directly sustitute the number at which the limit is to be find. After substitution, if we get a finite value, that is the limiting value of a function.

Example 4. $\lim_{x\to 1}(x^2 + 3x + 4) = 8.$

Example 5. $\lim_{x\to 1}(x^3 - 5x + 4) = 0.$

(ii) **Factorisation method**

In this method, we shall find out a common factor from the numerator as well as *a* denominator. Cancel the common factor and then directly substitute the number at which the limit is to be find.

Example 6. $\lim_{x\to 1}\left(\frac{x^2-1}{x-1}\right)$

$$= \lim_{x\to 1}\left(\frac{(x+1)(x-1)}{x-1}\right) = \lim_{x\to 1}(x+1) = 2$$

Example 7. $\lim_{x\to 0}\left(\frac{1-x^{-\frac{1}{3}}}{1-x^{-\frac{2}{3}}}\right)$

$$= \lim_{x\to 0}\left(\frac{1-x^{-\frac{1}{3}}}{\left(1+x^{-\frac{1}{3}}\right)\left(1-x^{-\frac{1}{3}}\right)}\right)$$

$$= \frac{1}{2}$$

(iii) **Rationalisation Method**

In this method, our first aim will be, remove the radical sign. This is particularly used when either the numerator or denominator or both involve the fractional powers.

Example 8. Evaluate: $\lim_{x\to 0}\left(\frac{\sqrt{x^2+x+1}-\sqrt{x^2+1}}{x}\right)$

$$= \lim_{x\to 0}\left(\frac{x^2+x+1-x^2-1}{x\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{x}{x\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{\left(\sqrt{x^2+x+1}+\sqrt{x^2+1}\right)}\right)$$

$$= \frac{1}{2}$$

(iv) **Standard Result method**

In this method we shall use the formula

$$\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}.$$

Example 9. $\lim_{x\to 2}\left(\frac{x^5-32}{x-2}\right)$

$$= \lim_{x\to 2}\left(\frac{x^5-2^5}{x-2}\right) = 5.2^{5-1} = 5 \times 16 = 80$$

Example 10. $\lim_{x\to 3}\left(\frac{x^5-243}{x^3-27}\right)$

$$= \lim_{x\to 3}\left(\frac{\frac{x^5-3^5}{x-3}}{\frac{x^3-3^3}{x-3}}\right) = \frac{5.3^{5-1}}{3.3^{3-1}} = 15$$

Example 11. Evaluate: $\lim_{x\to 0}\left(\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\right)$

We have $\lim_{x\to 0}\left(\frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}\right)$

$$= \frac{3-\sqrt{5}}{1-\sqrt{5}}.$$

(v) **Infinity Method**

(a) Form: $\left(\frac{\infty}{\infty}\right)$. In this method, we shall write down the given expression in the form of a rational function. i.e. $\frac{f(x)}{g(x)}$.

Then divide the numerator and denominator by the highest power of x and then use

$$\lim_{x\to\infty}\left(\frac{1}{x^n}\right) = 0,\ x > 1$$

Note: $\lim_{x\to\infty} x^n = \begin{cases} \infty & : x > 1 \\ 0 & : 0 < x < 1 \\ 1 & : x = 1 \\ \text{not defined} & : x < 0 \end{cases}$

(b) Form ($\infty \pm \infty$): In this method, first we reduce the function $f(x)$ into a rational function by rationalisation i.e. multiply the numerator and denominator by its conjugate and then apply $\lim_{x\to\infty}\left(\frac{1}{x^n}\right) = 0,\ x > 1$.

8.2 Non-algebraic Limit

Before solving the value of a limit, please read throughly each of the following formulae.

(a) **Trigonometric limit**

(i) $\lim_{x\to 0} \sin x = 0$

(ii) $\lim_{x\to 0} \cos x = 1$

(iii) $\lim_{x\to 0} \tan x = 0$

(iv) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$

(v) $\lim_{x\to 0}\left(\frac{\sin(x-a)}{x-a}\right) = 1$

(vi) $\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1$

(vii) $\lim_{x\to a}\left(\frac{\tan(x-a)}{(x-a)}\right) = 1$

(viii) $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right) = 0$

(ix) $\lim_{x\to\infty}\left(\frac{\cos x}{x}\right) = 0$

(x) $\lim_{x\to\infty}\left(\frac{\tan x}{x}\right) = 0$

(b) **Inverse Trigonometric limit**

(i) $\lim_{x\to 0}\left(\frac{\sin^{-1} x}{x}\right) = 1$

(ii) $\lim_{x\to 0}\left(\frac{\tan^{-1} x}{x}\right) = 1$

(c) **Exponential limit**

(i) $\lim_{x\to 0} a^x = 1,\ a \neq 1,\ a > 0.$

(ii) $\lim_{x\to 0} e^x = 1$

(iii) $\lim_{x\to 0}\left(\frac{a^x - 1}{x}\right) = \log_e a$

(iv) $\lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) = 1$

(d) **Logarithmic limit**

(i) $\lim_{x\to 0}\left(\frac{\log(1+x)}{x}\right) = 1$

(e) **Miscellaneous limit**

(i) ***L'Hospital Rule***

Let $f(x)$ and $g(x)$ be two real functions and $a \in$ R. If $f(a) = 0 = g(a)$ or $f(a) = \infty = g(a)$, then

$$\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to 0}\left(\frac{f'(x)}{g'(x)}\right) = \lim_{x\to 0}\left(\frac{f''(x)}{g''(x)}\right),$$

untill and unless the form of $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ is removed.

L'Hospital rule is applicable only when the limits are in the form of $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$.

> ***Note:*** Indeterminate form:
>
> (i) $\left(\frac{0}{0}\right)$ (ii) $\left(\frac{\infty}{\infty}\right)$
>
> (iii) $(\infty, -\infty)$ (iv) $(0, \infty)$
>
> (v) (∞^0) (vi) (0^0)
>
> (vii) (1^∞) (viii) (∞^0)

(ii) ***Advanced Exponential limit***

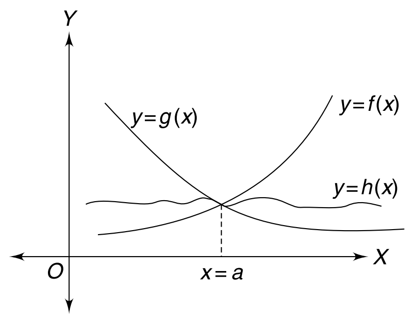
(a) $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

(b) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

(iii) ***Sandwich Theorem***

Let $f(x)$, $g(x)$ and $h(x)$ be three real functions and $a \in R$ such that $g(x) \le f(x) \le h(x)$.

If $\lim_{x\to a} g(x) = m$ and $\lim_{x\to a} h(x) = m$, then $\lim_{x\to 0} f(x) = m$.



(iv) ***Newton and Leibnitz Rule***

If the function $\varphi(x)$ and $\Psi(x)$ be defined on $[a, b]$ and differentiable at a point $x \in (a, b)$ and $f(t)$ is a continuous for $\varphi(a) \le t \le \varphi(b)$ then

$$\left\{\frac{d}{dt}\int_{\varphi(x)}^{\Psi(x)} f(t)\,dt\right\} = f(\Psi(x))\Psi'(x) - f(\varphi(x))\,\phi'(x)$$

> ***Note:*** 1. If $y = \int_a^{\varphi(x)} fx(t)\,dt$
>
> then $\frac{dy}{dx} = f(\varphi(x))\,\phi'(x)$
>
> 2. If $y = \int_{\psi(x)}^{a} f(t)\,dt$

then $\frac{dy}{dx} = -f(\psi(x))\,\psi'(x)$

Example 12. $\frac{d}{dx}\left\{\int_{x^3}^{x^7} \log t\, dt\right\}$

$= \left\{\log\left(x^7\right)\cdot 7x^6 - \log\left(x^3\right)\cdot 3x^2\right\}$

Example 13. $\frac{d}{dx}\left\{\int_{x^3}^{x^5} e^t\, dt\right\}$

$= \left\{e^{x^5}\cdot 5x^4 - e^{x^3}\cdot 3x^2\right\}$

(v) ***Definite Integral as the limit of a sum***

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=\varphi(x)}^{\Psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)\,dx,$$

where

(i) $\frac{r}{n}$ is replaced by x

(ii) $\frac{1}{n}$ is replaced by dx

(iii) $\lim_{n\to\infty} \Sigma$ is replaced by $\int$

(iv) $a = \lim_{n\to\infty}\left(\frac{\varphi(x)}{n}\right)$, & $b = \lim_{n\to\infty}\left(\frac{\Psi(x)}{n}\right)$.

(vi) **Form (0^0):** In this method, we shall get the function is in the form of $\lim_{x\to a} (f(x))^{g(x)}$.

If $\lim_{x\to a}(f(x)) = m$, $m > 0$ and $\lim_{x\to a}(g(x)) = n$ a finite quantity, then $\lim_{x\to a}(f(x))^{g(x)} = m^n$.

We write it in the form of

$\lim_{x\to a}(f(x))^{g(x)} = e^{\lim_{x\to a}(g(x)\log(f(x)))}$. Where the function $f(x)$ and $g(x)$ are defined in the neighbourhood of a and $f(x) > 0$.

(vii) **Form (∞^0)**

Exercises

Level I

(Problems Based on Fundamentals)

Direct Substitution Method (DSM)

1. Evaluate: $\lim_{x\to 1}(x^2 - 6x + 10)$
2. Evaluate: $\lim_{x\to 1}\left(x^{2016} - x^{2017} + 2\right)$

Factorisation Method (FM)

3. Evaluate: $\lim_{x\to 1}\left(\frac{x^2 - 3x + 2}{x^2 - 5x + 4}\right)$
4. Evaluate: $\lim_{x\to 1}\left(\frac{1 - x^{-\frac{1}{3}}}{1 - x^{-\frac{2}{3}}}\right)$
5. Evaluate: $\lim_{x\to 2}\left(\frac{x^3 + 7x^2 - 36}{x^2 + 2x - 8}\right)$
6. Evaluate: $\lim_{x\to 1}\left(\frac{x^2 + x\log x - \log x - 1}{x^3 - 1}\right)$

Rationalisation Method (RM)

7. Evaluate: $\lim_{x\to 0}\frac{x}{\sqrt{a+x} - \sqrt{a-x}}$
8. Evaluate: $\lim_{x\to 0}\left(\frac{\sqrt{1+x} - 1}{x}\right)$
9. Evaluate: $\lim_{x\to 0}\frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}}$
10. Evaluate: $\lim_{x\to 0}\frac{\sqrt{1+x} - \sqrt{1-x}}{2x}$
11. Evaluate: $\lim_{x\to 0}\frac{\sqrt{1+4x} - \sqrt{5+2x}}{x-2}$
12. Evaluate: $\lim_{x\to\sqrt{10}}\left(\frac{\sqrt{7+2x} - \sqrt{5} - \sqrt{2}}{x^2 - 10}\right)$
13. Evaluate: $\lim_{x\to a}\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$
14. Evaluate: $\lim_{x\to -1}\frac{\sqrt[3]{7-x} - 2}{(x+1)}$
15. Evaluate: $\lim_{x\to 1}\left(\frac{\sqrt{x^2+8} - \sqrt{10-x^2}}{\sqrt{x^2+3} - \sqrt{5-x^2}}\right)$

Standard Result Method (SRM)

16. Evaluate: $\lim_{x\to a}\frac{x\sqrt{x} - a\sqrt{a}}{x-a}$
17. Evaluate: $\lim_{x\to a}\frac{(x+2)^{\frac{5}{3}} - (a+2)^{\frac{5}{3}}}{x-a}$
18. Evaluate: $\lim_{x\to a}\frac{x^m - a^m}{x^n - a^n}$
19. Evaluate: $\lim_{x\to 0}\frac{(1-x)^n - 1}{x}$.

20. Evaluate: $\lim_{x\to 1} \frac{(x + x^2 + x^3 + ... + x^n) - n}{x - 1}$

21. Evaluate: $\lim_{x\to 1} \frac{x^{n+1} - (n + 1)x + n}{(x - 1)^2}$

22. Evaluate: $\lim_{x\to 0} \left(\frac{\sqrt[3]{1 + x^2} - \sqrt[4]{1 - 2x}}{x + x^2}\right)$

23. Evaluate: $\lim_{x\to 1} \left(\frac{\sqrt[3]{7 + x^3} - \sqrt[2]{3 + x^2}}{x - 1}\right)$

24. Evaluate: $\lim_{x\to 1} \left(\frac{\sqrt[3]{x} + \sqrt[4]{x} + \sqrt[5]{x} - 3}{x - 1}\right)$

25. Evaluate: $\lim_{x\to 1} \left(\frac{\left(\sum_{k=1}^{100} x^k\right) - 100}{x - 1}\right)$

Infinity Method (IM)

26. Evaluate: $\lim_{n\to\infty} (4^n + 5^n)^{\frac{1}{n}}$

27. Evaluate: $\lim_{n\to\infty} \left(\frac{n^2}{1 + 2 + 3 + ... + n}\right)$

28. Evaluate: $\lim_{n\to\infty} \left(\frac{x^n + y^n}{x^n - y^n}\right)$, where $0 < x < y$.

29. Evaluate: $\lim_{x\to\infty} (\sqrt{x^2 + 2x} - x)$

30. Evaluate: $\lim_{x\to-\infty} (\sqrt{x^2 + 8x} + x)$

31. Evaluate: $\lim_{x\to\infty} \left(\frac{x^2 + 2x - 1}{2x^2 - 3x - 2}\right)^{\frac{2x+5}{2x-7}}$

32. Evaluate:

$$\lim_{x\to\infty} \left(\frac{(x + 1)^{10} + (x + 2)^{10} + ... + (x + 2012)^{10}}{x^{10} + 2012^{10}}\right)$$

33. Evalaute:

$$\lim_{n\to\infty} \left(\frac{1\cdot n + 2\cdot(n - 1) + 3\cdot(n - 2) + ... + n\cdot 1}{n^3}\right)$$

34. Evaluate:

$$\lim_{n\to\infty} n^2 \left\{\sqrt{\left(1 - \cos\frac{1}{n}\right)\sqrt{\left(1 - \cos\frac{1}{n}\right)\sqrt{\left(1 - \cos\frac{1}{n}\right)...\infty}}}\right\}$$

35. If $\lim_{x\to\infty} \left(\frac{x^2 - 1}{x + 1} - ax - b\right) = 2$, find the values of a and b.

36. If $\lim_{x\to\infty} \left(\frac{x^2 + 1}{x + 1} - ax - b\right) = 0$, find the values of a and b.

37. If $\lim_{x\to 0} \left(\frac{x^2 + x + 1}{x + 1} - ax - b\right) = 4$, then find the values of a and b.

38. Evaluate $\lim_{x\to-\infty} \left(\frac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{1 + |x|^3}\right)$

Trigonometric Limit

39. Evaluate: $\lim_{x\to 0} \left(\frac{\sin 3x}{5x}\right)$

40. Evaluate: $\lim_{x\to 0} \left(\frac{\sin ax}{\sin bx}\right)$

41. Evaluate: $\lim_{x\to 0} \left(\frac{\sin 2x + 3x}{4x + \sin 6x}\right)$

42. Evaluate: $\lim_{x\to 0} \left(\frac{\sin 2x + \sin 5x}{\sin 4x + \sin 6x}\right)$

43. Evaluate: $\lim_{x\to 0} \left(\frac{1 - \cos 4x}{1 - \cos 6x}\right)$

44. Evaluate: $\lim_{x\to 0} \left(\frac{\sqrt{1 - \cos 2x}}{x^2}\right)$

45. Evaluate: $\lim_{x\to 0} \left(\frac{\cos 7x - \cos 9x}{\cos 3x - \cos 5x}\right)$

46. Evaluate: $\lim_{x\to 0} \left(\frac{1 - \cos x \sqrt{\cos 2x}}{x^2}\right)$

47. Evaluate: $\lim_{x\to 0} \left(\frac{\tan x - \sin x}{x^3}\right)$

48. Evaluate: $\lim_{x\to 0} \left(\frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}\right)$

49. Evaluate: $\lim_{x\to 0} \left(\frac{\sin^{-1}x - \tan^{-1}x}{x^3}\right)$

50. Evaluate: $\lim_{x\to 0} \left(\frac{x - \sin x}{x^3}\right)$

51. Evaluate: $\lim_{x\to 0} \left(\frac{x - \tan x}{x^3}\right)$

52. Evaluate: $\lim_{x\to\infty} \left(\frac{x + \sin x}{x + \cos x}\right)$

53. Evaluate: $\lim_{x\to 0} \left(\left[\frac{n \sin x}{x}\right] + \left[\frac{n \tan x}{x}\right]\right)$

54. If = min $\{x^2 + 2x + 3, x^2 + 4x + 10\}$ and b = $\lim_{x\to 0} \left(\frac{1 - \cos\theta}{\theta^2}\right)$, then find the value of $\sum_{r=0}^{n} (a^r b^{n-r})$.

55. If $\lim_{x\to 0}\left(\frac{a\cos x + bx\sin x - 5}{x^4}\right)$ exists and finite, then find the value of $a + 2b + 10$.

56. Evaluate: $\lim_{x\to 0}\left(\frac{\sin(\pi\cos^2 x)}{x^2}\right)$

Exponential Limit

57. Evaluate: $\lim_{x\to 0}\left(\frac{e^{4x} - 1}{5x}\right)$

58. Evaluate: $\lim_{x\to 0}\left(\frac{e^{3x} - 1}{e^{5x} - 1}\right)$

59. Evaluate: $\lim_{x\to 0}\left(\frac{e^{x} + e^{-x} - 2}{x^2}\right)$

60. Evaluate: $\lim_{x\to 0}\frac{e^{3+x} - \sin x - e^3}{x}$

61. Evaluate: $\lim_{x\to 0}\frac{e^{x} - e^{\sin x}}{x - \sin x}$

62. Evaluate: $\lim_{x\to 0}\left(\frac{e^{x} - e^{x\cos x}}{x + \sin x}\right)$

63. Evaluate: $\lim_{x\to 0}\left(\frac{e^{x} - 1 - x}{x^2}\right)$

64. Evaluate: $\lim_{x\to 0}\left(\frac{8^x - 4^x - 2^x + 1}{x^2}\right)$

65. Evaluate: $\lim_{x\to 0}\frac{9^x - 2.6^x + 4^x}{x^2}$

66. Evaluate: $\lim_{x\to a}\left(\frac{a^x - a^a}{x - a}\right)$, $a > 0$

67. Evaluate: $\lim_{x\to 0}\left(\frac{a^{x+h} + a^{x-h} - 2a^x}{h^2}\right)$

68. Evaluate: $\lim_{x\to 0}\left(\frac{8^x - 7^x}{6^x - 5^x}\right)$

69. Evaluate: $\lim_{x\to 0}\left(\frac{(5^x - 1)(4^x - 1)}{(3^x - 1)(6^x - 1)}\right)$

70. Evaluate: $\lim_{x\to 0}\left(\frac{x\cdot 2^x - x}{1 - \cos x}\right)$

71. Evaluate: $\lim_{x\to 0}\left(\frac{e^{x} - e^{-x} - 2x}{x - \sin x}\right)$

72. Evaluate: $\lim_{x\to 0}\left(\frac{e^{x^3} - 1 - x^3}{\sin^6 2x}\right)$

73. Evaluate: $\lim_{x\to 2}\left(\frac{(\cos\alpha)^x + (\sin\alpha)^x - 1}{x - 2}\right)$

74. Evaluate: $\lim_{x\to 0}\left(\frac{e^{x\cos x} - 1 - x}{\sin(x^2)}\right)$

75. If $a = \min\{x^2 + 4x + 6,\ x^2 + 2x + 8\}$ and $b = \lim_{x\to 0}\left(\frac{\sin x\cos x}{e^x - e^{-x}}\right)$, then find the value of $\sum_{r=0}^{n}(a^r b^{n-r})$

76. If $\lim_{x\to 0}\left(\frac{Ae^x - B\cos x + Ce^{-x}}{x\sin x}\right) = 2$ then find the value of $A + B + C + 10$.

Logarithmic Limit

77. Evaluate: $\lim_{x\to 0}\left(\frac{\log(1 + 3x)}{\sin 2x}\right)$

78. Evaluate: $\lim_{x\to 0}\left(\frac{\log(1 + 3x)}{\log(1 - 2x)}\right)$.

79. Evaluate: $\lim_{x\to e}\left(\frac{\log x - 1}{x - e}\right)$

80. Evaluate: $\lim_{x\to 5}\left(\frac{\log x - \log 5}{x - 5}\right)$.

81. Evaluate: $\lim_{x\to 5}\left(\frac{\log(x + 5) - \log(5 - x)}{x - 5}\right)$

82. Evaluate: $\lim_{x\to 0}\left(\frac{e^x - \log(x + e)}{e^x - 1}\right)$

83. Evaluate: $\lim_{x\to 0}\left(\frac{\ln(\cos x)}{\sqrt[4]{1 + x^2} - 1}\right)$.

84. Evaluate: $\lim_{x\to 0}\left(\frac{\ln(\cos(\sin x))}{x^2}\right)$

85. Evaluate: $\lim_{x\to\infty}\left(x - x^2\ln\left(1 + \frac{1}{x}\right)\right)$

86. Evaluate: $\lim_{x\to 1}\left(\frac{x^x - 1}{x\ln x}\right)$

87. Evaluate: $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x - 1}\right)$

88. Evaluate: $\lim_{x\to\frac{\pi}{4}}\left(\frac{\ln\cot x}{1 - \cot x}\right)$

L'Hospital Rule

89. Evaluate: $\lim_{x\to 0}\left(\frac{e^x - 1 - x}{x^2}\right)$

90. Evaluate: $\lim_{x\to 0}\left(\frac{e^{x^3}-1-x^3}{64x^6}\right)$

91. Evaluate: $\lim_{x\to 0}\left(\frac{e^x-e^{-x}-2x}{x-\sin x}\right)$

92. Evaluate: $\lim_{x\to e}\left(\frac{\ln x-1}{x-e}\right)$

93. Evaluate: $\lim_{x\to 0}\left(\frac{x+\tan 2x}{x-\tan 2x}\right)$

94. Evaluate: $\lim_{x\to e}\left(\frac{\sin x-x}{x^3}\right)$

95. Evaluate: $\lim_{x\to 0}\left(\frac{e^x+e^{-x}-x^2-2}{\sin^2 x-x^2}\right)$

96. Evaluate: $\lim_{x\to 0}\left(\frac{x\mathrm{e}^x-\log(1+x)}{x^2}\right)$

97. Evaluate: $\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)$

98. Evaluate: $\lim_{x\to\infty}\left(\frac{x}{e^x}\right)$

99. Evaluate: $\lim_{x\to\infty}\left(\frac{x^{2014}}{e^x}\right)$

100. Evaluate: $\lim_{x\to 0}\left(\frac{x^2+2\cos x-2}{x\sin^3 x}\right)$

101. Evaluate: $\lim_{x\to\frac{\pi}{2}}(\tan x\ \log_e \sin x)$

Advanced Exponential Limit

102. Evaluate: $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$

103. Evaluate: $\lim_{x\to\infty}\left(\frac{2x+4}{2x+3}\right)^{2x+10}$

104. Evaluate: $\lim_{x\to\infty}\left(\sin\left(\frac{1}{x}\right)+\cos\left(\frac{1}{x}\right)\right)^x$

105. Evaluate: $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)$

106. Evaluate: $\lim_{x\to 1}\left(2-\frac{1}{x}\right)^{\tan\left(\frac{\pi x}{2}\right)}$

107. Evaluate: $\lim_{x\to 1}\left(\tan\left(\frac{\pi}{4}+\ln x\right)\right)^{\frac{1}{\ln x}}$

108. Evaluate: $\lim_{x\to 0}\left(\frac{\ln\left(\tan\left(\frac{\pi}{4}+2x\right)\right)}{\sin 3x}\right)$

109. Evaluate: $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$

110. Evaluate: $\lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$

111. Evaluate: $\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n$

Sandwitch Theorem

112. Evaluate: $\lim_{x\to 0}\left(x^2\sin\left(\frac{1}{x}\right)\right)$

113. Evaluate: $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right)$

114. Evaluate: $\lim_{x\to\infty}\left(\frac{[x]}{x}\right)$

115. Evaluate: $\lim_{x\to\infty}\left(\frac{x}{[x]}\right)$

116. Evaluate: $\lim_{n\to\infty}\left(\frac{[x]+[2x]+[3x]+\ldots+[nx]}{n^2}\right)$

Newton and Leibnitz Rule

117. Evaluate: $\lim_{x\to 0}\left(\frac{x-\int_0^x \cos(t^2)dt}{x^3-6x}\right)$

118. Evaluate: $\lim_{x\to\infty}\left(\frac{\int_0^x \sqrt{4+t^4}dt}{x^3}\right)$

119. Evaluate: $\lim_{x\to 1}\left(\frac{\int_x^{x^2} e^{-t^2}dt}{x-1}\right)$

Definite Integral as the limit of a sum

120. Evaluate: $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}\right]$

121. Evaluate: $\lim_{n\to\infty}\left[\frac{1}{n^2+1^2}+\frac{1}{n^2+2^2}+\frac{1}{n^2+3^2}+\ldots+\frac{1}{2n}\right]$

122. Evaluate: $\lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{r}{\sqrt{n^2+r^2}}\right)$

123. Evaluate: $\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{2}{n}\left(\frac{2r}{n}+1\right)$

124. Evaluate: $\lim_{n\to\infty}\left(\frac{(2n)!}{n!\times n^n}\right)^{1/n}$

Advanced limit in the form of (0^0 and ∞^0)

125. Evaluate: $\lim_{x\to 0^+}(x-x^2)^{x^2}$

126. Evaluate: $\lim_{x\to 1^-}(1-x^2)^{\frac{1}{\log(1-x)}}$

127. Evaluate: $\lim_{x\to\infty}(3^x+4^x)^{\frac{1}{x}}$

128. Evaluate: $\lim_{x\to\infty} \left(x^{\frac{1}{x}}\right)$

129. Evaluate: $\lim_{x\to\infty} \left(\frac{x+1}{2x+1}\right)^{x^2+2014}$

130. Evaluate: $\lim_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}}$

131. Evaluate: $\lim_{x\to\infty} (2^x + 3^x)^{1/x}$

132. Evaluate: $\lim_{x\to\infty} (1^x + 2^x + 3^x + \ldots + 99^x)^{1/x}$

133. Evaluate: $\lim_{x\to 0} \left(1^{\operatorname{cosec}^2 x} + 2^{\operatorname{cosec}^2 x} + 3^{\operatorname{cosec}^2 x} + \ldots + n^{\operatorname{cosec}^2 x}\right)^{\sin^2 x}$

Level II

(Mixed Problems)

Choose the most appropriate one.

1. If $f(x) = \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$, then $\lim_{x\to 0} (f(x))$ is

(a) 0 (b) ∞
(c) 1 (d) None

2. If $f(x) = x \sin(1/x)$, $x \neq 0$, then $\lim_{x\to 0} (f(x))$

(a) 1 (b) 0
(c) -1 (d) None

3. $\lim_{x\to 0} \left(\frac{\sqrt{1-\cos 2x}}{x}\right) =$

(a) 1 (b) 0
(c) -1 (d) Non-existent

4. Let α and β be the roots of $ax^2 + bx + c$), then $\lim_{x\to 0} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is

(a) 0 (b) $\frac{1}{2}(\alpha-\beta)^2$
(c) $\frac{a^2}{2}(\alpha-\beta)^2$ (d) $-\frac{a^2}{2}(\alpha-\beta)^2$.

5. $\lim_{\alpha\to\beta} \frac{\sin^2\alpha - \sin^2\beta}{\alpha^2 - \beta^2}$ is

(a) 0 (b) 1
(c) $\frac{\sin\beta}{\beta}$ (d) $\frac{\sin 2\beta}{2\beta}$.

6. If α and β be the roots of $x^2 - ax + b = 0$, then $\lim_{x\to\alpha} \frac{e^{x^2-ax+b} - 1}{x-\alpha} =$

(a) $\alpha - \beta$ (b) $\beta - \alpha$
(c) 1 (d) None

7. $\lim_{x\to 0} \left(\frac{\sin(\pi\cos^2 x)}{x^2}\right) =$

(a) $-\pi$ (b) π
(c) $\pi/2$ (d) None

8. $\lim_{x\to\infty} x\cos\left(\frac{\pi}{4x}\right)\sin\left(\frac{\pi}{4x}\right) =$

(a) $\pi/4$ (b) $\pi/3$
(c) π (d) 0

9. $\lim_{x\to\infty} a^x \sin\left(\frac{b}{a^x}\right)$, where $a > 1$ is

(a) $b \log a$ (b) $a \log b$
(c) b (d) a

10. $\lim_{x\to 0} \frac{(a+x)^2 \sin(a+x) - a^2 \sin a}{x}$ is

(a) $a^2\cos a + a \sin a$ (b) $a^2\cos a + 2a \sin a$
(c) $2a^2\cos a + a \sin a$ (d) None

11. $\lim_{x\to 0} \frac{\sin nx[(a-n)nx - \tan x]}{x^2}$, where n is a non-zero positive integer, then a is

(a) $\left(\frac{n+1}{n}\right)$ (b) n^2
(c) $1/n$ (d) $n + 1/n$

12. The value of $\lim_{x\to\infty} \left(\cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{4}\right)\cdot\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right)\right)$ is

(a) 1 (b) $\sin x/x$
(c) $x/\sin x$ (d) None

13. $\lim_{x\to\infty} \left[\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x}\right]$ is equal to

(a) 0 (b) 1/2
(c) log 2 (d) None

14. $\lim_{x\to-\infty} \frac{x^4\sin(1/x) + x^2}{1+|x|^3} =$

(a) 1 (b) 2
(c) -1 (d) None

15. If $\lim_{x\to 0} \frac{x(1+a\cos x) - b\sin x}{x^3} = 1$, then a, b are

(a) 1/2, –3/2 (b) 5/2, 3/2
(c) –5/2, –3/2 (d) None

16. The value of $\lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{x^4}$ is equal to

(a) 1/5 (b) 1/6
(c) 1/4 (d) 1/2

17. If $\lim_{x\to 1} \sec^{-1}\left[\frac{\lambda^2}{\log x} - \frac{\lambda^2}{x-1}\right]$ exists, then λ lies in

(a) $(-\infty, \sqrt{2}]$ (b) $[\sqrt{2}, \infty)$
(c) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$ (d) None

18. The value of $\lim_{m\to\infty} \left[\cos\left(\frac{x}{m}\right)\right]^m$ is equal to

(a) 1 (b) e
(c) $1/e$ (d) None

19. $\lim_{n\to\infty} \left[\tan\left\{\frac{\pi - 4}{4} + \left(1 + \frac{1}{n}\right)\right\}^a\right]^n$ is equal to

(a) e^a (b) e^{2a}
(c) 1 (d) None

20. $\lim_{x\to 0} \frac{1}{x}\left[\int_y^a e^{\sin^2 t}dt - \int_{x+y}^a e^{\sin^2 t}dt\right]$ is equal to

(a) $e^{\sin^2 y}$ (b) $\sin^2 y e^{\sin^2 y}$
(c) 0 (d) None

21. $\lim_{x\to 0} \left(\frac{1 - 3^x - 4^x + 12x}{(\sqrt{(2\cos x + 7)} - 3)}\right)$ is

(a) 2 log4 · log3 (b) – 6 log4 · log3
(c) 3 log4 · log3 (d) None

22. $\lim_{x\to 0} \frac{64^x - 32^x - 16^x + 4^x + 2^x - 1}{(\sqrt{(15 + \cos x)} - 4)\sin x} =$

(a) $-96(\log 2)^3$ (b) $48(\log 2)^2$
(c) $\log 2$ (d) None

23. $\lim_{x\to 0} \frac{8^x - 4^x - 2^x + 1}{(\sqrt{2} - \sqrt{1 + \cos x})} =$

(a) $(4\sqrt{2})\log 2$ (b) $(8\sqrt{2})\log 2$
(c) $(8\sqrt{2})(\log 2)^2$ (d) None

24. $\lim_{x\to 0} \left\{\tan\left(\frac{\pi}{4} + x\right)\right\}^{1/x} =$

(a) e (b) e^2
(c) $\sqrt{e}$ (d) $\frac{1}{\sqrt{e}}$

25. If $\lim_{x\to\infty} \left[1 + \frac{a}{x} + \frac{b}{x^2}\right]^{2x} = e^2$, then (a, b) is

(a) (2, 1) (b) (1, 2)
(c) $(1, R)$ (d) (1, 1)

26. If a, b, c, d are positive, then $\lim_{x\to\infty} \left(1 + \frac{1}{a + bx}\right)^{c+dx}$

(a) $e^{d/b}$ (b) $e^{c/a}$
(c) $e^{(c+d)/(a+b)}$ (d) e

27. If $\lim_{x\to 0} (\cos x + a\sin bx)^{1/X} = e^2$, then (a, b) is equal to

(a) (1, 2) (b) (2, 1/2)
(c) $\left(2\sqrt{3}, \frac{1}{\sqrt{3}}\right)$ (d) (4, 2)

28. $\lim_{x\to\infty} \left(\frac{x+6}{x+1}\right)^{x+4} =$

(a) e (b) $e3$
(c) $e5$ (d) $e7$

29. $\lim_{x\to\infty} (4^n + 5^n)^{1/n}$ is equal to

(a) 4 (b) 5
(c) e (d) None

30. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then $\lim_{x\to 0} (ax^2 + bx + c + 1)^{\frac{2}{(x-\alpha)}}$ is

(a) $2a(\alpha - \beta)$ (b) $2\log|a(\alpha - \beta)|$
(c) $e^{2a(\alpha-\beta)}$ (d) None

31. $\lim_{n\to\infty} \frac{x^n + y^n}{x^n - y^n}$, where $x > y > 1$ is equal to

(a) 0 (b) 1
(c) –1 (d) None

32. If $f(x) = \left(\frac{x^2 + 5x + 3}{x^2 + x + 2}\right)^x$, then $\lim_{x\to\infty} f(x)$ is

(a) e^4 (b) e^3
(c) e^2 (d) 2^4

33. The value of $\lim_{x\to\infty} \left(\frac{x^2 - 2x + 1}{x^2 - 4x + 1}\right)^x$ is equal to

(a) e^2 (b) e^{-2}
(c) e^6 (d) None

34. $\lim_{x\to 0} \left(\frac{1 + \tan x}{1 + \sin x}\right)^{\operatorname{cosec} x}$ is equal to

(a) e (b) $1/e$
(c) 1 (d) None

35. $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x - \sin x}}$

(a) $1/e$ (b) e
(c) 1 (d) None

36. The integer n for which $\lim_{x\to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number is
(a) 1 (b) 2
(c) 3 (d) 4

37. Let $f: R \to R$ be such that $f(1) = 3$ and $f'(1) = 6$, then $\lim_{x\to 0} \left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ is
(a) 1 (b) $e^{1/2}$
(c) e^2 (d) None

38. For $x > 0$ $\lim_{x\to 0} \left((\sin x)^{1/x} + \left(\frac{1}{x}\right)^{\sin x}\right)$ is equal to
(a) –1 (b) 0
(c) 2 (d) 1

39. Suppose $f: R \to R$ is a differentiable function and $f(1) = 4$. Then the value of $\lim_{x\to 1} \frac{\int_4^{f(x)} 2t\,dt}{x-1}$ is
(a) $8f'(1)$ (b) $4f'(1)$
(c) $2f'(1)$ (d) $f'(1)$

40. If $f(x)$ is differentiable increasing function, then $\lim_{x\to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is equal to
(a) 2 (b) 1
(c) –1 (d) 0

41. The value of $\lim_{x\to 0} \left[\frac{x^2}{\sin x \tan x}\right]$, where [,] = G.I.F., is
(a) 0 (b) 1
(c) –1 (d) does not exist

42. The value of $\lim_{x\to 0} \{1 + x\}^{\frac{2}{x}}$, where {,} = F.P.F, is
(a) $(e^2 - 7)$ (b) $(e^2 - 8)$
(c) $(e^2 - 6)$ (d) $(e^2 - 10)$

43. The value of $\lim_{x\to 0} \left(\frac{2(\tan x - \sin x) - x^3}{x^5}\right)$ is
(a) 1/4 (b) 1/2
(c) 1/3 (d) None

44. The value of $\lim_{x\to\infty} x^2 \times \left(\tan^{-1}\left(\frac{2x^2+1}{x^2+2}\right) - \tan^{-1}2\right)$ is
(a) 3/5 (b) –3/5
(c) 5/3 (d) –5/3

45. The value of $\lim_{x\to\infty} \left(\frac{(\ln x)^{10}}{x^4}\right)$ is
(a) 1 (b) 0
(c) 1/2 (d) –1/2

46. The value of $\lim_{x\to\infty} \left(x + \sqrt{x^2 + x^2 \sin\left(\frac{1}{x}\right)}\right)$ is
(a) 0 (b) 2
(c) –2 (d) None

47. The value of $\lim_{x\to\infty} \left(x + \sqrt{x^2 + 3x \sin\left(\frac{1}{|x|}\right)}\right)$ is
(a) 3/2 (b) –3/2
(c) –1 (d) 0

48. If $\lim_{x\to 0} \left(\frac{x^{2n} \sin^n x}{x^{2n} - \sin^n x}\right)$ is a non-zero finite number, then the value of n must be
(a) 1 (b) 2
(c) 3 (d) 4

49. The value of $\lim_{x\to -\infty} \left(\frac{\sqrt[x]{p} + \sqrt[x]{q}}{2}\right)$, where $p > 0$, $q > 0$ is equal to
(a) 1 (b) $\sqrt{pq}$
(c) pq (d) $pq/2$

50. If $\lim_{x\to\infty} \frac{\int_0^{x^2} (\sin x^2)dx}{x^n}$ is a non-zero finite number, then the value of n is
(a) 1 (b) 3
(c) 5 (d) 4

51. If a, b, c are real numbers, then the value of $\lim_{x\to 0} \ln\left(\frac{1}{t}\int_0^t (1 + a \sin bx)^{\frac{e}{x}}\, dx\right)$ is
(a) abc (b) ab/c
(c) bc/a (d) ac/b

52. The value of $\lim_{x\to \frac{\pi}{2}} \left(\frac{\sin x - (\sin x)^{\sin x}}{1 - \sin x(\log \sin x)}\right)$ is
(a) 1 (b) 0
(c) 2 (d) 2/3

53. The value of $\lim_{x\to\infty} \left(\frac{1}{na} + \frac{1}{na+1} + \frac{1}{na+2} + \ldots \frac{1}{nb}\right)$ is
(a) $\log\left(\frac{a}{b}\right)$ (b) $\log\left(\frac{b}{a}\right)$

(c) $\log(ab)$ (d) None

54. The value of $\lim_{x\to 0}\left(\left[\frac{2000x}{\sin x}\right]+\left[\frac{13\sin x}{x}\right]\right)$ is, where [,] = G.I.F
(a) 2013 (b) 2012
(c) 2011 (d) None

55. The value of $\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)$ is
(a) e (b) $1/e$
(c) $1/2e$ (d) $1/3e$

56. The value of $\lim_{x\to 0}\left(\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}\right)$ is
(a) $-1/2$ (b) $1/2$
(c) $-1/3$ (d) $1/4$

57. The value of $\lim_{x\to\infty}\left(\frac{1}{x^2}-\frac{1}{\tan^2 x}\right)$ is
(a) 2/3 (b) 3/2
(c) 2/5 (d) 5/2

58. The value of $\lim_{x\to 0}(\operatorname{cosec} x)^{\frac{1}{\ln x}}$ is
(a) $1/e$ (b) e
(c) $2/e$ (d) $e/3$

59. The value of $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ is
(a) 1 (b) -1
(c) 0 (d) 2

60. The value of $\lim_{x\to 1}\left(\tan\left(\frac{\pi}{4}+\ln x\right)\right)^{\frac{1}{\ln x}}$ is
(a) e^4 (b) e^2
(c) e^3 (d) e^5

61. The value of $\lim_{x\to\frac{\pi}{4}}(1+[x])^{\frac{1}{\log(\tan x)}}$, where [,] = G.I.F, is equal to
(a) 0 (b) 1
(c) e (d) $1/e$

62. The value of $\lim_{x\to\infty}\left(\frac{\log x^n-[x]}{[x]}\right)$, $n\in N$, is
(a) -1 (b) 0
(c) 1 (d) 2

63. The value of $\lim_{x\to 0}\left(\frac{x\cos x-\log(1+x)}{x^2}\right)$ is equal to
(a) 1/2 (b) 0
(c) 1 (d) -1

64. The value of $\lim_{x\to 0}\left(\frac{e^x-e^{\sin x}}{x+\sin x}\right)$ is
(a) 0 (b) 1
(c) -1 (d) 2

65. The value of $\lim_{x\to 0}\left(\frac{e^x-e^{x\cos x}}{x+\sin x}\right)$ is
(a) 0 (b) 1
(c) 2 (d) -1

66. The value of $\lim_{x\to 1}\left(\tan\left(\frac{\pi}{4}+\log x\right)\right)^{\frac{1}{\log x}}$ is
(a) e (b) $1/e$
(c) e^2 (d) $1/2e$

67. The value of $\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)+\sin\left(\frac{1}{n}\right)\right)^n$ is
(a) e^{a-1} (b) e^{a+1}
(c) e^{-a+1} (d) e^{-a-1}

68. The value of $\lim_{x\to 0}\left(\frac{e-(1+x)^{\frac{1}{x}}}{\tan x}\right)$ is
(a) $e/2$ (b) $2/e$
(c) $3/e$ (d) $e/3$

69. The value of $\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n$, $n\in N$ is
(a) $\sqrt[a]{b}$ (b) $\sqrt[b]{a}$
(c) $\sqrt{b}$ (d) $\sqrt{a}$.

70. The value of $\lim_{x\to 0}\left[\frac{n\sin x}{x}\right]+\left[\frac{n\tan x}{x}\right]$, where [,] = G.I.F, is
(a) $2n$ (b) $2n+1$
(c) $2n-1$ (d) does not exist

71. The value of $\lim_{n\to\infty}\frac{n^k\sin^2(n!)}{n+1}$ is, where $0\le k<1$, is
(a) 0 (b) 1
(c) -1 (d) -2

72. The value of $\lim_{x\to 0}\left(\frac{(1+x)^{1/x}+ex-e}{\sin^{-1}x}\right)$ is
(a) $-e/2$ (b) $e/2$
(c) 1 (d) 0

73. The value of $\lim_{x\to 0^+}(-\ln(\{x\}+[x]))^{\{x\}}$ is
(a) 0 (b) 1
(c) $\ln 2$ (d) $\ln(1/2)$

74. The value of $\lim_{x\to\infty}\left(\frac{x^n + nx^{n-1} + 1}{[x]}\right)$, $n \in I$ is

(a) 1 (b) 0
(c) n (d) $n(n-1)$

75. The value of $\lim_{x\to\alpha^+}\left(\frac{\ln(x-a)}{\ln(e^x - e^a)}\right)$ is

(a) 1 (b) −1
(c) 0 (d) None

Level III

(Problems for JEE-Advanced)

1. Evaluate: $\lim_{x\to 0}\left[\frac{\sin x}{x}\right]$
2. Evaluate: $\lim_{x\to 0}\left[\frac{\tan x}{x}\right]$
3. Evaluate: $\lim_{x\to 0}\left[\frac{\sin^{-1}x}{x}\right]$
4. Evaluate: $\lim_{x\to 0}\left[\frac{\tan^{-1}x}{x}\right]$
5. Evaluate: $\lim_{x\to\frac{5\pi}{4}}[\sin x + \cos x]$
6. Evaluate: $\lim_{x\to 0}\left[\frac{x^2}{\sin x\,\tan x}\right]$
7. Evaluate: $\lim_{x\to 0}\left(\left[\frac{n\sin x}{x}\right] + \left[\frac{n\tan x}{x}\right]\right)$
8. Evaluate: $\lim_{x\to 0}\left(\left[2016\frac{x}{\sin x}\right] + \left[\frac{\tan x}{x}\right]\right)$
9. Evaluate: $\lim_{x\to 0}\left(\left[2016\frac{\sin^{-1}x}{x}\right] + \left[\frac{x}{\tan x}\right]\right)$
10. Evaluate: $\lim_{x\to 0}\left(\left[2016\ \frac{\tan^{-1}x}{x}\right] + \left[\frac{\tan x}{x}\right]\right)$
11. Evaluate: $\lim_{x\to 0}\left(\frac{\sin[\cos x]}{1 + [\cos x]}\right)$
12. Evaluate: $\lim_{x\to\infty}\frac{\log x}{[x]}$
13. Evaluate: $\lim_{x\to\infty}\left(\frac{\log(x^n) - [x]}{[x]}\right)$, $n \in N$
14. Evaluate: $\lim_{x\to\infty} {}^nC_x\left(\frac{m}{n}\right)^x\left(1 - \frac{m}{n}\right)^{n-x}$
15. Evaluate: $\lim_{n\to\infty}\left(\frac{n^k\sin(n!)}{n+1}\right)$, $(0 \le k \le 1)$
16. Evaluate: $\lim_{x\to 0}\left(\frac{\sin[x]}{[x]}\right)$
17. Evaluate: $\lim_{x\to\left(\frac{\pi}{2}\right)}\left[\sin^{-1}(\sin x)\right]$
18. Evaluate: $\lim_{x\to 0}\left(\left[\frac{100x}{\sin x}\right] + \left[\frac{99\sin x}{x}\right]\right)$
19. Evaluate: $\lim_{n\to\infty}\left(\frac{a - 1 + \sqrt[n]{b}}{a}\right)^n$, where $a, b > 0$
20. Evaluate: $\lim_{x\to 0}\left(\frac{(1-x)^{1/x} - e + \frac{ex}{2}}{x^2}\right)$.
21. Evaluate: $\lim_{x\to 0}\left(\frac{\sin\{x\}}{\{x\}}\right)$, where {,} = F.I.F
22. Evaluate: $\lim_{n\to\infty}\left[\tan\left\{\frac{\pi - 4}{4} + \left(1 + \frac{1}{n}\right)\right\}^a\right]^n$
23. Evaluate: $\lim_{n\to\infty}\left\{\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right)\right\}$
24. Evaluate: $\lim_{x\to\pi^+}\left(\frac{2^{\cot x} + 3^{\cot x} - 5^{1+\cot x} + 10}{(4^{\cot x})^{1/2} + (27^{\cot x})^{1/3} - 5^{\cot x} + 20}\right)$
25. Find a polynomial of the least degree such that
$$\lim_{x\to 0}\left(1 + \frac{x^2 + f(x)}{x^2}\right)^{1/x} = e^2.\text{t}$$
26. Evaluate:
$$\lim_{n\to\infty}\{(n^6 + 6n^5 + 12n^4 + 1)^{1/3} - (n^4 + 4n^2 + 6n + 1)^{1/2}\}$$
27. Evaluate:
$$\lim_{x\to 0}\left(\frac{(1 + 3x + 2x^2)^{1/x} - (1 + 3\text{x} - 2\text{x}^2)^{1/x}}{x}\right)$$
28. Evaluate: $\lim_{x\to 0}\left(\frac{1 - \cos(1 - \cos x)}{x^4}\right)$
29. Evaluate: $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x - \sin x}}$
30. Let $f(x)$ be a real function such that $\lim_{x\to 0}\left(\frac{f(x)}{x}\right) = 1$

If $\lim_{x\to 0}\left(\frac{x(1 + a\cos x) - b\sin x}{\{f(x)\}^3}\right) = 1$, then find the values of $(a + b + 10)$

Level IV

(Tougher Problems for JEE-Advanced)

1. Evaluate: $\lim_{x\to a}\left(\frac{x^x - a^a}{a^x - a^a}\right)$
2. Evaluate: $\lim_{x\to 0}\left(\frac{x + \ln(\sqrt{x^2 + 1} - x)}{x^3}\right)$

3. If $\lim_{x\to 0}\left(\frac{2}{x^3}(\sin^{-1}x-\tan^{-1}x)\right)^{\frac{2}{x^3}}=e^{-M}$, find the value of $2M+2013$.

4. For $n\in N$, let x_n be defined as $\left(1+\frac{1}{n}\right)^{n+x_n}=e$, find $\lim_{n\to\infty}(x_n)$

5. If α and β are the roots of the equation $ax^2+bx+c=0$, find the value of

$$\lim_{x\to\alpha}\left(\frac{\sqrt{1-\cos(ax^2+bx+x)}}{(x+\alpha)^2}\right)$$

6. Find the value of $a^3+b^3+c^3+10$

$$\lim_{x\to 0}\left(\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}\right)=2$$

7. If $\lim_{x\to 0}\left(\frac{1-\cos x+2\sin x-\sin^3 x-x^2+3x^4}{\tan^3 x-6\sin^2 x+x-5x^3}\right)=M$, find the value of $M+10$.

8. If $L=\lim_{x\to 0}\left[\frac{x^2}{\sin x\tan x}\right]$, find the value of $L+2013$.

9. Let $P(x)=a_1x+a_2x^2+\dots\dots+a_{100}x^{100}$, where $a_1=1$ and $a_1, a_2\dots a_{100}\in R$ such that

$$L=\lim_{x\to 0}\left(\frac{\sqrt[100]{1+P(x)}}{x}-1\right)$$

then find the value of $(1000L+2007)$

10. Let $L=\lim_{x\to 0}\left[\frac{100x}{\sin x}\right]$ and $M=\lim_{x\to 0}\left[\frac{99\sin x}{x}\right]$, find the value of $L+M+2$.

11. If $\lim_{x\to\infty}\left(\frac{729^x-243^x-81^x+9^x+3^x-1}{x^3}\right)=K(\log M)^N$, find the value of $K^2+M^3+N^2$.

12. If $\lim_{x\to 1^-}\left(\frac{(3+ax)^{5/2}-b\ln x+c\sin(x-1)}{(x-1)^2}\right)=2$, find the value of $a^2+b^2+c^2$.

13. Find the smallest integral value of n for which $\lim_{x\to 0}\left(\frac{(1-\cos x)(e^x-\cos x)}{x^n}\right)$ is a non-zero finite number.

14. If $\lim_{x\to 0}\left(\frac{\cos 4x+a\cos 2x+b}{x^4}\right)$ is a finite quantity, find the value of a^2+b^2+10.

15. If $\lim_{n\to\infty}\left(\sum_{k=0}^{n}\left(\frac{{}^nC_k}{n^k(k+3)}\right)\right)=Le+M$, find the value of $L+M+10$.

16. If $\lim_{n\to\infty}f(x)=\frac{1}{60}$ such that

$$f(x)=\frac{(1^a+2^a+3^a+\dots+n^a)\div(n+1)^{a-1}}{((na+1)+(na+2)+\dots\dots+(na+n))}$$

where $n\neq -1$ and $a\neq 0$, find the value of a.

17. If $L=\lim_{x\to-\infty}\left(\frac{x^4\sin\left(\frac{1}{x}\right)}{1+|x|^3}\right)$

and $M=\lim_{x\to 0}\left(\frac{\int_{\sqrt{x}}^{x^2}\tan^{-1}\left(\frac{t^2}{1+t^2}\right)dt}{\sin 2x}\right)$, find the value of $L+M+11$.

18. If $\lim_{n\to\infty}\left(\int_n^{2n}\frac{n^3x}{x^5+1}\right)dx=k$, find the value of $\left[\frac{1}{k}\right]+2013.$, where [ ] = G.I.F

19. Evaluate: $\lim_{x\to\infty}\left(\sqrt{(x^2+a^2)(x^2+b^2)}-\sqrt{(x^2+c^2)(x^2+d^2)}\right)$

20. Evaluate: $\lim_{x\to 1}\left\{\frac{(1-x)(1-x^2)\dots(1-x^{2n})}{\{(1-x)(1-x^2)\dots(1-x^n)\}^2}\right\}$

21. Evaluate: $\lim_{x\to\infty}\left(\frac{4^x+4^{-x}}{4^x-4^{-x}}\right)$

22. Evaluate: $\lim_{n\to\infty}\prod_{r=3}^{n}\left(\frac{r^3-8}{r^3+8}\right)$

23. Find the value of

$$\lim_{x\to 0}\left\{\frac{32}{x^8}\left(1-\cos\left(\frac{x^2}{2}\right)-\cos\left(\frac{x^2}{4}\right)+\cos\left(\frac{x^2}{2}\right)\cos\left(\frac{x^2}{4}\right)\right)\right\}$$

24. Find the value of

$$\lim_{x\to\infty}\left[\{(x+p)(x+q)(x+r)(x+s)\}^{1/4}-x\right]$$

25. Let $f(x)$ be a function such that $\lim_{x\to 0}\left(\frac{f(x)}{x}\right)=1$

If $\lim_{x\to 0}\left(\frac{x(1+a\cos x)-b\sin x}{\{f(x)\}^3}\right)=1$, find the value of $(a+b+10)$.

26. If $\lim_{x\to 0}\left(\frac{axe^x-b\log(1+x)+cxe^{-x}}{x^2\sin x}\right)=2$, find the value of $\left(\frac{a+b+c}{8}\right)$

27. Evaluate: $\lim_{x\to 0}\left(\frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\log(\sin x)}\right)$

28. Let $f(x)=\cos 2x\cos 4x\cos 8x\cos 10x$ and

$M=\lim_{x\to 0}\left(\frac{1-(f(x))^3}{5\tan^2 x}\right)$, where M is finite, find the value of $(\sqrt{M-2}+1)$

29. If $\lim_{x\to 0}\left(\left[\frac{\sin^{-1}x}{x}\right]+\left[\frac{2^2\sin^{-1}(2x)}{x}\right]+\left[\frac{3^2\sin^{-1}(3x)}{x}\right]+\dots+\left[\frac{n^2\sin^{-1}(nx)}{x}\right]\right)=100$, find the value of n, where [,] = G.I.F.

30. If $\lim_{x\to 0}\left(\frac{x+\sin x-x\cos x-\tan x}{x^n}\right)$ exists and have a non-zero finite value, then find n.

31. Find the value of

$$\lim_{n\to\infty}\left(\frac{n\cdot 1+(n-1)(1+2)+(n-2)(1+2+3)+\dots+1\sum_{r=1}^{n}}{n^4}\right)$$

32. Find the value of

$$\lim_{n\to\infty}\left(\frac{1^4+2^4+3^4+\dots+n^4}{(1^2+2^2+3^2+\dots+n^2)(1^3+2^3+3^3+\dots+n^3)}\right)$$

Integer Type Questions

1. If $\lim_{x\to 0}\left(\frac{\ln(1+x+x^2+\dots+x^n)}{nx}\right)=\frac{1}{5}$, find the value of n.

2. If $\lim_{x\to 0}\left(\frac{\tan x-\sin x}{x^3}\right)=\frac{a}{b}$, find the value of $(a+b+3)$

3. Let $P(x)=a_1x+a_2x^2+\dots+a_nx^n$, where $a_1=1$, $a_i\in R, i=1,2,3,\dots,64$ such that $\lim_{x\to 0}\left(\frac{\sqrt[64]{1+P(x)}-1}{x}\right)=\frac{a}{b}$, find the value of $\sqrt{b+a-1}$

4. If $\lim_{x\to 0}\left(\frac{x^{2n}\sin^n x}{x^{2n}-\sin^{2n}x}\right)$ is a non-zero finite number find n.

5. If $\lim_{x\to\infty}\left(\sqrt{x^2+x^2\sin\left(\frac{1}{x}\right)}-x\right)$ exists and equal to $\frac{a}{b}$, find the value of $(a+b)$.

6. If $\lim_{x\to 1}\left(\frac{\tan(x-1)\log_e(x^{x-1})}{(x-1)^3}\right)$ exists and equal to L, find the value of $(L+3)$.

7. If $\lim_{x\to\infty}\left(\frac{2(\tan x-\sin x)-x^3}{x^5}\right)=\frac{m}{n}$, find the value of $(m+n+2)$.

8. If $\lim_{x\to 0}\left(\frac{1+\sin x-\cos x-+\log(1-x)}{x\tan^2 x}\right)$ exists and is equal to $-\frac{m}{n}$, find the value of $(m+n)$.

9. If $\lim_{x\to 0}\left(\frac{e^x+e^{-x}+2\cos x-4}{x^4}\right)=\frac{p}{q}$, where $p, q\in N$, find the value of $(q-p)$.

10. If $P=\lim_{n\to\infty}(4^n+3^n)^{1/n}$ and $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{(x-1)}\right)=\frac{m}{n}$, where $m, n, p\in I^+$, find the value of $(m+n+p)$

11. If $\lim_{x\to 0}\left(\frac{2x-\sin 2x-\tan^{-1}\left(\frac{2x^3}{1+x^6}\right)}{x^3}\right)=-\frac{p}{q}$ where $p, q\in N$, find the value of $(p+q)$

12. If $\lim_{x\to\infty}\left\{x\left(\tan^{-1}\left(\frac{x+2}{x+1}\right)-\tan^{-1}\left(\frac{x}{x+2}\right)\right)\right\}$ exists and finite and equal to $\frac{m}{n}$, find $(m+n)$.

Comprehensive Link Passages (For JEE-Advanced Exam Only)

In these questions, a passage (paragraph) has been given followed by questions based on each of the passage. You have to answer the questions based on the passage given.

Passage I

Let $f(x)$ and $g(x)$ be two real functions and $a\in R$ such that $f(a)=0=g(a)$ and $f(a)=\infty=g(a)$, then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right)$$
$$=\lim_{x\to a}\left(\frac{f''(x)}{g''(x)}\right)$$

untill and unless the form of $\left(\frac{0}{0}\right)$ and $\left(\frac{\infty}{\infty}\right)$ is removed.

On the basis of above information, answer the following questions:

1. The value of $\lim_{x\to 0}\left(\dfrac{x^2\sin\left(\frac{1}{x}\right)}{\sin x}\right)$ is

(a) 0 (b) 1
(c) 1/2 (d) 1/4

2. The value of $\lim_{x\to 0}\left(\dfrac{x^2+2\cos x-2}{x\sin^3 x}\right)$ is

(a) 1/6 (b) 1/12
(c) 1/8 (d) 1/16

3. The limit of $\lim_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2}$ is

(a) $-\pi$ (b) π
(c) $\dfrac{\pi}{2}$ (d) None

Passage II

Let $f(x)$ and $g(x)$ be two real functions and

$a\in R$ such that $\lim_{x\to a}(f(x))=0$ and $\lim_{x\to a}(g(x))=\infty$

Then we can write $\lim_{x\to a}(f(x).g(x))$

$$=\lim_{x\to a}\left(\frac{f(x)}{1/g(x)}\right)\text{ form }\left(\frac{0}{0}\right)$$

or $\lim_{x\to a}\left(\dfrac{g(x)}{1/f(x)}\right)$ form $\left(\frac{\infty}{\infty}\right)$

On the basis of above information, answer the following questions:

1. The value of $\lim_{x\to \pi/2}(\cot x\cdot\log(\sec x))$ is

(a) 0 (b) 1
(c) –1 (d) None

2. The value of $\lim_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$ is

(a) 0 (b) 1
(c) 2 (d) –1

3. The value of $\lim_{x\to 0}\left(\dfrac{1}{\tan^2 x}-\dfrac{1}{x^2}\right)$ is

(a) 2/3 (b) –2/3
(c) 4/3 (d) –4/3

Passage III

Let $\varphi(x)$ and $\psi(x)$ be two differentiable functions, then

$$\frac{d}{dx}\left(\int_{\varphi(x)}^{\psi(x)} f(t)\,dt\right)$$

$$=f(\psi(x))\,\psi'(x)-f(\varphi(x))\,\varphi'(x)$$

Now $\quad\dfrac{d}{dx}\left(\int_{x^2}^{x^3} e^t\,dt\right)=(e^{x^3}\cdot 3x^2-e^{x^2}\cdot 2x)$

On the basis of above information, answers the following questions:

1. The value of $\lim_{x\to 0}\left(\dfrac{x-\int_0^{x^2}\cos(t^2)\,dt}{x^3-6x}\right)$ is

(a) –1/6 (b) 1/8
(c) –1/10 (d) 1/12

2. The value of $\lim_{x\to 0}\left(\dfrac{\int_0^{x^2}\sin\sqrt{t}dt}{x^3}\right)$

(a) 1/2 (b) –1/2
(c) 1/3 (d) Non-existent

3. The value of $\lim_{x\to 0}\dfrac{1}{x^3}\times\int_0^x\left(\dfrac{t\log(1+t)}{t^4+4}\right)dt$ is

(a) 0 (b) 1/12
(c) 1/24 (d) 1/64

Matrix Match
(For JEE-Advanced Exam Only)

1. **Match the following columns**

If $\lim_{x\to 0}\left(\dfrac{a\sin x-bx+cx^2+x^3}{2x^2\log_e(1+x)-2x^3+x^4}\right)$

exists and is finite.

Column I		Column II	
(A)	a	(P)	6
(B)	b	(Q)	0
(C)	c	(R)	12
(D)	$a+b+c$	(S)	5

2. **Match the following columns**

Column I		Column II	
(A)	$\lim_{x\to 0}\left(\dfrac{\{x\}}{\tan\{x\}}\right)$	(P)	$-\log_{16}e$
(B)	$\lim_{x\to 0}\left(\dfrac{\sqrt{1-x+x^2}-\sqrt{1+x^2}}{4^x-1}\right)$	(Q)	e^{-1}
(C)	$\lim_{x\to 0}\left(\dfrac{2e^{\sin x}-(1+\sin x)^2}{2[\tan^{-1}(\sin x)^2]}\right)$	(R)	1
(D)	$\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$	(S)	0

3. **Match the following columns**

Column I		Column II	
(A)	$\lim_{x\to0}\left\{\frac{\log(1+x)}{x^2}+\frac{x-1}{x}\right\}$	(P)	$\frac{1}{8}$
(B)	$\lim_{x\to0}\left(\frac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}\right)$	(Q)	$\frac{3}{2}$
(C)	$\lim_{x\to0}\left(\frac{e^{x^2}-\cos x}{x^2}\right)$	(R)	$8\sqrt{2}(\log 3)^2$
(D)	$\lim_{x\to0}\left(\frac{1-\cos(1-\cos x)}{x^4}\right)$	(S)	$\frac{1}{2}$

4. **Match the following columns**

Column I		Column II	
(A)	$\lim_{x\to0}\frac{1}{x}\left(\int_y^a e^{\sin^2 t}dt-\int_{x+y}^a e^{\sin^2 t}dt\right)$	(P)	1
(B)	$\lim_{x\to0}\left(\frac{\left(\int_0^{x+y} e^{t^2}dt\right)^2}{\int_0^{x+y} e^{2t^2}dt}\right)$	(Q)	$e^{\sin^2 y}$
(C)	$\lim_{x\to0}\left(\frac{\int_y^{x^3}\sin\sqrt{t}\,dt}{x^3}\right)$	(R)	0
(D)	$\lim_{x\to0}\left(\frac{\int_y^{x}\cos t^2\,dt}{x}\right)$	(S)	$\frac{2}{3}$

5. **Match the following columns**

Column I		Column II	
(A)	$\lim_{\theta\to0}\left(\frac{\sin\sqrt{\theta}}{\sqrt{\sin\theta}}\right)$	(P)	-2
(B)	$\lim_{x\to0}\left(\frac{\log_e\cos x}{x^2}\right)$	(Q)	1
(C)	$\lim_{x\to0}\left(\frac{\tan^3 x-3\tan x}{\cos\left(x+\frac{\pi}{6}\right)}\right)$	(R)	$-\frac{1}{2}$
(D)	$\lim_{x\to\infty}\left(\frac{(3x^4+2x^2)\sin\left(\frac{1}{x}\right)+\lvert x\rvert^2+5}{\lvert x\rvert^3+\lvert x\rvert^2+\lvert x\rvert+1}\right)$	(S)	$-24.$

6. **Match the following columns**

Column I		Column II	
(A)	$\lim_{x\to\infty}\left\{\frac{3-2x^2}{x+1}+ax+b\right\}=1$	(P)	$a=1$, $b=-2$
(B)	$\lim_{x\to0}\left(\int_0^X\frac{4t^2\,dt}{(bx-\sin 2x)\sqrt{t-a}}=1\right)$	(Q)	$a=1$, $b=2$
(C)	$\lim_{x\to\infty}\left(\frac{ae^{-\frac{1}{x}}-2e^{\frac{1}{x}}}{e^{-\frac{1}{x}}+be^{\frac{1}{x}}}=1\right)$	(R)	$a=2$, $b=-1$
(D)	$\lim_{x\to0}\left(\frac{b\sin x-a\sin 2x}{e^{-x}\times\cos 2x\times x^3}=1\right)$	(S)	$a=-1$, $b=2.$

Assertion (A) and Reason (R) Code

(A) Both A and R are true and R is the correct explanation of A

(B) Both A and R are true and R is not the correct explanation of A.

(C) A is true and R is true.

(D) A is false and R is true.

1. **Assertion (A):** The value of $\lim_{x\to0^+}\left[\frac{\sin x}{x}\right]=0$

   Reason (R): $\sin x\le x$ for all x in R

2. **Assertion (A):** The value of $\lim_{x\to\infty}\left(\frac{\sqrt{x}-\sin x}{x+\cos^2 x}\right)$ is 1.

   Reason (R): $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right)=0=\lim_{x\to\infty}\left(\frac{\cos x}{x}\right)$

3. **Assertion (A):** The limiting value of

   $\lim_{x\to0}\left(\frac{\sqrt{1-\cos 2x}}{x}\right)$ does not exist.

   Reason (R): The limiting value $\lim_{x\to0}\left(\frac{\lvert x\rvert}{x}\right)$ does not exist.

4. **Assertion (A):** The limiting value of

   $\lim_{x\to0}\left(\frac{\tan^{-1}x-\sin^{-1}x}{x^3}\right)$ is $-\frac{1}{2}$

   Reason (R): The limiting value of

   $\lim_{x\to0}\left(\frac{\cos(\sin x)-\cos x}{x^4}\right)$ is $\frac{1}{6}$

5. **Assertion (A):** The limiting value of

   $\lim_{x\to0}\left(\frac{\sin x-x}{x^3}\right)$ is $-\frac{1}{6}$

   Reason (R): The limiting value of

   $\lim_{x\to0}\left(\frac{e^{x^2}-\cos x}{x^2}\right)$ is 3/2

Problems asked in JEE Main Exams. From 2002-2014

1. $\lim_{n\to\infty}\left(\frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}}\right)$ equals

(a) $\frac{1}{p+1}$ (b) $\frac{1}{1-p}$

(c) $\left(\frac{1}{p} - \frac{1}{p-1}\right)$ (d) $\frac{1}{p+2}$

[JEE Main, 2002]

2. $\lim_{x\to\infty}\left(\frac{\log x^n - [x]}{[x]}\right)$, $n \in N$, [,] = G.I.F

(a) –1 (b) 0

(c) 1 (d) does not exist.

[JEE Main, 2002]

3. Let $f(2) = 4$ and $f'(2) = 4$, then

$\lim_{x\to 2}\left(\frac{xf(2) - 2f(x)}{x-2}\right)$ equals to

(a) 2 (b) –2

(c) –4 (d) 3

[JEE Main, 2002]

4. $\lim_{x\to\infty}\left(\frac{x^2 + 5x + 3}{x^2 + x + 3}\right)^x$ is equal to

(a) e^4 (b) e^2

(c) e^3 (d) e

[JEE Main, 2002]

5. The value of $\lim_{x\to 0}\left(\frac{\int_0^{x^2} \sec^2 t\, dt}{x \sin x}\right)$ is

(a) 3 (b) 2

(c) 1 (d) 0

[JEE Main, 2003]

6. If $\lim_{x\to 0}\left(\frac{\log(3+x) - \log(3-x)}{x}\right) = k$ then the value of k is

(a) 0 (b) –1/3

(c) 2/3 (d) –2/3

[JEE Main, 2003]

7. $\lim_{x\to\frac{\pi}{2}}\left(\frac{\left(1 - \tan\left(\frac{x}{2}\right)\right)}{\left(1 + \tan\left(\frac{x}{2}\right)\right)} \times \frac{(1 - \sin x)}{(\pi - 2x)^3}\right)$ is

(a) 1/8 (b) 0

(c) 1/32 (d) 1/16

[JEE Main, 2003]

8. The value of

$\lim_{n\to\infty}\left(\frac{1^4 + 2^4 + \dots + n^4}{n^5} - \frac{1^3 + 2^3 + \dots + n^3}{n^4}\right)$ is

(a) 0 (b) 1/4

(c) 1/5 (d) 1/30

[JEE Main, 2003]

9. Let $f(a) = g(a) = k$ and their nth derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n. Further, if

$\lim_{x\to a}\left(\frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)}\right) = 4$

then the value of k is

(a) 2 (b) 1

(c) 0 (d) 4

[JEE Main, 2003]

10. If $\lim_{x\to\infty}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x} = e^2$, then the values of a and b are

(a) $a \in R$, $b = 2$ (b) $a = 1$, $b \in R$

(c) $a \in R$, $b \in 2$ (d) $a = 1$, $b = 2$.

[JEE Main, 2004]

11. $\lim_{n\to\infty}\left(\frac{1}{n^2}\sec^2\left(\frac{1}{n^2}\right) + \frac{2}{n^2}\sec^2\left(\frac{4}{n^2}\right) + \dots + \frac{1}{n^2}\sec^2\left(\frac{n^2}{n^2}\right)\right)$

is equal to

(a) $\frac{1}{2}\tan(1)$ (b) $\tan(1) - \log(\sec 1)$

(c) $\frac{1}{2}\text{cosec}(1)$ (d) $\frac{1}{2}\sec(1)$

[JEE Main 2005]

12. Let $f : R \to R$ be a differentiable function having $f(2) = 6$, $f'(2) = \frac{1}{48}$, then $\lim_{x\to 2}\int_6^{f(x)}\left(\frac{4t^3}{x-2}\right)dt$ equals

(a) 18 (b) 12

(c) 36 (d) 24

[JEE Main 2005]

13. Let α and β be the distinct roots of $ax^2 + bx + c = 0$, then the limiting value of

$\lim_{x\to\alpha}\left(\frac{1 - \cos(ax^2 + bx + c)}{(x-\alpha)^2}\right)$ is

(a) $\frac{1}{2}(\alpha - \beta)^2$ (b) $-\frac{a^2}{2}(\alpha - \beta)^2$

(c) 0 (d) $\frac{a^2}{2}(\alpha - \beta)^2$

[JEE Main 2005]

14. No questions asked in 2006-2009.

15. Let $f : R \to R$ be a positive increasing function with $\lim_{x\to\infty}\left(\frac{f(3x)}{f(x)}\right) = 1$. Then $\lim_{x\to\infty}\left(\frac{f(2x)}{f(x)}\right)$ is

(a) 1 (b) 2/3

(c) 3/2 (d) 3

[JEE Main, 2010]

16. $\lim_{x\to 2}\left(\frac{\sqrt{1-\cos 2(x-2)}}{(x-2)}\right)$ is

(a) $-\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) does not exist (d) $\sqrt{2}$

[JEE Main, 2011]

17. No questions asked in 2012.

18. $\lim_{x\to 0}\left(\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}\right)$ is equal to

(a) 1/2 (b) 1

(c) 2 (c) –1/4

[JEE Main, 2013]

19. $\lim_{x\to 0}\left(\frac{\sin(\pi\cos^2 x)}{x^2}\right)$ is equal to

(a) $\frac{\pi}{2}$ (b) 1

(c) $-\pi$ (d) π

[JEE Main, 2014]

20. $\lim_{x\to 0}\left(\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}\right)$ is equal to

(a) 3 (b) 2

(c) 1/2 (d) 4

[JEE-Main, 2015]

21. If $p = \lim_{x\to 0^+}(1+\tan^2\sqrt{x})^{\frac{1}{2x}}$, then the value of $\log(p)$ is

(a) 1 (b) 1/2

(c) 1/4 (d) 2

[JEE-Main, 2016]

Questions asked in Past IIT-JEE Exams.

1. Evaluate:

$$\lim_{h\to 0}\left(\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}\right)$$

[IIT-JEE, 1980]

2. Prove that:

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{6n}\right)=\log_e 6$$

[IIT-JEE, 1981]

3. Evaluate: $\lim_{x\to 0}\left(\frac{2^x-1}{\sqrt{(1+x)}-1}\right)$

[IIT-JEE, 1982]

4. If $G(x) = -\sqrt{25-x^2}$, then $\lim_{x\to 1}\left(\frac{G(x)-G(1)}{x-1}\right)$ has the value

(a) $\frac{1}{\sqrt{24}}$ (b) $\frac{1}{5}$

(c) $-\sqrt{24}$ (d) None

[IIT-JEE, 1983]

5. If $f(a) = 2, f'(a) = 1, g(a) = -1, g'(a) = 2$, then the value of $\lim_{x\to a}\left(\frac{g(x)f(a)-g(a)f(x)}{x-a}\right)$ is

(a) –2 (b) 1/5

(c) 5 (d) None

[IIT-JEE, 1983]

6. $\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+\dots+\frac{n}{1-n^2}\right]$ is equal to

(a) 0 (b) –1/2

(c) 1/2 (d) None

[IIT-JEE, 1984]

7. If $f(x) = \begin{cases}\frac{\sin[x]}{[x]}, [x]\neq 0\\ 0, [x]=0\end{cases}$, where [,] = G.I.F,

then $\lim_{x\to 0} f(x)$ is equal to

(a) 1 (b) 0

(c) 1 (d) None

[IIT-JEE, 1984]

8. $\lim_{x\to 1}\left((1-x)\tan\left(\frac{\pi x}{2}\right)\right) = \dots$ **[IIT-JEE, 1984]**

9. The limiting value of $\lim_{x\to 1}\left(\frac{\sqrt{1-\cos 2(x-1)}}{x-1}\right)$ is

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) Non-existent (d) None

[IIT-JEE, 1985]

10. If $f(x) = \begin{cases}\sin x, x\neq n\pi, n\in Z\\ 2, \text{otherwise}\end{cases}$

and $g(x) = \begin{cases}x^2+1, x\neq 0, 2\\ 4, x=0\\ 5, x=2\end{cases}$,

then $\lim_{x\to 0}(g(f(x)))$ is...

[IIT-JEE, 1986]

11. $\lim_{x\to\infty}\left(\dfrac{x^4\sin\left(\frac{1}{4}\right)+x^2}{(1+|x|^3)}\right)=\ldots$ **[IIT-JEE, 1987]**

12. $\lim_{x\to 0}\left(\dfrac{\int_0^{x^2}\cos^2 t\,dt}{x\sin x}\right)=\ldots$ **[IIT-JEE, 1987]**

13. If $f(9)=9$, $f'(9)=4$, then

$\lim_{x\to 9}\left(\dfrac{\sqrt{f(x)}-3}{\sqrt{x}-3}\right)$ is equal to ... **[IIT-JEE, 1988]**

14. ABC is an isoceles triangle inscribed in a circle of radius r. If $AB = AC$ and h is the altitude from A to BC, then the triangle ABC has perimeter

$P=2\left(\sqrt{2hr-h^2}+\sqrt{2hr}\right)$ and area A is equal to ...

Also $\lim_{h\to 0}\left(\dfrac{A}{p^3}\right)=\ldots$ **[IIT-JEE, 1989]**

15. No questions asked in 1990.

16. The value of $\lim_{x\to 0}\left(\dfrac{\sqrt{\frac{1}{2}(1-\cos^2 x)}}{x}\right)$ is

(a) 1 (b) −1
(c) 0 (d) None

[IIT-JEE, 1991]

17. $\lim_{x\to\infty}\left(\dfrac{x+6}{x+1}\right)^{x+4}=\ldots$ **[IIT-JEE, 1991]**

18. $\lim_{x\to 0}\left(\dfrac{x}{\tan^{-1}2x}\right)$ equals **[IIT-JEE, 1992]**

19. Evaluate: $\lim_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{1/x}$. **[IIT-JEE, 1993]**

20. $\lim_{x\to 0}\left(\dfrac{5x^2+1}{3x^2+1}\right)^{\frac{1}{x^2}}=\ldots$ **[IIT-JEE, 1996]**

21. $\lim_{h\to 0}\left(\dfrac{\log(1+2h)-2\log(1+h)}{h^2}\right)=\ldots$

[IIT-JEE, 1997]

22. The value of $\lim_{n\to\infty}\left(\dfrac{1}{n}\sum_{r=1}^{2n}\dfrac{r}{\sqrt{n^2+r^2}}\right)$ is

(a) $1+\sqrt{5}$ (b) $-1+\sqrt{5}$
(c) $-1+\sqrt{2}$ (d) $1+\sqrt{2}$

[IIT-JEE, 1997]

23. $\lim_{x\to 1}\left(\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}\right)$

(a) exist and equals to $\sqrt{2}$
(b) exist and equals to $-\sqrt{2}$
(c) does not exist.
(d) None of these.

[IIT-JEE, 1998]

24. $\lim_{x\to 0}\left(\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}\right)$ equals to

(a) 2 (b) −2
(c) 1/2 (d) −1/2

[IIT-JEE, 1999]

25. For $x\in R$, $\lim_{x\to\infty}\left(\dfrac{x-3}{x+2}\right)^x$ is equal to

(a) e (b) $1/e$
(c) e^5 (d) e^{-5}

[IIT-JEE, 2000]

26. $\lim_{x\to 0}\left(\dfrac{\sin(\pi\cos^2 x}{x^2}\right)$ is equal to

(a) $-\pi$ (b) π
(c) $\dfrac{\pi}{2}$ (d) 1.

[IIT-JEE, 2001]

27. The integer n for which $\lim_{x\to 0}\left(\dfrac{(\cos x-1)(\cos x-e^x)}{x^n}\right)$ is a finite non-zero number is

(a) 1 (b) 2
(c) 3 (d) 4.

[IIT-JEE, 2002]

28. Let $f: R\to R$ be such that $f(1)=3$ and $f'(1)=6$, then $\lim_{x\to 0}\left(\dfrac{f(1+x)}{f(1)}\right)^{1/x}$ is

(a) 1 (b) $e^{1/2}$
(c) e^2 (d) e^3

[IIT-JEE, 2002]

29. $\lim_{h\to 0}\left(\dfrac{f(2+h+h^2)-f(2)}{f(h-h^2+1)-f(1)}\right)$

(given that $f'(2)=6$, $f'(1)=4$

(a) Does not exist (b) −3/2
(c) 3/2 (d) 3

[IIT-JEE, 2003]

30. If $\lim_{x\to 0}\left(\dfrac{[(a-n)nx-\tan x]\sin nx}{x^2}\right)=0$,

where n is non-zero real number, then a is

(a) 0 (b) $\frac{n}{n+1}$

(c) n (d) $n+\frac{1}{n}$

[IIT-JEE, 2003]

31. The value of $\lim_{x\to 0}\left(\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}\right)$ is

(a) 3 (b) 2

(c) 1 (d) 0

[IIT-JEE, 2003]

32. If $f(x)$ is differentiable and strictly increasing function, then the value of $\lim_{x\to 0}\left(\frac{f(x^2)-f(x)}{f(x)-0}\right)$ is

(a) 1 (b) 0

(c) –1 (d) 2.

[IIT-JEE, 2004]

33. Find the value of

$\lim_{n\to\infty}\left(\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n\right)$

[IIT-JEE, 2004]

34. No questions asked in 2005.

35. For $x > 0$, $\lim_{x\to 0}\left((\sin x)^{\frac{1}{x}}+\left(\frac{1}{x}\right)^{\sin x}\right)$ is

(a) 0 (b) –1

(c) 1 (d) 2

[IIT-JEE, 2006]

36. The value of $\lim_{x\to 0}\left(\frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2-\frac{\pi^2}{16}}\right)$ is equal to

(a) $\frac{8}{\pi}f(2)$ (b) $\frac{2}{\pi}f(2)$

(c) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ (d) $4f(2)$.

[IIT-JEE, 2007]

37. No questions asked in 2008.

38. Let $L=\lim_{x\to 0}\left(\frac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}\right)$, $a > 0$

If L is finite, then

(a) $a = 2$ (b) $a = 1$

(c) $L=\frac{1}{64}$ (d) $\frac{1}{32}$

[IIT-JEE, 2009]

39. The value of $\lim_{x\to 0}\frac{1}{x^3}\int_0^x\left(\frac{t\log(1+t)}{t^4+4}\right)dt$ is

(a) 0 (b) 1/12

(c) 1/24 (d) 1/64

[IIT-JEE, 2010]

40. If $\lim_{x\to 0}(1+x\log(1+b^2))^{\frac{1}{x}}=2b\tan^2\theta$, $b > 0$ and $\theta\in(-\pi,\pi)$, then the value of θ is

(a) $\pm\frac{\pi}{4}$ (b) $\pm\frac{\pi}{3}$

(c) $\pm\frac{\pi}{6}$ (d) $\pm\frac{\pi}{2}$

[IIT-JEE, 2011]

41. If $\lim_{x\to 0}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$, then

(a) $a = 1, b = 4$ (b) $a = 1, b = -4$

(c) $a = 2, b = -3$ (d) $a = 2, b = 3$

[IIT-JEE, 2012]

42. If

$$\lim_{n\to\infty}\left(\frac{1^a+2^a+\ldots+n^a}{(n+1)^{a-1}((na+1)+(na+2)+\ldots+(na+n))}\right)$$

$=\frac{1}{60}$, where $a\in R$, then the value of a is

(a) 7 (b) 1/7

(c) –7 (d) 1/5

[JEE Advanced, 2013]

43. The largest value of the non-negative integer a for which

$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$ is ...

[JEE-Advanced, 2014]

44. Let $f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all x in R

and $g(x)=\left(\frac{\pi}{2}\sin x\right)$, $\forall x\in R$, then $\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right)$ is ...

[JEE-Advanced, 2015]

45. Let m and n be two positive integers greater than 1.

If $\lim_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)-e}}{\alpha^m}\right)=-\left(\frac{e}{2}\right)$

then the value of $\left(\frac{m}{n}\right)$ is ... **[JEE-Advanced, 2015]**

46. Let α, β, $\in R$ be such that

$\lim_{x\to 0}\left(\frac{x^2\sin(\beta x)}{\alpha x-\sin x}\right)=1$. Then $6(\alpha+\beta)$ is ....

[JEE-Advanced, 2016]

Answers

LEVEL II

1. (c)	2. (b)	3. (d)	4. (c)	5. (d)
6. (a)	7. (b)	8. (a)	9. (c)	10. (b)
11. (d)	12. (b)	13. (b)	14. (c)	15. (c)
16. (b)	17. (a,b,c)	18. (a)	19. (b)	20. (a)
21. (c)	22. (b)	23. (a)	24. (c)	25. (b)
26. (c)	27. (a,c)	28. (c)	29. (c)	30. (b)
31. (b)	32. (a)	33. (a)	34. (c)	35. (a)
36. (c)	37. (c)	38. (d)	39. (a)	40. (c)
41. (a)	42. (a)	43. (a)	44. (b)	45. (b)
46. (d)	47. (b)	48. (b)	49. (b)	50. (d)
51. (a)	52. (b)	53. (b)	54. (b)	55. (b)
56. (a)	57. (a)	58. (a)	59. (c)	60. (b)
61. (b)	62. (a)	63. (a)	64. (b)	65. (a)
66. (*)	67. (c)	68. (a)	69. (a)	70. (c)
71. (a)	72. (b)	73. (d)	74. (b)	75. (a)

INTEGER TYPE QUESTIONS

1. 5	2. 6	3. 8	4. 2	5. 3
6. 4	7. 7	8. 3	9. 5	10. 7
11. 5	12. 3			

COMPREHENSIVE LINK PASSAGES

Passage I :	1. (a)	2. (b)	3. (b)
Passage II :	1. (a)	2. (a)	3. (b)
Passage III :	1. (a)	2. (d)	3. (b)

MATRIX MATCH

1. (A)→(P), (B)→(S), (C)→(Q), (D)→(R)
2. (A)→(Q), (B)→(A), (C)→(S), (D)→(Q)
3. (A)→(S), (B)→(R), (C)→(Q), (D)→(P)
4. (A)→(Q), (B)→(R), (C)→(S), (D)→(P)
5. (A)→(Q), (B)→(R), (C)→(S), (D)→(P)
6. (A)→(R), (B)→(S), (C)→(P), (D)→(Q)

ASSERTION AND REASON

1. A	2. A	3. B	4. B	5. B

Hints and Solutions

Level I

1. $\lim_{x\to 1}(x^2 - 6x + 10)$

$$= 1 - 6 + 10$$
$$= 11 - 6$$
$$= 5$$

2. $\lim_{x\to 1}(x^{2016} - x^{2017} + 2)$

$$= 1 - 1 + 2$$
$$= 2$$

3. $\lim_{x\to 1}\left(\frac{x^2 - 3x + 2}{x^2 - 5x + 4}\right)$

$$= \lim_{x\to 1}\left(\frac{(x-1)(x-2)}{(x-1)(x-4)}\right)$$
$$= \lim_{x\to 1}\left(\frac{(x-2)}{(x-4)}\right)$$
$$= \left(\frac{1-2}{1-4}\right) = \frac{-1}{-3} = \frac{1}{3}$$

4. $\lim_{x\to 1}\left(\frac{1 - x^{-\frac{1}{3}}}{1 - x^{-\frac{2}{3}}}\right)$

$$= \lim_{x\to 1}\left(\frac{1 - x^{-\frac{1}{3}}}{(1)^2 - \left(x^{-\frac{1}{3}}\right)^2}\right)$$
$$= \lim_{x\to 1}\left(\frac{\left(1 - x^{-\frac{1}{3}}\right)}{\left(1 + x^{-\frac{1}{3}}\right)\left(1 - x^{-\frac{1}{3}}\right)}\right)$$
$$= \lim_{x\to 1}\left(\frac{1}{\left(1 + x^{-\frac{1}{3}}\right)}\right) = \frac{1}{2}$$

5. $\lim_{x\to 2}\left(\frac{x^3 + 7x^2 - 36}{x^2 + 2x - 8}\right)$

$$= \lim_{x\to 2}\left(\frac{(x-2)(x+3)(x+6)}{(x+4)(x-2)}\right)$$
$$= \lim_{x\to 2}\left(\frac{(x+3)(x+6)}{(x+4)}\right)$$

$$= \frac{4 \times 7}{5} = \frac{28}{5}$$

6. $\lim_{x \to 1}\left(\frac{x^2 + x\log x - \log x - 1}{x^3 - 1}\right)$

$$= \lim_{x \to 1}\left(\frac{(x^2 - 1) + (x - 1)\log x}{x^3 - 1}\right)$$

$$= \lim_{x \to 1}\left(\frac{(x + 1) + \log x}{(x^2 + x + 1)}\right)$$

$$= \frac{2}{3}$$

7. $\lim_{x \to 0}\left(\frac{x}{\sqrt{a + x} - \sqrt{a - x}}\right)$

$$= \lim_{x \to 0}\left(\frac{x}{\sqrt{a + x} - \sqrt{a - x}} \times \frac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} + \sqrt{a - x}}\right)$$

$$= \lim_{x \to 0}\left(\frac{x(\sqrt{a + x} + \sqrt{a - x})}{(a + x - a + x)}\right)$$

$$= \lim_{x \to 0}\left(\frac{x(\sqrt{a + x} + \sqrt{a - x})}{2x}\right)$$

$$= \lim_{x \to 0}\left(\frac{(\sqrt{a + x} + \sqrt{a - x})}{2}\right)$$

$$= \lim_{x \to 0}\left(\frac{\sqrt{a} + \sqrt{a}}{2}\right) = \frac{2\sqrt{a}}{2} = \sqrt{a}$$

8. $\lim_{x \to 0}\left(\frac{\sqrt{1 + x} - 1}{x}\right)$

$$= \lim_{x \to 0}\left(\frac{\sqrt{1 + x} - 1}{x} \times \frac{\sqrt{1 + x} + 1}{\sqrt{1 + x} + 1}\right)$$

$$= \lim_{x \to 0}\left(\frac{1 + x - 1}{x(\sqrt{1 + x} + 1)}\right)$$

$$= \lim_{x \to 0}\left(\frac{x}{x(\sqrt{1 + x} + 1)}\right)$$

$$= \lim_{x \to 0}\left(\frac{1}{(\sqrt{1 + x} + 1)}\right)$$

$$= \frac{1}{2}$$

9. $\lim_{x \to 0}\frac{3 - \sqrt{5 + x}}{1 - \sqrt{5 - x}}$

$$= \left(\frac{3 - \sqrt{5}}{1 - \sqrt{5}}\right)$$

10. $\lim_{x \to 0}\left(\frac{\sqrt{1 + x} - \sqrt{1 - x}}{2x}\right)$

$$= \lim_{x \to 0}\frac{\sqrt{1 + x} - \sqrt{1 - x}}{2x} \times \frac{\sqrt{1 + x} + \sqrt{1 - x}}{\sqrt{1 + x} + \sqrt{1 - x}}$$

$$= \lim_{x \to 0}\left(\frac{1 + x - 1 + x}{2x(\sqrt{1 + x} + \sqrt{1 - x})}\right)$$

$$= \lim_{x \to 0}\left(\frac{2x}{2x(\sqrt{1 + x} + (\sqrt{1 - x})}\right)$$

$$= \lim_{x \to 0}\left(\frac{1}{(\sqrt{1 + x} + \sqrt{1 - x})}\right)$$

$$= \frac{1}{2}$$

11. $\lim_{x \to 2}\left(\frac{\sqrt{1 + 4x} - \sqrt{5 + 2x}}{x - 2}\right)$

$$= \lim_{x \to 2}\left(\frac{\sqrt{1 + 4x} - \sqrt{5 + 2x}}{x - 2} \times \frac{\sqrt{1 + 4x} + \sqrt{5 + 2x}}{\sqrt{1 + 4x} + \sqrt{5 + 2x}}\right)$$

$$= \lim_{x \to 2}\left(\frac{1 + 4x - 5 - 2x}{(x - 2)(\sqrt{1 + 4x} + \sqrt{5 + 2x})}\right)$$

$$= \lim_{x \to 2}\left(\frac{2x - 4}{(x - 2)(\sqrt{1 + 4x} + \sqrt{5 + 2x})}\right)$$

$$= \lim_{x \to 2}\left(\frac{2(x - 2)}{(x - 2)(\sqrt{1 + 4x} + \sqrt{5 + 2x})}\right)$$

$$= \lim_{x \to 2}\left(\frac{2}{\sqrt{1 + 4x} + \sqrt{5 + 2x}}\right)$$

$$= \frac{2}{\sqrt{9} + \sqrt{9}} = \frac{2}{6} = \frac{1}{3}$$

12. $\lim_{x \to \sqrt{10}}\left(\frac{\sqrt{7 + 2x} - (\sqrt{5} + \sqrt{2})}{x^2 - 10}\right)$

$$= \lim_{x \to \sqrt{10}}\left(\frac{\sqrt{7 + 2x} - (\sqrt{5} + \sqrt{2})}{x^2 - 10} \times \frac{\sqrt{7 + 2x} + (\sqrt{5} + \sqrt{2})}{\sqrt{7 + 2x} + (\sqrt{5} + \sqrt{2})}\right)$$

$$= \lim_{x \to \sqrt{10}}\left(\frac{7 + 2x - (\sqrt{5} + \sqrt{2})^2}{(x - \sqrt{10})(x + \sqrt{10})\sqrt{7 + 2x} + (\sqrt{5} + \sqrt{2})}\right)$$

$$= \lim_{x \to \sqrt{10}}\left(\frac{7 + 2x - (5 + 2 + 2\sqrt{10})}{(x - \sqrt{10})(x + \sqrt{10})(\sqrt{7} + 2x + (\sqrt{5} + \sqrt{2})}\right)$$

$$= \lim_{x \to \sqrt{10}}\left(\frac{2(x - \sqrt{10})}{(x - \sqrt{10})(x + \sqrt{10})(\sqrt{7} + 2x + (\sqrt{5} + \sqrt{2}))}\right)$$

$$= \lim_{x \to \sqrt{10}}\left(\frac{2}{(x + \sqrt{10})(\sqrt{7 + 2x} + (\sqrt{5} + \sqrt{2}))}\right)$$

$$= \left(\frac{2}{2\sqrt{10}(\sqrt{7+2\sqrt{10}} + (\sqrt{5}+\sqrt{2}))}\right)$$

$$= \left(\frac{1}{(\sqrt{10}((\sqrt{5}+\sqrt{2}) + (\sqrt{5}+\sqrt{2}))}\right)$$

$$= \left(\frac{1}{2\sqrt{10}(\sqrt{5}+\sqrt{2})}\right)$$

$$= \left(\frac{1}{2(\sqrt{50}+\sqrt{20})}\right)$$

13. $\lim_{x\to a}\left(\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right)$

$$= \lim_{x\to a}\left(\frac{(a+2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(3a+x-4x)\times(\sqrt{a+2x}+\sqrt{3x})}\right)$$

$$= \lim_{x\to a}\left(\frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{3(a-x)\times(\sqrt{a+2x}+\sqrt{3x})}\right)$$

$$= \lim_{x\to a}\left(\frac{(\sqrt{3a+x}+2\sqrt{x})}{3(\sqrt{a+2x}+\sqrt{3x})}\right)$$

$$= \frac{1}{3}\left(\frac{2\sqrt{a}+2\sqrt{a}}{(\sqrt{3a}+\sqrt{3a}}\right)$$

$$= \frac{1}{3}\left(\frac{4}{2\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$$

14. $\lim_{x\to -1} \frac{\sqrt[3]{7-x}-2}{(x+1)}$

$$= \lim_{x\to -1}\left(\frac{(7-x)^{1/3}-(8)^{1/3}}{(x+1)}\right)$$

$$= \lim_{x\to -1}\left(\frac{(7-x)^{1/3}-(8)^{1/3}}{(x+1)}\right.$$
$$\left.\times \frac{(7-x)^{2/3}+2(7-x)^{1/3}+8^{2/3}}{(7-x)^{2/3}+2(7-x)^{1/3}+8^{2/3})}\right)$$

$$= \lim_{x\to -1}\left(\frac{(7-x-8)}{(x+1)}\right.$$
$$\left.\times \frac{1}{(7-x)^{2/3}+2(7-x)^{1/3}+8^{2/3}}\right)$$

$$= \lim_{x\to -1}\left(\frac{-(1+x)}{(x+1)}\times\frac{1}{(7-x)^{2/3}+2(7-x)^{1/3}+8^{2/3}}\right)$$

$$= \lim_{x\to -1}\left(\frac{-1}{(7-x)^{2/3}+2(7-x)^{1/3}+8^{2/3}}\right)$$

$$= \left(\frac{-1}{(8)^{2/3}+2(8)^{1/3}+8^{2/3}}\right)$$

$$= \frac{-1}{4+4+4} = \frac{-1}{12}$$

15. $\lim_{x\to 1}\left(\frac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$

$$= \lim_{x\to 1}\left(\frac{(x^2+8-10+x^2)(\sqrt{x^2+3}+\sqrt{5-x^2}}{(x^2+3-5+x^2)(\sqrt{x^2+8}+\sqrt{10-x^2})}\right)$$

$$= \lim_{x\to 1}\left(\frac{(2x^2-2)(\sqrt{x^2+3}+\sqrt{5-x^2}}{(2x^2-2)(\sqrt{x^2+8}+\sqrt{10-x^2}}\right)$$

$$= \lim_{x\to 1}\left(\frac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})}\right)$$

$$= \frac{2+2}{3+3} = \frac{4}{6} = \frac{2}{3}$$

16. $\lim_{x\to a}\left(\frac{x\sqrt{x}-a\sqrt{a}}{x-a}\right)$

$$= \lim_{x\to a}\left(\frac{x^{2/3}-a^{3/2}}{x-a}\right)$$

$$= \frac{3}{2}(a)^{\frac{3}{2}-1}$$

$$= \frac{3\sqrt{a}}{2}$$

17. $\lim_{x\to a}\left(\frac{(x+2)^{\frac{5}{3}}-(a+2)^{\frac{5}{3}}}{x-a}\right)$

$$= \lim_{x+2\to a+2}\left(\frac{(x+2)^{\frac{5}{3}}-(a+2)^{\frac{5}{3}}}{(x+2)-(a+2)}\right)$$

$$= \frac{5}{3}(a+2)^{\frac{5}{3}-1}$$

$$= \frac{5}{3}(a+2)^{\frac{2}{3}}$$

18. $\lim_{x\to a}\left(\frac{x^m-a^m}{x^n-a^n}\right)$

$$= \lim_{x\to a}\left(\frac{\frac{x^m-a^m}{x-a}}{\frac{x^n-a^n}{x-a}}\right)$$

$$= \frac{ma^{m-1}}{na^{n-1}} = \left(\frac{m}{n}\right)a^{m-n}$$

19. $\lim_{x\to 0}\left(\frac{(1-x)^n-1}{x}\right)$

$= \lim_{1-x\to 1}\left(\frac{(1-x)^n-1}{(1-x)-1}\right)$

$= n(1)^{n-1} = n$

20. $\lim_{x\to 1}\left(\frac{(x+x^2+x^3+\ldots+x^n)-n}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{(x-1)+(x^2-1)+(x^3-1)+\ldots+(x^n-1)}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{(x-1)}{(x-1)}+\frac{(x^2-1)}{(x-1)}+\frac{(x^3-1)}{(x-1)}+\ldots+\frac{(x^n-1)}{x-1)}\right)$

$= (1+2+3+4+\ldots+n)$

$= \frac{n(n+1)}{2}$

21. $\lim_{x\to 1}\left(\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}\right)$

$= \lim_{x\to 1}\left(\frac{x^{n+1}-x-nx+n}{(x-1)^2}\right)$

$= \lim_{x\to 1}\left(\frac{x(x^n-1)-n(x-1)}{(x-1)^2}\right)$

$= \lim_{x\to 1}\left(\frac{x(x-1)(1+x+x^2+x^3+\ldots+x^{n-1})-n(x-1)}{(x-1)^2}\right)$

$= \lim_{x\to 1}\left(\frac{x(1+x+x^2+x^3+\ldots+x^{n-1})-n}{(x-1)}\right)$

$= \lim_{x\to 1}\left(\frac{(x+x^2+x^3+\ldots.+x^n)-n}{(x-1)}\right)$

$= \frac{n(n+1)}{2}$

22. $\lim_{x\to 0}\left(\frac{\sqrt[3]{1+x^2}-\sqrt[4]{1-2x}}{x+x^2}\right)$

$= \lim_{x\to 0}\left(\frac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\{(1+x^2)^{1/3}-1\}-\{(1-2x)^{1/4}-1\}}{x+x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\{(1+x^2)^{1/3}-1\}}{x+x^2}\right)-\lim_{x\to 0}\left(\frac{\{(1-2x)^{1/4}-1\}}{x+x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\{(1+x^2)^{1/3}-1\}}{x^2}\times\frac{x^2}{x+x^2}\right)$

$-\lim_{x\to 0}\left(\frac{\{(1-2x)^{1/4}-1\}}{-2x}\times\frac{-2x}{x+x^2}\right)$

$= \frac{1}{3}\times 0+\frac{1}{4}\times 2 = \frac{1}{2}$

23. $\lim_{x\to 1}\left(\frac{\sqrt[3]{7+x^3}-\sqrt[2]{3+x^2}}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{(7+x^3)^{1/3}-(3+x^2)^{1/2}}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{\{(7+x^3)^{1/3}-2\}-\{(3+x^2)^{1/2}-2\}}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{\{7+x^3)^{1/3}-(8)^{1/3}\}}{x-1}\right)$

$-\lim_{x\to 1}\left(\frac{\{(3+x^2)^{1/2}-(4)^{1/2}\}}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{\{(7+x^3-8)\}}{(x-1)\{(7+x)^{2/3}+8^{2/3}+8^{1/3}(7+x)^{1/3}\}}\right)$

$-\lim_{x\to 1}\left(\frac{\{(3+x^2-4)\}}{(x-1)(\sqrt{3+x^2}+4)}\right)$

$= \lim_{x\to 1}\left(\frac{(x^2+x+1)}{\{(7+x)^{2/3}+8^{2/3}+8^{1/3}(7+x)^{1/3}\}}\right)$

$-\lim_{x\to 1}\left(\frac{(x+1)}{(\sqrt{3+x^2}+2)}\right)$

$= \frac{3}{3.8^{2/3}}+\frac{1}{2}$

$= \frac{1}{4}+\frac{1}{2}$

$= \frac{3}{4}$

24. $\lim_{x\to 1}\left(\frac{\sqrt[3]{x}+\sqrt[4]{x}+\sqrt[5]{x}-3}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{(\sqrt[3]{x}-1)+(\sqrt[4]{x}-1)+(\sqrt[5]{x}-1)}{x-1}\right)$

$= \frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{20+15+12}{60} = \frac{47}{60}$

25. $\lim_{x\to 1}\left(\frac{\left(\sum_{k=1}^{100} x^k\right)-100}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{(x+x^2+x^3+\ldots+x^{100})-100}{x-1}\right)$

$= \lim_{x\to 1}\left(\frac{(x-1)+(x^2-1)+(x^3-1)+\ldots+(x^{100}-1)}{(x-1)}\right)$

$= \lim_{x\to 1}\left(\frac{(x-1)}{(x-1)}+\frac{(x^2-1)}{(x-1)}+\frac{(x^3-1)}{(x-1)}+\ldots+\frac{(x^{100}-1)}{(x-1)}\right)$

$= (1+2+3+\ldots.+100)$

$= \frac{100\times 101}{2} = 5050$

26. $\lim_{n\to\infty}(4^n+5^n)^{\frac{1}{n}}$

$= \lim_{n\to\infty}\left(5^n\left(1+\left(\frac{4}{5}\right)^n\right)\right)^{\frac{1}{n}}$

$= 5.$

27. $\lim_{n\to\infty}\left(\frac{n^2}{1+2+3+\ldots+n}\right)$

$= \lim_{n\to\infty}\left(\frac{2n^2}{n(n+1)}\right)$

$= \lim_{n\to\infty}\left(\frac{2}{(1+1/n)}\right)$

$= \lim_{n\to\infty}\left(\frac{2}{(1+1/n)}\right) = 2$

28. $\lim_{n\to\infty}\left(\frac{x^n+y^n}{x^n-y^n}\right)$, where $0<x<y$.

$= \lim_{n\to\infty}\left(\frac{\left(\frac{x}{y}\right)^n+1}{\left(\frac{x}{y}\right)^n-1}\right) = \left(\frac{0+1}{0-1}\right) = -1$

29. $\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$

$= \lim_{x\to\infty}\left(\frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}\right)$

$= \lim_{x\to\infty}\left(\frac{2}{\sqrt{1+\frac{2}{x}}+1}\right) = \frac{2}{2} = 1$

30. $\lim_{x\to\infty}\left(\sqrt{x^2+8x}+x\right)$

$= \lim_{y\to\infty}\left(\sqrt{y^2-8y}+y\right)$,

where $x=-y$, $x\to-\infty$, $y\to\infty$

$= \lim_{y\to\infty}\left(\frac{y^2-8y-y^2}{\sqrt{y^2-8y}+y}\right)$

$= \lim_{y\to\infty}\left(\frac{-8y}{\sqrt{y^2-8y}+y}\right)$

$= \lim_{y\to\infty}\left(\frac{-8}{\sqrt{1-\frac{8}{y}}+1}\right) = -4$

31. $\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+5}{2x-7}}$

$= \frac{1}{2}$, where $\lim_{x\to\infty}\left(\frac{2x+5}{2x-7}\right) = 1$

and $\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right) = \frac{1}{2}$

32. $\lim_{x\to\infty}\left(\frac{(x+1)^{10}+(x+2)^{10}+\ldots+(x+2012)^{10}}{x^{10}+2012^{10}}\right)$

$= \lim_{x\to\infty}\left(\frac{(1+1/x)^{10}+(1+2/x)^{10}+\ldots+(1+2012/x)^{10}}{1+(2012/x)^{10}}\right)$

$= 2012$

33. Let $t_r = (n-r+1)\cdot r$

$= (nr-(r-1)r)$

$= ((n-1)(r-r^2)$

Now, $S_n = \sum_{r-1}^{n} t_r$

$= \sum_{r-1}^{n}((n-1)r-r^2)$

$= \sum_{r-1}^{n}(n-1)r - \sum_{r=1}^{n} r^2$

$= (n-1)\left(\frac{n(n+1)}{2}\right) - \frac{n(n+1)(2n+1)}{6}$

$= \left(\frac{n(n+1)}{2}\right)\left(n-1-\frac{2n+1}{3}\right)$

$= \left(\frac{n(n+1)(n-2)}{6}\right)$

Therefore, $\lim_{n\to\infty}\left(\frac{n(n+1)(n-2)}{6n^3}\right)$

$$= \lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{6}\right)$$

$$= \frac{1}{6}$$

34. Put $\frac{1}{n} = x$

$$\lim_{x\to 0}\left\{\frac{\sqrt{1-\cos x)\sqrt{(1-\cos x)\sqrt{(1-\cos x)\ldots\infty}}}}{x^2}\right\}$$

$$= \lim_{x\to 0}\left(\frac{(1-\cos)^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\infty}}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{(1-\cos x)}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2(x/2)}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2(x/2)}{(x^2/4)/4}\right)$$

$$= \frac{1}{2}$$

35. Given $\lim_{x\to\infty}\left(\frac{x^2-1}{x+1} - ax - b\right) = 2$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{x^2 - 1 - ax^2 - ax - bx - b}{x+1}\right) = 2$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{(1-a)x^2 - (a+b)x - (1+b)}{x+1}\right) = 2$$

It is possible only when $1 - a = 0 \Rightarrow a = 1$

Also, $\lim_{x\to\infty}\left(\frac{-(a+b)x-(1+b)}{x+1}\right) = 2$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{-(a+b)-\frac{(1+b)}{x}}{1+\frac{1}{x}}\right) = 2$$

$\Rightarrow -(a+b) = 2$

$\Rightarrow (a+b) = -2$

$\Rightarrow b = -a - 2 = -1 - 2 = -3$

36. Given $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1} - ax - b\right) = 0$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{x^2 + 1 - ax^2 - ax - bx - b}{x+1}\right) = 0$$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{(1-a)x^2 - (a+b)x + (1-b)}{x+1}\right) = 0$$

It is possible only when, $1 - a = 0$ and $(a + b) = 0$

Thus, $a = 1$ and $b = -1$

37. Given $\lim_{x\to 0}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$

$$\Rightarrow \lim_{x\to 0}\left(\frac{x^2 + x + 1 - ax^2 - ax - bx - b}{x+1}\right) = 4$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{(1-a)x^2 + (1-a-b)x + (1-b)}{x+1}\right) = 4$$

It is possible only when

$\Rightarrow (1 - a) = 0,\ (1 - a - b) = 4$

$\Rightarrow a = 1,\ a + b = -3$

$\Rightarrow a = 1$ and $b = -4$

38. We have

$$\lim_{x\to-\infty}\left(\frac{x^4\sin\left(\frac{1}{x}\right) + x^2}{1 + |x|^3}\right)$$

$$= \lim_{x\to-\infty}\left(\frac{x^4\sin\left(\frac{1}{x}\right) + x^2}{1 - x^3}\right)$$

$$= \lim_{x\to-\infty}\left(\frac{x\sin\left(\frac{1}{x}\right) + \frac{1}{x}}{\frac{1}{x^3} - 1}\right)$$

$$= \left(\frac{1+0}{0-1}\right)$$

$$= -1$$

39. Limit $= \lim_{x\to 0}\left(\frac{\sin 3x}{5x}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin 3x}{3x} \times \frac{3x}{5x}\right)$$

$$= \frac{3}{5}$$

40. Limit $= \lim_{x\to 0}\left(\frac{\sin ax}{\sin bx}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx}\right)$$

$$= \frac{a}{b}$$

41. Limit $= \lim_{x\to 0}\left(\frac{\sin 2x + 3x}{4x + \sin 6x}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin 2x}{x} + 3}{4 + \frac{\sin 6x}{x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin 2x}{2x} \times 2 + 3}{4 + \frac{\sin 6x}{6x} \times 6}\right)$$

$$= \left(\frac{2 + 3}{4 + 6}\right)$$

$$= \frac{1}{2}$$

42. Limit $= \lim_{x\to 0}\left(\frac{\sin 2x + \sin 5x}{\sin 4x + \sin 6x}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin 2x}{x} + \frac{\sin 5x}{x}}{\frac{\sin 4x}{x} + \frac{\sin 6x}{x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin 2x}{2x} \times 2 + \frac{\sin 5x}{5x} \times 5}{\frac{\sin 4x}{4x} \times 4 + \frac{\sin 6x}{6x} \times 6}\right)$$

$$= \left(\frac{2 + 5}{4 + 6}\right)$$

$$= \frac{7}{10}$$

43. Limit $= \lim_{x\to 0}\left(\frac{1 - \cos 4x}{1 - \cos 6x}\right)$

$$= \lim_{x\to 0}\left(\frac{2\sin^2 2x}{2\sin^2 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin^2 2x}{4x^2} \times 4x^2}{\frac{\sin^2 3x}{9x^2} \times 9x^2}\right)$$

$$= \frac{4}{9}$$

44. Limit $= \lim_{x\to 0}\left(\frac{\sqrt{1 - \cos 2x}}{x^2}\right)$

$$= \lim_{x\to 0}\left(\frac{\sqrt{2\sin^2 x}}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sqrt{2}|\sin x|}{x}\right)$$

$$= \begin{cases} \lim_{x\to 0^+}\left(\frac{\sqrt{2}\,|\sin x|}{x}\right) \\ \lim_{x\to 0^-}\left(\frac{\sqrt{2}\,|\sin x|}{x}\right) \end{cases}$$

$$= \begin{cases} \lim_{x\to 0^+}\left(\frac{\sqrt{2}(\sin x)}{x}\right) \\ \lim_{x\to 0^-}\left(\frac{\sqrt{2}(\sin x)}{x}\right) \end{cases}$$

$$= \begin{cases} \sqrt{2} \\ -\sqrt{2} \end{cases}$$

Since R.H.L $\neq$ L.H.L, so limit does not exist.

45. Limit $= \lim_{x\to 0}\left(\frac{\cos 7x - \cos 9x}{\cos 3x - \cos 5x}\right)$

$$= \lim_{x\to 0}\left(\frac{2\sin 8x \sin x}{2\sin 4x \sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin 8x}{\sin 4x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin 8x}{8x} \times 8x}{\frac{\sin 4x}{4x} \times 4x}\right)$$

$$= 2$$

46. Limit $= \lim_{x\to 0}\left(\frac{1 - \cos x\sqrt{\cos 2x}}{x^2}\right)$

$$= \lim_{x\to 0}\left(\frac{(1 - \cos x\sqrt{\cos 2x})(1 + \cos x\sqrt{\cos 2x})}{x^2(1 + \cos x\sqrt{\cos 2x})}\right)$$

$$= \lim_{x\to 0}\left(\frac{(1 - \cos^2 x(\cos 2x))}{x^2(1 + \cos x\sqrt{\cos 2x})}\right)$$

$$= \lim_{x\to 0}\left(\frac{(1 - \cos^2 x(2\cos^2 x - 1))}{x^2(1 + \cos x\sqrt{\cos 2x})}\right)$$

$$= \lim_{x\to 0}\left(\frac{(1 - 2\cos^4 x + \cos^2 x)}{x^2(1 + \cos x\sqrt{\cos 2x})}\right)$$

$$= \lim_{x\to 0}\left(-\frac{(2\cos^4 x - \cos^2 x - 1)}{x^2(1 + \cos x\sqrt{\cos 2x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x(2\cos^2 x + 1)}{x^2(1 + \cos x\sqrt{\cos 2x})}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x}{x^2} \times \frac{(2\cos^2 x + 1)}{(1 + \cos x\sqrt{\cos 2x})}\right)$$

$$= \frac{2}{3}$$

47. Limit $= \lim_{x\to 0}\left(\frac{\tan x - \sin x}{x^3}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin x(1 - \cos x)}{x^3\cos x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{x} \times \frac{2\sin^2\left(\frac{x}{2}\right)}{\frac{x^2}{4} \times 4\cos x}\right)$$

$$= \frac{1}{2}$$

48. We have $\cos x \cos 2x \cos 3x$

$$= \frac{1}{2}(2\cos 3x\cos x)\cos 2x$$

$$= \frac{1}{2}(\cos 4x + \cos 2x)\cos 2x$$

$$= \frac{1}{4}(2\cos 4x\cos 2x + 2\cos^2 2x)$$

$$= \frac{1}{4}(\cos 6x + \cos 2x + 1 + \cos 4x)$$

$$= \frac{1}{4}(1 + \cos 2x + \cos 4x + \cos 6x)$$

Thus

Limit $= \lim_{x\to 0}\left(\frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}\right)$

$$= \lim_{x\to 0}\left(\frac{1 - \frac{1}{4}(1 + \cos 2x + \cos 4x + \cos 6x}{\sin^2 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{3 - (\cos 2x + \cos 4x + \cos 6x)}{4\sin^2 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{(1 - \cos 6x) + (1 - \cos 4x) + (1 - \cos 2x)}{4\sin^2 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2 3x + 2\sin^2 2x + 2\sin^2 x}{4\sin^2 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 3x + \sin^2 2x + \sin^2 x}{2\sin^2 2x}\right)$$

$$= \frac{9 + 4 + 1}{2 \times 4} = \frac{14}{8} = \frac{7}{4}$$

49. Limit $= \lim_{x\to 0}\frac{\sin^{-1}x - \tan^{-1}x}{x^3}$

$$= \lim_{x\to 0}\left(\frac{\tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right) - \tan^{-1}x}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\tan^{-1}\left(\dfrac{\frac{x}{\sqrt{1 - x^2}} - x}{1 + x\cdot\frac{x}{\sqrt{1 - x^2}}}\right)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\tan^{-1}\left(\frac{x - x\sqrt{1 - x^2}}{\sqrt{1 - x^2} + x^2}\right)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\tan^{-1}\left(\frac{x - x\sqrt{1 - x^2}}{\sqrt{1 - x^2} + x^2}\right)}{\left(\frac{x - x\sqrt{1 - x^2}}{\sqrt{1 - x^2} + x^2}\right)x^3} \times \left(\frac{x - x\sqrt{1 - x^2}}{\sqrt{1 - x^2} + x^2}\right)\right)$$

$$= \lim_{x\to 0}\left(\frac{x\left(1 - \sqrt{1 - x^2}\right)}{x^3\left(\sqrt{1 - x^2} + x^2\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{(1 - 1 + x^2)\left(\sqrt{1 - x^2} - x^2\right)}{x^2\left(1 + \sqrt{1 - x^2}\right)(1 - x^2 - x^2)}\right)$$

$$= \lim_{x\to 0}\left(\frac{(x^2)\left(\sqrt{1 - x^2} - x^2\right)}{x^2\left(1 + \sqrt{1 - x^2}\right)(1 - 2x^2)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(\sqrt{1 - x^2} - x^2\right)}{(1 + \sqrt{1 - x^2})(1 - 2x^2)}\right)$$

$$= \frac{1}{2}$$

50. Limit $= \lim_{x\to 0}\left(\frac{x - \sin x}{x^3}\right)$

$$= \lim_{x\to 0}\left(\frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7} - \dots\right)}{x^3}\right)$$

$$= \frac{1}{6}$$

51. Limit $= \lim_{x\to 0}\left(\frac{x - \tan x}{x^3}\right)$

$$= \lim_{x\to 0}\left(\frac{x - \left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\left(\frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\right)}{x^3}\right)$$

$$= -\frac{1}{3}$$

52. Limit $= \lim_{x\to \infty}\left(\frac{x + \sin x}{x + \cos x}\right)$

$$= \lim_{x\to \infty}\left(\frac{1 + \frac{\sin x}{x}}{1 + \frac{\cos x}{x}}\right)$$

$$= \left(\frac{1 + 0}{1 + 0}\right)$$

$$= 1$$

53. As we know that, $\sin x \le x$ and $\tan x \ge x$

Thus, $\frac{\sin x}{x} \le 1$ and $\frac{\tan x}{x} \ge 1$

Therefore,

$$\text{Limit} = \lim_{x\to 0}\left(\left[\frac{n\sin x}{x}\right] + \left[\frac{n\tan x}{x}\right]\right)$$

$$= (n - 1) + n$$

$$= 2n - 1.$$

54. We have $a = \min\{x^2 + 2x + 3,\ x^2 + 4x + 10\}$

$a = \min\{(x + 1)^2 + 2,\ (x + 2)^2 + 6\}$

$a = 2$

Also, $b = \lim_{x\to 0}\left(\frac{1 - \cos\theta}{\theta^2}\right)$

$$= \lim_{\theta\to 0}\left(\frac{2\sin^2\left(\frac{\theta}{2}\right)}{\left(\frac{\theta^2}{4}\right)\times 4}\right)$$

$$= \frac{1}{2}$$

Therefore, $\sum_{r=0}^{n}(a^r b^{n-r})$

$$= \sum_{r=0}^{n}\left(2^r\left(\frac{1}{2}\right)^{n-r}\right)$$

$$= \frac{1}{2^n}\sum_{r=0}^{n}(4^r)$$

$$= \frac{1}{2^n}(1 + 4 + 4^2 + \ldots + 4^n)$$

$$= \frac{1}{2^n}\left(\frac{4^{n+1} - 1}{3}\right)$$

$$= \left(\frac{4^{n+1} - 1}{3.2^n}\right)$$

55. Given $\lim_{x\to 0}\left(\frac{a\cos x + bx\sin x - 5}{x^4}\right)$

As $x\to 0$, $x^4\to 0$, the limit must be in the form of $\left(\frac{0}{0}\right)$

Thus, $\lim_{x\to 0}(a\cos x + bx\sin x - 5) = 0$

$\Rightarrow\quad a = 5 = 0$

$\Rightarrow\quad a = 5$

Now, Limit

$$= \lim_{x\to 0}\left(\frac{a\cos x + bx\sin x - 5}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{5\cos x + bx\sin x - 5}{x^4}\right)$$

$$= \lim_{x\to 0}\frac{5\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + bx\left(x - \frac{x^3}{3!} + \ldots\right) - 5}{x^4}$$

$$= \lim_{x\to 0}\frac{\left(b - \frac{5}{2}\right)x^2 + \left(\frac{5}{24} - \frac{b}{6}\right)x^4 + \ldots}{x^4}$$

It will provide finite limit only when, co-efficient of x^2 is zero.

Thus, $\left(b - \frac{5}{2}\right) = 0$

$\Rightarrow\quad b = \frac{5}{2}$

Also, the limit value

$$= \lim_{x\to 0}\frac{\left(\frac{5}{24} - \frac{b}{6}\right)x^4 + \ldots}{x^4}$$

$$= \left(\frac{5}{24} - \frac{b}{6}\right)$$

$$= \left(\frac{5}{24} - \frac{5}{12}\right)$$

$$= -\frac{5}{24}$$

Therefore, the value of $a + 2b + 10$

$= 5 + 5 + 10$

$= 20.$

56. Limit $= \lim_{x\to 0}\left(\frac{\sin(\pi\cos^2 x)}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\sin(\pi(1-\sin^2 x))}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\sin((\pi-\pi\sin^2 x))}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\sin(\pi\sin^2 x)}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\frac{\pi\sin^2 x}{x^2}\right)$

$= \pi$

57. Limit $= \lim_{x\to 0}\left(\frac{e^{4x}-1}{5x}\right)$

$= \lim_{x\to 0}\left(\frac{e^{4x}-1}{5x}\right)$

$= \lim_{x\to 0}\left(\frac{e^{4x}-1}{4x}\times\frac{4x}{5x}\right)$

$= \frac{4}{5}$

58. Limit $= \lim_{x\to 0}\left(\frac{e^{3x}-1}{e^{5x}-1}\right)$

$= \lim_{x\to 0}\left(\frac{\frac{e^{3x}-1}{3x}\times 3x}{\frac{e^{5x}-1}{5x}\times 5x}\right)$

$= \frac{3}{5}$

59. Limit $= \lim_{x\to 0}\left(\frac{e^x+e^{-x}-2}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{e^x+\frac{1}{e^x}-2}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{(e^x-1)^2}{e^x x^2}\right)$

$= \lim_{x\to 0}\left(\left(\frac{e^x-1}{x}\right)^2\times\frac{1}{e^x}\right)$

$= 1.$

60. Limit $= \lim_{x\to 0}\left(\frac{e^{3+x}-e^3}{x}-\frac{\sin x}{x}\right)$

$= \lim_{x\to 0}\left(\frac{e^3(e^x-1)}{x}-\frac{\sin x}{x}\right)$

$= (e^3-1)$

61. Limit $= \lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}$

$= \lim_{x\to 0} e^{\sin x}\left(\frac{e^{x-\sin x}-1}{x-\sin x}\right)$

$= e^0\times 1$

$= 1$

62. Limit $= \lim_{x\to 0}\left(\frac{e^x-e^{x\cos x}}{x+\sin x}\right)$

$= \lim_{x\to 0}\left(\frac{e^{x\cos x}(e^{x-x\cos x}-1)}{x+\sin x}\right)$

$= \lim_{x\to 0}\left(\frac{e^{x\cos x}(e^{x-x\cos x}-1)}{(x-x\cos x)}\times\frac{x(1-\cos x)}{x+\sin x}\right)$

$= \lim_{x\to 0}\left(\frac{e^{x\cos x}(e^{x-\cos x}-1)}{(x-x\cos x)}\times\frac{(1-\cos x)}{1+\frac{\sin x}{x}}\right)$

$= 0$

63. Limit $= \lim_{x\to 0}\left(\frac{e^x-1-x}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)-1-x}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)}{x^2}\right)$

$= \frac{1}{2}$

64. Limit $= \lim_{x\to 0}\frac{8^x-4^x-2^x+1}{x^2}$

$= \lim_{x\to 0}\left(\frac{4^x(2^x-1)-(2^x-1)}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{(4^x-1)(2^x-1)}{x^2}\right)$

$= \lim_{x\to 0}\left(\left(\frac{4^x-1}{x}\right)\left(\frac{2^x-1}{x}\right)\right)$

$= \log 4\times\log 2$

$= 2(\log 2)^2$

65. $\text{Limit} = \lim_{x\to 0} \dfrac{9^x - 2.6^x + 4^x}{x^2}$

$= \lim_{x\to 0}\left(\dfrac{(3^x)^2 - 2.3^x\cdot 2^x + (2^x)^2}{x^2}\right)$

$= \lim_{x\to 0}\left(\dfrac{(3^x - 2^x)^2}{x^2}\right)$

$= \lim_{x\to 0}\left(\dfrac{3^x - 2^x}{x}\right)^2$

$= \lim_{x\to 0}\left(\dfrac{3^x - 1}{x} - \dfrac{2^x - 1}{x}\right)^2$

$= (\log 3 - \log 2)^2$

$= \log^2\left(\dfrac{3}{2}\right)$

66. $\text{Limit} = \lim_{x\to a}\left(\dfrac{a^x - a^a}{x - a}\right),\ a > 0$

$= \lim_{x\to a}\left(\dfrac{a^a(a^{x-a} - 1)}{x - a}\right)$

$= (a^a\cdot\log a)$

67. $\text{Limit} = \lim_{x\to 0}\left(\dfrac{a^{x+h} + a^{x-h} - 2a^x}{h^2}\right)$

$= \lim_{x\to 0}\left(\dfrac{a^x(a^h + a^{-h} - 2)}{h^2}\right)$

$= \lim_{x\to 0}\left(\dfrac{a^x(a^{2h} - 2a^h + 1)}{a^h\times h^2}\right)$

$= \lim_{h\to 0}\left(\dfrac{a^x}{a^h}\times\left(\dfrac{(a^{2h} - 2a^h + 1)}{h^2}\right)\right)$

$= \lim_{h\to 0}\left(\dfrac{a^x}{a^h}\times\left(\dfrac{a^h - 1}{h}\right)^2\right)$

$= a^x\log a$

68. $\text{Limit} = \lim_{x\to 0}\left(\dfrac{8^x - 7^x}{6^x - 5^x}\right)$

$= \lim_{x\to 0}\left(\dfrac{(8^x - 1) - (7^x - 1)}{(6^x - 1) - (5^x - 1)}\right)$

$= \lim_{x\to 0}\left(\dfrac{\left(\dfrac{8^x - 1}{x}\right) - \left(\dfrac{7^x - 1}{x}\right)}{\left(\dfrac{6^x - 1}{x}\right) - \left(\dfrac{5^x - 1}{x}\right)}\right)$

$= \left(\dfrac{\log 8 - \log 7}{\log 6 - \log 5}\right)$

$= \left(\dfrac{\log(8/7)}{\log(6/5)}\right)$

69. $\text{Limit} = \lim_{x\to 0}\left(\dfrac{(5^x - 1)(4^x - 1)}{(3^x - 1)(6^x - 1)}\right)$

$= \lim_{x\to 0}\dfrac{\left(\dfrac{5^x - 1}{x}\right)\left(\dfrac{4^x - 1}{x}\right)}{\left(\dfrac{3^x - 1}{x}\right)\left(\dfrac{6^x - 1}{x}\right)}$

$= \left(\dfrac{\log 5\cdot\log 4}{\log 3\cdot\log 6}\right)$

70. $\text{Limit} = \lim_{x\to 0}\left(\dfrac{x\cdot 2^x - x}{1 - \cos x}\right)$

$= \lim_{x\to 0}\left(\dfrac{x(2^x - 1)}{1 - \cos x}\right)$

$= \lim_{x\to 0}\left(\dfrac{x^2\left(\dfrac{2^x - 1}{x}\right)}{\dfrac{2\sin^2\left(\dfrac{x}{2}\right)}{\dfrac{x^2}{4}}\times\dfrac{x^2}{4}}\right)$

$= 2\log 2$

71. $\text{Limit} = \lim_{x\to 0}\left(\dfrac{e^x - e^{-x} - 2x}{x - \sin x}\right)$

$$= \lim_{x\to 0}\left(\frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right) - 2x}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(\frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) - \left(-\frac{x^3}{3!} + \frac{x^4}{4!} - \ldots\right)}{-\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)}\right)$$

$$= \left(\frac{\frac{1}{3!} + \frac{1}{3!}}{\frac{1}{3!}}\right)$$

$= 2$

72. $\text{Limit} = \lim_{x\to 0}\left(\dfrac{e^{x^3} - 1 - x^3}{\sin^6 2x}\right)$

$= \lim_{x\to 0}\left(\dfrac{e^{x^3} - 1 - x^3}{\left(\dfrac{\sin^6 2x}{(2x^6)}\times(2x^6)\right)}\right)$

$= \lim_{x\to 0}\left(\dfrac{e^{x^3} - 1 - x^3}{(2x)^6}\right)\ \left(\because \dfrac{\sin^6 2x}{(2x)^6} = 1\right)$

$$= \lim_{t\to 0}\left(\frac{e^t - 1 - t}{2^6 \cdot t^2}\right)$$

(put $x^3 = t$, when $x \to 0$, $t \to 0$)

$$= \lim_{t\to 0}\left(\frac{1}{2^6} \times \left(\frac{e^t - 1 - t}{t^2}\right)\right)$$

$$= \frac{1}{2^6} \times \frac{1}{2}$$

$$= \left(\frac{1}{128}\right)$$

73. Limit $= \lim_{x\to 2}\left(\frac{(\cos\alpha)^x + (\sin\alpha)^x - 1}{x - 2}\right)$

$$= \lim_{x\to 2}\left(\frac{(\cos\alpha)^x + (\sin\alpha)^x - (\cos^2\alpha + \sin^2\alpha)}{x - 2}\right)$$

$$= \lim_{x\to 2}\left(\frac{\left((\cos\alpha)^x - \cos^2\alpha)\right) + (\sin\alpha)^x - \sin^2\alpha}{x - 2}\right)$$

$$= \lim_{t\to 0}\left(\frac{\left((\cos\alpha)^{t+2} - \cos^2\alpha)\right) + (\sin\alpha)^{t+2} - \sin^2\alpha}{t}\right)$$

$$= \lim_{t\to 0}\left(\cos^2\alpha\left(\frac{(\cos\alpha)^t - 1}{t}\right) + \sin^2\alpha\left(\frac{(\sin\alpha)^t - 1}{t}\right)\right)$$

$$= (\cos^2\alpha\log(\cos\alpha) + \sin^2\alpha\log(\sin\alpha))$$

74. Limit $= \lim_{x\to 0}\left(\frac{e^{x\cos x} - 1 - x}{\sin(x^2)}\right)$

$$= \lim_{x\to 0}\left(\frac{e^{x\cos x} - 1)}{\sin(x^2)} - \frac{x}{(\sin(x^2)}\right)$$

75. We have $a = \min\{x^2 + 4x + 6,\ x^2 + 2x + 8\}$

$a = \min\{(x + 2)^2 + 2,\ (x + 1)^2 + 5\}$

$a = 2$

Also, $b = \lim_{x\to 0}\left(\frac{\sin x\cos x}{e^x - e^{-x}}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin x}{x} \times \cos x}{\left(\frac{e^x - 1}{x}\right) - \left(\frac{e^{-x} - 1}{-x}\right)}\right) = \frac{1}{2}$$

Thus, $\sum_{r=0}^{n} (a^r b^{n-r})$

$$= \sum_{r=0}^{n}\left(2^r\left(\frac{1}{2}\right)^{n-r}\right)$$

$$= \frac{1}{2^n}\sum_{r=0}^{n}(4^r)$$

$$= \frac{1}{2^n}\ (1 + 4 + 4^2 + \ldots + 4^n)$$

$$= \frac{1}{2^n}\left(\frac{4^{n+1} - 1}{3}\right)$$

$$= \left(\frac{4^{n+1} - 1}{3\cdot 2^n}\right)$$

76. We have $\lim_{x\to 0}\left(\frac{Ae^x - B\cos x + Ce^{-x}}{x\sin x}\right)$

$$= \lim_{x\to 0}\left(\frac{A\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right) - B\left(1-\frac{x^2}{2!}+\ldots\right) + C\left(1-x-\frac{x^2}{2!}+\ldots\right)}{x\left(x - \frac{x^3}{3!} + \ldots\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{(A - B + C) + (A - C)x + \left(\frac{A + B + C}{2}\right)x^2 + \ldots}{x\left(x - \frac{x^3}{3!} + \ldots\right)}\right)$$

It will provide finite limit only when, $A - B + C = 0$,

$$A - C = 0 \text{ and } \frac{A + B + C}{2} = 2$$

On solving, we get, $A = 1$, $B = 2$, $C = 1$

Hence, the value of $A + B + C + 10 = 14$.

77. Limit $= \lim_{x\to 0}\left(\frac{\log(1 + 3x)}{\sin 2x}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{\log(1 + 3x)}{3x} \times 3x}{\frac{\sin 2x}{2x} \times 2x}\right)$$

$$= \frac{3}{2}$$

78. Limit $= \lim_{x\to 0}\left(\frac{\log(1 + 3x)}{\log(1 - 2x)}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{\log(1 + 3x)}{3x} \times 3x}{\frac{\log(1 - 2x)}{-2x} \times -2x}\right)$$

$$= -\frac{3}{2}$$

79. Limit $= \lim_{x\to e}\left(\frac{\log x - 1}{x - e}\right)$

$$= \lim_{x\to e}\left(\frac{\log x - \log e}{x - e}\right)$$

$$= \lim_{y \to 0}\left(\frac{\log(y+e)-\log e}{y+e-e}\right)$$

$$= \lim_{y \to 0}\left(\frac{\log\left(\frac{y+e}{e}\right)}{y}\right)$$

$$= \lim_{y \to 0}\left(\frac{\log\left(1+\frac{y}{e}\right)}{\frac{y}{e}\times e}\right)$$

$$= \frac{1}{e}$$

80. Limit $= \lim_{x \to 5}\left(\frac{\log x - \log 5}{x-5}\right)$

$$= \lim_{y \to 0}\left(\frac{\log(y+5)-\log 5}{y+5-5}\right)$$

$$= \lim_{y \to 0}\left(\frac{\log\left(\frac{y+5}{5}\right)}{y}\right)$$

$$= \lim_{y \to 0}\left(\frac{\log\left(1+\frac{y}{5}\right)}{\frac{y}{5}\times 5}\right)$$

$$= \frac{1}{5}$$

81. Limit $= \lim_{x \to 5}\left(\frac{\log(x+5)-\log(5-x)}{x-5}\right)$

$$= \lim_{x \to 5}\left(\frac{(\log(x+5)-\log 5)-(\log(5-x)-\log 5)}{x-5}\right)$$

$$= \lim_{x \to 5}\left(\left(\frac{\log(x+5)-\log 5}{x-5}\right)-\left(\frac{\log(5-x)-\log 5}{x-5}\right)\right)$$

$$= \lim_{x \to 0}\left(\left(\frac{(\log(x+5)-\log 5}{x-5}\right)+\left(\frac{\log(5-x)-\log 5}{5-x}\right)\right)$$

$$= \frac{1}{5}+\frac{1}{5}=\frac{2}{5}$$

82. Limit $= \lim_{x \to 0}\left(\frac{e^x-\log(x+e)}{e^x-1}\right)$

$$= \lim_{x \to 0}\left(\frac{e^x-\log\left\{e\left(1+\frac{x}{e}\right)\right\}}{e^x-1}\right)$$

$$= \lim_{x \to 0}\left(\frac{e^x-1-\left\{\log\left(1+\frac{x}{e}\right)\right\}}{e^x-1}\right)$$

$$= \lim_{x \to 0}\left(\frac{\frac{e^x-1}{x}-\left\{\frac{\log\left(1+\frac{x}{e}\right)}{\frac{x}{e}\times e}\right\}}{\frac{e^x-1}{x}}\right)$$

$$= \left(1-\frac{1}{e}\right)$$

83. Limit $= \lim_{x \to 0}\left(\frac{\ln(\cos x)}{\sqrt[4]{1+x^2}-1}\right)$

$$= \lim_{x \to 0}\left(\frac{\log\left(1-2\sin^2\left(\frac{x}{2}\right)\right)}{-2\sin^2\left(\frac{x}{2}\right)}\times\frac{-2\sin^2\left(\frac{x}{2}\right)}{\sqrt[4]{x^2+1}-1}\right)$$

$$= \lim_{x \to 0}\left(\frac{-2\sin^2\left(\frac{x}{2}\right)}{\sqrt[4]{x^2+1}-1}\right)$$

$$= \lim_{x \to 0}\left(\frac{-2\sin^2\left(\frac{x}{2}\right)\times\left(\sqrt[4]{x^2+1}+1\right)}{\left(\sqrt[4]{x^2+1}-1\right)\left(\sqrt[4]{x^2+1}+1\right)}\right)$$

$$= \lim_{x \to 0}\left(\frac{-2\sin^2\left(\frac{x}{2}\right)\times\left(\sqrt[4]{x^2+1}+1\right)\left(\sqrt[2]{x^2+1}+1\right)}{\left(\sqrt[2]{x^2+1}+-1\right)\left(\sqrt[2]{x^2+1}+1\right)}\right)$$

$$= \lim_{x \to 0}\left(\frac{-2\sin^2\left(\frac{x}{2}\right)\times\left(\sqrt[4]{x^2+1}+1\right)\left(\sqrt[2]{x^2+1}+1\right)}{(x^2)}\right)$$

$$= \lim_{x \to 0}\left(\frac{-2\sin^2\left(\frac{x}{2}\right)\times\left(\sqrt[4]{x^2+1}+1\right)\left(\sqrt[2]{x^2+1}+1\right)}{\left(\frac{x^2}{4}\right)\times 4}\right)$$

$$= -2$$

84. Limit $= \lim_{x \to 0}\left(\frac{\ln(\cos(\sin x))}{x^2}\right)$

$$= \lim_{x \to 0}\left(\frac{\log\left(1-2\sin^2\left(\frac{\sin x}{2}\right)\right)}{-2\sin^2\left(\frac{\sin x}{2}\right)}\times\frac{-2\sin^2\left(\frac{\sin x}{2}\right)}{x^2}\right)$$

$$= \lim_{x \to 0}\left(\frac{-2\sin^2\left(\frac{\sin x}{2}\right)}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{-2\sin^2\left(\frac{\sin x}{2}\right)}{\left(\frac{\sin^2 x}{4}\right)} \times \frac{\sin^2 x}{x^2} \times \frac{1}{4}\right)$$

$$= -\frac{1}{2}$$

85. $\text{Limit} = \lim_{x\to\infty}\left(x - x^2\ln\left(1 + \frac{1}{x}\right)\right)$

$$= \lim_{y\to 0}\left(\frac{1}{y} - \frac{1}{y^2}\log(1+y)\right)$$

$$= \lim_{y\to 0}\left(\frac{1}{y} - \frac{1}{y^2}\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots\right)\right)$$

$$= \lim_{y\to 0}\left(-\left(-\frac{1}{2} + \frac{y}{3} - \frac{y^2}{4} + \ldots\right)\right)$$

$$= \frac{1}{2}$$

86. $\text{Limit} = \lim_{x\to 1}\left(\frac{x^x - 1}{x\ln x}\right)$

$$= \lim_{y\to 0}\left(\frac{(1+y)^{1+y} - 1}{(1+y)\log(1+y)}\right)$$

$$= \lim_{y\to 0}\left(\frac{1 + y(1+y) + y\,(1+y)\frac{y^2}{2!} + \ldots -1}{(1+y)\frac{\log(1+y)}{y} \times y}\right)$$

$$= \lim_{y\to 0}\left(\frac{y(1+y) + y(1+y)\frac{y^2}{2!} + \ldots}{(1+y)y}\right)$$

$$= \lim_{y\to 0}\left(1 + \frac{y^2}{2!} + (y-1)\frac{y^3}{3!} + \ldots\right)$$

$$= 1.$$

87. $\text{Limit} = \lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

$$= \lim_{y\to 0}\left(\frac{1}{\log(1+y)} - \frac{1}{y}\right)$$

$$= \lim_{y\to 0}\left(\frac{y - \log(1+y)}{y\log(1+y)}\right)$$

$$= \lim_{y\to 0}\left(\frac{y - \log(1+y)}{y^2 \times \frac{\log(1+y)}{y}}\right)$$

$$= \lim_{y\to 0}\left(\frac{y - \log(1+y)}{y^2}\right)$$

$$= \lim_{y\to 0}\left(\frac{y - \left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots\right)}{y^2}\right)$$

$$= \lim_{y\to 0}\left(-\left(-\frac{1}{2} + \frac{y}{3} - \frac{y^2}{4} + \ldots\right)\right)$$

$$= \frac{1}{2}$$

88. $\text{Limit} = \lim_{x\to \pi/4}\left(\frac{\ln\cot x}{1 - \cot x}\right)$

$$= \lim_{y\to 0}\left(\frac{\log\left(\cot\left(\frac{\pi}{4} + y\right)\right)}{1 - \cot\left(\frac{\pi}{4} + y\right)}\right)$$

$$= \lim_{y\to 0}\left(\frac{\log\left(\frac{\cot y - 1}{\cot y + 1}\right)}{1 - \left(\frac{\cot y - 1}{\cot y + 1}\right)}\right)$$

$$= \lim_{y\to 0}\left(\frac{\log\left(1 + \left(\frac{\cot y - 1}{\cot y + 1} - 1\right)\right)}{-\left(1 - \left(\frac{\cot y - 1}{\cot y + 1}\right)\right)}\right)$$

$$= -1$$

89. $\text{Limit} = \lim_{x\to 0}\left(\frac{e^x - 1 - x}{x^2}\right)$

$$= \lim_{x\to 0}\left(\frac{e^x - 1}{2x}\right)$$

$$= \frac{1}{2}$$

90. $\text{Limit} = \lim_{x\to 0}\left(\frac{e^{x^3} - 1 - x^3}{64x^6}\right)$

$$= \lim_{y\to 0}\left(\frac{e^y - 1 - y}{64\cdot(y^2)}\right)$$

$$= \lim_{y\to 0}\frac{1}{64}\left(\frac{e^y - 1 - y}{(y^2)}\right)$$

$$= \lim_{y\to 0}\frac{1}{64}\left(\frac{e^y - 1}{2y}\right)$$

$$= \frac{1}{64} \times \frac{1}{2}$$

$$= \frac{1}{128}$$

91. Limit $= \lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2x}{x - \sin x}\right)$

$= \lim_{x\to 0}\left(\frac{e^x + e^{-x} - 2}{1 - \cos x}\right)$

$= \lim_{x\to 0}\left(\frac{e^x - e^{-x}}{\sin x}\right)$

$= \lim_{x\to 0}\left(\frac{e^x + e^{-x}}{\cos x}\right)$

$= 2$

92. Limit $= \lim_{x\to e}\left(\frac{\ln x - 1}{x - e}\right)$

$= \lim_{x\to e}\left(\frac{\frac{1}{x}}{1}\right)$

$= \frac{1}{e}$

93. Limit $= \lim_{x\to 0}\left(\frac{x + \tan 2x}{x - \tan 2x}\right)$

$= \lim_{x\to 0}\left(\frac{1 + \frac{\tan 2x}{x}}{1 - \frac{\tan 2x}{x}}\right)$

$= \lim_{x\to 0}\left(\frac{1 + 2\cdot\frac{\tan 2x}{2x}}{1 - 2\cdot\frac{\tan 2x}{2x}}\right)$

$= \left(\frac{1+2}{1-2}\right) = -3$

94. Limit $= \lim_{x\to 0}\left(\frac{\sin x - x}{x^3}\right)$

$= \lim_{x\to 0}\left(\frac{\cos x - 1}{3x^2}\right)$

$= \lim_{x\to 0}\left(\frac{-\sin x}{6x}\right)$

$= -\frac{1}{6}$

95. Limit $= \lim_{x\to 0}\left(\frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2}\right)$

$= \lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2x}{\sin 2x - 2x}\right)$

$= \lim_{x\to 0}\left(\frac{e^x + e^{-x} - 2x}{2\cos 2x - 2}\right)$

$= \lim_{x\to 0}\left(\frac{e^x - e^{-x}}{-4\sin 2x}\right)$

$= \lim_{x\to 0}\left(\frac{e^x + e^{-x}}{-8\cos 2x}\right)$

$= -\frac{2}{8} = -\frac{1}{4}$

96. Limit $= \lim_{x\to 0}\left(\frac{xe^x - \log(1+x)}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{e^x + xe^x - \frac{1}{1+x}}{2x}\right)$

$= \lim_{x\to 0}\left(\frac{e^x + e^x + xe^x + \frac{1}{(1+x)^2}}{2}\right)$

$= \frac{3}{2}$

97. Limit $= \lim_{x\to \infty}\left(\frac{\ln x}{x}\right)$

$= \lim_{x\to \infty}\left(\frac{\frac{1}{x}}{1}\right)$

$= 0$

98. Limit $= \lim_{x\to \infty}\left(\frac{x}{e^x}\right)$

$= \lim_{x\to \infty}\left(\frac{1}{e^x}\right)$

$= 0$

99. Limit $= \lim_{x\to \infty}\left(\frac{x^{2014}}{e^x}\right)$

$= \lim_{x\to \infty}\left(\frac{2014\cdot 2013 \ldots 3\cdot 2\cdot 1}{e^x}\right)$

$= \lim_{x\to \infty}\left(\frac{(2014)!}{e^x}\right)$

$= 0$

100. Limit $= \lim_{x\to 0}\left(\frac{x^2 + 2\cos x - 2}{x\sin^3 x}\right)$

$= \lim_{x\to 0}\left(\frac{x^2 + 2\cos x - 2}{x^4} \times \frac{x^3}{\sin^3 x}\right)$

$= \lim_{x\to 0}\left(\frac{x^2 + 2\cos x - 2}{x^4}\right)$

$= \lim_{x\to 0}\left(\frac{2x - 2\sin x}{4x^3}\right)$

$= \lim_{x\to 0}\left(\frac{2 - 2\cos x}{12x^2}\right)$

$= \lim_{x\to 0}\left(\frac{2\sin x}{24x}\right)$

$= \frac{1}{12}$

101. Limit $= \lim_{x\to\frac{\pi}{2}} (\tan x \log_e \sin x)$

$= \lim_{x\to\pi/2}\left(\frac{\log_e \sin x}{\cot x}\right)$

$= \lim_{x\to\pi/2}\left(\frac{\cot x}{-\operatorname{cosec}^2 x}\right)$

$= 0$

102. Limit $= \lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$

$= \lim_{x\to\infty}\left(1 + \frac{5}{x+1}\right)^{\frac{x+1}{5}\times\frac{x+4}{x+1}\times 5}$

$= e^{\lim_{x\to\infty}\frac{1+\frac{4}{x}}{1+\frac{1}{x}}\times 5}$

$= e^5$

103. Limit $= \lim_{x\to\infty}\left(\frac{2x+4}{2x+3}\right)^{2x+10}$

$= \lim_{y\to\infty}\left(\frac{y+4}{y+3}\right)^{y+10}$

$= \lim_{y\to\infty}\left(1 + \frac{1}{y+3}\right)^{\frac{y+3}{1}\times\frac{y+10}{y+3}}$

$= e^{\lim_{y\to\infty}\frac{y+10}{y+3}}$

$= e$

104. Limit $= \lim_{x\to\infty}\left(\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)\right)^x$

$= \lim_{y\to 0} (\sin y + \cos y)^{\frac{1}{y}}$

$= \lim_{y\to 0} (1 + (\sin y + \cos y - 1))^{\frac{1}{y}}$

$= e^{\lim_{y\to 0}((\sin y + \cos y - 1))^{\frac{1}{y}}}$

$= e^{\lim_{y\to 0}\left(\frac{(\sin y + \cos y - 1)}{y}\right)}$

$= e^{\lim_{y\to 0}((\cos y - \sin y))}$

$= e$

105. Limit $= \lim_{x\to 0}\left(\frac{a^x + b^x + c^x}{3}\right)^{2/x}$

$= \lim_{x\to 0}\left(1 + \left(\frac{a^x + b^x + c^x}{3} - 1\right)\right)^{2/x}$

$= \lim_{x\to 0}\left(1 + \left(\frac{a^x + b^x + c^x - 3}{3}\right)\right)^{2/x}$

$= e^{\lim_{x\to 0}\left(\left(\frac{a^x + b^x + c^x - 3}{3}\right)\times\frac{2}{x}\right)}$

$= e^{\lim_{x\to 0}\left(\left(\frac{(a^x - 1) + (b^x - 1) + (c^x - 1)}{x}\right)\times\frac{2}{3}\right)}$

$= e^{\left(\frac{2}{3}\log(abc)\right)}$

$= (abc)^{2/3}$

106. Limit $= \lim_{x\to 1}\left(2 - \frac{1}{x}\right)^{\tan\left(\frac{\pi x}{2}\right)}$

$= \lim_{y\to 0}\left(2 - \frac{1}{1+y}\right)^{\tan\left((1+y)\frac{\pi}{2}\right)}$

$= \lim_{y\to 0}\left(2 - \frac{1}{1+y}\right)^{-\cot\left(\frac{\pi y}{2}\right)}$

$= \lim_{y\to 0}\left(1 + \left(1 - \frac{1}{1+y}\right)\right)^{-\cot\left(\frac{\pi r}{2}\right)}$

$= \lim_{y\to 0}\left(1 + \left(\frac{y}{1+y}\right)\right)^{-\cot\left(\frac{\pi y}{2}\right)}$

$= e^{\lim_{y\to 0}\left(\frac{y}{1+y}\right)\times -\cot\left(\frac{\pi y}{2}\right)}$

$= e^{\lim_{y\to 0}\left(\frac{-1}{1+y}\right)\times\frac{\left(\frac{\pi y}{2}\right)}{\tan\left(\frac{\pi y}{2}\right)}\times\frac{2}{\pi}}$

$= e^{-\frac{2}{\pi}}$

107. Limit $= \lim_{x\to 1}\left(\tan\left(\frac{\pi}{4} + \ln x\right)\right)^{\frac{1}{\ln x}}$

$= \lim_{x\to 1}\left(\frac{1 + \tan(\log x)}{1 - \tan(\log x)}\right)^{\frac{1}{\log x}}$

$= \lim_{x\to 1}\left(1 + \frac{2\tan(\log x)}{1 - \tan(\log x)}\right)^{\frac{1}{\log x}}$

$$= e^{\lim_{x\to 1}\left(\frac{2\tan(\log x)}{1-\tan(\log x)}\right)\times\frac{1}{\log x}}$$

$$= e^2$$

108. Limit $= \lim_{x\to 0}\left(\frac{\ln\left(\tan\left(\frac{\pi}{4}+2x\right)\right)}{\sin 3x}\right)$

$$= \lim_{x\to 0}\left(\frac{\log\left(\frac{1+\tan 2x}{1-\tan 2x}\right)}{\sin 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\log\left(1+\left(\frac{1+\tan 2x}{1-\tan 2x}-1\right)\right)}{\sin 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\log\left(1+\left(\frac{2\tan 2x}{1-\tan 2x}\right)\right)}{\left(\frac{2\tan 2x}{1-\tan 2x}\right)}\times\frac{\left(\frac{2\tan 2x}{1-\tan 2x}\right)}{\sin 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(\frac{2\tan 2x}{1-\tan 2x}\right)}{\sin 3x}\right)$$

$$= \lim_{x\to 0}\left(\left(\frac{\tan 2x}{2x}\right)\times\frac{3x}{\sin 3x}\times\frac{2x}{3x}\times 2\right)$$

$$= \frac{4}{3}.$$

109. Limit $= \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$

$$= \lim_{x\to 0}\left(1+\left(\frac{\tan x}{x}-1\right)\right)^{1/x}$$

$$= \lim_{x\to 0}\left(1+\left(\frac{\tan x - x}{x}\right)\right)^{1/x}$$

$$= e^{\lim_{x\to 0}\left(\frac{\tan x - x}{x}\right)\times\frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\left(\frac{\tan x - x}{x^2}\right)}$$

$$= e^{\lim_{x\to 0}\left(\frac{\sec^2 x - 1}{2x}\right)}$$

$$= e^{\lim_{x\to 0}\left(\frac{2\sec^2 x\tan x}{2}\right)}$$

$$= 1$$

110. Limit $= \lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$

$$= \lim_{x\to 0}\left(1+\left(\frac{1+\tan x}{1+\sin x}-1\right)\right)^{\operatorname{cosec} x}$$

$$= \lim_{x\to 0}\left(1+\left(\frac{\tan x - \sin x}{1+\sin x}-x\right)\right)^{\operatorname{cosec} x}$$

$$= e^{\lim_{x\to 0}\left(\left(\frac{\tan x - \sin x}{1+\sin x}\right)\times\frac{1}{\sin x}\right)}$$

$$= e^{\lim_{x\to 0}\left(\sin x\left(\frac{\sec x - 1}{1+\sin x}\right)\times\frac{1}{\sin x}\right)}$$

$$= e^{\lim_{x\to 0}\left(\frac{\sec x - 1}{1+\sin x}\right)}$$

$$= 1$$

111. Limit $= \lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n$

$$= \lim_{n\to\infty}\left(1+\left(\frac{\sqrt[n]{b}-1}{a}\right)\right)^n$$

$$= e^{\lim_{n\to\infty}\left(\left(\frac{\sqrt[n]{b}-1}{a}\right)\times n\right)}$$

$$= e^{\lim_{n\to\infty}\left(\left(\frac{b^{\frac{1}{n}}-1}{a}\right)\times n\right)}$$

$$= e^{\lim_{n\to\infty}\left(\frac{b^{\frac{1}{n}}-1}{\frac{1}{n}}\right)\times\frac{1}{a}}$$

$$= e^{\frac{1}{a}\log b}$$

$$= e^{\log b^{\frac{1}{a}}}$$

$$= b^{\frac{1}{a}}$$

112. We have, $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

$\Rightarrow \quad -x^2 \le x^2\sin\left(\frac{1}{x}\right) \le x^2$

Now, $\lim_{x\to 0}(x^2) = 0$

and $\lim_{x\to 0}(-x^2) = 0$

Thus, $\lim_{x\to 0}\left(x^2\sin\left(\frac{1}{x}\right)\right) = 0$

113. We have, $-1 \le \sin x \le 1$

$\Rightarrow \quad -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

Now, $\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0$

and $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$

Thus, $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right) = 0$

114. We have, $x - 1 < [x] \le x$

$\Rightarrow \quad \frac{x-1}{x} < \frac{[x]}{x} \le \frac{x}{x}$

$\Rightarrow \quad \left(1 - \frac{1}{x}\right) < \frac{[x]}{x} \le 1$

Now, $\quad \lim_{x\to\infty} \left(1 - \frac{1}{x}\right) = 1$

and $\quad \lim_{x\to\infty} (1) = 1$

Thus, $\quad \lim_{x\to\infty} \left(\frac{[x]}{x}\right) = 1$

115. We have, $x - 1 < [x] \le x$

$\Rightarrow \quad \frac{1}{x} \le \frac{1}{[x]} < \frac{1}{x-1}$

$\Rightarrow \quad \frac{x}{x} \le \frac{x}{[x]} < \frac{x}{x-1}$

$\Rightarrow \quad 1 \le \frac{x}{[x]} < \frac{x}{x-1}$

Now, $\quad \lim_{x\to\infty} (1) = 1$

and $\quad \lim_{x\to\infty} \left(\frac{x}{x-1}\right) = 1$

Thus, $\quad \lim_{x\to\infty} \left(\frac{x}{[x]}\right) = 1$

116. We have, $x - 1 < [x] \le x$

$2x - 1 < [2x] \le 2x$

$3x - 1 < [3x] \le 3x$

... ...

... ...

... ...

$nx - 1 < [nx] \le nx$

By adding we get,

$(1 + 2 + 3 \dots + n)x - n < ([x] + [2x] + [3x] + \dots + ([nx]) \le (1 + 2 + 3 + \dots + n)x - n$

$$\Rightarrow \quad \frac{(1 + 2 + 3 + \dots + n)x - n}{n^2}$$

$$< \frac{([x] + [2x] + [3x] + \dots + [nx])}{n^2}$$

$$\le \frac{(1 + 2 + 3 + \dots + n)x}{n^2}$$

Now, $\quad \lim_{x\to\infty} \frac{(1 + 2 + 3 + \dots + n)x - n}{n^2}$

$$= \lim_{x\to\infty} \left(\frac{n(n+1)x}{2n^2} - \frac{1}{n}\right)$$

$$= \lim_{x\to\infty} \left(\left(1 + \frac{1}{n}\right)\frac{x}{2} - \frac{1}{n}\right)$$

$$= \frac{x}{2}$$

and $\quad \lim_{x\to\infty} \frac{(1 + 2 + 3 + \dots + n)x}{n^2}$

$$= \lim_{x\to\infty} \left(\frac{n(n+1)x}{2n^2}\right)$$

$$= \lim_{x\to\infty} \left(\left(1 + \frac{1}{n}\right)\frac{x}{2}\right)$$

$$= \frac{x}{2}$$

Thus, $\quad \lim_{x\to\infty} \left(\frac{[x] + [2x] + [3x] + \dots + [nx])}{n^2}\right) = \frac{x}{2}$

117. Limit $= \lim_{x\to\infty} \left(\frac{x - \int_0^x \cos(t^2)\,dt}{x^3 - 6x}\right)$

$$= \lim_{x\to\infty} \left(\frac{1 - \cos(x^2)1}{3x^2 - 6}\right)$$

$$= \left(\frac{1-1}{0-6}\right)$$

$$= 0$$

118. Limit $= \lim_{x\to\infty} \left(\frac{\int_0^x \sqrt{4 + t^4}\,dt}{x^3}\right)$

$$= \lim_{x\to\infty} \left(\frac{\sqrt{4 + x^4}}{3x^2}\right)$$

$$= \lim_{x\to\infty} \left(\sqrt{\frac{4 + x^4}{9x^4}}\right)$$

$$= \lim_{x\to\infty} \left(\sqrt{\frac{4}{9x^4} + \frac{1}{9}}\right)$$

$$= \frac{1}{3}$$

119. Limit $= \lim_{x\to 1} \left(\frac{\int_x^{x^2} e^{-t^2}\,dt}{x-1}\right)$

$$= \lim_{x\to 0} \left(\frac{e^{-x4} \cdot 2x - e^{-x} \cdot 1}{1}\right)$$

$$= -1$$

120. Limit $= \lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}\right]$

$$= \lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{n+n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{n}{n+r}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{1+\left(\frac{r}{n}\right)}\right)$$

$$= \int_0^1 \frac{dt}{1+x}$$

$$= \log 2$$

121. Limit $= \lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\ldots+\frac{1}{2n}\right]$

$$= \lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+1^2}+\ldots+\frac{n}{n^2+n^2}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{n}{n^2+r^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{n^2}{n^2+r^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{1+\left(\frac{r}{n}\right)^2}\right)$$

$$= \int_0^1 \frac{dt}{1+x^2}$$

$$= (\tan^{-1}x)\Big|_0^1$$

$$= (\tan^{-1}(1)-\tan^{-1}(0))$$

$$= \frac{\pi}{4}$$

122. Limit $= \lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{r}{\sqrt{n^2+r^2}}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\left(\frac{nr}{\sqrt{n^2+r^2}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\left(\frac{\frac{r}{n}}{\sqrt{1+\left(\frac{r}{n}\right)^2}}\right)$$

$$= \int_0^2\left(\frac{x}{\sqrt{1+x^2}}\right)dx$$

$$= \left(\sqrt{1+x^2}\right)\Big|_0^2$$

$$= (\sqrt{5}-1)$$

123. Limit $= \lim_{n\to\infty}\sum_{r=1}^{2n}\frac{2}{n}\left(\frac{2r}{n}+1\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}2\left(\frac{r}{n}\right)\left(2\frac{r}{n}+1\right)$$

$$= \int_0^2 2x(2x+1)dx$$

$$= \int_0^2 (4x^2+2x)dx$$

$$= \left(\frac{4x^2}{3}+x^2\right)\Bigg|_0^2$$

$$= \left(\frac{16}{3}+4\right) = \frac{28}{3}$$

124. Let $A = \lim_{n\to\infty}\left(\frac{(2n!)!}{n!\cdot n^n}\right)^{1/n}$

$$= \lim_{n\to\infty}\left(\frac{(2n(2n-1)(2n-2)\ldots(2n-(n-1)))n!}{n!n^n}\right)^{1/n}$$

$$= \lim_{n\to\infty}\left(\frac{(2n(2n-1)(2n-2)\ldots.(2n(n-1)))}{n^n}\right)^{1/n}$$

$\Rightarrow$ Log A

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\log\left(\frac{2n-r}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\log\left(2-\frac{r}{n}\right)$$

$$= \int_0^1 \log(2-x)dx$$

$$= -\int_2^1 \log y\,dy, \text{ where } (2-x) = y$$

$$= (y - y\log y)\Big|_2^1$$

$$= 1 - 2 + 2\log 2$$

$$= \log 4 - 1$$

$$= \log 4 - \log e$$

$$= \log\left(\frac{4}{e}\right)$$

Thus, $A = \frac{4}{e}$

125. $\lim_{x\to 0^+}(x-x^2)^{x^2} = e^{\lim_{x\to 0^+} x^2\log(x-x^2)}$

$$= e^{\lim_{x\to 0^+}\left(\frac{x}{x-1}\right)((x-x^2)\log(x-x^2))}$$

$$= e^0 \quad (\because \lim_{x\to 0^+} x\log x = 0) = 1$$

126. $\lim\limits_{x \to 1^-} (1 - x^2)^{\frac{1}{\log(1 - x)}}$

$$= e^{\lim\limits_{x \to 1^-} \frac{\log(1 - x^2)}{\log(1 - x)}}$$

$$= e^{e^{\lim\limits_{x \to 1^-} \frac{\log(1-x)}{\log(1-x)} + \frac{\log(1+x)}{\log(1-x)}}}$$

$$= e^{1+10} = e$$

127. $\lim\limits_{x \to \infty} (3^x + 4^x)^{\frac{1}{x}}$

$$= \lim_{x \to \infty} \left(4^x\left(1 + \left(\frac{3}{4}\right)^x\right)\right)^{\frac{1}{x}}$$

$$= \lim_{x \to \infty} 4\left(1 + \left(\frac{3}{4}\right)^x\right)^{\frac{1}{x}} = 4$$

128. $\lim\limits_{x \to \infty} \left(x^{\frac{1}{x}}\right)$

$$= e^{\lim\limits_{x \to \infty} \left(\frac{\log x}{x}\right)} = e^{\lim\limits_{x \to \infty} \left(\frac{\frac{1}{x}}{1}\right)} = e^0 = 1$$

129. Limit $= \lim\limits_{x \to \infty} \left(\dfrac{x + 1}{2x + 1}\right)^{x^2 + 2014}$

$$= \lim_{x \to \infty} \left(\frac{1 + \frac{1}{x}}{2 + \frac{1}{x}}\right)^{x^2 + 2014}$$

$$= \left(\frac{1}{2}\right)^{\infty}$$

$$= 0$$

130. Limit $= \lim\limits_{x \to \infty} x^{\frac{\ln 2}{1 + \ln x}}$

$$= e^{\lim\limits_{x \to \infty} \left(\frac{\log 2}{1 + \log x}\right) \log x}$$

$$= e^{\lim\limits_{x \to \infty} \left(\frac{\log 2}{\frac{1}{\log x} + 1}\right)}$$

$$= e^{\log 2} = 2$$

131. Limit $= \lim\limits_{x \to \infty} (2^x + 3^x)^{1/x}$

$$= \lim_{x \to \infty} \left(3^x\left(1 + \left(\frac{2}{3}\right)^x\right)\right)^{1/x}$$

$$= \lim_{x \to \infty} 3 \times \left(\left(1 + \left(\frac{2}{3}\right)^x\right)\right)^{1/x}$$

$$= (3(1 + 0)^0)$$

$$= 3$$

132. Limit $= \lim\limits_{x \to \infty} (1^x + 2^x + 3^x + \dots + 99^x)^{1/x}$

$$= \lim_{x \to \infty} (1^x + 2^x + 3^x + \dots + 99^x)^{1/x}$$

$$= \lim_{x \to \infty} \left(99^x\left(\left(\frac{1}{99}\right)^x + \left(\frac{2}{99}\right)^x + \left(\frac{3}{99}\right)^x + \dots + \left(\frac{99}{99}\right)^x\right)\right)^{1/x}$$

$$= 99(0 + 0 + 0 + \dots + 1)^0$$

$$= 99$$

133. Put $y = \operatorname{cosec}^2 x$

when $x \to 0$, $y \to \infty$

Limit $= \lim\limits_{y \to \infty} (1^y + 2^y + 3^y + \dots n^y)^{1/y}$

$$= \lim_{y \to \infty} \left(n^y\left(\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \left(\frac{3}{n}\right)^y + \dots + \left(\frac{n}{n}\right)^y\right)\right)^{1/y}$$

$$= (n(0 + 0 + \dots + 0 + 1))$$

$$= n$$

Level III

1. When $x > 0$, $\sin x < x \Rightarrow \dfrac{\sin x}{x} < 1$

Thus, R.H.L $= \lim\limits_{x \to 0^+} \left[\dfrac{\sin x}{x}\right] = 0$

When $x < 0$, $\sin x < x \Rightarrow \dfrac{\sin x}{x} < 1$

Thus, L.H.L $= \lim\limits_{x \to 0^-} \left[\dfrac{\sin x}{x}\right] = 0$

Hence, $\lim\limits_{x \to 0} \left[\dfrac{\sin x}{x}\right] = 0$

2. When $x > 0$, $\tan x > x \Rightarrow \dfrac{\tan x}{x} > 1$

Thus, R.H.L $= \lim\limits_{x \to 0^+} \left[\dfrac{\tan x}{x}\right] = 1$

When $x < 0$, $\tan x > x \Rightarrow \dfrac{\tan x}{x} > 1$

Thus, L.H.L $= \lim\limits_{x \to 0^-} \left[\dfrac{\tan x}{x}\right] = 1$

Hence, $\lim\limits_{x \to 0} \left[\dfrac{\tan x}{x}\right] = 1$

3. When $x > 0$, $\sin^{-1} x > x \Rightarrow \dfrac{\sin^{-1} x}{x} > 1$

Thus, R.H.L $= \lim\limits_{x \to 0^+} \left[\dfrac{\sin^{-1} x}{x}\right] = 1$

When $x < 0$, $\sin^{-1} x > x \Rightarrow \dfrac{\sin^{-1} x}{x} > 1$

Thus, L.H.L $= \lim\limits_{x \to 0^-} \left[\dfrac{\sin^{-1} x}{x}\right] = 1$

Hence, $\lim_{x\to 0}\left[\frac{\sin^{-1}x}{x}\right] = 1$

4. When $x > 0,\ \tan^{-1}x < x \Rightarrow \frac{\tan^{-1}x}{x} < 1$

Thus, R.H.L $= \lim_{x\to 0^+}\left[\frac{\tan^{-1}x}{x}\right] = 0$

When $x < 0,\ \tan^{-1}x < x \Rightarrow \frac{\tan^{-1}x}{x} < 1$

Thus, L.H.L $= \lim_{x\to 0^-}\left[\frac{\tan^{-1}x}{x}\right] = 0$

Hence, $\lim_{x\to 0}\left[\frac{\tan^{-1}x}{x}\right] = 0$

5. $\lim_{x\to\frac{5\pi}{4}} [\sin x + \cos x]$

$= \lim_{x\to\frac{5\pi}{4}}\left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right]$

$= \left[\sqrt{2}\sin\left(\frac{3\pi}{2}\right)\right]$

$= -2$

6. $\lim_{x\to 0}\left[\frac{x^2}{\sin x\,\tan x}\right]$

$= \lim_{x\to 0}\left[\frac{x}{\sin x}\,\frac{x}{\tan x}\right]$

7. As we know that, $\sin x < x$

$\Rightarrow \frac{\sin x}{x} < 1$

$\Rightarrow \lim_{x\to 0}\left[\frac{n\sin x}{x}\right] = n - 1$

Also, $\left(\frac{\tan x}{x}\right) > 1$

$\Rightarrow \lim_{x\to 0}\left[\frac{n\tan x}{x}\right] = n$

Thus, $\lim_{x\to 0}\left(\left[\frac{n\sin x}{x}\right] + \left[\frac{n\tan x}{x}\right]\right)$

$= n - 1 + n$

$= 2n - 1.$

8. We have $\lim_{x\to 0}\left(\left[2016\frac{x}{\sin x}\right] + \left[\frac{\tan x}{x}\right]\right)$

As we know that, $\sin x < x$ and $\tan x > x$

$\Rightarrow \frac{\sin x}{x} < 1$ and $\frac{\tan x}{x} > 1$

$\Rightarrow \frac{\sin x}{x} < 1$ and $1 < \frac{\tan x}{x} < 2$

$\Rightarrow \frac{x}{\sin x} > 1$ and $\frac{\tan x}{x} > 1$

Thus, $\lim_{x\to 0}\left(\left[2016\ \frac{x}{\sin x}\right] + \left[\frac{\tan x}{x}\right]\right)$

$= 2016 + 1$

$= 2017$

9. We have

$\lim_{x\to 0}\left(\left[2016\frac{\sin^{-1}x}{x}\right] + \left[\frac{x}{\tan x}\right]\right)$

As we know that,

$\sin^{-1}x > x$ and $\tan x > x$

$\Rightarrow \frac{\sin^{-1}x}{x} > 1$ and $\frac{x}{\tan x} < 1$

Thus, $\lim_{x\to 0}\left(\left[2016\frac{\sin^{-1}x}{x}\right] + \left[\frac{x}{\tan x}\right]\right)$

$= 2016 + 0$

$= 2016$

10. We have

$\lim_{x\to 0}\left(\left[2016\frac{\tan^{-1}x}{x}\right] + \left[\frac{\tan x}{x}\right]\right)$

As we know that

$\tan^{-1}x < x$ but $\tan x > x$

$\frac{\tan^{-1}x}{x} < 1$ but $\frac{\tan x}{x} > 1$

Thus, $\lim_{x\to 0}\left(\left[2016\frac{\tan^{-1}x}{x}\right] + \left[\frac{\tan x}{x}\right]\right)$

$= 2015 + 1$

$= 2016.$

11. As we know that

$\lim_{x\to 0^+} [\cos x] = 1 = \lim_{x\to 0^-} [\cos x]$

Thus, $\lim_{x\to 0}\left(\frac{\sin[\cos x]}{1 + [\cos x]}\right)$

$= \frac{\sin(1)}{1 + 1} = \frac{\sin(1)}{2}$

12. As we know that

$x \le [x] < (x - 1)$

$\Rightarrow \frac{1}{(x-1)} < \frac{1}{[x]} \le \frac{1}{x}$

$\Rightarrow \frac{\log x}{(x-1)} < \frac{\log x}{[x]} \le \frac{\log x}{x}$

Now, $\lim_{x\to\infty}\left(\frac{\log x}{(x-1)}\right)$

$$= \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right) = 0$$

Also, $\lim_{x\to\infty}\left(\frac{\log x}{x}\right)$

$$= \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right) = 0$$

By Sandwitch theorem, we get,

$$\lim_{x\to\infty}\left(\frac{\log x}{x}\right) = 0$$

13. Let $x = m + f,\ 0 \le f < 1$

Then $[x] = m$

when $x \to \infty$, then $m \to \infty$

Now, $\lim_{x\to\infty}\left(\frac{\log\ (x^n) - [x]}{[x]}\right),\ n \in N$

$$= \lim_{m\to\infty}\left(\frac{n\log(m+f) - m}{m}\right)$$

$$= \lim_{m\to\infty}\left(\frac{\frac{n}{(m+f)} - 1}{1}\right)$$

$$= \left(\frac{0-1}{1}\right) = -1$$

14. $\lim_{x\to\infty} {}^nC_x\left(\frac{m}{n}\right)^x\left(1-\frac{m}{n}\right)^{n-x}$

$$= \lim_{x\to\infty}\left(\frac{n!}{n!(n-x)!}\right)\left(\frac{m}{n}\right)^x \frac{\left(1-\frac{m}{n}\right)^n}{\left(1-\frac{m}{n}\right)^x}$$

$$= \lim_{x\to\infty}\left(\frac{m^x}{x!}\right)\frac{\left(1-\frac{m}{n}\right)^n}{\left(1-\frac{m}{n}\right)^x}\ \frac{n(n-1)(n-2)\ \dots\ (n-x+1)}{n^x}$$

$$= \left(\frac{m^x}{x!}\right)\cdot e^{-m}\cdot \lim_{x\to\infty}\left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ \dots\ \left(1-\frac{x-1}{n}\right)\right]$$

$$= \left(\frac{m^x}{x!}\right)\cdot e^{-m}\cdot 1$$

$$= e^{-m}\cdot\left(\frac{m^x}{x!}\right)$$

15. $\lim_{n\to\infty}\frac{n^k\sin(n!)}{n+1},\ (0 \le k \le 1)$

$$= \left(\lim_{n\to\infty}\frac{n^k}{n+1}\right)\times\left(\lim_{n\to\infty}\sin(n!)\right)$$

= 0 × (limit does not exist)

= 0

16. when $x > 0$, R.H.L $= \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$

when $x < 0$, L.H.L

$$= \lim_{x\to 0^-}\left(\frac{\sin x}{x}\right) = \left(\frac{\sin(-1)}{(-1)}\right) = \sin(1)$$

Thus, $\lim_{x\to 0}\left(\frac{\sin[x]}{[x]}\right)$ does not exist.

17. As we know that

$$\sin^{-1}(\sin x) = \begin{cases} x & : x < \frac{\pi}{2} \\ \pi - x & : x > \frac{\pi}{2} \end{cases}$$

Now, L.H.L

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-}[\sin^{-1}(\sin x)]$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-}[x] = \left[\frac{\pi}{2}\right] = 1$$

Also, R.H.L

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^+}[\sin^{-1}(\sin x)]$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^+}[x] = \left[\frac{\pi}{2}\right] = 1$$

Thus, $\lim_{x\to\left(\frac{\pi}{2}\right)}[\sin^{-1}(\sin x)] = 1$

18. As we know that $\sin x < x$

$\Rightarrow$ $\frac{\sin x}{x} < 1$

$\Rightarrow$ $\lim_{x\to 0}\left[\frac{99\sin x}{x}\right] = 98$

Also, $\left(\frac{x}{\sin x}\right) > 1$

$\Rightarrow$ $\lim_{x\to 0}\left[\frac{100x}{\sin x}\right] = 100$

Thus, $\lim_{x\to 0}\left(\left[\frac{100x}{\sin x}\right] + \left[\frac{99\sin x}{x}\right]\right)$

= 100 + 98

= 198

19. We have

$$\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^n, \text{ where } a, b > 0$$

$$= \lim_{n\to\infty}\left(1+\frac{\sqrt[n]{b}-1}{a}\right)^n$$

$$= e^{\lim_{n\to\infty}\left(\frac{\sqrt[n]{b}-1}{a}\right)^n}$$

$$= e^{\lim_{n\to\infty}\frac{1}{a}\left(\frac{b^{1/n}-1}{1/n}\right)}$$

$$= e^{\frac{1}{a}\log b} = e^{\log b^{1/a}} = b^{1/a}$$

20. We have

$$\lim_{x\to 0}\left(\frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{e\left(1+\frac{x}{2}+\frac{11}{24}x^2+\ldots\right) - e + \frac{ex}{2}}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{e(\frac{11}{24}x^2 + (\ldots)x^3 + (\ldots)x^4 + \ldots)}{x^2}\right)$$

$$= \frac{11e}{24}$$

21. We have $\lim_{x\to 0}\left(\frac{\sin\{x\}}{\{x\}}\right)$

Now, L.H.L.

$$= \lim_{x\to 0^-}\left(\frac{\sin\{x\}}{\{x\}}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin(0-h)}{(0-h)}\right)$$

$$= \left(\frac{\sin(-1)}{(-1)}\right) = \sin(1)$$

And, R.H.L

$$= \lim_{x\to 0^+}\left(\frac{\sin\{x\}}{\{x\}}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin(0+h)}{(0+h)}\right) = 1$$

Since R.H.L = L.H.L, so limit does not exist.

22. We have

$$= \lim_{n\to\infty}\left[\tan\left\{\frac{\pi-4}{4} + \left(1+\frac{1}{n}\right)^a\right\}\right]^n$$

$$= \lim_{n\to\infty}\left[\tan\left\{\left(\frac{\pi}{4}-1\right) + \left(1+\frac{1}{n}\right)^a\right\}\right]^n$$

Put $n = 1/x$

when $n \to \infty$, then $x \to 0$

$$= \lim_{x\to 0}\left[\tan\left\{\left(\frac{\pi}{4}-1\right) + (1+x)^a\right\}\right]^{1/x}$$

Let $A = \lim_{x\to 0}\left[\tan\left\{\left(\frac{\pi}{4}-1\right) + (1+x)^a\right\}\right]^{1/x}$

$$\log A = \lim_{x\to 0}\left\{\frac{\log\left[\tan\left\{\left(\frac{\pi}{4}-1\right)+(1+x)^a\right\}\right]}{x}\right\}$$

Applying L' Hospital rule, we get,

$$\log A = \lim_{x\to 0}\left\{\frac{1}{\left[\tan\left\{\left(\frac{\pi}{4}-1\right)+(1+x)^a\right\}\right]}\right\}$$

$$\times \sec^2\left\{\frac{\pi}{4} - 1 + (1+x)^a\right\} \times a(1+x)^{a-1}$$

$$\log A = 1\cdot(\sqrt{2})^2\cdot a = 2a$$

$$A = e^{2a}$$

23. We have

$$= \lim_{n\to\infty}\left\{\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right)\right\}$$

As we know that

$$\cos A\cdot\cos 2A\cdot\cos 2^2A\cdot\cos 2^3A \ldots \cos^{2n}A$$

$$= \frac{1}{2^n}\cdot\frac{\sin(2^nA)}{\sin A}$$

Let $\frac{x}{2^n} = A$

When $n\to\infty, A\to 0$

Thus, $\lim_{n\to\infty}\left\{\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\ldots\cos\left(\frac{x}{2^n}\right)\right\}$

$$= \lim_{A\to 0}\left\{\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\cos\left(\frac{x}{2^{n-2}}\right)\ldots\cos\left(\frac{x}{2}\right)\right\}$$

$$= \lim_{A\to 0}\{\cos(A)\cos(2A)\cos(2^2A)\ldots\cos(2^{n-1}A)\}$$

$$= \lim_{A\to 0}\left(\frac{1}{2^n}\cdot\frac{\sin(2^nA)}{\sin A}\right)$$

$$= -\lim_{n\to\infty}\left\{\frac{(n^4+4n^2+6n)}{(n^4+4n^2+6n+1)^{1/2}+1}\right\}$$

$$= \lim_{A\to 0}\left(\frac{A}{x}\cdot\frac{\sin(x)}{\sin A}\right)$$

$$= \lim_{A\to 0}\left(\frac{\sin(x)}{x}\cdot\frac{A}{\sin A}\right)$$

$$= \frac{\sin(x)}{x}$$

24. We have

$$\lim_{x\to\pi^+}\left(\frac{2^{\cot x}+3^{\cot x}-5^{1+\cot x}+10}{(4^{\cot x})^{1/2}+(27^{\cot x})^{1/3}-5^{\cot x}+20}\right)$$

when $x \to \pi^+, \cot x \to +\infty$

$$= \lim_{x\to\pi^+}\left(\frac{2^{\cot x}+3^{\cot x}-5.5^{\cot x}+10}{(2^{2\cot x})^{1/2}+(3^{3\cot x})^{1/3}-5^{\cot x}+20}\right)$$

$$= \lim_{x\to\pi^+}\left(\frac{2^{\cot x}+3^{\cot x}-5.5^{\cot x}+10}{(2^{\cot x})+(3^{\cot x})-5^{\cot x}+20}\right)$$

$$= \lim_{x\to\pi^+}\left(\frac{\left(\frac{2}{5}\right)^{\cot x}+\left(\frac{3}{5}\right)^{\cot x}-5+\frac{10}{5^{\cot x}}}{\left(\left(\frac{2}{5}\right)^{\cot x}\right)+\left(\left(\frac{3}{5}\right)^{\cot x}\right)-1+\frac{20}{5^{\cot x}}}\right)$$

$$= \left(\frac{0+0-5+0}{0+0-1+0}\right) = 5$$

25. We have

$$\lim_{x\to 0}\left(1+\frac{x^2+f(x)}{x^2}\right)^{1/x} = e^2.$$

It is possible only when

$$\left(\frac{x^2+f(x)}{x^2}\right) = 0$$

So, the least degree of $f(x)$ is 2

Let $f(x) = a_1x^2 + a_2x^3 + ...$

Now, $\lim_{x\to 0}\left(1+\frac{x^2+f(x)}{x^2}\right)^{1/x} = e^2$

$$e^{\lim_{x\to 0}\left(\frac{x^2+f(x)}{x^2}\right)\times\frac{1}{x}} = e^2$$

$$\lim_{x\to 0}\left(\frac{x^2+f(x)}{x^3}\right) = 2$$

$$\lim_{x\to 0}\left(\frac{x^2+a_1x^2+a_2x^3+...}{x^3}\right) = 2$$

$$\lim_{x\to 0}\left(\frac{(1+a_1)x^2+a_2x^3+...}{x^3}\right) = 2$$

It is possible only when

$(1 + a_1) = 0,\ a_2 = 2$

$a_1 = -1,\ a_2 = 2$

Hence, the polynomial $f(x) = -x^2 + 2x^3$.

26. We have

$$\lim_{n\to\infty}\{(n^6+6n^4+12n^3+1)^{1/3}-(n^4+4n^2+6n+1)^{1/2}\}$$

$$= \lim_{n\to\infty}\{(n^6+6n^4+12n^3+1)^{1/3}-n^2\}$$

$$- \lim_{n\to\infty}\{(n^4+4n^2+6n+1)^{1/2}-n^2\}$$

$$= \lim_{n\to\infty}\left\{\frac{(n^6+6n^4+12n^3+1)-n^6}{(n^6+6n^5+12n^4+1)^{2/3}+n^4+n^2(n^6+6n^5+12n^4+1)^{1/3}}\right\}$$

$$- \lim_{n\to\infty}\left\{\frac{(n^4+4n^2+6n+1)-n^4}{(n^4+4n^2+6n+1)^{1/2}+n^2}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{(6n^4+12n^3+1)}{(n^6+6n^4+12n^3+1)^{2/3}+n^4+n^2\ (n^6+6n^4+12n^3+1)^{1/3}}\right\}$$

$$- \lim_{n\to\infty}\left\{\frac{(4n^2+6n+1)}{(n^4+4n^2+6n+1)^{1/2}+n^2}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{\left(6+\frac{12}{n}+\frac{1}{n^4}\right)}{\left(1+\frac{6}{n^2}+\frac{12}{n^3}+\frac{1}{n^4}\right)^{2/3}+1+\left(1+\frac{6}{n^2}+\frac{12}{n^3}+\frac{1}{n^4}\right)^{1/3}}\right\}$$

$$- \lim_{n\to\infty}\left\{\frac{\left(4+\frac{6}{n}+\frac{1}{n^2}\right)}{\left(1+\frac{4}{n^2}+\frac{6}{n^3}+\frac{1}{n^4}\right)^{1/2}+1}\right\}$$

$$= \frac{6}{3}-\frac{4}{2}$$

$= 2 - 2$

$= 0$

27. We have

$$\lim_{x\to 0}\left(\frac{(1+3x+2x^2)^{1/x}-(1+3x-2x^2)^{1/x}}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\{(1+3x+2x^2)^{1/x}-e^3\}-\{(1+3x-2x^2)^{1/x}-e^3\}}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\{(1+3x+2x^2)^{1/x}-e^3\}}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\{(1+3x-2x^2)^{1/x}-e^3\}}{x}\right)$$

$= (L_1 - L_2)$ (say)

Now,

$$L_1 = \lim_{x\to 0}\left(\frac{\{(1+3x+2x^2)^{1/x} - e^3\}}{x}\right)$$

$$= \lim_{x\to 0}\frac{\left\{e^{\frac{\ln(1+3x+2x^2)}{x}} - e^3\right\}}{x}$$

$$= \lim_{x\to 0}\frac{\left\{e^{(3+2x)-\frac{x}{2}(3+2x)^2} - e^3\right\}}{x}$$

$$= \lim_{x\to 0}\frac{e^3\left\{e^{2x-\frac{x}{2}(3+2x)^2} - 1\right\}}{x}$$

$$= \lim_{x\to 0}\frac{e^3\left\{e^{2x-\frac{x}{2}(3+2x)^2} - 1\right\}}{\left\{2x-\frac{x}{2}(3+2x)^2\right\}} \times \left\{\frac{2x-\frac{x}{2}(3+2x)^2}{x}\right\}$$

$$= \lim_{x\to 0}\frac{e^3\left\{e^{2x-\frac{x}{2}(3+2x)^2} - 1\right\}}{\left\{2x-\frac{x}{2}(3+2x)^2\right\}} \times \left\{2-\frac{1}{2}(3+2x)^2\right\}$$

$$= e^3\left(2-\frac{9}{2}\right) = -\frac{5e^3}{2}$$

Again,

$$L_2 = \lim_{x\to 0}\left(\frac{\{(1+3x-2x^2)^{1/x} - e^3\}}{x}\right)$$

$$= \lim_{x\to 0}\frac{\left\{e^{\frac{\ln(1+3x-2x^2)}{x}} - e^3\right\}}{x}$$

$$= \lim_{x\to 0}\frac{\left\{e^{(3-2x)-\frac{x}{2}(3-2x)^2} - e^3\right\}}{x}$$

$$= \lim_{x\to 0}\frac{e^3\left\{e^{-2x-\frac{x}{2}(3-2x)^2} - 1\right\}}{x}$$

$$= \lim_{x\to 0}\frac{e^3\left\{e^{-2x-\frac{x}{2}(3-2x)^2} - 1\right\}}{\left\{-2x-\frac{x}{2}(3-2x)^2\right\}} \times \frac{\left\{-2x-\frac{x}{2}(3-2x)^2\right\}}{x}$$

$$= \lim_{x\to 0}\frac{e^3\left\{e^{-2x-\frac{x}{2}(3+2x)^2} - 1\right\}}{\left\{-2x-\frac{x}{2}(3-2x)^2\right\}} \times \left\{-2-\frac{1}{2}(3-2x)^2\right\}$$

$$= e^3\left(-2-\frac{9}{2}\right) = -\frac{13e^3}{2}$$

Now, $L_1 - L_2$

$$= -\frac{5e^3}{2} + \frac{13e^3}{2}$$

$$= \frac{8e^3}{2}$$

$$= 4e^3$$

28. We have

$$\lim_{x\to 0}\left(\frac{1-\cos(1-\cos x)}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\cos\left(2\sin^2\left(\frac{x}{2}\right)\right)}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2\left(\sin^2\left(\frac{x}{2}\right)\right)}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sin^2\left(\sin^2\left(\frac{x}{2}\right)\right)}{\sin^4\left(\frac{x}{2}\right)} \times \frac{\sin^4\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^4 \times 2^4}\right)$$

$$= \frac{2}{2^4} = \frac{1}{8}$$

29. We have

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$$

$$= \lim_{x\to 0}\left(1+\left(\frac{\sin x}{x}-1\right)\right)^{\frac{\sin x}{x-\sin x}}$$

$$= \lim_{x\to 0}\left(1+\left(\frac{\sin x - x}{x}\right)\right)^{\frac{\sin x}{x-\sin x}}$$

$$= e^{\lim_{x\to 0}\left(\frac{\sin x - x}{x}\right)\times\frac{\sin x}{x-\sin x}}$$

$$= e^{\lim_{x\to 0}\left(-\frac{\sin x}{x}\right)}$$

$$= e^{-1} = \frac{1}{e}$$

30. Given $\lim_{x\to 0}\left(\frac{f(x)}{x}\right) = 1$

We have

$$\lim_{x\to 0}\left(\frac{x(1+a\cos x) - b\sin x}{\{f(x)\}^3}\right) = 1$$

$$\lim_{x\to 0}\left(\frac{x(1+a\cos x)-b\sin x}{\left\{\frac{f(x)}{x}\right\}^3\times x^3}\right)=1$$

$$\lim_{x\to 0}\left(\frac{x+ax\cos x-b\sin x}{x^3}\right)=1$$

$$\lim_{x\to 0}\left(\frac{x+ax\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots\right)-b\left(x-\frac{x^3}{3!}+\ldots\right)}{x^3}\right)=1$$

$$\lim_{x\to 0}\left(\frac{(1+a-b)x+x^3\left(-\frac{a}{2!}+\frac{b}{3!}\right)+\ldots}{x^3}\right)=1$$

It is possible only when

$$(1+a-b)=0 \text{ and } \left(-\frac{a}{2!}+\frac{b}{3!}\right)=1$$

On solving, we get, $a=-\frac{5}{2}$, $b=-\frac{3}{2}$

Now, $(a+b+10)$

$$=-\frac{5}{2}-\frac{3}{2}+10$$

$$=-4+10$$

$$=6$$

Level IV

1. $\lim_{x\to a}\left(\frac{x^x-a^a}{a^x-a^a}\right)$

$$=\lim_{x\to a}\left(\frac{x^x(1+\log x)-0}{a^x\log a}\right)$$

$$=\left(\frac{a^a(1+\log a)}{a^a\log a}\right)$$

$$=\left(\frac{(1+\log a)}{\log a}\right)$$

$$=\left(\frac{1}{\log a}+1\right)$$

$$=(\log_a e+1)$$

2. $\lim_{x\to 0}\left(\frac{x+\ln\left(\sqrt{x^2+1}-x\right)}{x^3}\right)$

$$=\lim_{x\to 0}\left(\frac{1-\frac{1}{\sqrt{x^2+1}}}{3x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{\left(\sqrt{x^2+1}-1\right)}{\left(3x^2\sqrt{x^2+1}\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{x^2+1-1}{3x^2\sqrt{x^2+1}}\times\frac{1}{\left(\sqrt{x^2+1}+1\right)}\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{3\sqrt{x^2+1}}\times\frac{1}{\left(\sqrt{x^2+1}+1\right)}\right)$$

$$=\left(\frac{1}{3}\times\frac{1}{2}\right)=\frac{1}{6}$$

3. Given $\lim_{x\to 0}\left(\frac{2}{x^3}(\sin^{-1}x-\tan^{-1}x)\right)^{\frac{2}{x^3}}=e^{-m}$

Now, $\sin^{-1}x=x+\frac{x^3}{6}+\frac{3x^5}{40}+\ldots$

and $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$

$$\sin^{-1}x-\tan^{-1}x=\left(\frac{x^3}{6}+\frac{x^3}{3}\right)+\left(\frac{3x^5}{5}-\frac{x^5}{5}\right)+\ldots$$

Now, $\lim_{x\to 0}\left(\frac{2}{x^3}(\sin^{-1}x-\tan^{-1}x)\right)^{\frac{2}{x^3}}$ (1^∞ – from)

$$=e^{\lim_{x\to\infty}\frac{2}{x^2}\left(\frac{2}{x^3}(\sin^{-1}x-\tan^{-1}x)-1\right)}$$

$$=e^{2\lim_{x\to\infty}\left(\frac{(\sin^{-1}x-\tan^{-1}x)-x^3}{x^5}\right)}$$

$$=e^{2\times-\frac{1}{4}}=e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}$$

Thus, $M=-\frac{1}{2}$

Hence, the value of $2M+2013=2012$

4. We have $\left(1+\frac{1}{n}\right)^{n+x_n}=e$

$$\log\left(1+\frac{1}{n}\right)^{n+x_n}=\log(e)=1$$

$$(n+x_n)\log\left(1+\frac{1}{n}\right)=1$$

$$(n+x_n)=\frac{1}{\log\left(1+\frac{1}{n}\right)}$$

$$x_n=\frac{1}{\log\left(1+\frac{1}{n}\right)}-n$$

Let $\left(1+\frac{1}{n}\right)=y \Rightarrow ny=n+1$

$\Rightarrow$ $n=\frac{1}{y-1}$

Now, $\lim_{n\to\infty}(x_n)$

$$= \lim_{n\to 1}\left(\frac{1}{\log y} - \frac{1}{y-1}\right)$$

$$= \lim_{n\to 1}\left(\frac{y-1-\log y}{(y-1)\log y}\right)$$

$$= \lim_{n\to 1}\left(\frac{1-\frac{1}{y}}{\frac{y-1}{1} + \log y}\right)$$

$$= \lim_{n\to 1}\left(\frac{\frac{1}{y^2}}{\frac{1}{y^2} + \frac{1}{y}}\right) = \frac{1}{2}$$

5. We have $\lim_{x\to\alpha}\left(\sqrt{\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}}\right)$

$$= \lim_{x\to\alpha}\left(\sqrt{\frac{1-\cos\{a(x-\alpha)(x-\beta)\}}{(x-\alpha)^2}}\right)$$

$$= \lim_{x\to\alpha}\left(\sqrt{\frac{2\sin^2\left\{\frac{a(x-\alpha)(x-\beta)}{2}\right\}}{(x-\alpha)^2}}\right)$$

$$= \lim_{x\to\alpha}\left(\sqrt{\frac{2\sin^2\left\{\frac{a(x-\alpha)(x-\beta)}{2}\right\}}{\frac{(x-\alpha)^2(x-\beta)^2}{4}} \times \left(\frac{x-\beta}{2}\right)^2}\right)$$

$$= \sqrt{2\left(\frac{\alpha-\beta}{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}|(\alpha-\beta)|$$

6. We have $\lim_{x\to 0}\left(\frac{ae^x - b\cos x + ce^{-x}}{x\sin x}\right) = 2$

As $x \to 0$, numerator $(a - b + c) \to 0$ and the denominator $\to 0$

So, $(a - b + c) = 0$

$b = a + c$

Thus, $\lim_{x\to 0}\left(\frac{ae^x - (a+c)\cos x + ce^{-x}}{x\sin x}\right) = 2$

$$\lim_{x\to 0}\left(\frac{ae^x + (a+c)\sin x - ce^{-x}}{x\cos x + \sin x}\right) = 2$$

As $\lim_{x\to 0}$, numerator $(a - c) \to 0$ and the denominator $\to 0$

So, $a - c = 0$

$a = c$

Thus, $\lim_{x\to 0}\left(\frac{ae^x + 2a\sin x - ae^{-x}}{x\cos x + \sin x}\right) = 2$

$$\lim_{x\to 0}\left(\frac{ae^x + 2a\cos x + ae^{-x}}{\cos x - x\sin x + \cos x}\right) = 2$$

$$\frac{a + 2a + a}{2} = 2$$

$4a = 4$

$a = 1$

$c = 1,\ b = a + c = 2$

Hence, the value of $a^3 + b^3 + c^3 + 10$

$= 1 + 8 + 1 + 10$

$= 20$

7. We have

$$\lim_{x\to 0}\left(\frac{1-\cos x + 2\sin x - \sin^3 x - x^2 + 3x^4}{\tan^3 x - 6\sin^2 x + x - 5x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x + 2\cos x - 3\sin^2 x\cos x - 2x + 12x^3}{3\tan^2 x\sec^2 x - 6\sin 2x + 1 - 10x}\right)$$

$= 2$

Hence, the value of $M + 10 = 12$.

8. We have $L = \lim_{x\to 0}\left[\frac{x^2}{\sin x\tan x}\right]$

$$= \lim_{x\to 0}\left[\frac{x}{\sin x} \times \frac{x}{\tan x}\right]$$

As we know that

$$\frac{x}{\sin x} > 1 \text{ and } \frac{x}{\tan x} < 1$$

$$\Rightarrow \quad \frac{x}{\sin x}\cdot\frac{x}{\tan x} < 1$$

$$\Rightarrow \quad \left[\frac{x}{\sin x}\cdot\frac{x}{\tan x}\right] = 0$$

Thus, $\lim_{x\to 0}\left[\frac{x}{\sin x} \times \frac{x}{\tan x}\right] = 0$

Hence, the value of $L + 2013$ is 2013.

9. Given $P(x) = a_1 x + a_2 x^2 + \ldots\ldots + a_{100} x^{100}$

$$L = \lim_{x \to 0} \left(\frac{\sqrt[100]{1 + P(x)}}{x} - 1 \right)$$

$$= \lim_{x \to 0} \frac{\left(1 + \frac{1}{100} P(x) + \ldots\right) - 1}{x}$$

$$= \lim_{x \to 0} \frac{1}{100} \left(\frac{a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n}{x} \right)$$

$$= \lim_{x \to 0} \frac{1}{100} (a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1})$$

$$= \frac{a_1}{100}$$

$$= \frac{1}{100}$$

Hence, the value of $(1000\,L + 2007)$

$= 10 + 2007$

$= 2017$

10. We have $L = \lim_{x \to 0} \left[\frac{100x}{\sin x} \right]$

$= 100$, since $\left(\because \frac{x}{\sin x} > 1 \right)$

and $M = \lim_{x \to 0} \left[\frac{99 \sin x}{x} \right]$

$= 98$, since $\left(\frac{\sin x}{x} < 1 \right)$

Hence, the value of L + M + 2

$= 100 + 98 + 2$

$= 200.$

11. We have $\lim_{x \to \infty} \left(\frac{729^x - 243^x - 81^x + 9^x + 3^x - 1}{x^3} \right)$

$$= \lim_{x \to \infty} \left(\frac{243^x(3^x - 1) - 9^x(9^x - 1) + (3^x - 1)}{x^3} \right)$$

$$= \lim_{x \to \infty} \left(\frac{243^x(3^x - 1) - 9^x(3^x - 1) + (3^x - 1)}{x^3} \right)$$

$$= \lim_{x \to \infty} \left(\frac{(243^x - 9^x(3^x + 1) + 1)\ (3^x - 1)}{x^3} \right)$$

$$= \lim_{x \to \infty} \left(\frac{243^x - 9^x \cdot 3^x - (9^x - 1)(3^x - 1)}{x^3} \right)$$

$$= \lim_{x \to \infty} \left(\frac{(27^x(9^x - 1) - (9^x - 1))(3^x - 1)}{x^3} \right)$$

$$= \lim_{x \to \infty} \left(\frac{(27^x - 1)(9^x - 1)(3^x - 1)}{x^3} \right)$$

$$= \lim_{x \to \infty} \left(\left(\frac{27^x - 1}{x} \right) \left(\frac{9^x - 1}{x} \right) \left(\frac{3^x - 1}{x} \right) \right)$$

$= \log(27) \cdot \log(9) \cdot \log(3)$

$= \log(3^3) \cdot \log(3^2) \cdot \log(3)$

$= 6\{\log(3)\}^3$

So, $K = 6,\ M = 3,\ N = 3$

Hence, the value of $K^2 + M^2 + N^2$.

$= 36 + 9 + 9$

$= 54.$

12. We have

$$\lim_{x \to 1^-} \left(\frac{(3 + ax)^{5/2} - b \ln x + c \sin(x - 1)}{(x - 1)^2} \right) = 2$$

It is possible only when $(3 + a)^{5/2} = 0$

$(a + 3) = 0$

$a = -3$

$$\lim_{x \to 1^-} \left(\frac{(3 - 3x)^{5/2} - b \ln x + c \sin(x - 1)}{(x - 1)^2} \right) = 2$$

$$\lim_{x \to 1^-} \left(\frac{\frac{5}{2}(3 + ax)^{3/2} - \frac{b}{x} + c \cos(x - 1)}{2(x - 1)} \right) = 2$$

It is possible only when $-b + c = 0$

$b = c$

$$\lim_{x \to 1^-} \left(\frac{\frac{5}{2} \cdot \frac{3}{2} \cdot a^2 (3 + ax)^{12} + \frac{b}{x^2} + c \sin(x - 1)}{2} \right) = 2$$

It is possible only when $\frac{b}{2} = 2$

$b = 4$

Thus, $a = -3,\ b = 4 = c$

Hence, the value of $a^2 + b^2 + c^2$

$= 9 + 16 + 16$

$= 41$

13. Limit $= \lim_{x \to 0} \left(\frac{(\cos x - 1)(\cos x - e^x)}{x^n} \right)$

$$= \lim_{x \to 0} \left(\frac{\frac{2 \sin^2 \left(\frac{x}{2} \right)}{\frac{x^2}{4}} \left(\frac{(e^x - 1)}{x} + \frac{1 - \cos x}{x} \right)}{x^{n-3}} \right)$$

It will provide us a finite limit only when $n - 3 = 0$

Thus $n = 3$.

14. We have

$$\lim_{x\to 0}\left(\frac{\cos 4x + a\cos 2x + b}{x^4}\right)$$

Since it has a finite limit, so $a + b + 1 = 0$ ...(i)

$$\lim_{x\to 0}\left(\frac{-4\sin 4x - 2a\sin 2x}{4x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{-16\cos 4x - 4a\cos 2x}{12x^2}\right)$$

Since, it has a finite limit, so $-16 - 4a = 0$

$4a = -16$

$\Rightarrow$ $a = -4$

when $a = -4$, $b = 3$

Hence, the value of $a^2 + b^2 + 10$

$= 16 + 9 + 10$

$= 35$.

15. We have $\lim_{n\to\infty}\left(\sum_{k=0}^{n}\left(\frac{{}^nC_k}{n^k(k+3)}\right)\right)$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{{}^nC_k}{n^k}\right)\int_0^1 x^{k+2}dx$$

$$= \int_0^1 \lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{{}^nC_k}{n_k}\right)\cdot x^{k+2}dx$$

$$= \int_0^1 x^2\left\{\lim_{n\to\infty}\sum_{k=0}^{n}\left(\frac{{}^nC_k}{n^k}\right)\cdot x^k\right\}dx$$

$$= \int_0^1 x^2\left\{\lim_{n\to\infty}\sum_{k=0}^{n}\left({}^nC_k\left(\frac{x}{n}\right)^k\right)\right\}dx$$

$$= \int_0^1 x^2\left\{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right\}dx$$

$$= \int_0^1 x^2\left\{e^{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n}\right\}dx$$

$$= \int_0^1 x^2\left\{e^{\lim_{n\to\infty} n\left(1+\frac{x}{n}-1\right)}\right\}dx$$

$$= \int_0^1 x^2 e^x dx$$

$$= (x^2e^x)_0^1 - 2\int_0^1 xe^x dx$$

$$= (x^2e^x)_0^1 - 2(xe^x - e^x)_0^1$$

$$= e - 2(e - e) + 2(0 - 1)$$

$$= (e - 2).$$

Thus, $L = 1$, $M = -2$

Hence, the value of $L + M + 10$ is 9.

16. Limit

$$= \lim_{n\to\infty}\left(\frac{1^a + 2^a + \dots + n^a}{(n+1)^{a-1}((na+1) + (na+2)+\dots+(na+n))}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2\sum_{r=1}^{n} r^a}{(n+1)^{a-1}(2n^2a + n^2 + n)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{\left(1+\frac{1}{n}\right)^{a-1}\left(2a+1+\frac{1}{n}\right)}\right)$$

$$= \left(\frac{2\int_0^1 x^a dx}{2a+1}\right)$$

Thus, $\frac{2}{(2a+1)(a+1)} = \frac{1}{60}$

$\Rightarrow$ $(2a + 1)(a + 1) = 120$

$\Rightarrow$ $2a^2 + 3a - 119 = 0$

$\Rightarrow$ $2a^2 + 17a - 14a - 119 = 0$

$\Rightarrow$ $a(2a + 17) - 7(2a + 17) = 0$

$\Rightarrow$ $(2a + 17)(a - 7) = 0$

$\Rightarrow$ $a = 7, -\frac{17}{2}$.

17. Given

$$L = \lim_{x\to\infty}\left(\frac{x^4\sin\left(\frac{1}{x}\right)}{1+|x|^3}\right)$$

Put $x = -y$

when $x \to -\infty$, than $y \to \infty$

Thus, $L = \lim_{y\to\infty}\left(\frac{y^4\sin\left(-\frac{1}{y}\right)}{1+|-y|^3}\right)$

$$= \lim_{y\to\infty}\left(\frac{-y^4\sin\left(\frac{1}{y}\right)}{1+y^3}\right)$$

$$= \lim_{y\to\infty}\left(\frac{-y\sin\left(\frac{1}{y}\right)}{\frac{1}{y^3}+1}\right) = \frac{-1}{1} = -1$$

Also, $M = \lim_{x\to 0}\left(\frac{\int_{\sqrt{x}}^{x^2}\tan^{-1}\left(\frac{t^2}{1+t^2}\right)dt}{\sin 2x}\right)$

$$= \lim_{x\to 0}\left(\frac{\tan^{-1}\left(\frac{x^2}{1+x^2}\right)2x - \tan^{-1}\left(\frac{x}{1+x}\right)\cdot\frac{1}{2\sqrt{x}}}{2\cos(2x)}\right)$$

$$= \frac{0}{2} = 0$$

Hence, the value of $L + M + 11 = -1 + 0 + 11 = 10$.

18. Given $k = \lim_{n\to\infty}\left(\int_n^{2n}\frac{n^3x}{x^5+1}\right)dx$

put $n = \frac{1}{y}$

when $n\to\infty$, then $y \to 0$

$$\lim_{y\to 0}\left(\frac{\int_{1/y}^{2/y}\left(\frac{x}{x^5+1}\right)dx}{y^3}\right)$$

$$\lim_{y\to 0}\left(\frac{\frac{2/y}{32/y^5+1}\left(-\frac{2}{y^2}\right) - \frac{1/y}{(1/y^5)+1}\left(-\frac{1}{y^2}\right)}{3y^2}\right)$$

$$= \lim_{y\to 0}\left(\frac{\frac{-4}{32+y^5}+\frac{1}{1+y^5}}{3}\right)$$

$$= \frac{7}{24}$$

Now, $\left[\frac{1}{k}\right] + 2013$

$$= \left[\frac{24}{7}\right] + 2013$$

$$= 3 + 2013 = 2016$$

19. We have

$$\lim_{x\to\infty}\left(\sqrt{(x^2+a^2)(x^2+b^2)} - \sqrt{(x^2+c^2)(x^2+d^2)}\right)$$

$$= \lim_{x\to\infty}\left(\frac{((x^2+a^2)(x^2+b^2) - (x^2+c^2)(x^2+d^2))}{\sqrt{(x^2+a^2)(x^2+b^2)}+\sqrt{(x^2+c^2)(x^2+d^2)}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{((a^2+b^2)x^2+a^2b^2 - ((c^2+d^2)x^2+c^2d^2))}{\sqrt{(x^2+a^2)(x^2+b^2)}+\sqrt{(x^2+b^2)(x^2+d^2)}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\left((a^2+b^2)+\frac{a^2b^2}{x^2} - \left((c^2+d^2)+\frac{c^2d^2}{x^2}\right)\right)}{\sqrt{\left(1+\frac{a^2}{x^2}\right)\left(1+\frac{b^2}{x^2}\right)}+\sqrt{\left(1+\frac{c^2}{x^2}\right)\left(1+\frac{d^2}{x^2}\right)}}\right)$$

$$= (a^2+b^2) - (c^2+d^2)$$

20. $\lim_{x\to 1}\left\{\frac{(1-x)(1-x^2)\ldots(1-x^{2n})}{\{(1-x)(1-x^2)\ldots(1-x^n)\}^2}\right\}$

$$= \lim_{x\to 1}\left\{\frac{(1-x)(1-x^2)\ldots(1-x^n)(1-x^{n+2})\ldots(1-x^{2n})}{\{(1-x)(1-x^2)\ldots(1-x^n)\}^2}\right\}$$

$$= \lim_{x\to 1}\left\{\frac{(1-x^{n+1})(1-x^{n+2})\ldots(1-x^{2n})}{(1-x)(1-x^2)\ldots(1-x^n)}\right\}$$

$$= \lim_{x\to 1}\left\{\frac{(1-x^{n+1})(1-x^{n+2})\ldots(1-x^{2n})}{(1-x)(1-x)\ldots(1-x)}\right\}$$

$$\times \lim_{x\to 1}\left\{\frac{(1-x)(1-x)\ldots(1-x)}{(1-x)(1-x^2)\ldots(1-x^n)}\right\}$$

$$= (n+1)(n+2)\ldots(2n)\times 1\times\frac{1}{2}\times\frac{1}{3}\times\ldots\times\frac{1}{n}$$

$$= \frac{(n+1)(n+2)\ldots(2n)}{1\cdot 2\cdot 3\ \ldots\ n}$$

$$= \frac{\{(1\cdot 2\cdot 3\ldots n\cdot(n+1)(n+2)\ldots(2n)\}}{\{1\cdot 2\cdot 3\ldots n\}^2}$$

$$= \frac{(2n)!}{(n!)^2}$$

21. We have $\lim_{x\to\infty}\left(\frac{4^x+4^{-x}}{4^x-4^{-x}}\right)$

$$= \lim_{x\to\infty}\left(\frac{1+4^{-2x}}{1-4^{-2x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1+\left(\frac{1}{4}\right)^{2x}}{1-\left(\frac{1}{4}\right)^{2x}}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1+0}{1-0}\right) = 1$$

22. We have $\lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r^3-8}{r^3+8}\right)$

$$= \lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r^3-8}{r^3+8}\right)$$

$$= \lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r^3-2^3}{r^3+2^3}\right)$$

$$= \lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r-2}{r+2}\right)\left(\frac{r^2+2r+4}{r^2-2r+4}\right)$$

$$= \lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r-2}{r+2}\right) \times \lim_{n\to\infty} \prod_{r=3}^{n} \left(\frac{r^2+2r+4}{r-2r+4}\right)$$

$$= \lim_{n\to\infty} \left\{\frac{1}{5}\cdot\frac{2}{6}\cdot\frac{3}{7}\cdots\frac{(n-3)(n-1)}{(n+1)(n+3)}\right\}$$

$$\times \left\{\frac{19}{7} \times \frac{28}{12} \times \frac{39}{19} \times \ldots \times \frac{(n^2+3)}{(n^2-2n+7)} \times \frac{(n^2+2n+4)}{(n^2-2n+7)}\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1\cdot2\cdot3\cdot4\ldots(n^2+3)(n^2+2n+4)}{(n-1)n(n+1)(n+2)\ldots7.12}\right\}$$

$$= \frac{2}{7}\lim_{n\to\infty}\left\{\frac{(n^2+3)(n^2+2n+4)}{(n-1)n(n+1)(n+2)}\right\}$$

$$= \frac{2}{7}\lim_{n\to\infty}\left\{\frac{\left(1+\frac{3}{n^2}\right)\left(1+\frac{2}{n}+\frac{4}{n^2}\right)}{\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}\right\}$$

$$= \frac{2}{7}\left\{\frac{(1+0)(1+0+0)}{(1-0)(1+0)(1+0)}\right\}$$

$$= \frac{2}{7}$$

23. We have

$$\lim_{x\to0}\left\{\frac{32}{x^8}\left(1-\cos\left(\frac{x^2}{2}\right)-\cos\left(\frac{x^2}{4}\right)+\cos\left(\frac{x^2}{2}\right)\cos\left(\frac{x^2}{4}\right)\right)\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{32}{x^8}\left(1-\cos\left(\frac{x^2}{2}\right)-\cos\left(\frac{x^2}{4}\right)\left\{1-\cos\left(\frac{x^2}{4}\right)\right\}\right)\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{32}{x^8}\left(\left(1-\cos\left(\frac{x^2}{2}\right)\right)\left(1-\cos\left(\frac{x^2}{4}\right)\right)\right)\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{32}{x^8}\left(\left(2\sin^2\left(\frac{x^2}{4}\right)\right)\left(2\sin^2\left(\frac{x^2}{4}\right)\right)\right)\right\}$$

$$= \lim_{x\to\infty}\left\{32\left(\left(\frac{2\sin^2\left(\frac{x^2}{4}\right)}{\frac{x^4}{16}\times16}\right)\left(\frac{2\sin^2\left(\frac{x^2}{8}\right)}{\frac{x^4}{64}\times64}\right)\right)\right\}$$

$$= 32 \times 4 \times \frac{1}{16} \times \frac{1}{64}$$

$$= \frac{1}{8}$$

24. We have

$$\lim_{x\to\infty}\left[\{(x+p)(x+q)(x+r)(x+s)\}^{1/4}-x\right]$$

$$= \lim_{x\to\infty}\frac{(x+p)(x+q)(x+r)(x+s)-x^4}{\left\{\sqrt{(x+p)(x+q)(x+r)(x+s)}+x^2\right\}}$$

$$\times \frac{1}{\left[\{(x+p)(x+q)(x+r)(x+s)\}^{1/2}+x\right]}$$

$$= \lim_{x\to\infty}\frac{\{x^2+(p+q)x+pq\}\{x^2+(r+s)x+rs\}-x^4}{\left\{\sqrt{(x+p)(x+q)(x+r)(x+s)}+x^2\right\}}$$

$$\times \frac{1}{\left[\{(x+p)(x+q)(x+r)(x+s)\}^{1/2}+x\right]}$$

$$= \lim_{x\to\infty}\frac{(p+q+r+s)x^3+(\ldots)x^2+(\ldots)x+pqrs}{\left\{\sqrt{(x+p)(x+q)(x+r)(x+s)}+x^2\right\}}$$

$$\times \frac{1}{\left[\{(x+p)(x+q)(x+r)(x+s)\}^{12}+x\right]}$$

$$= \frac{p+q+r+s}{2\times2}$$

$$= \frac{(p+q+r+s)}{4}$$

25. We have $\lim_{x\to0}\left(\frac{f(x)}{x}\right)=1$

and $\lim_{x\to0}\left(\frac{x(1+a\theta\cos x)-b\sin x}{\{f(x)\}^3}\right)=1$

$$\lim_{x\to0}\left(\frac{x+a\cos x-b\sin x}{\left\{\frac{f(x)}{x^3}\right\}^3\times x^3}\right)=1$$

$$\lim_{x\to0}\left(\frac{x+ax\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\right)}{\left\{\frac{f(x)}{x}\right\}^3\times x^3}\right)=1$$

$$\lim_{x\to 0}\left(\frac{1+a-b)x+\left(-\frac{a}{2!}+\frac{b}{3!}\right)x^3+\ldots}{\left\{\frac{f(x)}{x}\right\}^3\times x^3}\right)=1$$

It is possible only when

$$(1+a-b)=0 \text{ and } \left(-\frac{a}{2!}+\frac{b}{3!}\right)=1$$

On solving, we get,

$$a=-\frac{5}{2},\ b=-\frac{3}{2}$$

Hence, the value of $(a+b+10)$

$$=-4+10=6.$$

26. We have

$$\lim_{x\to 0}\left(\frac{axe^x-b\log(1+x)+cxe^{-x}}{x^2\sin x}\right)=2$$

$$\lim_{x\to 0}\left(\frac{axe^x-b\log(1+x)+cxe^{-x}}{x^3\left(\frac{\sin x}{x}\right)}\right)=2$$

$$\lim_{x\to 0}\left(\frac{axe^x-b\log(1+x)+cxe^{-x}}{x^3}\right)=2$$

$$\lim_{x\to 0}\left(\frac{ax\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)-b\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\right)+cx\left(1-x+\frac{x^2}{2}-\frac{x^3}{3}!+\ldots\right)}{x^3}\right)=2$$

$$\lim_{x\to 0}\left(\frac{(a-b+x)x+\left(a\quad+\frac{b}{2}-c\right)x^2+\left(\frac{a}{2}-\frac{b}{3}+\frac{c}{2}\right)x^3+\ldots}{x^3}\right)=2$$

It is possible only when

$$(a-b+c)=0,\ \left(a+\frac{b}{2}-c\right)=0$$

and $\quad \frac{a}{2}-\frac{b}{3}+\frac{c}{2}=0$

On solving, we get, $a=3$, $b=12$, $c=9$

Hence, the value of $\left(\frac{a+b+c}{8}\right)=\frac{3+12+9}{8}=3$

27. We have

$$\lim_{x\to 0}\left(\frac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\log(\sin x)}\right)$$

Put $\quad \sin x=t$

when $\quad x\to 0,\ t\to 0$

$$\lim_{t\to 0}\left(\frac{t-t^t}{1-t+\log t}\right)$$

$$=\lim_{t\to 0}\left(\frac{1-t^t(1+\log t)}{-1+\frac{1}{t}}\right)$$

$$=\lim_{t\to 0}\left(\frac{0-t^t\times\frac{1}{t}-t^t(1+\log t)^2}{-\frac{1}{t^2}}\right)$$

$$=\frac{-1-1}{-1}=2$$

28. We have

$$M=\lim_{x\to 0}\left(\frac{1-(f(x))^3}{5\tan^2 x}\right)$$

$$=\lim_{x\to 0}\left(\frac{(1-f(x))(1+f(x)+f^2(x))}{5\left(\frac{\tan^2 x}{x^2}\right)x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{3(1-f(x))}{5x^2}\right)$$

$$=\frac{3}{5}\lim_{x\to 0}\left(\frac{1-\cos 2x\cos 4x\cos 6x\cos 8x\cos 10x}{x^2}\right)$$

$$=\frac{3}{5}\left(\frac{2^2+4^2+6^2+8^2+10^2}{2}\right)$$

$$=\frac{3}{5}\times 2^2\left(\frac{1^2+2^2+3^2+4^2+5^2}{2}\right)$$

$$=\frac{12}{5}\times\frac{5\cdot 6\cdot 11}{12}$$

$$=66$$

Hence, the value of $(\sqrt{M-2}+1)$

$$=\sqrt{66-2}+1=8+1=9$$

29. As we know that

$$\sin^{-1}x>x\Rightarrow\frac{\sin^{-1}x}{x}>1$$

we have

$$\lim_{x\to 0}\left(\left[\frac{\sin^{-1}x}{x}\right]+\left[\frac{2^2\sin^{-1}(2x)}{x}\right]+\left[\frac{3^2\sin^{-1}(3x)}{x}\right]+\ldots+\left[\frac{n^2\sin^{-1}(nx)}{x}\right]\right)=100$$

$(1^3 + 2^3 + 3^3 + ... + n^3) = 100$

$\left\{\frac{n(n+1)}{2}\right\}^2 = 100$

$\left\{\frac{n(n+1)}{2}\right\}^2 = 10$

$n(n+1) = 20$

$n = 4$

30. Now, $x + \sin x - x\cos x - \tan x$

$= x(1 - \cos x) - \tan x(1 - \cos x)$

$= (x - \tan x)(1 - \cos x)$

So, $\lim_{x\to 0}\left(\frac{(x - \tan x)(1 - \cos x)}{x^n}\right)$

$\lim_{x\to 0}\left(\frac{\left(\frac{x - \tan x}{x^3}\right)\left(\frac{1 - \cos x}{x^2}\right)}{x^{n-5}}\right)$

$= a$ non-zero finite value.

It is possible only when $n = 5$

31. We have

$$\lim_{n\to\infty}\left(\frac{n\cdot 1 + (n-1)(1+2) + (n-2)(1+2+3) + ... + 1\sum_{r=1}^{n} r}{n^4}\right)$$

Let $t_r = (n - (r-1))(1 + 2 + 3 + ... + r)$

$= (n + 1 - r)\left(\frac{r(r+1)}{2}\right)$

$= \frac{1}{2}(n+1)(r(r+1)) - \frac{1}{2}(r^2(r+1))$

$= \frac{1}{2}(n+1)(r^2 + r) - \frac{1}{2}(r^3 + r^2)$

Thus, $S_n = \frac{1}{2}(n+1)\sum_{r=1}^{n}(r^2 + r) - \frac{1}{2}\sum_{r=1}^{n}(r^3 + r^2)$

Now

$$\lim_{n\to\infty}\left(\frac{n\cdot 1 + (n-1)(1+2) + (n-2)(1+2+3) + ... + 1\sum_{r=1}^{n} r}{n^4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{2}(n+1)\sum_{r=1}^{n}(r^2 + r) - \frac{1}{2}\sum_{r=1}^{n}(r^3 + r^2)}{n^4}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{2}(n+1)\left\{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\} - \frac{1}{2}\left\{\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right\}}{n^4}\right)$$

$= \frac{1}{6} - \frac{1}{8} = \frac{4-3}{24} = \frac{1}{24}.$

32. We have

$$\lim_{n\to\infty}\left(\frac{1^4 + 2^4 + 3^4 + ... + n^4}{(1^2 + 2^2 + 3^2 + ... + n^2)(1^3 + 2^3 + 3^3 + ... + n^3)}\right)$$

INTEGER TYPE QUESTIONS

1. We have $\lim_{x\to 0}\left(\frac{\ln(1 + x + x^2 + ... + x^n)}{nx}\right) = \frac{1}{5}$

$\lim_{x\to 0}\left(\frac{\frac{1 + 2x + 3x^3 + ... + nx^{n-1}}{(1 + x + x^2 + ... + x^n)}}{n}\right) = \frac{1}{5}$

$\frac{1}{n} = \frac{1}{5}$

$n = 5.$

2. We have $\lim_{x\to 0}\left(\frac{\tan x - \sin x}{x^3}\right) = \frac{a}{b}$

$\lim_{x\to 0}\left(\frac{\tan x(1 - \cos x)}{x^3}\right) = \frac{a}{b}$

$\lim_{x\to 0}\left(\frac{\tan x}{x} \times \frac{2\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x^2}{4}\right) \times 4}\right) = \frac{a}{b}$

$\frac{a}{b} = \frac{1}{2}$

Hence, the value of $a + b + 3 = 1 + 2 + 3 = 6$

3. We have $\lim_{x\to 0}\left(\frac{\sqrt[64]{1 + P(x)} - 1}{x}\right)$

$= \lim_{x\to 0}\left(\frac{(1 + P(x))^{\frac{1}{64}} - 1}{x}\right)$

$= \lim_{x\to 0}\left(\frac{\left(1 + \frac{1}{64}P(x)\right) - 1}{x}\right)$

$= \lim_{x\to 0}\left(\frac{1}{64} \times \frac{P(x)}{x}\right)$

$$= \lim_{x\to 0}\left(\frac{1}{64}\times\frac{a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{64}\times\left(a_1 + a_2x + a_3x^2 + ... + a_nx^{n-1}\right)\right)$$

$$= \frac{a_1}{64} = \frac{1}{64}$$

So, $a = 1,\ b = 64$

Hence, the value of $\sqrt{b + a} - 1$

$$= \sqrt{64 + 1 - 1} = \sqrt{64} = 8$$

4. We have $\lim_{x\to 0}\left(\frac{x^{2n}\sin^n x}{x^{2n} - \sin^{2n} x}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin^n x}{1 - \frac{\sin^{2n} x}{x^{2n}}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)^n}{1 - \left(1 - \frac{x^2}{3!} + \frac{x^4}{6!} - \frac{x^6}{7!} + ...\right)^{2n}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(x - \frac{x^3}{3!}\right)^n}{1 - \left(1 - \frac{x^2}{3!}\right)^{2n}}\right)$$

$$= \lim_{x\to 0}\left(\frac{x^n\left(1 - \frac{x^2}{3!}\right)^n}{1 - \left(1 - \frac{x^2}{3!}\right)^{2n}}\right)$$

It provides a non-zero definite number only when $n = 2$.

5. We have

$$\lim_{x\to\infty}\left(\sqrt{x^2 + x^2\sin\left(\frac{1}{x}\right)} - x\right)$$

$$= \lim_{x\to\infty}\left(\frac{x^2 + x^2\sin\left(\frac{1}{x}\right) - x^2}{\sqrt{x^2 + x^2\sin\left(\frac{1}{x}\right)} + x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x^2\sin\left(\frac{1}{x}\right)}{\sqrt{x^2 + x^2\sin\left(\frac{1}{x}\right)} + x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{x\sin\left(\frac{1}{x}\right)}{\sqrt{1 + \sin\left(\frac{1}{x}\right)} + 1}\right)$$

$$= \frac{1}{2}$$

Hence, the value of $(a + b)$ is 3

6. We have $\lim_{x\to 1}\left(\frac{\tan(x - 1)\log_e(x^{x-1})}{(x - 1)^3}\right)$

$$= \lim_{x\to 1}\left(\frac{\tan(x - 1)(x - 1)\log_e(x)}{(x - 1)^3}\right)$$

$$= \lim_{x\to 1}\left(\frac{\tan(x - 1)\log_e(x)}{(x - 1)^2}\right)$$

$$= \lim_{x\to 1}\left(\frac{\tan(x - 1)}{(x - 1)}\times\frac{\log_e(x)}{(x - 1)}\right)$$

$$= \lim_{h\to 0}\left(\frac{\tan(1 + h - 1)}{(1 + h - 1)}\times\frac{\log_e(1 + h)}{(1 + h - 1)}\right)$$

$$= \lim_{h\to 0}\left(\frac{\tan(h)}{(h)}\times\frac{\log_e(1 + h)}{(h)}\right)$$

$$= 1$$

Thus, $L = 1$

Hence, the value of $(L + 3)$ is 4

7. We have $\lim_{x\to 0}\left(\frac{2(\tan x - \sin x) - x^3}{x^5}\right)$

$$= \lim_{x\to 0}\left(\frac{2\left(\left\{x + \frac{x^3}{3} + \frac{2x^5}{15} + ...\right\} - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right)\right)}{x^5}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\left(\left(\left(\frac{x^2}{3} + \frac{x^3}{3!}\right) + \left(\frac{2}{15}x^5 - \frac{x^5}{5!}\right) + ...\right)\right) - x^3}{x^5}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\left(\left(\frac{x^3}{2} + \frac{x^5}{8} + ...\right)\right) - x^3}{x^5}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(\left(x^3 + \frac{x^5}{4} + ...\right)\right) - x^3}{x^5}\right)$$

$$= \frac{1}{4}$$

Thus, $m = 1, n = 2$

Hence, the value of $m + n + 2 = 5$

8. We have

$$\lim_{x\to 0}\left(\frac{1 + \sin x - \cos x - + \log(1 - x)}{x\tan^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1 + \sin x - \cos x - + \log(1 - x)}{x^3\dfrac{\tan^2 x}{x^2}}\right)$$

$$= \lim_{x\to 0}\left(\frac{1 + \sin x - \cos - + \log(1 - x)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\cos x + \sin x - \dfrac{1}{(1 - x)}}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\sin x + \cos x - \dfrac{2}{(1 - x)^2}}{6x}\right)$$

$$= \lim_{x\to 0}\left(\frac{-\cos x - \sin x - \dfrac{2}{(1 - x)^3}}{6}\right)$$

$$= -\frac{3}{6} = -\frac{1}{2}$$

Thus, $m = 1, n = 2$

Hence, the value of $(m + n)$ is 3.

9. We have $\lim_{x\to 0}\left(\dfrac{e^x + e^{-x} + 2\cos x - 4}{x^4}\right)$

$$= \lim_{x\to 0}\left(\frac{e^x - e^{-x} - 2\sin x}{4x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^x + e^{-x} - 2\cos x}{12x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^x - e^{-x} + 2\sin x}{24x}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^x + e^{-x} + 2\cos x}{24}\right)$$

$$= \frac{4}{24} = \frac{1}{16}$$

Thus, $p = 1, q = 6$

Hence, the value of $(q - p)$ is $= 5$.

10. We have $P = \lim_{n\to\infty}(4^n + 3^n)^{1/n}$

$$= \lim_{n\to\infty}\left(4^n\left(1 + \left(\frac{3}{4}\right)^n\right)\right)^{1/n}$$

$$= \lim_{n\to\infty} 4\left(\left(1 + \left(\frac{3}{4}\right)^n\right)\right)^{1/n}$$

$$= 4((1 + 0))^0 = 4$$

Also, $\lim_{x\to 1}\left(\dfrac{1}{\ln x} - \dfrac{1}{(x - 1)}\right)$

$$= \lim_{x\to 1}\left(\frac{(x - 1) - \ln x}{(x - 1)\ln x}\right)$$

$$= \lim_{x\to 1}\left(\frac{\dfrac{1}{x^2}}{\dfrac{1}{x} + \dfrac{1}{x^2}}\right)$$

$$= \lim_{x\to 1}\left(\frac{1 - \dfrac{1}{x}}{\ln x + \left(\dfrac{x - 1}{x}\right)}\right)$$

$$= \lim_{x\to 1}\left(\frac{1 - \dfrac{1}{x}}{\ln x + \left(1 - \dfrac{1}{x}\right)}\right)$$

$$= \lim_{x\to 1}\left(\frac{\dfrac{1}{x^2}}{\dfrac{1}{x} + \dfrac{1}{x^2}}\right)$$

$$= \frac{1}{1 + 1} = \frac{1}{2}$$

Thus $m = 1$ and $n = 2$

Hence, the value of $(m + n + p)$

$= 4 + 1 + 2$

$= 7.$

11. We have

$$\lim_{x\to 0}\left(\frac{2x - \sin 2x - \tan^{-1}\left(\dfrac{2x^3}{1 + x^6}\right)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{2x - \sin 2x - 2\tan^{-1}(x^3)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{2 - 2\cos 2x - \dfrac{6x^2}{1 + x^6}}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{2(1 - \cos 2x) - \dfrac{6x^2}{1 + x^6}}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{4\sin^2 x - \dfrac{6x^2}{1+x^6}}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{4}{3}\frac{\sin^2 x}{x^2} - \frac{6}{3(1+x^6)}\right)$$

$$= \frac{4}{3} - 2 = -\frac{2}{3}$$

Thus, $p = 2$ and $q = 3$

Hence, the value of $(p + q)$ is 5.

12. We have

$$\lim_{x\to\infty}\left\{x\left(\tan^{-1}\left(\frac{x+1}{x+2}\right) - \tan^{-1}\left(\frac{x}{x+2}\right)\right)\right\}$$

$$= \lim_{x\to\infty}\left\{x\left(\tan^{-1}\left(\frac{\dfrac{x+2}{x+2} - \dfrac{x}{x+2}}{1+\left(\dfrac{x+1}{x+2}\times\dfrac{x}{x+2}\right)}\right)\right)\right\}$$

$$= \lim_{x\to\infty}\left\{x\left(\tan^{-1}\left(\frac{(x+2)}{(x+2)^2 + x(x+1)}\right)\right)\right\}$$

$$= \lim_{x\to\infty}\left\{x\left(\tan^{-1}\left(\frac{x+2}{2x^2+5x+4}\right)\right)\right\}$$

$$= \lim_{x\to\infty}\left(\frac{\tan^{-1}\left(\dfrac{x+2}{2x^2+5x+4}\right)}{\left(\dfrac{x+2}{2x^2+5x+4}\right)}\right)\times\left(\frac{x(x+2)}{2x^2+5x+4}\right)$$

$$= 1 \times \frac{1}{2} = \frac{1}{2}$$

Thus, $m = 1$ and $n = 2$

Hence, the value of $(m + n)$ is 3.

Problems asked in JEE Main Exams. From 2002-2014

1. Ans. (a)

We have

$$\lim_{n\to\infty}\left(\frac{1^p + 2^p + 3^p + \dots + n^p}{n^{p+1}}\right)$$

$$= \lim_{x\to\infty}\left\{\frac{1}{n}\left(\frac{1^p + 2^p + 3^p + \dots + n^p}{n^p}\right)\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{1}{n}\left(\left(\frac{1}{n}\right)^p + \left(\frac{2}{n}\right)^p + \left(\frac{3}{n}\right)^p + \dots + \left(\frac{n}{n}\right)^p\right)\right\}$$

$$= \lim_{x\to\infty}\left\{\frac{1}{n}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)^p\right\}$$

$$= \int_0^1 x^p\,dx$$

$$= \left(\frac{x^{p+1}}{p+1}\right)_0^1$$

$$= \frac{1}{p+1}$$

2. Ans.(a)

We have

$$\lim_{x\to\infty}\left(\frac{\log x^n - [x]}{[x]}\right),\ n \in N,\ [,] = \text{G.I.F}$$

$$= \lim_{x\to\infty}\left(\frac{\log x^n}{[x]} - \frac{[x]}{[x]}\right)$$

$= 0 - 1$

$= -1.$

3. Ans. (c)

$$\lim_{x\to 2}\left(\frac{xf(2) - 2f(x)}{x-2}\right)$$

$$= \lim_{x\to 2}\left(\frac{f(2) - 2f'(x)}{1}\right)$$

$= f(2) - 2f'(2)$

$= 4 - 8$

$= -4$

4. Ans. (a)

$$\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x$$

$$= \lim_{x\to\infty}\left(1 + \frac{4x}{x^2+x+3}\right)^x$$

$$= e^{\lim_{x\to\infty}\left(\frac{4x}{x^2+x+3}\right)^x}$$

$$= e^{\lim_{x\to\infty}\left(\frac{4x^2}{x^2+x+3}\right)}$$

$$= e^{\lim_{x\to\infty}\left(\frac{4}{1+\frac{1}{x}+\frac{3}{x^2}}\right)}$$

$= e^4$

5. Ans.(c)

$$\lim_{x\to 0}\left(\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sec^2(x^2)\,2x}{\sin x + x\cos x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sec^2(x^2)}{\frac{\sin x}{x} + \cos x}\right)$$

$$= \frac{2}{2} = 1$$

6. Ans. (c)

Given $\lim_{x\to 0}\left(\frac{\log(3+x) - \log(3-x)}{x}\right) = k$

$$k = \lim_{x\to 0}\left(\frac{\log\left(1+\frac{x}{3}\right) - \log\left(1+\left(-\frac{x}{3}\right)\right)}{x}\right)$$

$$k = \lim_{x\to 0}\left(\frac{\log\left(1+\frac{x}{3}\right)}{\frac{x}{3}\times 3} - \frac{\log\left(1+\left(-\frac{x}{3}\right)\right)}{\left(-\frac{x}{3}\right)\times(-3)}\right)$$

$$k = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

7. Ans. (c)

We have

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{\left(1-\tan\left(\frac{x}{2}\right)\right)}{\left(1+\tan\left(\frac{x}{2}\right)\right)} \times \frac{(1-\sin x)}{(\pi - 2x)^3}\right)$$

$$= \lim_{x\to\frac{\pi}{2}}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \times \frac{(1-\sin x)}{(\pi-2x)^3}\right)$$

$$= \lim_{h\to 0}\left(\tan\left(\frac{\pi}{4} - \frac{\pi}{4} - \frac{h}{2}\right) \times \frac{\left(1-\sin\left(\frac{\pi}{2}+h\right)\right)}{\left(\pi - 2\left(\frac{\pi}{2}+h\right)^3\right)}\right)$$

$$= \lim_{h\to 0}\left(\tan\left(-\frac{h}{2}\right) \times \frac{(1-\cos h)}{(-2h)^3}\right)$$

$$= \lim_{h\to 0}\left(\frac{\tan\left(-\frac{h}{2}\right)}{\left(-\frac{h}{2}\right)\times(-2)} \times \frac{2\sin^2\left(\frac{h}{2}\right)}{(-2)^3\left(\frac{h^2}{4}\right)\times 4}\right)$$

$$= \frac{1}{32}$$

8. Ans.

We have

$$\lim_{n\to\infty}\left(\frac{1^4 + 2^4 + \ldots + n^4}{n^5} - \frac{1^3 + 2^3 + \ldots + n^3}{n^4}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1^4+2^4+\ldots+n^4}{n^4} - \frac{1^3+2^3+\ldots+n^3}{n^3}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)^4 - \left(\frac{r}{n}\right)^3\right)$$

$$= \int_0^1 (x^4 - x^3)\,dx$$

$$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = -\frac{1}{20}$$

9. Ans. (d)

We have

$$\lim_{x\to a}\left(\frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)}\right) = 4$$

$$\lim_{x\to a}\left(\frac{f(a)g'(x) - g(a)f'(x)}{g'(x) - f'(x)}\right) = 4$$

$$\left(\frac{f(a)g'(a) - g(a)f'(a)}{g'(a) - f'(a)}\right) = 4$$

$$\left(\frac{kg'(a) - kf'(a)}{g'(a) - f'(a)}\right) = 4$$

$$k\left(\frac{g'(a) - f'(a)}{g'(a) - f'(a)}\right) = 4$$

$$k = 4$$

10. Ans. (b)

Given $\lim_{x\to\infty}\left(1 + \frac{a}{x} + \frac{b}{x^2}\right)^{2x} = e^2$

$$e^{\lim_{x\to\infty}\left(\frac{a}{x}+\frac{b}{x^2}\right)} = e^2$$

$$e^{\lim_{x\to\infty}\left(2a + \frac{2b}{x}\right)} = e^2$$

$$e^{2a} = e^2$$

$$a = 1,\ b \in R$$

11. We have

$$\lim_{n\to\infty}\left(\frac{1}{n^2}\sec^2\left(\frac{1}{n^2}\right) + \frac{2}{n^2}\sec^2\left(\frac{4}{n^2}\right) + \ldots + \frac{1}{n^2}\sec^2\left(\frac{n^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{r}{n^2}\sec^2\left(\frac{r^2}{n^2}\right)\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)\sec^2\left(\frac{r^2}{n^2}\right)\right)$$

$$= \int_0^1 x\sec^2 x\,dx$$

$$= x \int \sec^2 x \, dx \Big|_x^1 - \int_0^1 \tan x \, dx$$

$$= x \tan x |_0^1 - (\log \sec x)|_0^1$$

$$= \tan(1) - \log(\sec 1)$$

12. Ans. (a)

We have

$$\lim_{x \to 2} \int_6^{f(x)} \left(\frac{4t^3}{x-2}\right) dt$$

$$= \lim_{x \to 2} \left(\frac{\int_6^{f(x)} (4t^3)\, dt}{(x-2)}\right)$$

$$= \lim_{x \to 2} \left(\frac{4(f(x))^3 f'(x)}{1}\right)$$

$$= 4(f(2))^3 f'(2)$$

$$= 4 \times 216 \times \frac{1}{48}$$

$$= 18$$

13. Ans. (d)

We have

$$\lim_{x \to \alpha} \left(\frac{1 - \cos(ax^2 + bx + c}{(x-\alpha)^2}\right)$$

$$= \lim_{x \to \alpha} \left(\frac{1 - \cos\{a(x-\alpha)(x-\beta)\}}{(x-\alpha)^2}\right)$$

$$= \lim_{x \to a} \left(\frac{2\sin^2\left\{\frac{a(x-\alpha)(x-\beta)}{2}\right\}}{\left\{\frac{a(x-\alpha)(x-\beta)^2}{2}\right\}}\right) \times \frac{a^2(x-\beta)^2}{4}$$

$$= \frac{a^2(\alpha - \beta)^2}{2}$$

14. No questions asked in 2006-2009.

15. Since $f(x)$ be an increasing function, then we can write

$$f(x) \le f(2x) \le f(3x)$$

$$1 \le \frac{f(2x)}{f(x)} \le \frac{f(3x)}{f(x)}$$

$$\lim_{x \to \infty} 1 \le \lim_{x \to \infty} \frac{f(2x)}{f(x)} \le \lim_{x \to \infty} \frac{f(3x)}{f(x)}$$

$$1 \le \lim_{x \to \infty} \frac{f(2x)}{f(x)} \le 1$$

Thus, by Sandwitch theorem, we get,

$$\lim_{x \to \infty} \frac{f(2x)}{f(x)} = 1$$

16. Ans. (c)

We have

$$\lim_{x \to 2} \left(\frac{\sqrt{1 - \cos 2(x-2)}}{(x-2)}\right)$$

$$= \lim_{x \to 2} \left(\frac{\sqrt{2\sin^2(x-2)}}{(x-2)}\right)$$

$$= \lim_{x \to 2} \left(\frac{\sqrt{2}|\sin(x-2)|}{(x-2)}\right)$$

Now, L.H.L $= \lim_{x \to 2} \left(-\frac{\sqrt{2}\sin(x-2)}{(x-2)}\right) = -\sqrt{2}$

R.H.L $= \lim_{x \to 2^+} \left(\frac{\sqrt{2}\sin(x-2)}{(x-2)}\right) = \sqrt{2}$

Since R.H.L ≠ L.H.L, so limit does not exist.

17. No questions asked in 2012.

18. Ans. (c)

We have $\lim_{x \to 0} \left(\frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}\right)$

$$= \lim_{x \to 0} \left(2 \times \frac{\sin^2 x}{x^2} \times \frac{(3 + \cos x)}{\left(\frac{\tan 4x}{4x}\right) \times 4}\right)$$

$$= 2 \times \frac{1+3}{4} = 2$$

19. Ans. (d)

We have $\lim_{x \to 0} \left(\frac{\sin(\pi \cos^2 x)}{x^2}\right)$

$$= \lim_{x \to 0} \left(\frac{\sin(\pi(1 - \sin^2 x))}{x^2}\right)$$

$$= \lim_{x \to 0} \left(\frac{\sin(\pi - \pi \sin^2 x)}{x^2}\right)$$

$$= \lim_{x \to 0} \left(\frac{\sin(\pi \sin^2 x)}{x^2}\right)$$

$$= \lim_{x \to 0} \left(\frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{\pi \sin^2 x}{x^2}\right) = \pi$$

20. Ans. (b)

21. Ans. (b)

We have $p = \lim_{x \to 0^+} (1 + \tan^2 \sqrt{x})^{\frac{1}{2x}}$

$= e^{\lim_{x \to 0^+} (\tan^2 \sqrt{x})^{\frac{1}{2x}}}$

$= e^{\lim_{x \to 0^+} \frac{1}{2}\left(\frac{\tan^2 \sqrt{x}}{x}\right)}$

$= e^{\frac{1}{2}}$

Now, $\log(p) = \log(e^{1/2}) = \frac{1}{2}$

Questions asked in Past IIT-JEE Exams

1. Limit $= \lim_{h \to 0} \left(\frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}\right)$

$= \lim_{h \to 0} \left(\frac{2(a+h)\sin(a+h) + (a+h)^2 + \cos(a+h)}{1}\right)$

$= (2a \sin a + a^2 \cos a)$

2. Limit

$= \lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots + \frac{1}{6n}\right)$

$= \lim_{n \to \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots. + \frac{1}{n+5n}\right)$

$= \lim_{n \to \infty} \sum_{r=1}^{5n} \left(\frac{1}{n+r}\right)$

$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{5n} \left(\frac{n}{n+r}\right)$

$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{5n} \left(\frac{1}{1+(r/n)}\right)$

$= \int_0^5 \frac{dx}{1+x} = \log 6$

3. Limit $= \lim_{x \to 0} \left(\frac{2^x - 1}{\sqrt{(1+x)} - 1}\right)$

$= \lim_{x \to 0} \left(\frac{2^x - 1}{x} \times (\sqrt{1+x} + 1)\right)$

$= 2 \log 2$

4. Limit $= \lim_{x \to 1} \left(\frac{G(x) - G(1)}{x - 1}\right)$

$= \lim_{x \to 1} G'(x)$

$= G'(1)$

Now, $G(x) = -\sqrt{25 - x^2}$

$G'(x) = -\frac{(-2x)}{2\sqrt{25 - x^2}}$

$= \frac{x}{\sqrt{25 - x^2}}$

Thus, $G'(1) = \frac{1}{\sqrt{25 - 1}} = \frac{1}{2\sqrt{6}}$

5. Ans. (c)

Limit $= \lim_{x \to a} \left(\frac{g(x)f(a) - g(a)f(x)}{x - a}\right)$

$= \lim_{x \to a} \left(\frac{g'(x)f(a) - g(a)f'(x)}{1}\right)$

$= g'(a)f(a) - g(a)f'(a)$

$= 2 \cdot 2 + 1 \cdot 1$

$= 5$

6. Ans. (b)

Limit $= \lim_{n \to \infty} \left[\frac{1}{1-n^2} + \frac{2}{1-n^2} + \ldots + \frac{n}{1-n^2}\right]$

$= \lim_{n \to \infty} \left(\frac{1 + 2 + 3 + \ldots.. + n}{1 - n^2}\right)$

$= \lim_{n \to \infty} \left(\frac{n(n+1)}{2(1-n^2)}\right)$

$= \lim_{n \to \infty} \left(\frac{(n^2 + n)}{2(1-n^2)}\right)$

$= \lim_{n \to \infty} \left(\frac{\left(1 + \frac{1}{n}\right)}{2\left(\frac{1}{n^2} - 1\right)}\right)$

$= \left(\frac{(1+0)}{2(0-1)}\right)$

$= \left(-\frac{1}{2}\right)$

7. Ans. (d)

Limit $= \lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \left(\frac{\sin[x]}{[x]}\right)$

$= \begin{cases} \lim_{x \to 0^+} \left(\frac{\sin[x]}{[x]}\right) \\ \lim_{x \to 0^-} \left(\frac{\sin[x]}{[x]}\right) \end{cases}$

$$= \begin{cases} \lim_{x\to 0^+}\left(\dfrac{\sin(0)}{(0)}\right) = 1 \\ \lim_{x\to 0^-}\left(\dfrac{\sin[x]}{[x]}\right) = \left(\dfrac{\sin(-1)}{(-1)}\right) = \sin(1) \end{cases}$$

Since R.H.L ≠ L.H.L, so limit does not exist.

8. Limit $= \lim_{x\to 1}\left((1-x)\tan\left(\dfrac{\pi x}{2}\right)\right)$

$= \lim_{y\to 0}\left((1-1-y)\tan\left((1+y)\dfrac{\pi}{r}\right)\right)$

$= \lim_{y\to 0}(-y)\tan\left(\dfrac{\pi}{r} + \dfrac{\pi y}{2}\right)$

$= \lim_{y\to 0}(-y) \times -\cot\left(\dfrac{\pi y}{2}\right)$

$= \lim_{y\to 0}\left(\dfrac{\left(\dfrac{\pi y}{2}\right)}{\tan\left(\dfrac{\pi y}{2}\right)} \times \dfrac{2}{\pi}\right)$

$= \dfrac{2}{\pi}$

9. Ans. (c)

Limit $= \lim_{x\to 1}\left(\dfrac{\sqrt{1-\cos 2(x-1)}}{x-1}\right)$

$= \lim_{y\to 0}\left(\dfrac{\sqrt{1-\cos 2y}}{y}\right)$

$= \lim_{y\to 0}\left(\dfrac{\sqrt{2\sin^2 y}}{y}\right)$

$= \lim_{y\to 0}\left(\dfrac{\sqrt{2}|\sin y|}{y}\right)$

$$= \begin{cases} \lim_{y\to 0^+}\left(\dfrac{\sqrt{2}|\sin y|}{y}\right) \\ \lim_{y\to 0^-}\left(\dfrac{\sqrt{2}|\sin y|}{y}\right) \end{cases}$$

$$= \begin{cases} \lim_{y\to 0^+}\left(\sqrt{2} \times \dfrac{\sin y}{y}\right) \\ \lim_{y\to 0^-}\left(-\sqrt{2} \times \dfrac{\sin y}{y}\right) \end{cases}$$

$$= \begin{cases} \sqrt{2} \\ -\sqrt{2} \end{cases}$$

Since R.H.L ≠ L.H.L, so limit does not exist.

10. Limit $= \lim_{x\to 0}(g(f(x)))$

$= \lim_{x\to 0}(g(\sin x))$

$= \lim_{x\to 0}((\sin x)^2 + 1)$

$= 1$

11. Limit $= \lim_{x\to\infty}\left(\dfrac{x^4\sin\left(\dfrac{1}{x}\right) + x^2}{(1+|x|^3)}\right)$

$= \lim_{x\to\infty}\left(\dfrac{x^4\sin\left(\dfrac{1}{x}\right) + x^2}{(1-x^3)}\right)$

$= \lim_{x\to\infty}\left(\dfrac{x\sin\left(\dfrac{1}{x}\right) + \dfrac{1}{x}}{\left(\dfrac{1}{x^3} - 1\right)}\right)$

$= \left(\dfrac{1+0}{0-1}\right)$

$= -1$

12. Limit $= \lim_{x\to 0}\left(\dfrac{\int_0^{x^2}\cos^2 t\,dt}{x\sin x}\right)$

$= \lim_{x\to 0}\left(\dfrac{\cos^2 x \cdot 2x}{\sin x + x\cos x}\right)$

$= \lim_{x\to 0}\left(\dfrac{2\cos^2 x}{\dfrac{\sin}{x} + \cos x}\right)$

$= \left(\dfrac{2}{1+1}\right)$

$= 1$

13. Limit $= \lim_{x\to 9}\left(\dfrac{\sqrt{f(x)} - 3}{\sqrt{x} - 3}\right)$

$= \lim_{x\to 9}\left(\dfrac{\dfrac{1}{2\sqrt{f(x)}} \times f'(x)}{\dfrac{1}{2\sqrt{x}}}\right)$

$= \lim_{x\to 9}\left(\dfrac{f'(x)\sqrt{x}}{\sqrt{f(x)}}\right)$

$$= \left(\frac{3f'(9)}{\sqrt{f(9)}}\right)$$

$$= \left(\frac{3 \cdot 4}{\sqrt{9}}\right)$$

$$= 4$$

14.

15. No questions asked in 1990.

16. (d)

$$\text{Limit} = \lim_{x \to 0}\left(\frac{\sqrt{\frac{1}{2}(1 - \cos^2 x)}}{x}\right)$$

$$= \lim_{x \to 0}\left(\frac{1}{\sqrt{2}} \times \frac{|\sin x|}{x}\right)$$

$$= \begin{cases} \lim_{x \to 0^+}\left(\frac{1}{\sqrt{2}} \times \frac{|\sin x|}{x}\right) \\ \lim_{x \to 0^-}\left(\frac{1}{\sqrt{2}} \times \frac{|\sin x|}{x}\right) \end{cases}$$

$$= \begin{cases} \lim_{x \to 0^+}\left(\frac{1}{\sqrt{2}} \times \frac{\sin x}{x}\right) \\ \lim_{x \to 0^-}\left(-\frac{1}{\sqrt{2}} \times \frac{\sin x}{x}\right) \end{cases}$$

$$= \begin{cases} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{cases}$$

Since R.H.L ≠ L.H.L, so limit does not exist.

17. $$\text{Limit} = \lim_{x \to \infty}\left(\frac{x + 6}{x + 1}\right)^{x + 4}$$

$$= \lim_{x \to \infty}\left(1 + \frac{5}{x + 1}\right)^{x + 4}$$

$$= e^{\lim_{x \to \infty}\left(\frac{5(x + 4)}{x + 1}\right)}$$

$$= e^5$$

18. $$\text{Limit} = \lim_{x \to 0}\left(\frac{x}{\tan^{-1} 2x}\right)$$

$$= \lim_{x \to 0}\left(\frac{1}{2} \times \frac{2x}{\tan^{-1} 2x}\right)$$

$$= \frac{1}{2}$$

19. $$\text{Limit} = \lim_{x \to 0}\left(\tan\left(\frac{\pi}{4} + x\right)\right)^{\frac{1}{x}}$$

$$= \lim_{x \to 0}\left(\frac{1 + \tan x}{1 - \tan x}\right)^{\frac{1}{x}}$$

$$= \lim_{x \to 0}\left(1 + \left(\frac{1 + \tan x}{1 - \tan x} - 1\right)\right)^{\frac{1}{x}}$$

$$= \lim_{x \to 0}\left(1 + \left(\frac{2\tan x}{1 - \tan x}\right)\right)^{\frac{1}{x}}$$

$$= e^{\lim_{x \to 0}\left(\frac{2\tan x}{x(1 - \tan x)}\right)}$$

$$= e^2$$

20. $$\text{Limit} = \lim_{x \to 0}\left(\frac{5x^2 + 1}{3x^2 + 1}\right)^{\frac{1}{x^2}}$$

$$= \lim_{x \to 0}\left(\frac{5 + \frac{1}{x^2}}{3 + \frac{1}{x^2}}\right)^{\frac{1}{x^2}}$$

$$= \lim_{y \to \infty}\left(\frac{y + 5}{y + 3}\right)^{y}$$

$$= \lim_{y \to \infty}\left(1 + \frac{2}{y + 3}\right)^{y}$$

$$= e^{\lim_{y \to \infty}\left(\frac{2y}{y + 3}\right)}$$

$$= e^2$$

21. $$\text{Limit} = \lim_{h \to 0}\left(\frac{\log(1 + 2h) - 2\log(1 + h)}{h^2}\right)$$

$$= \lim_{h \to 0}\left(\frac{\left(\left(2h - \frac{(2h)^2}{2} + \frac{(2h)^3}{3} - \ldots\right) - 2\left(h - \frac{h^2}{2} + \frac{h^3}{3} - \ldots\right)\right)}{h^2}\right)$$

$$= \lim_{h \to 0}\left(\frac{\left(\left(-\frac{(2h)^2}{2} + \frac{(2h)^3}{3} - \ldots\right) - 2\left(-\frac{h^2}{2} + \frac{h^3}{3} - \ldots\right)\right)}{h^2}\right)$$

$$= \lim_{h \to 0}\left(\left(-2 + \frac{8h}{3} - \ldots\right) - \left(-1 + \frac{2h}{3} - \ldots\right)\right)$$

$$= -2 + 1$$

$$= -1$$

22. Ans. (b)

$$\text{Limit} = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{r=1}^{2n}\frac{\left(\frac{r}{n}\right)}{\sqrt{1+\left(\frac{r}{n}\right)^2}}\right)$$

$$= \int_0^2 \frac{x\,dx}{\sqrt{1+x^2}}$$

$$= \left(\sqrt{1+x^2}\right)\Big|_0^2$$

$$= (\sqrt{5}-1)$$

23. Ans. (c)

$$\text{Limit} = \lim_{x\to 1}\left(\frac{\sqrt{1-\cos 2(x-1)}}{x-1}\right)$$

$$= \lim_{y\to 0}\left(\frac{\sqrt{1-\cos 2y}}{y}\right)$$

$$= \lim_{y\to 0}\left(\frac{\sqrt{2\sin^2 y}}{y}\right)$$

$$= \lim_{y\to 0}\left(\frac{\sqrt{2}|\sin y|}{y}\right)$$

$$= \begin{cases}\lim\limits_{y\to 0^+}\left(\dfrac{\sqrt{2}|\sin y|}{y}\right)\\ \lim\limits_{y\to 0^-}\left(\dfrac{\sqrt{2}|\sin y|}{y}\right)\end{cases}$$

$$= \begin{cases}\lim\limits_{y\to 0^+}\left(\sqrt{2}\times\dfrac{\sin y}{y}\right)\\ \lim\limits_{y\to 0^-}\left(-\sqrt{2}\times\dfrac{\sin y}{y}\right)\end{cases}$$

$$= \begin{cases}\sqrt{2}\\ -\sqrt{2}\end{cases}$$

Since R.H.L ≠ L.H.L, so limit does not exist.

24. Ans. (b)

$$\text{Limit} = \lim_{x\to 0}\left(\frac{x\tan 2x - 2x\tan x}{1-(\cos 2x)^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{2x\tan x}{1-\tan^2 x} - 2x\tan x}{4\sin^4 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2x\tan x\left(\frac{1}{1-\tan^2 x}-1\right)}{4\sin^4 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2x\tan x\times\tan^2 x}{4\sin^4 x(1-\tan^2 x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{2x^4\times\frac{\tan x}{x}\times\frac{\tan^2 x}{x^2}}{4x^4\times\frac{\sin^4 x}{x^4}(1-\tan^2 x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\times\frac{\tan x}{x}\times\frac{\tan^2 x}{x^2}}{4\times\frac{\sin^4 x}{x^4}(1-\tan^2 x)}\right)$$

$$= \frac{1}{2}$$

25. Ans. (d)

$$\text{Limit} = \lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$$

$$= \lim_{x\to\infty}\left(1+\frac{-5}{x+2}\right)^x$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{-5x}{x+2}\right)}$$

$$= e^{-5}$$

26. Ans. (b)

$$\text{Limit} = \lim_{x\to 0}\left(\frac{\sin(\pi\cos^2 x)}{x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{\cos(\pi\cos^2 x)\times -\pi\sin 2x}{2x}\right)$$

$$= \lim_{x\to 0}\left(-\pi\cos(\pi\cos^2 x)\times\frac{\sin 2x}{2x}\right)$$

$$= (-\pi\cos(\pi))$$

$$= \pi$$

27. Ans. (c)

$$\text{Limit} = \lim_{x\to 0}\left(\frac{(\cos x-1)(\cos x-e^x)}{x^n}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{2\sin^2\left(\frac{x}{2}\right)}{\frac{x^2}{4}}\left(\frac{(e^x-1)}{x}+\frac{1-\cos x}{x}\right)}{x^{n-3}}\right)$$

It will provide us a finite limit only when

$n - 3 = 0$

Thus $n = 3$

28. Ans. (c)

$$\text{Limit} = \lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x}$$

$$= \lim_{x\to 0}\left(1+\frac{f(1+x)}{f(1)}-1\right)^{1/x}$$

$$= e^{\lim_{x\to 0}\left(\frac{f(1+x)-f(1)}{xf(1)}\right)}$$

$$= e^{\lim_{x\to 0}\left(\frac{f'(1+x)}{f(1)}\right)}$$

$$= e^{\frac{f'(1)}{f(1)}}$$

$$= e^{\frac{6}{3}} = e^2$$

29. Ans. (c)

$$\text{Limit} = \lim_{h\to 0}\left(\frac{f(2+h+h^2)-f(2)}{f(h-h^2+1)-f(1)}\right)$$

$$= \lim_{h\to 0}\left(\frac{f'(2+h+h^2)\cdot(1+2h)}{f'(h-h^2+1)\cdot(1-2h)}\right)$$

$$= \lim_{h\to 0}\left(\frac{f'(2)}{f'(1)}\right)$$

$$= \frac{6}{4} = \frac{3}{2}$$

30. Ans. (d)

We have, $\lim_{x\to 0}\left(\frac{[(a-n)nx-\tan x]\sin nx}{x^2}\right) = 0$

$$\lim_{x\to 0}\left(\left(\frac{(a-n)nx-\tan x}{x}\times\frac{\sin nx}{x}\right)\right) = 0$$

$$\lim_{x\to 0}\left(\left(\left((a-n)n-\frac{\tan x}{x}\right)\times\frac{\sin nx}{nx}\times n\right)\right) = 0$$

$$((a-n)n-1)n = 0$$

$$((a-n)n-1) = 0$$

$$(a-n)n = 1$$

$$(a-n) = \frac{1}{n}$$

$$a = n+\frac{1}{n}$$

31. Ans. (c)

$$\text{Limit} = \lim_{x\to 0}\left(\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sec^2 x\cdot 2x}{\sin x\cdot 1+x\cdot\cos x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\sec^2 x}{\frac{\sin x}{x}+\cos x}\right)$$

$$= \left(\frac{2}{1+1}\right) = 1$$

32. Ans.(c)

$$\text{Limit} = \lim_{x\to 0}\left(\frac{f(x^2)-f(x)}{f(x)-0}\right)$$

$$= \lim_{x\to 0}\left(\frac{f'(x^2)\cdot 2x-f'(x)}{f'(x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{0-f'(0)}{f'(0)}\right)$$

$$= -1$$

33. $\text{Limit} = \lim_{n\to\infty}\left(\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n\right)$

$$= \lim_{y\to 0}\left(\frac{2}{\pi}\left(\frac{1}{y}+1\right)\cos^{-1}(y)-\frac{1}{y}\right)$$

$$= \lim_{y\to 0}\left(\frac{2}{\pi}\cos^{-1}(y)+\frac{1}{y}(\cos^{-1}(y)-(1)\right)$$

$$= \lim_{y\to 0}\left(\frac{2}{\pi}\cos^{-1}(y)+\frac{(\cos^{-1}(y)-1)}{y}\right)$$

$$= \lim_{y\to 0}\left(\frac{2}{\pi}\cos^{-1}(y)+\frac{(\cos^{-1}(y)-\cos^{-1}(0))}{y}\right)$$

$$= \lim_{y\to 0}\left(\frac{2}{\pi}\cos^{-1}(y)+\frac{\cos^{-1}\left(y\cdot 0+\sqrt{1-y^2}\sqrt{1-0}\right)}{y}\right)$$

$$= \lim_{y\to 0}\left(\frac{2}{\pi}\cos^{-1}(y)+\frac{\cos^{-1}\left(\sqrt{1-y^2}\right)}{y}\right)$$

$$= \lim_{y\to 0}\left(\frac{2}{\pi}\cos^{-1}(y) + \frac{\sin^{-1}y}{y}\right)$$

$$= \left(\frac{2}{\pi}\times\cos^{-1}(0) + 1\right)$$

$$= \left(\frac{2}{\pi}\times\frac{\pi}{2} + 1\right)$$

$$= 1 + 1$$

$$= 2$$

34. No questions asked in 2005.

35. Ans. (c)

$$\text{Limit} = \lim_{x\to 0}\left((\sin x)^{\frac{1}{x}} + \left(\frac{1}{x}\right)^{\sin x}\right)$$

$$= e^{\lim_{x\to 0}\left(\frac{\log\sin x}{x}\right)} + e^{\lim_{x\to 0}\left(\frac{-\log x}{\operatorname{cosec} x}\right)}$$

$$= e^{-\infty} + e^{\lim_{x\to 0}\left(\frac{-\log x}{\operatorname{cosec} x}\right)}$$

$$(\because x\to 0^+,\ \log(\sin x)\to -\infty)$$

$$= e^{-\infty} + e^{\lim_{x\to 0}\left(-\frac{\frac{1}{x}}{-\operatorname{cosec} x\cot x}\right)}$$

$$= e^{-\infty} + e^{\lim_{x\to 0}\left(\frac{\tan x}{x}\times\sin x\right)}$$

$$= e^{-\infty} + e^{0}$$

$$= 0 + 1$$

$$= 1$$

36. Ans. (a)

$$\text{Limit} = \lim_{x\to\frac{\pi}{4}}\left(\frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}}\right)$$

$$= \lim_{x\to\frac{\pi}{4}}\left(\frac{f(\sec^2 x)\cdot 2\sec^2 x\tan x - 0}{2x}\right)$$

$$= \lim_{x\to\frac{\pi}{4}}\left(\frac{f(\sec^2 x)\cdot\sec^2 x\tan x}{x}\right)$$

$$= \left(\frac{2f(2)}{\frac{\pi}{4}}\right)$$

$$= \left(\frac{8f(2)}{\pi}\right)$$

$$= \left(\frac{8}{\pi}f(2)\right)$$

37. No questions asked in 2008.

38. Ans. (a, c)

$$L = \lim_{x\to 0}\left(\frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{a - a\left(1 - \frac{x^2}{a^2}\right)^{1/2} - \frac{x^2}{4}}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{a - a\left(1 - \frac{1}{2}\frac{x^2}{a^2} + \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{x^2}{a^2}\right)^2 - \ldots\right) - \frac{x^2}{4}\times\frac{1}{2!}}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{a - \left(a - \frac{1}{2}\frac{x^2}{a^2} + \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{x^4}{2a^3}\right) - \ldots\right) - \frac{x^2}{4}}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(\left(\frac{1}{2a} - \frac{1}{4}\right)x^2 + \frac{1}{8}\left(\frac{x^4}{a^3}\right) - \ldots\right)}{x^4}\right)$$

It will provide a finite limit only when

$$\left(\frac{1}{2a} - \frac{1}{4}\right) = 0 \text{ and } L = \frac{1}{8a^3}$$

$$\Rightarrow \quad 2a = 4 \text{ and } L = \frac{1}{8a^3}$$

$$\Rightarrow \quad a = 2 \text{ and } L = \frac{1}{8\times 2^3} = \frac{1}{64}$$

39. Ans. (b)

$$\text{Limit} = \lim_{x\to 0}\frac{1}{x^3}\int_0^x\left(\frac{t\log(1+t)}{t^4+4}\right)dt$$

$$= \lim_{x\to 0}\left(\frac{\int_0^x\left(\frac{t\log(1+t)}{t^4+4}\right)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(\frac{x\log(1+x)}{(x^4+4)}\right)}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{3}\times\frac{\log(1+x)}{x}\times\frac{1}{(x^4+4)}\right)$$

$$= \frac{1}{12}$$

40. Ans. (d)

$$\text{Limit} = \lim_{x\to 0}(1 + x\log(1 + b^2))^{\frac{1}{x}} = 2b\sin^2\theta$$

$$\Rightarrow \quad e^{\lim_{x\to 0}\left(\frac{x\log(1+b^2)}{x}\right)} = 2b\sin^2\theta$$

$$\Rightarrow \quad e^{\log(1+b^2)} = 2b\sin^2\theta$$

$$\Rightarrow \quad (1 + b^2) = 2b\sin^2\theta$$

$$\Rightarrow \quad \sin^2\theta = \left(\frac{1+b^2}{2b}\right)$$

$$\Rightarrow \quad \sin^2\theta = \frac{1}{2}\left(b + \frac{1}{b}\right) \geq 1$$

$$\Rightarrow \quad \sin^2\theta = 1$$

$$\Rightarrow \quad \theta = \pm\frac{\pi}{2}$$

41. Ans. (b)

Given $\lim_{x\to 0}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$

$$\lim_{x\to 0}\left(\frac{x^2+x+1-ax^2-ax-bx-b}{x+1}\right) = 4$$

$$\lim_{x\to 0}\left(\frac{(1-a)x^2+(1-a-b)x+(1-b)}{x+1}\right) = 4$$

It is possible only when

$$\Rightarrow \quad (1-a) = 0,\ (1-a-b) = 4$$

$$\Rightarrow \quad a = 1,\ a + b = -3$$

$$\Rightarrow \quad a = 1 \text{ and } b = -4$$

42. Ans. (a)

Limit

$$= \lim_{n\to\infty}\left(\frac{1^a + 2^a + \ldots + \cdot n^a}{(n+1)^{a-1}((na+1)+(na+2)+\ldots+(na+n))}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2\sum_{r=1}^{n} r^a}{(n+1)^{a-1}(2n^2a+n^2+n)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{\left(1+\frac{1}{n}\right)^{a-1}\left(2a+1+\frac{1}{n}\right)}\right)$$

$$= \left(\frac{2\int_0^1 x^a\,dx}{2a = 1}\right)$$

Thus, $\frac{2}{(2a+1)(a+1)} = \frac{1}{60}$

$$\Rightarrow \quad (2a+1)(a+1) = 120$$

$$\Rightarrow \quad 2a^2 + 3a - 119 = 0$$

$$\Rightarrow \quad 2a^2 + 17a - 14a - 119 = 0$$

$$\Rightarrow \quad a(2a+17) - 7(2a+17) = 0$$

$$\Rightarrow \quad (2a+17)(a-7) = 0$$

$$\Rightarrow \quad a = 7, -\frac{17}{2}$$

43. We have

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$

$$\lim_{x\to 1}\left\{\frac{\sin(x-1)-a(x-1)}{(x-1)+\sin(x-1)}\right\}^{\frac{(1+\sqrt{x})(1-\sqrt{x})}{1-\sqrt{x}}} = \frac{1}{4}$$

$$\lim_{x\to 1}\left\{\frac{\frac{\sin(x-1)}{(x-1)} - a}{1 + \frac{\sin(x-1)}{(x-1)}}\right\}^{(1+\sqrt{x})} = \frac{1}{4}$$

$$\left(\frac{1-a}{1+1}\right)^2 = \frac{1}{4}$$

$$\left(\frac{1-a}{2}\right)^2 = \frac{1}{4}$$

$$(1-a)^2 = 1$$

$$(1-a) = \pm 1$$

$$a = 1 + 1$$

$$a = 0, 2$$

The largest value of a is 2.

44. We have $\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right)$

$$= \lim_{x\to 0}\left(\frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x}\right)$$

$$= \lim_{x\to 0}\sin\left(\frac{\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)} \times \frac{\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\left(\frac{\pi}{2}\sin x\right)}\right)$$

$$= \frac{\pi}{6}$$

45. We have $\lim_{\alpha \to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\frac{e}{2}\right)$

$$\lim_{\alpha \to 0}\left(\frac{e(e^{\cos(\alpha^n)-1} - 1}{\cos(\alpha^n) - 1} \times \frac{\cos(\alpha^n) - 1}{\alpha^m}\right) = -\frac{e}{2}$$

$$\lim_{\alpha \to 0}\left(\frac{e\left(e^{\cos(\alpha^n)-1} - 1\right)}{\cos(\alpha^n) - 1} \times \frac{2\sin^2\left(\frac{\alpha^n}{2}\right)}{\left(\frac{\alpha^n}{2}\right)^2} \times \frac{\alpha^{2n}}{4\alpha^m}\right) = \frac{e}{2}$$

$$\lim_{\alpha \to 0}\left(\frac{e\left(e^{\cos(\alpha^n)-1} - 1\right)}{\cos(\alpha^n) - 1} \times \frac{\sin^2\left(\frac{\alpha^n}{2}\right)}{\left(\frac{\alpha^n}{2}\right)^2} \times \frac{\alpha^{2n-m}}{2}\right) = \frac{e}{2}$$

$$\lim_{\alpha \to 0}\left(\frac{e}{2} \times \alpha^{2n-m}\right) = \frac{e}{2}$$

It is possible only when $(2n - m) = 0$

$$\frac{m}{n} = 2$$

46. $\lim_{x \to 0}\left(\frac{x^2 \sin \beta x}{\alpha x - \sin x} \times \frac{\beta x}{\beta x}\right) = 1$

$$\lim_{x \to 0}\left(\frac{x^3\beta}{\alpha x - \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....\right]}\right) = 1$$

$$\lim_{x \to 0}\left(\frac{x^3\beta}{(\alpha - 1)x + \left[\frac{x^3}{3!} - \frac{x^5}{5!} + ...\right]}\right) = 1$$

It is possible only when

$\alpha - 1 = 0,\ \beta = \frac{1}{6}$

Thus, the value of $6(\alpha + \beta)$

$= 6\left(1 + \frac{1}{6}\right) = 7$

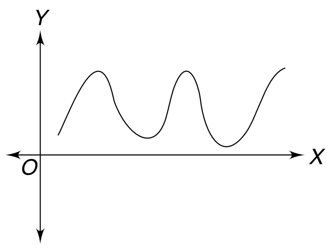
CHAPTER 4

The Continuity and Differentiability

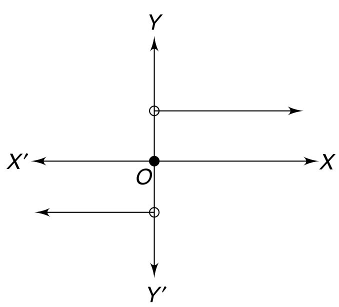
CONCEPT BOOSTER

1. Introduction

Graphically, a function is continuous at a point if its graph can be drawn at this point without raising the pen or pencil, otherwise it is discontinuous at that point.



(Continuous function)



(Discontinuous function)

But only graphical approach is not sufficient, because sometimes it is quite time taking (and in some cases it is even impossible) to draw the complete graph of a function.

So we must have an analytical approach to analyse the continuity of the function at any given point.

2. Continuity

A function $f(x)$ is said to be continuous at $x = c$ if

(i) $f(c)$ is defined

(ii) $\lim\limits_{x \to c} f(x)$ exists

(iii) $\lim\limits_{x \to c} f(x) = f(c)$

i.e $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^-} f(x) = f(c)$

It should be noted that continuity of a function at $x = c$ is meaningful only if the function is defined in the immediate neighbourhood of $x = c$, not necessarily at $x = c$.

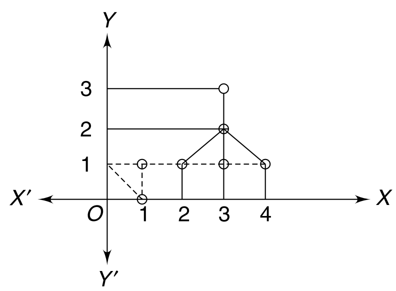
3. Reasons of Discontinuity

(i) $f(x)$ is not defined at $x = c$.

(ii) $\lim\limits_{x \to c} f(x)$ does not exist.

(iii) $\lim\limits_{x \to c} f(x) \neq f(c)$

Geometrically, the graph of the function will exhibit a break at $x = c$. The graph as shown is discontinuous at $x = 1$, 2 and 3.



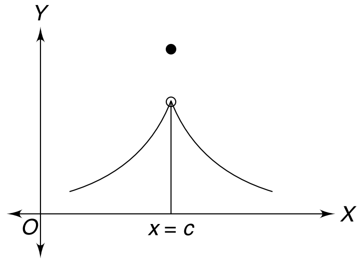
4. Types of Discontinuities

(i) Removal Discontinuity

(ii) Non-Removal Discontinuity

4.1 Removal Discontinuity

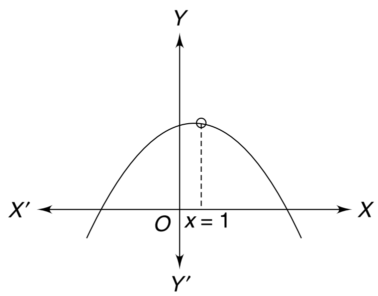
In case $\lim\limits_{x \to c} f(x)$ exists but is not equal to $f(c)$, then the function is said to have a removal discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim\limits_{x \to c} f(x) = f(c)$ and make it continuous at $x = c$



(a) Missing Point Discontinuity: Where $\lim_{x \to c} f(x)$ exists finitely but $f(c)$ is not defined.

For examples:

(i) $f(x) = \dfrac{(1 - x)(9 - x^2)}{(1 - x)}$ has a missing point discontinuity at $x = 1$.

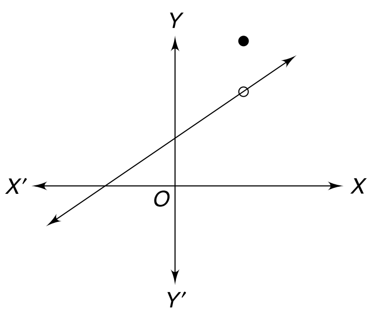


(ii) $f(x) = \dfrac{\sin x}{x}$ has a missing point discontinuity at $x = 0$.

(b) Isolated Point Discontinuity: Where $\lim_{x \to c} f(x)$ exists and $f(c)$ is also exists but $\lim_{x \to c} f(x) \neq f(c)$

For examples:

(i) $f(x) = \dfrac{x^2 - 16}{x - 4}$, $x \neq 4$ and $f(4) = 9$ has an isolated point of discontinuity at $x = 4$



(ii) $f(x) = [x] + [-x] = \begin{cases} 0\ , & x \in I \\ -1, & x \notin I \end{cases}$ has an isolated point of discontinuity at all integer points.

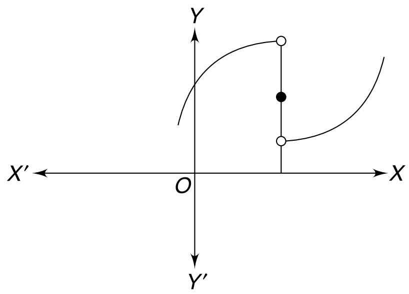
4.2 Non-Removal Discontinuity

In case $\lim_{x \to c} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such discontinuous are known as non-removal discontinuity or discontinuity of the 2nd kind.

(a) Finite Discontinuity:

$\lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x) \neq f(c)$

For examples:



(i) $f(x) = x - [x]$ at all integral x

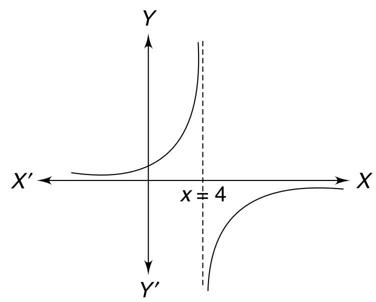
(ii) $f(x) = \tan^{-1}\left(\dfrac{1}{x}\right)$ at $x = 0$

(iii) $f(x) = \dfrac{1}{1 + 2^{\frac{1}{x}}}$ at $x = 0$

(Note that $f(0^+) = 0$, $f(0^-) = 1$).

(b) Infinite Discontinuity:

$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = f(c) = \infty$



For examples:

(i) $f(x) = \dfrac{1}{x - 4}$ at $x = 4$

(ii) $f(x) = \dfrac{1}{(x - 3)^4}$ at $x = 3$

(iii) $f(x) = 2^{\tan x}$ at $x = \dfrac{\pi}{2}$

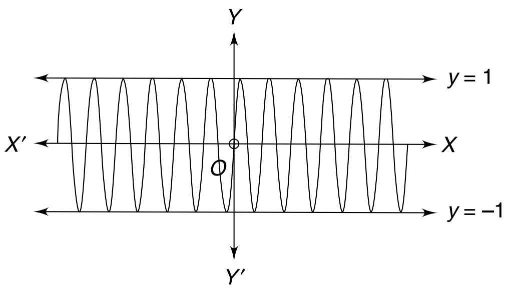
(iv) $f(x) = \dfrac{\cos x}{x}$ at $x = 0$.

(c) Oscillatory Discontinuity:

For example:

(i) $f(x) = \sin\left(\dfrac{1}{x}\right)$ at $x = 0$

> ***Note:*** In all these cases the value of $f(c)$ of the function $f(x)$ at $x = c$ (point of discontinuity) may or may not exist but in every case $\lim_{x \to c} f(x)$ does not exist.



In case of the discontinuity of the second kind, the non-negative difference between the value of the R.H.L at $x = c$ and L.H.L at $x = c$ is called the **Jump of Discontinuity.**

A function having a finite number of jumps in a given interval is called a **Piece-wise Continuous** or **Sectionally Continuous** function in this interval.

Very Important Points to Remember

(a) If $f(x)$ is continuous at $x = a$ and $g(x)$ is discontinuous at $x = a$, then both the functions $f(x) + g(x)$ and $f(x) - g(x)$ are discontinuous at $x = a$.

For example:

Let $f(x) = x$ and $g(x) = [x]$

Clearly, $f(x)$ is continuous at $x = 0$ and $g(x)$ is discontinuous at $x = 0$.

But both the function $x + [x]$ and $x - [x]$ are discontinuous at $x = 0$.

(b) If $f(x)$ is continuous at $x = a$ and $g(x)$ is discontinuous at $x = a$ then the product function $h(x) = f(x)g(x)$ is not necessarily be discontinuous at $x = a$.

For example:

Let $f(x) = x$ and $g(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & x \neq I \\ 0 \quad , & x = 0 \end{cases}$

Clearly $f(x)$ is continuous at $x = 0$ and $g(x)$ is discontinuous at $x = 0$.

Thus, the product $x \sin\left(\dfrac{1}{x}\right)$ is continuous at $x = 0$.

(c) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then the product function $h(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

For example:

Let $f(x) = [x]$ and $g(x) = [-x]$

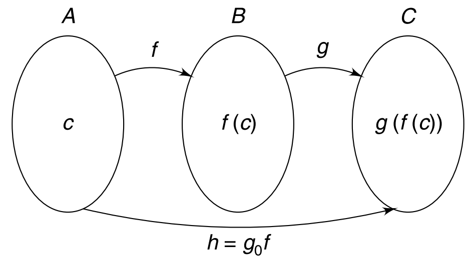
Clearly $f(x)$ and $g(x)$ are discontinuous at $x = 0$.

But the product $f(x) \cdot g(x) = [x] \cdot [-x]$ is continuous at $x = 0$.

(d) Point functions are to be treated as discontinuous.

(e) A continuous function whose domain is in closed interval must have a range also in a closed interval.

(f) If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$ then $g(f(x))$ is continuous at $x = c$.



Example-1. Let $f(x) = \sin x$ is continuous at $x = \dfrac{\pi}{2}$ and $g(x)$ is also continuous at $x = 1$. Then the function $(g_o f)(x) = \cos(\sin x)$ is continuous at $x = \dfrac{\pi}{2}$.

Example-2. Let $f(x) = |x - 2|$ is continuous at $x = 2$ and $g(x) = x^2 + 1$ is continuous at $x = 0$.

Then $(g_o f)(x) = |x - 2|^2 + 1$ is continuous at $x = 2$.

(g) If $f(x)$ is continuous, then $|f(x)|$ is also continuous.

Example-3. Let $f(x) = x$ is continuous everywhere

Then $g(x) = |x|$ is also continuous everywhere.

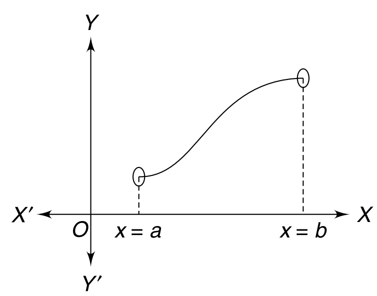
Example-4. Let $f(x) = \dfrac{x+1}{x^2+4}$ and $g(x) = |x - 2|$ are continuous for all x.

Then $(g_o f)(x) = \left|\dfrac{x+1}{x^2+4} - 1\right|$ is also continuous for all x.

5. Continuity of an Interval

Let $f(x)$ be a real function and a and b be two real numbers such that $a < b$

(i) A function $f(x)$ is said to be continuous in the open interval (a, b) if it is continuous to each point of (a, b).

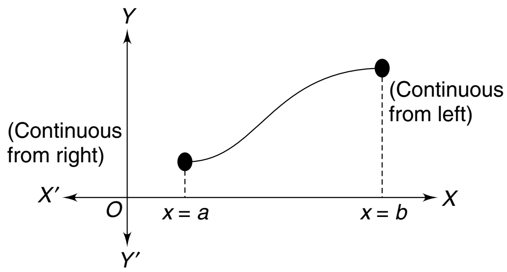


(ii) A function $f(x)$ is said to be continuous in the closed interval $[a, b]$ if

(a) it is continuous in (a, b)

(b) $\lim_{x \to a^+} f(x) = f(a)$ (or f is right continuous at $x = a$)

(c) $\lim_{x \to b^-} f(x) = f(b)$ (or f is left continuous at $x = b$).



6. Single Point Continuity

A function which are continuous only at one point are said to be exihibit single point continuity.

For example:

(i) $f(x) = \begin{cases} x\ , & x \in Q \\ -x, & x \notin Q \end{cases}$ is continuous at $x = 0$

(ii) $g(x) = \begin{cases} x, & x \in Q \\ 0, & x \notin Q \end{cases}$ is continuous at $x = 0$

7. Properties of Continuous Functions

Let $f(x)$ and $g(x)$ be two continuous functions at $x = c$ and k be a non-zero real number. Then

(i) $kf(x)$

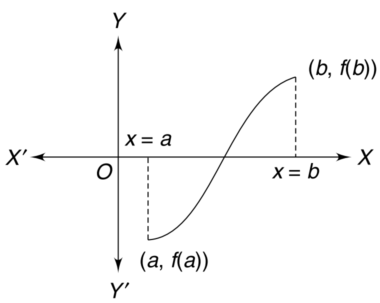
(ii) $f(x) + g(x)$

(iii) $f(x) - g(x)$

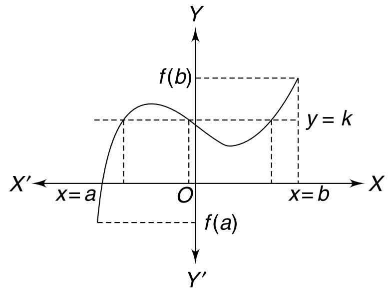
(iv) $f(x) \cdot g(x)$ are also continuous at $x = c$ and

(v) $f(x)/g(x)$ is also continuous, provided $g(c)$ is non-zero at $x = c$.

(vi) If $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).



(vii) If k is any real number between $f(a)$ and $f(b)$, then there exist atleast one solution of the equation $f(x) = k$ in the open interval (a, b). Which is known as **Intermediate Value Theorem.**



8. Differentiability

8.1. Introduction

In calculus (a branch of mathematics), a differentiable function of one real variable is a function whose derivative exists at each point in its domain. As a result, the graph of a differentiable function must have a non-vertical tangent line at each point in its domain, be relatively smooth, and cannot contain any breaks, bends, or cusps.

8.2 Differentiabilty of a Function at a Point

(i) The right hand derivative of $f(x)$ at $x = a$ is denoted by $f'(a^+)$ and is defined as

$f'(a^+) = \lim_{h \to 0^+}\left(\dfrac{f(a + h) - f(a)}{h}\right)$, provided the limit exists.

(ii) The right hand derivative of $f(x)$ at $x = a$ is denoted by $f'(a^-)$ and is defined as

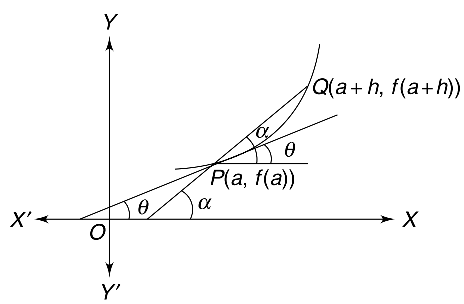
$f'(a^-) = \lim_{h \to 0^+}\left(\dfrac{f(a + h) - f(a)}{h}\right)$, provided the limit exists.

Thus, a function is said to be differentiable (finitely) at $x = a$, if $f'(a^+)$ = finite.

By definition, $f'(a^+) = \lim_{h \to 0^-}\left(\dfrac{f(a + h) - f(a)}{h}\right)$

i.e. $\lim_{h \to a^+}\left(\dfrac{f(a + h) - f(a)}{h}\right) = \lim_{h \to a^-}\left(\dfrac{f(a - h) - f(a)}{h}\right)$

8.3 Geometrical Meaning of a Derivative



$$m(PQ) = \tan\alpha = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$$

when $h \to 0$, the point Q moving along the curve tends to P. i.e. $Q \to P$

The chord PQ approaches the tangent line PT at the point P.

So $\alpha \to \theta$

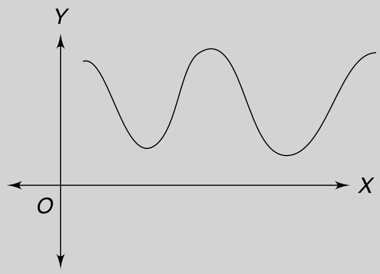
Thus, $\tan\theta = \lim\limits_{h\to 0}\left(\dfrac{f(a+h)-f(a)}{h}\right)$

$\Rightarrow \quad m = f'(a) = \lim\limits_{h\to 0}\left(\dfrac{f(a+h)-f(a)}{h}\right)$

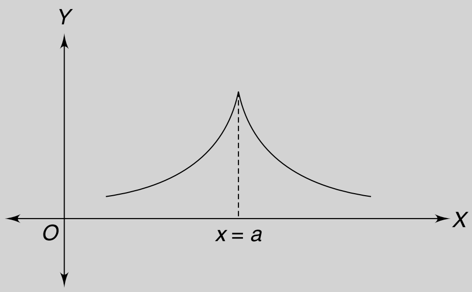
Thus, a function is differentiable at a point $x = a$, if there is a tangent at $x = a$.

Note:

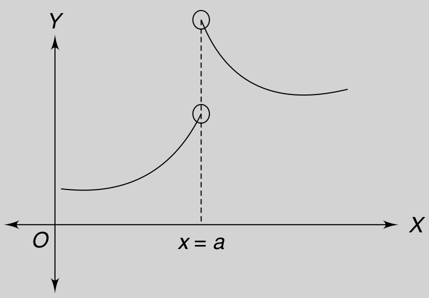
1. A function $f(x)$ is differentiable in its domain if it is a smooth curve.



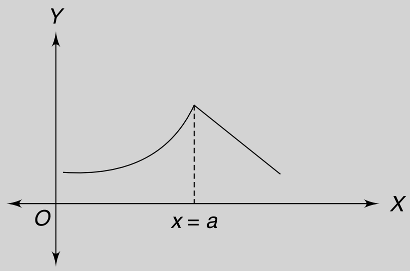
2. A function $f(x)$ is differentiable, if there is a no break point, hole point or corner point or a kink point on the given curve.



($f(x)$ is not differentiable at $x = a$)

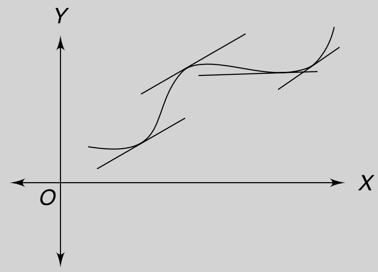


($f(x)$ is not differentiable at $x = a$)



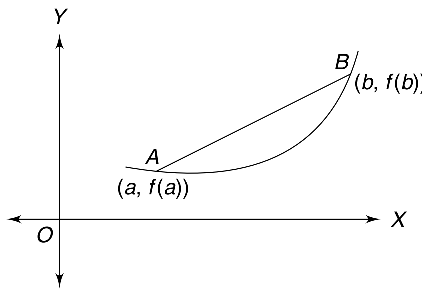
($f(x)$ is not differentiable at $x = a$)

3. A function $f(x)$ is differentiable if there exists a unique tangent to each point on the given curve.



8.4 Concept of Tangent and its Association with Derivability

Tangent: The tangent is defined as the limiting case of a chord or a secant.



Slope of the chord joining $(a, f(a))$ and

$(b, f(b)) = \dfrac{f(b)-f(a)}{b-a}$

Slope of the line joining $(a, f(a))$ and

$(a+h, f(a+h)) = \dfrac{f(a+h)-f(a)}{h}$

R.H.D $= f(a^+) = \lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

L.H.D $= f(a^-) = \lim\limits_{h\to 0^-}\dfrac{f(a-h)-f(a)}{-h}$

A function will have a tangent at point $x = a$ if $f'(a^+) = f'(a^-)$ (may or may be finite) and equation of tangent at $(a, f(a))$ is given by $y - f(a) = f'(a)(x - a)$

Notes:

1. $y = x^3$ has x-axis as tangent at origin.
2. Tangent is also defined as the line joining two infinitely small close points on a curve.
3. $y = \text{sgn}(x)$ will have a vertical tangent at $x = 0$
4. $y = |x|$ does not have tangent at $x = 0$ as L.H.D $\neq$ R.H.D.
5. Discontinuous function can also have vertical tangent, namely, $y = \text{sgn}(x)$ at $x = 0$
6. A function is said to be derivable at $x = a$ if their exist a tangent of finite slope at that point $f'(a^+) = f'(a^-)$ = finite value.
7. If a function $f(x)$ is differentiable at $x = a$, the graph of $f(x)$ will be such that there is a unique tangent to the graph at the corresponding point. But if $f(x)$ is non-differentiable at $x = a$, there will not be unique tangent at the corresponding point of the given curve.

8.5 Relation between the Continuity and Derivability

(i) If $f'(a)$ exists, then $f(x)$ is continuous at $x = a$

(ii) If $f(x)$ is derivable for every point of its domain, then it is continuous in that domain.

Proof: Let $f(x)$ be a real function and $a \in R$

Given $f'(a) = \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}\right)$

To prove $f(x)$ is continuous at $x = a$.

i.e. $\lim_{x\to a} f(x) = f(a)$

Now, $\lim_{x\to a} f(x) = \lim_{x\to a}((f(x) - f(a)) + f(a))$

$$= \lim_{x\to a}\left(\left(\frac{f(x) - f(a)}{x - a}\right) \times (x - a) + f(a)\right)$$

$$= \lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}\right) \cdot \lim_{x\to a} \times (x - a) + \lim_{x\to a} f(a)$$

$$= f'(a)0 + f(a)$$

$$= f(a)$$

Thus, $f(x)$ is continuous at $x = a$

Therefore, every differentiable function is continuous.

Note: The converse of the above result is not true. i.e. If 'f' is continuous at x, then 'f' is derivable at x is not true.

For example:
the function $f(x) = |x - 2|$ is continuous at $x = 2$ but not derivable at $x = 2$.

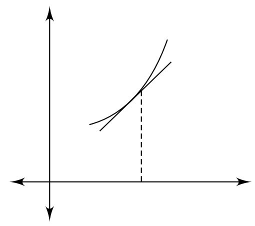
(iii) If a function is not differentiable but is continuous at $x = a$, it geometrically implies a sharp corner or kink at $x = a$.

If $f(x)$ is a function such that

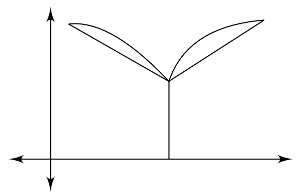
R.H.D = $f(a^+) = l$ and

L.H.D = $f(a^-) = m$

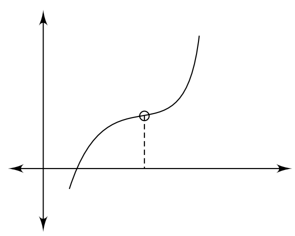
Case I: If $l = m$ = some finite value, then the function $f(x)$ is differentiable as well as continuous.



Case II: If $l \neq m$ = but both have some finite value, then the function $f(x)$ will not be differentiable but it will be continuous.



Case-III: If at-least one of the l or m is infinite, then the function is non differentiable but we can not say about continuity of $f(x)$

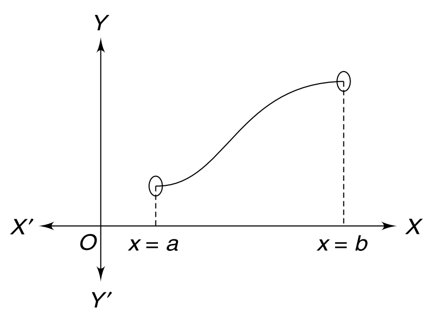


Notes:

1. Differentiable	$\Rightarrow$	Continuous
2. Continuous	$\Rightarrow$	Can be a differentiable.
3. Discontinuous	$\Rightarrow$	Non-differentiable.
4. Both one sided derivative exists	$\Rightarrow$	Continuous.

8.6. Derivability Over an Interval

(i) *Open Interval:* $f(x)$ is said to be derivable over an open interval (a, b) if it is derivable at each and every point of (a, b).



(ii) *Closed Interval:* $f(x)$ is said to be derivable over a closed interval $[a, b]$ if

(i) it is differentiable on (a, b)

(ii) it is right differentiable at left end.

i.e. $f'(a^+) = \lim_{h\to 0^+}\left(\dfrac{f(a+h)-f(a)}{h}\right)$

(iii) it is left differentiable at right end.

i.e. $f'(b^-) = \lim_{h\to 0^-}\left(\dfrac{f(b-h)-f(b)}{-h}\right)$

(iv) for any point c such that $a < c < b$, $f'(c^+)$ and $f'(c^-)$ exists finitely and are equal.

Note: For checking the derivability in an interval, then following points must kept in your mind.

1. All those point where discontinuity may arise
2. Modulus functions are also non-differentiable and hence should be checked at their critical points.
3. Every sine and cosine functions are differentiable everywhere.
4. $\tan^{-1}x$, $\cot^{-1}x$, all polynomials and exponential functions are differentiable everywhere.
5. logarithmic and trigonometric functions are differential in their domains.
6. Modulus function and signum function are non-differentiable at $x = 0$.

   Hence, $y = |f(x)|$ and $y = \text{sgn}(f(x))$ should be checked at points where $f(x) = 0$.
7. Power function $y = x^p$, $0 < p < 1$ is non-differentiable at $x = 0$. Hence $y = (f(x))^p$ should be checked at points where $f(x) = 0$.
8. The inverse trigonometric functions

   $y = \sin^{-1}x$, $\cos^{-1}x$, $\text{cosec}^{-1}x$, $\sec^{-1}x$

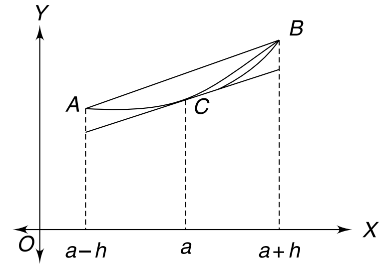
   are not differentiable at $x = \pm 1$

   Hence, $y = \sin^{-1}(f(x))$, $\cos^{-1}(f(x))$, $\text{cosec}^{-1}(f(x))$, $\sec^{-1}(f(x))$

   should be checked at points where $f(x) = \pm 1$.
9. Greatest integer function and fractional part functions are non-differentiable at all integral points of x. Hence, $y = [f(x)]$ and $y = \{f(x)\}$ should be checked at points where $f(x) = n$, $n \in I$.
10. The nth root function $f(x) = \sqrt[n]{x}$ is non-differentiable at $x = 0$
11. If $f(x)$ is differentiable at $x = a$, then the function $f(x)|f(x)|$ is also differentiable at $x = a$.

8.7 Centered Difference Quotient

The centered difference quotient is $\dfrac{f(a+h)-f(a-h)}{2h}$ and is used to approximate $f'(a)$ in numerical work.



Slope of AB $= \dfrac{f(a+h)-f(a-h)}{h}$

Slope of CB $= \dfrac{f(a+h)-f(a)}{h}$

(i) the limit of $\dfrac{f(a+h)-f(a-h)}{2h}$ as $h \to 0$ is $f'(a)$ when $f'(a)$ is exist.

(ii) it usually gives a better approximation of $f'(a)$ for a given value of h then

Fermat's difference quotient $\dfrac{f(a+h)-f(a)}{h}$

In fact, $\lim_{h\to 0}\dfrac{f(a+h)-f(a-h)}{2h}$

$= \dfrac{1}{2}\left(\lim_{h\to 0}\left(\left(\dfrac{f(a+h)-f(a)}{h}\right)+\left(\dfrac{f(a-h)-f(a)}{-h}\right)\right)\right)$

$= \dfrac{1}{2}(f'(a^+)+f'(a^-))$

We notice that it is equal to $f'(a)$ if f is differentiable at $x = a$.

8.8 Twice Differentiability

A function $f(x)$ is twice differentiable at $x = a$ if its derivative $f''(x)$ is differentiable at $x = a$

The limit $f''(a) = \lim_{h\to 0}\left(\dfrac{f'(a+h)-f'(a)}{h}\right)$ exists.

Alternatively, $f''(a) = \lim_{x\to 0}\left(\dfrac{f'(x)-f'(a)}{x-a}\right)$

Hence, if $f'(a)$ is exists then

$\lim_{h\to 0}\dfrac{f(a+h)-f(a-h)}{2h} = f'(a)$.

8.9 Differentiability of Composite Functions

Theorem: If $f(x)$ is differentiable at $x = a$ and $g(x)$ is differentiable at $x = f(a)$ then the composite function $(g_o f)(x)$ is differentiable at $x = a$.

A function of a function composed of a finite number of differentiable functions is a differentiable function.

Example-5. Let $f(x) = \sin x$ is differentiable at $x = \dfrac{\pi}{2}$ and

$g(x) = \begin{cases} x^2 - 1, & x \le 1 \\ 2x - 2, & x > 1 \end{cases}$ is differentiable at $x = f\left(\frac{\pi}{2}\right) = 1$

Hence the composite function $(g_o f)(x)$ is differentiable at $x = \frac{\pi}{2}$

Example-6. Let $f(x) = \frac{\sin x}{x^2 + 1}$ and $g(x) = x|x|$ are differentiable for all x. Hence the composite function $(g_o f)(x)$

$= \frac{\sin x}{x^2 + 1}\left|\frac{\sin x}{x^2 + 1}\right|$ is also differentiable for all x.

Theorem: If the function $f(x)$ is differentiable every where and the function $g(x)$ is also differentiable everywhere then the composite function $(g_o f)$ is also differentiable everywhere.

Example-7. Let $f(x) = \sin x$ and $g(x) = \cot s x$

Clearly, $f(x)$ and $g(x)$ is differentiable everywhere.

Thus, $f(g(x)) = \sin(\cos x)$ and $g(f(x)) = \cos(\sin x)$ is alo differentiable everywhere.

8.10 Algebra of Differentiable Functions

1. If $f(x)$ and $g(x)$ are differentiable functions at $x = a$, then
   (i) $cf(x)$ is differentiable at $x = a$, where c is any non-zero constant.
   (ii) $f(x) + g(x)$ is differentiable at $x = a$
   (iii) $f(x) - g(x)$ is differentiable at $x = a$
   (iv) $f(x) \cdot g(x)$ is differentiable at $x = a$
   (v) $f(x)/g(x)$ is differentiable at $x = a$, provided $g'(a)$ is non-zero.
2. If $f(x)$ is differentiable at $x = a$ and $g(x)$ is non-differentiable at $x = a$, then the following cases may arise.
   (i) Both the functions $f(x) + g(x)$ and $f(x) - g(x)$ is non-differentiable at $x = a$.

   Let $f(x) = x$ and $g(x) = |x|$.

   Obviously, $f(x)$ is differentiable at $x = 0$ and $g(x)$ is non-differentiable at $x = 0$.

   But the functions $x + |x|$, $x - |x|$ are non-differentiable at $x = 0$.

   (ii) $f(x) \cdot g(x)$ is not necessarily non-differentiable at $x = a$. We need to find the result by first principles.

   Example-8. Let $f(x) = x^2$, $g(x) = \text{sgn}(x)$

   Obviously, $f(x)$ is differentiable at $x = 0$ and $g(x)$ is non-differentiable at $x = 0$.

   But the product of the functions

$$f(x) \cdot g(x) = \begin{cases} x^2 & : x > 0 \\ 0 & : x = 0 \\ -x^2 & : x < 0 \end{cases}$$

   is differentiable at $x = 0$.

   Example-9. Let $f(x) = x$ and

$$g(x) = \begin{cases} x \sin\left(\frac{1}{2}\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

3. If $f(x)$ and $g(x)$ are two differentiable functions, Then the function $f(x)$ is differentiable at $x = 0$ but $g(x)$ is non-differentiable at $x = 0$

   Hence the product $f(x) \cdot g(x)$ is differentiable at $x = 0$.

   Example-10. However, the function $f(x) = x$ is differentiable at $x = 1$ and $g(x) = [x]$ is non-differentiable at $x = 1$ but the product $f(x) \cdot g(x)$ is non-differentiable at $x = 1$.
4. The quotient $f(x)/g(x)$ is not necessarily differentiable at $x = a$

   Let $f(x) = x^2(x^2 - 1)$ and $g(x) = \begin{cases} x + 1, & x \ge 0 \\ x - 1, & x < 0 \end{cases}$

   Here, $f(x)$ is differentiable at $x = 0$ and $g(x)$ is non-differentiable at $x = 0$

   Therefore, $f(x)/g(x) = \begin{cases} x^2(x - 1), & x \ge 0 \\ x^2(x + 1), & x < 0 \end{cases}$

   is differentiable at $x = 0$.
5. If $f(x)$ and $g(x)$ both are non-differentiable at $x = a$, then the follwing cases may arise:
   (i) The functions $f(x) + g(x)$ and $f(x) - g(x)$ are not necessarily non-differentiable at $x = a$.

   However, atmost one of $f(x) + g(x)$ and $f(x) - g(x)$ can be differentiable at $x = a$.

   Thus both of them can not be differentiable at $x = a$.

   Let $f(x) = [x]$ and $g(x) = \{x\}$. Here both $f(x)$ and $g(x)$ are non differentiable at $x = 0$.

   Thus the sum $f(x) + g(x)$ is differentiable at $x = 0$, however the difference $f(x) - g(x)$ is non-differentiable at $x = 0$.

   But this does not mean that one of the funtions $f(x) + g(x)$ and $f(x) - g(x)$ must be differentiable.

   Let $f(x) = 2[x]$ and $g(x) = \{x\}$. Here both $f(x)$ and $g(x)$ are non-differentiable at $x = 0$.

   Then the functions $f(x) + g(x)$ and $f(x) - g(x)$ are non-differentiable at $x = 0$.

(ii) $f(x) \cdot g(x)$ is not necessarily non-differentiable at $x = a$.

Let $f(x) = [x]$ and $g(x) = [-x]$. Here, both the functions $f(x)$ and $g(x)$ are non-differentiable at $x = 0$, but the product function $[x] \cdot [-x]$ is differentiable at $x = 0$.

Further more, let $f(x) = [x]$ and $g(x) = \{x\}$. Here both the functions $f(x)$ and $g(x)$ are non-differentiable at $x = 0$, but the product $[x].\ \{x\}$ is also non-differentiable at $x = 0$.

(iii) $f(x)/g(x)$ is not necessarily non-differentiable at $x = a$.

8.11 Functional Equations

We should follow the following steps to determine the functions which are differentiable

(i) First we write $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

(ii) We manipulate $f(x + h) - f(x)$ in such a way that the given functional rule is applicable.

Now we apply the functional rule and simplify the R.H.S to get $f'(x)$ as a function of x.

(iii) Then we integrate $f'(x)$ to get $f(x)$ as a function of x and a constant of integration.

(iv) Finally, we apply the boundary conditions to determine the value of the constant of integration.

Let $f(x) = \begin{cases} x^2 - 1, & x \ge 0 \\ x + 1, & x < 0 \end{cases}$ and

$g(x) = \begin{cases} x + 1, & x \ge 0 \\ x - 1, & x < 0 \end{cases}$

Then both the functions $f(x)$ and $g(x)$ are non differentiable at $x = 0$, but the function $f(x)/g(x)$ is differentiable at $x = 0$.

EXERCISES

Level I

(Problems Based on Fundamentals)

Type-I

1. Test the continuity of the function $f(x)$ at the origin

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

2. Show that the function $f(x)$ is given by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \neq 0 \\ 2 & x = 0 \end{cases}$$ is continuous at $x = 0$.

3. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{a^x + b^x + c^x - 3}{x} & : x \neq 0 \\ \log\ abc & : x = 0 \end{cases}$$ at $x = 0$.

4. Test the contnuty at $x = 0$ where

$$f(x) = \begin{cases} \dfrac{e^{3x} - 1}{\log(1 + 5x)} & : x \neq 0 \\ 5 & : x = 0 \end{cases}$$

5. Show that $f(x) = \begin{cases} \dfrac{x - |x|}{2} & : x \neq 0 \\ 2 & : x = 0 \end{cases}$ is discontinuous at $x = 0$.

6. Show that $f(x) = \begin{cases} \dfrac{x^{x-1}}{\log(1 + 2x)} & : x \neq 0 \\ 7 & : x = 0 \end{cases}$

is discontinuous at $x = 0$.

7. Show that $f(x) = \begin{cases} \dfrac{\sin 3x}{\tan 2x} & : x < 0 \\ \dfrac{3}{2} & : x = 0 \\ \dfrac{\log(1 + 3x)}{e^{2x} - 1} & : x > 0 \end{cases}$

is continuous at $x = 0$.

Type-II

8. For what value of k is the function

$$f(x) = \begin{cases} \dfrac{\sin 5x}{3x} & x \neq 0 \\ k & x = 0 \end{cases}$$ is continuous at $x = 0$?

9. Find the value of k, if $f(x)$ is continuous at $x = \dfrac{\pi}{2}$,

where $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x} & x \neq \dfrac{\pi}{2} \\ 3 & x = \dfrac{\pi}{2} \end{cases}$.

10. If $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16} & : x \neq 2 \\ k & : x = 2 \end{cases}$ is continuous at $x = 2$, then find k.

11. If $f(x) = \begin{cases} \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1} & : \quad x \neq 0 \\ k, & : \quad x = 0 \end{cases}$

is continuous at $x = 0$, then find k.

12. If $f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x} & : \quad x \neq 0 \\ \dfrac{1}{2} & : \quad x = 0 \end{cases}$ is continuous at $x = 0$, find k.

Type-III

13. Determine the values of a, b, c for which the function:

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & : \quad x < 0 \\ c & : \quad x = 0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{\frac{3}{2}}} & : \quad x > 0 \end{cases}$$ is continuous

at $x = 0$

14. If $f(x) = \begin{cases} \dfrac{(x-4)}{|x-4|} + a & : \quad x < 0 \\ a + b & : \quad x \neq 0 \\ \dfrac{(x-4)}{|x-4|} + b & : \quad x > 0 \end{cases}$

is continuous at $x = 4$, then find a, b.

15. Find the ordered pair (a, b) such that

$$f(x) = \begin{cases} \dfrac{be^x - \cos x - x}{x^2} & , \quad x > 0 \\ a & , \quad x = 0 \\ \dfrac{2\left(\tan^{-1}(e^x) - \dfrac{\pi}{4}\right)}{x} & , \quad x < 0 \end{cases}$$

is continuous at $x = 0$.

Type-IV

16. Let $f(x) = \left(\dfrac{\log\left(1+\dfrac{x}{a}\right) - \log\left(1-\dfrac{x}{b}\right)}{x}\right)$ $x \neq 0$. Find the value of $f(x)$ at $x = 0$ so that f becomes continuous at $x = 0$.

17. If $f(x)$ is continuous at $x = 0$ such that

$f(x) = \dfrac{\sin 3x + A \sin 2x + B \sin x}{x^5}$, $x \neq 0$,

then find $f(0)$.

18. If $f(x)$ is continuous at $x = 0$ for which the function $f(x) = \dfrac{e^{2x} - 1 - x(e^{2x}+1)}{x^3}$, $x \neq 0$, then find $f(0)$.

19. Let f be a continuous function on R such that $f\left(\dfrac{1}{4^n}\right) = (\sin e^n)e^{-n^2} + \dfrac{n^2}{n^2+1}$, then find $f(0)$.

20. Let $f(x) = \begin{cases} \dfrac{x^n \times e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}, & , \quad x < 0 \\ 0 & , \quad x = 0 \\ x^n \sin\left(\dfrac{1}{x}\right) & , \quad x > 0 \end{cases}$.

Find the smallest n in W such that $f(x)$ is continuous.

21. Let $f(x) = \left(\dfrac{1}{x} - \dfrac{2}{e^{2x}-1}\right)$.

If $f(x)$ is continuous at $x = 0$, then find $f(0)$.

22. Let $f(x) = \dfrac{2 - \sqrt[4]{x^2+16}}{\cos 2x - 1}$.

If $f(x)$ is continuous at $x = 0$, then find $f(0)$.

Properties of Continuous Functions

23. Prove that the equation $x - \cos x = 0$ has a root in $\left(0, \dfrac{\pi}{2}\right)$.

24. Prove that the equation $2\tan x + 5x - 2 = 0$ has at least one root in $\left(0, \dfrac{\pi}{4}\right)$.

25. Prove that the equation $x \cdot 2^x - 1 = 0$ has only one positive root in $(0, 1)$.

26. Prove that the equation $e^{2x} + e^x + 2\sin^{-1} x + x - \pi = 0$ has at least one real solution in $[0, 1]$.

Disconinuity

27. Prove that the function

$f(x) = \begin{cases} 2 - x : x \geq 1 \\ x + 2 : x < 1 \end{cases}$ is discontinuous at $x = 1$

28. Prove that the function

$f(x) = \begin{cases} \dfrac{\sin 3x + 2x}{\sin 7x + \sin 3x} & : \quad x \neq 0 \\ 2 & : \quad x = 0 \end{cases}$

is discontinuous at $x = 0$

29. Discuss the continuty of the function

$$f(x) = \begin{cases} \dfrac{e^{\sin x} - e^x}{x - \sin x} & : x \neq 0 \\ 2 & : x = 0 \end{cases} \text{ at } x = 0$$

30. Discuss the continuity of the function

$$f(x) = \begin{cases} \left(\dfrac{\sqrt{1 - \cos 4x}}{x}\right) & : x \neq 0 \\ 2 & : x = 0 \end{cases} \text{ at } x = 0.$$

Discontinuity

31. Let $f(x) = \begin{cases} \dfrac{|x|}{x} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

Find the length of the Jump.

32. Let $\lim\limits_{x \to 0}\left(\tan^{-1}\left(\dfrac{1}{x}\right)\right)$

Find the length of the Jump.

33. Let $\lim\limits_{x \to 3}\left(\dfrac{[x]}{x}\right)$

Find the length of the Jump.

34. Let $\lim\limits_{x \to 0} [\sin x]$

Find the length of the Jump.

35. Find the number of the points of discontinuity of each of the following functions

(i) $f(x) = \dfrac{1}{x - 1}$

(ii) $f(x) = \dfrac{1}{|x| - 1}$

(iii) $f(x) = \dfrac{1}{|x| + 2}$

(iv) $f(x) = \dfrac{x^2 + 1}{x^2 - 4}$

(v) $f(x) = \dfrac{1}{\log_e x}$

(vi) $f(x) = \dfrac{1}{\log_e |x|}$

(vii) $f(x) = \dfrac{1}{\log_e |x - 2|}$

(viii) $f(x) = \dfrac{1}{\log_e (x^2 - 1)}$

(ix) $f(x) = \dfrac{1}{\log_e |(x^2 - 1)|}$

(x) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

(xi) $f(x) = \dfrac{x^2 + 4x + 1}{2\sin x - 1}$

(xii) $f(x) = \dfrac{x^2 - 7x + 2014}{\sqrt{2}\cos x - 1}$

36. Discuss the continuity of the function $f(x) = [[x]] - [x - 1]$, where [,] denotes the greatest integer function.

37. Discuss the continuity of $f(x)$ in [0, 1] where $f(x) = [\sin \pi x]$.

38. Discuss the continuity for the function $f(x) = [x] + [-x]$.

39. Discuss the continuity of $f(x)$ in [–2, 2], where $f(x) = x + \{-x\} + [x]$, $x \in I$.

40. Find the number of points of discontinuity of each of the following functions.

(i) $f(x) = [\sin x]$, $\forall x \in [0, 2\pi]$

(ii) $f(x) = [\sqrt{2}\sin x]$, $\forall x \in [0, 2\pi]$

(iii) $f(x) = [\sin x + \cos x]$, $\forall x \in [0, 2\pi]$

(iv) $f(x) = [\sin \pi x]$, $\forall x \in [0, 2]$

(v) $f(x) = [2\cos x]$, $\forall x \in [0, 2\pi]$

41. Find the number of points of discontinuity of the function $f(u) = \dfrac{1}{u^2 + u - 2}$ where $u = \dfrac{1}{1 - x}$.

42. Find the number of points of discontinuity of the function $f(f(f(x)))$, where $f(x) = 1/1 - x$.

43. Discuss the continuity of the function $h(x) = f(g(x))$, where $f(x) = \dfrac{1}{x - 6}$ and $g(x) = x^2 + 5$.

44. Let $f(x) = \begin{cases} 1 + x & : 0 \le x \le 2 \\ 3 - x & : 2 < x \le 3 \end{cases}$

Discuss the continuity of the function $g(x)$, where $g(x) = f(f(x))$.

45. If $f(x) = \begin{cases} -1 & : x < 0 \\ 0 & : x = 0 \\ 1 & : x > 0 \end{cases}$ and $g(x) = x(1 - x^2)$

then discuss the continuity of the function $h(x)$, where $h(x) = f(g(x))$.

46. Discuss the continuity for the function $f(x)$, where $f(x) = |x + 1|(|x| + |x - 1|)$.

47. Let $f(x) = |x - 2| - 1$, $0 \le x \le 4$ and $g(x) = 2 - |x|$, $-1 \le x \le 3$

Then discuss the continuity of the function $(f_o g)(x)$.

48. Prove that the equation $\sqrt{x - 5} = \dfrac{1}{x + 3}$ has at-least one real root in (5, 6).

49. Show that the equation $x^5 + 3x^4 + x - 2 = 0$ has at-least one root in [0, 1].

50. Show that the equation $x^5 - 3x + 1 = 0$ has a real root in [1, 2].

51. Show that the equation $x^3 + x^2 - 3x - 3 = 0$ has root in [1, 2].

Intermediate Value Theorem

52. Show that the equation $x^5 + x = 1$ has a real root.
53. Show that the equation $x^5 + 3x^4 + x - 2 = 0$ has atleast one root in [0, 1].
54. Show that the equation $x^5 - 2x^3 + x^2 - 3x + 1 = 0$ has atleast one root in [1, 2].
55. Show that the equation $2x^3 + x^2 - x - 5 = 0$ has a solution in [1, 2].

Differentiability

56. Check the differentiability of the function $f(x) = |x - 2|$ at $x = 2$.
57. Check the differentiability of
$$f(x) = \begin{cases} x & : x < 1 \\ x^2 & : x \geq 1 \end{cases} \text{ at } x = 1.$$
58. Check the differentiability of the function $f(x) = \ln^2 x$ at $x = 1$.
59. Check the differentiability of the function $f(x) = e^{|x|}$ at $x = 0$.
60. Check the differentiability of the function
$$f(x) = \begin{cases} 3^x & : -1 \leq x \leq 1 \\ 4 - x & : 1 < x < 4 \end{cases} \text{ at } x = 1.$$
61. Check the differentiability of the function $f(x) = \sin x + |\sin x|$ at $x = 0$.
62. Let $f(x) = [x]\tan(\pi x)$, where [,] = G.I.F Find the R.H.D at $x = k$, where $k \in 1$
63. Let $f(x) = [x]\sin(\pi x)$, where [,] = G.I.F Find the L.H.D at $x = k$, where $k \in 1$
64. Check the differentiability of the function
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{2}\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases} \text{ at } x = 0.$$
65. Check the differentiability of the function
$$f(x) = \begin{cases} \dfrac{x}{1 + e^{\frac{1}{x}}} & : x \neq 0 \\ 0 & : x = 0 \end{cases} \text{ at } x = 0 .$$

Relation between Continuity and Differentiability

66. If $f(x) = \begin{cases} ax^2 + 1 & : x \leq 0 \\ x^2 + ax + b & : x > 0 \end{cases}$ is differentiable at $x = 1$, find a and b.
67. Let $f(x) = \begin{cases} \dfrac{1}{|x|} & : |x| \geq \dfrac{1}{2} \\ a + bx^2 & : |x| < \dfrac{1}{2} \end{cases}$

If $f(x)$ is differentiable at $x = 1/2$. Find the value of $a + b + 10$.

68. Let $f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

Examine the continuity and the differentiability at $x = 0$.

69. Let $f(x) = \begin{cases} \dfrac{xe^{\frac{1}{x}}}{1 + e^{\frac{1}{2}}} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

Examine the continuity and the differentiability at $x = 0$.

70. Let $f(x) = |x - 1|([x] - \{x\})$
Examine the continuity and differentiability at $x = 1$.
71. Let $f(x) = \begin{cases} \sin\left(\dfrac{\pi x}{2}\right) & : x < 0 \\ [2x - 3]x & : x \geq 0 \end{cases}$

Examine the continuity and the differentiability at $x = 1$.

72. Check the differentiability of the function $f(x) = e^{-|x|}$ in [−2, 2]
73. Check the differentiability of the function $f(x) = \left|\dfrac{1}{|x|} - 1\right|$ in [− 4, 4]
74. Check the differentiability of the function $f(x) = \sin x + \sin|x|$ in $[-2\pi, 2\pi]$.
75. Check the differentiability of the function $f(x) = \min\{|x + 1|, |x|, |x - 1|\}$ in [−4, 4].
76. Check the differentiability of the function $f(x) = \dfrac{x}{1 + |x|}$ in R.
77. Check the differentiability of the function $f(x) = \left|\dfrac{x}{x - 1}\right|$ in R.
78. Check the differentiability of the function $f(x) = |x| + |x^2 - 1|$ in R.
79. Check the differentiability of the function $f(x) = |x^2 - 1| + |x^2 - 4|$ in R.
80. Let $f(x) = \text{sgn}(x)$ and $g(x) = x(1 - x^2)$. Examine the differentiability of the function $f(g(x))$.
81. Let $f(x) = \sin^{-1}|\sin x|$. Examine the differentiability of the function $f(x)$ in R.

Centered difference Quotient

82. If $f'(2) = 5$, find the value of

$$\lim_{h\to 0}\left(\frac{f(2+h)-f(2-h)}{2h}\right)$$

83. Given $f'(2) = 6$ and $f'(1) = 4$, find the value of $\lim_{h\to 0}\left(\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}\right)$

Twice Differentiability

84. Let $f(x) = |x^3|$. Examine whether the function is twice differentiable or not.

85. Let $y = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

Examine whether the function is differentiable or not at $x = 0$.

86. Let $y = f(x) = \begin{cases} xe^x & : x \leq 0 \\ x + x^2 - x^3 & : x > 0 \end{cases}$

Examine whether the function $f(x)$ is twice differentiable or not.

Functional Equations

87. If $f(x + y) = f(x), f(y)$, $\forall\ x, y \in R$ and $f(x)$ is a differentiable function and $f(0) \neq 0$, find $f(x)$.

88. If $f(x + y) = f(x) + f(y)$, $\forall\ x, y \in R$ and $f(x)$ is a differentiable function, find $f(x)$.

89. Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all x, y in R. If $f'(0) = -1$, $f(0) = 1$, find $f(x)$.

90. If $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ for all x, y in R and $f'(2) = 2$, then find $f(x)$.

91. If $f\left(\frac{2x+3y}{5}\right) = \frac{2f(x)+3f(y)}{5}$ for all x, y in R and $f(0) = 1$, $f'(0) = -1$, find $f(x)$.

92. If $f(x + y + z) = f(x)\cdot f(y)\cdot f(z)$ for all x, y, z in R such that $f(2) = 4$, $f'(0) = 3$, find $f'(2)$.

Level II

(Mixed Problems)

Choose the most appropriate one

1. If $f(x) = \begin{cases} \frac{\sin[x]}{[x]} & : [x] \neq 0 \\ 0 & : x = 0 \end{cases}$, where $[,]$ = G.I.F, then $\lim_{x\to 0} f(x)$ equals

(a) 1 (b) 0
(c) –1 (d) None

2. $\lim_{x\to\infty}\left(\frac{\log[x]}{x}\right)$ where $[,]$ = G.I.F, is

(a) 1 (b) 0
(c) –1 (d) None

3. The left hand limit of $f(x) = \left\{\frac{|x|^3}{a} - \left[\frac{x}{a}\right]^3\right\}$, where $a > 0$ and $[,]$ = G.I.F, is

(a) a^2 (b) $a^2 - 1$
(c) $a^2 - 3$ (d) None

4. $\lim_{x\to t}\left(\frac{\int_1^x |t-1|\,dt}{\sin(x-1)}\right)$ is

(a) 0 (b) 1
(c) –1 (d) None

5. If $f(x) = \begin{cases} x & : x < 0 \\ 1 & : x = 0 \\ x^2 & : x > 0 \end{cases}$, then $\lim_{x\to 0} f(x)$ is

(a) 0 (b) 1
(c) 2 (d) None

6. If $[x]$ denotes the greatest integer less than or equal to x, then the value of $\lim_{x\to 0}(1 - x + [x - 1] + [1 - x])$ is

(a) 0 (b) 1
(c) –1 (d) None

7. $\lim_{x\to 0}(1 + \sin x)^{1/x}$ is equal to

(a) 0 (b) infinity
(c) e (d) Does not exist.

8. $\lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]}$, where $[,]$ = G.I.F

(a) 0 (b) 1
(c) Not exist (d) None

9. The number of points at which the function $f(x) = \frac{1}{\log|x|}$ is discontinuous is

(a) 1 (b) 2
(c) 3 (d) 4

10. The function $f(x) = \frac{\log(1+ax)-\log(1+bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$, so that it is continuous at $x = 0$ is

(a) $a - b$ (b) $1 + b$
(c) $\log a + \log b$ (d) none

11. If $f(x) = \dfrac{\cos^2 \pi x}{e^{2x} - 2ex}$, $x \neq \dfrac{1}{2}$, then the value of $f\left(\dfrac{1}{2}\right)$ so that $f(x)$ is continuous at $x = 1/2$ is

(a) $\dfrac{\pi}{2e^2}$ (b) $\dfrac{\pi}{2e}$
(c) $\dfrac{\pi^2}{2e^2}$ (d) $\dfrac{\pi^2}{2e}$

12. If the function $f(x) = \begin{cases} (\cos)^{1/x} & : x \neq 0 \\ k & : x = 0 \end{cases}$ is continuous at $x = 0$, then the value of k is

(a) 1 (b) –1
(c) 0 (d) e

13. Let $f(x) = \dfrac{1 - \tan x}{4x - \pi}$: $x \neq \dfrac{\pi}{4}$, $x \in \left(0, \dfrac{\pi}{2}\right)$. If $f(x)$ is continuous in $\left(0, \dfrac{\pi}{2}\right)$, then the value of $f\left(\dfrac{\pi}{4}\right)$ is

(a) 1 (b) 1/2
(c) –1/2 (d) –1

14. Let $f(x) = (x - 1)^{\frac{1}{2-x}}$ is not defined at $x = 2$. If $f(x)$ is continuous then $f(2)$ is

(a) e (b) $1/e$
(c) $1/e^2$ (d) 1

15. If $f(x) = x^{\alpha} \log x$ and $f(0) = 0$, then the value of α for which Rolles Theorem can be applied in [0, 1] is

(a) –2 (b) –1
(c) 0 (d) 1/2

16. The value of p for which the function

$$f(x) = \begin{cases} \dfrac{(4^x - 1)^3}{\sin\left(\dfrac{x}{p}\right)\log\left[1 + \left(\dfrac{x^2}{3}\right)\right]} & : x \neq 0 \\ 12(\log 4)^3 & : x = 0 \end{cases},$$

may be continuous at $x = 0$ is

(a) 1 (b) 2
(c) 3 (d) 4

17. Let $f(x) = \begin{cases} \dfrac{1 - \cos 4x}{x^2} & : x < 0 \\ a & : x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{(16 + \sqrt{x})} - 4} & : x > 0 \end{cases}$.

If the function is continuous at $x = 0$, then a is

(a) 4 (b) 6
(c) 8 (d) 10

18. In order that the function $f(x) = (1 + x)^{\cot x}$ is continuous at $x = 0$, then $f(0)$ must be defined as

(a) 0 (b) e
(c) $1/e$ (d) None

19. Let $f(x) = (\sin 2x)^{\tan 2x}$ is not defined at $x = \dfrac{\pi}{4}$. If $f(x)$ is continuous at $x = \dfrac{\pi}{4}$, then $f\left(\dfrac{\pi}{4}\right)$ is equal to

(a) 1 (b) 2
(c) $e^{1/2}$ (d) None

20. Let $f(x) = \begin{cases} -2\sin x & : x \leq \dfrac{-\pi}{2} \\ a\sin x + b & : -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \cos x & : x \geq \dfrac{\pi}{2} \end{cases}$.

If $f(x)$ is continuous everywhere, then (a, b) is

(a) (0, 1) (b) (1, 1)
(c) (–1, 1) (d) (–1, 0)

21. The value of $f(0)$ so that the function $f(x) = \dfrac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$ is continuous at each point on its domain is

(a) 2 (b) $\dfrac{1}{3}$
(c) $\dfrac{2}{3}$ (d) $-\dfrac{1}{3}$

22. If $f(x) = \begin{cases} \dfrac{36^x - 9^x - 4^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}} & : x \neq 0 \\ k & : x = 0 \end{cases}$ is continuous at $x = 0$, then k is

(a) $16 \log 2 \log 3$ (b) $16\sqrt{2} \log 6$
(c) $16\sqrt{2} \log 2 \log 3$ (d) None

23. Let $f(x) = \begin{cases} \dfrac{x - 4}{|x - 4|} + a & : x < 4 \\ a + b & : x = 4 \\ \dfrac{x - 4}{|x - 4|} + b & : x > 4 \end{cases}$. Then $f(x)$ is continuous at $x = 4$, when

(a) $a = b = 0$ (b) $a = b = 1$
(c) $a = -1, b = 1$ (d) $a = 1, b = -1$.

24. If $f(x) = \begin{cases} \dfrac{\sin(a + 1)x + \sin x}{x} & : x < 0 \\ c & : x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}} & : x > 0 \end{cases}$ is continuous at $x = 0$, then

(a) $a = -3/2,\ b = 0,\ c = 1/2$
(b) $a = -3/2,\ b = 1,\ c = -1/2$
(c) $a = -3/2,\ b = R,\ c = 1/2$
(d) None

25. Let $f(x) = \begin{cases} \dfrac{ae^{|\sin x|} - b\cos x - |x|}{x^2} & : x \neq 0 \\ c & : x = 0 \end{cases}$, then
(a) discontinuous at $x = 0$
(b) continuous at $x = 0$, if $a = b = c = 1$
(c) $f'(0) = 1$
(d) continuous but non-differentiable at $x = 0$.

26. Let $f(x) = \begin{cases} x^p \sin\left(\dfrac{1}{x}\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$. Then $f(x)$ is continuous but not differentiable at $x = 0$ if
(a) $p \in (0, 1]$ (b) $p \in [1, \infty)$
(c) $p \in (-\infty, 0)$ (d) $p = 0$

27. The value of k for which the function
$f(x) = \begin{cases} \sin(1/x) & : x \neq 0 \\ k & : x = 0 \end{cases}$ makes continuous at $x = 0$ is
(a) 8 (b) 1
(c) –1 (d) None

28. Given the function $f(x) = \dfrac{1}{1-x}$. The points of discontinuity of the composite function $y = f(f(f(x))$ are at $x =$
(a) 0 (b) 1
(c) 2 (d) –1

29. Let $f(x) = x - |x - x^2|$, where x lies in $[-1, 1]$. Then the number of points of discontinuity is
(a) 0 (b) 1
(c) 2 (d) None

30. The function $f(x) = [x]^2 - [x^2]$, where [,] = G.I.F., is discontinuous as
(a) all integers
(b) all integers except 0 and 1
(c) all integers except 0
(d) all integers except 1.

31. The function $f(x) = [x] \cos\left(\dfrac{2x-1}{2}\right)\pi$, where [,] = G.I.F., is discontinuous at
(a) all x
(b) all integer points
(c) no x
(d) x which is not an integer.

32. The number of points where $f(x) = [\sin x + \cos x]$, where [,] = G.I.F., $x = (0, 2\pi)$ is discontinuous is
(a) 3 (b) 4
(c) 5 (d) 6

33. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x.
Then g is
(a) onto if f is onto
(b) one-one if f is one-one
(c) Continuous if f is continuous
(d) differentiable if f is differentiable.

34. If $f(x) = \begin{cases} xe^{-1\left[\frac{1}{|x|}+\frac{1}{x}\right]} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$, then $f(x)$ is
(a) continuous and diff for all real x
(b) continuous for all x but not differentiable at $x = 0$
(b) neither differentiable not continuous at $x = 0$
(d) discontinous everywhere.

35. The value of the derivative of $|x - 1| + |x - 2|$ at $x = 2$ is
(a) –2 (b) 0
(c) 2 (d) not defined

36. Let $f(x) = [\tan^2 x]$, where [,] = G.I.F., then
(a) $\lim_{x \to 0} f(x)$ does not exist
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) $f'(0) = 1$.

37. The set of all points where the function $f(x) = \dfrac{x}{1 + |x|}$ is differentiable is
(a) R (b) $(0, \infty)$
(c) $(-\infty, 0)$ (d) None.

38. The set of all the points where the function
$f(x) = \begin{cases} 0 & : x = 0 \\ \dfrac{x}{1 + e^{1/x}} & : x \neq 0 \end{cases}$, is differentiable is
(a) $(0, \infty)$ (b) $(-\infty, \infty) - \{0\}$
(c) $(-\infty, 0)$ (d) R.

39. The set of all points of differentiability of the function $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$ is
(a) $(0, \infty)$ (b) $(-\infty, \infty) - \{0\}$
(c) $(-\infty, 0)$ (d) R.

40. The function $f(x)$ is defined as

$$f(x) = \begin{cases} \frac{1}{3} - x & : x < \frac{1}{3} \\ \left(\frac{1}{3} - x\right)^2 & : x \geq \frac{1}{3} \end{cases},$$

then in the interval (0, 1), the mean value theorem is not true, because

(a) $f(x)$ is not continuous

(b) $f(x)$ is not differentiable

(c) $f(0) = f(1)$

(d) None.

41. Let $f(x) = [x \sin \pi x]$, where [,] = G.I.F., then $f(x)$ is

(a) continuous at $x = 0$

(b) continuous in $(-1, 0)$

(c) differentable at $x = 1$

(d) differentable in $(-1, 1)$

42. The function $f(x) = \begin{cases} |2x - 3|[x] & : x \geq 1 \\ \sin\left(\frac{\pi x}{2}\right) & : x < 1 \end{cases}$

(a) is continuous at $x = 2$

(b) is differentiable at $x = 1$

(c) is continuous but not differentiable at $x = 1$

(d) None.

43. The function $f(x)$ is defined as under:

$f(x) = \begin{cases} 3^x & : -1 \leq x \leq 1 \\ 4 - x & : 1 < x < 4 \end{cases}$. The function is

(a) continuous at $x = 1$

(b) differentiable at $x = 1$

(c) continuous but not differentiable at $x = 1$

(d) None.

44. A function is defined as follows:

$f(x) = \begin{cases} x^2 & : x^2 < 1 \\ x & : x^2 \geq 1 \end{cases}$. The function is

(a) continuous at $x = 1$

(b) differentiable at $x = 1$

(c) continuous but not differentiable at $x = 1$

(d) None.

45. The left hand derivative of $f(x) = [x] \sin(\pi x)$ at $x = k$, where k is an integer, is

(a) $(-1)^k (k - 1)\pi$ (b) $(-1)^{k-1} (k - 1)\pi$

(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$.

46. Let $f: R \to R$ be a function is defined by $f(x) = \max \{x, x^3\}$. The set of all points where the function $f(x)$ is not differentiable is

(a) $\{-1, 1\}$ (b) $\{-1, 0\}$

(c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

47. Let $f(x) = \begin{cases} -\frac{1}{|x|} & : |x| \geq 1 \\ ax^2 - b & : |x| < 1 \end{cases}$. If $f(x)$ is continuous and differentiable at any point, then

(a) $a = 1/2, b = -3/2$ (b) $a = 1/2, b = 3/2$

(c) $a = 1, b = -1$ (d) None

51. The number of points at which the function $f(x) = \left|x - \frac{1}{2}\right| + |x - 1| + \tan x$ does not have a derivative in the interval (0, 2),

(a) 1 (b) 2

(c) 3 (d) 4

52. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is not differentiable at $x =$

(a) -1 (b) 0

(c) 1 (d) 2.

53. If $f(x) = x^2 + \frac{x^2}{(1 + x^2)^2} + \frac{x^2}{(1 + x^2)^3} + \dots + \frac{x^2}{(1 + x^2)^n}$ + ... to $-\infty$, then at $x = 0$

(a) $f(x)$ has no limit

(b) $f(x)$ is discontinuous

(c) $f(x)$ is continuous but not differentiable

(d) $f(x)$ is differentiable.

54. Let $f(x + y) = f(x) f(y)$ for all x and y. Suppose that $f(3) = 3$ and $f'(0) = 11$, then $f'(3)$ is given by

(a) 22 (b) 44

(c) 28 (d) 33.

55. A function $f: R \to R$ satisfies the equation $f(x + y) = f(x) f(y)$ for all values of x and y and for any $x \in R, f(x) \neq 0$. Suppose the function is differentiable at $x = 0$ and $f'(0) = 2$, then for all $x = R$, $f(x) =$

(a) e^x (b) e^{2x}

(c) e^{-x} (d) None

56. Let $F(x) = (f(x))^2 + \left(g\left(\frac{x}{2}\right)^2\right)$, $F(5) = 5$ and $f''(x) = -f(x)$, $g(x) = f'(x)$, then $F(10)$ is equal to

(a) 5 (b) 10

(c) 0 (d) None

57. Let f be a differentiable for every x. If $f(1) = -2$ and f' for all x in [1, 6], then

(a) $f(6) < 5$ (b) $f(6) = 5$

(c) $f(6) \geq 8$ (d) $f(6) < 8$

58. If f is a real valued differentiable function satisfying $|f(x) - f(y) \leq (x - y)^2$, for all x, y in R, then $f(1)$ is equal to

(a) 2 (b) 1
(c) –1 (d) 0.

Level III

(Problems for JEE-Advanced)

1. Discuss the continuity of the function $f(x)$, where
$f(x) = \lim_{n\to\infty} \left(\frac{\sin x}{1 + (2\sin x)^{2n}}\right)$ in $(0, \pi)$.
2. Determine the set of all points where the function $f(x) = \frac{x^3}{4 + |x|}$ is continuous.
3. Examine the continuity at $x = 0$ of the function $f(x) = \frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \ldots$
4. Let $f(x) = \begin{cases} x\exp\left(-\left(\frac{1}{|x|}+\frac{1}{x}\right)\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$
Discuss the continuity of the function $f(x)$ at $x = 0$.
5. Let $f(x) = \begin{cases} \frac{a(1 - x\sin x) + b\cos x + 5}{x^2} & : x < 0 \\ 3 & : x = 0 \\ \left(1 + \left(\frac{cx + dx^3}{x^2}\right)\right)^{1/x} & : x > 0 \end{cases}$
If f is continuous at $x = 0$, then find the values of a, b, c and d.
6. Let $f(x) = \left[x - \frac{1}{3}\right] + [x] + \left[x + \frac{1}{3}\right]$, $x = [-1, 0]$.
Discuss the continuity of the function $f(x)$ at $[-1, 0]$.
7. If the function $f(x) = \left[\frac{(x-2)^3}{a}\right]\sin(x-2) + a\cos(x-2)$ is continuous in (4, 6), then find the set of all values of a, where [,] = G.I.F.
8. Let $f(x) = \min\{1, \cos x, 1 - \sin x\}$, $-\frac{\pi}{2} \le x \le \pi$
Discuss the continuity for the function $f(x)$ at $x = 0, \frac{\pi}{2}$
9. Let $f(x) = \max\left\{\text{sgn}(x), -\sqrt{9 - x^2}, x^3\right\}$
Discuss the continuity of the function $f(x)$ at $x = 0$.
10. Let $f(x) = \begin{cases} x^3 & : x^2 < 1 \\ x & : x > 1 \end{cases}$. Discuss the continuity of the function $f(x)$.
11. Let $f(x) = \begin{cases} 3^x & : x^2 \le 1 \\ 4 - x & : 1 < x < 4 \end{cases}$
Discuss the continuity for the function $f(x)$ at $x = 1$.
12. Let $f(x) = x - [x - x^2]$. Find the number of points of discontinuity of $f(x)$.
13. Let $f(x) = [x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right]$. Find the number of points of discontinuity of $f(x)$ in $[-1, 1]$.
14. If the graph of the continuous function $y = f(x)$ passes through $(a, 0)$, find
$\lim_{x\to a}\left(\frac{\ln(1 + 6f^2(x) - 3f(x))}{3f(x)}\right)$
15. Find the number of points of discontinuity of $f(x) = [4x] + \{3x\}$ in [0, 5].
16. Find the number of points of discontinuity of $f(x) = \left|\frac{x}{x^2+1}\right|$.
17. Find the number of points of discontinuity of $f(x) = \text{sgn}(x^2 - 1)$.
18. Find the number of points of discontinuity of the function $f(x) = [x] + \{2x\} + [3x]$ in [0, 1]
19. Determine the set of all points, where the function $f(x) = \frac{x}{1 + |x|}$ is continuous.
20. Let $f(x) = \begin{cases} e^x & : x \le 0 \\ |x - 1| & : x > 0 \end{cases}$
Discuss the continuity for the function $f(x)$ at $x = 0, 1$.
21. Discuss the continuity of $f(x) = \lfloor\tan^{-1}x\rfloor$.
22. Discuss the continuity of $f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$
23. Discuss the continuity of the function
$f(x) = |x + 1|(|x| - |x - 1|)$ in $[-2, 2]$.
24. Discuss the continuity of the function
$f(x) = \begin{cases} (1 - x) & : x < 1 \\ (1 - x)(2 - x) & : 1 \le x \le 2 \\ (3 - x) & : x > 2 \end{cases}$
25. Discuss the continuity of the function $f(x) = \frac{x}{1 - |x|}$.
26. Discuss the continuity of the function
$f(x) = [x] + [-x]$.
27. Discuss the continuity of the function
$f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$.
28. Discuss the continuity of each of the following functions.
(i) $f(x) = [\log_e x]$

(ii) $f(x) = \lfloor \sin^{-1} x \rfloor$

(iii) $f(x) = \left[\dfrac{2}{x^2 + 1}\right]$.

29. Discuss the continuity of the function

$$f(x) = \left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{4}\right] \text{ in } [0, 4]$$

30. Let $f(x) = \dfrac{\sin(3x) + a\sin(2x) + b\sin x}{x^5}$, $x \neq 0$

If $f(x)$ is continuous at $x = 0$, find $f(0)$

Level 10

(Tougher Problems for JEE-Advanced)

1. Let $f(x) = \begin{cases} \dfrac{1 + a\cos x + b\cos 4x}{x^2 \sin^2 x} & : x \neq 0 \\ c & : x = 0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, find the value of $\left(a + b + c + \dfrac{1}{3}\right)$

2. Let $f(x) = \begin{cases} (\sin x + \cos x)^{\operatorname{cosec} x} & : -\dfrac{1}{2} \leq x < 0 \\ a & : x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/|x|}}{ae^{2/x} + be^{3/|x|}} & : 0 < x \leq \dfrac{1}{2} \end{cases}$

If $f(x)$ is continuous at $x = 0$, find the value of $\{e(a + b) + 2\}$.

3. Let $f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x + c}{2}\right) & : -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2} & : x = 0 \\ \dfrac{e^{ax/2} - 1}{x} & : 0 < x < \dfrac{1}{2} \end{cases}$

If $f(x)$ is differentiable at $x = 0$, find the value of a and hence prove that $64b^2 + c^2 = 4$

4. Let $f(x) = \begin{cases} \dfrac{|x|\left(3e^{\frac{1}{|x|}} + 4\right)}{2 - e^{\frac{1}{|x|}}} & : x \neq 0 \\ a & : x = 0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, find the value of $(a^2 + a + 10)$

5. Let $f(x) = \begin{cases} \dfrac{ae^{\frac{1}{|x+2|}} - 1}{2 - e^{\frac{1}{|x+2|}}} & : -3 < x < -2 \\ b & : x = -2 \\ \sin\left(\dfrac{x^4 - 16}{x^5 + 32}\right) & : -x < 0 \end{cases}$

If $f(x)$ is continuous at $x = -2$, find the value of $(a + b + 2)$.

6. Discuss the continuity of

$$f(x) = \lim_{n \to \infty}\left(\frac{x^{2n} - 1}{x^{2n} + 1}\right)$$

7. Discuss the continuity of $f(x)$, where

$$f(x) = \lim_{n \to \infty}(\cos^{2n} x).$$

8. Discuss the continuity of $f(x)$ in $[0, 2]$,

where $f(x) = \lim_{n \to \infty}\left(\sin\left(\dfrac{\pi x}{2}\right)\right)^{2n}$

9. Discuss the continuity of the function

$$f(x) = \lim_{n \to \infty}\left(\frac{\log(2 + x) - x^{2n}\sin x}{1 + x^{2n}}\right) \text{ at } x = 1.$$

10. Let $f(x) = \lim_{n \to \infty}\left(\dfrac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}\right)$

If $f(x)$ is continuous for all x in R, find the value of $2a + 3b + 10$.

11. Let $f(x) = \lim_{n \to \infty}\left(\dfrac{|a + \sin \pi x|^n - 1}{|a + \sin \pi x|^n + 1}\right)$, $x \in (0, 6)$

Find the number of discontinuity, when $a = 0$ and 1.

12. Discuss the continuity of the function $f(x)$,

where $f(x) = \lim_{n \to \infty}\left(\dfrac{x}{1 + (2\sin x)^{2n}}\right)$.

13. Discuss the continuity of the function $f(x)$ at $x = 1$,

where $f(x) = \lim_{n \to \infty}\left(\dfrac{\cos(\pi x) - x^{2n}\sin(x - 1)}{1 + x^{2n+1} + \; - x^{2n}}\right)$

14. Discuss the continuity of the function $f(x)$ in $[0, 2]$, where

$$f(x) = \begin{cases} |2x - 3|[x] & : x \geq 1 \\ \sin\left(\dfrac{\pi x}{2}\right) & : x < 1 \end{cases}$$

15. Discuss the continuity of the function $f(x)$ in [0, 2], where

$$f(x) = \begin{cases} [\cos \pi x] & : 0 \le x \le 1 \\ |2x-3|[x-2] & : 1 < x \le 2 \end{cases}$$

16. Discuss the continuity of the function $f(x)$ in [0, 2), where

$$f(x) = \begin{cases} |1-4x^2| & : 0 \le x < 1 \\ [x^2-2x] & : 1 \le x < 2 \end{cases}$$

17. Discuss the continuity of the function $f(x)$ in [0, 2], where

$$f(x) = \begin{cases} |\sin(\pi x)| & : -1 \le x < 0 \\ 1-\{x\} & : 0 \le x < 1 \\ 1+\left[\cos\left(\frac{\pi x}{2}\right)\right] & : 1 < x \le 2 \end{cases}$$

18. If $f\left(\frac{x+y}{2}\right) = \frac{f(x)+(y)}{2}$, $x, y \in R$,

$f(0) = 1, f'(0) = -1$, then show that

$\{(f(0))^2 + (f(1))^2 + (f(2))^2 + ... + (f(n))^2\}$

$= 1 + \frac{n(n-1)(2n-1)}{6}$.

19. Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \begin{cases} \max.\{f(t): 0 \le t \le x\} & : 0 \le x \le 1 \\ 3-x & : 1 < x \le 2 \end{cases}$$

Discuss the continuity and differentiability of the function $g(x)$ in (0, 2).

20. Let $f(x) = x^4 - 8x^3 + 22x^2 - 24x - 55$ and

$$g(x) = \begin{cases} \min\{f(t): x \le t \le x+1\} & : -1 \le x \le 1 \\ x-10 & : x > 1 \end{cases}$$

Discuss the continuity and differentiability of $g(x)$ in $[-1, \infty)$.

21. Let a function $y = f(x)$ be defined by

$$f(x) = \begin{cases} \varphi(x) & : x \le 0 \\ \left(\frac{e^x + x}{1+2x}\right) & : x > 0 \end{cases}.$$

If $\varphi(x) = a\sin x + b\cos x$ and $f(x)$ is continuous and satisfies $f'(1) = f(-\pi/2)$, find the values of a and b.

22. Let $f(x)$ be defined in the interval [–1, 1] such that

$$f(x) = \begin{cases} x-1 & : -1 < x < 0 \\ x^2 & : 0 \le x \le 1 \end{cases}$$

and $g(x) = \sin x$, then discuss the continuity and differentiability of $h(x)$ in $[-1,1]$, where $h(x) = f(|g(x)|) + |f(g(x))|$

Integer Type Questions

1. Let $f: [1, 7] \to Q$ be a continuous function such that $f(1) = 7$, find the value of $f(7)$.
2. Let f be a continuous function on [1, 3] which takes rational values for all x. If $f(2) = 5$, find the value of $f(2.5)$.
3. Let m is the number of non-differentiable points of $f(x) = ||x| - 1|$ and n is the number of points of differeniable of $g(x) = \frac{1}{\log|x|}$, find the value of $(m + n)$.
4. If m is the number of differentiable points of $f(x) = \frac{1}{\log|x^2 - 4|}$ and the value of n for which $\lim_{x\to 0} \left(\frac{x^n - \sin(x^n)}{x - \sin^n x}\right)$ has a non-zero finite value, find the value of $(m + n)$.
5. Find the number of points of discontinuity of $f(x) = [3 + 4\sin x], x \in [\pi, 2\pi]$
6. If p is the number of discontinuity points of $f(x) = [[x]] - [x - 2]$, where [,] = G.I.F. and q is the limiting value of $\lim_{x\to\infty}\left(\frac{3x^4 - 2}{\sqrt{x^8 + 3x + 4}}\right)$, find the value of $(p + q + 2)$.
7. If m is the limiting value of $\lim_{x\to 0}\left(\frac{\tan 2x - 2\sin x}{x^3}\right)$ and n is the number of points of disc of $f(x) = [\sin x + \cos x]$ in $(0, 2\pi)$ and p is the number of non-differentiable points of $f(x) = |\log|x||$, find the value of $(m + n - p)$.
8. If f be a differentiable function such that $f'(2) = 6$, $f'(1) = 4$, find the value of

$$\lim_{x\to 0}\left(\frac{f(h^2 + 2h + 2) - f(2)}{f(h - h^2 + 1) - f(1)}\right).$$

9. Let $f(x) = \begin{cases} a + bx^2 & : x < 0 \\ 3ax - b + 2 & : 0 \ge 1 \end{cases}$

If $f(x)$ is differentiable at $x = 1$, find the value of $(a + b + 1)$.

10. Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$. If p is the number of discontinuity points and q is the number of non-differentiable points of $f(x)$, find the value of $(p + q + 2)$.

11. Let $f(x) = [3x] - \{2x\}, x = \in \left[0, \frac{3}{2}\right)$. Find the number of points of discontinuity.

12. Let $f(x) = (x - 2)$. Find the number of non-differentiable points of $g(x) = \tan^{-1}(|f(|x|)|)$.

Comprehensive Link Passage

Passage I

If two functions $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$, $f(x) - g(x)$, $f(x).\ g(x)$, $|f(x)|$ are continuous at $x = a$ and $\frac{f(x)}{g(x)}$ is continuous at $x = a$, provided $g(a) \neq 0$.

If one of $f(x)$ and $g(x)$ is continuous but other is discontinuous at any point $x = a$, then $f(x) + g(x)$ is discontinuous at $x = a$.

1. Number of points where $f(x) = \frac{|x + 2|}{\tan^{-1}(x + 2)}$ is discontinuous is/are

(a) 0 (b) 1
(c) 2 (d) infinite

2. $f(x) = |x + 1|(|x| + |x - 1|)$ is discontinuous at

(a) no where (b) $x = -1$
(c) $x = \{0, -1\}$ (d) $x = \{-1, 0, 1\}$

3. Number of points in the interval $(-1, 1)$ where the function $f(x) = [x]/\sin^{-1}x - \pi/6)$ is discontinuous is/are

(a) 0 (b) 1
(c) 2 (d) infinite

Passage II

Let a function is defined as

$$f(x) = \begin{cases} [x] & : -2 \le x \le \frac{-1}{2} \\ 2x^2 - 1 & : \frac{-1}{2} < x \le 2 \end{cases}, \text{ where [,] = G. I. F}$$

Answer the following question by using the above information:

1. The number of points of discontinuity of $f(x)$ is

(a) 1 (b) 2
(c) 3 (d) None

2. The function $f(x - 1)$ is discontinuous at the points

(a) $-1, -1/2$ (b) $-1/2, 1$
(c) $0, 1/2$ (d) $0, 1$

3. Number of points where $|f(x)|$ is not differentiable is

(a) 1 (b) 2
(c) 3 (d) 4.

Passage III

In certain problem the differentiation of $\{f(x) \cdot g(x)\}$ appears. One student commits mistake and differentiate as $\frac{df}{dx} \frac{dg}{dx}$, but he gets correct result if $f(x) = x^3$ and $g(x)$ is a decreasing function for which $g(0) = 1/3$.

1. The function $g(x)$ is

(a) $\frac{3}{(x - 3)^3}$ (b) $\frac{4}{(x - 3)^3}$
(c) $\frac{9}{(x - 3)^3}$ (d) $\frac{27}{(x - 3)^3}$.

2. Derivative of $\{f(x–3) \cdot g(x)\}$ with respect to x at $x = 100$ is

(a) 0 (b) 1
(c) -1 (d) 2

3. $\lim_{x \to 0} \frac{f(x) \cdot g(x)}{x(1 + g(x))}$ will be

(a) 0 (b) -1
(c) 1 (d) 2

Passage IV

A function $f(x)$ is said to have a jump discontinuity at a point $x = a$, if both of the limits L.H.L and R.H.L exists and finite at $x = a$, but not equal and $f(a)$ may be equal to either of these limits. The value of |L.H.L – R.H.L| is known as jump discontinuity.

1. Jump of discontinuity of $y = 2[x]$ at $x = 2$, where [,] = G.I.F, is

(a) 1 (b) 3
(c) 2 (d) -2.

2. If $f(x) = \begin{cases} x^2 + 1 & : x \le 1 \\ 2x + 5 & : x > 1 \end{cases}$, then jump of discontinuity of $f(x)$ at $x = 1$ is\

(a) 4 (b) 3
(c) 7 (d) None

3. Number of jump discontinuities in $y = f(x) \cdot g(x)$, where

$$f(x) = \begin{cases} x + 1 & : x \ge 1 \\ x^2 & : x < 1 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} \sin x & : x < 1 \\ 2x^2 & : x \ge 1 \end{cases}$$

(a) 1 (b) 2
(c) 3 (d) None

Passage V

Suppose that f is continuous on the closed interval $[a, b]$ and let k be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$.

Then there exist a number c in (a, b) such that $f(c) = k$.

On the basis of the above information answer the following questions.

1. The number of real root of $x.2^x - 1 = 0$ in $(0, 1)$ is
   (a) 2 (b) 3
   (c) 0 (d) 1
2. The number of real root of $2 \tan x + 5x - 2 = 0$ in $\left[0, \frac{\pi}{4}\right]$ is
   (a) 1 (b) 2
   (c) 3 (d) 4
3. The number of real root of $x - \cos x = 0$ in $\left(0, \frac{\pi}{2}\right)$ is
   (a) 1 (b) 2
   (c) 0 (d) 3

Passage VI

Let $f(x) = \frac{A \cos x + B\ x \sin x - 5}{x^4}$, $x \neq 0$ is continuous at $x = 0$. Then

1. The value of A is
   (a) 2 (b) 5
   (c) 7 (d) 10
2. The value of B is
   (a) 1/2 (b) 3/2
   (c) 5/2 (d) 7/2
3. The value of $f(0)$ is
   (a) –1/24 (b) –5/24
   (c) –7/24 (d) –11/24

Passage VII

Let $f(x) = \frac{\sin 3x + a \sin 2x + b \sin x}{x^5}$, $x \neq 0$ is continuous at $x = 0$. Then

1. The value of A is
   (a) 2 (b) –2
   (c) 4 (d) –4
2. The value of B is
   (a) 3 (b) 5
   (c) 7 (d) 9
3. The value of $f(0)$ is
   (a) 1 (b) 0
   (c) –1 (d) –2

Matrix Match

1. **Observe the following Columns:**

	Column I		Column II
(A)	The Nnumber of points of discontinuity of $f(x) = \tan^2 x - \sec^2 x$ in the interval $(0, 2\pi)$ is	(P)	1
(B)	The number of points at which $f(x) = \sin^{-1} x + \tan^{-1} x + \cot^{-1} x$ is non-differentiable in $(-1, 1)$ is	(Q)	2
(C)	Number of points of discontinuity of $y = [\sin x]$ $x \in [0.2\pi]$, where [,] = G.I.F	(R)	0
(D)	Number of points where $y = \lvert(x-1)^3\rvert + \lvert(x-2)^5\rvert + \lvert x-3\rvert$ is non-differentiable	(S)	3

2. **Observe the following columns:**

	Column I		Column II
(A)	Points of discontinuity of $y = \frac{1}{t^2 - t - 2}$, where $t = \frac{1}{x+1}$	(P)	–1/2
(B)	Points of continuity of $y = [x] + [-x]$	(Q)	–2
(C)	$y = \{\sin(\pi x)\}$ is non-differentiable at	(R)	–1
(D)	$y = \lvert 2x+1\rvert + \lvert x+2\rvert - \lvert x+1\rvert - \lvert x-4\rvert$ is non-differentiable at	(S)	4

3. **Observe the following Columns:**

	Column I		Column II
(A)	$f(x) = \{\sin(\pi x)\}$ is disc for $x \in$	(P)	[0, 1)
(B)	$g(x) = \left\{\frac{\sin x}{x}\right\}$ is disc for $x \in$	(Q)	{1, 2}
(C)	$h(x) = \frac{\{\sin x\}}{\{x\}}$ is non-differentiable for $x \in$	(R)	{0}
(D)	$p(x) = \frac{(\sin x)}{[x]}$ is disc for $x \in$	(S)	$\left\{\frac{1}{2}\right\}$

4. **Observe the following Columns:**
   The number of points of discontinuity of the functions

Column I		Column II	
(A)	$y = f(x) = \dfrac{x}{\lvert x\rvert + 1}$	(P)	2
(B)	$y = f(x) = \dfrac{x}{\lvert x\rvert - 1}$	(Q)	3
(C)	$y = f(x) = \dfrac{1}{\log\lvert x\rvert}$	(R)	1
(D)	$y = f(x) = \dfrac{1}{\log\lvert x^2 + 1\rvert}$	(S)	0

Problems asked in Roorkee - JEE Exam

1. Sketch the function $y = |x - 2|$ in $[-1, 2]$
   Is the function (i) Continuous (ii) differentiable at $x = 2$?
   [Roorkee-JEE, 1984]

2. Let $$f(x) = \begin{cases} x-1 & : x<0 \\ \dfrac{1}{4} & : x=0 \\ x^2 & : x>0 \end{cases}$$
   Discuss the continuity of $f(x)$ at $x = 0$.
   [Roorkee-JEE, 1988]

3. Draw the graph of the function $f(x) = x - |x - x^2|$, $-1 \le x \le 1$ and discuss the continuity and discontinuity of f in the interval $[-1, 1]$.
   [Roorkee-JEE, 1989]

4. If $f(x) = -1 + |x - 1|$, $-1 \le x \le 2$, $g(x) = 2 - |x + 1|$, $-2 \le x \le 2$, then find $(f_o g)(x)$ and $(g_o f)(x)$. Draw their graphs. Discuss the continuity $(f_o g)(x)$ at $x = -1$, and the differentiability of $(g_o f)(x)$ at $x = 1$.
   [Roorkee-JEE, 1990]

5. The function f is defined by $y = f(x)$, where $x = 2t - |t|$, $y = t^2 + t|t|$, $t \in R$. Draw the graph of f for the interval $-1 \le x \le 1$. Also, discuss the continuity and differentiability at $x = 0$.
   [Roorkee-JEE, 1991]

6. If $f(x) = \sqrt{|x - 1|}$ and $g(x) = \sin x$, then calculate $(f_o g)(x)$ and $(g_o f)(x)$ and discuss the differentiability of $(g_o f)(x)$ at $x = 1$.
   [Roorkee-JEE, 1992]

7. A function is defined as follows:
   $$f(x) = \begin{cases} x^3 & : x^2 < 1 \\ x & : x^2 \ge 1 \end{cases}.$$
   Draw the graph of the function and discuss limit, continuity and differentiability at $x = 1$
   [Roorkee-JEE, 1993]

8. Draw the graph of the following function and discuss its continuity and differentiability at $x = 1$
   $$f(x) = \begin{cases} 3^x & : -1 \le x \le 1 \\ 4 - x & : 1 < x < 4 \end{cases}$$
   [Roorkee-JEE, 1994]

9. Discuss the limit, continuity and differentiability of the function at $x = 0$
   $$f(x) = \begin{cases} \dfrac{x(3e^{1/x} + 4)}{(2 - e^{1/x})} & : x \ne 0 \\ 0 & : x = 0 \end{cases}$$
   [Roorkee-JEE, 1995]

10. Find the value of $f(0)$ so that the function $f(x) = \dfrac{1}{x} - \dfrac{1}{e^{2x} - 1}$, $x \ne 0$ is continuous at $x = 0$ and then examine the differentiability of $f(x)$ at $x = 0$.
    [Roorkee-JEE, 1996]

11. Discuss the continuity and differentiability of the function
    $$f(x) = \begin{cases} 2 + \sqrt{1 - x^2} & : |x| \le 1 \\ 2e^{(1-x)^2} & : |x| > 1 \end{cases}$$
    [Roorkee-JEE, 1998]

12. Determine the constants a, b and c for which the function
    $$f(x) = \begin{cases} (1 + ax)^{1/x} & : x < 0 \\ b & : x = 0 \\ \dfrac{(x + c)^{1/3} - 1}{(x + 1)^{1/2}\ 1} & : x > 0 \end{cases}$$
    is continuous at $x = 0$.
    [Roorkee-JEE, 1999]

13. Discuss the continuity and differentiability of the function
    $$f(x) = \begin{cases} \dfrac{x}{1 + |x|} & : |x| \ge 1 \\ \dfrac{x}{1 - |x|} & : |x| < 1 \end{cases}$$
    [Roorkee-JEE, 2000]

14. Discuss the continuity of the function
    $$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x-1}} - 2}{e^{\frac{1}{x-1}} + 2} & : x \ne 1 \\ 1 & : x = 1 \end{cases} \quad \text{at } x = 1$$
    [Roorkee-JEE, 2001]

Problems asked in Previous Years' IIT-JEE Exam

1. If $f(x) = x(\sqrt{x} - \sqrt{x+1})$, then
   (a) $f(x)$ is continuous but not differentiable at $x = 0$
   (b) $f(x)$ is differentiable at $x = 0$
   (c) $f(x)$ is not differentiable at $x = 0$
   (d) None of these

[IIT-JEE, 1985]

2. Let $f(x) = x^3 - x^2 + x + 1$ and

$$g(x) = \begin{cases} \max\{f(t): 0 \le t \le x & : 0 \le x \le 1 \\ 3 - x & : 1 \le x \le 2 \end{cases}.$$

Discuss the continuity and differentiability of the function $g(x)$ in (0, 2).

[IIT-JEE, 1985]

3. The function $f(x) = 1 + |\sin x|$ is
   (a) Continuous nowhere
   (b) Continuous everywhere
   (c) differentiable nowhere
   (d) not differentiable at $x = 0$.

[IIT-JEE, 1986]

4. Let $[x]$ denote the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is
   (a) Continuous at $x = 0$
   (b) Continuous in $(-1, 0)$
   (c) differentiable at $x = 1$
   (d) differentiable in $(-1, 1)$

[IIT-JEE, 1986]

5. Let $f(x)$ be defined in the interval $[-2, 2]$ such that

$$f(x) = \begin{cases} -1 & : -2 \le x \le 0 \\ x - 1 & : 0 < x \le 2 \end{cases}$$

and $g(x) = f(|x|) + |f(x)|$. Test the differentiability of $g(x)$ in $(-2, 2)$

[IIT-JEE, 1986]

6. The set of all points where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable is
   (a) $(-\infty, \infty)$ (b) $[0, \infty)$
   (c) $(-\infty, 0) \cup (0, \infty)$ (d) $(0, \infty)$

[IIT-JEE, 1987]

7. Let $f(x)$ be a function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, then find its value.

[IIT-JEE, 1987]

8. Let $f(x)$ be a continuous function and $g(x)$ be a discontinuous function, then prove that $f(x) + g(x)$ is a discontinuous function.

[IIT-JEE, 1987]

9. Let $g(x)$ be a polynomial of degree one and $f(x)$ be defined by

$$f(x) = \begin{cases} g(x) & : x \le 0 \\ \left(\dfrac{1+x}{2+x}\right)^{\frac{1}{x}} & : x > 0 \end{cases}$$

Find the continuous function $f(x)$ satisfying $f'(1) = f(-1)$

[IIT-JEE, 1987]

10. Let R be the set of real numbers and $f: R \to R$ be such that for all x and y in R, $|f(x) - f(y)| \le (x - y)^2$. Prove that $f(x)$ is a constant.

[IIT-JEE, 1988]

11. The function $f(x) = \begin{cases} |x-3| & : x \ge 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} & : x < 1 \end{cases}$ is
    (a) continuous at $x = 1$
    (b) differentiable at $x = 1$
    (c) continuous at $x = 3$
    (d) differentiable at $x = 3$.

[IIT-JEE, 1988]

12. Find the values of a and b so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & : 0 \le x < \dfrac{\pi}{4} \\ 2x\cot x + b & : \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \\ a\cos 2x - b\sin x & : \dfrac{\pi}{2} < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$

[IIT-JEE, 1989]

13. If $f(x) = \left(\dfrac{x}{2} - 1\right)$, then on the interval $[0, \pi]$
    (a) tan $(f(x))$ and $1/f(x)$ are both continuous
    (b) tan $(f(x))$ and $1/f(x)$ are both discontinuous
    (c) tan $(f(x))$ and $f^{-1}(x)$ are both continuous
    (d) tan $(f(x))$ is continuous but $1/f(x)$ is not continuous.

[IIT-JEE, 1989]

14. Let

$$f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & : x < 0 \\ a & : x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4} & : x > 0 \end{cases}$$

Determine the value of a, if possible, so that the function is continuous at $x = 0$.

[IIT-JEE, 1989]

15. Draw a graph of the function $y = [x] + |1 - x|$ for all x in $[-1, 2]$,
[,] = G.I.F, Determine the points, if any, where this function is not differentiable.
[IIT-JEE, 1989]

16. A function $f: R \to R$ satisfies the equation $f(x + y) = f(x).\ f(y)$ for all x, y in R and $f(x)$ is non-zero for any x in R. Let the function is differentiable at $x = 0$ and $f'(0) = 2$. Show that $f'(x) = 2f(x)$ for all x in R.
Hence determine $f(x)$.
[IIT-JEE, 1990]

17. The function f is defined by $y = f(x)$ where $x = 2t - |t|$ and $y = t^2 + |t|$, $t \in R$.
Draw the graph of f for the interval $-1 \le x \le 1$. Discuss the differentiability of the function $y = f(x)$ at $x = 0$.
[IIT-JEE, 1991]

18. The following functions are continuous on $(0, \pi)$
(a) $\tan x$
(b) $\int_0^x t \sin\left(\frac{1}{t}\right) dt$
(c) $\begin{cases} 1 & : 0 < x \le \frac{3\pi}{4} \\ 2\sin\left(\frac{2}{9}\right)x & : \frac{3\pi}{4} < x < \pi \end{cases}$
(d) $\begin{cases} x + \sin x & : 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}\sin(\pi + x) & : \frac{\pi}{2} < x < \pi \end{cases}$
[IIT-JEE, 1991]

19. Each entry in Column-I is related to exactly in Column-II. Write the correct letter from Column-II against the entry number in Column-I.

Column I		Column II	
(i)	$\sin(\pi[x])$	(A)	differentiable everywhere
(ii)	$\sin\{\pi(x - [x])\}$	(B)	nowhere differentiable
		(C)	not differentiable at 1 and-1 where [,] = G.I.F

[IIT-JEE, 1992]

20. Let $f(x) = x|x|$. The set of points where $f(x)$ is twice differentiable is ...
[IIT-JEE, 1992]

21. Let $f(x) = [\tan^2 x]$, where [,] = G. I. F, then
(a) $\lim_{x \to 0} f(x)$ does not exist
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) $f'(0) = 1$.
[IIT-JEE, 1993]

22. Let $f(x) = \begin{cases} (1 + |\sin x|)^{\frac{a}{|\sin x|}} & : -\frac{\pi}{6} < x < 0 \\ b & : x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & : 0 < x < \frac{\pi}{6} \end{cases}$
Determine a and b such that f is continuous at $x = 0$.
[IIT-JEE, 1994]

23. Let $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(x)}{2}$ for all real x and y. If $f'(0)$ exists and equals to -1 and $f(0) = 1$, find $f(2)$.
[IIT-JEE, 1995]

24. Let $f(x) = [x]\sin\left(\frac{\pi}{[x+1]}\right)$, where [,] = G.I.F. The domain of f is ... and the points of discontinuity of f in the domain are ...
[IIT-JEE, 1996]

25. Let $f(x)$ be a continuous function defined for $1 \le x \le 3$. If $f(x)$ takes rational values for all x and $f(10) = 10$, then $f(1.5) = ...$
[IIT-JEE, 1997]

26. Let $f(x) = \begin{cases} e^{\left\{-\left(\frac{1}{|x|} + \frac{1}{x}\right)\right\}} & : x \ne 0 \\ 0 & : x = 0 \end{cases}$
Test whether
(a) $f(x)$ is continuous at $x = 0$
(b) $f(x)$ is differentiable at $x = 0$
[IIT-JEE, 1997]

27. Determine the values of x for which the following function fails to be continuous or differentiable
$$f(x) = \begin{cases} 1 - x & : x < 1 \\ (1-2)(2-x) & : 1 \le x \le 2 \\ 3 - x & : x > 2 \end{cases}$$
Justify your answer. **[IIT-JEE, 1997]**

28. Let $h(x) = \min\{x, x^2\}$ for every real number of x. Then
(a) h is continuous for all x
(b) h is differentiable for all x
(c) $h'(x) = 1$ for all $x > 1$
(d) h is not differentiable at two values of x.
[IIT-JEE, 1998]

29. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is not differentiable at

(a) -1 (b) 0
(c) 1 (d) 2

[IIT-JEE, 1999]

30. The function $f(x) = [x]^2 - [x^2]$, where [,] = G. I. F. is discontinuous at
 (a) all integers
 (b) all integer except 0 and 1
 (c) all integers except 0
 (d) all integers except 1.

[IIT-JEE, 1999]

31. Let $f: R \to R$ be any function. Define $g: R \to R$ by $g(x) = |f(x)|$ for all x. Then g is
 (a) onto if f is onto
 (b) one-one if f is one–one
 (c) continuous if f is continuous
 (d) differentiable if f is differentiable.

[IIT-JEE, 2000]

32. Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ If $|p(x)| \le |e^{x-1} - 1|$ for all $x \le 0$, then prove that $|a_1 + 2a_2 + 3a_3 + \ldots + na_n| \le 1$

[IIT-JEE, 2000]

33. Discuss the continuity and differentiability of the function

$$f(x) = \begin{cases} \dfrac{x}{1+|x|} & : \ |x| \ge 1 \\ \dfrac{x}{1-|x|} & : \ |x| < 1 \end{cases}$$

[IIT-JEE, 2000]

34. Let $f(x)$, $x \ge 0$, be a non-negative continuous function and let

$$F(x) = \int_0^x f(t)\,dt, \ x \ge 0.$$

If for some $c > 0$, $f(x) \le cF(x)$ for all $x > 0$, then show that $f(x) = 0$ for all $x \ge 0$.

[IIT-JEE, 2001]

35. Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x-1}} - 2}{e^{\frac{1}{x-1}} + 2} & : \ x \ne 1 \\ 1 & : \ x = 1 \end{cases} \quad \text{at } x = 1$$

[IIT-JEE, 2001]

36. Which of the following functions is differentiable at $x = 0$?
 (a) $\cos(|x|) + |x|$ (b) $\cos(|x|) - |x|$
 (c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

[IIT-JEE, 2001]

37. Let $f: R \to R$ be a function is defined by $f(x) = \max\{x, x^2\}$. The set of all points where $f(x)$ is not differentiable is
 (a) $\{-1, 1\}$ (b) $\{-1, 0\}$
 (c) $\{0, 1\}$ (d) $\{-1, 0, 1\}$

[IIT-JEE, 2001]

38. The left hand derivative of $f(x) = [x]\sin(\pi x)$ at $x = k$, where k is an integer, is
 (a) $(-1)^k(k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$
 (c) $(-1)^k k\pi$ (d) $(-1)^{k-1}k\pi$

[IIT-JEE, 2001]

39. Let $\alpha \in R$. Prove that a function $f: R \to R$ is differentiable at $x = \alpha$ and only if there is a function $g: R \to R$ which is continuous at $x = \alpha$ and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all $x \in R$

[IIT-JEE, 2001]

40. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & : \ |x| \le 1 \\ \dfrac{1}{2}(|x| - 1) & : \ |x| > 1 \end{cases} \text{ is}$$

 (a) $R - \{0\}$ (b) $R - \{1\}$
 (c) $R - \{-1\}$ (d) $R - \{-1, 1\}$

[IIT-JEE, 2002]

41. Let $f(x) = \begin{cases} x + a & : \ x < 0 \\ |x - 1| & : \ x \ge 0 \end{cases}$ and

$g(x) = \begin{cases} x + 1 & : \ x < 0 \\ (x-1)^2 + b & : \ x \ge 0 \end{cases}$ where a and b are non-negative real numbers.

Determine the composite function g_of.

If $(g_of)(x)$ is continuous for all real x, determine the values of a and b. Further for these values of a and b, is goo differentiable at $x = 0$? Justify your answer

[IIT-JEE, 2002]

42. If $f: [-2a, 2a] \to R$ be an odd function such that left hand derivative at $x = a$ is zero and $f(x) = f(2a - x)$, $x \in (a, 2a)$, then find left hand derivative of f at $x = -a$.

[IIT-JEE, 2003]

43. If $f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right) & : \ -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2} & : \ x = 0 \\ \dfrac{e^{ax/2} - 1}{x} & : \ 0 < x < \dfrac{1}{2} \end{cases}$

is differentiable at $x = 0$. Find the value of a and also prove that $64b^2 = 4 - c^2$.

[IIT-JEE, 2004]

44. The function is given by $y = ||x| - 1|$ is differentiable for all real numbers except the points

(a) $\{-1, 0, 1\}$ (b) $\{-1, 1\}$

(c) 1 (d) -1

[IIT-JEE, 2005]

45. If $f(x)$ is continuous and differentiable function and $f(1/n) = 0$ for every $n \geq 1$ and $n \in I$, then

(a) $f(x) = 0, x \in (0, 1]$

(b) $f(0) = 0$ and $f'(0) = 0$

(c) $f(0) = 0 = f'(0), x \in (0, 1]$

(d) $f(0) = 0$ and $f'(0)$ need to be zero.

[IIT-JEE, 2005]

46. Let f be a twice differentiable function satisfying $f(1) = 1, f(2) = 4, f(3) = 9$, then

(a) $f''(x) = 2, \forall \in R$

(b) $f'(x) = 5 = f''(x)$ for some $x \in (1, 3)$

(c) there exists atleast one $x \in (1, 3)$ such that $f''(x) = 2$

(d) None of the above.

[IIT-JEE, 2005]

47. If $f(x - y) = f(x).\ g(y) - f(y).\ g(x)$ and $g(x - y) = g(x).\ g(y) + f(x).\ f(y)$ for all $x, y \in R$, if right hand derivative at $x = 0$ exists for $f(x)$. Then find the derivative of $g(x)$ at $x = 0$.

[IIT-JEE, 2005]

48. Let $f(x) = \min\{1, x^2, x^3\}$, then

(a) $f(x)$ is continuous for all $x \in R$

(b) $f'(x) = 0$ for every $x > 1$

(c) continuous but not differentiable for all $x \in R$

(d) differentiable everywhere.

[IIT-JEE, 2006]

49. If $f''(x) = -f(x)$, where $f(x)$ is a continuous double differentiable function and $g(x) = f'(x)$

If $$F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$$

and $F(5) = 5$ then $F(10)$ is

(a) 0 (b) 5

(c) 10 (d) 25.

[IIT-JEE, 2006]

50. Match the functions in Column-I with the properties in Column-II

Column-I		Column-II	
(a)	$x\|x\|$	(p)	Cont. in (–1, 1)
(b)	$\sqrt{\|x\|}$	(q)	differentiable in (–1, 1)
(c)	$x + [x]$	(r)	stictly inc in (–1, 1)
(d)	$\|x - 1\| + \|x + 1\|$	(s)	not differentiable atleast at one point in (–1, 1)

[IIT-JEE, 2007]

51. Let $g(x) = \left(\dfrac{(x-1)^n}{\log\{\cos^m(x-1)\}}\right) : 0 < 2$, m and n are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x - 1|$ at $x = 1$. If $\lim_{x \to 1}(g(x)), = p$, then

(a) $n = 1, m = 1$ (b) $n = 1, m = -1$

(c) $n = 2, m = 2$ (d) $n > 2, m = n$.

[IIT-JEE, 2008]

52. Let f and g be real valued functions defined on $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0, f'(0) = 0$ and $f(x) = g(x)\sin x$

Statement-I: $\lim_{x \to 0} (g(x)\cot x - g(0)\text{cosec} x) = f''(0)$

Statement-II: $f'(0) = g(0)$

[IIT-JEE, 2008]

53. Let $f(x)$ be a non constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$ and $f\left(\frac{1}{4}\right) = 0$. Then

(a) $f'(x)$ vanishes at least twice on [0, 1]

(b) $f'\left(\frac{1}{2}\right) = 0$

(c) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right)\sin x\,dx = 0$

(d) $\int_{-1/2}^{1/2} f(t)e^{\sin(\pi t)}dt = \int_{0}^{1/2} f(1-t)e^{\sin(\pi t)}dt$

[IIT-JEE, 2008]

54. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$. **[IIT-JEE, 2009]**

55. Let f be a real valued function defined on the interval $(0, \infty)$ by

$$f(x) = \ln x + \int_0^x \sqrt{1 + \sin t\,dt}$$

Then which one of the following statements is true.

(a) $f''(x)$ exists for all $x \in (0, \infty)$

(b) $f'(x)$ exists for all $x \in (0, \infty)$ and and f' is continuous on $(0, \infty)$ but not differentiable on $(0, \infty)$

(c) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(d) there exists $\beta > 0$ such that $|f'(x)| + |f(x)| \leq \beta$ for all $x \in (0, \infty)$

[IIT-JEE, 2010]

56. If $f(x) = \begin{cases} -x - \frac{\pi}{2} & : x \le -\frac{\pi}{2} \\ -\cos x & : -\frac{\pi}{2} < x \le 0 \\ x - 1 & : 0 < x < 1 \\ \ln x & : x > 1 \end{cases}$

then

(a) $f(x)$ is cont at $x = -\frac{\pi}{2}$

(b) $f(x)$ is not differentiable at $x = 0$

(c) $f(x)$ is differentiable at $x = 1$

(d) $f(x)$ is differentiable at $x = -3/2$

[IIT-JEE, 2011]

57. Let $f: R \to R$ be a function such that $f(x + y) = f(x) + f(y)$ for all x, y in R.

If $f(x)$ is differentiable at $x = 0$, then

(a) $f(x)$ is differentiable only in a finite interval containing zero.

(b) $f(x)$ is continuous $\forall\ x \in R$

(c) is constant $\forall\ x \in R$

(d) $f(x)$ is differentiable except at finitely many points.

[IIT-JEE, 2011]

58. Let $f(x) = \begin{cases} x^2 \left|\cos\left(\frac{\pi}{x}\right)\right| & : x \ne 0 \\ 0 & : x = 0 \end{cases}$

then f is

(a) differentiable both at $x = 0$, $x = 2$

(b) differentiable at $x = 0$ but not differentiable at $x = 2$.

(c) differentiable at $x = 2$ but not differentiable at $x = 0$.

(d) neither differentiable at $x = 0$ nor at $x = 2$.

[IIT-JEE, 2012]

59. Let $f: [0, 1] \to R$ be a function. Suppose the function f is twice differentiable,

$f(0) = 0 = f(1)$ and satisfies

$$f''(x) - 2f'(x) + f(x) \ge e^x,\ x \in [0, 1]$$

Which of the following is true.

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

[IIT-JEE, 2013]

60. Let $f(x) = x\sin(\pi x)$, $x > 0$. Then for all natural number n, $f'(x)$ vanishes at

(a) a unique point in $\left(n,\ n + \frac{1}{2}\right)$

(b) a unique point in $\left(n + \frac{1}{2},\ n + 1\right)$

(c) a unique point in $(n,\ n + 1)$

(d) two points in $(n,\ n + 1)$

[IIT-JEE, 2013]

61. Let $f: [a, b] \to [1, \infty)$ be a continuous function and $g: R \to R$ be defined as

$$g(x) = \begin{cases} 0 & : x < a \\ \int_a^x f(t)dt & : x \le x \le b \\ \int_a^b f(t)dt & : x > b \end{cases}$$

(a) $g(x)$ is continuous but not differentiable at $x = a$.

(b) $g(x)$ is differentiable on R.

(c) $g(x)$ is continuous but not differentiable at $x = b$.

(d) $g(x)$ is continuous and differentiable at either $x = a$ or $x = b$ but not both.

[IIT-JEE, 2014]

62. Let $f: R \to R$ and $g: R \to R$ be respectively given by $f(x) = |x| + 1$ and

$g(x) = x^2 + 1$. Define $h: R \to R$ by

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & : x \le 0 \\ \min\{f(x), g(x)\} & : x > 0 \end{cases}.$$

Then the number of points at which $h(x)$ is not differentiable is...

[IIT-JEE, 2014]

63. Let $g: R \to R$ be a differential function

$g(0) = 0$, $g'(0) = 0$ and $g'(1) \ne 0$

Let $f(x) = \begin{cases} \frac{xg(x)}{|x|} & : x \ne 0 \\ 0 & : x = 0 \end{cases}$

and $h(x) = e^{|x|}$ for all x in R

Let $(f_0 h)(x)$ denote $f(h(x))$

and $(h_0 f)(x)$ denote $h(f(x))$

Then which of the following is (are) true?

(a) f is differentiable at $x = 0$

(b) h is differentiable at $x = 0$

(c) $(f_0 h)$ is differentiable at $x = 0$

(d) $(h_0 f)$ is differentiable at $x = 0$.

[IIT-JEE, 2015]

ANSWERS

LEVEL-II

1. (a)	2. (b)	3. (a)	4. (a)	5. (a)
6. (c)	7. (d)	8. (a)	9. (c)	10. (d)
11. (d)	12. (a)	13. (c)	14. (b)	15. (d)
16. (d)	17. (c)	18. (b)	19. (c)	20. (c)
21. (b)	22. (c)	23. (d)	24. (c)	25. (b)
26. (a)	27. (d)	28. (a,b)	29. (a)	30. (d)
31. (b)	32. (c)	33. (c)	34. (b)	35. (d)
36. (b)	37. (a)	38. (b)	39. (b)	40. (b,c)
41. (a,b)	42. (c)	43. (a,c)	44. (a,c)	45. (a)
46. (d)	47. (b)	48. (c)	49. (c)	50. (b)
51. (c)	52. (d)	53. (b)	54. (d)	55. (b)
56. (a)	57. (c)	58. (c)		

INTEGER TYPE QUESTIONS

1. 7	2. 5	3. 5	4. 7	5. 8
6. 5	7. 5	8. 3	9. 6	10. 4
11. 5	12. 3.			

COMPREHENSIVE LINK PASSAGES

Passage I :	1. (b)	2. (a)	3. (c)
Passage II :	1. (b)	2. (c)	3. (c)
Passage III :	1. (c)	2. (a)	3. (a)
Passage IV :	1. (c)	2. (a)	3. (a)
Passage VI :	1. (d)	2. (a)	3. (a)
Passage VI :	1. (b)	2. (c)	3. (b)
Passage VII:	1. (d)	2. (b)	3. (a)

MATRIX MATCH

1. (A) → (S), (B) → (R),
   (C) → (S), (D) → (S)
2. (A) → (P, R), (B) → (Q, R, S),
   (C) → (Q, R, S), (D) → (P, Q, R, S)
3.
4. (A) → (S), (B) → (P),
   (C) → (Q), (D) → (R)

HINTS AND SOLUTIONS

Level I

Type I

1. (i) $f(0) = 1$

(ii) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{|x|}{x|}\right)$

Now, $\lim_{x\to 0^+}\left(\frac{x}{x}\right) = 1$

and $\lim_{x\to 0^-}\left(\frac{-x}{x}\right) = -1$

Thus, limit does not exist.

Hence, $f(x)$ is discontinuous at $x = 0$

2. (i) $f(0) = 2$

(ii) $\lim_{x\to 0} f(x)$

$= \lim_{x\to 0}\left(\frac{\sin x}{x} + \cos x\right)$

$= 1 + 1$

$= 2$

(iii) $\lim_{x\to 0} f(x) = 2 = f(0)$

Thus, $f(x)$ is continuous at $x = 0$

3. (i) $f(0) = \log(abc)$

(ii) $\lim_{x\to 0} f(x)$

$= \lim_{x\to 0}\left(\frac{a^x + b^x + c^x - 3}{x}\right)$

$= \lim_{x\to 0}\left(\frac{(a^x - 1) + (b^x - 1) + (c^x - 1)}{x}\right)$

$= \lim_{x\to 0}\left(\frac{(a^x - 1)}{x} + \frac{(b^x - 1)}{x} + \frac{(c^x - 1)}{x}\right)$

$= \log a + \log b + \log c$

$= \log(abc)$

(iii) $\lim_{x\to 0} f(x) = \log(abc) = f(0)$

Thus, $f(x)$ is continuous at $x = 0$

4. (i) $f(0) = 5$

(ii) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{e^{3x} - 1}{\log(1 + 5x)}\right)$

$= \lim_{x\to 0}\left(\frac{\frac{e^{3x} - 1}{3x}}{\frac{\log(1 + 5x)}{5x}} \times \frac{3x}{5x}\right)$

$= \frac{3}{5}$

(iii) $\lim_{x\to 0} f(x) = \frac{3}{5} \neq f(0)$

Thus, $f(x)$ is discontinuous at $x = 0$

5. (i) $f(0) = 2$

(ii) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x-|x|}{2}\right)$

Now, $\lim_{x\to 0+}\left(\frac{x-x}{2}\right) = 0$

and $\lim_{x\to 0^-}\left(\frac{x+x}{2}\right) = 0$

Thus, $\lim_{x\to 0} f(x) = 0$

(iii) $\lim_{x\to 0} f(x) = 0 \neq f(0)$

Thus, $f(x)$ is discontinuous at $x = 0$

6. (i) $f(0) = 7$

(ii) $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{e^x - 1}{\log(1 + 2x)}\right)$

$$= \lim_{x\to 0}\left(\frac{\frac{e^x - 1}{x}}{\frac{\log(1 + 2x)}{2x}} \times \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

(iii) $\lim_{x\to 0} f(x) = \frac{1}{2} \neq f(0)$

Thus, $f(x)$ is discontinuous at $x = 0$

7. (i) $f(0) = \frac{3}{2}$

(ii) $\lim_{x\to 0} f(x)$

Now, $\lim_{x\to 0^+}\left(\frac{\log(1 + 3x)}{e^{2x} - 1}\right)$

$$= \lim_{x\to 0^+}\left(\frac{\frac{\log(1 + 3x)}{3x}}{\frac{e^{2x} - 1}{2x}} \times \frac{3}{2}\right)$$

$$= \frac{3}{2}$$

Also, $\lim_{x\to 0^-}\left(\frac{\sin 3x}{\tan 2x}\right)$

$$= \lim_{x\to 0^-}\left(\frac{\frac{\sin 3x}{3x}}{\frac{\tan 2x}{2x}} \times \frac{3}{2}\right)$$

$$= \frac{3}{2}$$

(iii) $\lim_{x\to 0} f(x) = \frac{3}{2} = f(0)$

Thus, $f(x)$ is continuous at $x = 0$

Type-II

8. Since $f(x)$ is continuous at $x = 0$, so

$\lim_{x\to 0} f(x) = f(0) = k$

$\Rightarrow \quad k = \lim_{x\to 0} f(x)$

$\Rightarrow \quad k = \lim_{x\to 0}\left(\frac{\sin 5x}{3x}\right)$

$\Rightarrow \quad k = \lim_{x\to 0}\left(\frac{\sin 5x}{5x} \times \frac{5}{3}\right)$

$\Rightarrow \quad k = \frac{5}{3}$

9. Since $f(x)$ is continuous at $x = \frac{\pi}{2}$

so, $\lim_{x\to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right) = 3$

$\Rightarrow \quad \lim_{x\to \frac{\pi}{2}}\left(\frac{k\cos x}{\pi - 2x}\right) = 3$

$\Rightarrow \quad \lim_{x\to \frac{\pi}{2}}\left(\frac{-k\sin x}{-2}\right) = 3$

$\Rightarrow \quad \frac{k}{2} = 3$

$\Rightarrow \quad k = 6$

Hence, the value of k is 6.

10. Since $f(x)$ is continuous at $x = 2$, so

$\lim_{x\to 2} f(x) = f(2) = k$

$k = \lim_{x\to 2}\left(\frac{2^{x+2} - 16}{4^x - 16}\right)$

$k = \lim_{x\to 2}\left(\frac{4(2^x - 4)}{(2^x - 4)(2^x + 4)}\right)$

$k = \lim_{x\to 2}\left(\frac{4}{(2^x + 4)}\right)$

$k = \frac{4}{8} = \frac{1}{2}$

Hence, the value of k is $\frac{1}{2}$

11. Since $f(x)$ is continuous at $x = 0$, so

$\lim_{x\to 0} f(x) = f(0) = k$

$k = \lim_{x\to 0} f(x)$

$k = \lim_{x\to 0}\left(\frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2 + 1} - 1}\right)$

$k = \lim_{x\to 0}\left(\frac{\cos 2x - 1}{\sqrt{x^2 + 1} - 1}\right)$

$k = \lim_{x\to 0}\left(-\frac{\sin^2 x}{x^2} \times \left(\sqrt{x^2 + 1} + 1\right)\right)$

$k = 2$

Hence, the value of k is 2.

12. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0) = \frac{1}{2}$$

$$\lim_{x\to 0} f(x) = \frac{1}{2}$$

$$\lim_{x\to 0}\left(\frac{1-\cos kx}{x\sin x}\right) = \frac{1}{2}$$

$$\lim_{x\to 0}\left(\frac{k\sin(kx)}{\sin x + x\cos x}\right) = \frac{1}{2}$$

$$\lim_{x\to 0}\left(\frac{\frac{k\sin(kx)}{x}}{\frac{\sin x}{x} + \cos x}\right) = \frac{1}{2}$$

$$\lim_{x\to 0}\left(\frac{\frac{k^2\sin(kx)}{kx}}{\frac{\sin x}{x} + \cos x}\right) = \frac{1}{2}$$

$$\frac{k^2}{2} = \frac{1}{2}$$

$$k^2 = 1$$

$$1 = \pm 1$$

Type III

13. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0) = c$$

$$\lim_{x\to 0} f(x) = c$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = c$$

$$\lim_{x\to 0^+}\left(\frac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{\sin(a+1)x + \sin x}{x}\right) = c$$

$$\lim_{x\to 0^+}\left(\frac{x + bx^2 - x}{bx^{3/2}(\sqrt{x + bx^2} + \sqrt{x})}\right) = (a+2) = c$$

$$\lim_{x\to 0^+}\left(\frac{bx^2}{bx^{3/2}(\sqrt{x + bx^2} + \sqrt{x})}\right) = (a+2) = c$$

$$\lim_{x\to 0^+}\left(\frac{\sqrt{x}}{(\sqrt{x + bx^2} + \sqrt{x})}\right) = (a+2) = c$$

$$\lim_{x\to 0^+}\left(\frac{1}{(\sqrt{1 + bx} + 1)}\right) = (a+2) = c$$

$$\frac{1}{2} = (a + c) = c$$

$$a = -\frac{3}{2},\ c = \frac{1}{2},\ b \in R - \{0\}$$

14. Since $f(x)$ is continuous at $x = 4$, so

$$\lim_{x\to 4} f(x) = f(4)$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) = (a + b)$$

$$\lim_{x\to 4^+}\left(\frac{(x-4)}{(x-4)} + b\right) = \lim_{x\to 4^-}\left(-\frac{(x-4)}{(x-4)} + a\right) = (a + b)$$

$$1 + b = -I + a = (a + b)$$

$$a = 1,\ b = -1$$

15. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0) = a$$

$$\lim_{x\to 0^+}\left(\frac{be^x - \cos x - x}{x^2}\right) = \lim_{x\to 0^-}\left(\frac{2\tan^{-1}(e^x) - \frac{\pi}{4}}{x}\right) = a$$

$$\lim_{x\to 0^+}\left(\frac{be^x + \sin x - 1}{2x}\right) = \lim_{x\to 0^-}\left(\frac{2e^x}{1 + e^{2x}}\right) = a$$

$$\lim_{x\to 0^+}\left(\frac{be^x + \cos x}{2}\right) = 1 = a$$

$$\frac{b+1}{2} = 1 = a$$

$$a = 1 = b$$

Type IV

16. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0)$$

$$f(0) = \lim_{x\to 0}\left(\frac{\log\left(1 + \frac{x}{a}\right) - \log\left(1 - \frac{x}{b}\right)}{x}\right)$$

$$f(0) = \lim_{x\to 0}\left(\frac{\log\left(1 + \frac{x}{a}\right)}{x} - \frac{\log\left(1 - \frac{x}{b}\right)}{x}\right)$$

$$f(0) = \lim_{x\to 0}\left(\frac{\log\left(1 + \frac{x}{a}\right)}{\frac{x}{a}\times a} - \frac{\log\left(1 - \frac{x}{b}\right)}{-\frac{x}{b}\times -b}\right)$$

$$f(0) = \frac{1}{a} + \frac{1}{b}$$

17. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0)$$

$$f(0) = \lim_{x\to 0}\left(\frac{\sin 3x + A\sin 2x + B\sin x}{x^5}\right)$$

$$= \lim_{x\to 0}\left[\frac{1}{x^5}\left(3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \ldots\right)\right.$$

$$+ A\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots\right)$$

$$+ B\left(x - \frac{(x)^3}{3!} + \frac{(x)^5}{5!} - \ldots\right)$$

Since $f(x)$ is continuous at $x = 0$, it must

$$2A + 3 + B = 0$$

have $$\frac{8A}{6} + \frac{B}{6} + \frac{27}{6} = 0$$

On solving, we get, $A = -4$, $B = 5$

Thus, $$f(0) = \frac{3^5}{5!} + \frac{A \cdot 2^5}{5!} + \frac{B}{5!}$$

$$= \frac{3^5}{5!} - \frac{2^7}{5!} + \frac{5}{5!}$$

$$= \frac{243 - 128 + 5}{120}$$

$$= \frac{120}{120} = 1$$

18. Since $f(x)$ is continuous at $x = 0$, so $f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0}\left(\frac{e^{2x} - 1 - x(e^{2x} + 1)}{x^3}\right)$$

$$= \lim_{x\to 0}\left(\frac{2e^{2x} - (e^{2x} + 1) - 2xe^{2x}}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^{2x} - 2xe^{2x} - 1}{3x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{2e^{2x} - 2e^{2x} - 4xe^{2x}}{6x}\right)$$

$$= \lim_{x\to 0}\left(\frac{-4xe^{2x}}{6x}\right)$$

$$= \lim_{x\to 0}\left(\frac{-4e^{2x} - 8xe^{2x}}{6}\right)$$

$$= -\frac{2}{3}$$

19. Since f is continuous in R, so it is continuous for all n in R.

Now, $$f(0) = \lim_{n\to\infty} f\left(\frac{1}{4^n}\right)$$

$$= \lim_{n\to\infty}\left((\sin(e^n))e^{-n^2} + \frac{n^2}{n^2 + 1}\right)$$

$$= \lim_{n\to\infty}\left((\sin(e^n))e^{-n^2} + \frac{1}{1 + \frac{1}{n^2}}\right)$$

$$= \left(0 + \frac{1}{1 + 0}\right)$$

$$= 1$$

20. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\lim_{x\to 0^+} x^n \sin\left(\frac{1}{x}\right) = \lim_{x\to 0^-}\left(\frac{x^n e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}\right) = 0$$

It is possible only when n is natural number. Hence, the smallest value of n is 1.

21. Since $f(x)$ is continuous at $x = 0$, so

$$f(0) = \lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0}\left(\frac{1}{x} - \frac{2}{e^{2x} - 1}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^{2x} - 1 - 2x}{x(e^{2x} - 1)}\right)$$

$$= \lim_{x\to 0}\left(\frac{1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots - 1 - 2x}{2x^2\left(\frac{e^{2x} - 1}{2x}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots}{2x^2}\right)$$

$$= 1$$

22. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0)$$

$$f(0) = \lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0}\left(\frac{2 - \sqrt[4]{x^2 + 16}}{\cos 2x - 1}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sqrt[4]{x^2 + 16} - 2}{1 - \cos 2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{2\left(\sqrt[4]{1 + \frac{x^2}{16}} - 1\right)}{2\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(\sqrt[4]{1 + \frac{x^2}{16}} - 1\right)}{\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\left(1 + \frac{x^2}{4} + \ldots - 1\right)}{\sin^2 x}\right)$$

$$= \lim_{x\to 0}\left(\frac{x^2}{4\sin^2 x}\right) = \frac{1}{4}$$

Properties of Continuous Functions

23. Let $f(x) = x - \cos x$

Now, $f(0) = 0 - 1 = -1$

and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2} > 0$

Thus, $f(x) = 0$ has a root in $\left(0, \frac{\pi}{2}\right)$

24. Let $f(x) = 2\tan x + 5x - 2$

Now, $f(0) = -2 < 0$

and $f\left(\frac{\pi}{4}\right) = 2 + \frac{5\pi}{4} - 2 = \frac{5\pi}{5} > 0$

Thus, $f(x) = 0$ has a root in $\left(0, \frac{\pi}{4}\right)$

25. Let $f(x) = x.2^x - 1$

Now, $f(0) = -1 < 0$

and $f(1) = 1.2 - 1 = 1 > 0$

Thus, $f(x) = 0$ has a root in (0, 1)

26. Let $f(x) = e^{2x} + e^x + 2\sin^{-1}x + x - \pi$

Now, $f(0) = 1 + 1 - \pi = 2 - \pi < 0$

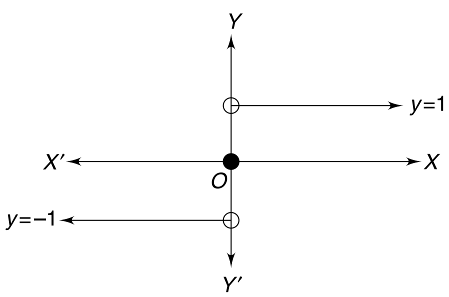
and $f(1) = e^2 + e + \pi + 1 - \pi = e^2 + e + 1 > 0$

Thus, $f(x) = 0$ has a root in [0, 1]

Disconinuity

27. Do yourself
28. Do yourself
29. Do yourself
30. Do yourself.

31. Let $f(x) = \begin{cases} \frac{|x|}{x} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$



Thus, the graph of the function $f(x) = sgn(x)$ makes a jump of 2 units at the point $x = 0$

Jump = (1 – (–1)) = 2

32. Let $\lim_{x\to 0}\left(\tan^{-1}\left(\frac{1}{x}\right)\right)$

Then $\lim_{x\to 0^+}\left(\tan^{-1}\left(\frac{1}{x}\right)\right) = \frac{\pi}{2}$

and $\lim_{x\to 0^-}\left(\tan^{-1}\left(\frac{1}{x}\right)\right) = -\frac{\pi}{2}$

Thus, Jump $= \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi$

33. Let $\lim_{x\to 3}\left(\frac{[x]}{x}\right)$

Then $\lim_{x\to 3^+}\left(\frac{[x]}{x}\right) = \frac{3}{3} = 1$

and $\lim_{x\to 3^-}\left(\frac{[x]}{x}\right) = \frac{2}{3}$

Thus, Jump $= \left(1 - \frac{2}{3}\right) = \frac{1}{3}$

34. Let $\lim_{x\to 0}[\sin x]$

Then $\lim_{x\to 0^+}[\sin x] = 0$ and $\lim_{x\to 0^-}[\sin x] = -1$

Thus, Jump = (0 – (–1)) = 1

35. A function is discontinuous at all such points where it is undefined.

(i) $f(x) = \frac{1}{x-1}$

$f(x)$ is discontinuous when $x - 1 = 0$

i.e. $x = 1$

Thus, the number of points of discontinuity is 1 at $x = 1$.

(ii) $f(x) = \frac{1}{|x| - 1}$

$f(x)$ is discontinuous when $|x| - 1 = 0$

$|x| = 1$

$x = \pm 1$

Thus, the number of points of discontinuity is 2 at $x = -1, 1$.

(iii) $f(x) = \frac{1}{|x| + 2}$

Here, denominator of a function is defined for all real values of x.

Thus, $f(x)$ is continuous for all real x.

(iv) $f(x) = \frac{x^2+1}{x^2-4}$

$f(x)$ is discontinuous when $x^2 - 4 = 0$

$\Rightarrow x = \pm 2$

Thus, the number of points of discontinuity is 2 at $x = -2, 2$.

(v) $f(x) = \frac{1}{\log_e x}$.

$f(x)$ is discontinuous for

$x < 0$, $x = 0$ and $x = 1$

Thus, the number of points of discontinuity is infinite.

(vi) $f(x) = \dfrac{1}{\log_e |x|}$

$f(x)$ is discontinuous for $x = 0, x = -1, x = 1$

Thus, the number of points of discontinuity is 3 at $x = -1, 0, 1$.

(vii) $f(x) = \dfrac{1}{\log_e |x-2|}$

$f(x)$ is discontinuous for $|x - 2| = 0, |x - 2| = 1$

$\Rightarrow \quad x = 2, x - 2 = \pm 1$

$\Rightarrow \quad x = 2, x = 3, 1$

Thus, the number of points of discontinuity is 3 at $x = 1, 2, 3$.

(viii) $f(x) = \dfrac{1}{\log_e (x^2 - 1)}$.

$f(x)$ is discontinuous for $(x^2 - 1) < 0$,

$\Rightarrow \quad x^2 - 1 = 0, x^2 - 1 = 1$

$\Rightarrow \quad -1 < x < 1, x = \pm 1, x = 0$

Thus, the number of points of discontinuity is infinite.

(ix) $f(x) = \dfrac{1}{\log_e |(x^2 - 1)|}$

$f(x)$ is discontinuous for

$|x^2 - 1| = 0, |x^2 - 1| = 1$

$\Rightarrow \quad x^2 - 1 = 0, x^2 - 1 = \pm 1$

$\Rightarrow \quad x^2 = 1, x^2 = 1 \pm 1$

$\Rightarrow \quad x = \pm 1, x = 0 \; x = \pm\sqrt{2}$

Thus, the number of points of discontinuity is 5 at $x = -\sqrt{2}, -1, 0, 1, \sqrt{2}$

(x) $f(x) = \dfrac{1}{x^2 - 3|x| + 2}$

$f(x)$ is discontinuous for $x^2 - 3\,|x| + 2 = 0$

$\Rightarrow \quad |x^2| - 3|x| + 2 = 0$

$\Rightarrow \quad (|x| - 1)(|x| - 2) = 0$

$\Rightarrow \quad (|x| - 1) = 0, (|x| - 2) = 0$

$\Rightarrow \quad x = \pm 1, x = \pm 2$

Thus, the number of points of discontinuity is 4 at $x = -2, -1, 1, 2$

(xi) $f(x) = \dfrac{x^2 + 4x + 1}{2\sin x - 1}$

$f(x)$ is discontinuous for $2\sin x - 1 = 0$

$\Rightarrow \quad \sin x = \dfrac{1}{2}$

$\Rightarrow \quad x = n\pi + (-1)^n \left(\dfrac{\pi}{6}\right), n \in I.$

Thus, the number of points of discontinuity is infinite.

(xii) $f(x) = \dfrac{x^2 - 7x + 2014}{\sqrt{2}\cos x - 1}$.

$f(x)$ is discontinuous for $\sqrt{2}\cos x - 1 = 0$

$\Rightarrow \quad \cos x = \dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad x = 2n\pi \pm \dfrac{\pi}{4}, n \in I$

Thus, the number of points of discontinuity is infinite.

36. We have $f(x) = [[x]] - [x - 1]$

$$= [x] - [x - 1]$$
$$= [x] - ([x] - 1)$$
$$= 1$$

Thus, $f(x)$ is continuous for all real x.

37. As we know that $[x]$ is discontinuous for all integral values of x.

Thus, the function $f(x)$ should be checked at all integral values of x for which $[\sin \pi x] \in$ I.

Now, $\quad \sin \pi x = 0 \Rightarrow \pi x = 0 \Rightarrow x = 0$ and $\sin \pi x$

$$= 1 \;\; \pi x = 1 \Rightarrow \pi x = \frac{\pi}{x} \Rightarrow x = \frac{1}{2}$$

Thus, $f(x) = [\sin \pi x]$ is continuous for

all $x \in [0, 1] - \left\{0, \dfrac{1}{2}\right\}$.

38. We have, $f(x) = [x] + [-x]$

$$= \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$$

Thus, $f(x)$ is discontinuous for all $x \in I$.

39. We have $f(x) = x + \{-x\} + [x], x \in I$

$$= x + [x] + \{-x\}$$
$$= \{x\} + \{-x\}$$
$$= \begin{cases} 1, & x \notin I \\ 0, & x \in I \end{cases}$$

Thus, $f(x)$ is discontinuous at

$$x = -2, -1, 0, 1, 2.$$

40. (i) We have $f(x) = [\sin x]$

The function $f(x)$ is discontinuous at all such points where it is broken.

$\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

$\sin x = 1 \Rightarrow x = \dfrac{\pi}{2}$

$\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$

Thus, the number of points of discontinuity are 4 at $x = 0, \frac{\pi}{2}, \pi, 2\pi$ (it is continuous at $x = \frac{3\pi}{2}$

(ii) We have $f(x) = \lfloor\sqrt{2}\sin x\rfloor, \forall\ x \in [0, 2\pi]$

The function $f(x)$ is discontinuous at all such points where it is broken.

$$\sqrt{2}\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$$

$$\sqrt{2}\sin x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\sqrt{2}\sin x = -1 \Rightarrow x = \frac{5\pi}{4}, \frac{7\pi}{4}$$

Thus, the number of points of discontinuity

$$= 7 \text{ at } x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi.$$

(iii) We have $f(x) = [\sin x + \cos x], \forall x \in [0, 2\pi]$

$$= \left[\sqrt{2}\left(\sin\left(x + \frac{\pi}{4}\right)\right)\right]$$

The function $f(x)$ is discontinuous at all such points where it is broken.

$$\sqrt{2}\left(\sin\left(x + \frac{\pi}{4}\right)\right) = 0 \Rightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$\sqrt{2}\left(\sin\left(x + \frac{\pi}{4}\right)\right) = 1 \Rightarrow x = 0, \frac{\pi}{2}$$

$$\sqrt{2}\left(\sin\left(x + \frac{\pi}{4}\right)\right) = -1 \Rightarrow x = \pi, \frac{3\pi}{4}$$

Thus, the number of points of discontinuity

$$= 7 \text{ at } x = 0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi.$$

(iv) We have $f(x) = [\sin \pi x], \forall x \in [0, 2]$

The function $f(x)$ is discontinuous at all such points where it is broken.

$$\sin \pi x = 0 \Rightarrow x = 0, 1, 2$$

$$\sin \pi x = 1 \Rightarrow x = \frac{1}{2}$$

$$\sin \pi x = -1 \Rightarrow x = \frac{3}{2}$$

Thus, the number of points of discontinuity is 4 at $x = 0, \frac{1}{2}, 1, 2$ $\left(\text{it is continuous at } x = \frac{3}{2}\right)$.

(v) We have $f(x) = [2\cos x], \forall\ x \in [0, 2\pi]$

The function $f(x)$ is discontinuous at all such points where it is broken.

$$2\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$2\cos x = 1 \Rightarrow x = \frac{\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

$$2\cos x = -1 \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$2\cos x = -2 \Rightarrow x = \pi$$

$$2\cos x = 2 \Rightarrow x = 0, 2\pi$$

Thus, the number of points of discontinuity

$$= 8, \text{ at } x = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi.$$

21. We have $f(u) = \frac{1}{u^2 + u - 2}$

$$= \frac{1}{(u+2)(u-1)}$$

Now, $u + 2 = 0$

$\Rightarrow$ $u = -2$

$\Rightarrow$ $\frac{1}{x-1} = -2$

$\Rightarrow$ $x - 1 = -\frac{1}{2}$

$\Rightarrow$ $x = 1 - \frac{1}{2} = \frac{1}{2}$

Also, $u - 1 = 0$

$\Rightarrow$ $u = 1$

$\Rightarrow$ $\frac{1}{x-1} = 1$

$\Rightarrow$ $x - 1 = 1$

$\Rightarrow$ $x = 2$

Thus, the number of points of discontinuity

$$= 3 \text{ at } x = \frac{1}{2}, 1, 2$$

22. We have, $f(f(f(x)))$

$$= f\left(f\left(\frac{1}{1-x}\right)\right)$$

$$= f(f(a)), \text{ where } a = \frac{1}{1-x}$$

$$= f\left(\left(\frac{1}{1-a}\right)\right)$$

$$= f\left(\frac{1}{1 - \frac{1}{1-x}}\right)$$

$$= f\left(\frac{1-x}{1-x-1}\right)$$

$$= f\left(\frac{x-1}{x}\right)$$

$$= f(b), \text{ where } b = \frac{x-1}{x}$$

$$= \frac{1}{1-b}$$

$$= \frac{1}{1 - \frac{x-1}{x}}$$

$$= \frac{x}{x - x + 1}$$

$$= x$$

Thus, the number of points of discontinuity

$= 2$ at $x = 1$ and $x = 0$

Note: If $x \neq 0, 1$ then the function

$$f(f(f(x))) = \frac{1}{1 - \frac{x-1}{x}} = x$$

is continuous everywhere.

23. We have $h(x) = f(g(x))$

$$= f(x^2 + 5)$$

$$= \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$$

It is continuous at $x \in R - \{-1, 1\}$.

24. We have $g(x) = f(f(x))$

$$= \begin{cases} 1 + f(x) : 0 \le f(x) \le 2 \\ 3 - f(x) : 2 < f(x) \le 3 \end{cases}$$

$$= \begin{cases} 1 + (1 + x) : 0 \le 1 + x \le 2, 0 \le x \le 2 \\ 3 - (1 + x) : 2 < 1 + x \le 3, 0 \le x \le 2 \\ 1 + (3 - x) : 0 \le 3 - x \le 2, 2 < x \le 3 \\ 3 - (3 - x) : 2 < 3 - x \le 3, 2 < x \le 3 \end{cases}$$

$$= \begin{cases} 2 + x : -1 \le x \le 1, 0 \le x \le 2 \\ 2 - x : 1 < x \le 2, 0 \le x \le 2 \\ 4 - x : 2 \le x \le 3, 2 < x \le 3 \\ x \quad : 0 < x \le 1, 2 < x \le 3 \end{cases}$$

$$= \begin{cases} 2 + x : -1 \le x \le 1 \\ 2 - x : 1 < x \le 2 \\ 4 - x : 2 < x \le 3 \\ x \quad : x \in \varphi \end{cases}$$

$$= \begin{cases} 2 + x : -1 \le x \le 1 \\ 2 - x : 1 < x \le 2 \\ 4 - x : 2 < x \le 3 \end{cases}$$

Thus, $g(x)$ is discontinuous at $x = 1$ and $x = 2$.

25. Clearly $f(x)$ is a signum function of x

i.e. $f(x) = \text{sgn}(x)$.

Now, $f(g(x)) = \text{sgn}(x(1 - x^2))$

$$= \begin{cases} 1 \ : x(1 - x^2) > 0 \\ 0 \ : x(1 - x^2) = 0 \\ -1 : x(1 - x^2) < 0 \end{cases}$$

$$= \begin{cases} 1 \ : x \in (-\infty, -1) \cup (0, 1) \\ 0 \ : x = 0, \pm 1 \\ -1 : x \in (-1, 0) \cup (1, \infty) \end{cases}$$

Thus, $f(g(x))$ is discontinuous at $x = -1, 0, 1$.

Note: $g(f(x))$ is continuous for all real x.

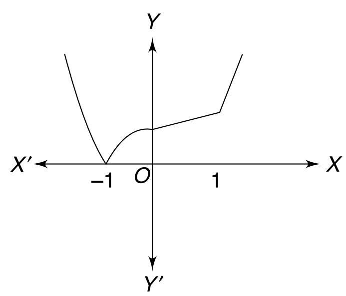
since $g(f(x)) = \text{sgn}(x)(1 - (\text{sgn}(x))^2)$

$$= \begin{cases} \frac{|x|}{x}\left(1 - \left(\frac{|x|}{x}\right)^2\right) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

$$= \begin{cases} 0 : x > 0 \\ 0 : x = 0 \\ 0 : x < 0 \end{cases}$$

26. We have $f(x) = |x + 1|(|x| + |x - 1|)$

$$= \begin{cases} (x + 1)(2x - 1) & : -2 \le x < -1 \\ -(x + 1)(2x - 1) & : -1 \le x < 0 \\ x + 1 & : 0 \le x < 1 \\ (x + 1)(2x - 1) & : 1 \le x \le 2 \end{cases}$$



Thus $f(x)$ is continuous everywhere.

27. We have $f(x) = \begin{cases} 1 - x & : 0 \le x \le 2 \\ x - 3 & : 2 < x \le 4 \end{cases}$

and $g(x) = \begin{cases} 2 + x & : -1 \le x \le 0 \\ 2 - x & : 0 < x \le 3 \end{cases}$

Now $(f_0 g)(x)$

$$= f(g(x))$$

$$= \begin{cases} 1 - g(x) & : 0 \le g(x) \le 2 \\ g(x) - 3 & : 2 < g(x) \le 4 \end{cases}$$

$$= \begin{cases} 1 - (2 + x) & : 0 \le 2 + x \le 2, -1 \le x \le 0 \\ 2 + (x - 3) & : 2 < 2 + x \le 4, -1 \le x \le 0 \\ 1 - (2 - x) & : 0 \le 2 - x \le 2, 0 < x \le 3 \\ 2 - (x + 3) & : 2 < 2 - x \le 4, 0 < x \le 3 \end{cases}$$

$$= \begin{cases} -1-x & : -2 \le x \le 0, -1 \le x \le 0 \\ x-1 & : 0 < x \le 2, -1 \le x \le 0 \\ x-1 & : 0 \le x \le 2, 0 < x \le 3 \\ -x-1 & : -2 < x \le 0, 0 < x \le 3 \end{cases}$$

$$= \begin{cases} -1-x & : -1 \le x \le 0 \\ x-1 & : x \in \phi \\ x-1 & : 0 \le x \le 2 \\ -x-1 & : x \in \phi \end{cases}$$

$$= \begin{cases} -1-x & : -1 \le x \le 0 \\ x-1 & : 0 \le x \le 2 \end{cases}$$

Thus, $f(x)$ is continuous in $[-1, 2]$.

28. Let $f(x) = \sqrt{x-5} - \frac{1}{x+3}$.

Now, $f(5) = \sqrt{5-5} - \frac{1}{5+3} = -\frac{1}{8} < 0$

Also, $f(6) = \sqrt{6-5} - \frac{1}{6+3} = 1 - \frac{1}{9} = \frac{8}{9} > 0$

Thus, $f(x)$ has a real root in (5, 6).

29. Let $f(x) = x^5 + 3x^4 + x - 2$

Now, $f(0) = -2 < 0$

Also, $f(1) = 1 + 3 + 1 - 2 = 4 - 2 = > 0$

Thus, $f(x)$ has a real root in [0, 1].

30. Let $f(x) = x^5 - 3x + 1$

Now, $f(1) = 1 - 3 + 1 = 2 - 3 = -1 < 0$

Also, $f(2) = 32 - 6 + 1 = 27 > 0$

Thus, $f(x)$ has a real root in [1, 2].

31. Let $f(x) = x^3 + x^2 - 3x - 3$

Now, $f(1) = 1 + 1 - 3 - 3 = -4 < 0$

Also, $f(2) = 8 + 4 - 6 - 3 = 3 > 0$

Thus, $f(x)$ has a real root in [1, 2].

32. Let $f(x) = x^5 + x$

Clearly f is continuous for all real x.

Now, $f(0) = 0 < 1$

and $f(1) = 2 > 1$

Hence, from the Intermediate value theorem there exist a number c in (0, 1) such that $f(c) = 1$.

33. Let $f(x) = x^5 + 3x^4 + x$

Clearly f is continuous for all real x.

Now, $f(0) = 0 < 2$ and

$f(1) = 1 + 3 + 1 = 5 > 2$.

Hence, from the Intermediate value theorem there exist a number c in (0, 1) such that $f(c) = 2$.

34. Let $f(x) = x^5 - 2x^3 + x^2 - 3x$

Clearly, f is continuous for all real x.

Now, $f(1) = 1 - 2 + 1 - 3 = -3 < -1$

and $f(2) = 32 - 16 + 4 - 6 = 16 > -1$

Hence, from the Intermediate value theorem there exist a number c in (1, 2) such that $f(c) = -1$.

35. Let $f(x) = 2x^3 + x^2 - x$

Clearly, f is continuous for all real x.

Now, $f(1) = 2 + 1 - 1 = 2 < 5$

and $f(2) = 16 + 4 - 2 = 18 > 5$

Hence, from the Intermediate value theorem there exist a number c in (1, 2) such that $f(c) = 5$.

36. We have $f(x) = |x-2| = \begin{cases} (x-2) & : x \ge 2 \\ -(x-2) & : x < 2 \end{cases}$

$$\text{L.H.D} = f'(2^+) = \lim_{x \to 2^+} \left(\frac{f(x) - f(2)}{(x-2)} \right)$$

$$= \lim_{x \to 2^+} \left(\frac{(x-2) - 0}{(x-2)} \right)$$

$$= 1.$$

Also, $$\text{R.H.D} = f'(2^-) = \lim_{x \to 2^-} \left(\frac{f(x) - f(2)}{(x-2)} \right)$$

$$= \lim_{x \to 2^-} \left(\frac{-(x-2) - 0}{(x-2)} \right)$$

$$= -1$$

Since L.H.D ≠ R.H.D, so, $f(x)$ is not differentiable at $x = 2$.

37. Now, R.H.D

$$= f'(1^+) = \lim_{x \to 1^+} \left(\frac{f(x) - f(1)}{(x-1)} \right)$$

$$= \lim_{x \to 1^+} \left(\frac{x^2 - 1}{(x-1)} \right)$$

$$= \lim_{x \to 1^+} ((x+1))$$

$$= 2$$

Also, L.H.D

$$= f'(1^-) = \lim_{x \to 1^-} \left(\frac{x-1}{(x-1)} \right)$$

$$= 1.$$

Since L.H.D ≠ R.H.D, so, $f(x)$ is not differentiable at $x = 1$.

38. We have, $f'(1)$

$$= \lim_{x \to 1} \left(\frac{f(x) - f(1)}{x-1} \right)$$

$$= \lim_{x \to 1} \left(\frac{\ln^2 x - 1}{x-1} \right)$$

$= \lim_{x \to 1} \left(\frac{2\ln x \cdot \frac{1}{x}}{1} \right)$

$= \lim_{x \to 1^+} \left(\frac{2\ln x.}{x} \right)$

$= 0$

Thus, $f(x)$ is differentiable at $x = 0$.

39. We have

$f(x) = e^{|x|}$

$= \begin{cases} e^x & : x \geq 0 \\ e^{-x} & : x < 0 \end{cases}$

Now, R.H.D

$= f'(0^+)$

$= \lim_{x \to 0^+} \left(\frac{f(x) - f(0)}{x - 0} \right)$

$= \lim_{x \to 0^+} \left(\frac{e^x - 1}{x} \right)$

$= 1$

Also, L.H.D

$= f'(0^-)$

$= \lim_{x \to 0^-} \left(\frac{f(x) - f(0)}{x - 0} \right)$

$= \lim_{x \to 0^-} \left(\frac{e^{-x} - 1}{x} \right)$

$= -1.$

Since L.H.D ≠ R.H.D, so, $f(x)$ is not differentiable at $x = 0$.

40. We have, R.H.D

$= f'(1^+)$

$= \lim_{x \to 1^+} \left(\frac{f(x) - f(1)}{x - 1} \right)$

$= \lim_{x \to 1^+} \left(\frac{4 - x - 3}{x - 1} \right)$

$= \lim_{x \to 1^+} \left(\frac{1 - x}{x - 1} \right)$

$= -1$

Also, L.H.D

$= f'(1^-)$

$= \frac{\lim}{x \to 1^-} \left(\frac{f(x) - f(1)}{x - 1} \right)$

$= \lim_{x \to 1^-} \left(\frac{3^x - 3}{x - 1} \right)$

$= \lim_{x \to 1^+} \left(\frac{3^x \log 3}{1} \right)$

$= 3 \log 3$

Since L.H.D ≠ R.H.D, so, $f(x)$ is not differentiable at $x = 1$.

41. We have $f(x) = \sin x + |\sin x|$

$= \begin{cases} \sin x + \sin x & : x \geq 0 \\ \sin x - \sin x & : x < 0 \end{cases}$

$= \begin{cases} 2\sin x & : x \geq 0 \\ 0 & : x < 0 \end{cases}$

Now, R.H.D

$= f'(0^+)$

$= \lim_{x \to 0^+} \left(\frac{f(x) - f(0)}{x - 0} \right)$

$= \lim_{x \to 0^+} \left(\frac{2\sin x - 0}{x - 0} \right)$

$= 2$

Also, L.H.D

$= f'(0^-)$

$= \lim_{x \to 0^-} \left(\frac{f(x) - f(0)}{x - 0} \right)$

$= \lim_{x \to 0^-} \left(\frac{0 - 0}{x - 0} \right) = 0$

Since L.H.D ≠ R.H.D, so, $f(x)$ is not differentiable at $x = 0$.

42. We have $f(x) = [x] \tan(\pi x)$

Now, R.H.D

$= f'(k^+)$

$= \lim_{x \to k^+} \left(\frac{f(x) - f(k)}{x - k} \right)$

$= \lim_{h \to 0^+} \left(\frac{f(k + h) - f(k)}{(k + h) - k} \right)$

$= \lim_{h \to 0^+} \left(\frac{f(k + h) - f(k)}{h} \right)$

$= \lim_{h \to 0^+} \left(\frac{[k + h] \tan(k\pi + \pi h) - \tan(\pi k)}{h} \right)$

$= \lim_{h \to 0^+} \left(\frac{[k + h] \tan(\pi h) - 0}{h} \right)$

$= \lim_{h \to 0^+} \left(\frac{k \tan(\pi h)}{h} \right)$

$= \lim_{h \to 0^+} \left(\frac{\pi k \tan(\pi h)}{\pi h} \right)$

$= \pi k.$

43. We have $f(x) = [x]\sin(\pi x)$

Now, L.H.D

$$= f'(k^-)$$

$$= \lim_{x \to k^-} \left(\frac{f(x) - f(k)}{x - k} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{f(k-h) - f(k)}{(k-h)/k} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{f(k-h) - f(k)}{-h} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{[k-h]\,\sin(k\pi - \pi h) - \sin(\pi k)}{-h} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{(k-1)(-1)^{k-1}\sin(\pi h) - 0}{-h} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{\pi(k-1)(-1)^{k}\sin(\pi h)}{\pi h} \right)$$

$$= (-1)^k\,(k-1)\pi$$

44. We have $f'(0)$

$$= \lim_{x \to 0} \left(\frac{f(x) - f(0)}{x - 0} \right)$$

$$= \lim_{x \to 0} \left(\frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x - 0} \right)$$

$$= \lim_{x \to 0} \left(\frac{x^2 \sin\left(\frac{1}{x}\right)}{x} \right)$$

$$= \lim_{x \to 0} \left(x \sin\left(\frac{1}{x}\right) \right)$$

$$= 0$$

Thus, $f(x)$ is differentiable at $x = 0$.

45. We have $f'(0)$

$$= \lim_{x \to 0} \left(\frac{f(x) - f(0)}{x - 0} \right)$$

$$= \lim_{x \to 0} \left(\frac{\frac{x}{1 + e^{\frac{1}{x}}} - 0}{x - 0} \right)$$

$$= \lim_{x \to 0} \left(\frac{1}{1 + e^{\frac{1}{x}}} \right)$$

$$= \begin{cases} \lim_{x \to 0^+} \left(\dfrac{1}{1 + e^{\frac{1}{x}}} \right) \\ \lim_{x \to 0^+} \left(\dfrac{1}{1 + e^{\frac{1}{x}}} \right) \end{cases}$$

$$= \begin{cases} \lim_{x \to 0^+} \left(\dfrac{1}{1 + e^{\frac{1}{x}}} \right) \\ \lim_{x \to 0^+} \left(\dfrac{e^{-\frac{1}{x}}}{e^{-\frac{1}{x}} + 1} \right) \end{cases}$$

$$= \begin{cases} \left(\dfrac{1}{1 + e^{\infty}} \right) \\ \left(\dfrac{e^{-\infty}}{e^{-\infty} + 1} \right) \end{cases}$$

$$= \begin{cases} \dfrac{1}{\infty} \\ \left(\dfrac{0}{0 + 1} \right) \end{cases}$$

$$= \begin{cases} 0 \\ 0 \end{cases}$$

$$= 0$$

Thus, $f(x)$ is differentiable at $x = 0$.

46. Given $f(x)$ is differentiable at $x = 1$,

so, $f'(1^+) = f(1^-)$

$$\Rightarrow \lim_{x \to 1^+} \left(\frac{f(x) - f(1)}{x - 1} \right) = \lim_{x \to 1^{-1}} \left(\frac{f(x) - f(1)}{x - 1} \right)$$

$$\Rightarrow \lim_{x \to 1^+} \left(\frac{x^2 + ax + b - a - 1}{x - 1} \right)$$

$$= \lim_{x \to 1^{-1}} \left(\frac{ax^2 + 1 - a - 1}{x - 1} \right)$$

$$\Rightarrow \lim_{x \to 1^+} \left(\frac{(x^2 - 1) + a\,(x - 1) + b}{x - 1} \right)$$

$$= \lim_{x \to 1^{-1}} \left(\frac{a(x^2 - 1)}{x - 1} \right)$$

$$\Rightarrow \lim_{x \to 1^+} \left(a + (x + 1) + \frac{b}{x - 1} \right) = \lim_{x \to 1^{-1}} (a(x + 1))$$

$$\Rightarrow \lim_{x \to 1^+} \left(a + (x + 1) + \frac{b}{x - 1} \right) = 2a \quad \text{...(i)}$$

As we know that, every differentiable function is continuous.

So, $f(x)$ is continuous at $x = 1$.

$\Rightarrow \quad \lim_{x\to 1} f(x) = f(1)$

$\Rightarrow \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

$\Rightarrow \quad \lim_{x\to 1^+} (x^2 + ax + b) = \lim_{x\to 1^-} (ax^2 + 1) = a + 1$

$\Rightarrow \quad (1 + a + b) = (a + 1) = a + 1$

$\Rightarrow \quad b = 0$

Put $b = 0$ in (i), we get,

$\Rightarrow \quad \lim_{x\to 1^+} (a + (x + 1)) = 2a$

$\Rightarrow \quad (a + (1 + 1)) = 2a$

$\Rightarrow \quad a = 2.$

Hence, the values of a and b are 2 and 0.

47. Since $f(x)$ is differentiable at $x = \frac{1}{2}$,

so, $\quad f'\left(\frac{1^+}{2}\right) = f'\left(\frac{1^-}{2}\right)$

$$\Rightarrow \quad \lim_{x\to 1/2^+}\left(\frac{f(x) - f\left(\frac{1}{2}\right)}{x - \frac{1}{2}}\right) = \lim_{x\to 1/2^-}\left(\frac{f(x) - f\left(\frac{1}{2}\right)}{x - \frac{1}{2}}\right)$$

$$\Rightarrow \quad \lim_{x\to 1/2^+}\left(\frac{\frac{1}{|x|} - 2}{x - \frac{1}{2}}\right) = \lim_{x\to 1/2^-}\left(\frac{(a + bx^2) - 2}{x - \frac{1}{2}}\right)$$

$$\Rightarrow \quad \lim_{x\to 1/2^-}\left(\frac{(a + bx^2) - 2}{x - \frac{1}{2}}\right) = -1 \qquad \text{...(i)}$$

As we know that every differentiable function is continuous, so it is continuous at $x = 1/2$

$\Rightarrow \quad \lim_{x\to 1/2} f(x) = f\left(\frac{1}{2}\right)$

$\Rightarrow \quad \lim_{x\to 1/2^+} f(x) = \lim_{x\to 1/2^-} f(x) = f\left(\frac{1}{2}\right)$

$\Rightarrow \quad \lim_{x\to 1/2^+} \left(\frac{1}{|x|}\right) = \lim_{x\to 1/2^-} (a + bx^2) = 2$

$$\Rightarrow \quad \left(a + \frac{b}{4}\right) = 2 \qquad \text{...(ii)}$$

From (i) and (ii), we get,

$$\lim_{x\to 1/2^-}\left(\frac{2 - \frac{b}{4} + bx^2 - 2}{x - \frac{1}{2}}\right) = -1$$

$$\Rightarrow \quad \lim_{x\to 1/2^-}\left(\frac{b\left(x^2 - \frac{1}{4}\right)}{\left(x - \frac{1}{2}\right)}\right) = -1$$

$$\Rightarrow \quad \lim_{x\to 1/2^-}\left(b\left(x + \frac{1}{2}\right)\right) = -1$$

$b = -1$

From (ii), we get,

$\Rightarrow \quad a = 2 + \frac{1}{4} = \frac{9}{4}$

Hence, the values of a and b are $a = 9/4$ and $b = -1$.

48. To check continuity:

$\lim_{x\to 0} f(x) = f(0) = 0$

Now, $\quad \lim_{x\to 0} f(x)$

$= \lim_{x\to 0} \left(x \sin\left(\frac{1}{x}\right)\right)$

$= 0$

Thus, $f(x)$ is continuous at $x = 0$.

To check differentiability

$$f'(0) = \lim_{x\to 0}\left(\frac{f(x) - f(0)}{x - 0}\right)$$

$$= \lim_{x\to 0}\left(\frac{x \sin\left(\frac{1}{x}\right) - 0}{x - 0}\right)$$

$$= \lim_{x\to 0}\left(\sin\left(\frac{1}{x}\right)\right)$$

= Oscillating a finite value between –1 to 1.

Thus, $f(x)$ is not differentiable at $x = 0$.

49. To check the differentiability

$$f'(0^+) = \lim_{x\to 0^+}\left(\frac{f(x) - f(0)}{x - 0}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{\frac{xe^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}} - 0}{x - 0}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{1}{1 + e^{-\frac{1}{x}}}\right)$$

$$= \left(\frac{1}{1 + e^{-\infty}}\right)$$

$$= \left(\frac{1}{1 + 0}\right) = 1$$

Also, $\quad f'(0^-) = \lim_{x\to 0^-}\left(\frac{f(x) - f(0)}{x - 0}\right)$

$$= \lim_{x\to 0^-}\left(\frac{\dfrac{xe^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}-0}{x-0}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}}\right)$$

$$= \left(\frac{e^{-\infty}}{1+e^{-\infty}}\right)$$

$$= 0$$

So, $f(x)$ is not differentiable at $x = 0$

Since, both one sided derivative exists (though they are unequal), so $f(x)$ is continuous at $x = 0$.

50. To check the differentiability

$$f'(1^+) = \lim_{x\to 1^+}\left(\frac{f(x)-f(1)}{x-1}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{1+h-1}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(1+h)-f(1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{|h|([1+h]-\{1+h\})-0}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h(1-h)}{h}\right)$$

$$= 1$$

Also, $$f'(1^-) = \lim_{x\to 1^-}\left(\frac{f(x)-f(1)}{x-1}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(1-h)-f(1)}{1-h-1}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(1-h)-0}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{|-h|([1+h]-\{1+h\})}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-h(0-(1+h))}{-h}\right)$$

$$= -1$$

Thus, $f(x)$ is not differentiable at $x = 0$.

Since, both one sided derivative exists (though they are unequal), so $f(x)$ is continuous at $x = 0$.

51. To check the differentiability.

$$f'(1^+) = \lim_{x\to 1^+}\left(\frac{f(x)-f(1)}{x-1}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{[2x-3]x-(-1)}{x-1}\right)$$

$$= \lim_{h\to 0}\left(\frac{[2(1+h)-3](1+h)+1}{1+h-1}\right)$$

$$= \lim_{h\to 0}\left(\frac{[2h-1](1+h)+1}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-(1+h)+1}{h}\right)$$

$$= -1$$

Also, $$\lim_{x\to 1^-}\left(\frac{f(x)-f(1)}{x-1}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(1-h)-f(1)}{1-h-1}\right)$$

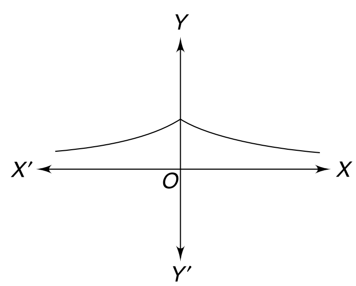
$$= \lim_{h\to 0}\left(\frac{\sin\left((1-h)\frac{\pi}{2}\right)+1}{-h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cos\left(\frac{\pi}{2}\right)h+1}{-h}\right)$$

$$= -\infty$$

Thus, $f(x)$ is not differentiable at $x = 1$.

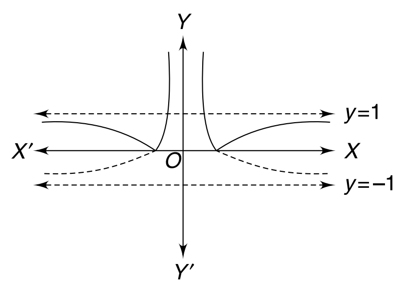
52.



Clearly $f(x)$ is not differentiable at $x = 0$

So $f(x)$ is not differentiable in $[-2, 2]$

53.



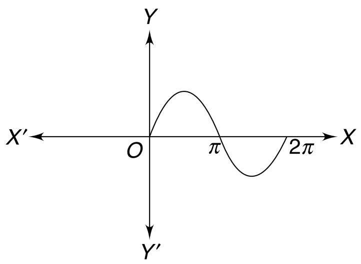
Clearly, $f(x)$ is not differentiable at $x = -1, 0, 1$.

Thus, $f(x)$ is not differentiable in; $-4, 4]$

54. We have $f(x) = \sin x + \sin|x|$

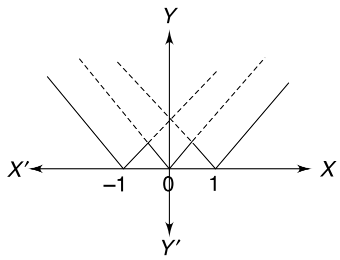
$$f(x) = \begin{cases}\sin x+\sin x & : x\ge 0\\ \sin x-\sin x & : x<0\end{cases}$$

$$= \begin{cases}2\sin x & : x\ge 0\\ 0 & : x<0\end{cases}$$



Clearly $f(x)$ is not differentiable at $x = 0$.

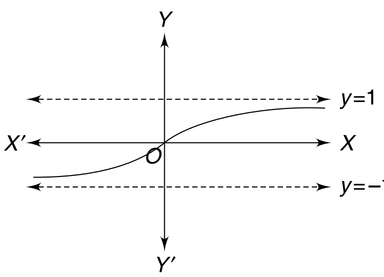
55.



Clearly, $f(x)$ is not differentiable at $x = –1, 0, 1$.

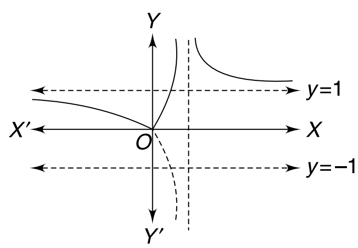
Thus, $f(x)$ is not differentiable in; $-4, 4]$

56. We have, $f(x) = \dfrac{x}{1 + |x|}$



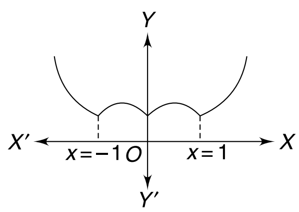
Clearly, $f(x)$ is differentiable everywhere.

57.



Clearly $f(x)$ is not differentiable at $x = 0$.

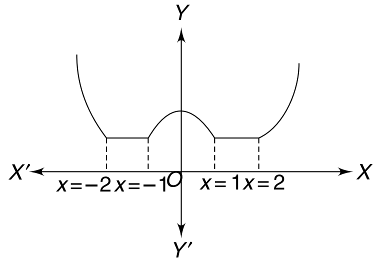
58. We have, $f(x) = |x| + |x^2 - 1|$



Thus, $f(x)$ is not differentiable at $x = -1, 0, 1$

So $f(x)$ is not differentiable in R.

59. We have $f(x) = |x^2 - 1| + |x^2 - 4|$



Clearly, $f(x)$ is not differentiable at $x = -2, -1, 1, 2$.

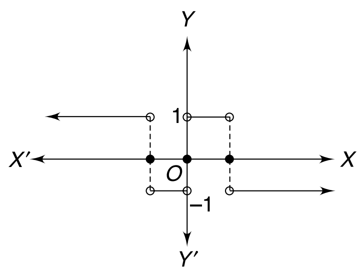
Thus, $f(x)$ is not differentiable in R.

60. We have $f(g(x))$

$$= f(x(1 - x^2))$$

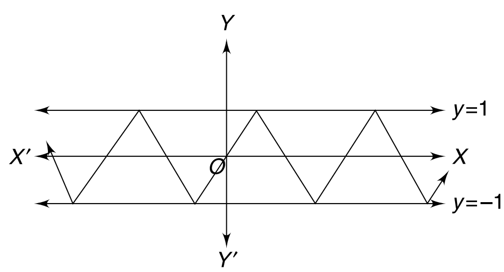
$$= \begin{cases} 1 & : g(x) > 0 \\ 0 & : g(x) = 0 \\ -1 & : g(x) < 0 \end{cases}$$

$$= \begin{cases} 1 & : x \in (-\infty, -1) \cup (0, 1) \\ 0 & : x = -1, 0, 1 \\ -1 & : (-1, 0) \cup (1, \infty) \end{cases}$$



Clearly, $f(x)$ is not differentiable at $x = -1, 0, 1$.

61. We have $f(x) = \sin^{-1}|\sin x|$



Clearl, $f(x)$ is not differentiable at $x = (2n + 1)\dfrac{\pi}{2}$

Thus, $f(x)$ is non-differentiable in R.

62. We have $\lim\limits_{h \to 0} \left(\dfrac{f(2 + h) - f(2 - h)}{h}\right)$

$= f'(2)$

$= 5.$

63. We have $\lim_{h\to 0}\left(\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}\right)$

$$= \lim_{h\to 0}\left[\left(\frac{f(2h+2+h^2)-f(2)}{2h+h^2}\right)\right.$$

$$\left.\times\left(\frac{h-h^2}{f(h-h^2+1)-f(1)}\right)\times\left(\frac{2h+h^2}{h-h^2}\right)\right]$$

$$= \lim_{h\to 0}\left(f'(2)\times\left(\frac{h(2+h)}{h(1-h)}\right)\times\frac{1}{f'(x)}\right)$$

$$= 6\times 2\times\frac{1}{4}$$

$= 3.$

64. We have $f(x) = |x^3|$

$$= \begin{cases} x^3 & : x \ge 0 \\ -x^3 & : x < 0 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} 3x^2 & : x \ge 0 \\ -3x^2 & : x < 0 \end{cases}$$

$$\Rightarrow \quad f''(x) = \begin{cases} 6x & : x \ge 0 \\ -6x & : x < 0 \end{cases}$$

Clearly, $f''(0^+) = 0 = f''(0^-)$

Thus, $f(x)$ is twice differentiable at other values of x.

65. We have $y = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & : x \ne 0 \\ 0 & : x = 0 \end{cases}$

Here, $f'(0) = \lim_{h\to 0}\left(\frac{f(x)-f(0)}{x-0}\right)$

$$= \lim_{x\to 0}\left(\frac{x^2\sin\left(\frac{1}{x}\right)}{x}\right)$$

$$= \lim_{x\to 0}\left(x\sin\left(\frac{1}{x}\right)\right)$$

$= 0$

Thus, $y = f(x)$ is differentiable at $x = 0$

$$y' = \begin{cases} 2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right) & : x \ne 0 \\ 0 & : x = 0 \end{cases}$$

Now $f''(0) = \lim_{x\to 0}\left(\frac{f'(x)-f'(0)}{x-0}\right)$

$$= \lim_{x\to 0}\left(\frac{2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)}{x}\right)$$

Thus, the limit does not exist at $x = 0$.

Clearly, $y = f(x)$ is not twice differentiable at $x = 0$.

66. We have $y = f(x) = \begin{cases} xe^x & : x \le 0 \\ x + x^2 - x^3 & : x > 0 \end{cases}$

$$\Rightarrow \quad f'(x) = \begin{cases} xe^x + e^x & : x < 0 \\ 1 + 2x - 3x^2 & : x > 0 \end{cases}$$

Clearly, $f'(0^+) = 1 = f'(0^-)$

Therefore, $f(x)$ is differentiable at $x = 0$.

Also, $f''(x) = \begin{cases} xe^x + 2e^x & : x < 0 \\ 2 - 6x & : x > 0 \end{cases}$

Clearly, $f''(0^+) = 2 = f''(0^-)$

Thus, $f(x)$ is twice differentiable at $x = 0$.

67. We have $f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{f(x)\cdot f(h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(x)\,(f(h)-1)}{h}\right)$$

$$= \lim_{h\to 0}\left(f(x)\times\left(\frac{(f(h)-1)}{h}\right)\right) \quad \text{...(i)}$$

Also, $f(x+y) = f(x)\cdot f(y)$, $\forall\ x \in R$

Put $x = 0 = y$, then we get,

$$f(0) = f(0)\cdot f(0)$$

$$\Rightarrow \quad f(0) = f^2(0)$$

$$\Rightarrow \quad f(0)(f(0)-1) = 0$$

$$\Rightarrow \quad f(0) = 1,\ (\because f(0) \ne 0)$$

From (i), we get,

$$f'(x) = \lim_{h\to 0}\left(f(x)\times\left(\frac{(f(h)-f(0))}{h}\right)\right)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \lim_{h\to 0}\left(\left(\frac{(f(h)-f(0))}{h}\right)\right)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = f'(0) = k$$

On integration, we get,

$$\log|f(x)| = kx + c$$

put $x = 0$, $\log|f(0)| = \log|1| = 0$, so $c = 0$.

$\Rightarrow \quad \log|f(x)| = kx$

$\Rightarrow \quad f(x) = e^{kx}$.

Hence, the required function is $f(x) = e^{kx}$.

68. Given $\quad f(x + y) = f(x) + f(y)$

Put $\quad x = 0 = y, f(0) = 0$

Now, $\quad f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{f(x) + f(h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(h)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(h) - f(0)}{h}\right)$$

$$= f'(0) = k$$

On integration, we get, $f(x) = kx + c$

If $x = 0$, $f(0) = 0$, then $c = 0$.

Thus, $\quad f(x) = kx$

69. We have $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$

$\Rightarrow f'\left(\frac{x+y}{2}\right)\cdot\frac{1}{2} = \frac{1}{2}f'(x)$, y as a constant

$\Rightarrow \quad f'\left(\frac{x+y}{2}\right) = f'(x)$

Replacing x by 0 and y by $2x$, we get,

$$f'(x) = f'(0) = -1$$

On integration, we get,

$$f(x) = -x + c$$

When $x = 0$, $c = 1$.

Thus, $f(x) = -x + 1$

70. We have $f\left(\frac{x+y}{3}\right) = \frac{2 + f(x) + f(y)}{3}$

$\Rightarrow \quad f'\left(\frac{x+y}{3}\right)\cdot\frac{2}{5} = \frac{2}{5}f'(x)$, y as a constant

$\Rightarrow \quad f'\left(\frac{x+y}{3}\right) = f'(x)$

Replacing x by 0 and y by $3x$, we get,

$$f'(x) = f'(0) = k(\text{say})$$

when $x = 2$, then $k = f'(2) = 2$

Thus, $\quad f'(x) = 2$

On integration, we get,

$$f(x) = 2x + c$$

when $x = 0$, then $c = f(0) = 2$

Hence, the given function is $f(x) = 2x + 2$.

71. We have $f\left(\frac{2x+3y}{5}\right) = \frac{2f(x) + 3f(y)}{5}$

$\Rightarrow f'\left(\frac{2x+3y}{5}\right)\cdot\frac{2}{5} = \frac{2}{5}\cdot f'(x)$, y as a constant.

$\Rightarrow f'\left(\frac{2x+3y}{5}\right) = f'(x)$

Replacing x by 0 and y by $\frac{5}{3}x$, we get,

$$f'(x) = f'(0) = -1$$

On integration, we get,

$$f(x) = -x + c$$

when $x = 0$, then $c = f(0) = 1$

Hence, the function is, $f(x) = -x + 1$

72. If $f(x + y + z) = f(x)\cdot f(y)\cdot f(z)$ for all x, y, z in R such that $f(2) = 4$, $f'(0) = 3$, then find $f'(2)$. Ans. 12

HINTS AND SOLUTIONS

Level I

(Problems for JEE-Advanced)

1. We have $f(x) = \lim_{n\to\infty}\left(\frac{\sin x}{1 + (2\sin x)^{2n}}\right)$

$$= \begin{cases} \sin x & : 0 < x < \frac{5\pi}{6} \text{ or } \frac{5\pi}{6} < x < \pi \\ \frac{1}{4} & : x = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \\ 0 & : \frac{\pi}{6} < x < \frac{5\pi}{6} \end{cases}$$

Thus $f(x)$ is discontinuous at $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

2. We have $f(x) = \frac{x^3}{4 + |x|}$

Here $f(x)$ is defined for all real values of x.

Thus, $f(x)$ is continuous for all real x.

3. We have $f(0) = 0$.

Now, $\quad \lim_{x\to 0} f(x)$

$$\lim_{x\to 0}\left(\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\ldots\right)$$

$$\lim_{x\to 0}\left[\left(1-\frac{1}{x+1}\right)+\left(\frac{1}{(x+1)}-\frac{1}{(2x+1)}\right)+\left(\frac{1}{(2x+1)}-\frac{1}{(3x+1)}\right)+\left(\frac{1}{((n-1)x+1)}-\frac{1}{nx+1}\right)\right]$$

$$=\lim_{x\to 0}\left(1-\frac{1}{nx+1}\right)$$

$$=1$$

Thus, $f(x)$ is discontinuous at $x = 0$.

4. We have

$$f(x)=\begin{cases} x\exp\left(-\left(\frac{1}{|x|}+\frac{1}{x}\right)\right) & : x\neq 0\\ 0 & : x=0\end{cases}$$

$$\Rightarrow \quad f(x)=\begin{cases} xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)} & : x>0\\ xe^{-\left(-\frac{1}{x}+\frac{1}{x}\right)} & : x<0\\ 0 & : x=0\end{cases}$$

$$\Rightarrow \quad f(x)=\begin{cases} xe^{-\frac{2}{x}} & : x>0\\ x & : x<0\\ 0 & : x=0\end{cases}$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)=f(0)=0$$

Thus, $f(x)$ is continuous at $x = 0$.

5. Now, $\lim_{x\to 0^+} f(x)$

$$=\lim_{x\to 0^+}\left(1+\left(\frac{cx+dx^3}{x^2}\right)\right)^{1/x}$$

$$=\lim_{x\to 0^+}(1+dx)^{1/x} \text{(since } c=0 \text{ for a finite limit)}$$

$$=e^d$$

Also, $\lim_{x\to 0^-}\left(\frac{a(1-x\sin x)+b\cos x+5}{x^2}\right)$

$$=\lim_{x\to 0^-}\left(\frac{a-ax\sin x+b\cos x+5}{x^2}\right)$$

(since it has a finite limit, so $a + b + 5 = 0$)

$$=\lim_{x\to 0^-}\left(\frac{0-a\sin x-ax\cos x-b\sin x}{2x}\right)$$

$$=\lim_{x\to 0^-}\left(\frac{0-a\cos x-a\cos x+ax\sin x-b\cos x}{2}\right)$$

$$=\frac{-2a-b}{2}$$

Also, $a + b + 5 = 0$

and $\frac{-2a-b}{2}=3$

On solving we get, $a = –$, $b = -4$, $c = 0$ and $d = \log_e 3$.

6. We have, $f(x)=\left[x-\frac{1}{3}\right]+[x]+\left[x+\frac{1}{3}\right]$

$$=\begin{cases} -2-1-1=-4 & : -1\leq x<-\frac{2}{3}\\ -1-1-1=-3 & : -\frac{2}{3}\leq x<-\frac{1}{3}\\ -1-1+0=-2 & : -\frac{1}{3}\leq x<0\\ -1+0+0=-1 & : x=0\end{cases}$$

Thus, $f(x)$ is discontinuous at

$$x=-1,-\frac{2}{3},-\frac{1}{3},0$$

7. As we know that $[x]$ is discontinuous at all integral values of x.

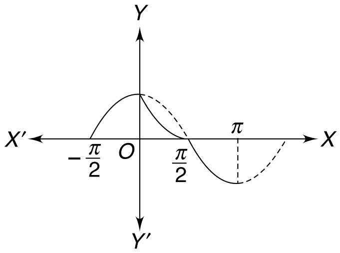
Therefore the function $f(x)$ will be continuous in (4, 6) only when $\left[\frac{(x-2)^3}{a}\right]=0$

$$\Rightarrow \quad a\geq 64$$

$$\Rightarrow \quad a\in[64,\infty)$$

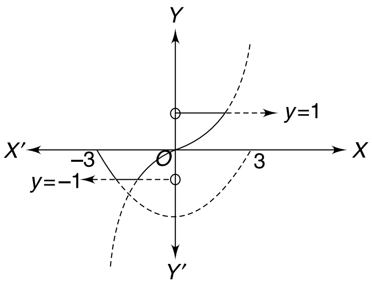
8. We have $f(x) = \min\{1, \cos x, 1 - \sin x\}$

$$=\begin{cases} \cos x & : -\frac{\pi}{2}\leq x\leq 0\\ 1-\sin x & : 0<x\leq\frac{\pi}{2}\\ \cos x & : \frac{\pi}{2}<x\leq\pi\end{cases}$$



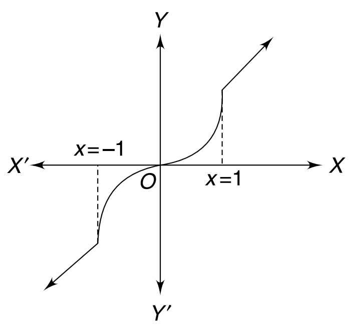
Thus, $f(x)$ is continuous at $x = 0, \frac{\pi}{2}$

9. We have $f(x) = \max\left\{\text{sng}(x), -\sqrt{9-x^2}, x^3\right\}$

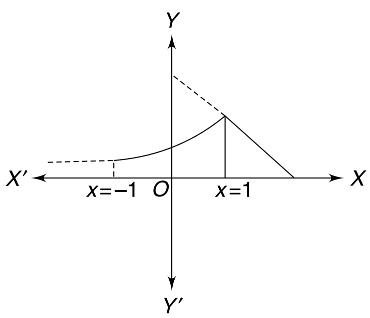


Clearly, $f(x)$ is discontinuous at $x = 0$.

10. We have $f(x) = \begin{cases} x^3 & : x^2 < 1 \\ x & : x > 1 \end{cases}$



Clearly $f(x)$ is continuous at all real x.

11. 

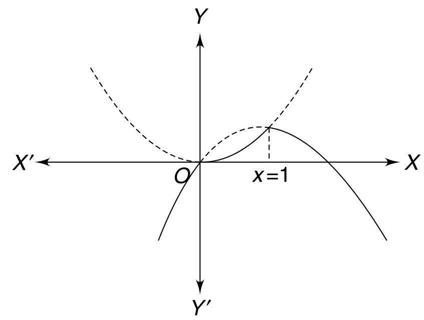
Clearly $f(x)$ is continuous at $x = 1$.

12. We have $f(x) = x - |x - x^2|$

$$\Rightarrow \quad f(x) = \begin{cases} x^2 & : x - x^2 \geq 0 \\ 2x - x^2 & : x - x^2 < 0 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} x^2 & : x^2 - x \leq 0 \\ 2x - x^2 & : x^2 - x > 0 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} x^2 & : 0 \leq x \leq 1 \\ 2x - x^2 & : x \in (-\infty, 0] \cup [1, \infty) \end{cases}$$



Clearly $f(x)$ is continuous for all real x.

13. We have $f(x) = [x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right]$

$$= \begin{cases} -1-1-1=-3 & : -1 \leq x < -\frac{2}{3} \\ -1-1=-2 & : -\frac{2}{3} \leq x < -\frac{1}{3} \\ -1+0+0=-1 & : -\frac{1}{3} \leq x < 0 \\ 0+0+0=0 & : 0 \leq x < \frac{1}{3} \\ 0+0+1=1 & : \frac{1}{3} \leq x < \frac{2}{3} \\ 0+1+1=2 & : \frac{2}{3} \leq x < 1 \\ 1+1+1=3 & : x = 1 \end{cases}$$

Thus the number of points of discontinuity is 7.

14. Since $f(x)$ is continuous at $x = a$, so $f(a) = 0$

Now, $\lim_{x \to a} \left(\frac{\ln(1 + 6f^2(x) - 3f(x))}{3f(x)}\right)$

$$= \lim_{x \to a} \left(\frac{\log(1 + 6f^2(x) - 3f(x))}{6f^2(x) - 3f(x) - 3f(x)} \times \frac{6f^2(x) - 3f(x)}{3f(x)}\right)$$

$$= \lim_{x \to a} \left(\frac{6f^2(x) - 3f(x)}{3f(x)}\right)$$

$$= \lim_{x \to a} (2f(x) - 1)$$

$$= (2f(a) - 1)$$

$$= 0 - 1$$

$$= -1.$$

15. Let $f(x) = g(x) + h(x)$

where $g(x) = [4x]$ and $h(x) = \{3x\}$

Now, $g(x) = [4x]$ is discontinuous at

$$x \in \left[0, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{3}{4}\right), \left[\frac{3}{4}, 1\right), \left[1, \frac{5}{4}\right) \ldots \left[5, \frac{21}{4}\right)$$

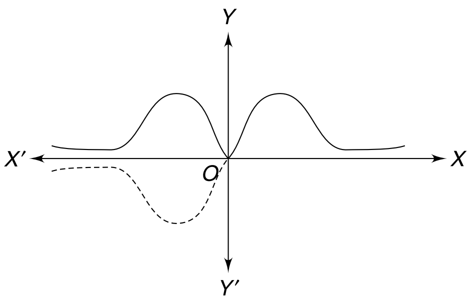
There are 25 points, where $g(x)$ is discontinuous

Also, $h(x) = \{3x\}$ is discontinuous at $x = 1, 2, 3, 4, 5$

Thus, the number discontinuous points is 5.

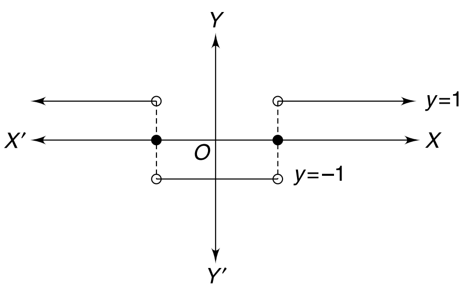
Therfore, the number of discontinuous points of $f(x)$ is 30.

16.



Clearly, the function $f(x)$ is continuous everywhere.

17.

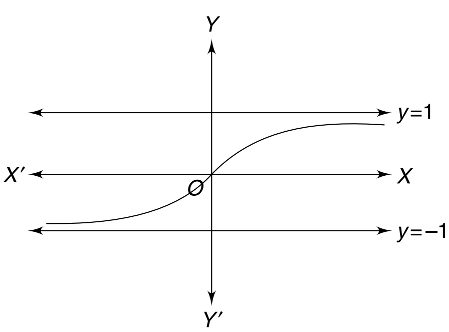


Thus the number of points of discontinuity is 2 at $x = -1, 1$.

18. We have $f(x) = [x] + \{2x\} + [3x]$

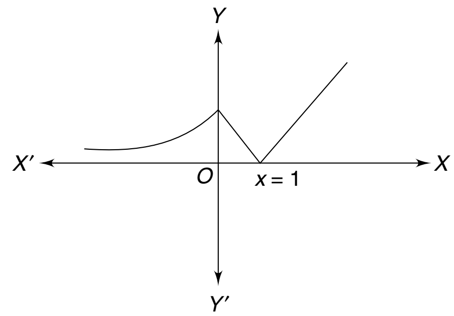
Thus, the number of points of discontinuous is 4.

19.



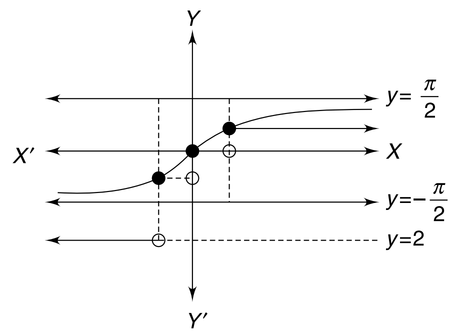
Clearly, $f(x)$ is continuous for all real x.

20.



Clearly $f(x)$ is continuous at $x = 0, 1$.

21. We have $f(x) = [\tan^{-1} x]$



22. The function $f(x)$ is undefined only when

$$[x + 1] = 0$$

$$\Rightarrow \quad 0 \le x + 1 < 1$$

$$\Rightarrow \quad -1 \le x < 0$$

Therefore, $D_f = R - [-1, 0)$

Thus, the function $f(x)$ is continuous at $x \in D_f = R - [-1, 0)$.

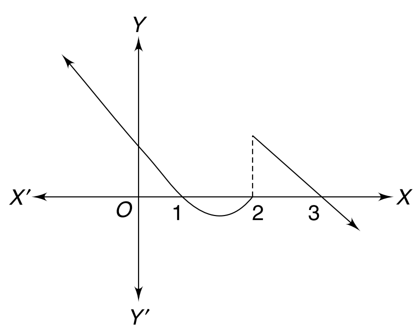
23. We have

$f(x) = |x + 1|(|x| - |x - 1|)$

$$= \begin{cases} (x+1)(2x-1) & : -2 \le x < -1 \\ -(x+1)(2x-1) & : -1 \le x < 0 \\ (x+1) & : 0 \le x < 1 \\ (x+1)(2x-1) & : -1 \le x < 0 \end{cases}$$

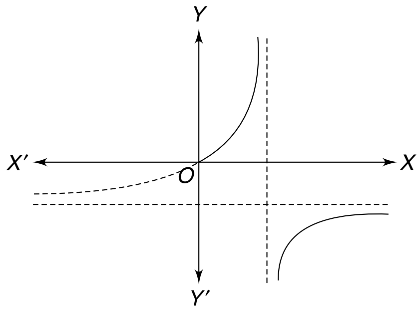
Clearly, f is continuous everywhere but not differentiable at $x = -1, 0, 1$

24.

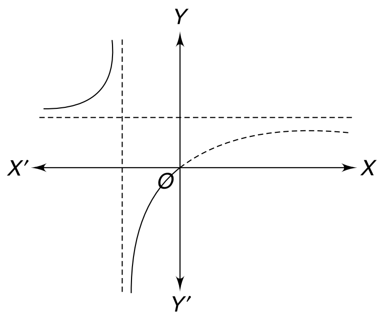


Clearly, f is discontinuous at $x = 2$.

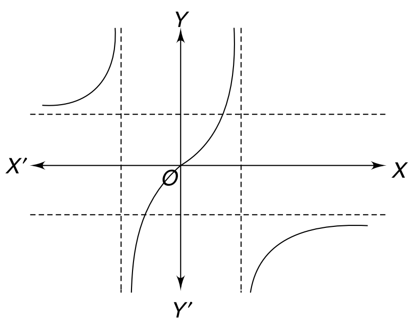
25. **Case-I:** When $x \geq 0$



Case-II: When $x < 0$.

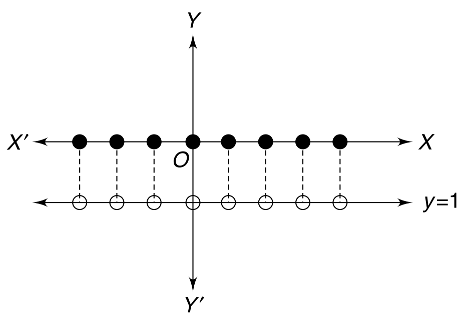


Thus,



Thus, $f(x)$ is discontinuous at $x = -1, 1$.

26.



Thus, the number of points of discontinuity is infinite at $x \in I$.

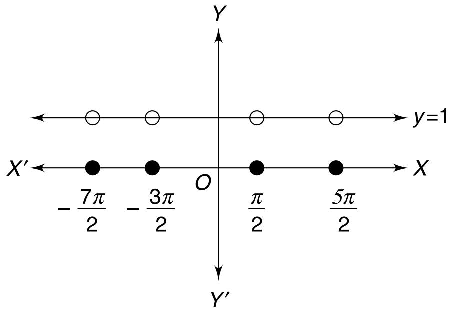
27. We have $f(x) = \text{sgn}(\cos 2x - 2\sin x + 3)$

$$= \text{sgn}(1 - 2\sin^2 x - 2\sin x + 3)$$

$$= \text{sgn}(2 - 2\sin^2 x - 2\sin x)$$

$$= \text{sgn}\{2(1 - \sin x)(2 + \sin x)\}$$

$$= \begin{cases} 0 : & x = 2n\pi + \dfrac{\pi}{2} \\ 1 : & x \neq 2n\pi + \dfrac{\pi}{2} \end{cases}$$



Thus the function has discontinuous at $x = 2n\pi + \dfrac{\pi}{2}$, $n \in I$

28.

(i) We have $f(x) = [\log_e x]$

It is discontinuous, where $\log_e x$ is an integer.

i.e. $\log_e x = k$

i.e. $x = e^k$, $k \in I$

Thus, the number of points of discontinuity is infinite at $x = ..., e^{-3}, e^{-2}, e^{-1}, 1, e, e^2, e^3,...$

(ii) We have $f(x) = \lfloor \sin^{-1} x \rfloor$

It is discontinuous, where $\sin^{-1} x$ is an integer.

i.e. $\sin^{-1} x = -1, 0, 1$

i.e. $x = -\sin 1, 0, \sin 1$

Thus, the number of points of discontinuity is 3 at $x = -\sin 1, 0, \sin 1$

(iii) We have $f(x) = \left[\dfrac{2}{x^2 + 1}\right]$

It is discontinuous, where $\dfrac{2}{x^2 + 1}$ is an integer.

i.e. $\dfrac{2}{x^2 + 1} = 1, 2$

i.e. $x = 0, 1$.

Thus, the number of points of discontinuity is 2 at $x = 0, 1$.

29. We have $f(x) = \left[\dfrac{x}{2}\right] + \left[\dfrac{x}{3}\right] + \left[\dfrac{x}{4}\right]$

It is discontinuous, where $f(x)$ is an integer.

Thus, $f(x) = \begin{cases} 1 & : 0 \leq x < 2 \\ 1 & : 2 \leq x < 3 \\ 2 & : 3 \leq x < 4 \\ 3 & : \quad x = 2 \end{cases}$

Therefore, the number of points of discontinuity is infinite.

Level-II

(Tougher Problems for JEE-Advanced)

1. Since f is continuous at $x = 0$, so

$\lim_{x\to 0} f(x) = f(0) = c$

$$c = \lim_{x\to 0}\left(\frac{1 + a\cos 2x + b\cos 4x}{x^2\sin^2 x}\right)$$

Since it has a finite limit, so $a + b + 1 = 0$ ...(i)

$$\text{Thus, } c = \lim_{x\to 0}\left(\frac{-a - b + a\cos 2x + b\cos 4x}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{-a(1 - \cos 2x) - b(1 - \cos 4x)}{x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{-a\left(\frac{2\sin^2 x}{x^2}\right) - b\left(\frac{2\sin^2(2x)}{x^2}\right)}{x^2}\right)$$

Since it has a finite limit, so $(2a + 8b) = 0$

$a = -4b$ ...(ii)

On solving (i) and (ii), we get,

$$a = -\frac{4}{3},\quad b = \frac{1}{3}$$

$$\text{Thus, } \lim_{x\to 0}\left(\frac{4(1 - \cos 2x) - (1 - \cos 4x)}{3x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{8\sin^2 x - 8\sin^2 x\cos^2 x}{3x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{8\sin^2 x(1 - \cos^2 x)}{3x^4}\right)$$

$$= \lim_{x\to 0}\left(\frac{8}{3}\times\frac{\sin^2 x}{x^2}\times\frac{\sin^2 x}{x^2}\right)$$

$$= \frac{8}{3}$$

Hence, the value of $\left(a + b + c + \frac{1}{3}\right)$

$$= \left(-\frac{4}{3} + \frac{1}{3} + \frac{8}{3} + \frac{1}{3}\right)$$

$= 2.$

2. We have

$\lim_{x\to 0} (\sin x + \cos x)^{\operatorname{cosec} x}$

$$= \lim_{x\to 0^-} (1 + (\sin x + \cos x - 1))^{\operatorname{cosec} x}$$

$$= e^{\lim_{x\to 0^-}\frac{(\sin x + \cos x - 1)}{\sin x}}$$

$$= e^{\lim_{x\to 0^-}\left(1 - \frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}\right)}$$

$$= e^{\lim_{x\to 0^-}\left(1 - \tan\left(\frac{x}{2}\right)\right)}$$

$= e$

$$\text{Now, } \lim_{x\to 0^-}\left(\frac{e^{1/x} + e^{2/x} + e^{3/|x|}}{ae^{2/x} + be^{3/|x|}}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{e^{1/x} + e^{2/x} + e^{3/x}}{ae^{2/x} + be^{3/x}}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{e^{3/x}(e^{-2/x} + e^{-1/x} + 1)}{e^{3/x}(ae^{-1/x} + b)}\right)$$

$$= \frac{1}{b}$$

Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0^+} f(x) = = \lim_{x\to 0^-} f(x) = f(0)$$

$$\frac{1}{b} = e = a$$

Thus, $a = e,\ b = \frac{1}{e}$

Hence, the value of $\{e(a + b) + 2\}$

$$= \left\{e\left(e + \frac{1}{e}\right) + 2\right\}$$

$= (e^2 + 3)$

3. Since $f(x)$ is differentiable at $x = 0$, so it is continuous at $x = 0$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{e^{ax/2} - 1}{x}\right) = \lim_{x\to 0^-}\left(b\sin^{-1}\left(\frac{x + c}{2}\right)\right) = \frac{1}{2}$$

$$\Rightarrow \frac{a}{2} = b\sin^{-1}\left(\frac{c}{2}\right) = \frac{1}{2}$$

$\Rightarrow a = 1$

Also, it is differentiable at $x = 0$, so $f'(0^+) = f'(0^-)$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{f(x) - f(0)}{x - 0}\right) = \lim_{x\to 0^-}\left(\frac{f(x) - f(0)}{x - 0}\right)\lim_{x\to 0^-} e^{\left\{\frac{2}{x}\right\}}$$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{e^{ax/2} - \frac{1}{2}}{x}\right) = \lim_{x\to 0^-}\left(\frac{b\sin^{-1}\left(\frac{x + c}{2}\right) - \frac{1}{2}}{x}\right)$$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{e^{x/2} - \frac{1}{2}}{x}\right) = \lim_{x\to 0^-}\left(\frac{b\sin^{-1}\left(\frac{x + c}{2}\right) - \frac{1}{2}}{x}\right)$$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{1 + \left(\frac{x}{2}\right) + \frac{1}{2}\left(\frac{x}{2}\right)^2 + \dots - \frac{1}{2}}{x}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{b\dfrac{1}{\sqrt{1-\left(\dfrac{x+c}{2}\right)^2}}\times\dfrac{1}{2}}{1}\right)$$

$$\Rightarrow \frac{1}{8} = \frac{b}{2\sqrt{1-\dfrac{c^2}{4}}}$$

$$\Rightarrow \frac{1}{8} = \frac{b}{\sqrt{1-\dfrac{c^2}{4}}}$$

$$\Rightarrow \frac{1}{8} = \frac{b}{\sqrt{4-c^2}}$$

$$\Rightarrow 64b^2 = 4 - c^2$$

Hence, the result.

4. Since $f(x)$ is continuous at $x = 0$,

so $\lim_{x\to 0} f(x) = f(0)$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = a$$

Now, $\lim_{x\to 0^+} f(x)$

$$= 0$$

$$= \lim_{x\to 0^+}\left(\frac{x(3e^{1/x}+4)}{2-e^{1/x}}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{x\left(3+4e^{-\frac{1}{x}}\right)}{2e^{-\frac{1}{x}}-1}\right)$$

$$= 0$$

Also, $\lim_{x\to 0^-} f(x)$

$$= \lim_{x\to 0^-}\left(\frac{-x\left(3e^{-\frac{1}{x}}+4\right)}{\left(2-e^{-1/x}\right)}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{-x\left(3+4e^{1/x}\right)}{(2e^{1/x}-1)}\right)$$

$$= 0$$

Thus, $f(x)$ is continuous at $x = 0$

Therefore, the value of a is 0.

Hence, the value of $(a^2 + a + 10)$ is 10.

5. Since $f(x)$ is continuous at $x = -2$

so, $\lim_{x\to -2} f(x) = f(-2)$

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^-} f(x) = f(-2)$$

Now,

$$= \lim_{x\to -2^+} f(x) = \lim_{x\to -2^+}\left\{\sin\left(\frac{x^4-16}{x^5+32}\right)\right\}$$

$$= \lim_{x\to -2^+}\left\{\sin\left(\frac{4x^3}{5x^4}\right)\right\}$$

$$= \sin\left(\frac{4\times -8}{5\times 16}\right)$$

$$= -\sin\left(\frac{2}{5}\right)$$

Also, $\lim_{x\to -2^-} f(x)$

$$= \lim_{x\to -2^-}\left(\frac{ae^{\frac{-1}{(x+2)}}-1}{2-e^{\frac{-1}{(x+2)}}}\right)$$

$$= \lim_{x\to -2}\left(\frac{a-e^{\frac{1}{(x+2)}}}{2e^{\frac{1}{(x+2)}}-1}\right)$$

$$= -a$$

Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = f(0)$$

$$-a = -\sin\left(\frac{2}{5}\right) = b$$

$$a = \sin\left(\frac{2}{5}\right),\ b = -\sin\left(\frac{2}{5}\right)$$

Hence, the value of $(a + b + 2) = 2$.

Note: As we know that

$$\lim_{n\to\infty} x^{2n} = \begin{cases} 0 & : 0 < x < 1 \\ 1 & : \quad x = 1 \\ \infty & : \quad x > 1 \end{cases}.$$

6. We have $f(x) = \lim_{n\to\infty}\left(\dfrac{x^{2n}-1}{x^{2n}+1}\right)$

$$= \begin{cases} -1 & : 0 \le x^2 < 1 \\ 0 & : x^2 = 1 \\ 1 & : x > 1 \end{cases}$$

$$= \begin{cases} -1 & : \quad -1 < x < 1 \\ 0 & : \quad x = \pm 1 \\ 1 & : x < -1, x > 1 \end{cases}$$

Thus, the number of discontinuity is 2 at $x = -1, 1$.

7. We have $f(x) = \lim_{n\to\infty} (\cos^{2n} x)$

$$= \begin{cases} 0 : & 0 \le \cos^2 x < 1 \\ 1 : & \cos^2 x = 1 \end{cases}$$

$$= \begin{cases} 0 : & x \ne n\pi, n \in I \\ 1 : & x = n\pi, n \in I \end{cases}$$

Thus, the number of discontinuity is infinite at $x = n\pi,\ n \in I$.

8. We have $f(x) = \lim_{n\to\infty}\left(\sin\left(\frac{\pi x}{2}\right)\right)^{2n}$

$$= \begin{cases} 0 & : \quad 0 \le \sin\left(\frac{\pi x}{2}\right) < 1 \\ 1 & : \quad \sin\left(\frac{\pi x}{2}\right) = 2 \end{cases}$$

$$= \begin{cases} 0 & : \quad x \ne (2n+1), n \in I \\ 1 & : \quad x = (2n+1), n \in I \end{cases}$$

Here $x \in [0, 2]$

Thus, $f(x)$ is discontinuous at $x = 1$.

9. **Case-I:** When $x < 1$

We have $\lim_{n\to\infty}\left(\frac{\log(2+x) - x^{2n}\sin x}{x^{2n}+1}\right)$

$$= \left(\frac{\log(2+x) - 0}{1+0}\right)$$

$$= \log(2+x)$$

Case-II: When $x > 1$

We have $\lim_{n\to\infty}\left(\frac{\log(2+x) - x^{2n}\sin x}{x^{2n}+1}\right)$

$$= \lim_{n\to\infty}\left(\frac{\frac{\log(2+x)}{x^{2n}} - \sin x}{1 + \frac{1}{x^{2n}}}\right)$$

$$= \left(\frac{0 - \sin x}{1+0}\right) = -\sin x$$

Thus, $f(x)$ is discontinuous at $x = 1$.

10. We have $f(x) = \lim_{n\to\infty}\left(\frac{x^{2n-1} + ax^2 + bx}{x^{2n}+1}\right)$

$$= \begin{cases} ax^2 + bx & : -1 < x < 1 \\ \frac{a-b-1}{2} & : x = -1 \\ \frac{a+b+1}{2} & : x = 1 \\ \frac{1}{x} & : x > 1 \text{ or } x < -1 \end{cases}$$

For continuity at $x = 1$,

$$\frac{a+b+1}{2} = a + b$$

$$a + b = 1 \qquad \text{...(i)}$$

For continuity at $x = -1$

$$a - b = -1 \qquad \text{...(ii)}$$

On solving (i) and (ii), we get, $a = 0,\ b = 1$

Thus, the value of $2a + 3b + 10$

$$= 0 + 3 + 10 = 13.$$

11. Given function is discontinuous when

$$a + \sin\pi x = 1$$

Now, if $a = 1$

$\Rightarrow \quad \sin\pi x = 0$

$\Rightarrow \quad x = 1, 2, 3, 4, 5.$

When $a = 0$

$\Rightarrow \quad \sin\pi x = 1$

$\Rightarrow \quad x = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}.$

Thus, the total number of points of discontinuity $= 8$.

12. We have $f(x) = \lim_{n\to\infty}\left(\frac{x}{1 + (2\sin x)^{2n}}\right)$

$$= \lim_{n\to\infty}\left(\frac{x}{1 + (4\sin^2 x)^n}\right)$$

$$= \begin{cases} 0 & : \quad 0 \le 4\sin^2 x < 1 \\ 1 & : \quad 4\sin^2 x = 1 \end{cases}$$

$$= \begin{cases} 0 & : \quad x \ne n\pi \pm \frac{\pi}{6}, n \in I \\ 1 & : \quad x = n\pi \pm \frac{\pi}{6}, n \in I \end{cases}$$

Thus, $f(x)$ is discontinuous at $x = n\pi \pm \frac{\pi}{6},\ n \in I$

13. **Case-I:** When $x < 1$

We have $f(x) = \lim_{n\to\infty}\left(\frac{\cos(\pi x) - x^{2n}\sin(x-1)}{1 + x^{2n+1} - x^{2n}}\right)$

$$= \lim_{x\to 1^-}\left(\frac{\cos(\pi x) - 0}{1 + 0 - 0}\right)$$

$$= \cos\pi = -1$$

Case-II: When $x > 1$

We have $f(x)$

$$= \lim_{n\to\infty}\left(\frac{\frac{\cos(\pi x)}{x^{2n+1}} - \frac{\sin(x-1)}{x}}{1 + \frac{1}{x^{2n+1}} - \frac{1}{x}}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{0 - \frac{\sin(x-1)}{x}}{1 + 0 - \frac{1}{x}}\right)$$

$$= \lim_{x\to 1^+}\left(-\frac{\sin(x-1)}{x-1}\right)$$

$$= -1$$

Thus, $f(x)$ is continuous at $x = 1$.

14. We have $f(x) = \begin{cases} |2x-3|[x] : & x \geq 1 \\ \sin\left(\frac{\pi x}{2}\right) : & x < 1 \end{cases}$

$$f(x) = \begin{cases} \sin\left(\frac{\pi x}{2}\right) & : 0 \leq x < 1 \\ (3-2x)[x] & : 1 \leq x < \frac{3}{2} \\ (2x-3)[x] & : \frac{3}{2} \leq x \leq 2 \end{cases}$$

$$f(x) = \begin{cases} \sin\left(\frac{\pi x}{2}\right) & : 0 \leq x < 1 \\ (3-2x) & : 1 \leq x < \frac{3}{2} \\ (2x-3) & : \frac{3}{2} \leq x < 2 \\ 2 & : x = 2 \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = 2$.

15. We have

$$f(x) = \begin{cases} [\cos \pi x] & : 0 \leq x \leq 1 \\ |2x-3|[x-2] & : 1 < x \leq 2 \end{cases}$$

$$f(x) = \begin{cases} 1 & : x = 0 \\ 0 & : 0 < x \leq \frac{1}{2} \\ -1 & : \frac{1}{2} < x \leq 1 \\ (2x-3) & : 1 < x \leq \frac{3}{2} \\ (3-2x) & : \frac{3}{2} < x < 2 \\ 0 & : x = 2 \end{cases}$$

Thus $f(x)$ is discontinuous at $x = 0, 1/2, 2$.

16. Since $1 \leq x < 2$

$\Rightarrow \quad 0 \leq x - 1 < 1$

$\Rightarrow \quad [x^2 - 2x] = \lfloor (x-1)^2 - 1 \rfloor$

$= \quad \lfloor (x-1)^2 \rfloor - 1 = 0 - 1 = -1.$

We have

$$\Rightarrow \quad f(x) = \begin{cases} |1-4x^2| & : 0 \leq x < 1 \\ [x^2 - 2x] & : 1 \leq x < 2 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} (1-4x^2) & : 0 \leq x < \frac{1}{2} \\ (4x^2-1) & : \frac{1}{2} \leq x < 1 \\ -1 & : 1 \leq x < 2 \end{cases}$$

Thus, $f(x)$ is discontinuous at $x = 1$.

17. We have

$$f(x) = \begin{cases} |\sin(\pi x)| & : -1 \leq x < 0 \\ 1 - \{x\} & : 0 \leq x < 1 \\ 1 + \left[\cos\left(\frac{\pi x}{2}\right)\right] & : 1 < x \leq 2 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -(\sin(\pi x)) & : -1 \leq x < 0 \\ 1 - x & : 0 \leq x < 1 \\ 1 - 1 & : 1 < x \leq 2 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -(\sin(\pi x)) & : -1 \leq x < 0 \\ 1 - x & : 1 \leq x < 1 \\ 0 & : 1 < x \leq 2 \end{cases}$$

Clearly, $f(x)$ is discontinuous at $x = 0$.

18. Given $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$

Differentiate w.r.t x where y as constant

$$f'\left(\frac{x+y}{2}\right) \times \frac{1}{2} = \frac{f'(x)}{2}$$

Put $y = -x$, we get,

$$f'\left(\frac{x-x}{2}\right) = f'(x)$$

$$f'(x) = f'(0) = -1$$

On integration, we get,

$$f(x) = -x + c$$

when $x = 0$, $f(0) = 1$, then $c = 1$

Thus, $\quad f(x) = 1 - x$

Now,

$$\{(f(0))^2 + (f(1))^2 + (f(2))^2 + \ldots + (f(n))^2\}$$

$$= 1^2 + 0^2 + 1^2 + 2^2 + \ldots + (n-1)^2$$

$$= 1 + \frac{n(n-1)(2n-1)}{6}$$

Hence, the result.

19. Given $\quad f(x) = x^3 - x^2 + x + 1$

$$f'(x) = 3x^2 - 2x + 1$$

Since its D is negative,

so, $f'(x) > 0, \forall x \in R$

Thus, f is strictly increasing in (0, 2)

Now, $g(x) = \begin{cases} f(x) & : 0 \le x \le 1 \\ 3 - x & : 1 < x \le 2 \end{cases}$

$= \begin{cases} x^3 - x^2 + x + 1 & : 0 \le x \le 1 \\ 3 - x & : 1 < x \le 2 \end{cases}$

Clearly, $g(x)$ is continuous in (0, 2) but not differentiable at $x = 1$

20. We have

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x - 55$$

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$

$$= 4(x^3 - 6x^2 + 11x - 6)$$

$$= 4(x - 1)(x - 2)(x - 3)$$

Clearly, f is increasing in $[1, 2] \cup [3, \infty)$ and decreasing in $(-\infty, 1] \cup [2, 3]$

Now, $\min\{f(t) : x \le t \le x + 1\} : -1 \le x \le 1$

$$= \begin{cases} f(x+1) & : -1 \le x \le 0 \\ f(1) & : 0 < x \le 1 \end{cases}$$

Thus, $g(x) = \begin{cases} f(x+1) & : -1 \le x \le 0 \\ f(1) & : 0 < x \le 1 \\ (x - 10) & : x > 1 \end{cases}$

$$= \begin{cases} x^4 - 4x^3 + 4x^2 - 64 & : -1 \le x \le 0 \\ -64 & : 0 < x \le 1 \\ (x - 10) & : x > 1 \end{cases}$$

Now, $\lim_{x \to 0} g(x) = g(0)$

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^-} g(x) = g(0)$$

$$-64 = -64 = -64$$

So, $g(x)$ is continuous at $x = 0$

Also, $g(x)$ is continuous at $x = 1$

Thus, $g(x)$ is continuous in $[-1, \infty)$

Now, $g'(x) = \begin{cases} 4x^3 - 12x^2 + 8x & : -1 \le x \le 0 \\ 0 & : 0 \le x \le 1 \\ 1 & : x > 1 \end{cases}$

Clearly, $g'(0^+) = g'(0^-) = 0$

and $g'(1^+) \ne g'(1^-)$

Thus, $f(x)$ is not differentiable at $x = 1$

So, $f(x)$ is not differentiable in $[-1, \infty)$.

21. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\lim_{x \to 0^-} (a \sin x + b \cos x) = \lim_{x \to 0^+} \left(\frac{e^x + x}{1 + 2x}\right)^{1/x} = b$$

$$b = \lim_{x \to 0^+} \left(\frac{e^x + x}{1 + 2x}\right)^{1/x}$$

$$= \lim_{x \to 0^+} \left(1 + \left(\frac{e^x + x}{1 + 2x} - 1\right)\right)^{1/x}$$

$$= \lim_{x \to 0^+} \left(1 + \left(\frac{e^x - 1 - x}{1 + 2x}\right)\right)^{1/x}$$

$$= e^{\lim_{x \to 0^+} \left(\left(\frac{e^x - 1 - x}{1 + 2x}\right)\right)\frac{1}{x}}$$

$$= e^{\lim_{x \to 0^+} \left(\left(\frac{e^x - 1 - x}{x(1 + 2x)}\right)\right)}$$

$$= e^{\lim_{x \to 0^+} \left(\left(\frac{e^x - 1}{2x + (1 + 2x)}\right)\right)}$$

$$= e^0 = 1$$

Again, for $x > 0$, we have

$$f(x) = \left(\frac{e^x + x}{1 + 2x}\right)^{1/x}$$

$$= e^{\frac{1}{x}\ln\left(\frac{e^x + x}{1 + 2x}\right)}$$

$$= e^{\frac{1}{x}\ln(e^x + x) - \ln(1 + 2x)}$$

$$f'(x) = e^{\frac{1}{x}\ln\left(\frac{x + e^x}{1 + 2x}\right)} \left[\left\{\frac{1}{x}\left(\frac{e^x + 1}{e^x + x}\right) - \frac{2}{1 + 2x}\right\} - \frac{1}{x^2}\{\ln(e^x + x) - \ln(1 + 2x)\}\right]$$

$$= e^{\frac{1}{x}\ln\left(\frac{x + e^x}{1 + e^x}\right)} \left[\frac{1}{x}\frac{\{e^x + 2xe^x + 1 + 2x - 2e^x - 2x\}}{(e^x + x)(1 + 2x)} - \frac{1}{x^2}\ln\left(\frac{e^x + 1}{1 + 2x}\right)\right]$$

$$= e^{\frac{1}{x}\ln\left(\frac{x + e^x}{1 + 2x}\right)} \left[\frac{1}{x}\frac{(2xe^x - e^x + 1)}{(e^x + x)(1 + 2x)} - \frac{1}{x^2}\ln\left(\frac{e^x + x}{1 + 2x}\right)\right]$$

Now, $f'(1) = e^{\ln\left(\frac{1 + e}{3}\right)} \left\{\frac{1}{3} - \ln\left(\frac{1 + e}{3}\right)\right\}$

$$= \left(\frac{1 + e}{3}\right)\left\{\frac{1}{3} - \ln\left(\frac{1 + e}{3}\right)\right\}$$

For $x = -\frac{\pi}{2}$, $f(x) = \varphi(x)$

$$f\left(-\frac{\pi}{2}\right) = \varphi\left(\frac{\pi}{2}\right) = -a + 0 = -a$$

Thus, $a = -\left(\frac{e+1}{3}\right)\left\{\frac{1}{3} - \ln\left(\frac{e+1}{3}\right)\right\}$

$$= \left(\frac{e+1}{3}\right)\left\{\ln\left(\frac{e+1}{3}\right) - \frac{1}{3}\right\}$$

Therefore,

$$a = \left(\frac{e+1}{3}\right)\left\{\ln\left(\frac{e+1}{3}\right) - \frac{1}{3}\right\}, b = 1$$

22. If $x \in [-1, 1]$, then

$$f(g(x)) = \begin{cases} \sin x - 1 & : -1 \le x < 0 \\ \sin^2 x & : 0 \le x \le 1 \end{cases}$$

Now, $|g(x)| = \begin{cases} -\sin x & : -1 \le x < 0 \\ \sin x & : 0 \le x \le 1 \end{cases}$

$f(|g(x)|) = \sin^2 x, \forall x \in [-1, 1]$

Also, $|f(g(x))| = \begin{cases} 1 - \sin x & : -1 \le x < 0 \\ 2\sin^2 x & : 0 \le x < 1 \end{cases}$

Now, L.H.L $= \lim_{x \to 0^-} (\sin^2 x - \sin x + 1) = 1$

and R.H.L $= \lim_{x \to 0^+} (2\sin^2 x) = 0$

Thus $h(x)$ is not continuous at $x = 0$

Again, $h'(x) = \begin{cases} \sin(2x) - \cos x - 1 & : -1 \le x < 0 \\ 2\sin(2x) & : 0 \le x \le 1 \end{cases}$

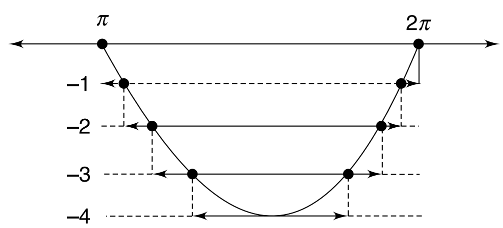
Now, L.H.D $= \lim_{x \to 0^-} (\sin 2x - \cos x - 1) = -2$

and R.H.D $= \lim_{x \to 0^+} (2\sin(2x)) = 0$

Therfore $h(x)$ is not differentiable at $x = 0$.

Integer Type Questions

1. Since $f : [1, 7] \to Q$ be a continuous function where $f(1) = 7$, so it is a constant function.
   Thus, $f(7) = 7$.
2. Clearly, f be a constant function, where $f(2) = 5$
   Thus, $f(2.5) = 5$.
3. Given $f(x) = ||x| - 1|$
   Clearly, f is non differentiable at $x = -1, 0, 1$
   Thus, $m = 3$
   Also, $g(x) = \frac{1}{\log|x|}$
   So, $g(x)$ is not defined at $x = -1, 0, 1$
   Thus $g(x)$ is discontinuous at $x = -1, 0, 1$
   So, $n = 3$
   Hence, the value of $(m + n)$ is 6.
4. We have $f(x) \frac{1}{\log|x^2 - 4|}$
   f is not defined when
   $$(x^2 - 4) = 0, |x^2 - 4| = 1$$
   $$(x^2 - 4) = 0, (x^2 - 4) = \pm 1$$
   $$x^2 = 4, x^2 = 4 \pm 1$$
   $$x = \pm 2, \pm\sqrt{5}, \pm\sqrt{3}$$
   Thus $m = 6$
   Also, $= \lim_{x \to 0}\left(\frac{x^n - \sin(x^n)}{x - \sin^n x}\right)$
   Its limit exists only when $n = 1$
   Hence, the value of $(m + n)$ is 7.
5. We have
   $$f(x) = [3 + 4\sin x], x \in [\pi, 2\pi]$$
   $$= 3 + [4\sin x]$$



   Thus, the number of points of disc is 8.
6. We have
   $$f(x) = [[x]] - [x - 2]$$
   $$= [x] - [x - 2]$$
   $$= [x] - [x] + 2$$
   $$= 2$$
   So $f(x)$ is a constant function.
   It is continuous everywhere.
   So, p = the number of discontinuous points = 0
   and $q = \lim_{x \to \infty}\left(\frac{3x^4 - 2}{\sqrt{x^8 + 3x + 4}}\right)$
   $$= \lim_{x \to \infty}\left(\frac{3 - \frac{2}{x^4}}{\sqrt{1 + \frac{3}{x^7} + \frac{4}{x^8}}}\right)$$
   $$= 3$$
   Hence, the value of $(p + q + 2)$ is 5.
7. We have m
   $$= \lim_{x \to 0}\left(\frac{\tan 2x - 2\sin x}{x^3}\right)$$

$$= \lim_{x \to 0} \left(\frac{2x + \frac{(2x)^3}{3} + \frac{2(2x)^5}{15} + \ldots - 2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}{x^3} \right)$$

$$= \left(\frac{8}{3} + \frac{2}{6}\right)$$

$$= \left(\frac{8}{3} + \frac{1}{3}\right) = \frac{9}{3} = 3$$

Also, $f(x) = [\sin x + \cos x]$, $\forall\ x \in (0, 2\pi)$

Thus, the number of disc continuous points of f is 5, where $x = \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, \frac{7\pi}{4}$.

So, $n = 5$

Again, $f(x) = |\log|x||$

Clearly, f is not differentiable at $x = -1, 0, 1$

So, $p = 3$

Hence, the value of $(m + n - p)$

$= 3 + 5 - 3$

$= 5.$

8. We have

$$\lim_{h \to 0} \left(\frac{f(h^2 + 2h + 2) - f(2)}{f(h - h^2 + 1) - f(1)} \right)$$

$$= \lim_{h \to 0} \left(\frac{f'(h^2 + 2h + 2)(2h + 2)}{f'(h - h^2 + 1)(1 - 2h)} \right)$$

$$= \frac{2f'(2)}{f'(1)}$$

$$= \frac{2 \times 6}{4} = 3$$

9. As we know that, every differentiable function is continuous, so $f(x)$ is continuous at $x = 1$

Thus, $\lim_{x \to 1} f(x) = f(1)$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

$$\lim_{x \to 1^+} (3ax - b + 2) = \lim_{x \to 1^-} (a + bx^2) = 3a - b + 2$$

$$(3a - b + 2) = (a + b) = 3a - b + 2$$

$$(a + b) = 3a - b + 2$$

$$2a - 2b + 2 = 0$$

$$a - b + 1 = 0$$

$$b = a + 1 \qquad \ldots\text{(i)}$$

Also, $f(x)$ is differentiable at $x = 1$

So, $$\lim_{x \to 1^+} \left(\frac{f(x) - f(1)}{x - 1} \right) = \lim_{x \to 1^-} \left(\frac{f(x) - f(1)}{x - 1} \right)$$

$$\lim_{x \to 1^+} \left(\frac{(3ax - b + 2) - (3a - b + 2)}{x - 1} \right)$$

$$= \lim_{x \to 1^-} \left(\frac{(a + bx^2) - (3a - b + 2)}{x - 1} \right)$$

$$\lim_{x \to 1^+} \left(\frac{3a(x - 1)}{x - 1} \right)$$

$$= \lim_{x \to 1^-} \left(\frac{(a + (a + 1)x^2) - (3a - a - 1 + 2)}{x - 1} \right)$$

$$\lim_{x \to 1^-} \left(\frac{(a + (a + 1)x^2) - (2a + 1)}{x - 1} \right) = 3a$$

$$\lim_{x \to 1^-} \left(\frac{(a + 1)x^2 - a - 1)}{x - 1} \right) = 3a$$

$$\lim_{x \to 1^-} \left(\frac{((a + 1)(x^2 - 1))}{x - 1} \right) = 3a$$

$$\lim_{x \to 1^-} ((x + 1)(a + 1)) = 3a$$

$$2(a + 1) = 3a$$

$$2a + 2 = 3a$$

$$a = 2,\ b = a + 1 = 2 + 1 = 3$$

Hence, the value of $(a + b + 1)$ is 6.

10. We have $f(x) = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$

Clearly, $f(x)$ is non-differentiable at $x = -1$ and $x = 1$

But it is continuous everywhere.

So, $p = 0$ and $q = 2$

Hence, the value of $(p + q + 2)$ is 4.

11. We have

$$f(x) = [3x] - \{2x\},\ x \in \left[0, \frac{3}{2}\right).$$

Clearly, f is discontinuous at

$$x = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{4}{3}, 1$$

Thus, the number of points of discontinuity is 5.

12. We have $h(x) = |f(|x|)| = ||x| - 2|$

Clearly, h is not differentiable at $x = -2, 0, 2$

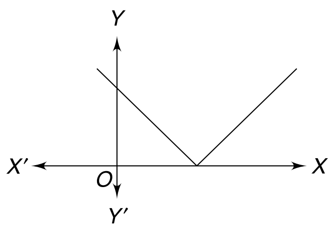
Thus, $g(x)$ is non-differentiable at $x = -2, 0, 2$

Hence, the number of non-differentiable points is 3.

HINTS AND SOLUTIONS

Questions asked in Roorkee-JEE Exams

1. Given $y = |x - 2|$



Clearly, $y = f(x)$ is continuous everywhere but not differentiable at $x = 2$.

2. We have $\lim_{x\to 0} f(x) = f(0)$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\lim_{x\to 0^+} (x^2) = \lim_{x\to 0^-} (x - 1) = \frac{1}{4}$$

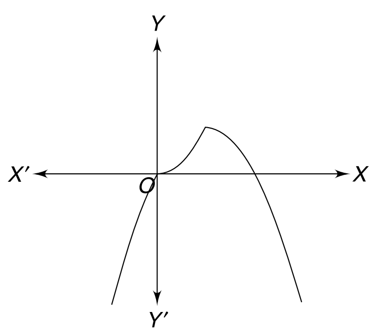
Thus, $f(x)$ is not continuous at $x = 0$.

3. We have $f(x) = x - |x - x^2|$

$$= \begin{cases} x-(x-x^2) & : x-x^2 \geq 0 \\ x+(x-x^2) & : x-x^2 < 0 \end{cases}$$

$$= \begin{cases} x^2 & : x^2-x \leq 0 \\ (2x-x^2) & : x^2-x > 0 \end{cases}$$

$$= \begin{cases} x^2 & : 0 \leq x \leq 1 \\ (2x-x^2) & : x \in (-\infty, 0) \cup (1, \infty) \end{cases}$$



Clearly $f(x)$ is continuous in $[-1, 1]$ but not differentiable at $x = -1$ and $x = 1$.

4. We have $f(x) = \begin{cases} -1-(x-1) & : -1 \leq x \leq 1 \\ -1+(x-1) & : 1 \leq x \leq 3 \end{cases}$

$$f(x) = \begin{cases} -x & : -1 \leq x \leq 1 \\ x-2 & : 1 \leq x \leq 3 \end{cases}$$

and $g(x) = \begin{cases} 2+(x+1) & : -2 \leq x \leq -1 \\ 2-(x+1) & : -1 \leq x \leq 2 \end{cases}$

$$= \begin{cases} 3+x & : -2 \leq x \leq -1 \\ 1-x & : -1 \leq x \leq 2 \end{cases}$$

Now, $(f_o g)(x) = f(g(x))$

$$= \begin{cases} -g(x) & : -1 \leq g(x) \leq 1 \\ g(x)-2 & : 1 \leq g(x) \leq 3 \end{cases}$$

$$= \begin{cases} -(3+x) & : -2 \leq x \leq -1, -1 \leq (3+x) \leq 1 \\ (3+x)-2 & : -2 \leq x \leq -1, 1 \leq (3+x) \leq 3 \\ (x-1) & : -1 \leq x \leq 2, -1 \leq (1-x) \leq 1 \\ (1-x)-2 & : -2 \leq x \leq -1, 1 \leq (1-x) \leq 3 \end{cases}$$

$$= \begin{cases} -(3+x) & : -2 \leq x \leq -1, -4 \leq x \leq -2 \\ (1+x) & : -2 \leq x \leq -1, -2 \leq x \leq 0 \\ (x-1) & : -1 \leq x \leq 2, 0 \leq x \leq 2 \\ -(1+x) & : -2 \leq x \leq -1, -2 \leq x \leq 0 \end{cases}$$

Clearly, it is continuous at $x = -1$.

5. **Case-I:** When $t \geq 0$

Then $x = 2t - t = t$ and $y = t^2 + t^2 = 2t^2$

Thus, $y = 2x^2$

Case-II: When $t < 0$

Then $x = 2t + t = 3t$ and $y = 0$

Thus, $y = 0$, $x < 0$.

The given function is defined as

$$f(x) = \begin{cases} 2x^2 & : 0 \leq x \leq 1 \\ 0 & : -1 \leq x < 0 \end{cases}$$

Now, $f'(0^-) = 0$

and $f'(0^+) = \lim_{x\to 0^+} \left(\frac{f(x) - f(0)}{x - 0}\right)$

$$= \lim_{x\to 0^+} \left(\frac{2x^2 - 0}{x - 0}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{2x^2}{x}\right)$$

$= \lim_{x \to 0^+} (2x)$

$= 0$

Thus, $f(x)$ is differentiable at $x = 0$

Hence, it is also continuous at $x = 0$.

6. We have $(f_o g)(x)$

$= f(g(x))$

$= f(\sin x)$

$= \sqrt{|\sin x - 1|}$

Also, $(g_o f)(x)$

$= g(f(x))$

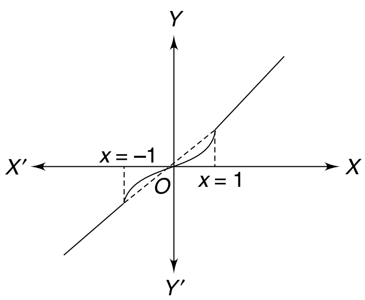
$= g(\sqrt{|x - 1|})$

$= (\sqrt{\sin|x - 1|})$

Clearly, $(g_o f)(x)$ is not differentiable at $x = 1$.

7. We have $f(x) = \begin{cases} x^3 & : x^2 < 1 \\ x & : x^2 \geq 1 \end{cases}$

$= \begin{cases} x^3 & : -1 < x < 1 \\ x & : x \geq 1 \text{ and } x \leq -1 \end{cases}$

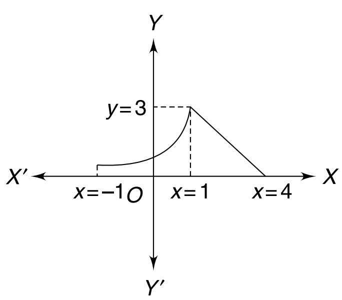


limit = 1

Function $f(x)$ is continuous at $x = 1$

but not differentiable at $x = 1$

8. We have $f(x) = \begin{cases} 3^x & : -1 \leq x \leq 1 \\ 4 - x & : 1 < x < 4 \end{cases}$



Thus, $f(x)$ is continuous in $[-1, 4]$ but not differentiable at $x = 1$.

9. We have $f(0) = 0$

Now, $\lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} \left(\frac{x(3e^{1/x} + 4)}{(2 - e^{1/x})} \right)$

$= \lim_{x \to 0^+} \left(\frac{x(3 + 4e^{-1/x})}{(2e^{-1/x} - 1)} \right)$

$= \lim_{x \to 0^+} \left(\frac{0(3 + 4e^{-\infty})}{(2e^{-\infty} - 1)} \right) = 0$

Also, $\lim_{x \to 0^-} \left(\frac{x(3e^{1/x} + 4)}{2 - e^{1/x}} \right)$

$= \lim_{x \to 0^-} \left(\frac{0(3e^{-\infty} + 4)}{2 - e^{-\infty}} \right)$

$= 0$

Thus, $f(x)$ is continuous at $x = 0$

Now, $f'(0^+)$

$= \lim_{x \to 0^+} \left(\frac{f(x) - f(0)}{x - 0)} \right)$

$= \lim_{x \to 0^+} \left(\frac{\frac{x(3e^{1/x} + 4)}{(2 - e^{1/x})} - 0}{x - 0} \right)$

$= \lim_{x \to 0^+} \left(\frac{(3e^{1/x} + 4)}{(2 - e^{1/x})} \right)$

$= \lim_{x \to 0^+} \left(\frac{(3 + 4e^{-1/x})}{(2 - e^{-1/x})} \right)$

$= \lim_{x \to 0^+} \left(\frac{(3 + 4e^{-\infty})}{(2 - e^{-\infty})} \right)$

$= \frac{3}{2}$

Also, $f'(0^-)$

$= \lim_{x \to 0^-} \left(\frac{f(x) - f(0)}{x - 0} \right)$

$= \lim_{x \to 0^+} \left(\frac{\frac{x(3e^{1/x} + 4)}{(2 - e^{1/x})}}{x} \right)$

$= \lim_{x \to 0^-} \left(\frac{(3e^{1/x} + 4)}{(2 - e^{1/x})} \right)$

$= \lim_{x \to 0^-} \left(\frac{(3e^{-\infty} + 4)}{(2 - e^{-\infty})} \right)$

$$= \frac{4}{2} = 2$$

Thus, $f(x)$ is not differentiable at $x = 0$.

10. We have $f(0)$

$$= \lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0}\left(\frac{1}{x} - \frac{2}{e^{2x}-1}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^{2x}-1-2x}{x(e^{2x}-1)}\right)$$

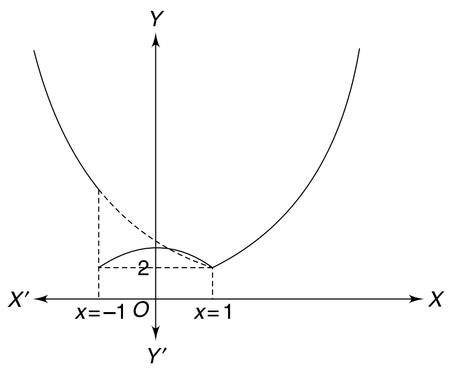
$$= \lim_{x\to 0}\left(\frac{1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots-1-2x}{2x^2\left(\frac{e^{2x}-1}{2x}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{4x^2}{2!}+\frac{8x^3}{3!}+\ldots}{2x^2\left(\frac{e^{2x}-1}{2x}\right)}\right)$$

$$= 1$$

11. We have $f(x) = \begin{cases} 2+\sqrt{1-x^2} & : |x| \le 1 \\ 2e^{(1-x)^2} & : |x| > 1 \end{cases}$

$$= \begin{cases} 2+\sqrt{1-x^2} & : -1 \le x \le 1 \\ 2e^{(1-x)^2} & : x > 1 \text{ and } x < -1 \end{cases}$$



Clearly $f(x)$ is discontinuous at $x = -1$ and not differentiable at $x = -1$ and $x = 1$.

12. Since $f(x)$ is continuous at $x = 0$, so

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = b \quad \ldots(i)$$

Now, $\lim_{x\to 0^-} f(x)$

$$= \lim_{x\to 0^-} (1 + ax)^{1/x}$$

$$= e^{\lim_{x\to 0^-}\left(\frac{ax}{x}\right)} = e^a$$

Also, $\lim_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+}\left(\frac{(x+c)^{1/3}-1}{(x+1)^{1/2}-1}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{((x+c)-1)(\sqrt{x+1}+1)}{((x+1)-1)((x+c)^{2/3}+(x+c)^{1/3}+1)}\right)$$

Its limit exists only when $c = 1$

So, R.H.L

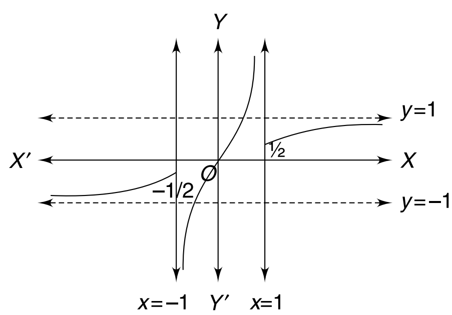
$$= \lim_{x\to 0^+}\left(\frac{((x+1)-1)(\sqrt{x+1}+1)}{((x+1)-1)((x+1)^{2/3}+(x+1)^{1/3}+1)}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{(\sqrt{x+1}+1)}{((x+1)^{2/3}+(x+1)^{1/3}+1)}\right)$$

$$= \frac{2}{3}$$

Thus, $e^a = \frac{2}{3} = b,\ c, = 1$

Hence, $a = \log_e\left(\frac{2}{3}\right),\ b = \frac{2}{3},\ c = 1$

13. 

Clearly the function $f(x)$ is not continuous at $x = 1$ and $x = -1$

So, $f(x)$ is not differentiable at $x = -1$ and $x = 1$.

14. We have

$$\lim_{x\to 1^+}\left(\frac{e^{\frac{1}{x-1}}-2}{e^{\frac{1}{x-1}}+2}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{1-2e^{-\frac{1}{x-1}}}{1+2e^{-\frac{1}{x-1}}}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{1-2e^{-\infty}}{1+2e^{-\infty}}\right)$$

$$= \left(\frac{1-2.0}{1+2.0}\right) = 1$$

Also, $\lim_{x\to 1^-}\left(\frac{e^{\frac{1}{x-1}}-2}{e^{\frac{1}{x-1}}+2}\right)$

$= \lim_{x\to 1^-}\left(\frac{1e^{-\infty}-2}{e^{-\infty}+2}\right)$

$= \left(\frac{1.0-2}{0+2}\right)$

$= -1$

Since R.H.L ≠ L.H.L., so limit does not exist.

Thus, $f(x)$ is not continuous at $x = 1$.

Problems asked in Past IIT-JEE Exams

1. Ans. (a, c)

We have $f(x) = x(\sqrt{x} - \sqrt{x+1})$

Clearly, $\lim_{x\to 0} f(x) = 0 = f(0)$

Thus, $f(x)$ is differentiable at $x = 0$

Also, $f'(x) = (\sqrt{x} - \sqrt{x+1}) + x\left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x+1}}\right)$

Clearly, $f(x)$ is not differentiable at $x = 0$.

2. We have, $f(x) = x^3 - x^2 + x + 1$

$f'(x) = 3x^2 - 2x + 1 > 0, \forall x \in R$

$\lfloor \because a = 3 > 0, D = b^2 - 4ac = 4 - 12 = -8 < 0 \rfloor$

Thus, $f(x)$ is strictly increasing in (0. 2).

$$g(x) = \begin{cases} f(x) & : 0 \le x \le 1 \\ 3-x & : 1 < x \le 2 \end{cases}$$

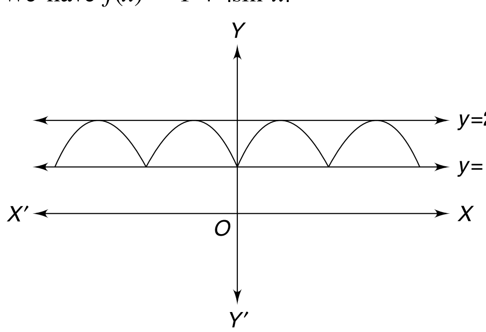
$$g(x) = \begin{cases} x^3 - x^2 + x + 1 & : 0 \le x \le 1 \\ 3-x & : 1 < x \le 2 \end{cases}$$

$$g'(x) = \begin{cases} 3x^2 - 2x + 1 & : 0 \le x \le 1 \\ -1 & : 1 < x \le 2 \end{cases}$$

Thus, $g(x)$ is continuous for all x in [0, 2] but not differentiable at $x = 1$.

3. Ans. (a, d)

We have $f(x) = 1 + |\sin x|$



Clearly $f(x)$ is continuous everywhere but not differentiable at $x = n\pi$, $n \in I$.

4. We have $f(x) = [x \sin \pi x]$

Here, $0 < x \sin \pi x \le 1$

Thus, $f(x) = [x \sin \pi x] = 0$

Therefore, $f(x)$ is continuous and differentiable at $(-1, 1)$.

5. We have, $f(x) = \begin{cases} -1 & : -2 \le x \le 0 \\ x-1 & : 0 < x \le 2 \end{cases}$

Now, $f(|x|) = \begin{cases} -1 & : -2 \le |x| \le 0 \\ |x|-1 & : 0 < |x| \le 2 \end{cases}$

$\Rightarrow \quad f(|x|) = |x| - 1 : 0 < |x| \le 2$

$\Rightarrow \quad f(|x|) = \begin{cases} -x-1 & : -2 \le x \le 0 \\ x-1 & : 0 < x \le 2 \end{cases}$

Also, $|f(x)| = \begin{cases} 1 & : -2 \le x \le 0 \\ |x-1| & : 0 < x \le 2 \end{cases}$

$\Rightarrow \quad |f(x)| = \begin{cases} 1 & : -2 \le x \le 0 \\ 1-x & : 0 < x \le 1 \\ x-1 & : 1 < x \le 2 \end{cases}$

Therefore, $g(x) = f(|x|) + |f(x)|$

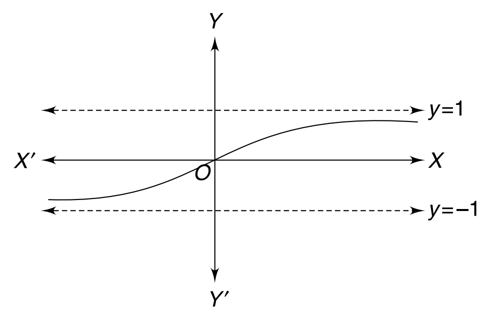
$$= \begin{cases} -x & : -2 \le x \le 0 \\ 0 & : 0 < x \le 1 \\ 2x-2 & : 1 < x \le 2 \end{cases}$$

$\Rightarrow \quad g'(x) = \begin{cases} -1 & : -2 \le x \le 0 \\ 0 & : 0 < x \le 1 \\ 2 & : 1 < x \le 2 \end{cases}$

Clearly, $g(x)$ is continuous but not differentiable at $x = 0$ and $x = 1$.

6. Ans. (a).

We have $f(x) = \frac{x}{1+|x|}$



Clearly, $f(x)$ is differentiable everywhere.

7. Let $f(x) = x^2$, since $f(-x) = f(x)$.

Now, $\quad f'(x) = 2x$

Thus, $\quad f'(0) = 0$

8. Let $f(x) = x$ and $g(x) = [x]$
Clearly $f(x)$ is continuous everywhere and $g(x)$ is discontinuous function.
Thus, $f(x) + g(x) = x + [x]$ is a discontinuous function.

9. Let $g(x) = ax + b$

Then $f(x) = \begin{cases} ax+b & x \le 0 \\ \left(\dfrac{1+x}{2+x}\right)^{1/x} & x > 0 \end{cases}$

Since $f(x)$ is continuous, so it is continuous at $x = 0$.

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(ax + b) = b$

Also, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\dfrac{x+1}{x+2}\right)^{1/x} = \left(\dfrac{1}{2}\right)^{\infty} = 0$

Thus. $b = 0$.
Now, $f(-1) = -a$
We have, $f(x) = \left(\dfrac{x+1}{x+2}\right)^{1/x}$

$$= e^{\log\left(\frac{x+1}{x+2}\right)^{1/x}}$$

$$= e^{\frac{1}{x}\log\left(\frac{x+1}{x+2}\right)}$$

$$\Rightarrow \quad f'(x) = e^{\frac{1}{x}\log\left(\frac{x+1}{x+2}\right)} \times$$

$$\left(-\frac{1}{x^2}\log\left(\frac{x+1}{x+2}\right) + \frac{1}{x}\left(\frac{x+2}{x+1}\right)\left(\frac{1}{(x+2)^2}\right)\right)$$

Thus, $f'(1) = \left(\dfrac{2}{3}\right)\left(-\log\left(\dfrac{2}{3}\right) + \dfrac{3}{2}\times\dfrac{1}{9}\right)$

$$= \left(\frac{2}{3}\right)\left(-\log\left(\frac{2}{3}\right) + \frac{1}{6}\right)$$

$$= \left(\frac{2}{3}\log\left(\frac{3}{2}\right) + \frac{1}{9}\right)$$

Since, $f'(1) = f(-1)$, so

$$a = -\left(\frac{2}{3}\log\left(\frac{3}{2}\right) + \frac{1}{9}\right)$$

Thus, the required function is

$$f(x) = \begin{cases} -\left(\dfrac{2}{3}\log\left(\dfrac{3}{2}\right)+\dfrac{1}{9}\right)x & : x \le 0 \\ \left(\dfrac{x+1}{x+2}\right)^{1/x} & : x > 0 \end{cases}$$

10. We have $|f(x) - f(y)| \le (x - y)^2$

$$\Rightarrow \quad \left|\frac{f(x)-f(y)}{(x-y)}\right| \le (x - y)$$

$$\Rightarrow \quad \lim_{x\to y}\left|\frac{f(x)-f(y)}{(x-y)}\right| \le \lim_{x\to y}(x - y)$$

$$\Rightarrow \quad |f'(y)| \le 0$$

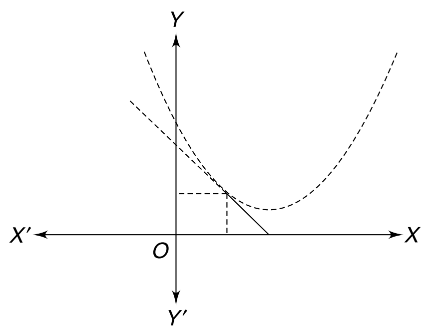
$$\Rightarrow \quad |f'(y)| = 0$$

$$\Rightarrow \quad f'(y) = 0$$

Thus, $f(x)$ is a constant function.

11. We have $f(x) = \begin{cases} |x-3| & : x \ge 0 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} & : x < 0 \end{cases}$

$$= \begin{cases} |x-3| & : x \ge 1 \\ \dfrac{1}{4}(x-3)^2 + 1 & : x < 1 \end{cases}$$



Clearly $f(x)$ is continuous at $x = 1$ and $x = 3$. and $f(x)$ is not differentiable at $x = 3$.

Also, $\quad f(x) = \begin{cases} |x-3| & : x \ge 1 \\ \dfrac{1}{4}(x-3)^2 + 1 & : x < 1 \end{cases}$

$$f'(x) = \begin{cases} -1 & : x < 1 \\ 1 & : 1 \le x < 3 \\ \dfrac{1}{2}(x-3) & : x < 1 \end{cases}$$

Thus, $f'(1) = -1$
Therefore, $f(x)$ is differentiable at $x = 1$.

12. We have

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & : 0 \le x < \dfrac{\pi}{4} \\ 2x\cot x + b & : \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \\ a\cos 2x - b\sin x & : \dfrac{\pi}{2} < x \le \pi \end{cases}$$

At $x = \dfrac{\pi}{4}$

$$\lim_{x \to \frac{\pi^+}{4}} f(x) = \lim_{x \to \frac{\pi^-}{4}} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\lim_{x \to \frac{\pi^+}{4}} (2x \cot x + b) = \lim_{x \to \frac{\pi^-}{4}} (x + a\sqrt{2} \sin x)$$

$$= \left(2 \cdot \frac{\pi}{4} \cot\left(\frac{\pi}{4}\right) + b\right)$$

$$\frac{\pi}{4} + a\sqrt{2} \cdot \sin\left(\frac{\pi}{4}\right) = \left(2 \cdot \frac{\pi}{4} \cot\left(\frac{\pi}{4}\right) + b\right)$$

$$\frac{\pi}{4} + a\sqrt{2} \cdot \frac{1}{\sqrt{2}} = \left(2 . \frac{\pi}{4} + b\right)$$

$$a - b = \frac{\pi}{4} \quad \text{...(i)}$$

At $x = \frac{\pi}{2}$

$$\lim_{x \to \frac{\pi^+}{2}} f(x) = \lim_{x \to \frac{\pi^-}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\lim_{x \to \frac{\pi^+}{2}} f(x) = \lim_{x \to \frac{\pi^-}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\lim_{x \to \frac{\pi^+}{2}} (a \cos 2x - b \sin x) = \lim_{x \to \frac{\pi^-}{2}} (2x \cot x + b) = b$$

$(-a - b) = b$

$a = 0.$

Thus, $b = -\frac{\pi}{4}$

Hence, the values of $a = 0$, $b = -\frac{\pi}{4}$.

13. Ans. (b)

Given $f(x) = \left(\frac{x}{2} - 1\right)$ in $[0, \pi]$

Thus, $$[f(x)] = \begin{cases} -1 & : 0 \le x < 2 \\ 0 & : 2 \le x \le \pi \end{cases}$$

$$\tan([f(x)]) = \begin{cases} \tan(-1) & : 0 \le x < 2 \\ \tan(0) & : 2 \le x \le \pi \end{cases}$$

Clearly $\tan(f(x))$ is not continuous at $x = 2$

Now, $f(x) = \left(\frac{x}{2} - 1\right) = \left(\frac{x - 2}{2}\right)$

$$\frac{1}{f(x)} = \frac{2}{(x - 2)}$$

Thus, $\frac{1}{f(x)}$ is also not continuous at $x = 2$.

So, $\tan[f(x)]$ & $\tan\left[\frac{1}{f(x)}\right]$ are both discontinuous at $x = 2$.

14. It is given that $f(x)$ is continuous at $x = 0$,

so, $\lim_{x \to 0} f(x) = f(0)$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

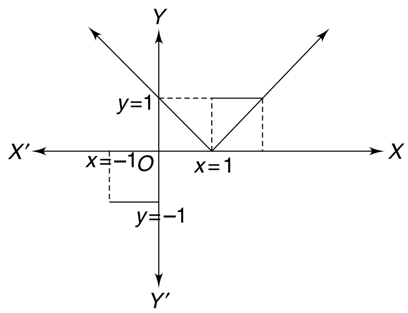
$$\lim_{x \to 0^+} \left(\frac{\sqrt{x}}{\sqrt{16} + \sqrt{x} - 4}\right) = \lim_{x \to 0^-} \left(\frac{1 - \cos 4x}{x^2}\right) = a$$

$$\lim_{x \to 0^+} \left(\frac{\sqrt{x}(\sqrt{16} + \sqrt{x} + 4)}{16 + \sqrt{x} - 16}\right) = \lim_{x \to 0^-} \left(\frac{2\sin^2 2x}{x^2}\right) = a$$

$$\lim_{x \to 0^+} ((\sqrt{16} + \sqrt{x} + 4)) = \lim_{x \to 0^-} \left(\frac{8\sin^2 2x}{4x^2}\right) = a$$

$a = 8$

15.



Clearly, the function is not differentiable at $x = -1$, 0, 1, 2.

16. Given $f(x + y) = f(x) . f(y)$

Put $x = 0 = y$, $f(0) = 0$, then $f(0) = 1$

It is given that $f'(0) = 2$

$$\lim_{h \to 0} \left(\frac{f(0 + h) - f(0)}{h - 0}\right) = 2$$

$$\lim_{h \to 0} \left(\frac{f(0) . f(h) - f(0)}{h}\right) = 2$$

$$\lim_{h \to 0} \left(\frac{f(h) - 1}{h}\right) = 2 \quad \text{...(i)}$$

Now, $$f'(x) = \lim_{h \to 0} \left(\frac{f(x + h) - f(x)}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{f(x) . f(h) - f(x)}{h}\right)$$

$$= \lim_{h \to 0} \left(\frac{f(x)(f(h - 1)}{h}\right)$$

$$= \lim_{h \to 0} f(x) \times \left(\frac{f(h) - f(0)}{h}\right)$$

$= 2f(x)$, from (i)

Thus, $\frac{f'(x)}{f(x)} = 2$

On integration, we get,

$$\log|f(x)| = 2x + c$$

If $x = 0$, $f(0) = 1$, then $c = 0$.

Thus, $\log|f(x)| = 2x$.

$\Rightarrow \quad f(x) = e^{2x}$

17. **Case-I:** When $t \geq 0$

Then $x = 2t - t = t$ and $y = t^2 + t^2 = 2t^2$

Thus, $y = 2x^2$

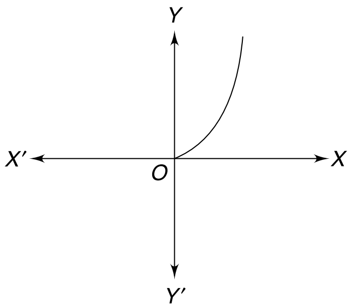
Case-II: When $t < 0$

Then $x = 2t + t = 3t$ and $y = 0$

Thus, $y = 0$, $x < 0$.

The given function is defined as

$$f(x) = \begin{cases} 2x^2 & : 0 \leq x \leq 2 \\ 0 & : -1 \leq x < 0 \end{cases}$$



Now, $f'(0^-) = 0$

and $f'(0^+) = \lim_{x\to 0^+}\left(\frac{f(x) - f(0)}{x - 0}\right)$

$= \lim_{x\to 0^+}\left(\frac{2x^2 - 0}{x - 0}\right)$

$= \lim_{x\to 0^+}\left(\frac{2x^2}{x}\right)$

$= \lim_{x\to 0^+}(2x)$

$= 0$

Thus, $f(x)$ is differentiable at $x = 0$

Hence, it is also continuous at $x = 0$.

18. Ans. (b, c)

(a) Since $\tan x$ is not defined at $x = \frac{\pi}{2}$

So, it is not continuous in $(0, \pi)$

(b) Let $g(x) = x\sin\left(\frac{1}{x}\right)$

which is continuous in $(0, \pi)$.

So, the integral function of a continuous function is also a continuous function.

Thus, $\int_0^x t \sin\left(\frac{1}{t}\right)dt$ is continuous in $(0, \pi)$

(c) Let $f(x) = \begin{cases} 1 & : 0 < x \leq \frac{3\pi}{4} \\ 2\sin\left(\frac{2}{9}\right)x & : \frac{3\pi}{4} < x < \pi \end{cases}$

Now, $\lim_{x\to \frac{3\pi^+}{4}} f(x) = \lim_{x\to \frac{3\pi^+}{4}} f(x) = 1$

Thus $f(x)$ is continuous at $x = \frac{3\pi}{4}$

Also, $f(x)$ is continuous at all other points in $(0, \pi)$

(d) Let $f(x) = \begin{cases} x + \sin x & : 0 < x \leq \frac{\pi}{2} \\ \frac{\pi}{2}\sin(\pi + x) & : \frac{\pi}{2} < x < \pi \end{cases}$

Now, $\lim_{x\to \frac{\pi^+}{2}} = \lim_{x\to \frac{\pi^-}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$\lim_{x\to \frac{\pi^+}{2}} f(x) = \lim_{x\to \frac{\pi^+}{2}}\left(\frac{\pi}{2}\sin(\pi + x)\right)$

$= -\frac{\pi}{2}$

And, $\lim_{x\to \frac{\pi^-}{2}} f(x) = \lim_{x\to \frac{\pi^-}{2}}(x + \sin x)$

$= \left(\frac{\pi}{2} + 1\right)$

Thus, $f(x)$ is not continuous at $x = \frac{\pi}{2}$

So, $f(x)$ is not continuous in $(0, \pi)$.

19. (i) Let $f(x) = \sin(\pi[x])$

Since $[x]$ provides us integer, so $f(x) = 0$

It is continuous and differentiable everywhere

(ii) Let $f(x) = \sin(\pi(x - [x]))$

$= \sin(\pi\{x\})$

As we know that, $\{x\}$ is discontinuous at every integral points, so $f(x)$ is disc at all integers

Thus $f(x)$ is non-differentiable at $x = -1$ and 1.

20. Given $f(x) = x|x|$

$$= \begin{cases} x^2 & : x \geq 0 \\ -x^2 & : x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x & : x \geq 0 \\ -2x & : x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 2 & : x \geq 0 \\ -2 & : x < 0 \end{cases}$$

Clearly, the function $f(x)$ is not twice differentiable at $x = 0$.

Thus, $f(x)$ is twice differentiable at $x \in R - \{0\}$

21. Ans. (a, b, c)

We have $f(x) = \lfloor \tan^2 x \rfloor$

Now, to find $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$

$\lim_{x \to 0^+} [\tan^2 x] = \lim_{x \to 0^-} [\tan^2 x] = 0 = f(0)$

Thus, $f(x)$ is continuous at $x = 0$.

Also, $f'(0) = \lim_{x \to 0} \left(\frac{f(x) - f(0)}{x - 0} \right)$

$= \lim_{x \to 0} \left(\frac{[\tan^2 x] - 0}{x} \right)$

$= \lim_{x \to 0} \left(\frac{0}{x} \right)$

$= 0$

Thus, $f(x)$ is also differentiable at $x = 0$.

22. Since f is continuous at $x = 0$, so

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = b$$

Now, $\lim_{x \to 0^-} f(x)$

$= \lim_{x \to 0^-} e^{\frac{\tan 2x}{\tan 3x}}$

$= e^{2/3}$

Also, $\lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} (1 + |\sin x|)^{\frac{a}{|\sin x|}}$

$= \lim_{x \to 0^+} (1 + \sin x)^{\frac{a}{\sin x}}$

$= \lim_{x \to 0^+} e^{\sin x \times \frac{a}{\sin x}} = e^a$

Thus, $b = e^{\frac{2}{3}} = e^a$

$a = \frac{2}{3}, \; b = e^{\frac{2}{3}}$

23. We have $f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}$

$\Rightarrow \quad f'\left(\frac{x + y}{2}\right) \cdot \frac{1}{2} = \frac{1}{2} f'(x)$, y as a constant

$\Rightarrow \quad f'\left(\frac{x + y}{2}\right) = f'(x)$

Replacing x by 0 and y by $2x$, we get,

$f'(x) = f'(0) = -1$

On integration, we get,

$f(x) = -x + c$

When $x = 0$, $c = 1$.

Thus, $\quad f(x) = -x + 1$

Therefore, $f(2) = -2 + 1 = -1$.

24. Now, $[x + 1] = 0$

$0 \le (x + 1) < 1$

$-1 \le x < 0$

$x \in [-1, 0)$

Thus, $D_f = R - [-1, 0)$

25. Here f is a constant function, where $f(10) = 10$

Thus, $f(1.5) = 10$

26. (a) $\lim_{x \to 0^+} f(x)$

$= \lim_{x \to 0^+} e^{\left\{-\frac{1}{|x|} + \frac{1}{x}\right\}}$

$= \lim_{x \to 0^+} e^{\left\{-\frac{1}{x} + \frac{1}{x}\right\}} = 1$

Also, $\lim_{x \to 0^-}$

$= \lim_{x \to 0^-} e^{\left\{-\frac{1}{|x|} + \frac{1}{x}\right\}}$

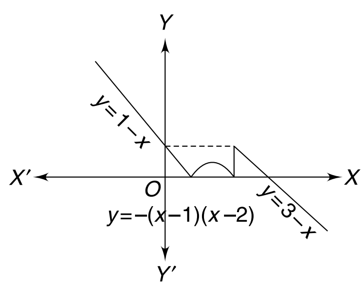
$= \lim_{x \to 0^-} e^{\left\{-\frac{1}{|x|} + \frac{1}{x}\right\}}$

$= \lim_{x \to 0^-} e^{\left\{\frac{1}{x} + \frac{1}{x}\right\}}$

$= \lim_{x \to 0^-} e^{\left\{\frac{2}{x}\right\}} = 0$

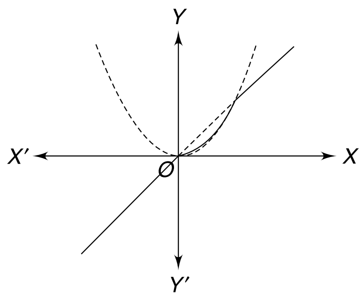
Thus, $f(x)$ is disc at $x = 0$

(b) As we know that, every disc function is not differentiable. So, $f(x)$ is not differentiable at $x = 0$.

27. 

Clearly, $f(x)$ is not continuous at $x = 2$ and not differentiable at $x = 1$ and $x = 2$.

28. Ans. (a, c, d)



29. Ans. (c, d)

We have

$f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$

$f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(x)$

$f(x) = (x^2 - 1)|(x - 1)(x - 2)| + \cos(x)$

Thus, $f(x)$ is not differentiable at $x = 1, 2$

30. The function $f(x) = [x]^2 - [x^2]$, where [,] = G.I.F., is discontinuous at

(a) all integers
(b) all integer except 0 and 1
(c) all integers except 0
(d) all integers except 1.

[IIT-JEE, 1999]

31. Ans. (c)

Let $f(x) = x$

Then $g(x) = |f(x)| = |x|$

Clearly $f(x)$ is continuous everywhere, so its modulus is also continuous everywhere.

Thus, $g(x)$ is continuous, whenever f is continuous.

32. Given

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$$

$$\Rightarrow\ p'(x) = 0 + a_1 + 2a_2x + ... + na_nx^{n-1}$$

$$\Rightarrow\ p'(1) = 0 + a_1 + 2a_2 + ... + na_n$$

Now, $|p(x)| \le |e^{x-1} - 1|$

$$\Rightarrow\ |p(1)| \le |e^{1-1} - 1| = |e^0 - 1| = |1 - 1| = 0$$

$$\Rightarrow\ |p(1)| \le 0$$

$$\Rightarrow\ p(1) = 0$$

As $|p(x)| \le |e^{x-1} - 1|$

$$\Rightarrow\ |p(1 + h)| \le |e^h - 1|$$

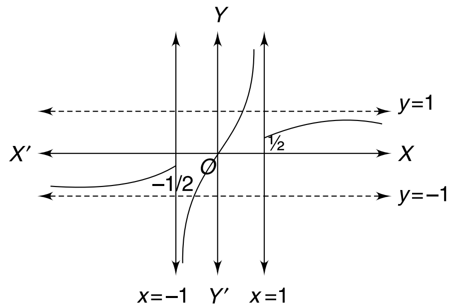
$$\Rightarrow\ |p(1 + h) - p(1)| \le |e^h - 1|\ (\because p(1) = 0)$$

$$\Rightarrow\ \left|\frac{p(1 + h) - p(1)}{h}\right| \le \left|\frac{e^h - 1}{h}\right|$$

$$\Rightarrow\ \lim_{h\to 0}\left|\frac{p(1 + h) - p(1)}{h}\right| \le \lim_{h\to 0}\left|\frac{e^h - 1}{h}\right|$$

$$\Rightarrow\ |p'(1)| \le 1$$

$$\Rightarrow\ |a_1 + 2a_2 + 3a_2\ ... + na_n| \le 1$$

33. 

Clearly the function $f(x)$ is not continuous at $x = 1$ and $x = -1$

So, $f(x)$ is not differentiable at $x = -1$ and $x = 1$.

34. We are given, for $x \ge 0$,

$$F(x) = \int_0^x f(t)\, dt$$

$$F(0) = \int_0^0 f(t)\, dt = 0$$

As $f(x) \le cF(x)$, for all $x \ge 0$, we get,

$$f(0) \le F(0)$$

$$f(0) \le 0$$

Since $f(x) \ge 0$, $\forall\ x \ge 0$, we get, $f(0) \ge 0$

$f(0) = 0$

Since f is continuous in $(0, \infty)$ and F is differentiable in $(0, \infty)$, so $F'(x) = f(x)\ \forall\ x \ge 0$

Since, $f(x) \le cF(x)$, $\forall\ x \ge 0$, we have

$$F'(x) - cF(x) \le 0$$

Multiplying both sides by e^{-cx}, we get,

$$F(x)\, e^{-cx} - cF(x)\, e^{-cx} \le 0$$

$$\frac{d}{dx}(e^{-cx} F(x)) \le 0$$

Thus, $g(x) = e^{-cx} F(x)$ is a decreasing function on $(0, \infty)$

Therefore $g(x) \le g(0)$, for each $x \ge 0$

But $g(0) = e^{-c0} F(0) = 0$

Thus, $g(x) \le 0,\ \forall\ x \ge 0$

$e^{-cx} F(x) \le 0,\ \forall\ x \ge 0$

$F(x) \le 0,\ \forall\ x \ge 0$

Thus, $f(x) \le cF(x) \le 0\ \forall\ x \ge 0$

But, it is given that, $f(x) \ge 0\ \forall\ x \ge 0$

Hence, $f(x) = 0,\ \forall\ x \ge 0$

35. We have

$$\lim_{x \to 1^+} \left(\frac{e^{\frac{1}{x-1}} - 2}{e^{\frac{1}{x-1}} + 2} \right)$$

$$= \lim_{x \to 1^+} \left(\frac{1 - 2e^{-\frac{1}{x-1}}}{1 + 2e^{-\frac{1}{x-1}}} \right)$$

$$= \lim_{x \to 1^+} \left(\frac{1 - 2e^{-\infty}}{1 + 2e^{-\infty}} \right)$$

$$= \left(\frac{1 - 2.0}{1 + 2.0} \right)$$

$$= 1$$

Also, $\lim_{x \to 1^-} \left(\frac{e^{\frac{1}{x-1}} - 2}{e^{\frac{1}{x-1}} + 2} \right)$

$$= \lim_{x \to 1^-} \left(\frac{1e^{-\infty} - 2}{e^{-\infty} + 2} \right)$$

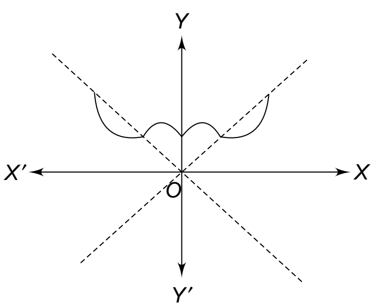
$$= \left(\frac{1.0 - 2}{0 + 2} \right)$$

$$= -1$$

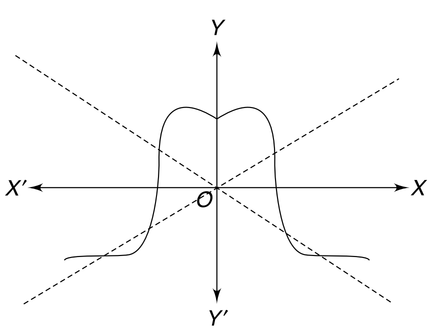
Since R.H.L ≠ L.H.L, so limit does not exist.

Thus, $f(x)$ is not continuous at $x = 1$.

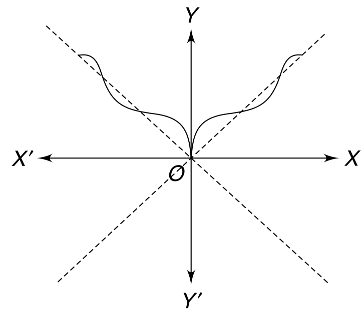
36. Ans. (d)

(a) 

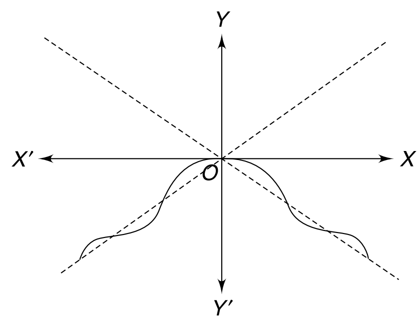
Clearly $\cos(|x|) + |x|$ is not differentiable at $x = 0$

(b) 

Clearly, $\cos(|x|) - |x|$ is not differentiable at $x = 0$

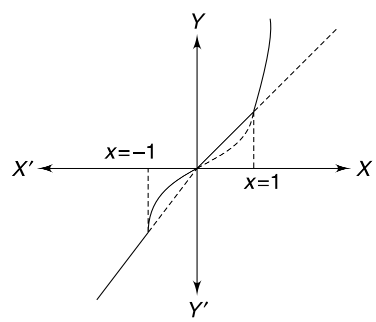
(c) 

Clearly, $\sin(|x|) + |x|$ is not differentiable at $x = 0$

(d) 

Clearly, $\sin(|x|) - |x|$ is differentiable at $x = 0$.

37. Ans. (d)



38. Ans. (a)

Now, $f'(k^-)$

$$= \lim_{x \to k^-} \left(\frac{f(x) - f(k)}{x - k} \right)$$

$$= \lim_{x \to k^-} \left(\frac{[x] \sin(\pi k) - 0}{x - k} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{[k - h] \sin(\pi k - \pi h)}{k - h - k} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{[k - 1](-1)^{k-1} \sin(\pi h)}{-h} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{[k - 1]\pi(-1)^{k} \sin(\pi h)}{\pi h} \right)$$

$$= (k - 1)\pi\,(-1)^k$$

39. Since $g(x)$ is continuous at $x = \alpha$

so, $\lim_{x\to\alpha} g(x) = g(\alpha)$

$$\lim_{x\to\alpha}\left(\frac{f(x) - f(\alpha)}{x - \alpha}\right) = g(\alpha)$$

$$f' = g(a)$$

Thus, $f(x)$ is differentiable at $x = (\alpha)$

Conversely, let $f(x)$ is differentiable at $x = \alpha$

i.e. $f'(\alpha) = \lim_{x\to\alpha}\left(\frac{f(x) - (\alpha)}{x - \alpha}\right)$

Let $g(x) = \begin{cases}\left(\dfrac{f(x) - f(\alpha)}{x - \alpha}\right) & : x \neq \alpha \\ f'(\alpha) & : x = \alpha\end{cases}$

Clearly, $\lim_{x\to\alpha} g(x) = f'(\alpha)$

$g(x)$ is continuous at $x = \alpha$

Hence, $f(x)$ is differentiable at $x = \alpha$, iff $g(x)$ is continuous at $x = \alpha$.

40. We have $f(x) = \begin{cases}\tan^{-1} x & : |x| \le 1 \\ \dfrac{1}{2}(|x| - 1) & : |x| > 1\end{cases}$

$$f(x) = \begin{cases}\dfrac{1}{2}(-x - 1) & : x < -1 \\ \tan^{-1} x & : -1 \le x \le 1 \\ \dfrac{1}{2}(x - 1) & : x > 1\end{cases}$$

Thus, $f(x)$ is discontinuous at $x = 1$, $x = -1$

Hence, the domain of $f'(x)$ is $R - \{-1, 1\}$.

41. We have $(g_o f)(x)$

$$= g(f(x))$$

$$= \begin{cases} f(x) + 1 & : f(x) < 0 \\ (f(x) - 1)^2 + b & : f(x) > 0\end{cases}$$

$$= \begin{cases} x + a + 1 & : x < -a \\ (x + a - 1)^2 + b & : -a \le x < 0 \\ (|x - 1| - 1)^2 + b & : x \ge 0\end{cases}$$

As $g_o f$ is continuous at $x = -a$

i.e. $(g_o f)(-a^+) = (g_o f)(-a^+) = (g_o f)(-a)$

$$1 + b = 1 + b = 1$$

$$b = 0$$

Also, $g_o f$ is continuous at $x = 0$

$$(g_o f)(0^+) = (g_o f)(0^-) = (g_o f)(0)$$

$$(a - 1)^2 + b = b = b$$

$$a = 1$$

Thus, $(g_o f)(x) = \begin{cases} x + 2 & : x < -1 \\ x^2 & : -1 \le x < 0 \\ (|x - 1| - 1)^2 & : x \ge 0\end{cases}$

In the neighbourhood of $x = 0$, $(g_o f)(x) = x^2$ which is also differentiable at $x = 0$.

42. Given $f'(a^-) = 0$

$$\lim_{h\to 0^+}\left(\frac{f(a - h) - f(a)}{-h}\right) = 0$$

Now, $f'(-a^-)$

$$= \lim_{h\to 0^-}\left(\frac{f(-a - h) - f(-a)}{-h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{-f(a + h) + f(a)}{-h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{f(a + h) - f(a)}{h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{f(2a - (a + h)) - f(a)}{h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{f(a - h) - f(a)}{h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{f(a - h) - f(a)}{-h}\right)$$

$$= -f'(a^-)$$

$$= 0$$

43. Since $f(x)$ is differentiable at $x = 0$, so it is continuous at $x = 0$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{e^{ax/2} - 1}{x}\right) = \lim_{x\to 0^-}\left(b\sin^{-1}\left(\frac{x + c}{2}\right)\right) = \frac{1}{2}$$

$$\Rightarrow \frac{a}{2} = b\sin^{-1}\left(\frac{c}{2}\right) = \frac{1}{2}$$

$$\Rightarrow a = 1$$

Also, it is differentiable at $x = 0$, so $f'(0^+) = f'(0^-)$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{f(x) - f(0)}{x - 0}\right) = \lim_{x\to 0^-}\left(\frac{f(x) - f(0)}{x - 0}\right)$$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{e^{ax/2} - \frac{1}{2}}{x}\right) = \lim_{x\to 0^-}\left(\frac{b\sin^{-1}\left(\frac{x + c}{2}\right) - \frac{1}{2}}{x}\right)$$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{e^{x/2} - \frac{1}{2}}{x}\right) = \lim_{x\to 0^-}\left(\frac{b\sin^{-1}\left(\frac{x + c}{2}\right) - \frac{1}{2}}{x}\right)$$

$$\Rightarrow \lim_{x\to 0^+}\left(\frac{1+\left(\frac{x}{2}\right)+\frac{1}{2}\left(\frac{x}{2}\right)^2+\ldots-\frac{1}{2}}{x}\right)$$

$$= \lim_{x\to 0^-}\left(\frac{b\,\frac{1}{\sqrt{1-\left(\frac{x+c}{2}\right)^2}}\times\frac{1}{2}}{1}\right)$$

$$\Rightarrow \frac{1}{8}=\frac{b}{2\sqrt{1-\frac{c^2}{4}}}$$

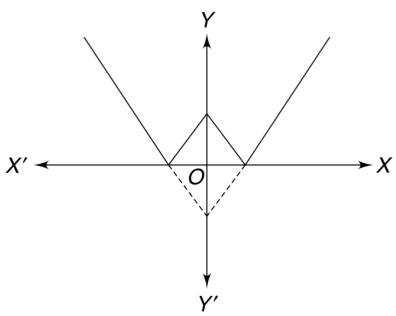
$$\Rightarrow \frac{1}{8}=\frac{b}{\sqrt{1-\frac{c^2}{4}}}$$

$$\Rightarrow \frac{1}{8}=\frac{b}{\sqrt{4}-c^2}$$

$$\Rightarrow 64b^2 = 4-c^2$$

Hence, the result.

44. Ans (a)



45. Ans. (b)

Given

$$f(1)=f\left(\frac{1}{2}\right)=f\left(\frac{1}{3}\right)=\ldots \lim_{n\to\infty} f\left(\frac{1}{n}\right)=0$$

$$\Rightarrow \lim_{n\to\infty} f\left(\frac{1}{n}\right)=0$$

$$\Rightarrow f(0)=0$$

Since there are infinitely many points in the neighbourhood of $x = 0$, so

$$f(x)=0$$

$$\Rightarrow f'(x)=0$$

$$\Rightarrow f'(0)=0$$

Thus, $f(0) = 0 = f'(0)$

46. Ans. (c)

Let $g(x) = f(x) - x^2$

By mean value theorem, we can say that, $g(x)$ has three roots at $x = 1, 2, 3$.

$\Rightarrow g'(x)$ has atleast 2 real roots in $x \in (1, 3)$

$\Rightarrow g''(x)$ has atleast 1 real root in $x \in (1, 3)$

$\Rightarrow f''(x) - 2 = 0$ for atleast 1 real root in $x \in (1, 3)$

$\Rightarrow f''(x) = 2$ has 1 real root in $x \in (1, 3)$

47. It is given that, $f(x-y) = f(x)g(y) - g(x)f(y)$

and $g(x-y) = g(x)g(y) + f(x)f(y)$

Now, $f(0) = f(x-x)$

$= f(x)g(x) - g(x)f(x)$

$= 0$

Also, $g(x) = g(x-0)$

$= g(x)g(0) + f(x)f(0)$

$= g(x)g(0)$

$g(x) - g(x)g(0) = 0$

$g(x)(1 - g(0)) = 0$

Case-I: When $g(x) = 0$

We can say that, $g'(x) = 0$ for all x in R

Thus, $g'(0) = 0$

Case-II: When $g(x) \neq 0$

Then, $1 - g(0) = 0$

$g(0) = 1$

Now, $f'(0^+)$

$$= \lim_{h\to 0^+}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{f(0)g(-h)-g(0)f(-h)-0}{h}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{0.g(-h)-1.f(-h)}{h}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{-f(-h)}{h}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{f(-h)-f(0)}{-h}\right)$$

$$= f'(0^-)$$

Thus f is derivable at $x = 0$

So f is continuous at $x = 0$

Now, $g(0) = g(x-x)$

$= g(x)g(x) + f(x)f(x)$

$= (g(x))^2 + (f(x))^2$

Thus, $(g(x))^2 + (f(x))^2 = 1$

Since f is continuous at $x = 0$, so g is also continuous at $x = 0$.

We have $g'(0)$

$$= \lim_{h\to 0}\left(\frac{g(h)-g(0)}{h}\right)$$

$= \lim_{h\to 0}\left(\frac{g(h)-1}{h}\right)$

$= \lim_{h\to 0}\left(\frac{g(h)-1}{h}\times\frac{g(h)+1}{g(h)+1}\right)$

$= \lim_{h\to 0}\left(\frac{(g(h))^2-1}{h(g(h)+1)}\right)$

$= \lim_{h\to 0}\left(\frac{-(f(h))^2}{h}\times\frac{1}{(g(h)+1)}\right)$

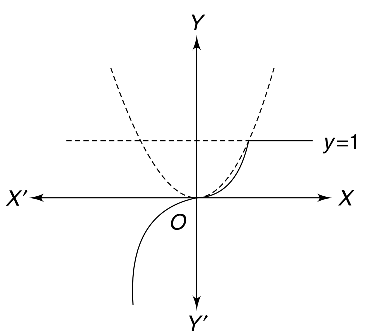
$= -\lim_{h\to 0}\left(\frac{f(h)}{h}\times\frac{f(h)}{(g(h)+1)}\right)$

$= -\lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\times\frac{f(h)}{(g(h)+1)}\right)$ $(\because f(0)=0)$

$= f'(0)\times\left(\frac{0}{g(0)+1}\right)$

$= 0$

48. Ans. (a, b, c)



Clearly $f(x)$ is continuous for all x in R.

Now $f(x) = 1$ for all $x > 1$

$f'(x) = 1$

So, $f(x)$ is continuous for all x in R but not differentiable at $x = 1$.

49. Ans. (b)

Given $f''(x) = -f(x)$

$\Rightarrow \frac{d}{dx}(f'(x)) = -f(x)$

$\Rightarrow \frac{d}{dx}(g(x)) = -f(x)$

$\Rightarrow g'(x) = -f(x)$

Also, $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$

$\Rightarrow F'(x) = 2f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right).\frac{1}{2} + 2g\left(\frac{x}{2}\right).g'\left(\frac{x}{2}\right).\frac{1}{2}$

$\Rightarrow F'(x) = f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right).g'\left(\frac{x}{2}\right)$

$\Rightarrow F'(x) = f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right).f'\left(\frac{x}{2}\right) = 0$

$\Rightarrow F'(x) = 0$

$\Rightarrow F(x) = \text{constant}$

$\Rightarrow F(10) = F(5) = 5.$

50. Ans.

(a) $\rightarrow p, q$, r

(b) $\rightarrow p, s$

(c) $\rightarrow r, s$

(d) $\rightarrow p, q$

(a) Let $f(x) = x|x|$

$= \begin{cases} x^2 & : x \ge 0 \\ -x^2 & : x < 0 \end{cases}$

$f'(x) = \begin{cases} 2x & : x \ge 0 \\ -2x^2 & : x < 0 \end{cases}$

$f(x)$ is differentiable everywhere.

So, $f(x)$ is differentiable as well as continuous in $(-1, 1)$

(b) Let $g(x) = \sqrt{|x|}$

$g(x) = \begin{cases} \sqrt{x} & : x \ge 0 \\ \sqrt{-x} & : x < 0 \end{cases}$

$g'(x) = \begin{cases} \frac{1}{2\sqrt{x}} & : x \ge 0 \\ \frac{-1}{2\sqrt{-x}} & : x < 0 \end{cases}$

Clearly, $f(x)$ is not differentiable at $x = 0$.

However, it is continuous at $x = 0$.

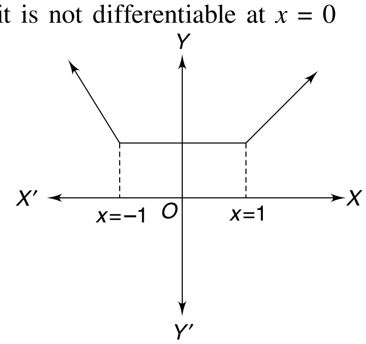
(c) Let $h(x) = x + [x]$

$= \begin{cases} x-1 & : -1 < x < 0 \\ x & : 0 \le x < 1 \end{cases}$

Clearly, $f(x)$ is not continuous at $x = 0$

So it is not differentiable at $x = 0$

(d)



Clearly $f(x)$ is differentiable and continuous in $(-1, 1)$

51. Ans. (c)

Given $\lim_{x\to 1^+}(g(x)) = p = -1$

$\Rightarrow \quad \lim_{h\to 0}\left(\frac{(1+h-1)^n}{\log(\cos^m(1+h-1))}\right) = -1$

$\Rightarrow \quad \lim_{h\to 0}\left(\frac{h^n}{\log(\cos^m h)}\right) = -1$

$\Rightarrow \quad \lim_{h\to 0}\left(\frac{h^n}{m\log(\cos h)}\right) = -1$

$\Rightarrow \quad \lim_{h\to 0}\left(\frac{nh^{n-1}}{-m\tan h}\right) = -1$

$\Rightarrow \quad \lim_{h\to 0}\left(-\frac{n}{m}\right)\left(\frac{h^{n-2}}{-m\frac{\tan h}{h}}\right) = -1$

$\Rightarrow \quad \lim_{h\to 0}\left(\frac{n}{m}\right)\left(\frac{h^{n-2}}{m\frac{\tan h}{h}}\right) = 1$

It is possible only when $n-2=0$ and $\left(\frac{n}{m}\right) = 1$

$n = 2 = m$

52. **Statement-I**

We have $\lim_{x\to 0}\left(\frac{g(x)\cos x - g(0)}{\sin x}\right)$

$= \lim_{x\to 0}\left(\frac{g'(x)\cos x - g(x)\sin x}{\cos x}\right)$

$= 0$

Also, $f(x) = g(x)\sin x$

$f'(x) = g'(x)\sin x + g(x)\cos x$

$f''(x) = g''(x)\sin x + 2g'(x)\cos x - g(x)\sin x$

$f''(0) = 0$

Thus, $\lim_{x\to 0}(g'(x)\cot x - g(0)\operatorname{cosec} x)$

$0 = f''(0)$

$\Rightarrow$ Statement is true

Satetement-II

$f'(x) = g'(x)\sin x + g(x)\cos x$

$f'(0) = g(0)$

It is not a correct explanation of statement-I.

53. Ans. (a, b, c, d)

Given $f(x) = f(1-x)$

$f'(x) = f'(1-x)\times -1$

Put $x = 1/2$

$f'\left(\frac{1}{2}\right) = -f'\left(1-\frac{1}{2}\right)$

$f'\left(\frac{1}{2}\right) = -f'\left(\frac{1}{2}\right)$

$2f'\left(\frac{1}{2}\right) = 0$

$f'\left(\frac{1}{2}\right) = 0$

Also, $f'\left(\frac{1}{4}\right) = 0$

Thus, $f'(x) = 0$ at two points in [0, 1]

Let $g(x) = f\left(x+\frac{1}{2}\right)\sin x$

$g(-x) = f\left(-x+\frac{1}{2}\right)\sin(-x)$

$= -f\left(\frac{1}{2}-x\right)\sin x$

$= -f\left(1-\left(\frac{1}{2}-x\right)\right)\sin x$

$= -f\left(x+\frac{1}{2}\right)\sin x$

$= -g(x)$

Thus, $g(x)$ is an odd function.

$$\int_{-1/2}^{1/2} f\left(x+\frac{1}{2}\right)\sin x\,dx = 0$$

Again, $\int_{1/2}^{1} f(1-t)e^{\sin(\pi t)}dt$

Put $1-t = u$

$= \int_0^{1/2} f(u)e^{\sin\pi(1-u)}du$

$= \int_0^{1/2} f(u)e^{\sin(\pi-\pi u)}du$

$= \int_0^{1/2} f(u)e^{\sin\pi u}du$

$= \int_0^{1/2} f(t)e^{\sin\pi t}dt$

54. We have $g(x) = f^{-1}(x)$

$\Rightarrow \quad f(g(x)) = x$

$\Rightarrow \quad f'(g(x)).g'(x) = 1$ ...(i)

Now, $f(0) = 1$

From (i), we get, $f'(g(1)).g'(1) = 1$

$\Rightarrow \quad g'(1) = \frac{1}{f'(g(1))}$

$\Rightarrow \quad g'(1) = \frac{1}{f'(0)} = \frac{1}{1/2} = 2$

[Let $g(1) = k$

Also, $f(g(1)) = 1$

$f(k) = 1 = f(0)$

$k = 0$

Thus, $g(1) = k = 0]$

55. Ans. (b, c)

We have, $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt$

$$f'(x) = \frac{1}{x} + \sqrt{1 + \sin x}\, dx$$

Clearly, $f'(x)$ exists for all $x > 0$

Thus, $f'(x)$ is continuous for all $x > 0$ but not differentiable at $x > 0$.

More-over, $f'(x), f(x) > 0\ \forall\ x \in (1, \infty)$

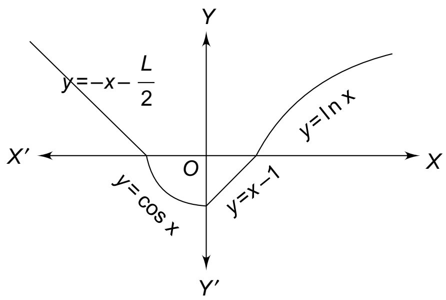
$$\ln x + \int_0^x \sqrt{1 + \sin t}\, dt$$

$$> \frac{1}{x} + \sqrt{1 + \sin x}\, dx\ \forall\ x \in (\pi, \infty)$$

Thus, $\frac{1}{x}$ is not bounded.

56. Ans. (a, b, c, d)

We have



Clearly $f(x)$ is continuous at $x = \frac{\pi}{2}$ and $f(x)$ is not differentiable at $x = 0$.

Also, $f(x)$ is differentiable at $x = 3/2$

Now, $f'(1^+) = \lim_{x\to 1^+}\left(\frac{f(x) - f(1)}{x - 1}\right)$

$$= \lim_{x\to 1^+}\left(\frac{\ln x - 0}{x - 1}\right)$$

$$= \lim_{x\to 1^+}\left(\frac{\frac{1}{x}}{1}\right) = 1$$

Again, $f'(1^-) = \lim_{x\to 1^-}\left(\frac{f(x) - f(1)}{x - 1}\right)$

$$= \lim_{x\to 1^-}\left(\frac{x - 1 - 0}{x - 1}\right) = 1$$

Thus, $f(x)$ is differentiable at $x = 1$.

57. Ans. (b, c)

Given $f(x + y) = f(x) + f(y)$

Put $x = 0 = y$, $f(0) = 0$

Now, $f'(x) = \lim_{h\to 0}\left(\frac{f(x + h) - f(x)}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{f(x) + f(h) - f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(h)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(h) - f(0)}{h}\right)$$

$$= f'(0) = k$$

On integration, we get, $f(x) = kx + c$

If $x = 0$, $f(0) = 0$, then $c = 0$.

Thus, $f(x) = kx$.

$\Rightarrow \quad f'(x) = k$

58. We have $f(x) = \begin{cases} x^2\left|\cos\left(\frac{\pi}{x}\right)\right| & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

At $x = 0$,

$$f'(0^+) = \lim_{x\to 0^+}\left(\frac{f(x) - f(0)}{x - 0}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{x^2|\cos\left(\frac{\pi}{x}\right)| - 0}{x - 0}\right)$$

$$= \lim_{x\to 0^+}\left(x\left(\cos\left(\frac{\pi}{x}\right)\right)\right) = 0$$

Also, $f'(0^-) = \lim_{x\to 0^-}\left(\frac{f(x) - f(0)}{x - 0}\right)$

$$= \lim_{x\to 0^-}\left(\frac{x^2\left(\cos\left(\frac{\pi}{x}\right)\right)}{x}\right)$$

$$= \lim_{x\to 0^-}\left(x\left(\cos\left(\frac{\pi}{x}\right)\right)\right) = 0$$

Thus, $f(x)$ is differentiable at $x = 0$

At $x = 2$

$$f'(2^+) = \lim_{x\to 2^+}\left(\frac{f(x) - f(2)}{x - 2}\right)$$

$$= \lim_{x\to 2^+}\left(\frac{x^2\left|\cos\left(\frac{\pi}{x}\right)\right| - 0}{x - 2}\right)$$

$$= \lim_{x\to 2^+}\left(\frac{-x^2\cos\left(\frac{\pi}{x}\right)}{x - 2}\right)$$

$$= \lim_{x\to 2^+}\left(\frac{x^2\sin\left(\frac{\pi}{x}\right)\times -\frac{\pi}{x^2} - 2x\cos\left(\frac{\pi}{x}\right)}{1}\right)$$

$$= 4\sin\left(\frac{\pi}{2}\right)\times\left(-\frac{\pi}{4}\right) = -\pi$$

Also, $f'(2^-) = \lim_{x\to 2^-}\left(\frac{f(x)-f(2)}{x-2}\right)$

$$= \lim_{x\to 2^-}\left(\frac{x^2\cos\left(\frac{\pi}{x}\right) - 0}{x-2}\right)$$

$$= \lim_{x\to 2^-}\left(\frac{2x\cos\left(\frac{\pi}{x}\right) + x^2\sin\left(\frac{\pi}{x}\right)\times\frac{\pi}{x^2}}{1}\right)$$

$$= 4\sin\left(\frac{\pi}{2}\right)\times\frac{\pi}{4} = \pi$$

Thus, $f(x)$ is differentiable at $x = 0$ but not differentiable at $x = 2$.

59. Ans. (d)

We have, $f''(x) - 2f'(x) + f(x) \geq e^x$

$\Rightarrow \quad f''(x).e^x - f'(x).e^x - f'(x)\cdot e^{-x} + f(x)e^{-x} \geq 1$

$\Rightarrow \quad \frac{d}{dx}(f'(x).e^{-x}) - \frac{d}{dx}(f(x).e^{-x}) \geq 1$

$\Rightarrow \quad \frac{d}{dx}(f'(x).e^{-x} - (f(x)e^{-x})) \geq 1$

$\Rightarrow \quad \frac{d^2}{dx^2}(f(x)e^{-x})) \geq 1$

$\Rightarrow \quad \frac{d^2}{dx^2}(\varphi(x)) \geq 1$, where $\varphi(x) = f(x)e^{-x}$

$\Rightarrow \quad \varphi(x)$ is concave upward.

$f(0) = 0 = f(1)$

$\Rightarrow \quad \varphi(0) = 0 = \varphi(1)$

$\Rightarrow \quad \varphi(x) < 0$

$\Rightarrow \quad f(x) < 0$

60. Ans. (b, c)

We have $f(x) = x\sin(\pi x)$

$$f'(x) = \sin(\pi x) + \pi x\cos(\pi x)$$

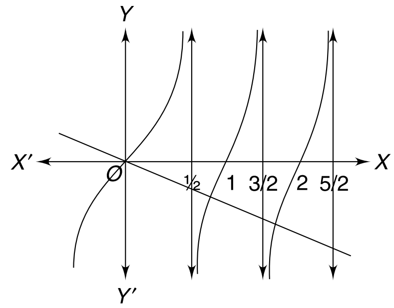
Now, $f'(x) = 0$ gives

$$\sin(\pi x) + \pi x\cos(\pi x) = 0$$

$$\sin(\pi x) = -\pi x\cos(\pi x)$$

$$\frac{\sin(\pi x)}{\cos(\pi x)} = -\pi x$$

$$\tan(\pi x) = -\pi x$$



Clearly, it will intersect at

$$\frac{1}{2} < x < 1 \text{ or } \frac{3}{2} < x < 2 \text{ or } \frac{5}{2} < x < 3$$

Thus, it will intersect at unique point in

$$\left(n + \frac{1}{2}, n + 1\right) \text{ or } (n, n + 1)$$

61. Ans. (a, c)

Since $f(x) \geq 1\ \forall\ x \in [a, b]$

For $g(x)$

L.H.D at $x = a$ is zero.

and R.H.D at $x = a$

$$= \lim_{x\to a^+}\left(\frac{\int_a^x f(t)\,dt - 0}{x-a}\right)$$

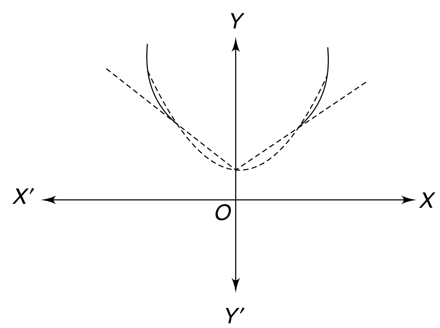
$$= \lim_{x\to a^+} f(x) \geq 1$$

Thus, $g(x)$ is not differentiable at $x = a$

Similarly, L.H.D at $x = b$ is greater than 1.

So, $g(x)$ is not differentiable at $x = b$.

62.



Clearly $f(x)$ is not differentiable at $x = -1, 0, 1$.

Thus, the number of points non-differentiable points = 3.

63. Differentiability at $x = 0$

L.H.D $= f'(0^-) = \lim_{\theta\to 0}\left(\frac{f(0) - f(0-\theta)}{\theta}\right)$

$$= \lim_{\theta \to 0}\left(\frac{0 + g(-\theta)}{\theta}\right) = 0$$

$$\text{RHD} = f'(0^+) = \lim_{\theta \to 0}\left(\frac{f(0) - f(0+\theta)}{\theta}\right)$$

$$= \lim_{\theta \to 0}\left(\frac{g(\theta)}{\theta}\right) = 0$$

Thus, $f(x)$ is differentiable at $x = 0$

Differentiability of $h(x)$ at $x = 0$

$h'(0^+) = 1$, $h(x)$ is an even function.

Hence, non differentiable at $x = 0$

Differentiability of $f(h(x))$ at $x = 0$

$f(h(x)) = g(e^{|x|}) \forall\ x \in R$

$$\text{LHD} = f'(h(0^-)) = \lim_{\theta \to 0}\left(\frac{f(h(0)) - f(h(0-\theta))}{\theta}\right)$$

$$= \lim_{\theta \to 0}\left(\frac{g(1) - g(e^{\theta})}{\theta}\right) = -g'(1)$$

$$\text{RHD} = f'(h(0^+)) = \lim_{\theta \to 0}\left(\frac{f(h(0+\theta)) - f(h(0))}{\theta}\right)$$

$$= \lim_{\theta \to 0}\left(\frac{g(e^{\theta}) - g(1)}{\theta}\right) = g'(1)$$

since $g'(1) \neq 0$, so $f(h(x))$ is non differentiable at $x = 0$

Differentiability of $h(f(x))$ at $x = 0$

$$h(f(x)) = \begin{cases} e^{|f(x)|} & : \quad x \neq 0 \\ 1 & : \quad x = 0 \end{cases}$$

$$\text{LHD} = h'(f(0 - \theta))$$

$$= \lim_{\theta \to 0}\left(\frac{h(f(0)) - h(f(0-\theta))}{\theta}\right)$$

$$= \lim_{\theta \to 0}\left(\frac{1 - e^{|g(-\theta)|}}{|g(-\theta)|} \times \frac{g(-\theta)}{\theta}\right) = 0$$

$$\text{Also, RHD} = h'(f(0 + \theta))$$

$$= \lim_{\theta \to 0}\left(\frac{h(f(0+\theta)) - h(f(0))}{\theta}\right)$$

$$= \lim_{\theta \to 0}\left(\frac{e^{|g(\theta)|} - 1}{|g(\theta)|} \times \frac{g(-\theta)}{\theta}\right) = 0$$

CHAPTER

5 Differentiation

CONCEPT BOOSTER

1. Introduction

In mathematics, differential calculus is a subfield of calculus concerned with the study of the rates at which quantities change. It is one of the two traditional divisions of calculus, the other being integral calculus.

The primary objects of study in differential calculus are the derivative of a function, related notions such as the differential, and their applications. The derivative of a function at a chosen input value describes the rate of change of the function near that input value. The process of finding a derivative is called differentiation. Geometrically, the derivative at a point is the slope of the tangent line to the graph of the function at that point, provided that the derivative exists and is defined at that point. For a real-valued function of a single real variable, the derivative of a function at a point generally determines the best linear approximation to the function at that point.

Differential calculus and integral calculus are connected by the fundamental theorem of calculus, which states that differentiation is the reverse process to integration.

Differentiation has applications to nearly all quantitative disciplines. For example, in physics, the derivative of the displacement of a moving body with respect to time is the velocity of the body, and the derivative of velocity with respect to time is acceleration. Newton's second law of motion states that the derivative of the momentum of a body equals the force applied to the body. The reaction rate of a chemical reaction is a derivative. In operations research, derivatives determine the most efficient ways to transport materials and design factories.

Calculus was created by Isaac Newton, a British scientist, as well as Gottfried Leibniz, a self-taught German mathematician, in the 17th century. It has been long disputed who should take credit for inventing calculus first, but both independently made discoveries that led to what we know now as calculus. Newton discovered the inverse relationship between the derivative (slope of a curve) and the integral (the area beneath it), which deemed him as the creator of calculus. Thereafter, calculus was actively used to solve the major scientific dilemmas of the time, such as:

1. calculating the slope of the tangent line to a curve at any point along its length.
2. determining the velocity and acceleration of an object given a function describing its position, and designing such a position function given the object's velocity or acceleration.
3. calculating arc lengths and the volume and surface area of solids.
4. calculating the relative and absolute extrema of objects, especially projectiles.

For Newton, the applications for calculus were geometrical and related to the physical world-such as describing the orbit of the planets around the sun.

For Leibniz, calculus was more about analysis of change in graphs. Leibniz's work was just as important as Newton's, and many of his notations are used today, such as the notations for taking the derivative and the integral.

2. Definitions

The derivative of the function $y = f(x)$ with respect to the variable x is the function $f'(x)$ is given by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right)$$

i.e

$$f'(x) = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x}\right)$$

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x + h) - f(x)}{h}\right)$$

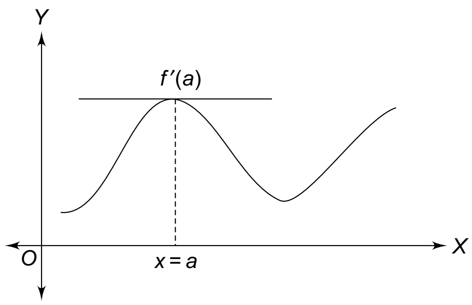
The function f' is called the derivative of f, since it is derived from f by the limiting operation.

We have a variety of notations for the derivative of the function $f(x)$

$$\frac{d}{dx}(f(x)),\ \frac{df}{dx},\ D(f(x)),\ f'(x),\ f',\ y',\ y_1$$

The process of finding the derivative of function's is called the **differentiation.**

Mathematically, derivative is equal to the slope of the tangent to a curve at a point.



Also, $$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

is known as first principle of differentiation or *ab-initio* method of differentiation.

Rules to find out the Differentiation by the First Principle

1. Let $y = f(x)$
2. Write $f(x+h) - f(x)$
3. Find $\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$
4. Which is the value of $f'(x)$ or $\frac{dy}{dx}$

3. Derivative of Some Standard Functions

Step-I

1. (i) $\frac{d}{dx}(c) = 0$

(ii) $\frac{d(x)}{dx} = 0$

(iii) $\frac{d(kx)}{dx} = k$

(iv) $\frac{d(x^n)}{dx} = nx^{n-1}$

(v) $\frac{d(e^x)}{dx} = e^x$

(vi) $\frac{d(a^x)}{dx} = a^x \log a$

(vii) $\frac{d(\log_e x)}{dx} = \frac{1}{x}$

(viii) $\frac{d(\log_a x)}{dx} = \frac{1}{x\log_e a}$

Step-II

2. (i) $\frac{d(\sin x)}{dx} = \cos x$

(ii) $\frac{d(\cos x)}{dx} = -\sin x$

(iii) $\frac{d(\tan x)}{dx} = \sec^2 x$

(iv) $\frac{d(\cot x)}{dx} = -\text{cosec}^2 x$

(v) $\frac{d(\sec x)}{dx} = \sec x \tan x$

(vi) $\frac{d(\text{cosec}\, x)}{dx} = -\text{cosec}\, x \cot x$

Step-III

3. (i) $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$

(iv) $\frac{d(\cot^{-1} x)}{dx} = -\frac{1}{1+x^2}$

(v) $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

(vi) $\frac{d(\text{cosec}^{-1} x)}{dx} = -\frac{1}{|x|\sqrt{x^2-1}}$

Step-IV

4. (i) $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(ii) $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}$

(iii) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

(iv) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2x\sqrt{x}}$

Step-V

5. (i) The Sum Rule:

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

(ii) The Difference Rule:

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

(iii) The product Rule:

$$\frac{d}{dx}(u \times v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

(iv) The quotient Rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{v^2}$$

4. Differentiation of Composite Function

If y is a differentiable function of u and u is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Thus chain rule can aslo be difined as

$\frac{d}{dx}(f(g(x))) = f(g(x))\cdot g'(x)$, where

$y = f(u)$ and $u = g(x)$

5. Differentiation by Inverse Trigonometric Function

We shall use the following standard result to find the derivative of inverse trigonometric functions.

1. (i) $\sin^{-1}(-x) = -\sin^{-1}(x)$
   (ii) $\tan^{-1}(-x) = -\tan^{-1}(x)$
   (iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$
2. (i) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$
   (ii) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$
   (iii) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$
3. (i) $\sin(\sin^{-1}x) = x : -1 \le x \le 1$
   (ii) $\cos^{-1}(\cos x) = x : -1 \le x \le 1$
   (iii) $\tan^{-1}(\tan x) = x : x \in R$
   (iv) $\cot^{-1}(\cot x) = x : x \in R$
   (v) $\text{cosec}^{-1}(\text{cosec}\, x) = x : x \in (-\infty, 1] \cup [1, \infty)$
   (vi) $\sec(\sec^{-1}x) = x : x \in (-\infty, 1] \cup [1, \infty)$
4. (i) $\sin^{-1}(\sin x)$

$$= \begin{cases} x & : -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x & : \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ -\pi - x & : -\frac{3\pi}{2} \le x \le -\frac{\pi}{2} \end{cases}$$

   (ii) $\cos^{-1}(\cos x)$

$$= \begin{cases} x & : 0 \le x \le \pi \\ 2\pi - x & : \pi \le x \le 3\pi \\ -x & : -\pi \le x \le 0 \end{cases}$$

   (iii) $\tan^{-1}(\tan x)$

$$= \begin{cases} x & : -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x - \pi & : \frac{\pi}{2} < x < \frac{3\pi}{2} \\ x + \pi & : -\frac{3\pi}{2} < x < \frac{\pi}{2} \end{cases}.$$

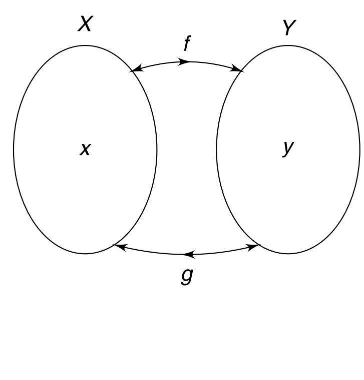
5. (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, x \in [-1, 1]$
   (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in R$
   (iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2} : x \in (-\infty, 1] \cup [1, \infty)$
6. (i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1 - y^2} + y\sqrt{1 - x^2})$
   (ii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1 - x^2}\sqrt{1 - y^2})$
   (iii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1 - y^2} - y\sqrt{1 - x^2})$
   (iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}(xy + \sqrt{1 - x^2}\sqrt{1 - y^2})$
7. (i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$
   (ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right)$
8. $2\tan^{-1}x$

$$= \tan^{-1}\left(\frac{2x}{1 - x^2}\right) = \sin^{-1}\left(\frac{2x}{1 + x^2}\right) = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$$

Some Important Substitutions

Expressions	Substitutions
(i) $\sqrt{a^2 - x^2}$	$x = a\sin\theta,\ a\cos\theta$
(ii) $\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ a\cot\theta$
(iii) $\sqrt{x^2 - a^2}$	$x = a\sec\theta,\ a\,\text{cosec}\,\theta$
(iv) $\sqrt{a - x}$	$x = a\sin^2\theta,\ a\cos^2\theta$
(v) $\sqrt{a + x}$	$x = a\tan^2\theta,\ a\cot^2\theta$
(vi) $\sqrt{\frac{a - x}{a + x}}$	$x = a\cos\theta,\ a\cos 2\theta$

6. Formula for Differentiation of Inverse Function



If g is the inverse of f, then we can write

$$y = f(x) \Leftrightarrow g(y) = x$$

So, $\dfrac{dy}{dx} = f'(x)$ and $g'(y)\dfrac{dy}{dx} = 1$

$\Rightarrow$ $g'(y)\dfrac{dy}{dx} = 1$

$\Rightarrow$ $g'(y) \times f'(x) = 1$

$\Rightarrow$ $g'(y) = \dfrac{1}{f'(x)}$

7. Differentiation of Implicit Functions

When y is expressed directly in terms of x, then it is known as explicit function.

So, when y is not expressed directly in terms of x or conversely, then it is known as implicit function.

For examples, $x^3 + y^3 = xy$, $\sin(xy) = e^{x+y}$, $\log(xy) = x + y^2$ etc.

Here, we shall use $\dfrac{d(y)}{dx} = \dfrac{dy}{dx}, \dfrac{d(y^2)}{dx} = 2y \cdot \dfrac{dy}{dx},$

$\dfrac{d(y^3)}{dx} = 3y^2 \cdot \dfrac{dy}{dx}, \dfrac{d(y^4)}{dx} = 4y^3 \cdot \dfrac{dy}{dx}$ and etc.

8. Logarithmic Differentiation

We shall use the logarithmic differentiation, only when, if we get, a function which is the product or quotient of a number of functions or a function is of the form $(f(x))^{g(x)}$ where f and g both are differentiable.

9. Differentiation of Infinite Series

Here, we shall find out the derivative of an infite series., If we take one term from an infinite series, it remains unchanged.

Let $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots \text{ to } \infty}}}$

$\Rightarrow$ $y = \sqrt{f(x) + y}$

$\Rightarrow$ $y^2 = (f(x) + y)$

$\Rightarrow$ $2y\dfrac{dy}{dx} = f'(x) + \dfrac{dy}{dx}$

$\Rightarrow$ $(2y - 1)\dfrac{dy}{dx} = f'(x)$

$\Rightarrow$ $\dfrac{dy}{dx} = \dfrac{f'(x)}{(2y - 1)}$

10. Differentiation of Parametric Functions

Let a function y of x be represented by the parametric equations $x = f(t)$ and $y = g(t)$

Now, we shall find $\dfrac{dx}{dy}$

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{f'(t)}{g'(t)}$$

11. Differentiation of a Function w.r.t Another Function

Let $u = f(x)$ and $v = g(x)$,

then $\dfrac{du}{dy} = \dfrac{\frac{du}{dx}}{\frac{dv}{dx}} = \dfrac{f'(x)}{g'(x)}.$

For example, if $u = \sin x$ and $v = \sqrt{x}$

then $\dfrac{du}{dv} = \dfrac{\cos x}{\frac{1}{2\sqrt{x}}} = 2\sqrt{x}\,\cos x$

12. Higher Order Derivatives

Let $y = f(x)$ be a differentiable function of x.

Then, $y' = \dfrac{dy}{dx} = f'(x)$ is the first derivative or first order derivative of y w.r.t x.

If y' is differentiable, then its derivative

$y'' = \dfrac{d}{dx}(y') = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ is the second order derivative of y w.r.t x

If y'' is differentiable, then its derivative

$y''' = \dfrac{d(y'')}{dx} = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d^3y}{dx^3}$ is the third order derivative of y w.r.t. x.

Thus, if y is a function of x, then its several derivatives are $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \ldots, \dfrac{d^ny}{dx^n}$

i.e. $y', y'', y''', \ldots, y^n$

13. Differentiation of a Determinant

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where $f(x)$,

$g(x)$, $h(x)$, $l(x)$, $m(x)$, $n(x)$, $u(x)$, $v(x)$ and $w(x)$ are all functions of x and differentiable then $F'(x)$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

or

$$F'x = \begin{vmatrix} f'(x) & g(x) & h(x) \\ l'(x) & m(x) & n(x) \\ u'(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g'(x) & h(x) \\ l(x) & m'(x) & n(x) \\ u(x) & v'(x) & w(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h'(x) \\ l(x) & m(x) & n'(x) \\ u(x) & v(x) & w'(x) \end{vmatrix}$$

14. Leibnitz Rules for Differentiation

If u and v are two functions of x, then

$$(uv)_n = u_n v + C_1 u_{n-1} v_1 + C_2 u_{n-2} v_2 + C_3 u_{n-3} v_3 + \ldots + C_r u_{n-r} v_r + \ldots + C_n u v_n,$$

where

$${}^nC_r = C_r,\ (uv)_n, = \frac{d^n}{dx^n}(uv)$$

$$(u_{n-1}v) = \left\{\frac{d^{n-1}}{dx^{n-1}}(u)\right\}\frac{dv}{dx} \text{ and so on.}$$

Some Standard Results:

(i) $\frac{d^n}{dx^n}\{\sin(ax+b)\} = a^n \cdot \sin\left(ax + b + \frac{n\pi}{2}\right),\ n \in N$

(ii) $\frac{d^n}{dx^n}\{\cos(ax+b)\}$

$= a^n \cdot \cos\left(ax + b + \frac{n\pi}{2}\right),\ n \in N$

(iii) $\frac{d^n}{dx^n}\{(ax+b)^m\}$

$= \frac{m!}{(m-n)!} \cdot a^n \cdot (ax+b)^{m-n},\ m \in N$

(iv) $\frac{d^n}{dx^n}\{\log(ax+b)\}$

$= \frac{(-1)^{n-1}(n-1)!}{(ax+b)^n} \cdot a^n,\ n \in N$

(v) $\frac{d^n}{dx^n}\{a^x\} = (\log_e a)^n a^x,\ n \in N$

(vi) $\frac{d^n}{dx^n}\{e^{mx}\} = m^n \cdot e^{mx},\ n \in N.$

(vii) $\frac{d^n}{dx^n}\{e^{ax} \cdot \sin(bx+c)\}$

$= r^n \cdot e^{ax} \cdot \sin(bx + c + n\varphi)\ n \in N$

(viii) $\frac{d^n}{dx^n}\{e^{ax} \cdot \cos(bx+c)\}$

$= r^n \cdot e^{ax} \cdot \cos(bx + c + n\varphi),\ n \in N,$

where $r = \sqrt{a^2 + b^2},\ \varphi = \tan^{-1}\left(\frac{b}{a}\right)$

Exercises

Level 1

(Problems Based on Fundamentals)

First Principle of Differentiation

1. Find the derivative of $y = e^x$, using first principle.
2. Find the derivative of $y = f(x) = \log_e x$, using first principle.
3. Find the derivative of $y = f(x) = \log_a x$, using first principle.
4. Find the derivative of $y = f(x) = \sin x$, using first principles.
5. Find the derivative of $y = f(x) = \tan^{-1} x$, using first principle of differentiation.
6. Find the derivative of $y = f(x) = e^{\sin x}$, using first principle.
7. Find the derivative of $y = f(x) = \sin(x^2)$, using first principle.

ABC of Differentiation

8. If $y = \log_x x + 10$, find $\frac{dy}{dx}$
9. If $y = 5^{\log_3 x} - x^{\log_3 5}$, find $\frac{dy}{dx}$
10. If $y = \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}$, find $\frac{dy}{dx}$
11. If $y = \sqrt{x} + x\sqrt{x} + x^2\sqrt{x} + x^3\sqrt{x}$, find $\frac{dy}{dx}$
12. If $y = \left(1 + \tan\left(\frac{\pi}{8} - x\right)\right)\left(1 + \tan\left(x + \frac{\pi}{8}\right)\right)$, find $\frac{dy}{dx}$

13. If $y = \dfrac{\sec x + \tan x - 1}{\sec x - \tan x + 1}$, find $\dfrac{dy}{dx}$ at $x = 0$

14. If $\dfrac{x^4 + x^2 + 1}{x^2 - x + 1}$ such that $\dfrac{dy}{dx} = ax + b$, find the, value of $a + b + 10$

15. If $y = \left(\sin\dfrac{x}{2} + \cos\dfrac{x}{2}\right)^2$, find $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{6}$

16. If $y = \left(\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{a}{x}}\right)$, prove that $2xy\dfrac{dy}{dx} = \dfrac{x}{a} - \dfrac{a}{x}$

Differentiation of Composite Functions

17. If $y = \log(\sin(3x + 5))$, find $\dfrac{dy}{dx}$

18. If $y = \log(x + \sqrt{x^2 + 1})$, find $\dfrac{dy}{dx}$

19. If $y = \log(\sqrt{x - 1} - \sqrt{x + 1})$, find $\dfrac{dy}{dx}$

20. If $y = (\sin x)^x$, find $y = (\sin x)^x$.

21. If $y = \log(\sin x + \cos x)$, find $\dfrac{dy}{dx}$

22. If $f(x) = \dfrac{1}{x - 1}$, find $\dfrac{d((f(f(x))))}{dx}$

23. If $f(x) = \dfrac{x - 1}{x + 1}$, find $\dfrac{d(f(f(f(x))))}{dx}$

24. Let f be a function for which $f'(x) = \dfrac{1}{x^2 + 1}$

If $g(x) = f(3x - 1)$, find $g'(x)$.

25. Let f be a function for which $f'(x) = x^2 + 1$

If $y = f(\sin(x^3))$, find $\dfrac{dy}{dx}$.

26. If $f(x) = |x - 1| + |x - 3|$, find $f'(2)$

27. If $f(x) = |x^2 - 1| + |x^2 - 4|$, find the value of $f'\left(\dfrac{3}{2}\right)$.

28. If $f(x^2) = x^4 + x^3 + 1$, find $f'(x^4)$

29. If g is the inverse of f and $f'(x) = \cos 2x$, find $g'(x)$.

30. If $f(x) = x + \tan x$ and f is the inverse of g, find $g'(x)$.

31. If $y = \dfrac{x^4 - x^2 + 1}{x^2 + \sqrt{3}x + 1}$ such that $\dfrac{dy}{dx} = px + q$ then prove that $p - q = \tan\left(\dfrac{5\pi}{12}\right)$

32. If $y = \sqrt{x - 1} + \sqrt{x + 1}$, prove that $\sqrt{x^2 - 1}\,\dfrac{dy}{dx} = \dfrac{1}{2}y$.

33. If $y = \dfrac{x}{x + 2}$, prove that $x\dfrac{dy}{dx} = (1 - y)y$.

34. If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, prove that $2x\dfrac{dy}{dx} = \sqrt{x} - \dfrac{1}{\sqrt{x}}$.

35. If $y = e^x \cos x$, prove that $\dfrac{dy}{dx} = \sqrt{2}\,e^x \cos\left(x + \dfrac{\pi}{4}\right)$.

36. If $y = \sqrt{x^2 + a^2}$, prove that $y\dfrac{dy}{dx} - x = 0$.

37. If $y = e^x + e^{-x}$, prove that $\dfrac{dy}{dx} = \sqrt{y^2 - 4}$.

38. If $xy = 4$, prove that $x\left(\dfrac{dy}{dx} + y^2\right) = 3y$.

39. Prove that

$$\frac{d}{dx}\left\{\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right\} = \sqrt{a^2 - x^2}$$

Differentiation by Inverse Trigonometric Function

40. Differentiate the following functions w.r.t. x:

(i) $\tan^{-1}(\sqrt{1 + x^2} + x)$

(ii) $\tan^{-1}(\sqrt{1 + x^2} - x)$

(iii) $\tan^{-1}\left\{\dfrac{\sqrt{1 + x^2} - 1}{x}\right\}$ $x \neq 0$

(iv) $\tan^{-1}\left\{\dfrac{\sqrt{1 + x^2} + 1}{x}\right\}$ $x \neq 0$

(v) $\cot^{-1}(\sqrt{1 + x^2} + x)$

(vi) $\tan^{-1}\left\{\dfrac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right\}$, $0 < x < \pi$

41. Differentiate the following functions w.r.t. x:

(i) $\tan^{-1}\left(\dfrac{a + x}{1 - ax}\right)$

(ii) $\tan^{-1}\left\{\dfrac{a\cos x - b\sin x}{a\cos x + b\sin x}\right\}$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

(iii) $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$, $-\dfrac{1}{\sqrt{3}} < \dfrac{x}{a} < \dfrac{1}{\sqrt{3}}$

(iv) $\tan^{-1}\left(\sqrt{\dfrac{a - x}{a + x}}\right)$, $-a < x < a$

(v) $\tan^{-1}\left(\dfrac{\sin x}{1 + \cos x}\right)$, $-\pi < x < \pi$.

(vi) $\tan^{-1}\left(\dfrac{a + b\tan x}{b - a\tan x}\right)$

(vii) $\tan^{-1}\left(\dfrac{a + bx}{b - ax}\right)$

(viii) $\tan^{-1}\left(\dfrac{x}{1 + 6x^2}\right)$

(ix) $\sin^{-1}\left(\dfrac{\sin x + \cos x}{\sqrt{2}}\right)$, $-\dfrac{3\pi}{4} < x < \dfrac{3\pi}{4}$

(x) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, $x \in R$

(xi) $\sin^{-1}\left(\frac{x-1}{x+1}\right) + \cos^{-1}\left(\frac{x-1}{x+1}\right)$

(xii) $\sin^{-1}\left(\frac{3\sin x + 4\cos x}{5}\right)$

(xiii) If $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$, find $\frac{dy}{dx}$

(xiv) If $y = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$, find $\frac{dy}{dx}$

42. If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$,

$0 < x < 1$, prove that, $\frac{dy}{dx} = \frac{4}{1+x^2}$.

43. If $y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{x^2+1}}\right)$,

$0 < x < \infty$, prove that, $\frac{dy}{dx} = \frac{2}{1+x^2}$

44. If $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\}$, find $\frac{dy}{dx}$.

45. If $y = \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$,

then prove that $\frac{dy}{dx}$ is independent of x.

46. If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, $x > 0$,

then prove that $\frac{dy}{dx} = \frac{4}{1+x^2}$.

47. If $y = \sec^{-1}\left(\frac{x-1}{x+1}\right) + \sin^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, $x > 0$,

prove that $\frac{dy}{dx} = 0$.

48. If $y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$, find $\frac{dy}{dx}$.

49. If $y = \cos^{-1}(2x) + 2\cos^{-1}(\sqrt{1-4x^2})$,

$0 < x < \frac{1}{2}$, find $\frac{dy}{dx}$.

50. If $y = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$,

find $\frac{dy}{dx}$.

51. If $y = \cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right)$

$\frac{1}{2} < x < 1$, find $\frac{dy}{dx}$.

52. If $y = \sin^{-1}(x\sqrt{1-x} - \sqrt{x-x^3})$, find $\frac{dy}{dx}$

53. If $y = f(x) = x^3 + x^5 + x^7$ and g is the inverse of f, then find $g'(3)$

54. If $y = f(x) = x^5 + 2x^3 + 2x$ and g is the inverse of f, find $g'(-5)$

55. If $f(x) = x^3 + 2x^2 + 3x + 4$ and $g(x)$ is the inverse of $f(x)$, find $g'(4)$.

56. If f and g are inverse of each other and $f'(x) = \frac{1}{1+x^n}$

find $g'(x)$.

Differentiation of Implicit Functions

57. Find $\frac{dy}{dx}$, if $2x^2 + 3xy + 3y^2 = 1$

58. If $e^x + e^y = e^{x+y}$, prove that, $\frac{dy}{dx} + e^{y-x} = 0$

59. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

60. If $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$ prove that

$\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$

61. If $\sin y = x \sin(a+y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

62. If $\log(x+y) = 2xy$, then prove that $y'(0) = 1$.

63. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$ prove that

$\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$

64. If $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$ then prove that $\frac{dy}{dx} = \frac{1}{x^3y}$.

65. If $\sec\left(\frac{x+y}{x-y}\right) = a$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

66. If $\tan^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = a$ prove that $\frac{dy}{dx} = \frac{x(1-\tan a)}{y(1+\tan a)}$.

67. If $xy = 1$, prove that $\frac{dy}{dx} + y^2 = 0$

68. If $y = x\sin y$, prove that $\frac{dy}{dx} = \frac{y}{x(1-x\cos y)}$.

69. If $\cos y = x\cos(a+y)$ with $\cos a \neq \pm 1$,

prove that $\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$

Logarithmic Differentiation

70. Find $\frac{dy}{dx}$, if $y = x^{\sin x}$

71. Find $\frac{dy}{dx}$, if $y = (\sin x)^{\cos x}$

72. Find $\frac{dy}{dx}$, if $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

73. If $x^m y^n = (x + y)^{m+n}$, prove that, $\frac{dy}{dx} = \frac{y}{x}$

74. If $y = \left(1 + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$, find $\frac{dy}{dx}$ at $x = 1$

75. If $y = \sin x \cdot \sin 2x \cdot \sin 3x \ldots \sin(2014)x$, find $\frac{dy}{dx}$.

76. If $x^y = e^{x-y}$, prove that, $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

77. If $x^y = y^x$, find $\frac{dy}{dx}$.

78. If $e^y = y^x$, prove that $\frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$.

79. If $x^m y^n = 1$, prove that, $\frac{dy}{dx} = -\frac{my}{nx}$

80. If $e^{x+y} - x = 0$, prove that $\frac{dy}{dx} = \frac{1 - x}{x}$.

81. Find $\frac{dy}{dx}$, if $y = x^{\sin x - \cos x} + \frac{x^2 - 1}{x^2 + 1}$.

82. Find $y'(0)$, if

$$y = (x + 1)(x + 2)(x + 3) \ldots (x + 2012)$$

83. Find $y'(0)$, if
$y = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8) \ldots (1 + x^{1006})$

Differentiation of Infinite Series

84. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \text{ to } \infty}}}$

then find $\frac{dy}{dx}$.

85. If $y = \frac{\sin x}{1+} \frac{\cos x}{1+} \frac{\sin x}{1+} \frac{\cos x}{1+} \ldots$, find $\frac{dy}{dx}$

86. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}$ to ∞, prove that

$\frac{dy}{dx} = \frac{1}{2y - 1}$.

87. If $y = x^{x^{x \ldots \text{to} \infty}}$, prove that $\frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$.

88. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x}}} + \ldots$ to ∞, prove that

$\frac{dy}{dx} = \frac{\cos x}{2y - 1}$

89. If $y = e^{x + e^{x + e^{x + \ldots \text{to} \infty}}}$, prove that $\frac{dy}{dx} = \frac{y}{1 - y}$

90. If $y = (\tan)^{(\tan)^{(\tan) \ldots \text{to } x}}$, prove that $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.

91. If $y = x + \frac{1}{x+} \frac{1}{x+} \frac{1}{x+} \ldots$ to ∞, then prove that,

$\frac{dy}{dx} = \frac{y}{2y - x}$.

92. If $y = \frac{\sin x}{1+} \cdot \frac{\cos x}{1+} \cdot \frac{\sin x}{1+} \cdot \frac{\cos x}{1+} \ldots$ to ∞,

then prove that, $y'(0) = \frac{1}{2}$.

93. If $y = \frac{x}{1+} \cdot \frac{x}{2+} \cdot \frac{x}{1+} \cdot \frac{x}{2+} \ldots$ to ∞, then find $\frac{dy}{dx}$.

Differentiation of Parametric Functions

94. Find $\frac{dy}{dx}$, when

$x = a(t - \sin t)$, $y = a(l - \cos t)$.

95. If $x^2 - y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$

then prove that $x^3 y \frac{dy}{dx} + 1 = 0$

96. If $x = a\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$,

then prove that $\frac{dy}{dx} = \frac{x}{y}$.

97. If $y = \cos^{-1}\left(\frac{5t + 12\sqrt{1 - t^2}}{13}\right)$

and $x = \cos^{-1}\left(\frac{1 - t^2}{1 + t^2}\right)$, find $\frac{dy}{dx}$.

98. If $x = \sin^{-1}\left(\frac{3\sin t + 4\cos t}{5}\right)$

and $y = \sin^{-1}\left(\frac{6\cos t + 8\sin t}{10}\right)$, find $\frac{dy}{dx}$.

99. If $x = \cos t$, and $y = \sin t$, prove that $\frac{dy}{dx} = \frac{1}{\sqrt{3}}$ at

$x = \frac{2\pi}{3}$.

100. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = \frac{y\log x}{x\log y}$

101. If $x = x = 2\cos\theta - \cos(2\theta)$ and
$y = 2\sin\theta - \sin 2\theta$, prove that

$$\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$$

102. If $x = a\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$, then prove that $\frac{dy}{dx} = \frac{x}{y}$

103. If $x = \sin^{-1}\left(\frac{2t}{1 + t^2}\right)$ and $y = \tan^{-1}\left(\frac{2t}{1 - t^2}\right)$, $t > 1$, then prove that $\frac{dx}{dy} = -1$

104. If $y = \cos^{-1}\left(\frac{5t + 12\sqrt{1 - t^2}}{13}\right)$ and $x = \cos^{-1}\left(\frac{1 - t^2}{1 + t^2}\right)$, find $\frac{dy}{dx}$.

105. If $x = \sin^{-1}\left(\frac{3\sin t + 4\cos t}{5}\right)$ and $y = \sin^{-1}\left(\frac{6\cos t + 8\sin t}{10}\right)$, find $\frac{dy}{dx}$.

106. If $x = \tan^{-1}\left(\frac{\sqrt{1 + \sin t} + \sqrt{1 - \sin t}}{\sqrt{1 + \sin t} - \sqrt{1 - \sin t}}\right)$ and $y = \tan^{-1}\left(\frac{\sqrt{1 + t^2} - 1}{t}\right)$, find $\frac{dy}{dx}$.

Differentiation of a Function with Respect to Another Function

107. Differentiate $f(x^2 + 2012)$. w.r.t $f(x^3 + 2013)$.
108. Differentiate x^x w.r.t. $x \log x$.
109. Differentiate $\tan^{-1}\left\{\frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}\right\}$ w.r.t. $\cos^{-1} x^2$.
110. Differentiate $\sin^{-1}\left(\frac{2x}{1 + x^2}\right)$ w.r.t. $\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$
111. Differentiate $f(\sin x)$ w.r.t. $f(\cos x)$

Higher Order Derivatives

112. If $y = c_1 e^x + c_2 e^{-x}$, then prove that $\frac{d^2y}{dx^2} - y = 0$.
113. If $x = at^2$, $y = 2at$, find $\frac{d^2y}{dx^2}$
114. If $x = a\cos\theta$, $y = b\sin\theta$, find $\frac{d^2y}{dx^2}$
115. Prove that $\frac{d^2y}{dx^2} = -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$
116. If $y = e^{2x}$, find $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$.
117. If $(a + bx)e^{\frac{x}{y}} = x$, then prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.
118. If $x^2 + y^2 = a^2$ and $k = \frac{1}{a}$, then prove that $k = \frac{|y''|}{\sqrt{(1 + y'^2)^3}}$.
119. If $y = e^{ax}\sin bx$, prove that, $y^2 - 2ay_1 + (a^2 + b^2)y = 0$
120. If $y = 2\sin x + 3\cos x$, prove that, $\frac{d^2y}{dx^2} + y = 0$
121. If $y = x + \tan x$, prove that $\cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0$
122. If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1 - \sin x)^2}$
123. If $y = A\cos(\log x) + B\sin(\log x)$, then prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$
124. If $y = \tan^{-1} x$, prove that $(1 + x^2)\, y_2 + 2xy_1 = 0$
125. If $y = e^x(\sin x + \cos x)$, prove that $y_2 - 2y_1 + 2y = 0$
126. If $y = \sin^{-1} x$, prove that $(1 - x^2)\, y_2 - xy_1 = 0$
127. Find $\frac{d^2y}{dx^2}$, if
 (i) $x = at^2$, $y = 2at$
 (ii) $x = a\cos^3\theta$, $y = a\sin^3\theta$
 (iii) $x = a\cos\theta$, $y = b\sin\theta$
128. If $x = a\sec\theta$, $y = b\tan\theta$, prove that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.
129. If $y = \sin^{-1} x$, prove that $(1 - x^2)\, y_2 - xy_1 = 0$
130. Find $\frac{d^2y}{dx^2}$, if
 (i) $x = at^2$, $y = 2at$
 (ii) $x = a\cos^3\theta$, $y = a\sin^3\theta$
 (iii) $x = a\cos\theta$, $y = b\sin\theta$
131. If $x = a\sec\theta$, $y = b\tan\theta$ prove that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.
132. If $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ prove that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$
133. If $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

134. If $y = x\log\left(\frac{x}{a+bx}\right)$, prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

135. If $\sqrt{y+x} + \sqrt{y-x} = c$, then prove that $\frac{d^2y}{dx^2} = \frac{2}{c^2}$

136. If $\frac{\cos^4\alpha}{x} + \frac{\sin^4\alpha}{y} = \frac{1}{x+y}$, then prove that $\frac{dy}{dx} = \tan^2\alpha$.

137. If $y = x\sin x$, then prove that,
$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = 0.$$

138. If $f: R \to R$ be a function is defined by $y = f(x) = x^2$, then prove that, $\left(\frac{d^2y}{dx^2}\right) \times \left(\frac{d^2y}{dx^2}\right) = -\frac{1}{2x^3}$.

139. There is a polynomial $P(x) = ax^3 + bx^2 + cx + d$ such that $P(0) = P(1) = -2$, $P'(0) = -1$, then find the value of a + $b + c + d + 10$.

140. If $f(x) = x + \tan x$ and g is the inverse of f, then prove that $g'(x) = \frac{1}{2+\tan^2(g(x))}$.

141. If g is the inverse of f such that $f(x) = e^x + x^3 - 1$, then prove that $g''(e) = -\frac{e+6}{(e+3)^3}$.

142. Let $f(x) = 1 + x^3$. If $g(x) = f^{-1}(x)$, then prove that $g'''(2) = \frac{8}{3}$.

143. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + (f(x))^2$, then prove that $g'(x) = \frac{1}{1+x^2}$.

Differentiation of a Determinant

144. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is a constant then prove that $\frac{d^3}{dx^3}(f(x))$ at $x = 0$ is 0.

145. Find the coefficient of x in the determinant
$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$$
with the help of determinant.

146. Let $f(x) = \begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix}$

Then prove that $f(x)$ is a linear polynomial of x.

147. Let $f(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ x & x^2 & x^2 \\ 0 & 2 & 6x \end{vmatrix}$

Then find the value of $f'(1)$.

148. Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 2\cos 3x \end{vmatrix}$

Then find the value of $f'\left(\frac{\pi}{2}\right)$.

Leibnitz Rules for Differentiation

149. If $y = e^x$, find the value of $y_n(0)$.

150. If $y = a^x$, find the value of $y_n(0)$.

151. If $y = \frac{1}{(x+1)}$, find the value of $y_n(0)$.

152. If $y = \sin x$, find the value of $y_n(0)$.

153. If $y = x^n$, find the value of $y_n(1)$.

154. If $y = A\sin x + B\cos x$, then prove that
(i) $y_2 + y = 0$
(ii) $y_{n+2} + y_n = 0$.

155. If $y = \tan^{-1}x$, then prove that
(i) $(1+x^2)y_2 + 2xy_1 = 0$
(ii) $(1+x^2)y_{n+2} + 2(n+2)xy_{n+1} + n(n+1)y_n = 0$.

156. If $y = \sin^{-1}x$, then prove that
(i) $(1-x^2)y_2 - xy_1 = 0$
(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

157. If $y = a\cos(\log x) + b\sin(\log x)$, then prove that $x^2 \cdot y_{n+2} + (2n+3)x \cdot y_{n+1} + (n^2+1)y_n = 0$.

Level II

(Mixed Problems)

1. If $y = \cot^{-1}(\sqrt{\cos 2x})$, then the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{6}$ will be

(a) $\left(\frac{2}{3}\right)^{1/2}$ (b) $\left(\frac{1}{3}\right)^{1/2}$
(c) $(3)^{1/2}$ (d) $(6)^{1/2}$

2. If $y = \sqrt{\frac{1+x}{1-x}}$, then $\frac{dy}{dx} =$

(a) $\frac{2}{(1+x)^{1/2}(1-x)^{3/2}}$

(b) $\frac{1}{(1+x)^{1/2}(1-x)^{3/2}}$

(c) $\dfrac{1}{2(1+x)^{1/2}(1-x)^{3/2}}$

(d) $\dfrac{2}{(1+x)^{3/2}(1-x)^{1/2}}$

3. $\dfrac{d}{dx}(\sqrt{\sec^2 x + \operatorname{cosec}^2 x}) =$

(a) $4\operatorname{cosec}2x\cdot\cot 2x$ (b) $-4\operatorname{cosec}2x\cdot\cot 2x$
(c) $-4\operatorname{cosec}x\cdot\cot 2x$ (d) None of these

4. $\dfrac{d}{dx}(\log_{\sqrt{x}}(1/x)) =$

(a) $-\dfrac{1}{2\sqrt{x}}$ (b) -2
(c) $-\dfrac{1}{x^2\sqrt{x}}$ (d) 0

5. If $y = \log\left\{\dfrac{x+\sqrt{(a^2+x^2)}}{a}\right\}$, then the value of $\dfrac{dy}{dx}$ is

(a) $\sqrt{a^2-x^2}$ (b) $a\sqrt{a^2+x^2}$
(c) $\dfrac{1}{\sqrt{a^2+x^2}}$ (d) $x\sqrt{a^2+x^2}$

6. If $y = \tan^{-1}\left(\dfrac{\sqrt{x}-x}{1+x^{3/2}}\right)$, then $y'(1)$ is

(a) 0 (b) $\dfrac{1}{2}$
(c) –1 (d) $-\dfrac{1}{4}$

7. If $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) + \sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{4}{1-x^2}$ (b) $\dfrac{1}{1+x^2}$
(c) $\dfrac{4}{1+x^2}$ (d) $\dfrac{-4}{1+x^2}$

8. If $y = \tan^{-1}\left(\dfrac{\sqrt{a}-\sqrt{x}}{1+\sqrt{ax}}\right)$, then $\dfrac{dx}{dy} =$

(a) $\dfrac{1}{2(1+x)\sqrt{x}}$ (b) $\dfrac{1}{(1+x)\sqrt{x}}$
(c) $-\dfrac{1}{(1+x)\sqrt{x}}$ (d) None of these

9. If $y^x + x^y = a^b$, then $\dfrac{dy}{dx} =$

(a) $-\dfrac{yx^{y-1}+y^x\log y}{xy^{x-1}+x^y\log x}$ (b) $\dfrac{yx^{y-1}+y^x\log y}{xy^{x-1}+x^y\log x}$
(c) $-\dfrac{yx^{y-1}+y^x}{xy^{x-1}+x^y}$ (d) $\dfrac{yx^{y-1}+y^x}{xy^{x-1}+x^y}$

10. $y = (\tan x)^{(\tan x)^{\tan x}}$, then at $x = \dfrac{\pi}{4}$, the value of $\dfrac{dy}{dx} =$

(a) 0 (b) 1
(c) 2 (d) None of these

11. If $x = \sin^{-1}(3t-4t^3)$ and $y = \cos^{-1}(\sqrt{1-t^2})$ then $\dfrac{dy}{dx} =$

(a) 1/2 (b) 2/5
(c) 3/2 (d) 1/3

12. If $y = e^{x+e^{x+e^{x+\dots\infty}}}$ then $\dfrac{dx}{dy} =$

(a) $\dfrac{y}{1-y}$ (b) $\dfrac{1}{1-y}$
(c) $\dfrac{y}{1+y}$ (d) $\dfrac{y}{y-1}$

13. The derivative of $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ w.r.t. $\cot^{-1}\left(\dfrac{1-3x^2}{3x-x^2}\right)$ is

(a) 1 (b) $\dfrac{3}{2}$
(c) $\dfrac{2}{3}$ (d) $\dfrac{1}{2}$

14. If $y = x\log\left(\dfrac{x}{a+bx}\right)$, then $x^3\dfrac{d^2y}{dx^2} =$

(a) $x\dfrac{dy}{dx}-y$ (b) $\left(x\dfrac{dy}{dx}-y\right)^2$
(c) $y\dfrac{dy}{dx}-x$ (d) $\left(y\dfrac{dy}{dx}-x\right)^2$

15. If $x = e^t\sin t$, $y = e^t\cos t$, t is a parameter, then $\dfrac{d^2y}{dx^2}$ at (1, 1) is equal to

(a) –1/2 (b) –1/4
(c) 0 (d) 1/2

16. If $\sqrt{(1-x^6)} + \sqrt{(1-y^6)} = a^3(x^3-y^3)$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{x^2}{y^2}\sqrt{\dfrac{1-x^6}{1-y^6}}$ (b) $\dfrac{y^2}{x^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$
(c) $\dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$ (d) None of these

17. If $x = f(m)\cos m - f'(m)\sin m$ and $y = f(m)\sin m + f'(m)\cos m$, then

$\left(\dfrac{dy}{dm}\right)^2 + \left(\dfrac{dx}{dm}\right)^2 =$

(a) $[f(m) + f''(m)]^2$ (b) $[f(m) - f''(m)]^2$

(c) $\{f(m)\}^2 + \{f''(m)\}^2$ (d) $\dfrac{\{f(m)\}^2}{\{f'(m)\}^2}$

18. If $f(x) = \cos^{-1}\left[\dfrac{1-(\log x)^2}{1+(\log x)^2}\right]$, then the value of $f'(e) =$

(a) 1 (b) $\dfrac{1}{e}$

(c) $\dfrac{2}{e}$ (d) $\dfrac{2}{e^2}$

19. If $y = \cfrac{x}{a + \cfrac{x}{b + \cfrac{x}{a + \cfrac{x}{b + \dots}}}}$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{b}{a(b+2y)}$ (b) $\dfrac{b}{b+2y}$

(c) $\dfrac{a}{b(b+2y)}$ (d) None of these

20. The derivative of $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = 0$, is

(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{4}$

(c) $\dfrac{1}{2}$ (d) 1

21. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$, then $xy_2 + \dfrac{1}{2}y_1 - \dfrac{1}{4}y =$

(a) 0 (b) 1

(c) -1 (d) 2

22. $f(x) = \begin{vmatrix} x^3 & x^2 & 3x^2 \\ 1 & -6 & 4 \\ p & p^2 & p^3 \end{vmatrix}$, here p is a constant,

then $\dfrac{d^3}{dx^3}(f(x))$ is

(a) Proportional to x^2 (b) Proportional to x

(c) Proportional to x^3 (d) A constant

23. If $x^2 + y^2 = a^2$ and $k = \dfrac{1}{a}$ then k is equal to

(a) $\dfrac{y''}{\sqrt{1+y'}}$ (b) $\dfrac{|y''|}{\sqrt{(1+y'^2)^3}}$

(c) $\dfrac{2y''}{\sqrt{1+y'^2}}$ (d) $\dfrac{y''}{2\sqrt{(1+y'^2)^3}}$

24. If $y = (x\log x)^{\log\log x}$, then $\dfrac{dy}{dx} =$

(a) $(x\log x)^{\log\log x}\left\{\dfrac{1}{x\log x}(\log x + \log(\log x)) + (\log(\log x))\left(\dfrac{1}{x} + \dfrac{1}{x\log x}\right)\right\}$

(b) $(x\log x)^{x\log x}\log(\log x)\left[\dfrac{2}{\log x} + \dfrac{1}{x}\right]$

(c) $(x\log x)^{x\log x}\dfrac{\log(\log x)}{x}\left[\dfrac{1}{\log x} + 1\right]$

(d) None of these

25. If $f''(x) = -f(x)$, where $f(x)$ is a continuous double differentiable function and $g(x) = f'(x)$.

If $F(x) = \left(f\left(\dfrac{x}{2}\right)\right)^2 + \left(g\left(\dfrac{x}{2}\right)\right)^2$ and

$F(5) = 5$, then $F(10)$ is:

(a) 0 (b) 5

(c) 10 (d) 25

26. If $x^2 + y^2 = 1$, then:

(a) $yy'' - 2(y')^2 + 1 = 0$

(b) $yy'' + (y')^2 + 1 = 0$

(c) $yy'' + (y')^2 - 1 = 0$

(d) $yy'' + 2(y')^2 + 1 = 0$

27. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is constant.

Then $\dfrac{d^3}{dx^3}(f(x))$ at $x = 0$ is

(a) p (b) $p + p^2$

(c) $p + p^3$ (d) Independent of p

28. There exists a function $f(x)$ satisfying $f(0) = 1$, $f'(0) = -1$, $f(x) > 0$ for all x and

(a) $f''(x) < 0$ for all x

(b) $-1 < f''(x) < 0$ for all x

(c) $-2 \le f''(x) \le -1$ for all x

(d) $f''(x) < -2$ for all x

29. If $y^2 = P(x)$, a polynomial of degree 3, then

$2\dfrac{d}{dx}\left(y^3\dfrac{d^2y}{dx^2}\right) =$

(a) $P'''(x) + P''(x)$ (b) $P''(x).\ P'''(x)$

(c) $P(x)\ P'''(x)$ (d) A constant

30. If y is a function of x and $\log(x + y) = 2xy$, then the value of $y'(0)$ is equal to
(a) 1 (b) -1
(c) 2 (d) 0

31. If $f(x) = (\log_{\tan x}(\cot x))(\log_{\cot x}\tan x)^{-1}$, then the value of $f'(1)$ is
(a) 0 (b) 1
(c) $\frac{\pi}{4}$ (d) $\pi/2$.

32. Let $f(x) = |x| + |x - 2|$, then the value of $f'(1)$ is
(a) 2 (b) 0
(c) 1 (d) not exist

33. If $e^x + e^y = e^{x+y}$, then the value of $\left(\frac{dy}{dx}\right)_{/(1,1)}$ is
(a) 0 (b) 1
(c) -1 (d) not defined

34. If $y = f\left(\frac{x-1}{x+1}\right)$ and $f'(x) = x^2 \frac{dy}{dx}$ at $x = 0$ is
(a) 1 (b) 0
(c) 2 (d) -2

35. Let $f(x) = \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x$,
Then the value of $f'\left(\frac{\pi}{4}\right)$ is
(a) 1 (b) 0
(c) 2 (d) $\sqrt{2}$

36. If $x = \sec\theta - \tan\theta$ and $y = \sec^n\theta - \tan^n\theta$ such that $(x^2 + A)\left(\frac{dy}{dx}\right)^2 = n^2(y^2 + B)$, then the value of $a + b + 2$ is
(a) 4 (b) 6
(c) 8 (d) 10

37. Let $f(x) = \log_x(\log_e x)$, then $f'(e)$ is
(a) e (b) $-e$
(c) $1/e$ (d) $1/e^2$.

38. Let $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all x in R, where $f'(1) = a$, $f''(2) = b$, $f'''(3) = c$ then the value of a + $b + c + 10$ is
(a) 10 (b) 11
(c) 12 (d) 13

39. If $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ and $y = xf(x)$, then the value of $y'(1)$ is
(a) 14 (b) 7/8
(c) 1 (d) 10

40. Let $f(x)$
$= \cos^{-1}\left(\frac{2\cos x - 3\sin x}{\sqrt{13}}\right) + \sin^{-1}\left(\frac{2\sin x + 3\cos x}{\sqrt{13}}\right)$,
then the value of $f'(2012)$ is
(a) 0 (b) 2
(c) 1 (d) -1

41. The differential co-efficient of the function $f(x) = |x - 1| + |x - 3|$ at $x = 2$ is
(a) 1 (b) 3
(c) 0 (d) not defined

42. Let $f(x) = \max\left\{\sin x, \frac{1}{2}, \cos x\right\}$. Then the value of $f'\left(\frac{\pi}{4}\right)$ is
(a) 1 (b) 0
(c) -1 (d) not defined

43. If $x^m \cdot y^n = (x + y)^{m+n}$, then $\frac{dy}{dx}$ is
(a) $\frac{1}{x} + \frac{1}{y}$ (b) $\frac{1}{x} - \frac{1}{y}$
(c) $\frac{y}{x}$ (d) $\frac{1}{x} - \frac{1}{y}$

44. If y is a function of x and $\log(x + y) = 2xy$, then the value of $y'(0)$ is
(a) 1 (b) -1
(c) 2 (d) 0

45. Let $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$. Then the value of $f'(1)$ is
(a) 0 (b) 1
(c) -1 (d) not defined

46. Let $y = y(x)$ and it follows the relations $x\cos y + y\cos x = \pi$, then $y''(0)$ is
(a) 1 (b) -1
(c) π (d) $-\pi$

47. If $xe^{xy} = y + \sin^2 x$, then the value of $y'(0)$ is
(a) -1 (b) 2
(c) 0 (d) 1

48. Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.
Then the value of $y'(1)$ is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) $-\frac{\pi}{2}$ (d) not defined

49. If $e^{\tan^{-1}\left(\frac{y}{x}\right)} = \sqrt{x^2 + y^2}$, then $y''(0)$ is
(a) $-2e^{\frac{\pi}{2}}$ (b) $-2e^{-\frac{\pi}{2}}$
(c) $-2e^{\pi}$ (d) $-2e^{-\pi}$.

50. Let g be a differentiable function of x.
If $f(x) = \frac{g(x)}{x^2}$, $x > 0$, such that $g(2) = 3$, $g'(2) = -2$, then the value of $f'(2)$ is

(a) −4/5 (b) 4/5
(c) −5/4 (d) 5/4

51. If $x^y = e^{x-y}$, then the value of $\frac{dy}{dx}$ at (1, 2) is
(a) −1 (b) 2
(c) 1 (d) −2

52. Let $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$. If the differential co-efficients of $f(x)$ at $x = -2, 0, 2$ are a, b and c respectively, then the value of $a + b + c + \frac{4}{5}$ is
(a) 1 (d) 2
(c) 3 (d) 4

53. Let $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.
If the differential co-efficients of $f(x)$ at $x = -1$ and 1 are p and q respectively, then the value of $p + q + 10$ is
(a) 0 (b) 5
(c) 10 (d) 15

54. Let $f(x) = \min\{|x + 1|, |x|, |x - 1|\}$.
Then the value of $f'\left(\frac{1}{2}\right)$ is
(a) 1 (b) –1
(c) 0 (d) not defined

55. Let $f(x) = \begin{cases} 2^x, & x < 1 \\ 3 - x & x \ge 1 \end{cases}$.
Then the value of $f'(1)$ is
(a) $2\log 2$ (b) −1
(c) $-2\log 2$ (d) not defined

56. Let $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ and $x\sin\theta - y\cos\theta = 0$, then the value of $\frac{dy}{dx}$ at (2, 1) is
(a) 2 (b) 1
(c) −2 (d) −1

57. Let $y = (\sin^2 5° + \sin^2 10° + ... + \sin^2 90°)\, x$, then the value of $\frac{dy}{dx}$ at (2011) is
(a) $6\frac{1}{2}$ (b) $7\frac{1}{2}$
(c) $8\frac{1}{2}$ (d) $9\frac{1}{2}$...

58. Let $y = \left(1 + \tan\left(\frac{\pi}{8} - x\right)\right)\left(\left(1 + \tan\left(x + \frac{\pi}{8}\right)\right)\right)$.
Then the value of $y'(2012)$ is
(a) 2012 (b) 2010
(c) 0 (d) 1

59. Let $y = (1 + \tan(x + 1°))(1 + \tan(x + 2°))$.
$(1 + \tan(x + 3°))(1 + \tan(42° - x))$.
$(1 + \tan(43° - x))(1 + \tan(44° - x))$.
Then the value of $\frac{d}{dx}((1 + y)^{1+y})$ is
(a) 1 (b) 2
(c) −1 (d) 0

60. If $f(x) = \cot^{-1}\left(\frac{x^x - x^{-x}}{2}\right)$, then $f'(1)$ is
(a) 1 (b) 2
(c) −1 (d) 0

61. If $\frac{\cos^4\alpha}{x} + \frac{\sin^4\alpha}{y} = \frac{1}{x+y}$, then
(a) $y = x\tan^2\alpha$ (b) $y = x\cot^2\alpha$
(c) $\frac{dy}{dx} = \tan^2\alpha$ (d) $\frac{dy}{dx} = \cot^2\alpha$

62. If $\sqrt{x+y} + \sqrt{y-x} = c$, then
(a) $\frac{dy}{dx} = \frac{2x}{c^2}$ (b) $\frac{d^2y}{dx^2} = \frac{2}{c^2}$
(c) $\frac{d^2y}{dx^2} = \frac{4}{c^2}$ (d) $\frac{dy}{dx} = \frac{4x}{c^2}$

63. If $y = \frac{x^4 - x^2 + 1}{x^2 + \sqrt{3}x + 1}$ and $\frac{dy}{dx} = cx + d$ then the value of $(c - d)$ is
(a) $\cot\left(\frac{\pi}{8}\right)$ (b) $\cot\left(\frac{5\pi}{12}\right)$
(c) $\tan\left(\frac{5\pi}{12}\right)$ (d) $\tan\left(\frac{5\pi}{8}\right)$

64. Let $f(x) = |x - 1| + |x - 2|$. Then
(a) $f'(1\cdot 5) = 0$ (b) $f'(2)$ = not defined
(c) $f'(3) = 2$ (d) $f'(0) = -2$

65. Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$. Then
(a) $f'(2) = -\frac{2}{5}$ (b) $f'(0) = 2$
(c) $f'\left(\frac{1}{2}\right) = \frac{8}{5}$ (d) $f'(1) = 0$

Level III

(Problems for JEE-Advanced)

1. If $y = \left\{\log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)\right\}^2 + k\log\left(x + \sqrt{x^2 - a^2}\right)$
then prove that $(x^2 - a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2a$

2. If $(a + bx)e^{\frac{y}{x}} = x$, then prove that
$$x^3 \cdot \left(\frac{d^2y}{dx^2}\right) = \left(x\frac{dy}{dx} - y\right)^2$$

3. $y = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left[\left\{\sqrt{\frac{a-b}{a+b}}\right\} \tan\frac{x}{2}\right]$, then prove that

(i) $\frac{dy}{dx} = \frac{1}{a + b\cos x}$.

(ii) $\frac{d^2y}{dx^2} = \frac{b\sin x}{(a + b\cos x)^2}$.

4. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$ then prove that $(x^2 + 4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2 + 4)$.

5. If $y = (\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ then prove that $\left(\frac{dy}{dx}\right)$ at $x = \frac{\pi}{4}$ is $8\left(\frac{4}{\pi^2 + 16} - \frac{1}{\log_e 2}\right)$

6. If $y = \log_n(|\cos 4x|) + |\sin x|$, where $x = \sec 2x$, find $\frac{dy}{dx}$ at $= -\frac{\pi}{4}$

7. If $y = 2\tan^{-1}\left(\frac{x\sqrt{2}}{1 - x^2}\right) + \log\left(\frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2}\right)$

then prove that $\frac{dy}{dx} = \frac{4\sqrt{2}}{1 + x^4}$

8. Prove that If $\prod_{r=1}^{n} \cos\left(\frac{x}{2^r}\right) = \frac{\sin x}{2^n \cdot \sin\left(\frac{x}{2^n}\right)}$,

(i) $\sum_{r=1}^{n} \frac{1}{2^r}\tan\left(\frac{x}{2^r}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - \cot x$

(ii) $\sum_{r=1}^{n} \left(\frac{1}{2^r}\sec x\left(\frac{x}{2^r}\right)\right) = \operatorname{cosec} x^n - \frac{1}{2^{2n}} \operatorname{cosec}^n \left(\frac{x}{2^{2n}}\right)$

9. If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, then prove that
$$p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}.$$

10. If $z = \cos^7 x$, $y = \sin x$, then prove that,
$$\frac{d^3z}{dy^3} = \frac{105}{4}\sin 4x$$

11. If $y^2 = p(x)$ is a polynomial of degree $n \geq 3$, then prove that $2 \cdot \frac{d}{dx}\left\{y^3 \cdot \frac{d^2y}{dx^2}\right\} = p(x).\ p'''(x)$.

12. If $\varphi(x) = f(x) \cdot g(x)$, where $f'(x) \cdot g'(x) = c$ then prove that $\frac{\varphi''}{\varphi} = \frac{f''}{f} + \frac{g''}{g} + \frac{2c}{f.g}$.

13. If $x < 1$, using differentiation, prove that
$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots \text{ to } \infty = \frac{1}{1-x}$$

14. If $a = x\sin\theta + y\cos\theta c$ and $b = x\cos\theta - y\sin\theta i$ then prove that,
$$\frac{d^3x}{d\theta^3} \cdot \frac{d^2y}{d\theta^2} - \frac{d^2y}{d\theta^2} \cdot \frac{d^3y}{d\theta^3} = a^2 + b^2.$$

15. If $x^2 + y^2 + z^2 - 2xyz = 1$ then prove that
$$\frac{dx}{\sqrt{1 - x^2}} + \frac{dy}{\sqrt{1 - y^2}} + \frac{dz}{\sqrt{1 - z^2}} = 0$$

16. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$ then prove that,
$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$$

17. If f, g, h are differential functions of x and
$$\Delta = \begin{vmatrix} f & g & h \\ (xf)^I & (xg)^I & (xh)^I \\ (x^2f)^{II} & (x^2g)^{II} & (x^2h)^{II} \end{vmatrix}$$ then prove that
$$\Delta^I = \begin{vmatrix} f & g & h \\ f^I & g^I & h^I \\ (x^3f^{II})^I & (x^3g^{II})^I & (x^3h^{II})^I \end{vmatrix}$$

18. $\frac{\cos^4\alpha}{x} + \frac{\sin^4\alpha}{y} = \frac{1}{x+y}$, prove that $\frac{dy}{dx} = \tan^2\alpha$

19. If $y = \log\left(\sqrt{\frac{x^2 + x + 1}{x^2 - x + 1}}\right) + \frac{1}{2\sqrt{3}}\left\{\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right\}$

then prove that, $\frac{dy}{dx} = \frac{1}{x^4 + x^2 + 1}$.

20. If $y = \tan^{-1}\left(\frac{1}{x^2 + x + 1}\right) + \tan^{-1}\left(\frac{1}{x^2 + 3x + 3}\right) + \tan^{-1}\left(\frac{1}{x^2 + 5x + 7}\right) + \ldots$ to n terms.

Then prove that, $\frac{dy}{dx} = \frac{1}{1 + (x+n)^2} - \frac{1}{x^2 + 1}$

21. If $y = \tan^{-1}\left(\frac{1}{\sin^2 x + \sin x + 1}\right)$

$+ \tan^{-1}\left(\frac{1}{\sin^2 + 3\sin x + 3}\right)$

$+ \tan^{-1}\left(\frac{1}{\sin^2 x + 5\sin x + 7}\right)$

+ ... to n-terms. then prove that

$$\frac{dy}{dx} = \frac{\cos x}{1 + (\sin x + n)^2} - \frac{\cos x}{1 + (\sin x)^2}$$

22. For a function $y = f(x)$, then prove that

$$\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} + \frac{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}} = 0.$$

23. If $y = \cos^{-1}(8x^4 - 8x^2 + 1)$, then prove that

$$\frac{dy}{dx} + \frac{4}{\sqrt{1 - x^2}} = 0.$$

24. If $f(x) = \frac{a}{a + x} + \frac{b}{b + x} - \frac{c}{c + x} - \frac{d}{d + x}$ and $f(x)$ is divisible by x^2, then find the value of $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} - \frac{1}{d}$.

25. If $y = (\sin^{-1} x)^2 + (\cos^{-1} x)^2$, then prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 4$$

26. If $\left(\frac{x + b}{2}\right) = a\tan^{-1}(a\ln y)$, $a > 0$ show that

$$y\frac{d^2y}{dx^2} - (y\ln y)\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$$

27. If $y = \frac{1}{x}$, then show that $\frac{dy}{\sqrt{1 + y^4}} + \frac{dx}{\sqrt{1 + x^4}} = 0$

28. If $2x = y^{1/3} + y^{-1/3}$, prove that

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$$

29. If a curve is represented parametrically by the equations $x = f(t)$, $y = g(t)$, then prove that

$$\left(\frac{d^2y}{dx^2}\right)/\frac{d^2x}{dy^2} = -\left(\frac{g'(t)}{f'(t)}\right)^3$$

30. If $y = x^5(\cos(\ln x) + \sin(\ln x))$, prove that

$$x^2\frac{d^2y}{dx^2} - 9x\frac{dy}{dx} + 26y = 0$$

Level II

(Tougher Problems for JEE-Advanced)

1. If f is a function from R to R such that

$$f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$$

for all x in R, show that $f(2) + f(0) = f(1)$

2. If $y = \frac{ax + b}{x^2 + c}$, then show that

$$\left(2x\frac{dy}{dx} + y\right)\frac{d^3y}{dx^3} = 3\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right)\frac{d^2y}{dx^2}$$

3. If $y = \tan^{-1}\left(\frac{x}{1.2 + x^2}\right) + \tan^{-1}\left(\frac{x}{2.3 + x^2}\right)$

$+ \tan^{-1}\left(\frac{x}{3.4 + x^2}\right)$ + ... to n-terms then prove that

$$\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{n + 1}{x^2 + (n + 1)^2}$$

4. If $f(x) = (ax + b)\sin x + (cx + d)\cos x$, find the values of a, b, c, and d such that

$$f'(x) = x\cos x$$

5. If $g(x) = (ax^2 + bx + c)\sin x + (dx^2 + ex + f)\cos x$ then find the values of a, b, c, d, e and f such that $g'(x) = x^2 \sin x$.

6. If $y = \frac{ax + b}{Ax + B}$ and $z = \frac{ay + b}{Ay + B}$, prove that

$$\frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = 0 = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$$

7. Find the value of

$$\lim_{x \to 0}\left(\frac{1 - \cos x\cos 2x\cos 3x}{x^2}\right)$$

8. If $y = e^{\tan^{-1}x}$, show that

$$(x^2 + 1)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$$

9. If $y = x^{n-1}\ln x$, then prove that

$$x^2\left(\frac{d^2y}{dx^2}\right) + (3 - 2n)x\frac{dy}{dx} + (n - 1)^2 y = 0$$

10. If $y = (C_1 + C_2 x)\sin x + (C_3 + C_4 x)\cos x$,

show that $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$

11. Let $f(x) = x^3 f(1) + x^2 f'(2) + xf''(3) + f'''(4)$
Then prove that $f(x)$ is an independent of x.

12. If $y = \cos^{-1}\left\{\frac{7}{2}(1 + \cos 2x) + \sqrt{\sin^2 x - 48\cos^2 x}\,\sin x\right\}$

for all x in $\left(0, \frac{\pi}{2}\right)$ then prove that

$$\frac{dy}{dx} = 1 + \frac{\sin x}{\sqrt{\sin^2 x - 48\cos^2 x}}.$$

13. If $x = \tan\left(\frac{y}{2}\right) - \left(\frac{(1 + \tan(y/2))^2}{\tan(y/2)}\right)$, prove that $2\frac{dy}{dx} = -\sin y(1 + \sin y + \cos y)$

14. If $y = \cos^{-1}\left(\sqrt{\frac{\cos 3x}{\cos^3 x}}\right)$, prove that $\frac{dy}{dx} = \sqrt{\frac{6}{\cos 2x + \cos 4x}}$

15. If $\sqrt{x^2 + y^2} = ae^{\tan^{-1}}x$, where $a > 0$, $y(0) \neq 0$ then find the value of $y''(0)$.

Integer Type Questions

1. If $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$, $x > 0$, find the value of $y'(0)$.
2. If $y = y(x)$ and it follows the relation $4xe^{xy} = y + 5\sin^2 x$, find $y'(0)$.
3. If $\sqrt{x + y} + \sqrt{x - y} = c$, find the value of $\left(c^2 \frac{d^2y}{dx^2} + 3\right)$.
4. Let $f'(x) = g(x)$ and $g'(x) = f(x)$ for all x
   Also, $f(3) = 5$ and $f'(3) = 4$. Then find the value of $[f(10)]^2 - [g(10)]^2$.
5. If $\sin x = \frac{2t}{1 + t^2}$ and $\cot y = \frac{1 - t^2}{2t}$, find the value of $\left(\frac{d^2y}{dx^2} + 3\right)$.
6. If $2x = (y^{1/3} + y^{-1/3})$, find the value of
   $$\left(\frac{x^2 - 1}{y}\right)\frac{d^2y}{dx^2} + \frac{x}{y}\frac{dy}{dx}.$$
7. Let g be a differentiable function of x such that $f(x) = \frac{g(x)}{x}$. If $g'(2) = 6$, $g(2) = 4$, find the value of $f'(2)$.
8. Suppose that $F(x) = f(g(x))$ and $g(3) = 6$
   $g'(3) = 4$, $f'(3) = 2$, $f'(6) = 2$. Find $F'(3)$.
9. If $f(x) = x^2 + x^3$ and g is the inverse of f then find the value of $(5g'(2) + 3)$.
10. If $x^y = e^{x-y}$, then find the value of $2\left(\frac{dy}{dx}\right)$ at $x = e$.
11. Let f be twice differentiable function such that
    $$f'(x) = g(x), f''(x) = -f(x).$$
    If $h(x) = \{f(x)^2\} + \{g(x)\}^2$ and $h(5) = 7$, then find the value of $h(2016)$.
12. If $f''(x) = 10$, $f'(1) = 6$ and $f(1) = 4$, then find the value of $f(0) + 2$.
13. There is a polynomial $P(x) = ax^3 + bx^2 + cx + d$ such that $P(0) = -2 = P(1)$, $P'(0) = -1$ and $P''(0) = 10$, then find the value of $(a + b + c + d + 10)$.

Comprehensive Link Passages

Passage I

If f and g both are differentiable functions and $F = f_0 g$ is the composite function defined by $F(x) = f(g(x))$ then f is differentiable and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

(i) Let $f(x) = \sin x$, $g(x)[x + 1]$ and $g\{f(x)\} = h(x)$, where {.} is the G.I.F, then the value of $h'\left(\frac{\pi}{2}\right)$ is

(a) non-existent (b) 1
(c) −1 (d) None

(ii) If $f(x) = |3 - x|$ and $g(x) = f\{f(x)\}$, then $g'(2)$ is

(a) 1 (b) −1
(c) 3 (d) −3

(iii) If $y = f\left(\frac{x - 1}{x + 1}\right)$ and $f'(x) = x^2$, then $\frac{dy}{dx}$ at $x = 0$ is

(a) 0 (b) 2
(c) −2 (d) 1

Passage II

Let x be the independent variable and y be a dependent variable defined as $y = f(x)$.

Assuming $y^{1/m} + y^{-1/m} = 2x$, where $m \in R - \{0\}$, then

(i) The value of y is given by

(a) $(x - \sqrt{x^2 - 1})^m$ (b) $(x + \sqrt{x^2 - 1})^m$
(c) $(x + \sqrt{x^2 + 1})^m$ (d) $(x - \sqrt{x^2 + 1})^m$

(ii) $\frac{dy}{dx}$ is equal to

(a) $\frac{my}{\sqrt{x^2 + 1}}$ (b) $\frac{my}{\sqrt{x^2 - 1}}$
(c) $\frac{\sqrt{x^2 - 1}}{my}$ (d) $-\frac{\sqrt{x^2 - 1}}{my}$

(iii) $(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} =$

(a) my (b) m^2/y
(c) m^2y (d) my^2

Passage III

Since the graph of $f(x) = |x - \alpha|$ has a turning point at $x = \alpha$, so $f(x) = |x - \alpha|$ is not differentiable at $x = \alpha$ and $f'(x)\ \frac{|x-\alpha|}{x-\alpha}$, $\alpha \neq 0$, for example $f(x) = |x|$ is not differentiable at $x = 0$ and $f'(x) = \frac{|x|}{x}$, $x \neq 0$.

(i) If $f(x) = e^{||x|-1|}$, then $f'\left(\frac{1}{2}\right)$ is equal to

(a) $\sqrt{e}$ (b) $1/\sqrt{e}$

(c) $-\sqrt{e}$ (d) $-1/\sqrt{e}$

(ii) If $f(x) = |x|^{|\tan x|}$, then $f'\left(-\frac{\pi}{6}\right)$ is equal to

(a) $\left(\frac{\pi}{6}\right)^{1/\sqrt{3}}\left\{\frac{2\sqrt{3}}{\pi} - \frac{4}{3}\log\frac{6}{\pi}\right\}$

(b) $-\left(\frac{\pi}{6}\right)^{1/\sqrt{3}}\left\{\frac{2\sqrt{3}}{\pi} - \frac{4}{3}\log\frac{6}{\pi}\right\}$

(c) $\left(\frac{\pi}{6}\right)^{1/\sqrt{3}}\left\{\frac{2\sqrt{3}}{\pi} + \frac{4}{3}\log\frac{6}{\pi}\right\}$

(d) None

(iii) If $f(x) = |\cos x| + |\sin x|$, then $f'\left(\frac{2\pi}{3}\right)$ is equal to

(a) $\frac{1-\sqrt{3}}{2}$ (b) 0

(c) $\frac{\sqrt{3}-1}{2}$ (d) None

(iv) If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{4}\right)$ is equal to

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) 0 (d) None

Matrix Match

Given below are Matching type questions, with two columns (each having some items) each. Each item of column I has to be matched with the items of Column II, by encircling the correct match(es).

> ***Note:*** An item of Column I can be matched with more than one items of Column II. All the items of Column II have to be matched.

1. **Match the following columns:**

	Column I		Column II
(A)	If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ then $\frac{dy}{dx}$ is	(P)	$\frac{y}{x}$
(B)	If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ then $\frac{dy}{dx}$ is	(Q)	$xy\,(1 + 2\log x)$
(C)	If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is	(R)	$\frac{4}{1+x^2}$
(D)	If $x^p y^q = (x+y)^{p+q}$, then $\frac{dy}{dx}$ is	(S)	$\frac{\log x}{(1+\log x)^2}$
(E)	If $y = (x^x)^x$, then $\frac{dy}{dx}$ is	(T)	$-\frac{x}{\sqrt{1-x^4}}$

2. **Match the following columns:**

	Column I		Column II
(A)	If $x = t - \frac{1}{t}$ and $y = t + \frac{1}{t}$ then dy/dx is	(P)	$\sin 2y\,(\cot x - 2\sin 2x)$
(B)	If $y = \left[\cos^2\left(\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right)\right]$ then dy/dx is	(Q)	x/y
(C)	If $y = \sqrt{\tan y} = e^{\cos 2x} \times \sin x$ then dy/dx is	(R)	1/2
(D)	If $xy = (x+y)^p$ and $dy/dx = y/x$, then p is	(S)	2

Questions asked in Roorkee-JEE Exams

1. Differentiate $\sin^{-1}\left(\frac{1-x}{1+x}\right)$ w.r.t. $\sqrt{x}$

[Roorkee-JEE, 1984]

2. Find $\frac{dy}{dx}$, if $(\tan^{-1}x)^y + y^{\cot x} = 1$

[Roorkee-JEE, 1987]

3. If $y = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}\right)\right)$ then prove that $\frac{d^2y}{dx^2} = \frac{b\sin x}{(a+b\cos x)^2}$

[Roorkee-JEE, 1988]

4. Find $\frac{dy}{dx}$, when

$$\sqrt{1-y^2} + \sqrt{1-t^2} = a(y-t) \text{ and}$$

$$x = \sin^{-1}\left(t\sqrt{1-t} + \sqrt{t}\sqrt{1-t^2}\right)$$

[Roorkee-JEE, 1990]

5. If f, g and h are differentiable functions of x and

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}, \text{ prove that}$$

$$\Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^2 f'')' & (x^2 g'')' & (x^2 h'')' \end{vmatrix}$$

[Roorkee-JEE, 1991]

6. If $\sqrt{1 - x^6} + \sqrt{1 - y^6} = a^3(x^3 - y^3)$ prove that

$$\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1 - y^6}{1 - x^6}}$$

[Roorkee-JEE, 1994]

7. Find the differential co-efficient of $\log_{(1-\sqrt{x})}(\sin^{-1}(1 - \sqrt{x}))$ with respect to $2^{2^{(1-\sqrt{x})}}$

[Roorkee-JEE, 1996]

8. Find the differential co-efficient of $f(x) = \log_x(\sin(x^2)) + (\sin x^2)^{\log_e x}$ with respect to $\sqrt{x + 1}$

[Roorkee-JEE, 1997]

Questions asked in Previous Years' IIT-JEE Exams

1. Find the derivative of $\sin(x^2 + 1)$ w.r.t x from first principle. **[IIT-JEE, 1978]**

2. Find the derivative of

$$f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5} & : x \neq 1 \\ -\dfrac{1}{3} & : x = 1 \end{cases} \quad \text{at } x = 1$$

[IIT-JEE, 1979]

3. Given $y = \dfrac{5x}{3\sqrt{(1 - x)^2}} + \cos^2(2x + 1)$.

Find $\dfrac{dy}{dx}$. **[IIT-JEE, 1980]**

4. Let $y = e^{x\sin x^3} + (\tan x)^x$, find $\dfrac{dy}{dx}$

[IIT-JEE, 1981]

5. Let f be a twice differentiable function such that $f''(x) = -f(x), f'(x) = g(x)$ and $h(x) = [f(x)]^2 + [g(x)]^2$, find $h(10)$ if $h(5) = 11$ **[IIT-JEE, 1982]**

6. If $y = f\left(\dfrac{2x - 1}{x^2 + 1}\right)$ and $f'(x) = \sin(x^2)$ then $\dfrac{dy}{dx} = ...$

[IIT-JEE, 1982]

7. If $(a + bx)e^{y/x} = x$, then prove that

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

[IIT-JEE, 1983]

8. Find the derivative w.r.t x of the function $(\log_{\sin x}\cos x)(\log_{\cos x}\sin x)^{-1} + \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right)$ at $x = \dfrac{\pi}{4}$

[IIT-JEE, 1984]

9. If $f(x) = \log x(\ln x)$, then $f'(x)$ at $x = e$ is...

[IIT-JEE, 1985]

10. The derivative of $\sin^{-1}\left(-\dfrac{1}{2x^2 - 1}\right)$ w.r.t $\sqrt{1 - x^2}$ at $x = \dfrac{1}{2}$ is...

[IIT-JEE, 1986]

11. No questions asked in 1987.

12. If $y^2 = P(x)$, a polynomial of degree 3, then $2\dfrac{d}{dx}\left(y^3\dfrac{d^2y}{dx^2}\right)$ equals

(a) $P'''(x) + P'(x)$ (b) $P''(x)\cdot P'''(x)$
(c) $P(x)\cdot P'''(x)$ (d) a constant.

[IIT-JEE, 1988]

13. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$ then prove that $(x^2 + 4)\left(\dfrac{dy}{dx}\right)^2 = n^2(y^2 + 4)$

[IIT-JEE, 1989]

14. If $f(x) = |x - 2|$ and $g(x) = f(f(x))$, then $g'(x) = ...$ for $x > 2$

[IIT-JEE, 1990]

15. Let $f(x)$ be a quadratic expression which is +ve for all real values of x.
If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x,

(a) $g(x) < 0$ (b) $g(x) > 0$
(c) $g(x) = 0$ (d) $g(x) \geq 0$

[IIT-JEE, 1990]

16. Find $\dfrac{dy}{dx}$ at $x = -1$, when

$(\sin y)^{\sin(\pi x/2)} + \dfrac{\sqrt{3}}{2}\sec^{-1}(2x) + 2^x\tan(\ln(x + 2)) = 0$

[IIT-JEE, 1991]

17. No questions asked in 1992.

18. The derivative of an even function is always an odd function. Is it true or false? **[IIT-JEE, 1993]**

19. If $y = (\sin x)^{\tan x}$, then $\dfrac{dy}{dx}$ is

(a) $\sin x^{\tan x}(1 + \sec^2 x \log\sin x)$
(b) $\tan x(\sin x)^{\tan x - 1}\cdot\cos x$
(c) $\sin x^{\cos x}\cdot\sec^2 x \log\sin x$
(d) $\tan x\cdot(\sin x)^{\tan x - 1}$

[IIT-JEE, 1994]

20. No questions asked in 1995.

21. If $xe^{xy} = y + \sin^2 x$, then find $\frac{dy}{dx}$ at $x = 0$

[IIT-JEE, 1996]

22. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

where p is a constant.

Then $\frac{d^3}{dx^3}(f(x))$ at $x = 0$ is

(a) p (b) $p + p^3$

(c) $p - p^3$ (d) independent of p.

[IIT-JEE, 1997]

23. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$, then prove that

$$\frac{dy}{dx} = \frac{y}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$$

[IIT-JEE, 1998]

24. No questions asked in 1999.

25. If $x^2 + y^2 = 1$, then

(a) $yy'' - 2(y')^2 + 1 = 0$

(b) $yy'' + (y')^2 + 1 = 0$

(c) $yy'' - (y')^2 - 1 = 0$

(d) $yy'' + 2(y')^2 + 1 = 0$

[IIT-JEE, 2000]

26. Let $f: (0, \infty) \to R$ and $F(x) = \int_0^x f(t)\,dt$

If $F(x^2) = x^2(1 + x)$, then $f(4)$ is

(a) 5/4 (b) 7

(c) 4 (d) 2

[IIT-JEE, 2001]

27. No questions asked in between 2002-2003.

28. If y is a function of x and $\log(x + y) = 2xy$, then the value of $y'(0)$ is

(a) 1 (b) –1

(c) 2 (d) 0

[IIT-JEE, 2004]

29. No questions asked in between 2005-2006.

30. $\frac{d^2x}{dy^2}$ is

(a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-1}$

(c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

[IIT-JEE, 2007]

31. Let $f(x) = 2 + \cos x$ for all real x.

Assertion (A): For each real t, there exist a point c in $[t, t + \pi]$ such that $f'(c) = 0$

Reason (R): Because $f(t) = f(t + 2\pi)$ for each real t.

[IIT-JEE, 2007]

32. Let f and g be real valued functions defined on $(-1, 1)$ such that $g''(x)$ is continuous $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$ and $f(x) = g(x)\sin x$

Assertion (A): $\lim_{x\to 0}(g(x)\cot x - g(0)\operatorname{cosec} x) = f''(0)$

Reason (R): Because $f'(0) = g(0)$

[IIT-JEE, 2008]

33. Let $g(x) = \log(f(x))$, where $f(x)$ is twice differentiable function on $(0, \infty)$ such that $f(x + 1) = xf(x)$. Then $N = 1, 2, 3...$

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(a) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N+1)^2}\right\}$

[IIT-JEE, 2008]

34. No questions asked in between 2009-2010

35. Let $f: (0, 1) \to R$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

(a) f is not invertible on (0, 1)

(b) $f \neq f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f(0)}$

(c) $f = f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f(0)}$

(d) f^{-1} is differentiable on (0, 1),

[IIT-JEE, 2011]

36. Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is...

[IIT-JEE, 2011]

37. No questions asked in between 2012-2014.

ANSWERS

LEVEL II

1. (a)	2. (b)	3. (b)	4. (d)	5. (c)
6. (d)	7. (c)	8. (c)	9. (a)	10. (c)
11. (d)	12. (a)	13. (c)	14. (b)	15. (a)
16. (c)	17. (a)	18. (b)	19. (a)	20. (b)
21. (a)	22. (d)	23. (b)	24. (a)	25. (c)
26. (b)	27. (d)	28. (a)	29. (c)	30. (a)
31. (a)	32. (b)	33. (c)	34. (c)	35. (d)
36. (d)	37. (c)	38. (b)	39. (c)	40. (b)
41. (d)	42. (d)	43. (c)	44. (a)	45. (d)
46. (a)	47. (d)	48. (d)	49. (b)	50. (c)
51. (a)	52. (d)	53. (c)	54. (d)	55. (d)
56. (c)	57. (d)	58. (c)	59. (d)	60. (c)
61. (a, c)	62. (a, b)	63. (a, c)	64. (a, b, c, d)	
65. (a, b, c)				

INTEGER TYPE QUESTIONS

1. 2	2. 4	3. 5	4. 9	5. 3
6. 9	7. 2	8. 8	9. 4	10. 1
11. 7	12. 5	13. 8.		

COMPREHENSIVE LINK PASSAGES

Passage I :	(i) (b)	(ii) (d)	(iii) (a)	
Passage II :	(i) (c)	(ii) (b)	(iii) (c)	
Passage III :	(i) (c)	(ii) (b)	(iii) (c)	(iv) (d)

MATRIX MATCH

1. (A)→(P, S), (B)→(R), (C)→(Q, T)
2. (A)→(P, S), (B)→(Q, T), (C)→(P, R)

HINTS AND SOLUTIONS

Level I

1. We have $f'(x)$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{e^{x+h}-e^x}{h}\right)$$

$$= \lim_{h\to 0}\left(e^x\cdot\frac{e^h-1}{h}\right)$$

$$= \lim_{h\to 0} e^x\cdot\left(\frac{e^h-1}{h}\right)$$

$$= e^x$$

2. We have $f'(x)$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log(x+h)-\log x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(\frac{x+h}{h}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(1+\frac{h}{x}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{x}\cdot\left(\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\right)\right)$$

$$= 1/x$$

3. We have $f'(x)$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log_a(x+h)-\log_a x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log_a\left(\frac{x+h}{x}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log_a\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x}\right)$$

$$= \lim_{h\to 0}\frac{1}{x}\left(\frac{\log_e\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot\log_e a}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{x\cdot\log_e a}\left(\frac{\log_e\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\right)\right)$$

$$= \frac{1}{x\cdot\log_e a}$$

4. We have $f'(x)$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}\cdot 2}\right)$$

$$= \cos x$$

5. We have $f'(x)$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\tan^{-1}(x+h)-\tan^{-1}x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\tan^{-1}\left(\frac{h}{1+x(x+h)}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\tan^{-1}\left(\frac{h}{1+x(x+h)}\right)}{\frac{h}{1+x(x+h)}\cdot(1+x(x+h))}\right)$$

$$= \frac{1}{1+x^2}$$

6. We have $f'(x)$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{e^{\sin(x+h)}-e^{\sin x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{e^{\sin}(e^{\sin(x+h)-\sin x}-1)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{e^{\sin}(e^{\sin(x+h)-\sin x}-1)}{(\sin(x+h)-\sin x)}\times\frac{\sin(x+h)-\sin x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{e^{\sin}\times(\sin(x+h)-\sin x)}{h}\right)$$

$$= (e^{\sin}\times\cos x)$$

7. We have $f'(x)$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin^2(x+h)-\sin(x)^2}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cos\left(\frac{(x+h)^2+x^2}{2}\right)\sin\left(\frac{(x+h)^2-x^2}{2}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(\frac{(x+h)^2+x^2}{2}\right)\sin\left(\frac{(2hx+h^2)}{2}\right)}{\left(\frac{2hx+h^2}{2}\right)\left(\frac{2h}{2hx+h^2}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(\frac{(x+h)^2+x^2}{2}\right)\sin\left(\frac{(2hx+h^2)}{2}\right)}{\left(\frac{2hx+h^2}{2}\right)\left(\frac{2}{2x+h}\right)}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cos\left(\frac{(x+h)^2+x^2}{2}\right)}{\left(\frac{1}{2x+h}\right)}\right)$$

$$= 2x\cos(x^2).$$

8. We have $y = \log_x x + 10 = 1 + 10 = 11$

$$\Rightarrow \quad \frac{dy}{dx} = 0.$$

9. We have $y = 5^{\log_3 x} - x^{\log_3 5}$

$$\Rightarrow \quad y = x^{\log_3 x} - x^{\log_3 5} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = 0$$

10. We have $y = \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2} - \frac{4}{x^3} - \frac{9}{x^4}$$

11. We have $y = \sqrt{x} + x\sqrt{x} + x^2\sqrt{x} + x^3\sqrt{x}$

$$= x^{\frac{1}{2}} + x^{\frac{3}{2}} + x^{\frac{5}{2}} + x^{\frac{7}{2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{3}{2}\sqrt{x} + \frac{5}{2}x\sqrt{x} + \frac{7}{2}x^2\sqrt{x}$$

12. We have

$$y = \left(1+\tan\left(\frac{\pi}{8}-x\right)\right)\left(1+\tan\left(x+\frac{\pi}{8}\right)\right)$$

$\Rightarrow \quad y = 2$

$\Rightarrow \quad \frac{dy}{dx} = 0$

13. We have $y = \frac{\sec x + \tan x - 1}{\sec x - \tan x + 1}$

$$= \frac{\sec x + \tan x - (\sec x^2 - \tan x^2)}{\sec x - \tan x + 1}$$

$$= \frac{(\sec x + \tan x)(1 + \sec x - \tan x)}{(\sec x - \tan x + 1)}$$

$$= (\sec x + \tan x)$$

Thus, $\frac{dy}{dx} = (\sec x \tan x + \sec^2 x)$

$$\left(\frac{dy}{dx}\right)_{x=0} = 0 + 1 = 1$$

14. We have $y = \frac{x^4 + x^2 + 1}{x^2 - x + 1}$

$$= \frac{(x^2 - x + 1)(x^2 + x + 1)}{(x^2 - x + 1)}$$

$$= (x^2 + x + 1)$$

Thus, $\frac{dy}{dx} = 2x + 1$

Given $\frac{dy}{dx} = ax + b$

Comparing the co-efficients of x and the constant term, we get, a = 2 and $b = 1$

Thus, $a + b + 10 = 2 + 1 + 10 = 13$.

15.

16.

17. Given $y = \log(\sin(3x + 5))$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin(3x + 5)} \cdot \cos(3x + 5) \cdot 3$$

$$\Rightarrow \frac{dy}{dx} = 3\cot(3x + 5)$$

18. Given $y = \log(x + \sqrt{x^2 + 1})$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(x + \sqrt{x^2 + 1})} \times \left(1 + \frac{1 \times 2x}{2\sqrt{x^2 + 1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(x + \sqrt{x^2 + 1})} \times \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(x + \sqrt{x^2 + 1})} \times \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

19. Given $y = \log(\sqrt{x - 1} + \sqrt{x + 1})$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(\sqrt{x - 1} - \sqrt{x + 1})} \times \left(\frac{1}{2\sqrt{x - 1}} - \frac{1}{2\sqrt{x + 1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(\sqrt{x - 1} - \sqrt{x + 1})} \times \left(\frac{\sqrt{x + 1} - \sqrt{x - 1}}{2\sqrt{x^2 - 1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2\sqrt{x^2 - 2}}$$

20. Given $y = (\sin x)^x$

$$\Rightarrow y = e^{\log(\sin x)^x} = e^{x\log(\sin x)}$$

$$\Rightarrow \frac{dy}{dx} = e^{x\log(\sin x)}\left(\log\sin x + x \cdot \frac{\cos x}{\sin x}\right)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x\ (\log\sin x + x \cdot \cot x)$$

21. Given $y = \log(\sin x + \cos x)$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x - \sin x}{\sin x + \cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - \tan x}{1 + \tan x}$$

$$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{\pi}{4} - x\right)$$

22. We have $f(x) = \frac{1}{x - 1}$

Now, $f(f(x)) = f\left(\frac{1}{x - 1}\right)$

$$= \frac{1}{\frac{1}{x - 1} - 1}$$

$$= \frac{x - 1}{1 - x + 1}$$

$$= \frac{x - 1}{2 - x}$$

Thus, $\frac{d((f(f(x))))}{dx}$

$$= \frac{d}{dx}\left(\frac{x - 1}{2 - x}\right)$$

$$= \frac{(2 - x) \cdot 1 - (x - 1) \cdot (-1)}{(2 - x)^2}$$

$$= \frac{(2 - x + x - 1)}{(2 - x)^2} = \frac{1}{(2 - x)^2}$$

23. Given $f(x) = \frac{x - 1}{x + 1}$

$$f(f(f(x))) = f\left(f\left(\frac{x-1}{x+1}\right)\right)$$

$$= f\left(\frac{\left(\frac{x-1}{x+1}\right)-1}{\left(\frac{x-1}{x+1}\right)+1}\right)$$

$$= f\left(\frac{x-1-x-1}{x-1+x+1}\right)$$

$$= f\left(\frac{-2}{2x}\right)$$

$$= f\left(\frac{-1}{x}\right)$$

$$= \left(\frac{-\frac{1}{x}-1}{-\frac{1}{x}+1}\right)$$

$$= \left(\frac{-1-x}{1-x}\right) = \left(\frac{x+1}{x-1}\right)$$

Thus, $\dfrac{d(f(f(f(x))))}{dx}$

$$= \frac{d}{dx}\left(\frac{x+1}{x-1}\right)$$

$$= \frac{(x-1)\cdot 1-(x+1)\cdot 1}{(x-1)^2}$$

$$= \left(\frac{x-1-x-1}{(x-1)^2}\right) = \frac{-2}{(x-1)^2}$$

24. Given $g(x) = f(3x-1)$

$$g'(x) = 3f'(3x-1)$$

$$= 3 \times \frac{1}{(3x-1)^2+1}$$

$$= \frac{3}{9x^2-6x+2}$$

25. Given $y = f(\sin(x^3))$

$$\Rightarrow \quad \frac{dy}{dx} = f'(\sin(x^3))\cdot\cos(x^3)\cdot 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2\cdot\cos(x^3)\cdot((\sin(x^3))^2+1)$$

26. Given $f(x) = |x-1| + |x-3|$

$$= \begin{cases} -(x-1)-(x-3) & : x<1 \\ (x-1)-(x-3) & : 1 \le x < 3 \\ (x-1)+(x-3) & : x \ge 3 \end{cases}$$

$$= \begin{cases} -2x+2 & : x<1 \\ 2 & : 1 \le x < 3 \\ 2x-4 & : x \ge 3 \end{cases}$$

Thus, $f'(2) = 0$

27. Given $f(x) = |x^2-1| + |x^2-4|$

$$= \begin{cases} -(x^2-1)-(x^2-4) & : x<-2 \\ (x^2-1)-(x^2-4) & : -2 \le x < -1 \\ (1-x^2)+(4-x^2) & : -1 \le x < 1 \\ (x^2-1)+(4-x^2) & : 1 \le x < 2 \\ (x^2-1)+(x^2-4) & : x \ge 2 \end{cases}$$

Thus, $f'\left(\frac{3}{2}\right) = 0$

28. Given $f(x^2) = x^4 + x^3 + 1$

$$\Rightarrow \quad f'(x^2)\cdot 2x = 4x^3 + 3x^2$$

$$\Rightarrow \quad f'(x^2) = \frac{4x^3+3x^2}{2x} = 2x^2 + \frac{3}{2}x$$

Replacing x by x^2, we get,

$$f'(x^4) = 2x^4 + \frac{3}{2}x^2$$

29. We have $g(x) = f^{-1}(x)$

$$\Rightarrow \quad f(g(x)) = x$$

$$\Rightarrow \quad f(g(x))\cdot g'(x) = 1$$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$$

$$\Rightarrow \quad g'(x) = \frac{1}{\cos(2g(x))}$$

30. We have $f(x) = x + \tan x$

Given $f(x) = g^{-1}(x)$

$$\Rightarrow \quad g(f(x)) = x$$

$$\Rightarrow \quad g'(f(x))\cdot f'(x) = 1.$$

$$\Rightarrow \quad g'(f(x)) = \frac{1}{f'(x)} = \frac{1}{1+\sec^2 x}$$

$$\Rightarrow \quad g'(f(x)) = \frac{1}{2+\tan^2 x}$$

$$\Rightarrow \quad g'(f(x)) = \frac{1}{2+(f(x)-x)^2}$$

$$\Rightarrow \quad g'(x) = \frac{1}{2+(g(x)-x)^2}$$

31. Given $y = \dfrac{x^4-x^2+1}{x^2+\sqrt{3x}+1}$

$$= \frac{(x^4+1)-x^2}{x^2+\sqrt{3x}+1}$$

$$= \frac{(x^2+1)^2 - 3x^2}{x^2 + \sqrt{3}x + 1}$$

$$= \frac{(x^2 + \sqrt{3}x + 1)(x^2 - \sqrt{3}x + 1)}{(x^2 + \sqrt{3}x + 1)}$$

$$= (x^2 - \sqrt{3}x + 1)$$

$$\frac{dy}{dx} = 2x - \sqrt{3}$$

Given $\quad \frac{dy}{dx} = px + q$

Comparing the co-efficients of x and constant term, we get, $p = 2$, $q = -\sqrt{3}$

Thus, $p - q$

$$= 2 + \sqrt{3}$$

$$= \cot\left(\frac{\pi}{12}\right)$$

$$= \tan\left(\frac{\pi}{2} - \frac{\pi}{12}\right)$$

$$= \tan\left(\frac{5\pi}{12}\right)$$

32. We have $y = \sqrt{x-1} + \sqrt{x+1}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}} + \frac{1}{2\sqrt{x+1}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x+1} + \sqrt{x-1}}{2\sqrt{x-1}\sqrt{x+1}}$$

$$\frac{dy}{dx} = \frac{y}{2\sqrt{x^2-1}}$$

$$\sqrt{x^2-1}\,\frac{dy}{dx} = \frac{y}{2}$$

Hence, the result.

33. Given $\quad y = \frac{x}{x+2} = \left(1 - \frac{1}{(x+2)}\right)$

$$\frac{dy}{dx} = \frac{1}{(x+2)^2}$$

$$x\frac{dy}{dx} = \frac{x}{(x+2)^2}$$

$$x\frac{dy}{dx} = \frac{1}{(x+2)}\ \frac{x}{(x+2)}$$

$$x\frac{dy}{dx} = \left(1 - \frac{x}{(x+2)}\right)\frac{x}{(x+2)}$$

$$x\frac{dy}{dx} = (1-y)\,y$$

34. Given $\quad y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

$$2x\frac{dy}{dx} = \frac{2x}{2\sqrt{x}} - \frac{2x}{2x\sqrt{x}}$$

$$2x\frac{dy}{dx} = \sqrt{x} - \frac{1}{\sqrt{x}}$$

Hence, the result.

35. Given $\quad y = e^x \cos x$

$$\frac{dy}{dx} = e^x \cos x - e^x \sin x$$

$$\frac{dy}{dx} = e^x(\cos x - \sin x)$$

$$\frac{dy}{dx} = \sqrt{2}e^x\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$$

$$\frac{dy}{dx} = \sqrt{2}e^x \cos\left(x + \frac{\pi}{4}\right)$$

36. Given $\quad y = \sqrt{x^2 + a^2}$

$$\frac{dy}{dx} = \frac{1 \times 2x}{2\sqrt{x^2+a^2}}$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+a^2}}$$

$$\sqrt{x^2+a^2}\,\frac{dy}{dx} = x$$

$$y\,\frac{dy}{dx} - x = 0$$

37. Given $\quad y = e^x + e^{-x}$

$$\frac{dy}{dx} = e^x - e^{-x}$$

$$\frac{dy}{dx} = \sqrt{(e^x - e^{-x})^2}$$

$$\frac{dy}{dx} = \sqrt{(e^x - e^{-x})^2 - 4}$$

$$\frac{dy}{dx} = \sqrt{y^2 - 4}$$

38. Given $\quad xy = 4$

$$y = \frac{4}{x}$$

$$\frac{dy}{dx} = -\frac{4}{x^2}$$

$$\frac{dy}{dx} + y^2 = -\frac{4}{x^2} + y^2$$

$$\frac{dy}{dx} + y^2 = -\frac{4}{x^2} + \frac{16}{x^2} = \frac{12}{x^2}$$

$$x\left(\frac{dy}{dx} + y^2\right) = x \times \frac{12}{x^2} = \frac{12}{x}$$

$$x\left(\frac{dy}{dx} + y^2\right) = 3\left(\frac{4}{x}\right) = 3y$$

39. Let $y = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{a^2 - x^2} + \frac{x}{2} \times \frac{1 \times -2x}{2\sqrt{a^2 - x^2}}$$

$$+ \frac{a^2}{2} \times \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} \times \frac{1}{a}$$

$$= \frac{1}{2}\sqrt{a^2 - x^2} - \frac{1}{2} \times \frac{x^2}{\sqrt{a^2 - x^2}} + \frac{a^2}{2} \times \frac{1}{\sqrt{a^2 - x^2}}$$

$$= \frac{a^2 - 2x^2 + a^2}{2\sqrt{a^2 - x^2}}$$

$$= \frac{2(a^2 - x^2)}{2\sqrt{a^2 - x^2}}$$

$$= \sqrt{a^2 - x^2}$$

Hence, the result.

40.

(i) $y = \tan^{-1}(\sqrt{1 + x^2} + x)$

Put $x = \tan\theta$

$$y = \tan^{-1}(\sec\theta + \tan\theta)$$

$$y = \tan^{-1}\left(\frac{1 + \sin\theta}{\cos\theta}\right)$$

$$y = \tan^{-1}\left(\frac{1 + \cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)}\right)$$

$$y = \tan^{-1}\left(\frac{2\cos^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}\right)$$

$$y = \tan^{-1}\left(\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)$$

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)\right)$$

$$y = \left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{\pi}{4} + \frac{\tan^{-1}x}{2}$$

$$\frac{dy}{dx} = \frac{1}{2(1 + x^2)}$$

(ii) $y = \tan^{-1}(\sqrt{1 + x^2} - x)$

Put $x = \tan\theta$

$$y = \tan^{-1}(\sec\theta - \tan\theta)$$

$$y = \tan^{-1}\left(\frac{1 - \sin\theta}{\cos\theta}\right)$$

$$y = \tan^{-1}\left(\frac{1 - \cos\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right)}\right)$$

$$y = \tan^{-1}\left(\frac{2\sin^2\left(\frac{\pi}{2} - \frac{\theta}{2}\right)}{2\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\cos\left(\frac{\pi}{4} - \frac{\theta}{2}\right)}\right)$$

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)$$

$$y = \left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\pi}{4} - \frac{\tan^{-1}x}{2}$$

$$\frac{dy}{dx} = -\frac{1}{2(1 + x^2)}$$

(iii) $y = \tan^{-1}\left\{\frac{\sqrt{1 + x^2} - 1}{x}\right\}, x \neq 0$

Put $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$y = \tan^{-1}\left(\frac{1 - \cos\theta}{\sin\theta}\right)$$

$$y = \tan^{-1}\left(\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$$

$$y = \tan^{-1}(\tan(\theta/2)) = \frac{\theta}{2}$$

$$y = \frac{\tan^{-1}x}{2}$$

$$\frac{dy}{dx} = \frac{1}{2(1 + x^2)}$$

(iv) $y = \tan^{-1}\left\{\frac{\sqrt{1 + x^2} + 1}{x}\right\}, x \neq 0$

Put $x = \tan\theta$

$$y = \tan^{-1}\left(\frac{\sec\theta + 1}{\tan\theta}\right)$$

$$y = \tan^{-1}\left(\frac{1 + \cos\theta}{\sin\theta}\right)$$

$$y = \tan^{-1}\left(\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$$

$$y = \tan^{-1}(2\cos(\theta/2))$$

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)$$

$$y = \left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

$$y = \left(\frac{\pi}{2} - \frac{\tan^{-1}x}{2}\right)$$

$$\frac{dy}{dx} = -\frac{1}{2(1 + x^2)}$$

(v) $y = \cot^{-1}(\sqrt{1 + x^2} + x)$

Put $x = \tan\theta$

$$y = \cot^{-1}\left(\frac{\sec\theta + 1}{\tan\theta}\right)$$

$$y = \cot^{-1}\left(\frac{1 + \cos\theta}{\sin\theta}\right)$$

$$y = \cot^{-1}\left(\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$$
$$y = \cot^{-1}(\cot(\theta/2))$$
$$y = \frac{\theta}{2} = \frac{\tan^{-1}x}{2}$$
$$\frac{dy}{dx} = \frac{1}{2(1 + x^2)}$$

(vi) $y = \tan^{-1}\left\{\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right\}, 0 < x < \pi$
$$y = \tan^{-1}\left(\frac{\cos(x/2)}{\sin(x/2)}\right)$$
$$y = \tan^{-1}\left(\cot\left(\frac{x}{2}\right)\right)$$
$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \left(\frac{x}{2}\right)\right)\right)$$
$$y = \frac{\pi}{2} - \frac{x}{2}$$
$$\frac{dy}{dx} = -\frac{1}{2}$$

41.

(i) Given $y = \tan^{-1}\left(\frac{a + x}{1 - ax}\right)$
$$y = \tan^{-1}(x) + \tan^{-1}(a)$$
$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

(ii) $y = \tan^{-1}\left\{\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right\} - \frac{\pi}{2} < x < \frac{\pi}{2}$
$$y = \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right)$$
$$y = \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x)$$
$$y = \tan^{-1}\left(\frac{a}{b}\right) - x$$
$$\frac{dy}{dx} = -1$$

(iii) $y = \tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), -\frac{1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}}$
$$y = \tan^{-1}\left(\frac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right)$$
$$y = \tan^{-1}\left(\frac{3x}{a}\right)$$
$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{3x}{a}\right)^2} \times \frac{3}{a}$$
$$\frac{dy}{dx} = \frac{3a}{9x^2 + a^2}$$

(iv) $y = \tan^{-1}\left(\sqrt{\frac{a - x}{a + x}}\right)$

Put $x = a\cos\theta$
$$y = \tan^{-1}\left(\sqrt{\frac{a(1 - \cos\theta)}{a(1 + \cos\theta)}}\right)$$
$$y = \tan^{-1}\left(\frac{\sin(\theta/2)}{\cos(\theta/2)}\right) = \tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right)$$
$$y = \left(\frac{\theta}{2}\right) = \frac{\cos^{-1}x}{2}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{1 - x^2}}$$

(v) $y = \tan^{-1}\left(\frac{\sin x}{1 + \cos x}\right), -\pi < x < \pi.$
$$y = \tan^{-1}\left(\frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right)$$
$$y = \tan^{-1}\left(\tan\left(\frac{x}{2}\right)\right) = \frac{x}{2}$$
$$\frac{dy}{dx} = \frac{1}{2}$$

(vi) $y = \tan^{-1}\left(\frac{a + b\tan x}{b - a\tan x}\right)$
$$y = \tan^{-1}\left(\frac{\frac{a}{b} + \tan x}{1 - \frac{a}{b}\tan x}\right)$$
$$y = \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}(\tan x)$$
$$y = \tan^{-1}\left(\frac{a}{b}\right) + x$$
$$\frac{dy}{dx} = 1$$

(vii) $y = \tan^{-1}\left(\frac{a + bx}{b - ax}\right)$
$$y = \tan^{-1}\left(\frac{\frac{a}{b} + x}{1 - \frac{a}{b}x}\right) = \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}(x)$$
$$y = \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}(x)$$
$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

(viii) $y = \tan^{-1}\left(\frac{x}{1 + 6x^2}\right)$
$$y = \tan^{-1}\left(\frac{3x - 2x}{1 + 6x}\right)$$
$$y = \tan^{-1}(3x) - \tan^{-1}(2x)$$
$$\frac{dy}{dx} = \frac{3}{1 + 9x^2} - \frac{2}{1 + 4x^2}$$

(ix) Given $y = \sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$
$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)$$

$= \sin^{-1}\left(\sin\left(x + \frac{\pi}{4}\right)\right)$

Thus, $y = \left(x + \frac{\pi}{4}\right)$

$\Rightarrow \frac{dy}{dx} = 1$

(x) $y = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) + \sec^{-1}\left(\frac{1 + x^2}{1 - x^2}\right), x \in R$

$y = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) + \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$

$y = \frac{\pi}{2}$

$\frac{dy}{dx} = 0$

(xi) Given $y = \sin^{-1}\left(\sqrt{\frac{x - 1}{x + 1}}\right) + \cos^{-1}\left(\sqrt{\frac{x - 1}{x + 1}}\right)$

$\Rightarrow y = \frac{\pi}{2}$

$\Rightarrow \frac{dy}{dx} = 0$

(xii) Given $y = \sin^{-1}\left(\frac{3\sin x + 4\cos x}{5}\right)$

$\Rightarrow y = \sin^{-1}\left(\frac{3}{5}\sin x + \frac{4}{5}\cos x\right)$

$\Rightarrow y = \sin^{-1}(\cos\alpha \sin x + \sin\alpha \cos x)$

$\Rightarrow y = \sin^{-1}(\sin(x + \alpha)) = (x + \alpha)$

$\Rightarrow \frac{dy}{dx} = 1$, where a $= \tan^{-1}\left(\frac{4}{3}\right)$

(xiii) Given $y = \tan^{-1}\left(\frac{1 - \cos x}{\sin x}\right)$

$= \tan^{-1}\left(\frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right)$

$= \tan^{-1}\left(\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right)$

$= \tan^{-1}\left(\tan\left(\frac{x}{2}\right)\right) = \frac{x}{2}$

Thus, $\frac{dy}{dx} = \frac{1}{2}$

(xiv) Given $y = \tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right)$

$= \tan^{-1}\left(\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right)$

$= \tan^{-1}\left(\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right)$

$= \tan^{-1}\left(\cot\left(\frac{x}{2}\right)\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right)$

$\Rightarrow y = \left(\frac{\pi}{2} - \frac{x}{2}\right)$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}$

42. Given $y = \sin^{-1}\left(\frac{2x}{1 + x^2}\right) + \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$

$y = 2\tan^{-1}x + 2\tan^{-1}x = 4\tan^{-1}x$

$\frac{dy}{dx} = \frac{4}{1 + x^2}$

Hence, the result.

43. Given $y = \sin^{-1}\left(\frac{x}{\sqrt{1 + x^2}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{x^2 + 1}}\right)$

Put $x = \tan\theta$

Then $y = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{1 + \tan^2\theta}}\right)$

$y = \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right) + \cos^{-1}\left(\frac{1}{\sec\theta}\right)$

$y = \sin^{-1}(\sin\theta) + \cos^{-1}(\cos\theta)$

$y = \theta + \theta = 2\theta = 2\tan^{-1}x$

$\frac{dy}{dx} = \frac{2}{1 + x^2}$

44. Given $y = \tan^{-1}\left\{\frac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}\right\}$

Put $x^2 = \cos(2\theta)$

Thus, $y = \tan^{-1}\left(\frac{\sqrt{1 + \cos(2\theta)} + \sqrt{1 - \cos(2\theta)}}{\sqrt{1 + \cos(2\theta)} - \sqrt{1 - \cos(2\theta)}}\right)$

$y = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}\right)$

$y = \tan^{-1}\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right)$

$y = \tan^{-1}\left(\frac{1 + \tan\theta}{1 - \tan\theta}\right)$

$y = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \theta\right)\right) = \left(\frac{\pi}{4} + \theta\right)$

$y = \left(\frac{\pi}{4} + \theta\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}(x^2)$

$\frac{dy}{dx} = -\frac{1 \times 2x}{2\sqrt{1 - x^4}} = -\frac{x}{\sqrt{1 - x^4}}$

45. Given $y = \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$

$$= \cot^{-1}\left\{\frac{\sqrt{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)^2}+\sqrt{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)^2}}{\sqrt{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)^2}-\sqrt{\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)^2}}\right\}$$

$$= \cot^{-1}\left\{\frac{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)+\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}{\left(\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right)-\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}\right\}$$

$$= \cot^{-1}\left\{\frac{2\cos\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}\right\}$$

$$= \cot^{-1}\left\{\cot\left(\frac{x}{2}\right)\right\} = \frac{x}{2}$$

$$\frac{dy}{dx} = \frac{1}{2} = \text{independent of } x$$

Hence, the result.

46. Given $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$y = 2\tan^{-1}x + 2\tan^{-1}x = 4\tan^{-1}x$$

$$\frac{dy}{dx} = \frac{4}{1+x^2}$$

47. Given $y = \sec^{-1}\left(\frac{x-1}{x+1}\right) + \sin^{-1}\left(\frac{x+1}{x-1}\right)$

$$y = \cos^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x+1}{x-1}\right)$$

$$y = \frac{\pi}{2}$$

$$\frac{dy}{dx} = 0$$

48. Given $y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$

Put $x = \cos\theta$

$$y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}\right]$$

$$y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\right\}\right]$$

$$y = \sin\left[2\tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right)\right]$$

$$y = \sin\left[2\left(\frac{\theta}{2}\right)\right] = \sin\theta$$

$$y = \sqrt{1-\cos^2\theta} = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1\times(-2x)}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

49. Given $y = \cos^{-1}(2x) + 2\cos^{-1}(\sqrt{1-4x^2})$

Put $2x = \sin\theta$

Thus, $y = \cos^{-1}(\sin\theta) + 2\cos^{-1}\sqrt{1-\sin^2\theta}$

$$y = \cos^{-1}(\sin\theta) + 2\cos^{-1}(\cos\theta)$$

$$y = \cos^{-1}\left(\cos\left(\frac{\pi}{2}-\theta\right)\right) + 2\cos^{-1}(\cos\theta)$$

$$y = \left(\frac{\pi}{2}-\theta\right) + 2\theta$$

$$y = \left(\frac{\pi}{2}+\theta\right) = \frac{\pi}{2} + \sin^{-1}(2x)$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$$

50. Given $y = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x})}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Put $x = \cos\theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

$$y = \tan^{-1}\left(\frac{\cos(\theta/2)-\sin(\theta/2)}{\cos(\theta/2)+\sin(\theta/2)}\right)$$

$$y = \tan^{-1}\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)$$

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right) = \left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$y = \left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \frac{\pi}{2} - \frac{\cos^{-1}x}{2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

51. $y = \cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right)$

$$y = \cos^{-1}x + \cos^{-1}\left(\frac{1}{2}\cdot x + \frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)$$

$$y = \cos^{-1}x + \cos^{-1}\left(\frac{1}{2}\right) - \cos^{-1}(x)$$

$$y = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\frac{dy}{dx} = 0$$

52. We have $y = \sin^{-1}(x\sqrt{1-x} - \sqrt{x - x^3})$

$\Rightarrow \quad y = \sin^{-1}(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2})$

$\Rightarrow \quad y = \sin^{-1}x - \sin^{-1}(\sqrt{x})$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{\sqrt{x}}{2\sqrt{1-x}}$

53. Given $\quad y = f(x) = x^3 + x^5 + x^7$

$\Rightarrow \quad \dfrac{dy}{dx} = 3x^2 + 5x^4 + 7x^6$

Now, $\quad g'(y) = \dfrac{dx}{dy} = \dfrac{1}{3x^2 + 5x^4 + 7x^6}$

When $\quad y = 3,\ 3 = x^3 + x^5 + x^7$

Thus, $\quad x = 1$

Therefore, $g'(3) = \dfrac{1}{3+5+7} = \dfrac{1}{15}$

54. Given $\quad y = x^5 + 2x^3 + 2x$

$\dfrac{dy}{dx} = 5x^4 + 6x^2 + 2$

Now, $\quad g'(y) = \dfrac{dx}{dy} = \dfrac{1}{5x^4 + 6x^2 + 2}$

when $\quad y = -5$, then $x = -1$

Therefore, $g'(-5) = \dfrac{1}{5+6+2} = \dfrac{1}{13}$

55. Given $\quad y = f(x) = x^3 + 2x^2 + 3x + 4$

$\Rightarrow \quad \dfrac{dy}{dx} = 3x^2 + 4x + 3$

Now, $\quad g'(y) = \dfrac{dx}{dy} = \dfrac{1}{3x^2 + 4x + 3}$

When $\quad y = 4,\ x^3 + 2x^2 + 3x + 4 = 4$

Thus, $\quad x = 0$

Therefore, $g'(4) = \dfrac{1}{0+0+3} = \dfrac{1}{3}$

56. Given f and g are inverse of each other

Thus, $\quad f(g(x)) = g(f(x)) = x$

$\Rightarrow \quad f'(g(x)) \cdot g'(x) = 1$

$\Rightarrow \quad g'(x) = \dfrac{1}{f'(g(x))}$

$\Rightarrow \quad g'(x) = \dfrac{1}{1 + (g(x))^n}$

57. We have $2x^2 + 3xy + 3y^2 = 1$

$\Rightarrow \quad 4x + 3x\dfrac{dy}{dx} + 3y + 6y^2\dfrac{dy}{dx} = 0$

$\Rightarrow \quad (3x + 6y^2)\dfrac{dy}{dx} = -(4x + 3y)$

$\Rightarrow \quad \dfrac{dy}{dx} = -\left(\dfrac{4x + 3y}{3x + 6y^2}\right)$

58. We have, $e^x + e^y = e^{x+y}$

$\Rightarrow e^x + e^y\dfrac{dy}{dx} = e^{x+y}\left(1 + \dfrac{dy}{dx}\right) = e^{x+y} + e^{x+y}\dfrac{dy}{dx}$

$\Rightarrow e^x + e^y\dfrac{dy}{dx} = e^{x+y} + e^{x+y}\dfrac{dy}{dx}$

$\Rightarrow (e^y - e^{x+y})\dfrac{dy}{dx} = (e^{x+y} - e^x)$

$\Rightarrow \dfrac{dy}{dx} = \left(\dfrac{e^{x+y} - e^x}{e^y - e^{x+y}}\right) = \left(\dfrac{e^x + e^y - e^x}{e^y - e^x - e^y}\right)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{e^y}{e^x} = -e^{y-x}$

59. We have $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$

$\Rightarrow \quad x^2(1+y) = y^2(1+x)$

$\Rightarrow \quad x^2 + x^2y = y^2 + xy^2$

$\Rightarrow \quad x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow \quad (x+y)(x-y) = -xy(x-y)$

$\Rightarrow \quad (x+y) = -xy$

$\Rightarrow \quad y + xy = -x$

$\Rightarrow \quad y(1+x) = -x$

$\Rightarrow \quad y = -\dfrac{x}{1+x} = -\left(1 - \dfrac{1}{1+x}\right)$

$\Rightarrow \quad y = \left(-1 - \dfrac{1}{1+x}\right)$

$\Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$

60. We have $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1 \qquad$ ...(i)

Put $\quad x = \sin\theta$ and $y = \sin\varphi$

(i) reduces to $\sin\varphi\cos\theta + \sin\theta\cos\varphi = 1$

$\Rightarrow \quad \sin(\theta + \varphi) = 1$

$\Rightarrow \quad \theta + \varphi = \sin^{-1}(1)$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = \sin^{-1}(1)$

$\Rightarrow \quad \dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-y^2}}\dfrac{dy}{dx} = 0$

$\Rightarrow \quad \dfrac{1}{\sqrt{1-y^2}}\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

Hence, the result.

61. Given $\sin y = x \sin (a + y)$

$$\Rightarrow \quad x = \frac{\sin y}{\sin(a+y)}$$

Differentiate w.r.t y, we get,

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin(a+y)\cdot\cos y - \sin y\cdot\cos(a+y)}{\{\sin(a+y)\}^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin(a+y-y)}{\{\sin(a+y)\}^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{1}{\frac{dx}{dy}} = \frac{\sin^2(a+y)}{\sin a}.$$

62. We have $\log(x + y) = 2xy$

$$\Rightarrow \quad \frac{1}{(x+y)}\left(1 + \frac{dy}{dx}\right) = 2\left(x\frac{dy}{dx} + y\cdot 1\right) \quad \text{...(i)}$$

when $x = 0$, then $y = 1$

Put $x = 0$ and $y = 1$ in (i), we get,

$$\Rightarrow \quad \frac{1}{(0+1)}\left(1 + \frac{dy}{dx}\right) = 2\left(0\cdot\frac{dy}{dx} + 1.1\right)$$

$$\Rightarrow \quad \left(1 + \frac{dy}{dx}\right) = 2$$

$$\Rightarrow \quad \frac{dy}{dx} = 1.$$

63. Given $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$

Put $x^3 = \sin\theta,\ y^3 = \sin\varphi$

The given equation reduces to

$$\cos\theta + \cos\varphi = a(\sin\theta - \sin\varphi)$$

$$\Rightarrow \quad 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$$

$$= 2a\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

$$\Rightarrow \quad \cos\left(\frac{\theta-\varphi}{2}\right) = a\sin\left(\frac{\theta-\varphi}{2}\right)$$

$$\Rightarrow \quad \cot\left(\frac{\theta-\varphi}{2}\right) = a$$

$$\Rightarrow \quad \left(\frac{\theta-\varphi}{2}\right) = \cot^{-1}(a)$$

$$\Rightarrow \quad (\theta - \varphi) = 2\cot^{-1}(a)$$

$$\Rightarrow \quad \sin^{-1}(x^3) - \sin^{-1}(y^3) = 2\cot^{-1}(a)$$

Differentiating w.r.t x, we get,

$$\Rightarrow \quad \frac{3x^2}{\sqrt{1-x^6}} - \frac{3y^2}{\sqrt{1-y^6}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{3y^2}{\sqrt{1-y^6}}\frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}$$

64. Given, $x^2 + y^2 = t - \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$

Now, $x^2 + y^2 = t - \frac{1}{t}$

$$\Rightarrow \quad (x^2 + y^2)^2 = \left(t - \frac{1}{t}\right)^2$$

$$\Rightarrow \quad (x^2 + y^2)^2 = t^2 + \frac{1}{t^2} - 2$$

$$\Rightarrow \quad x^4 + y^4 + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$$

$$\Rightarrow \quad t^2 + \frac{1}{t^2} + 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$$

$$\Rightarrow \quad 2x^2y^2 = -2$$

$$\Rightarrow \quad x^2y^2 = -1$$

$$\Rightarrow \quad y^2 = \frac{-1}{x^2}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = \frac{2}{x^3}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^3y}$$

65. Given $\sec\left(\frac{x+y}{x-y}\right) = a$

$$\Rightarrow \quad \left(\frac{x+y}{x-y}\right) = \frac{\sec^{-1}(a)}{1}$$

$$\Rightarrow \quad \left(\frac{x+y+x-y}{x+y-x+y}\right) = \frac{\sec^{-1}(a)+1}{\sec^{-1}(a)-1}$$

$$\Rightarrow \quad \frac{x}{y} = \frac{\sec^{-1}(a) + 1}{\sec^{-1}(a) - 1}$$

$$\Rightarrow \quad \frac{y}{x} = \frac{\sec^{-1}(a) - 1}{\sec^{-1}(a) + 1}$$

$$\Rightarrow \quad y = \left(\frac{\sec^{-1}(a) - 1}{\sec^{-1}(a) + 1}\right)x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sec^{-1}(a) - 1}{\sec^{-1}(a) + 1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

Hence, the result.

66. Given $\tan^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = a$

$$\Rightarrow \quad \left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \frac{\tan a}{1}$$

$$\Rightarrow \quad \left(\frac{x^2 - y^2 + x^2 + y^2}{x^2 - y^2 - x^2 - y^2}\right) = \frac{\tan a + 1}{\tan a - 1}$$

$$\Rightarrow \quad -\frac{x^2}{y^2} = \frac{\tan a + 1}{\tan a - 1}$$

$$\Rightarrow \quad \frac{y^2}{x^2} = \frac{1 - \tan a}{\tan a + 1}$$

$$\Rightarrow \quad y^2 = \left(\frac{1 - \tan a}{\tan a + 1}\right)x^2$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2\left(\frac{1 - \tan a}{\tan a + 1}\right)x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}\left(\frac{1 - \tan a}{1 + \tan a}\right)$$

Hence, the result.

67. Given $xy = 1$

$$\Rightarrow \quad y = \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = -y^2$$

$$\Rightarrow \quad \frac{dy}{dx} + y^2 = 0$$

Hence, the result.

68. Given $y = x\sin y$

$$\Rightarrow \quad \frac{dy}{dx} = \sin y + x\cos y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}(1 - x\cos y) = \sin y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin y}{(1 - x\cos y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x(1 - x\cos y)}$$

Hence, the result.

69. Given $\cos y = x\cos(a + y)$

$$\Rightarrow \quad x = \frac{\cos y}{\cos(a + y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-\cos(a + y)\sin y + \cos y\sin(a + y)}{\cos^2(a + y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin(a + y - y)}{\cos^2(a + y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin(a)}{\cos^2(a + y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos^2(a + y)}{\sin(a)}$$

Hence, the result.

70. Given $y = x^{\sin x}$

$$\Rightarrow \quad \log(y) = \log\left(x^{\sin x}\right)$$

$$\Rightarrow \quad \log(y) = \sin x\log x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \cos x\log x$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{\sin x}{x} + \cos x\log x\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x\log x\right)$$

71. Given $y = (\sin x)^{\cos x}$

$$\Rightarrow \quad y = e^{\log(\sin x)^{\cos x}}$$

$$\Rightarrow \quad y = e^{\cos x\log(\sin x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\cos x\log(\sin x)}\left(\cos x\cdot\frac{\cos x}{\sin x} - \sin x\log\sin x\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos x}(\cos x\cdot\cot x - \sin x\log\sin x)$$

72. Given $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

$$\Rightarrow y = e^{\log(\tan x)^{\cot x}} + e^{\log(\cot x)^{\tan x}}$$

$$\Rightarrow y = e^{\cot x \log(\tan x)} + e^{\tan x \log(\cot x)}$$

$$\Rightarrow \frac{dy}{dx} = e^{\cot x \log(\tan x)}\left(\cot x \cdot \frac{\sec^2 x}{\tan x} - \operatorname{cosec}^2 x \cdot \log(\tan x)\right)$$

$$+ e^{\tan x \log(\cot x)}\left(\tan x \cdot \frac{-\operatorname{cosec}^2 x}{\cot x} + \sec^2 x \cdot \log(\cot x)\right)$$

$$\Rightarrow \frac{dy}{dx} = (\tan x)^{\cot x}(\operatorname{cosec}^2 x - \operatorname{cosec}^2 x \cdot \log(\tan x))$$

$$+ (\cot x)^{\tan x}(-\sec^2 x + \sec^2 x \cdot \log(\cot x))$$

73. Given $x^m x^n = (x+y)^{m+n}$

$$\Rightarrow \log(x^m x^n) = \log((x+y)^{m+n})$$

$$\Rightarrow m\log x + n\log y = (m+n)\log(x+y)$$

$$\Rightarrow \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{(x+y)}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{n}{y} - \frac{m+n}{x+y}\right)\frac{dy}{dx} = \left(\frac{m+n}{x+y} - \frac{m}{x}\right)$$

$$\Rightarrow \left(\frac{nx + ny - my - ny}{y(x+y)}\right)\frac{dy}{dx}$$

$$= \left(\frac{mx + nx - mx - my}{x(x+y)}\right)$$

$$\Rightarrow \left(\frac{nx - my}{y(x+y)}\right)\frac{dy}{dx} = \left(\frac{nx - my}{x(x+y)}\right)$$

$$\Rightarrow \left(\frac{1}{y}\right)\frac{dy}{dx} = \left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

74. Given $y = \left(1 + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

$$\Rightarrow y = e^{x\log\left(1+\frac{1}{x}\right)} + e^{\left(1+\frac{1}{x}\right)\log x}$$

$$\Rightarrow \frac{dy}{dx} = e^{x\log\left(1+\frac{1}{x}\right)}\left(\log\left(x + \frac{1}{x}\right) + x\frac{1}{\left(1+\frac{1}{x}\right)} \cdot -\frac{1}{x^2}\right)$$

$$+ e^{\left(1+\frac{1}{x}\right)\log x}\left(\left(1 + \frac{1}{x}\right)\frac{1}{x} - \frac{1}{x^2}\log x\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(1 + \frac{1}{x}\right)^x\left(\log\left(x + \frac{1}{x}\right) - \frac{1}{(1+x)}\right)$$

$$+ x^{\left(1+\frac{1}{x}\right)}\left(\left(1 + \frac{1}{x}\right)\frac{1}{x} - \frac{1}{x^2}\log x\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 2\left(\log 2 - \frac{1}{2}\right) + 2 = \log 4 + 1$$

75. Given $y = \sin x \cdot \sin 2x \cdot \sin 3x \ldots \sin(2014)x$

$$\log y = \log(\sin x \cdot \sin 2x \ldots \sin(2014)x)$$

$$\log y = \log\sin x + \log\sin x2x + \ldots + \log\sin(2014)x$$

$$\frac{1}{y}\frac{dx}{dy} = \cot x + 2\cot 2x + \ldots + 2014\cot(2014)x$$

$$\frac{dy}{dx} = y(\cot x + 2\cot 2x + \ldots + 2014\cot(2014)x)$$

76. Given $x^y = e^{x-y}$

$$\Rightarrow y\log x = (x - y)$$

$$\Rightarrow y(1 + \log x) = x$$

$$\Rightarrow y = \frac{x}{(1+\log x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\log x) - x \cdot \frac{1}{x}}{(1+\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\log x) - 1}{(1+\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

Hence, the result.

77. Given $x^y = y^x$

$$\Rightarrow y\log x = x\log y$$

$$\Rightarrow \frac{y}{x} + \log x\frac{dy}{dx} = \log y + \frac{x}{y}\frac{dy}{dx}$$

$$\Rightarrow \left(\log x - \frac{x}{y}\right)\frac{dy}{dx} = \left(\log y - \frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(\log y - \frac{y}{x}\right)}{\left(\log x - \frac{x}{y}\right)} = \frac{y}{x}\left(\frac{x\log y - y}{y\log x - x}\right)$$

78. Given $e^y = y^x$,

$$\Rightarrow \log(e^y) = \log(y^x)$$

$$\Rightarrow y\log(e) = x\log(y)$$

$$\Rightarrow y = x\log(y)$$

$$\Rightarrow \frac{dy}{dx} = \log(y) + \frac{x}{y}\frac{dx}{dy}$$

$$\Rightarrow \frac{dy}{dx}\left(1 - \frac{x}{y}\right) = \log(y)$$

$$\Rightarrow \quad \frac{dy}{dx}\left(1 - \frac{1}{\log(y)}\right) = \log(y)$$

$$\Rightarrow \quad \frac{dy}{dx}(\log(y) - 1) = \{\log(y)\}^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\{\log(y)\}^2}{(\log(y) - 1)}$$

79. Given $\quad x^m y^n = 1$

$$\Rightarrow \quad \log(x^m y^n) = \log(1) = 0$$

$$\Rightarrow \quad m\log x + n\log y = 0$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{my}{nx}$$

Hence, the result.

80. Given $\quad e^{x+y} = x$

$$\Rightarrow \quad (x + y) = \log x$$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} - 1 = \frac{1 - x}{x}$$

82. Given $y = (x + 1)(x + 2)(x + 3)\ldots(x + 2012)$

$\log y = \log(x + 1) + \log(x + 2) + \ldots + \log(x + 2012)$

Differentiating w.r.t x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{(x + 1)} + \frac{1}{(x + 2)} + \ldots + \frac{1}{(x + 2012)}$$

$$\frac{dy}{dx} = y\left(\frac{1}{(x + 1)} + \frac{1}{(x + 2)} + \ldots + \frac{1}{(x + 2012)}\right)$$

Put $x = 0$, we get,

$$\frac{dy}{dx} = (2012)!\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2012}\right)$$

83. Given $y = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\ \ldots\ (1 + x^{1006})$

$\log y = \log(1 + x) + \log(1 + x^2) + \log(1 + x^4) + \log(1 + x^8) + \ldots + \log(1 + x^{1006})$

Differentiating w.r.t x, we get,

$$\frac{1}{y}\frac{dx}{dy} = \frac{1}{(1 + x)} + \frac{2x}{1 + x^2} + \frac{4x^3}{1 + x^4} + \frac{8x^7}{1 + x^8} \ldots + \frac{1006x^{1005}}{1 + x^{1006}}$$

$$\frac{dy}{dx} = y\left(\frac{1}{1 + x} + \frac{2x}{1 + x^2} + \ldots + \frac{1006x^{1005}}{1 + x^{1006}}\right)$$

$$\left(\frac{dy}{dx}\right)_{x=0} = 1(1 + 1 + 1 + \ldots + 1)(1006\text{-times}) = 1006.$$

84. We have

$$y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \text{ to } \infty}}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{f'(x)}{(2y - 1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sin x}{2y - 1}$$

85. We have $y = \frac{\sin x}{1+}\frac{\cos x}{1+}\frac{\sin x}{1+}\frac{\cos x}{1+}\ldots$

$$\Rightarrow \quad y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1 + \ldots}}} = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + y}}$$

$$\Rightarrow \quad y = \frac{(1 + y)\cos x}{1 + y + \cos x}$$

$$\Rightarrow \quad y + y^2 + y\cos x = \sin x + y\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} + 2y\frac{dy}{dx} + \cos x\frac{dy}{dx} - y\sin x = \cos x + \sin x\frac{dy}{dx}$$

$$\Rightarrow \quad (1 + 2y + \cos x - \sin x)\frac{dy}{dx} = \cos x + y\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos x + y\sin x}{(1 + 2y + \cos x - \sin x)}$$

86. Given $\quad y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots \text{ to } \infty}}}$

$$\Rightarrow \quad y = \sqrt{x + y}$$

$$\Rightarrow \quad y^2 = x + y$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \quad (2y - 1)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{(2y - 1)}$$

Hence, the result.

87. Given $y = x^{x^{\ldots \text{to}\infty}}$

$\Rightarrow$ $y = x^y$

$\Rightarrow$ $\log(y) = y\log x$

$\Rightarrow$ $\frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + \log x\frac{dy}{dx}$

$\Rightarrow$ $\left(\frac{1}{y} - \log(x)\right)\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow$ $(1 - y\log(x))\,\frac{dy}{dx} = \frac{y^2}{x}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y^2}{x(1 - y\log(x))}$

Hence, the result.

88. Given $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x \ldots \text{to}\infty}}}$

$\Rightarrow$ $y = \sqrt{\sin x + y}$

$\Rightarrow$ $y^2 = \sin x + y$

$\Rightarrow$ $2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$

$\Rightarrow$ $(2y - 1)\frac{dy}{dx} = \cos x$

$\Rightarrow$ $\frac{dy}{dx} = \frac{\cos x}{(2y - 1)}$

Hence, the result.

89. Given $y = e^{x+e^{x+e^{x+\text{to}\infty}}}$

$\Rightarrow$ $y = e^{x+y}$

$\Rightarrow$ $\frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow$ $\frac{dy}{dx} = y\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow$ $(1 - y)\frac{dy}{dy} = y$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y}{1 - y}$

90. Given $y = (\tan x)^{(\tan x)^{(\tan x)\ldots\text{to}\infty}}$

$\Rightarrow$ $y = (\tan x)^y$

$\Rightarrow$ $\log(y) = y\log(\tan x)$

$\Rightarrow$ $\frac{1}{y}\frac{dy}{dx} = \log(\tan x)\frac{dy}{dx} + \left(\frac{\sec^2 x}{\tan x}\right)^y$

when $x = \frac{\pi}{4}$, $y = 1$, then

$\Rightarrow$ $\frac{dy}{dx} = 0\cdot\frac{dy}{dx} + 2$

$\Rightarrow$ $\frac{dy}{dx} = 2$

91. Given $y = x + \frac{1}{x+}\frac{1}{x+}\frac{1}{x+}$ .... to ∞

$\Rightarrow$ $y = x + \frac{1}{y}$

$\Rightarrow$ $y^2 = xy + 1$

Differentiating w.r.t x, we get,

$\Rightarrow$ $2y\frac{dy}{dx} = y + x\frac{dy}{dx}$

$\Rightarrow$ $(2y - x)\frac{dy}{dx} = y$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y}{(2y - x)}$

Hence, the result.

92. Given $y = \frac{\sin x}{1+}\cdot\frac{\cos x}{1+}\cdot\frac{\sin x}{1+}\cdot\frac{\cos x}{1+}$ ... to ∞

$\Rightarrow$ $y = \frac{\sin x}{1 + \frac{\cos x}{1 + y}}$

$\Rightarrow$ $y = \frac{(1 + y)\sin x}{(1 + y) + \cos x}$

$\Rightarrow$ $y\{(1 + y) + \cos x\} = (1 + y)\sin x$

$\Rightarrow$ $y'\{(1 + y) + \cos x\} + y(y' - \sin x)$

$= (1 + y)\cos x + y'\sin x$

when $x = 0$ and $y = 0$, then

$\Rightarrow$ $y'(1 + 1) + 0 = 1 + 0$

$\Rightarrow$ $2y' = 1$

$\Rightarrow$ $y' = \frac{1}{2}$

$\Rightarrow$ $\frac{dy}{dx} = \frac{1}{2}$

93. Given $y = \frac{x}{1+}\cdot\frac{x}{2+}\cdot\frac{x}{1+}\cdot\frac{x}{2+}$ .... to ∞

$\Rightarrow$ $y = \frac{x}{1 + \frac{x}{2 + y}} = \frac{x(2 + y)}{2 + y + x}$

$\Rightarrow$ $y(2 + y + x) = x(2 + y)$

$\Rightarrow$ $2y + y^2 + xy = 2x + xy$

$\Rightarrow$ $2y + y^2 = 2x$

$\Rightarrow$ $2\frac{dy}{dx} + 2y\frac{dy}{dx} = 2$

$\Rightarrow$ $\frac{dy}{dx} + y\frac{dy}{dx} = 1$

$\Rightarrow$ $(1 + y)\frac{dy}{dx} = 1$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{(1+y)}$$

94. We have $x = a(t - \sin t)$, $y = a(1 - \cos t)$

$$\Rightarrow \quad \frac{dx}{dy} = a(1 - \cos t) \text{ and } \frac{dy}{dt} = a\sin t$$

Thus, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a(1-\cos t)}{a\sin t}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin^2\left(\frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)} = \tan\left(\frac{1}{2}\right)$$

95. We have

$$\Rightarrow \quad x^2 - y^2 = t - \frac{1}{t} \text{ and } x^4 + y^4 = t^2 + \frac{1}{t^2}$$

$$\Rightarrow \quad (x^2 - y^2)^2 = \left(t - \frac{1}{t}\right)^2$$

$$\Rightarrow \quad x^4 + y^4 - 2x^2y^2 = t^2 + \frac{1}{t^2} - 2$$

$$\Rightarrow \quad x^4 + y^4 - 2x^2y^2 = x^4 + y^4 - 2$$

$$\Rightarrow \quad -2x^2y^2 = -2$$

$$\Rightarrow \quad x^2y^2 = 1$$

$$\Rightarrow \quad y^2 = \frac{1}{x^2}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -\frac{2}{x^3}$$

$$\Rightarrow \quad y\frac{dy}{dx} = -\frac{1}{x^3}$$

$$\Rightarrow \quad x^3y\frac{dy}{dx} + 1 = 0$$

96. Given $x = a\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$

$$\Rightarrow \quad \frac{dx}{dy} = a\left(1 - \frac{1}{t^2}\right) \& \frac{dy}{dt} = a\left(1 + \frac{1}{t^2}\right)$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\left(1 + \frac{1}{t^2}\right)}{a\left(1 - \frac{1}{t^2}\right)} = \frac{a\left(t + \frac{1}{t}\right)}{a\left(t - \frac{1}{t}\right)} = \frac{x}{y}$$

97. Given $y = \cos^{-1}\left(\frac{5t + 12\sqrt{1-t^2}}{13}\right)$

$$\Rightarrow \quad y = \cos^{-1}\left(\frac{5}{13}t + \frac{12}{13}\sqrt{1-y^2}\right)$$

$$\Rightarrow \quad y = \cos^{-1}\left(\frac{5}{13}\right) - \cos^{-1} t$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$$

Also, $x = \cos^{-1}\left(\frac{1-t^2}{1+t^2}\right)$

$$\Rightarrow \quad x = 2\tan^{-1} t$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{2}{1+t^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{\sqrt{1-t^2}}}{\frac{2}{1+t^2}} = \frac{1+t^2}{2\sqrt{1-t^2}}$$

98. Given $x = \sin^{-1}\left(\frac{3\sin t + 4\cos t}{5}\right)$

$$\Rightarrow \quad x = \sin^{-1}\left(\frac{3}{5}\sin t + \frac{4}{5}\cos t\right)$$

$$\Rightarrow \quad x = \sin^{-1}(\sin(t+\alpha)) = t + \alpha$$

$$\Rightarrow \quad \frac{dx}{dt} = 1$$

Also, $y = \sin^{-1}\left(\frac{6\cos t + 8\sin t}{10}\right)$

$$\Rightarrow \quad y = \sin^{-1}\left(\frac{6}{10}\cos t + \frac{8}{10}\sin t\right)$$

$$\Rightarrow \quad y = \sin^{-1}\left(\frac{3}{5}\cos t + \frac{4}{5}\sin t\right)$$

$$\Rightarrow \quad y = \sin^{-1}(\sin(t+\beta))$$

$$\Rightarrow \quad y = (t + \beta)$$

$$\Rightarrow \quad \frac{dy}{dt} = 1$$

Thus, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1}{1} = 1.$$

99. $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t}$$

$$\frac{dy}{dx} = -\cot(t)$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{2\pi}{3}} = -\cot\left(\frac{2\pi}{3}\right)$$

$$\left(\frac{dy}{dx}\right)_{x=\frac{2\pi}{3}} = \cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$$

Hence, the result.

100. $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^{\sin 2t} \times 2\cos(2t)}{e^{\cos 2t} \times -2\sin(2t)}$$

$$\frac{dy}{dx} = -\frac{y \times \log(x)}{x \times \log(y)}$$

101. $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{2\sin(2\theta) - 2\sin(\theta)}{2\cos(\theta) - 2\cos(2\theta)}$$

$$= \frac{\sin(2\theta) - \sin(\theta)}{\cos(\theta) - \cos(2\theta)}$$

$$= \frac{2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{\cos\left(\frac{3\theta}{2}\right)}{\sin\left(\frac{3\theta}{2}\right)} = \cot\left(\frac{3\theta}{2}\right)$$

102. Given $x = a\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a\left(1 + \frac{1}{t^2}\right)}{a\left(1 - \frac{1}{t^2}\right)} = \frac{a\left(t + \frac{1}{t}\right)}{a\left(t - \frac{1}{t}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a\left(t + \frac{1}{t}\right)}{a\left(t - \frac{1}{t}\right)} = \frac{x}{y}$$

103. Given $x = \sin^{-1}\left(\frac{2t}{1 + t^2}\right)$ and $y = \tan^{-1}\left(\frac{2t}{1 - t^2}\right)$

$$\Rightarrow \quad x = \pi - 2\tan^{-1}(t),\ y = 2\tan^{-1}(t)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{1}{(1 + t^2)}}{\frac{1}{(1 + t^2)}} = -1$$

Hence, the result.

104. Given $y = \cos^{-1}\left(\frac{5t + 12\sqrt{1 - t^2}}{13}\right)$

and $x = \cos^{-1}\left(\frac{1 - t^2}{1 + t^2}\right)$, then find $\frac{dy}{dx}$

$$\Rightarrow \quad y = \cos^{-1}(t) - \cos^{-1}\left(\frac{5}{13}\right),\ x = 2\tan^{-1}(t)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{-\frac{1}{\sqrt{1 - t^2}}}{\frac{2}{1 + t^2}} = -\frac{1 + t^2}{2\sqrt{1 - t^2}}.$$

105. Given $x = \sin^{-1}\left(\frac{3\sin t + 4\cos t}{5}\right)$

and $y = \sin^{-1}\left(\frac{6\cos t + 8\sin t}{10}\right)$,

$$\Rightarrow \quad y = \sin^{-1}(\sin(\alpha + \sin t)),\ \alpha = \tan^{-1}\left(\frac{3}{4}\right)$$

and $\quad x = \sin^{-1}(\sin(\beta + \sin t)),\ \beta = \tan^{-1}\left(\frac{3}{4}\right)$

$$\Rightarrow \quad y = \alpha + \sin(t),\ x = \beta + \sin(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{\cos t} = 1$$

Hence, the result.

106. Given $x = \tan^{-1}\frac{(\sqrt{1 + \sin t} + \sqrt{1 - \sin t}}{\sqrt{1 + \sin t} - \sqrt{1 - \sin t})}$

and $y = \tan^{-1}\left(\frac{\sqrt{1 + t^2} - 1}{t}\right)$.

$$\Rightarrow \quad x = \tan^{-1}\left(\cot\left(\frac{t}{2}\right)\right)$$

and $y = \tan^{-1}\left(\tan\left(\frac{\theta}{2}\right)\right)$, $t = \tan(\theta)$

$$\Rightarrow \quad x = \frac{t}{2},\ y = \frac{\theta}{2} = \frac{\tan^{-1}(t)}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2(1 + t^2)}}{\frac{1}{2}} = \frac{1}{(t^2 + 1)}$$

107. Let $u = f(x^2 + 2012)$ and $v = f(x^3 + 2013)$

Thus, $\frac{du}{dv} = \frac{d(f(x^2 + 2012))}{d(f(x^3 + 2013))}$

$$\Rightarrow \quad \frac{du}{dv} = \frac{f'(x^2 + 2012)\cdot 2x}{f'(x^3 + 2013)\cdot 3x^2}$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{2}{3x}\ \frac{f'(x^2 + 2012)}{f'(x^2 + 2013))}.$$

108. Let $u = x^x$ and $v = x\log x$

Thus, $\frac{du}{dv} = \frac{d(x^x)}{d(x\log x)}$

$$= \frac{x^x(1 + \log x)}{(1 + \log x)} = x^x$$

109. Let $u = \tan^{-1}\left\{\frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}\right\}$, $v = \cos^{-1}(x^2)$

$$\Rightarrow \quad u = \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right),\ x^2 = \cos(2\theta)$$

$$\Rightarrow \quad u = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right)$$

$$\Rightarrow \quad u = \left(\frac{\pi}{4} - \theta\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}(x^2)$$

$$\Rightarrow \quad u = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}(x^2);\ v = \cos^{-1}(x^2)$$

$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = -\frac{1}{2}$

110. Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \quad u = 2\tan^{-1}(x)$ and $v = 2\tan^{-1}(x)$

$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{2/(1+x^2)}{2/(1+x^2)} = 1$

111. Let $u = (\sin x)$, $v = f(\cos x)$

$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{dv/dx}$

$\Rightarrow \quad \frac{du}{dv} = \frac{du/dx}{du/dx} = -\frac{f'(\sin x)\cos x}{f'(\cos x)\sin x}$

112. We have $y = c_1 e^x + c_2 e^{-x}$

$\Rightarrow \quad \frac{dy}{dx} = c_1 e^x - c_2 e^{-x}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = c_1 e^x + c_2 e^{-x} = y$

$\Rightarrow \quad \frac{d^2y}{dx^2} - y = 0.$

113. We have $x = \text{at}^2$, $y = 2$ at

$\Rightarrow \quad \frac{dx}{dt} = 2\text{at}$ & $\frac{dy}{dt} = 2a$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$\Rightarrow \quad = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$

$\Rightarrow \quad = \frac{d}{dt}\left(\frac{1}{t}\right)\cdot\frac{dt}{dx}$

$\Rightarrow \quad = -\frac{1}{t^2}\cdot\frac{1}{2at}$

$\Rightarrow \quad = -\frac{1}{2at^3}$

114. We have $x = a\cos\theta$, $y = b\sin\theta$

$\Rightarrow \quad \frac{dx}{d\theta} = -a\sin\theta$ & $\frac{dy}{d\theta} = b\cos\theta$

$\Rightarrow \quad \frac{dx}{d\theta} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{b}{a}\cot\theta\right)$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\frac{d\theta}{dx}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{b}{a}\text{cosec}^2\theta\right)\cdot\frac{-1}{a\sin\theta}$

$= \left(-\frac{b}{a^2}\text{cosec}^3\theta\right)$

$= \left(-\frac{b}{a}\cdot\frac{b^3}{y^3}\right)$

$= -\frac{b^4}{ay^3}$

115. $\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right)\cdot\frac{dx}{dy}$

$= \frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)\cdot\frac{dx}{dy}$

$= \frac{-1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\frac{dy}{dx}}$

$= -\left(\frac{dy}{dx}\right)^{-3}\cdot\left(\frac{d^2y}{dx^2}\right)$

116. Given $y = e^{2x}$

$\Rightarrow \quad \frac{dy}{dx} = 2e^{2x}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = 4e^{2x}$

$\Rightarrow \quad \frac{d^2x}{d^2y} = \frac{d}{dy}\left(\frac{dx}{dy}\right)\cdot\frac{dx}{dy} = \frac{d}{dx}\left(\frac{dx}{dy}\right)\cdot\frac{dx}{dy}$

$= \frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)\cdot\frac{dx}{dy}$

$= \frac{-1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\frac{dy}{dx}}$

$= -\left(\frac{dy}{dx}\right)^{-3}\cdot\left(\frac{d^2y}{dx^2}\right)$

Thus, $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$.

$= \frac{d^2y}{dx^2}\cdot -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}$

$= \left(\frac{d^2y}{dx^2}\right)^2\left(\frac{dy}{dx}\right)^{-3}$

$= (4e^{2x})^2(2e^{2x})^{-3}$

$$= \frac{16e^{4x}}{8e^{6x}}$$

$$= \frac{2}{e^{2x}}$$

$$= \frac{2}{y}.$$

117. We have $(a + bx)e^{\frac{y}{x}} = x$

$\Rightarrow \quad e^{\frac{y}{x}} = \frac{x}{a + bx}$

$\Rightarrow \quad \frac{y}{x} = \log\left(\frac{x}{a + bx}\right)$

$\Rightarrow \quad \frac{y}{x} = \log x - \log(a + bx)$

$\Rightarrow \quad \frac{x \cdot \frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{b}{a + bx} = \frac{a + bx - bx}{x(a + bx)}$

$\Rightarrow \quad x\frac{dy}{dx} - y = \frac{ax^2}{x(a + bx)} = \frac{ax}{a + bx}$

$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 - \frac{dy}{dx} = \frac{(a + bx)\cdot a - ax\cdot b}{(a + bx)^2}$

$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a + bx)^2}$

$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a + bx)^2} = \left(\frac{ax}{a + bx}\right)^2$

$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$

Hence, the result.

118. We have $x^2 + y^2 = a^2$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{b}{a^2}\cot^3\theta = -\frac{b}{a^2}\left(\frac{b}{y}\right)^3 = \frac{a^2b^4}{a^2y^3}$

$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \quad x + y\frac{dy}{dx} = 0$

$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$

$\Rightarrow \quad -y\frac{d^2y}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2$

$\Rightarrow \quad \left|\frac{d^2y}{dx^2}\right| = \left|\frac{1 + \left(\frac{dy}{dx}\right)^2}{-y}\right| = \frac{1 + \left(\frac{dy}{dx}\right)^2}{|y|}$

$\Rightarrow \quad |y''| = \frac{(1 + y'^2)}{|y|} = \frac{(1 + y'^2)^{3/2}}{a}$

$\Rightarrow \quad \frac{1}{a} = \frac{|y''|}{(1 + y'^2)^{3/2}}$

$\Rightarrow \quad k = \frac{|y''|}{(1 + y'^2)^{3/2}}$

$\left[\because x + yy' = 0 \Rightarrow y' = -\frac{x}{y} \Rightarrow y'^2 = \frac{x^2}{y^2}\right.$

$\Rightarrow \quad 1 + y'^2 = 1 + \frac{x^2}{y^2} = \frac{x^2 + y^2}{y^2} = \frac{a^2}{y^2}$

$\Rightarrow \quad \left. y^2 = \frac{a^2}{1 + y'^2} \Rightarrow |y| = \frac{a}{\sqrt{1 + y'^2}}\right]$

119. We have $y = e^{ax}\sin bx$

$\Rightarrow \quad \frac{dy}{dx} = ae^{ax}\sin bx + e^{ax}\cdot b\cos bx$

$\Rightarrow \quad \frac{d^2y}{dx^2} = a^2e^{ax}\sin bx + abe^{ax}\cos bx + abe^{ax}\cos bx - b^2e^{ax}\sin bx$

$y^2 - 2ay_1 + (a^2 + b^2)y$

$= a^2e^{ax}\sin bx + abe^{ax}\cos bx$

$+ abe^{ax}\cos bx - b^2e^{ax}\sin bx$

$-2a(ae^{ax}\sin bx + be^{ax}\cos bx)$

$+ (a^2 + b^2)e^{ax}\sin bx$

$= e^{ax}(a^2\sin bx + 2ab\cos bx - b^2\sin bx$

$-2a^2\sin bx - 2ab\cos bx + (a^2 + b^2)\sin bx)$

$= 0.$

Hence, the result.

120. Given $y = 2\sin x + 3\cos x$

$\Rightarrow \quad \frac{dy}{dx} = 2\cos x - 3\sin x$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -2\sin x - 3\cos x$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -2(\sin x + 3\cos x) = -y$

$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0$

Hence, the result.

121. Given $y = x + \tan x$

$\Rightarrow \quad \frac{dy}{dx} = 1 + \sec^2 x$

$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\sec^2 x\tan x$

$\Rightarrow \quad \cos^2 x\frac{d^2y}{dx^2} = 2\tan x$

Now, $\quad \cos^2 x\frac{d^2y}{dx^2} - 2y + 2x$

$= 2\tan x - 2(x + \tan x) + 2x$

$= 2\tan x - 2x - 2\tan x + 2x$
$= 0.$

122. Given $y = \tan x + \sec x$

$\Rightarrow \frac{dy}{dx} = \sec^2 x + \sec x \tan x$

$\Rightarrow \frac{dy}{dx} = \frac{1 + \sin x}{\cos^2 x}$

$\Rightarrow \frac{dy}{dx} = \frac{1 + \sin x}{1 - \sin^2 x} = \frac{1}{1 - \sin x}$

$\Rightarrow \frac{dy}{dx} = \frac{-\cos x}{(1 - \sin x)^2} = \frac{\cos x}{(1 - \sin x)^2}$

Hence, the result.

123. Given $y = A \cos(\log x) + B \sin(\log x)$

$\Rightarrow \frac{dy}{dx} = -\frac{A\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$

$\Rightarrow x\frac{dy}{dx} = -A\sin(\log x) + B\cos(\log x)$

$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{A\cos(\log x)}{x} - \frac{B\sin(\log x)}{x}$

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -(A\cos(\log x) + B\sin(\log x))$

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$

$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Hence, the result.

124. Given $y = \tan^{-1} x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1 + x^2}$

$\Rightarrow (1 + x^2)\frac{dy}{dx} = 1$

$\Rightarrow (1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

$\Rightarrow (1 + x^2)y_2 + 2xy_1 = 0$

125. Given $y = e^x (\sin x + \cos x)$,

$\Rightarrow \frac{dy}{dx} = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$

$\Rightarrow \frac{dy}{dx} = 2e^x \cos x$

$\Rightarrow \frac{d^2y}{dx^2} = 2e^x(\cos x - \sin x)$

Now, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$

$= 2e^x(\cos x - \sin x) - 4e^x\cos x + 2e^x(\sin x + \cos x)$
$= 2e^x(\cos x - \sin x - 2\cos x + \sin x + \cos x)$
$= 2e^x(0)$
$= 0.$

Hence, the result.

126. Given $y = \sin^{-1}(x)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{(2x)}{2(1 - x^2)\sqrt{1 - x^2}}$

$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} = \frac{x}{\sqrt{1 - x^2}} = x\frac{dy}{dx}$

$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

Hence, the result.

127.

(i) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{t^2} \times \frac{1}{2at} = -\frac{1}{2at^3}$

(ii) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta}$

$\Rightarrow \frac{dy}{dx} = -\tan\theta$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\tan\theta)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{3a\cos^2\theta\sin\theta}$

$= \frac{a^3}{3x(axy)^{1/3}} = \frac{a^{8/3}}{3(x^4y)^{1/3}} = \frac{1}{3}\left(\frac{a^8}{x^4y}\right)^{1/3}$

128. Given $x = a\sec\theta$, $y = b\tan\theta$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a}\operatorname{cosec}\theta\cot\theta \times \frac{1}{a\sec\theta\tan\theta}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{b}{a^2}\cot^3\theta = -\frac{b}{a^2}\left(\frac{b}{y}\right)^3 = -\frac{b^4}{a^2y^3}$

129. Given $y = \sin^{-1} x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{(-2x)}{2(1-x^2)\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} = \frac{x}{\sqrt{1-x^2}} = x\frac{dy}{dx}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1-x^2)y_2 - xy_1 = 0$$

Hence, the result.

130. **

131. **

132. Given $x = a(\theta + \sin\ \theta)$, $y = a(1 + \cos\theta)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-a\sin\theta}{a(1+\cos\theta)} = -\frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)} = -\tan\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) \times \frac{1}{a(1+\cos\ \theta)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{a}\ \frac{1}{(1+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{a} \times \frac{a^2}{y^2} = -\frac{a}{y^2}$$

Hence, the result.

133. Given $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$,

$$\frac{dy}{dx} = \frac{a(1+\cos\theta)}{a\sin\theta}$$

$$\frac{dy}{dx} = -\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} = \cot\left(\frac{\theta}{2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\cot\left(\frac{\theta}{2}\right)\right) \times \frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\text{cosec}^2\left(\frac{\theta}{2}\right) \times \frac{1}{a\sin\theta}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2a}\ \frac{\text{cosec}^2\left(\frac{\theta}{2}\right)}{\sin\theta}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{2}} = -\frac{1}{2a} \times \frac{2}{1} = -\frac{1}{a}$$

134. Given $y = x\log\left(\frac{x}{a+bx}\right)$

$$\Rightarrow \quad \frac{y}{x} = \log\left(\frac{x}{a+bx}\right)$$

$$\Rightarrow \quad \frac{y}{x} = \log x - \log(a+bx)$$

$$\Rightarrow \quad \frac{x\cdot\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{b}{a+bx} = \frac{a+bx-bx}{x(a+bx)}$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = \frac{ax^2}{x(a+bx)} = \frac{ax}{a+bx}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 - \frac{dy}{dx} = \frac{(a+bx)a\cdot - ax\cdot b}{(a+bx)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(\frac{ax}{a+bx}\right)^2$$

$$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$

135. Given $\sqrt{x+y} + \sqrt{y-x} = c$

$$\Rightarrow \quad \frac{1}{2\sqrt{x+y}}\left(1 + \frac{dy}{dx}\right) + \frac{1}{2\sqrt{y-x}}\left(\frac{dy}{dx} - 1\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(\sqrt{x+y} - \sqrt{y-x})(\sqrt{x+y} + \sqrt{y-x}\ )}{c(\sqrt{x+y} + \sqrt{y-x})}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x+y-y+x)}{c^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{c^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2}{c^2}$$

136. We have $\frac{\cos^4\alpha}{x} + \frac{\sin^4\alpha}{y} = \frac{1}{x+y}$

$$\Rightarrow \quad \left(\frac{x+y}{x}\right)\cos^4\alpha + \left(\frac{x+y}{y}\right)\sin^4\alpha = 1$$

$$\Rightarrow \quad \left(1 + \frac{y}{x}\right)\cos^4\alpha + \left(1 + \frac{x}{y}\right)\sin^4\alpha = 1$$

$$\Rightarrow \quad \left(\frac{y}{x}\right)\cos^4\alpha + \left(\frac{x}{y}\right)\sin^4\alpha + (\sin^4\alpha + \cos^4\alpha) = 1$$

$$\Rightarrow \quad \left(\frac{y}{x}\right)\cos^4\alpha + \left(\frac{x}{y}\right)\sin^4\alpha + 1 - 2\sin^2\alpha\ \cos^2\alpha = 1$$

$$\Rightarrow \quad \left(\frac{y}{x}\right)\cos^4\alpha + \left(\frac{x}{y}\right)\sin^4\alpha - 2\ \sin^2\alpha\ \cos^2\alpha = 0$$

$$\Rightarrow \quad \left(\sqrt{\frac{y}{x}}\cos^2\alpha\right)^2 + \left(\sqrt{\frac{x}{y}}\sin^2\alpha\right)^2 - 2\left(\sqrt{\frac{x}{y}}\sin^2\alpha\right)\left(\sqrt{\frac{y}{x}}\cos^2\alpha\right) = 0$$

$$\Rightarrow \left(\sqrt{\frac{x}{y}}\sin^2\alpha - \sqrt{\frac{y}{x}}\cos^2\alpha\right)^2 = 0$$

$$\Rightarrow \left(\sqrt{\frac{x}{y}}\sin^2\alpha - \sqrt{\frac{y}{x}}\cos^2\alpha\right) = 0$$

$$\Rightarrow \sqrt{\frac{x}{y}}\sin^2\alpha = \sqrt{\frac{y}{x}}\cos^2\alpha$$

$$\Rightarrow y = (x)\tan^2\alpha$$

$$\Rightarrow \frac{dy}{dx} = \tan^2\alpha$$

137. Given $y = x\sin x$

$$\Rightarrow \frac{dy}{dx} = x\cos x + \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \cos x - x\sin x + \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\cos x - x\sin x$$

Now, $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2 + 2)y$

$$x^2(2\cos x - x\sin x) - 2x(x\cos x + \sin x) + (x^2 + 2)x\sin x$$

$$= x^2(2\cos x - x\sin x + x\sin x - 2\cos x) + x(2\sin x - 2\sin x)$$

$$= x^2(0) + x(0) = 0$$

Hence, the result.

138. Given $y = f(x) = x^2$

$$\Rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2$$

Also, $x^2 = y$

$$\Rightarrow 2x\frac{dx}{dy} = 1$$

$$\Rightarrow 2x\frac{d^2x}{dy^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = 0$$

$$\Rightarrow x\frac{d^2x}{dy^2} + \frac{1}{4x^2} = 0$$

$$\Rightarrow \frac{d^2x}{dy^2} = -\frac{1}{4x^3}$$

Now, $\left(\frac{d^2y}{dx^2}\right) \times \left(\frac{d^2x}{dy^2}\right) = 2 \times -\frac{1}{4x^3} = -\frac{1}{2x^3}$

139. Given $P(x) = ax^3 + bx^2 + cx + d$

$$P'(x) = 2ax^2 + 2bx + c$$

$$P''(x) = 4ax + 2b$$

$$P''(0) = 10 \Rightarrow 2b = 10 \Rightarrow b = 5$$

$$P(0) = -2 \Rightarrow d = -2$$

$$P(1) = -2 \Rightarrow a + b + c + d = -2$$

$$a + 5 + c - 2 = -2$$

$$a + c = -5 \quad \text{...(i)}$$

$$P'(0) = -1 \Rightarrow c - 1$$

From (i), we get, $a = -4$

Hence, the value of $(a + b + c + d + 10)$

$$= -4 + 5 - 1 - 2 + 10$$

$$= 8$$

140. Given $f(x) = x + \tan x$

Since f is the inverse of g, so $(f_0 g)(x) = x$

$$f(g(x)) = x$$

$$f'(g(x))g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g'(x) = \frac{1}{1 + \sec^2(g(x))} = \frac{1}{1 + 1 + \tan^2(g(x))}$$

$$g'(x) = \frac{1}{2 + \tan^2(g(x))}$$

141. Given $f(x) = e^x + x^3 - 1$

$$f'(x) = e^x + 3x^2$$

$$f''(x) = e^x + 6x$$

when $y = f(x) = e$, then $x = 1$

$$g''(x) = -\frac{f''(x)}{\{f'(x)\}^3}$$

$$g''(e) = -\frac{f''(e)}{\{f'(e)\}^3} = -\frac{e + 6}{(e + 3)^3}$$

142. Given $f(x) = 1 + x^3$

$$\Rightarrow f'(x) = 3x^2$$

$$\Rightarrow f''(x) = 6x$$

$$\Rightarrow f'''(x) = 6$$

when $y = 2$, then $x = 1$

Now, $g''(y) = -\frac{f''(x)}{\{f'(x)\}^2}$

$$\Rightarrow g'''(y) = -\frac{f'(x)f'''(x) - 3\{f''(x)\}^2}{\{f'(x)\}^3}$$

$$\Rightarrow g'''(2) = -\frac{f'(1)f'''(1) - 3\{f''(1)\}^2}{\{f'(1)\}^3}$$

$\Rightarrow \quad g'''(2) = -\dfrac{3 \times 6 - 3(6)^2}{(3)^3}$

$\Rightarrow \quad g'''(2) = \dfrac{72}{27} = \dfrac{8}{3}$

143. Given $f(g(x)) = x$

$\Rightarrow \quad f'(g(x))\,g'(x) = 1$

$\Rightarrow \quad g'(x) = \dfrac{1}{f'(g(x))}$

$\Rightarrow \quad g'(x) = \dfrac{1}{1 + (f(g(x)))^2}$

$\Rightarrow \quad g'(x) = \dfrac{1}{1 + x^2}$

Hence, the result.

144 We have $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$\Rightarrow \quad f'(x) = \begin{vmatrix} 3x^3 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$\Rightarrow \quad f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$\Rightarrow \quad f'''(x) = \begin{vmatrix} 6 & -\cos x & \sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$\Rightarrow \quad f'''(x) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$

Thus, $\dfrac{d^3(0)}{dx^3} = 0$

145. Let $f(x) = \begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}$

Differentiating both sides w.r.t x and then putting $x = 0$, we get, the co-efficient of x

$$\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix}$$

$= 0 + 0 + 0$

$= 0.$

146. We have $f(x) = \begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix}$

$\Rightarrow \quad f'(x) = \begin{vmatrix} 1 & 1 & 1 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix}$

$$+ \begin{vmatrix} x+1 & x+2 & x+3 \\ 1 & 1 & 1 \\ x+7 & x+8 & x+9 \end{vmatrix}$$

$$+ \begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ 1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow \quad f''(x) = 0$

On integration, we get,

$\Rightarrow \quad f(x) = c_1$

Again, integrating, we get,

$\Rightarrow \quad f(x) = c_1x + c_2$

Hence, the result.

147. We have $\begin{vmatrix} 1 & 2x & 3x^2 \\ x & x^2 & x^3 \\ 0 & 2 & 6x \end{vmatrix}$

$\Rightarrow \quad f'(x) = \begin{vmatrix} 0 & 2 & 6x \\ x & x^2 & x^3 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$

$$+ \begin{vmatrix} 1 & 2x & 3x^2 \\ x & x^2 & x^3 \\ 0 & 0 & 6 \end{vmatrix}$$

$\Rightarrow \quad f'(x) = 0 + 0 + 6(x^2 - 2x^2) = -6x^2$

$\Rightarrow \quad f'(1) = -6$

148. We have

$$f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 2\cos 3x \end{vmatrix}$$

$$\Rightarrow \quad f'(x) = \begin{vmatrix} -\sin x & \cos x & -\sin x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 2\cos 3x \end{vmatrix}$$

$$+ \begin{vmatrix} \cos x & \sin x & \cos x \\ -2\sin 2x & 2\cos 2x & -4\sin 2x \\ \cos 3x & \sin 3x & 2\cos 3x \end{vmatrix}$$

$$+ \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ -3\sin 3x & 3\cos 3x & -6\sin 3x \end{vmatrix}$$

$$\Rightarrow \quad f'\left(\frac{\pi}{2}\right) = \begin{vmatrix} -1 & 0 & -1 \\ -1 & 0 & -2 \\ 0 & -2 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & -2 \\ 3 & 0 & 6 \end{vmatrix}$$

$$\Rightarrow \quad f'\left(\frac{\pi}{2}\right) = (2-1) - 1(6-6) = 1$$

149. We have $y = x^n$

$\Rightarrow \quad y_1 = e^x$

$\Rightarrow \quad y_2 = e^x$

$\Rightarrow \quad y_3 = e^x$

... ...

... ...

$\Rightarrow \quad y_n = e^x$

Thus, $y_n(0) = e^0 = 1$

150. We have $y = a^x$

$\Rightarrow \quad y_1 = a^x \log a$

$\Rightarrow \quad y_2 = a^x (\log a)^2$

$\Rightarrow \quad y_3 = ax(\log a)^3$

... ...

... ...

$\Rightarrow \quad y_n = a^x (\log a)^n$

Thus, $y_n(0) = a^0 (\log a)^n = (\log a)^n$

151. We have $y = \dfrac{1}{(x+1)}$

$$\Rightarrow \quad y_1 = -\frac{1}{(x+1)^2}$$

$$\Rightarrow \quad y_2 = \frac{2}{(x+1)^3}$$

$$\Rightarrow \quad y_3 = -\frac{6}{(x+1)^4} = \frac{(-1)^3\, 3!}{(x+1)^{3+1}}$$

$$\Rightarrow \quad y_4 = \frac{(-1)^4\, 4!}{(x+1)^{4+1}}$$

... ...

... ...

$$\Rightarrow \quad y_n = \frac{(-1)^n (n)!}{(x+1)^{n+1}}$$

Thus, $y_n(0) = \dfrac{(-1)^n (n)!}{(0+1)^{n+1}} = (-1)^n (n)!$

152. We have $y = \sin x$

$\Rightarrow \quad y_1 = \cos x = \sin\left(\dfrac{\pi}{2} + x\right)$

$\Rightarrow \quad y_2 = -\sin x = \sin\left(2 \cdot \dfrac{\pi}{2} + x\right)$

$\Rightarrow \quad y_3 = -\cos x = \sin\left(3 \cdot \dfrac{\pi}{2} + x\right)$

... ...

... ...

... ...

$\Rightarrow \quad y_n = \sin\left(\dfrac{n\pi}{2} + x\right)$

Thus, $y_n(0) = \sin\left(\dfrac{n\pi}{2} + 0\right) = \sin\left(\dfrac{n\pi}{2}\right)$

$$= \begin{cases} 1 & : n = (4k+1), k \in 1 \\ -1 & : n = (4k-1), k \in 1 \\ 0 & : n = 2k, k \in 1 \end{cases}$$

153. We have $y = x^n$

$y_1 = nx^{n-1}$

$y_2 = n(n-1)x^{n-2}$

$y_3 = n(n-1)(n-2)x^{n-3}$

... ...

... ...

$y_k = n(n-1)(n-2)...(n-(k-1))x^{n-k+1}$

Thus, $\quad y_k(1) = n(n-1)(n-2)...(n-(k-1))$

$= (n-(k-1))!$

154. We have $y = A\sin x + B\cos x$

$\Rightarrow \quad y_1 = A\cos x - B\sin x$

$\Rightarrow \quad y_2 = -A\sin x - B\cos x = -y$

$\Rightarrow \quad y_2 + y = 0$

By Leibnitz theorem, we get,

$\Rightarrow \quad y_{n+2} + (0)y_{n+1} + (0)y_n + y_n = 0$

$\Rightarrow \quad y_{n+2} + y_n = 0$

155. We have $y = \tan^{-1}x$

$\Rightarrow \quad y_1 = \dfrac{1}{(1+x^2)}$

$\Rightarrow \quad (1+x^2)y_1 = 1$

$\Rightarrow \quad (1+x^2)y_2 + 2xy_1 = 0$

By Leibnitz theorem, we get,

$$(1+x^2)y_{n+2} + (2(n+1)x)y_{n+1} + \left(\frac{2n(n-1)}{2}\right)y_n$$

$$+ 2xy_{n+1} + (2n)y_n = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + (2(n+2)x)y_{n+1} + n(n+1)y_n = 0$$

156 We have $y = \sin^{-1}x$

$\Rightarrow \quad y_1 = \dfrac{1}{\sqrt{1-x^2}}$

$\Rightarrow \quad y_2 = -\dfrac{-2x}{2(1-x^2)\sqrt{1-x^2}} = \dfrac{xy_1}{(1-x^2)}$

$\Rightarrow \quad (1-x^2)y_2 = xy_1$

$\Rightarrow \quad (1-x^2)y_2 - xy_1 = 0$

By Leibnitz theorem, we get,

$$(1-x^2)y_{n+2} - 2(n+1)xy_{n+1} - n(n-1)y_n$$

$$-xy_{n+1} - ny_n = 0$$

$$\Rightarrow \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$$

157. We have $y = a\cos(\log x) + b\sin(\log x)$

$\Rightarrow \quad y_1 = -a\sin(\log x)\dfrac{1}{x} + b\cos(\log x)\cdot\dfrac{1}{x}$

$\Rightarrow \quad xy_1 = -a\sin(\log x) + b\cos(\log x)$

$\Rightarrow \quad xy_2 + y_1 = -a\cos(\log x)\cdot\dfrac{1}{x} - b\sin(\log x)\cdot\dfrac{1}{x}$

$\Rightarrow \quad x^2y_2 + xy_1 = -y$

$\Rightarrow \quad x^2y_2 + xy_1 + y = 0$

By Leibnitz theorem, we get,

$$x^2y_{n+2} + 2(n+1)xy_{n+1} + n(n-1)y_n$$

$$+ xy_{n+1} + ny_n + y_n = 0$$

$$\Rightarrow \quad x^2\cdot y_{n+2} + (2n+3)x\cdot y_{n+1} + (n^2+1)y_n = 0$$

Hints and Solutions

Level III

1. We have $y = \left\{\log\left(\dfrac{x+\sqrt{x^2-a^2}}{a}\right)\right\}^2$

$$+ k\log(x + \sqrt{x^2-a^2})$$

$\Rightarrow \quad \dfrac{dy}{dx} = 2\log\left(\dfrac{x+\sqrt{x^2-a^2}}{a}\right)$

$$\times\left(\frac{a}{x+\sqrt{x^2-a^2}}\right)\times\left(1+\frac{x}{\sqrt{x^2-a^2}}\right)$$

$$+ \frac{k}{(x+\sqrt{x^2-a^2})}\times\left(1+\frac{x}{\sqrt{x^2-a^2}}\right)$$

$$= 2\log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)\times\frac{a}{\sqrt{x^2-a^2}} + \frac{k}{\sqrt{x^2-a^2}}$$

$\Rightarrow \quad \left(\sqrt{x^2-a^2}\right)\dfrac{dy}{dx} = 2a\log\left(\left(\dfrac{x+\sqrt{x^2-a^2}}{a}\right)\right) + k$

$\Rightarrow \quad \left(\sqrt{x^2-a^2}\right)\dfrac{d^2y}{dx^2} + \dfrac{x}{\sqrt{x^2-a^2}}\dfrac{dy}{dx}$

$$= \frac{2a^2}{\left(x+\sqrt{x^2-a^2}\right)}\times\left(1+\frac{x}{\sqrt{x^2-a^2}}\right)$$

$$= \frac{2a^2}{\sqrt{x^2-a^2}}\times\frac{1}{a}$$

$$(x^2-a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2a$$

Hence, the result.

2. We have $(a+bx)^{e^{y/x}} = x$

$\Rightarrow \quad e^{\frac{y}{x}} = \dfrac{x}{a+bx}$

$\Rightarrow \quad \dfrac{y}{x} = \log\left(\dfrac{x}{a+bx}\right)$

$\Rightarrow \quad \dfrac{y}{x} = \log x - \log(a+bx)$

$\Rightarrow \quad \dfrac{x\cdot\dfrac{dy}{dx} - y}{x^2} = \dfrac{1}{x} - \dfrac{b}{a+bx} = \dfrac{a+bx-bx}{x(a+bx)}$

$\Rightarrow \quad x\dfrac{dy}{dx} - y = \dfrac{ax^2}{x(a+bx)} = \dfrac{ax}{a+bx}$

$\Rightarrow \quad x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\cdot 1 - \dfrac{dy}{dx} = \dfrac{(a+bx)\cdot a - ax\cdot b}{(a+bx)^2}$

$\Rightarrow \quad x\dfrac{d^2y}{dx^2} = \dfrac{a^2}{(a+bx)^2}$

$$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(\frac{ax}{a+bx}\right)^2$$

$$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$

Hence, the result.

3. If $y = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\left\{\sqrt{\frac{a-b}{a+b}}\right\}\tan\frac{x}{2}\right]$, then prove that

(i) $\frac{dy}{dx} = \frac{1}{a+b\cos x}$

(ii) $\frac{d^2y}{dx^2} = \frac{b\sin x}{(a+b\cos x)^2}$

3.(i) Given $y = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\left\{\sqrt{\frac{a-b}{a+b}}\right\}\tan\frac{x}{2}\right]$

$$\frac{dy}{dx} = \frac{2}{\sqrt{a^2-b^2}} \times \frac{1}{1+\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{x}{2}\right)} \times \frac{1}{2}\sqrt{\frac{a-b}{a+b}}\sec^2\left(\frac{x}{2}\right)$$

$$= \frac{1}{\sqrt{a^2-b^2}} \times \frac{(a+b)}{(a+b)+(a-b)\tan^2\left(\frac{x}{2}\right)} \times \sqrt{\frac{a-b}{a+b}}\sec^2\left(\frac{x}{2}\right)$$

$$= \frac{\sec^2(x/2)}{(a+b)+(a-b)\tan^2\left(\frac{x}{2}\right)}$$

$$= \frac{1+\tan^2\left(\frac{x}{2}\right)}{a\left(1+\tan^2\left(\frac{x}{2}\right)\right)+b\left(1-\tan^2\left(\frac{x}{2}\right)\right)}$$

$$= \frac{1}{a+b\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right)}$$

$$= \frac{1}{a+b\cos x}$$

(ii) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dx}\left(\frac{1}{a+b\cos x}\right)$$

$$= -\frac{1 \times -b\sin x}{(a+b\cos x)^2}$$

$$= \frac{b\sin x}{(a+b\cos x)^2}$$

4. We have $x = \sec\theta - \cos\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta = \tan\theta(\sec\theta + \cos\theta)$$

Also, $y = \sec^n\theta - \cos^n\theta$

$$\Rightarrow \quad \frac{dy}{d\theta} = n\sec^n\theta\tan\theta + n\cos^{n-1}\theta\sin\theta$$

$$\Rightarrow \quad \frac{dy}{d\theta} = n\tan\theta\,(\sec^n\theta + \cos^n\theta)$$

Now, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$= \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}$$

$$= \frac{n(\sec^n\theta + \cos^n\theta)}{(\sec\theta + \cos\theta)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{n^2\{(\sec^n\theta - \cos^n\theta)^2 + 4\}}{\{(\sec\theta - \cos\theta)^2 + 4\}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{n^2(y^2+4)}{(x^2+4)}.$$

$$\Rightarrow \quad (x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

Hence, the result.

5. We have

$$y = (\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$= \frac{\log(\sin x)}{\log(\cos x)} \times \frac{\log(\sin x)}{\log(\cos x)} + 2\tan^{-1}x$$

$$= \frac{\{\log(\sin x)\}^2}{\{\log(\cos x)\}^2} + 2\tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\{\log(\cos x)\}^2\, 2\{\log(\sin x)\}\cot x + \{\log(\sin x)\}^2 2\{\log(\cot x)\}\tan x}{\{\log(\cos x)\}^4} + \frac{2}{1+x^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}}$$

$$= \frac{2\left\{\log\left(\frac{1}{\sqrt{2}}\right)\right\}^3 + 2\left\{\log\left(\frac{1}{\sqrt{2}}\right)\right\}^3}{\left\{\log\left(\frac{1}{\sqrt{2}}\right)\right\}^4} + \frac{2}{1+\left(\frac{\pi}{4}\right)^2}$$

$$= \frac{4\left\{\log\left(\frac{1}{\sqrt{2}}\right)\right\}^3}{\left\{\log\left(\frac{1}{\sqrt{2}}\right)\right\}^4} + \frac{32}{\pi^2 + 16}$$

$$= \frac{4}{\left\{\left(\frac{1}{\sqrt{2}}\right)\right\}} + \frac{32}{\pi^2 + 16}$$

$$= \frac{4}{-\frac{1}{2}\log(2)} + \frac{32}{\pi^2 + 16}$$

$$= \frac{32}{\pi^2 + 16} - \frac{8}{\log(2)}$$

6. If $y = \log_u(|\cos 4x|) + |\sin x|$,

   where $u = \sec 2x$, then find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$

7. We have

$$y = 2\tan^{-1}\left(\frac{x\sqrt{2}}{1 - x^2}\right) + \log\left(\frac{1 + x\sqrt{2} + x^2}{1 - x\sqrt{2} + x^2}\right)$$

$$= 2\tan^{-1}\left(\frac{x\sqrt{2}}{1 - x^2}\right) + \log(1 + x\sqrt{2} + x^2) - \log(1 - x\sqrt{2} + x^2)$$

$$\frac{dy}{dx} = \frac{2}{1 + \left(\frac{2x^2}{(1 - x^2)^2}\right)} \cdot \frac{d}{dx}\left(\frac{x\sqrt{2}}{1 - x^2}\right) + \frac{2x + \sqrt{2}}{1 + x^2 + x\sqrt{2}} - \frac{2x - \sqrt{2}}{1 + x^2 - x\sqrt{2}}$$

$$= \frac{2(1 - x^2)^2}{1 + x^4} \cdot \frac{\sqrt{2}(1 + x^2)}{(1 - x^2)^2} + \frac{2(-2\sqrt{2}x^2) + 2\sqrt{2}(1 + x^2)}{(1 + x^2)^2 - 2x^2}$$

$$= \frac{2\sqrt{2}(1 + x^2)}{(1 + x^4)} + \frac{2\sqrt{2}(1 - x^2)}{(1 + x^4)}$$

$$= \frac{2\sqrt{2}(1 + x^2) + 2\sqrt{2}(1 - x^2)}{(1 + x^4)}$$

$$= \frac{2\sqrt{2}(1 + x^2 + 1 - x^2)}{(1 + x^4)}$$

$$= \frac{4\sqrt{2}}{(1 + x^4)}$$

8. We have $\prod_{r=1}^{n} \cos\left(\frac{x}{2^r}\right) = \frac{\sin x}{2^n \cdot \sin\left(\frac{x}{2^n}\right)}$,

$$\cos\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2^2}\right) \cdot \cos\left(\frac{x}{2^3}\right) \ldots \cos\left(\frac{x}{2^n}\right) = \frac{\sin x}{2^n \cdot \sin\left(\frac{x}{2^n}\right)},$$

Taking logarithm of both the sides, we get,

$$\log\left(\cos\left(\frac{x}{2}\right)\right) + \log\left(\cos\left(\frac{x}{2^2}\right)\right) + \log\left(\cos\left(\frac{x}{2^3}\right)\right) + \ldots + \log\cos\left(\frac{x}{2^n}\right) = \log\left(\frac{\sin x}{2^n \cdot \sin\left(\frac{x}{2^n}\right)}\right)$$

Differentiating w.r.t x, we get,

$$-\frac{1}{2}\tan\left(\frac{x}{2}\right) - \frac{1}{2^2}\tan\left(\frac{x}{2^2}\right) - \frac{1}{2^3}\tan\left(\frac{x}{2^3}\right) - \ldots - \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)$$

$$= \cot x - \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)$$

$$\frac{1}{2}\tan\left(\frac{x}{2}\right) + \frac{1}{2^2}\tan\left(\frac{x}{2^2}\right) + \frac{1}{2^3}\tan\left(\frac{x}{2^3}\right) + \ldots + \frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) = \frac{1}{2}\cot\left(\frac{x}{2^n}\right) - \cot x$$

$$\sum_{r=1}^{n} \frac{1}{2^r}\tan\left(\frac{x}{2^r}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - \cot(x)$$

(ii) Differentiating w.r.t x, we get,

$$\frac{1}{2^2}\sec^2\left(\frac{x}{2}\right) + \frac{1}{2^4}\sec^2\left(\frac{x}{2^2}\right) + \frac{1}{2^6}\sec^2\left(\frac{x}{2^3}\right) + \ldots + \frac{1}{2^{2n}}\sec^2\left(\frac{x}{2^n}\right) = \operatorname{cosec}^2 x - \frac{1}{2^{2n}}\operatorname{cosec}^2\left(\cdot\frac{x}{2^n}\right)$$

$$\sum_{r=1}^{n} \frac{1}{2^{2r}}\sec^2\left(\frac{x}{2^r}\right) = \operatorname{cosec}^2 x - \frac{1}{2^{2n}}\operatorname{cosec}^2\left(\frac{x}{2^n}\right)$$

9. We have $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$

$\Rightarrow\ 2p^2 = a^2(2\cos^2\theta) + b^2(2\sin^2\theta)$

$\Rightarrow\ 2p^2 = a^2(1 + \cos 2\theta) + b^2(1 - \cos 2\theta)$

$\Rightarrow\ 2p^2 = (a^2 + b^2) + (a^2 - b^2)\cos 2\theta$

Differentiate w.r.t θ, we get,

$$\Rightarrow\ 4p\frac{dp}{d\theta} = 2(a^2 - b^2)\sin 2\theta$$

$$\Rightarrow\ 2p\frac{dp}{d\theta} = (a^2 - b^2)\sin 2\theta$$

Again, differentiating w.r.t θ, we get,

$$\Rightarrow\ p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (a^2 - b^2)\cos 2\theta$$

$$\Rightarrow\ p^3\frac{d^2p}{d\theta^2} + p^2\left(\frac{dp}{d\theta}\right)^2 = p^2(a^2 - b^2)\cos 2\theta$$

$$\Rightarrow\ p^3\frac{d^2p}{d\theta^2} = p^2(a^2 - b^2)\cos 2\theta - p^2\left(\frac{dp}{d\theta}\right)^2$$

$$\Rightarrow\ p^4 + p^3\frac{d^2p}{d\theta^2} = p^4 + p^2(a^2 - b^2)\cos 2\theta - p^2\left(\frac{dp}{d\theta}\right)^2$$

$$= p^2(p^2 + (a^2 - b^2))\cos 2\theta - (b^2 - a^2)\sin^2\theta\cos^2\theta$$

$= (a^2\cos^2\theta + b^2\sin^2\theta)(b^2\cos^2\theta + a^2\sin^2\theta)$
$\quad - (b^2 - a^2)\sin^2\theta\cos^2\theta$
$= a^2b^2(\cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta)$
$= a^2b^2(\sin^2\theta + \cos^2\theta)^2$
$= a^2b^2$

Thus, $p + \dfrac{d^2p}{d\theta^2} = \dfrac{a^2p^2}{p^3}$

10. We have $z = \cos^7 x$

$\Rightarrow \dfrac{dz}{dx} = -7\cos^6 x\sin x$

Also, $y = \sin x$

$\Rightarrow \dfrac{dy}{dx} = \cos x$

Now, $\dfrac{dz}{dy} = -\dfrac{7\cos^6 x\sin x}{\cos x} = -7\cos^5 x\sin x$

$$\Rightarrow \frac{d^2z}{dy^2} = \frac{d}{dy}\left(\frac{dz}{dy}\right)$$
$$= \frac{d}{dx}\left(\frac{dz}{dy}\right)\frac{dx}{dy}$$
$$= \frac{d}{dx}\{(-7\cos^5 x\sin x)\}\cdot\frac{1}{\cos x}$$
$$= (35\cos^4 x\sin^2 x - 7\cos^6 x)\frac{1}{\cos x}$$
$$= (35\cos^3 x\sin^2 x - 7\cos^5 x)$$
$$\Rightarrow \frac{d^3y}{dz^3} = \frac{d}{dx}\left(\frac{d^2z}{dy^2}\right)\left(\frac{dz}{dy}\right)$$
$$= \frac{d}{dx}\{(35\cos^3 x\sin^2 x - 7\cos^5 x)\}\cdot\frac{1}{\cos x}$$
$$= -105\cos x + \sin^3 x + 105\cos^3 x\cdot\sin x$$
$$= \frac{105}{4}[4\cos^3 x\sin x - 4\cos x\sin^3 x]$$
$$= \frac{105}{4}(4\cos x\sin x)[\cos^2 x - \sin^2 x]$$
$$= \frac{105}{4}(2\sin 2x\cos 2x)$$
$$= \frac{105}{4}(\sin 4x)$$

11. Given $y^2 = p(x)$

Differentiate w.r.t x, we get,

$$\Rightarrow \quad 2y\frac{dy}{dx} = p'(x)$$

Again, differentiate w.r.t x, we get,

$$\Rightarrow \quad 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = p''(x)$$
$$\Rightarrow \quad 2y^3\frac{d^2y}{dx^2} + 2y^2\left(\frac{dy}{dx}\right)^2 = y^2p''(x)$$
$$\Rightarrow \quad 2y^3\frac{d^2y}{dx^2} = y^2p''(x) - 2y^2\left(\frac{dy}{dx}\right)^2$$
$$\Rightarrow \quad 2y^3\frac{d^2y}{dx^2} = y^2p''(x) - \frac{1}{2}(p'(x))^2$$

Again, differentiate w.r.t.x, we get,

$$\Rightarrow \quad 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$$
$$= 2y\left(\frac{dy}{dx}\right)p''(x) + y^2p'''(x) - p'(x)p''(x)$$
$$= p''(x)p''(x) + y^2p'''(x) - p'(x)p''(x)$$
$$= y^2p'''(x)$$
$$= p(x)p'''(x)$$

12. We have $\varphi(x) = f(x)g(x)$

Differentiate w.r.t.x, we get,

$$\Rightarrow \quad \varphi'(x) = f(x)g'(x) + f'(x)g(x)$$
$$\Rightarrow \quad \varphi' = fg' + f'g$$
$$\Rightarrow \quad \varphi'' = f'g' + fg'' + f''g + f'g'$$
$$\Rightarrow \quad \varphi'' = f''g + 2f'g' + g''f$$
$$\Rightarrow \quad \frac{\varphi''}{\varphi} = \frac{f''g + 2f'g' + g''f}{fg}$$
$$= \frac{f''}{f} + \frac{2f'g'}{fg} + \frac{g''}{g}$$
$$= \frac{f''}{f} + \frac{2c}{fg} + \frac{g''}{g}$$

Thus, $\dfrac{\varphi''}{\varphi} = \dfrac{f''}{f} + \dfrac{2c}{fg} + \dfrac{g''}{g}$

13. We have, $(1 - x)(1 + x) = (1 - x^2)$

$(1 - x)(1 + x)(1 + x^2) = 1 - x^4$

$(1 - x)(1 + x)(1 + x^2)(1 + x^4) = 1 - x^8$

... ...

... ...

... ...

$(1 - x)(1 + x)(1 + x^2)...(1 + x^{2n-1}) = 1 - x^{2n}$

when $n \to \infty$, $x < 1$, then $x^{2n} \to 0$

Thus, $(1 - x)(1 + x)(1 + x^2)...(1 + x^{2n-1}) = 1$

$\log\{(1 - x)(1 + x)(1 + x^2)...(1 + x^{2n-1})\} = \log(1) = 0$

$\log\{(1 - x)(1 + x)(1 + x^2)...(1 + x^{2n-1})\} = 0$

$$\log(1-x) + \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8) + \ldots = 0$$

Differentiate w.r.t x, we get,

$$-\frac{1}{1-x} + \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots = 0$$

$$\Rightarrow \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots = \frac{1}{1-x}$$

Hence, the result.

14. Given $x\sin\theta + y\cos\theta = a$

$x\cos\theta - y\sin\theta = b$

On solving, we get,

$x = a\sin\theta + b\cos\theta$

$y = a\cos\theta - b\sin\theta$

Now, put $a = r\sin\varphi$, $b = r\cos\varphi$

Therefore, $r^2 = a^2 + b^2$, $\varphi = \tan^{-1}\left(\frac{a}{b}\right)$

Then, $x = r\cos(\theta - \varphi)$, $y = r\sin(\varphi - \theta)$

$$\text{Hence, } \frac{d^3y}{dx^3} = r\cos\left(\frac{3\pi}{2} + \theta - \varphi\right)$$

$$= r\cos\left(\pi + \frac{\pi}{2} + \theta - \varphi\right)$$

$$= -\cos\left(\frac{\pi}{2} + \theta - \varphi\right)$$

$$= r\sin(\theta - \varphi)$$

$$\frac{d^2z}{d\theta^2} = r\cos\left(2\frac{\pi}{2} + \theta - \varphi\right)$$

$$= -r\cos(\theta - \varphi)$$

$$\frac{d^3y}{d\theta^3} = r\sin\left(\frac{3\pi}{2} + \theta - \varphi\right) = r\cos(\varphi - \theta)$$

$$\frac{d^2y}{d\theta^2} = r\sin\left(2\cdot\frac{\pi}{2} + \theta - \varphi\right) = -r\sin(\varphi - \theta)$$

$$\text{Now, } \frac{d^3x}{d\theta^3}\cdot\frac{d^2y}{d\theta^2} - \frac{d^2y}{d\theta^2}\cdot\frac{d^3y}{d\theta^3}$$

$$= r^2[\sin(\theta - \varphi)\sin(\theta - \varphi) + \cos(\theta - \varphi)\cos(\theta - \varphi)]$$

$$= r^2[\sin^2(\theta - \varphi) + \cos^2(\theta - \varphi)$$

$$= r^2 = (a^2 + b^2)$$

15. We have $x^2 + y^2 + z^2 - 2xyz = 1$

$$\Rightarrow 2xdx + 2ydy + 2zdz - 2yzdx - 2xzdy - 2xydz = 0$$

$$\Rightarrow xdx + ydy + zdz - yzdx - xzdy - xydz = 0$$

$$\Rightarrow (x - yz)dx + (y - zx)dy + (z - xy)dz = 0$$

$$\Rightarrow \frac{dx}{(y - zx)(z - xy)} + \frac{dy}{(x - yz)(z - xy)} + \frac{dz}{(y - zx)(x - yz)} = 0 \quad \ldots\text{(i)}$$

Now, $(y - zx)^2$

$$= y^2 + z^2x^2 - 2xyz$$

$$= z^2x^2 + (y^2 - 2xyz)$$

$$= z^2x^2 + 1 - x^2 - z^2$$

$$= 1 - x^2 - z^2 + z^2x^2$$

$$= (1 - x^2)(1 - z^2)$$

Similarly, $(z - xy)^2 = (1 - x^2)(1 - y^2)$,

$(x - zy)^2 = (1 - z^2)(1 - y^2)$

From (i), we get,

$$\frac{dx}{\sqrt{1 - x^2}} + \frac{dy}{\sqrt{1 - y^2}} + \frac{dz}{\sqrt{1 - z^2}} = 0$$

16. We have $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c} + 1.$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{x}{(x-c)}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx + x^2 - bx}{(x-b)(x-c)}$$

$$= \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x^2}{(x-b)(x-c)}$$

$$= \frac{ax^2 + x^3 - ax^2}{(x-a)(x-b)(x-c)}$$

$$= \frac{x^3}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow \log(y) = \log\left(\frac{x^3}{(x-a)(x-b)(x-c)}\right)$$

$$\Rightarrow \log(y) = 3\log x - \log((x-a)(x-b)(x-c))$$

$$\Rightarrow \log(y) = 3\log x - \log(x-a) - \log(x-b) - \log(x-c)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \frac{1}{(x-a)} - \frac{1}{(x-b)} - \frac{1}{(x-c)}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x} - \frac{1}{(x-a)}\right) + \left(\frac{1}{x} - \frac{1}{(x-b)}\right) + \left(\frac{1}{x} - \frac{1}{(x-c)}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left(\frac{x-a-x}{x(x-a)}\right) + \left(\frac{x-b-x}{x(x-b)}\right) + \left(\frac{x-c-x}{x(x-c)}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left(\frac{-a}{x(x-a)}\right) + \left(\frac{-b}{x(x-b)}\right) + \left(\frac{-c}{x(x-c)}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\left(\frac{a}{(a-x)} + \frac{b}{(b-x)} + \frac{c}{(c-x)}\right)$$

17. We have $\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}$

$$= \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ (x^2 f'' + 4xf') & (x^2 g'' + 4xg') & (x^2 h'' + 4xh') \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ (x^2 f'' + 4xf') & (x^2 g'' + 4xg') & (x^2 h'' + 4xh') \end{vmatrix}$$

$(R_2 \to R_2 - R_1)$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix} (R_3 \to R_3 - R_1)$$

$$= x^3 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

Thus, $\Delta' = 3x^2 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} + x^3 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f''' & g''' & h''' \end{vmatrix}$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ 3x^2 f'' & 3x^2 g'' & 3x^2 h'' \end{vmatrix} + x^3 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f''' & x^3 g''' & x^3 h''' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f''') & (x^3 g''') & (x^3 h''') \end{vmatrix}$$

Hence, the result.

18. We have $\frac{\cos^4\alpha}{x} + \frac{\sin^4\alpha}{y} = \frac{1}{x+y}$

$$\Rightarrow \left(\frac{x+y}{x}\right)\cos^4\alpha + \left(\frac{x+y}{y}\right)\sin^4\alpha = 1$$

$$\Rightarrow \left(1 + \frac{y}{x}\right)\cos^4\alpha + \left(1 + \frac{x}{y}\right)\sin^4\alpha = 1$$

$$\Rightarrow \left(\frac{y}{x}\right)\cos^4\alpha + \left(\frac{x}{y}\right)\sin^4\alpha + (\sin^4\alpha + \cos^4\alpha) = 1$$

$$\Rightarrow \left(\frac{y}{x}\right)\cos^4\alpha + \left(\frac{x}{y}\right)\sin^4\alpha + 1 - 2\sin^2\alpha\cos^2\alpha = 1$$

$$\Rightarrow \left(\frac{y}{x}\right)\cos^4\alpha + \left(\frac{x}{y}\right)\sin^4\alpha - 2\sin^2\alpha\cos^2\alpha = 0$$

$$\Rightarrow \left(\sqrt{\frac{y}{x}}\cos^2\alpha\right)^2 + \left(\sqrt{\frac{x}{y}}\sin^2\alpha\right)^2 - 2\left(\sqrt{\frac{x}{y}}\sin^2\alpha\right)\left(\sqrt{\frac{y}{x}}\cos^2\alpha\right) = 0$$

$$\Rightarrow \left(\sqrt{\frac{x}{y}}\sin^2\alpha - \sqrt{\frac{y}{x}}\cos^2\alpha\right)^2 = 0$$

$$\Rightarrow \left(\sqrt{\frac{x}{y}}\sin^2\alpha - \sqrt{\frac{y}{x}}\cos^2\alpha\right) = 0$$

$$\Rightarrow \sqrt{\frac{x}{y}}\sin^2\alpha = \sqrt{\frac{y}{x}}\cos^2\alpha$$

$$\Rightarrow y = (x)\tan^2\alpha$$

$$\Rightarrow \frac{dy}{dx} = \tan^2\alpha$$

19. If $y = \log\left(\sqrt{\frac{x^2+x+1}{x^2-x+1}}\right) + \frac{1}{2\sqrt{3}}\left\{\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right\}$,

then prove that, $\frac{dy}{dx} = \frac{1}{x^4 + x^2 + 1}$.

20. We have $t_1 = \tan^{-1}\left(\frac{1}{x^2+x+1}\right)$

$$= \tan^{-1}\left(\frac{(x+1)-x}{1+x(x+1)}\right) = \tan^{-1}(x+1) - \tan^{-1}(x)$$

$$t_2 = \tan^{-1}\left(\frac{1}{x^2+3x+3}\right)$$

$$= \tan^{-1}\left(\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right)$$

$$= \tan^{-1}(x+2) - \tan^{-1}(x+1)$$

$$t^3 = \tan^{-1}\left(\frac{1}{x^2+5x+7}\right)$$

$$= \tan^{-1}\left(\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}\right)$$

$= \tan^{-1}(x+3) - \tan^{-1}(x+2)$

....

....

....

$t_n = \tan^{-1}(x+n)\tan^{-1}(x+(n-))$

Thus, $S_n = t_1 + t_2 + t_3 + \dots + t_n$

$$y = \tan^{-1}(x+n) - \tan^{-1}(x)$$

$$\frac{dy}{dx} = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

21. We have

$$y = \tan^{-1}(\sin x + 1) - \tan^{-1}(\sin x) + \tan^{-1}(\sin x + 2) - \tan^{-1}(\sin x + 1) + \tan^{-1}(\sin x + 3) - \tan^{-1}(\sin x + 2) + \dots + \tan^{-1}(\sin x + n) - \tan^{-1}(\sin x + (n-1))$$

$$= -\tan^{-1}(\sin x + n) - \tan^{-1}(\sin x)$$

Differentiating w.r.t x, we get,

$$\frac{dy}{dx} = \frac{\cos x}{1+(\sin x + n)^2} - \frac{\cos x}{1+(\sin x)^2}$$

22. Given $y = f(x)$

Differentiating w.r.t x, we get,

$$\frac{dy}{dx} = f'(x),\ \frac{d^2y}{dx^2} = f''(x)$$

$$\frac{dx}{dy} = \frac{1}{f'(x)},\ \frac{d^2x}{dy^2} = -\frac{f''(x)}{\{f'(x)\}^2}\frac{dx}{dy} = -\frac{f''(x)}{\{f'(x)\}^3}$$

Now, $$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} + \frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}$$

$$= \frac{\{1+\{f'(x)\}^2\}^{3/2}}{f''(x)} + \frac{\left\{1+\left\{\frac{1}{f'(x)}\right\}^2\right\}^{3/2}}{-\frac{f''(x)}{\{f'(x)\}^3}}$$

$$= \frac{\{1+\{f'(x)\}^2\}^{3/2}}{f''(x)} - \frac{\{1+\{f'(x)\}^2\}^{3/2}}{f''(x)}$$

$$= 0$$

23. We have $y = \cos^{-1}(8x^4 - 8x^2 + 1)$

Put $x = \cos\theta$

$$y = \cos^{-1}(\cos 4\theta)$$

$$y = 4\theta = 4\cos^{-1}x$$

$$\frac{dy}{dx} = -\frac{4}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} + \frac{4}{\sqrt{1-x^2}} = 0$$

Note: 1. $\cos(4\theta) = 2(\cos(2\theta))^2 - 1$

$= 2(2\cos^2\theta - 1)^2 - 1$

$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$

$= (8\cos^4\theta - 8\cos^2\theta + 2) - 1$

$= (8\cos^4\theta - 8\cos^2\theta + 1)$

$= (8x^4 - 8x^2 + 1),\ x = \cos\theta$

24. As we know that, if $f(x)$ is divisible by x and $f'(x)$ is also divisible by x, then $f(x)$ is also divisible by x^2.

Given $$f(x) = \frac{a}{a+x} + \frac{b}{b+x} - \frac{c}{c+x} - \frac{d}{d+x}$$

$$f'(x) = -\frac{a}{(a+x)^2} - \frac{b}{(b+x)^2} + \frac{c}{(c+x)^2} + \frac{d}{(d+x)^2}$$

Since $f'(x)$ is divisible by x, then $f'(x) = 0$

$$-\frac{a}{(a+x)^2} - \frac{b}{(b+x)^2} + \frac{c}{(c+x)^2} + \frac{d}{(d+x)^2} = 0$$

Put $x = 0$, we get,

$$-\frac{1}{a} - \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 0$$

$$\frac{1}{a} + \frac{1}{b} - \frac{1}{c} - \frac{1}{d} = 0$$

25. We have $y = (\sin^{-1}x)^2 + (\cos^{-1}x)^2$

$$\frac{dy}{dx} = \frac{2\sin^{-1}x}{\sqrt{1-x^2}} - \frac{2\cos^{-1}x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{2(\sin^{-1}x - \cos^{-1}x)}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{2\sqrt{1-x^2}\left(\frac{2}{\sqrt{1-x^2}}\right) - 2(\sin^{-1}x - \cos^{-1}x)\frac{(-2x)}{2\sqrt{1-x^2}}}{(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = 4 + 2(\sin^{-1}x - \cos^{-1}x)\frac{x}{\sqrt{1-x^2}}$$

$$(1-x^2)\frac{d^2y}{dx^2} = 4 + x\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 4$$

Hence, the result.

26. We have $\left(\frac{x+b}{2}\right) = a\tan^{-1}(a\ln y), a > 0$

$$a\ln y = \tan\left(\frac{x+b}{2a}\right)$$

Differentiating w.r.t x, we get,

$$\frac{a}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2a}\sec^2\left(\frac{x+b}{2a}\right)$$

$$\frac{2a^2}{y}\left(\frac{dy}{dx}\right) = 1 + \tan^2\left(\frac{x+b}{2a}\right)$$

$$\frac{2a^2}{y}\left(\frac{dy}{dx}\right) = 1 + a^2(\ln y)^2$$

$$\frac{2}{y}\left(\frac{dy}{dx}\right) = \frac{1}{a^2} + (\ln y)^2$$

Again differentiating both sides w.r.t x, we get,

$$\frac{2}{y}\left(\frac{d^2y}{dx^2}\right) - \frac{2}{y^2}\left(\frac{dy}{dx}\right)^2 = \frac{2(\ln y)}{y}\left(\frac{dy}{dx}\right)$$

$$\frac{yd^2y}{dx^2} - (y\ln y)\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$$

Hence, the result.

27. We have $y = \frac{1}{x}$

$$\frac{dy}{dx} = -\frac{1}{x^2} \quad ...(i)$$

Now, $\sqrt{1 + y^4}$

$$= \sqrt{1 + \left(\frac{1}{x}\right)^4}$$

$$= \sqrt{\frac{x^4 + 1}{x^4}}$$

$$= \sqrt{\frac{x^4 + 1}{x^2}}$$

$$\frac{\sqrt{1 + y^4}}{\sqrt{1 + x^4}} = \frac{1}{x^2} \quad ...(ii)$$

From (i) and (ii), we get,

$$\frac{dy}{dx} = -\frac{\sqrt{1 + y^4}}{\sqrt{1 + x^4}}$$

$$\frac{dy}{\sqrt{1 + y^4}} = -\frac{dx}{\sqrt{1 + x^4}}$$

$$\frac{dy}{\sqrt{1 + y^4}} + \frac{dx}{\sqrt{1 + x^4}} = 0$$

Hence, the result.

28. We have $2x = y^{1/3} + y^{-1/3}$

$$\Rightarrow \quad 2xy^{1/3} = (y^{1/3})^2 + 1$$

$$\Rightarrow \quad (y^{1/3})^2 - 2xy^{1/3} + 1 = 0$$

$$\Rightarrow \quad (y^{1/3}) = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$\Rightarrow \quad (y^{1/3}) = \left(x \pm \sqrt{x^2 - 1}\right)$$

$$\Rightarrow \quad \frac{1}{3}\log(y) = \log\left(x \pm \sqrt{x^2 - 1}\right)$$

Differentiating w.r.t x, we get,

$$\Rightarrow \frac{1}{3y}\frac{dy}{dx} = \frac{1}{\left(x \pm \sqrt{x^2 - 1}\right)} \times \left(1 \pm \frac{1 \times 2x}{2\sqrt{x^2 - 1}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3y}{\sqrt{x^2 - 1}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{9y^2}{(x^2 - 1)}$$

$$\Rightarrow \quad (x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 9y^2$$

Again differentiating w.r.t x, we get,

$$\Rightarrow \quad 2(x^2 - 1)\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 18y\frac{dy}{dx}$$

$$\Rightarrow \quad (x^2 - 1)\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 9y\frac{dy}{dx}$$

$$\Rightarrow \quad (x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 9y$$

Hence, the result.

29. As we know that $\frac{dx}{dy} = \frac{1}{(dy/dx)}$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{(dy/dx)}\right)$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{d}{dx}\left(\frac{1}{(dy/dx)}\right)\frac{dx}{dy}$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = -\frac{1}{\left(\frac{dy}{dx}\right)^2} \times \frac{d^2y}{dx^2} \times \frac{dx}{dy}$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

$$\Rightarrow \quad \frac{\frac{d^2y}{dx^2}}{\frac{d^2x}{dy^2}} = -\left(\frac{dy}{dx}\right)^3 = -\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^3 = -\left(\frac{g'(t)}{f'(t)}\right)^3$$

Hence, the result.

30. We have

$y = x^5\,(\cos(\ln x) + \sin(\ln x))$

$$\Rightarrow \quad \frac{dy}{dx} = 5x^4(\cos(\ln x) + \sin(\ln x)) + x^5\left(-\frac{\sin(\ln x)}{x} + \frac{\cos(\ln x)}{x}\right)$$

$$\Rightarrow \quad x\frac{dy}{dx} = 5y + x^5(\cos(\ln x) - \sin(\ln x))$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 5\frac{dy}{dx} + 5x^4(\cos(\ln x) - \sin(\ln x)) + x^5\left(-\frac{\sin(\ln x)}{x} - \frac{\cos(\ln x)}{x}\right)$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 5\left(x\frac{dy}{dx} - 5y\right) - y$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} - 9x\frac{dy}{dx} = -26y$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} - 9x\frac{dy}{dx} + 26y = 0$$

LEVEL IV

1. Given $f(x) = x^3 + x^2 f'(1) + xf''(2) + f'''(3)$
Let $a = f'(1)$, $b = f''(2)$, $c = f'''(3)$
Then $f(x) = x^3 + ax^2 + bx + c$

$$\Rightarrow \quad f'(x) = 3x^2 + 2ax + b$$

$$\Rightarrow \quad f''(x) = 6x + 2a$$

$$\Rightarrow \quad f'''(x) = 6$$

$$\Rightarrow \quad f'''(3) = \Rightarrow c = 6$$

$$\Rightarrow \quad f''(2) = 12 + 2a \Rightarrow b = 12 + 2a$$

$$\Rightarrow \quad f'(1) = 3 + 2a + b \Rightarrow a = 3 + 2a + b$$

On solving, we get, $a = -5$, $b = 2$
Therefore, $f(x) = x^3 - 5x^2 + 2x + 6$
Now, $f(2) + f(0)$

$$= 8 - 20 + 4 + 6 + 6$$
$$= 24 - 20$$
$$= 4$$
$$= 1 - 5 + 2 + 6$$
$$= f(1)$$

2. Given $y = \frac{ax + b}{x^2 + c}$

Differentiating both sides w.r.t x, we get,
$(x^2 + c)y' + 2xy = a$
Again differentiating w.r.t x, we get,

$$\Rightarrow \quad (x^2 + c)y'' + 2xy' + 2y + 2xy' = 0$$

$$\Rightarrow \quad (x^2 + c)y'' + 4xy' + 2y = 0$$

$$\Rightarrow \quad (x^2 + c)y'' = -4xy' - 2y$$

$$\Rightarrow \quad -(x^2 + c) = (4xy' + 2y)/y''$$

Again, differentiating, w.r.t x, we get,

$$\Rightarrow \quad -2x = \frac{y''\{(4xy'' + 4y') + 2y'\} - (4xy' + 2y)y'''}{y''}$$

$$\Rightarrow \quad -x(y'')^2 = 2x(y'')^2 + 3y'y'' - (2xy' + y)y'''$$

$$\Rightarrow \quad (2xy' + y)y''' = 3(xy'' + y')y''$$

$$\Rightarrow \quad \left(2x\frac{dy}{dx} + y\right)\frac{d^3y}{dx^3} = 3\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right)\frac{d^2y}{dx^2}$$

Hence, the result.

3. Given $y = \tan^{-1}\left(\frac{x}{1.2 + x^2}\right) + \tan^{-1}\left(\frac{x}{2.3 + x^2}\right)$
$+ \tan^{-1}\left(\frac{x}{3.4 + x^2}\right) + \ldots$to n-terms

Let $t_n = \tan^{-1}\left(\frac{x}{n(n+1) + x^2}\right)$

$$= \tan^{-1}\left(\frac{\frac{x}{n(n+1)}}{1 + \frac{x^2}{n(n+1)}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x}{n} - \frac{x}{(n+1)}}{1 + \frac{x}{n}\cdot\frac{x}{(n+1)}}\right)$$

$$= \tan^{-1}\left(\frac{x}{n}\right) - \tan^{-1}\left(\frac{x}{n+1}\right)$$

Thus, $y = S_n = t_1 + t_2 + t_3 + \ldots + t_n$

$$= \tan^{-1}x - \tan^{-1}\left(\frac{x}{n+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{1}{1 + \left(\frac{x}{n+1}\right)^2}\left(\frac{1}{n+1}\right)$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2} - \frac{n+1}{(n+1)^2 + x^2}$$

4. Given $f(x) = (ax + b)\sin x + (cx + d)\cos x$

$$\Rightarrow \quad f'(x) = a\sin x + (ax + b)\cos x + c\cos x - (cx + d)\sin x$$

$$\Rightarrow \quad x\cos x = a\sin x + (ax + b)\cos x + c\cos x - (cx + d)\sin x$$

$= (a - cx - d)\sin x + (ax + b + c)\cos x$

Comparing the co-efficients of $\cos x$ and $\sin x$ we get,
$(a - cx - d) = 0$, $(ax + b + c) = x$

$$\Rightarrow \quad a - d = 0,\ c = 0;\ a = 1,\ b + c = 0$$

$$\Rightarrow \quad a = d = 1,\ b = 0 = c$$

5. Given $g(x) = (ax^2 + bx + c)\sin x + (dx^2 + ex + f)\cos x$

$$\Rightarrow \quad g'(x) = (2ax + b)\sin x + (ax^2 + bx + c)\cos x + (2dx + e)\cos x - (dx^2 + ex + f)\sin x$$

$$= (2ax + b - dx^2 - ex - f)\sin x$$

$+ (ax^2 + bx + c + 2dx + e)\cos x$

Comparing the co-efficients of sin x and cos x, we get, $(2ax + b - dx^2 - ex - f) = x^2$

and $(ax^2 + bx + c + 2dx + e) = 0$

$$\Rightarrow \begin{cases} d = -1, & b = f, \quad 2a = e \\ c = 0 = e, & a = 0, \quad b = -2d \end{cases}$$

$$\Rightarrow \begin{cases} a = 0, & b = 2, \quad c = 0 \\ d = -1, & e = 0, \quad f = 2 \end{cases}$$

6. Given $y = \dfrac{ax + b}{Ax + B}$

$$(Ax + B)y = ax + b$$

Differentiating w.r.t x, we get,

$$(Ax + B)y' + Ay = a \qquad ...(i)$$

$$(Ax + B)y'' + 2Ay' = 0 \qquad ...(ii)$$

$$(Ax + B)y''' + 3Ay'' = 0 \qquad ...(iii)$$

Dividing (iii) by (ii), we get,

$$\frac{y'''}{y''} = \frac{3y''}{2y'}$$

$$\frac{y'''}{y''y'} = \frac{3y''}{2(y')^2}$$

$$\frac{y'''}{y'} = \frac{3(y'')^2}{2(y')^2}$$

$$\frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = 0$$

Hence, the result

Also, $z = \dfrac{ay + b}{Ay + B} = \dfrac{a\left(\dfrac{ax + b}{Ax + B}\right) + b}{A\left(\dfrac{ax + b}{Ax + B}\right) + B}$

$$= \frac{(a^2 + Ab)x + (ab + Bb)}{(aA + AB) + (Ab + B^2)} = \frac{cx + b}{Cx + D}$$

which is same as of the form $y = \dfrac{ax + b}{Ax + B}$

So, we can easily prove that

$$\frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2 = 0$$

Thus, $\dfrac{y'''}{y'} - \dfrac{3}{2}\left(\dfrac{y''}{y'}\right)^2 = 0 = \dfrac{z'''}{z'} - \dfrac{3}{2}\left(\dfrac{z''}{z'}\right)^2$

Hence, the result.

7. We have $\lim\limits_{x\to 0}\left(\dfrac{1 - \cos x\cos 2x\cos 3x}{x^2}\right)$

Applying L'Hospital Rule, we get,

$$\lim_{x\to 0}\left(\frac{0 + \sin x \cos 2x \cos 3x + 2\cos x \sin 2x \cos 3x + 3\cos x \cos 2x \sin 3x}{2x}\right)$$

$$= \lim_{x\to 0}\frac{1}{2}\left(\frac{\sin x \cos 2x \cos 3x}{x} + \frac{2\cos x \sin 2x \cos 3x}{x} + \frac{3\cos x \cos 2x \sin 3x}{2x}\right)$$

$$= \frac{1}{2}(1 + 2^2 + 3^3)$$

$$= \frac{1}{2}(14) = 7$$

8. Given $y = e^{\tan^{-1}x}$

$$\frac{dy}{dx} = e^{\tan^{-1}x} \times \frac{1}{(1 + x^2)}$$

$$(x^2 + 1)\frac{dy}{dx} = e^{\tan^{-1}x}$$

$$(x^2 + 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = e^{\tan^{-1}x} \times \frac{1}{x^2 + 1}$$

$$(x^2 + 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{dy}{dx}$$

$$(x^2 + 1)\frac{d^2y}{dx^2} + (2x - 1)\frac{dy}{dx} = 0$$

Hence, the result.

9. Given $y = x^{n-1}\ln x \qquad ...(i)$

$$\frac{dy}{dx}\; x^{n-1}\cdot\frac{1}{x} + (n - 1)x^{n-2}\ln x$$

$$\frac{dy}{dx} = x^{n-2} + (n - 1)x^{n-2}\ln x$$

$$\frac{d^2y}{dx^2} = (n - 2)x^{n-3} + (n - 1)x^{n-2}\cdot\frac{1}{x} + (n - 1)(n - 2)x^{n-3}\ln x$$

$$\frac{d^2y}{dx^2} = (n - 2)x^{n-3} + (n - 1)x^{n-3} + (n - 1)(n - 2)x^{n-3}\ln x$$

$$\frac{x^2d^2y}{dx^2} = (n - 2)x^{n-1} + (n - 1)x^{n-1} + (n - 1)(n - 2)x^{n-1}\ln x \qquad ...(ii)$$

$$(3 - 2n)x\frac{dy}{dx} = (3 - 2n)x^{n-1} + (3 - 2n)(n - 1)x^{n-1}\ln x \qquad ...(iii)$$

From (i), (ii) and (iii), we get,

$$\frac{x^2d^2y}{dx^2} + (3 - 2n)x\frac{dy}{dx} + (n - 1)^2y$$

$= \{(n-2+n-1)+(n^2-3n+2)\ln x\}x^{n-1}$
$\quad + \{(3-2n)+(3-2n)(n-1)\ln x\}x^{n-1}$
$\quad + \{(n-1)^2\ln x\}x^{n-1}$
$= \{(2n-3)+(3-2n)\}x^{n-1}$
$\quad + \{(n^2-3n+2)+(n^2-2n+1)$
$\quad + (5n-2n^2-3)\}x^{n-1}\ln x$
$= (0)x^{n-1} + (0)x^{n-1}\ln x$
$= 0.$

10. Given $= (C_1 + C_2x)\sin x + (C_3 + C_4x)\cos x$ ...(i)

$\frac{dy}{dx} = (C_1 + C_2x)\cos x + C_2\sin x$
$\quad - (C_3 + C_4x)\sin x + C_4\cos x$

$= (C_1 + C_2x + C_4)\cos x + (C_2 - C_3 - C_4x)\sin x$

$\frac{d^2y}{dx^2} = C_2\cos x - (C_1 + C_2x + C_4)\sin x$
$\quad + (C_2 - C_3 - C_4x)\cos x - C_4\sin x$

$= (2C_2 - C_3 - C_4x)\cos x - (C_1 + C_2x + 2C_4)\sin x$ ...(ii)

$\frac{d^3y}{dx^3} = -(2C_2 - C_3 - C_4x)\sin x - C_4\cos x$
$\quad - (C_1 + C_2x + 2C_4)\cos x - C_2\sin x$

$= -(3C_2 - C_3 - C_4x)\sin x - (C_1 + C_2x + 3C_4)\cos x$

$\frac{d^4y}{dx^4} = -(3C_2 - C_3 - C_4x)\cos x + C_4\sin x$
$\quad + (C_1 + C_2x + 3C_4)\sin x - C_2\cos x$

$= -(4C_2 - C_3 - C_4x)\cos x + (C_1 + C_2x + 4C_4)\sin x$ ...(iii)

Now, $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y$

$= \{-(4C_2 - C_3 - C_4x) + 2(2C_2 - C_3 - C_4x)$
$\quad + (C_3 + C_4x)\}\cos x + \{(C_1 + C_2x + 4C_4)$
$\quad - 2(C_1 + C_2x + 2C_4) + (C_1 + C_2x)\}\sin x$
$= (0)\cos x + (0)\sin x$
$= 0.$

11. Given $f(x) = x^3f(1) + x^2f'(2) + xf''(3) + f'''(4)$

Let $f(1) = a, f'(2) = b, f''(3) = c, f''(4) = d$

Then $f(x) = ax^3 + bx^2 + cx + d$

$\Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$
$\Rightarrow \quad f''(x) = 6ax + 2b$
$\Rightarrow \quad f'''(x) = 6a$
$\Rightarrow \quad f'''(4) = d = 6a$
$\Rightarrow \quad f''(3) = c = 18a + 2b$
$\Rightarrow \quad f'(2) = b = 12a + 4b + c$
$\Rightarrow \quad b = 12a + 4b + 18a + 2b$
$\Rightarrow \quad b = 30a + 6b$
$\Rightarrow \quad -5b = 30a$
$\Rightarrow \quad b = -6a,\ c = 18a + 2b = 18a - 12a = 6a$

Now, $f(1) = a + b + c + d$
$\quad = a - 6a + 6a + 6a = 7a$
$\Rightarrow \quad f(1) = 7a$
$\Rightarrow \quad a = 7a$
$\Rightarrow \quad a = 0$

Thus, $f(x) = ax^3 + bx^2 + cx + d$
$\quad = ax^3 - 6ax^2 + 6ax + 6a$
$\quad = 0$

Thus, $f(x)$ is an independent of x.

12. $y = \cos^{-1}\left\{\frac{7}{2}(1 + \cos 2x) + \sqrt{\sin^2 x - 48\cos^2 x}\ \sin x\right\}$

$= \cos^{-1}\left(7\cos^2 x + \sqrt{\sin^2 x - 48\cos^2 x}\ \sin x\right)$

$= \cos^{-1}(7\cos^2 x + \sqrt{1 - 49\cos^2 x}\ \sin x)$

$= \cos^{-1}(7\cos x.\ \cos x + \sqrt{1 - 49\cos^2 x}\ \sin x)$

$= \cos^{-1}(\cos x) - \cos^{-1}(7\cos x)$

$= x - \cos^{-1}(7\cos x)$

$\frac{dy}{dx} = 1 + \frac{\sin x}{\sqrt{1 - 49\cos^2 x}}$

$\Rightarrow \quad \frac{dy}{dx} = 1 + \frac{\sin x}{\sqrt{\sin^2 x - 48\cos^2 x}}$

Hence, the result.

13. Given $x = \tan\left(\frac{y}{2}\right) - \ln\left(\frac{(1 + \tan(y/2))^2}{\tan(y/2)}\right)$

$= \tan\left(\frac{y}{2}\right) - 2\ln\left(1 + \tan\left(\frac{y}{2}\right)\right) - \ln\left(\tan\left(\frac{y}{2}\right)\right)$

$\Rightarrow \quad \frac{dx}{dy} = \frac{1}{2}\sec^2\left(\frac{y}{2}\right) - \frac{\sec^2(y/2)}{\left(1 + \tan\left(\frac{y}{2}\right)\right)}$
$\quad - \frac{1}{\left(\tan\left(\frac{y}{2}\right)\right)} \cdot \frac{1}{2}\sec^2\left(\frac{y}{2}\right)$

$\Rightarrow 2\frac{dx}{dy} = \sec^2\left(\frac{y}{2}\right)\left(1 - \frac{2}{1 + \tan(y/2)} - \frac{1}{\tan(y/2)}\right)$

$\Rightarrow 2\frac{dx}{dy} = \sec^2\left(\frac{y}{2}\right)\left(\frac{1 - \tan(y/2)}{1 + \tan(y/2)} - \frac{1}{\tan(y/2)}\right)$

$\Rightarrow \quad 2\dfrac{dx}{dy} = \sec^2\left(\dfrac{y}{2}\right)\left(\dfrac{-\tan^2(y/2) - 1}{(1 + \tan(y/2))\tan(y/2)}\right)$

$\Rightarrow \quad 2\dfrac{dx}{dy} = -\left(\dfrac{\sec^4(y/2)}{\{1 + \tan(y/2)\}\tan(y/2)}\right)$

$\Rightarrow \quad \dfrac{1}{2}\dfrac{dy}{dx} = -\left(\dfrac{\{1 + \tan(y/2)\}\tan(y/2)}{\sec^4(y/2)}\right)$

$\Rightarrow \quad 2\dfrac{dy}{dx} = -\left(\dfrac{4\{1 + \tan(y/2)\}\tan(y/2)}{\sec^4(y/2)}\right)$

$\Rightarrow \quad 2\dfrac{dy}{dx} = -\dfrac{2\tan(y/2)}{(1 + \tan^2(y/2))}$

$\left(1 + \dfrac{2\tan(y/2)}{1 + \tan^2(y/2)} + \dfrac{1-\tan^2(y/2)}{1+\tan^2(y/2)}\right)$

$\Rightarrow \quad 2\dfrac{dy}{dx} = -\sin y(1 + \sin y + \cos y)$

Hence, the result.

14. Given $y = \cos^{-1}\left(\sqrt{\dfrac{\cos 3x}{\cos^3 x}}\right)$

$\Rightarrow \quad \cos(y) = \sqrt{\dfrac{\cos 3x}{\cos^3 x}}$

$\Rightarrow \quad \cos^2(y) = \dfrac{\cos 3x}{\cos^3 x} = \dfrac{4\cos^3 x - 3\cos x}{\cos^3 x}$

$\Rightarrow \quad \cos^2(y) = 4 - 3\sec^2 x$

$\Rightarrow \quad \cos^2(y) = 4 - 3(1 + \tan^2 x)$

$\Rightarrow \quad \cos^2(y) = 1 - 3\tan^2 x$

$\Rightarrow \quad 1 - \sin^2 y = 1 - 3\tan^2 x$

$\Rightarrow \quad \sin^2 y = 3\tan^2 x$

$\Rightarrow \quad \sin y = \sqrt{3}\tan x$

Differentiating both sides w.r.t x, we get,

$\Rightarrow \quad \cos y\dfrac{dy}{dx} = \sqrt{3}\sec^2 x$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\sqrt{3}\sec^2 x}{\cos y} = \dfrac{\sqrt{3}}{\cos y\cos^2 x}$

$\Rightarrow \quad \dfrac{dy}{dx} = \sqrt{\dfrac{3}{\cos^2 y\cos^4 x}}$

$= \sqrt{\dfrac{3}{(1 - 3\tan^2 x)\cos^4 x}}$

$= \sqrt{\dfrac{3}{\cos^2 x(\cos^2 x - 3\sin^2 x)}}$

$= \sqrt{\dfrac{3}{\cos^2 x(4\cos^2 x - 3)}}$

$= \sqrt{\dfrac{3}{\cos x(4\cos^3 x - 3\cos x)}}$

$= \sqrt{\dfrac{3}{\cos x\cos(3x)}}$

Hence, the result.

15. Given $\sqrt{x^2 + y^2} = a e^{\tan^{-1}x}$

$\Rightarrow \quad (x^2 + y^2) = a^2 e^{2\tan^{-1}x}$

$\Rightarrow \quad 2x + 2y\dfrac{dy}{dx} = a^2 e^{2\tan^{-1}x} \times \dfrac{2}{1 + x^2}$

$\Rightarrow \quad x + y\dfrac{dy}{dx} = \dfrac{a^2 e^{2\tan^{-1}x}}{(1 + x^2)}$

$\Rightarrow \quad x(1 + x^2) + y(1 + x^2)\dfrac{dy}{dx} = (x^2 + y^2)$

when $x = 0$, then $y^2 = a^2$

when $x = 0$, $y^2 = a^2$, then $\dfrac{dy}{dx} = a$

Again differentiating both sides w.r.t x, we get

$(1 + x^2) + 2x^2 + (1 + x^2)\left(\dfrac{dy}{dx}\right)^2 + 2xy\dfrac{dy}{dx}$

$\Rightarrow \quad y(1 + x^2)\dfrac{d^2y}{dx^2} = 2x + 2y\dfrac{dy}{dx}$

Put $x = 0$, $y = a$, $\dfrac{dy}{dx} = a$, then

$\Rightarrow \quad 1 + a^2 + 0 + a(1 + 0)\dfrac{d^2y}{dx^2} = 0 + 2a^2$

$\Rightarrow \quad 1 + a^2 + a\dfrac{d^2y}{dx^2} = 2a^2$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = \left(\dfrac{a^2 - 1}{a}\right)$

Hence, the value of $y''(0)$ is $\left(\dfrac{a^2 - 1}{a}\right) = \left(a - \dfrac{1}{a}\right)$

Integer Type Questions

1. We have $y = \cos^{-1}\left(\dfrac{x - x^{-1}}{x + x^{-1}}\right)$, $x > 0$

$\Rightarrow \quad y = \cos^{-1}\left(\dfrac{x^2 - 1}{x^2 + 1}\right) = \cos^{-1}\left(-\dfrac{1 - x^2}{x^2 + 1}\right)$

$\Rightarrow \quad y = \pi - \cos^{-1}\left(\dfrac{1 - x^2}{x^2 + 1}\right)$

$\Rightarrow \quad y = \begin{cases} 2 + \tan^{-1}x & : x < 0 \\ 2 - \tan^{-1}x & : x > 0 \end{cases}$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2} & : x<0 \\ \text{non existent} & : x=0 \\ -\dfrac{2}{1+x^2} & : x>0 \end{cases}$$

Thus, $\left(\dfrac{dy}{dx}\right)_{x=0} = 2$

2. Given $4xe^{xy} = y + 5\sin^2 x$

$$\Rightarrow \quad 4\left(e^{xy} + xe^{xy}\left(y + x\frac{dy}{dx}\right)\right) = \frac{dy}{dx} + 5\sin 2x$$

when $x = 0$, then $y = 0$

Now, put $x = 0$ and $y = 0$ in (i), we get,

$$4\left(1 + 0\left(0 + 0\cdot\frac{dy}{dx}\right)\right) = \frac{dy}{dx} + 0$$

$$\Rightarrow \quad \frac{dy}{dx} = 4$$

3. Given $\sqrt{x+y} + \sqrt{x-y} = c$

$$\Rightarrow \frac{1}{2\sqrt{x+y}}\left(1 + \frac{dy}{dx}\right) + \frac{1}{2\sqrt{y-x}}\left(\frac{dy}{dx} - 1\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\sqrt{x+y} - \sqrt{y-x})(\sqrt{x+y} + \sqrt{y-x})}{c(\sqrt{x+y} + \sqrt{y-x})}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x+y-y+x)}{c^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{c^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2}{c^2}$$

$$\Rightarrow \quad \left(c^2\frac{d^2y}{dx^2} + 3\right) = 2 + 3 = 5$$

4. Let $h(x) = [f(x)]^2 - [g(x)]^2$

$h'(x) = 2f(x)f'(x) - 2g(x)g'(x)$

$h'(x) = 2f(x)g(x) - 2g(x)f(x) = 0$

$h(x) = c$

$h(10) = [f(10)]^2 - [g(10)]^2$

$= [f(3)]^2 - [g(3)]^2$

$= [f(3)]^2 - [f'(3)]^2$

$= 25 - 16 = 9$

5. Given $\sin x = \dfrac{2t}{1+t^2}$ and $\cot y = \dfrac{1-t^2}{2t}$

$$\Rightarrow \quad x = \sin^{-1}\left(\frac{2t}{1+t^2}\right),\ y = \cot^{-1}\left(\frac{1-t^2}{2t}\right)$$

$$\Rightarrow \quad x = \sin^{-1}\left(\frac{2t}{1+t^2}\right),\ y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$$

$$\Rightarrow \quad x = 2\tan^{-1}t,\ y = 2\tan^{-1}t$$

$$\Rightarrow \quad \frac{dx}{dy} = 1$$

$$\Rightarrow \quad \frac{d^2x}{dy^2} + 3 = 0 + 3 = 3$$

6. Given $2x = (y^{1/3} + y^{-1/3})$

$$\Rightarrow \quad (y^{1/3})^2 - 2xy^{1/3} + 1 = 0$$

$$\Rightarrow \quad (y^{1/3}) = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\Rightarrow \quad (y^{1/3}) = x \pm \sqrt{x^2-1}$$

$$\Rightarrow \quad \ln(y) = 3\ln(x \pm \sqrt{x^2-1})$$

Differentiating w.r.t x, we get,

$$\frac{(dy/dx)}{y} = \frac{3}{(x \pm \sqrt{x^2-1})}\left(1 \pm \frac{2x}{2\sqrt{x^2-1}}\right)$$

$$\Rightarrow \quad \frac{(dy/dx)}{y} = \frac{3}{(x \pm \sqrt{x^2-1})}\left(\frac{x \pm \sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow \quad \frac{(dy/dx)}{y} = \frac{3}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3y}{\sqrt{x^2-1}}$$

$$\Rightarrow \quad (x^2-1)\left(\frac{dy}{dx}\right)^2 = 9y^2$$

$$\Rightarrow \quad 2x\left(\frac{dy}{dx}\right)^2 + 2(x^2-1)\frac{dy}{dx}\frac{d^2y}{dx^2} = 18y\frac{dy}{dx}$$

$$\Rightarrow \quad x\left(\frac{dy}{dx}\right) + (x^2-1)\frac{d^2y}{dx^2} = 9y$$

$$\Rightarrow \quad \left(\frac{x^2-1}{y}\right)\frac{d^2y}{dx^2} + \frac{x}{y}\left(\frac{dy}{dx}\right) = 9$$

7. Given $f(x) = \dfrac{g(x)}{x}$

$$f'(x) = \frac{xg'(x) - g(x)}{x^2}$$

$$f'(2) = \frac{2g'(2) - g(2)}{4}$$

$$f'(2) = \frac{2 \times 6 - 4}{4} = \frac{8}{4} = 2$$

8. Given $F(x) = f(g(x))$

$$\Rightarrow \quad F'(x) = f'(g(x))g'(x)$$

$\Rightarrow \quad F'(3) = f'(g(3))g'(3)$

$= f'(6)g'(3) = 2 \times 4 = 8$

9. Given $f(x) = x^2 + x^3$

As we know that, if g is the inverse of f, then $y = f(x) \Leftrightarrow x = g(y)$

Now, $\frac{dy}{dx} = 2x + 3x^2$

$\Rightarrow \quad g'(y) = \frac{dx}{dy} = \frac{1}{2x + 3x^2}$

when $y = 2$, then $x = 1$

Now, $g'(2) = \left(\frac{dx}{dy}\right)_{\substack{x=1\\y=2}} = \frac{1}{2+3} = \frac{1}{5}$

Hence, the value of $(5g'(2) + 3)$

$= 1 + 3$

$= 4.$

10. Given $x^y = e^{x-y}$

$\Rightarrow \quad \log(x^y) = (x - y)\log(e)$

$\Rightarrow \quad y\log(x) = (x - y)$

$\Rightarrow \quad \frac{y}{x} + \log(x)\frac{dy}{dx} = 1 - \frac{dy}{dx}$ ...(i)

when $x = e$, then $y = 0$

Put $x = e$ and $y = 0$ in (i), we get,

$$0 + 1 \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$2\frac{dy}{dx} = 1$$

11. Given $h(x) = \{f(x)\}^2 + \{g(x)\}^2$

$h'(x) = 2f(x)f'(x) + 2g(x)g'(x)$

$h'(x) = 2f(x)g(x) + 2g(x)g'(x)$

$h'(x) = 2f(x)g(x) + 2g(x)f''(x)$

$h'(x) = 2f(x)g(x) - 2g(x)f(x) = 0$

$h'(x) = 0$

$h(x) = c$

$h(5) = 7$

$h(x) = 7$

$h(2016) = 7.$

12. Clearly, $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$f''(x) = 2a.$

Thus, $2a = 10 \Leftrightarrow a = 5$

$f'(1) = 6 \Rightarrow 2a + b = 6$

$b = 6 - 2a = 6 - 10 = -4$

$f(1) = 4 \Rightarrow a + b + c = 4$

$c = 4 - a - b = 4 - 5 + 4 = 3$

Hence, the value of $f(0) + 2 = 3 + 2 = 5$

13. Given $P(x) = ax^3 + bx^2 + cx + d$

$P'(x) = 2ax^2 + 2bx + c$

$P''(x) = 4ax + 2b$

$P''(0) = 10 \Rightarrow 2b = 10 \Rightarrow b = 5$

$P(0) = -2 \Rightarrow d = -2$

$P(1) = -2 \Rightarrow a + b + c + d = -2$

$a + 5 + c - 2 = -2$

$a + c = -5$ ...(i)

$P'(0) = -1 \Rightarrow c = -1$

From (i), we get, $a = -4$

Hence, the value of $(a + b + c + d + 10)$

$= -4 + 5 - 1 - 2 + 10$

$= 8$

Hints and Solutions

Questions asked in Roorkee - JEE Exams

1. Let $\sin^{-1}\left(\frac{1-x}{1+x}\right)$ and $v = \sqrt{x}$

Then

$$\frac{du}{dx} = \frac{1}{\sqrt{1 - \left(\frac{1-x}{1+x}\right)^2}} \times \frac{(1+x).(-1) - (1-x).1}{(1+x)^2}$$

$$= \frac{(1+x)}{\sqrt{(1+x)^2 - (1-x)^2}} \times \frac{-2}{(1+x)^2}$$

$$= \frac{-2}{\sqrt{4x}} \times \frac{1}{(1+x)}$$

$$= \frac{-1}{(1+x)\sqrt{x}}$$

Also, $\frac{dv}{dx} = \frac{1}{2\sqrt{x}}$

Thus, $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-1}{(1+x)\sqrt{x}}}{\frac{1}{2\sqrt{x}}} = -\frac{2}{(1+x)}$

2. We have $(\tan^{-1}x)^y + y^{\cot x} = 1$

$$\Rightarrow e^{y\log\tan^{-1}x} + e^{\cot x\log y} = 1$$

$$\Rightarrow e^{y\log\tan^{-1}x}\left(\log\tan^{-1}x.\frac{dy}{dx} + y.\frac{1}{\tan^{-1}x}.\frac{1}{1+x^2}\right)$$

$$+ e^{\cot x\log y}\left(-\operatorname{cosec}^2 x.\log y + \cot x.\frac{1}{y}.\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (\tan^{-1}x)^y\left(\log\tan^{-1}x\cdot\frac{dy}{dx} + y\cdot\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}\right)$$

$$+ y^{\cot x}\left(-\operatorname{cosec}^2 x\cdot\log y + \cot x\cdot\frac{1}{y}\cdot\frac{dy}{dx}\right) = 0$$

$$\Rightarrow (\log(\tan^{-1}x)\tan^{-1}x)^y + \cot x\cdot y^{\cot x-1})\frac{dy}{dx}$$

$$= \left(y^{\cot x}\operatorname{cosec}^2 x.\log y - (\tan^{-1}x)^{y-1}\left(\frac{y}{1+x^2}\right)\right)$$

$$\frac{dy}{dx} = \left(\frac{\left(y^{\cot x}\operatorname{cosec}^2 x.\log y - (\tan^{-1}x)^{y-1}\left(\frac{y}{1+x^2}\right)\right)}{\log(\tan^{-1}x)(\tan^{-1}x)^y + \cot x.y^{\cot x-1}}\right)$$

3. We have

$$y = \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\left(\frac{x}{2}\right)\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{a^2-b^2}}\frac{1}{1+\left(\frac{a-b}{a+b}\right)\tan^2\left(\frac{x}{2}\right)}$$

$$\cdot\sqrt{\frac{a-b}{a+b}}\cdot\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{a^2-b^2}}\cdot\frac{(a+b)}{(a+b)+(a-b)\tan^2\left(\frac{x}{2}\right)}$$

$$\cdot\sqrt{\frac{a-b}{a+b}}\cdot\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1}{(a^2-b^2)}\times\frac{a-b}{a+b}\times(a+b)^2}$$

$$\frac{\left(1+\tan^2\left(\frac{x}{2}\right)\right)}{a\left(1+\tan^2\left(\frac{x}{2}\right)\right)+b\left(1-\tan^2\left(\frac{x}{2}\right)\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a+b\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{a+b\cos x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{-b\sin x}{(a+b\cos x)^2} = \frac{b\sin x}{(a+b\cos x)^2}$$

Hence, the result.

4. We have $\sqrt{1-y^2} + \sqrt{1-t^2} = a(y-t)$

put $y = \sin\theta$ and $t = \sin\varphi$

$\cos\theta + \cos\varphi = a(\sin\theta - \sin\varphi)$

$$2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$$

$$= a\cdot 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)\cot\left(\frac{\theta+\varphi}{2}\right) = \frac{1}{a}$$

$$\left(\frac{\theta-\varphi}{2}\right) = \cot^{-1}\left(\frac{1}{a}\right)$$

$$\theta - \varphi = 2\cot^{-1}\left(\frac{1}{a}\right)$$

$$\sin^{-1}y - \sin^{-1}t = 2\cot^{-1}\left(\frac{1}{a}\right)$$

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dt} - \frac{1}{\sqrt{1-t^2}} = 0$$

$$\frac{dy}{dt} = \frac{\sqrt{1-y^2}}{\sqrt{1-t^2}} = \sqrt{\frac{1-y^2}{1-t^2}}$$

Also, $x = \sin^{-1}\left(t\sqrt{1-t} + \sqrt{t}\sqrt{1-t^2}\right)$

$$= \sin^{-1}t - \sin^{-1}\sqrt{t}$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} - \frac{1}{\sqrt{1-t}}\cdot\frac{1}{2\sqrt{t}}$$

$$= \frac{2\sqrt{t}-\sqrt{1-t}}{2\sqrt{1-t^2}\sqrt{t}}$$

Thus, $\frac{dy}{dx} = \frac{\left(\sqrt{\frac{1-y^2}{1-t^2}}\right)}{\left(\frac{2\sqrt{t}-\sqrt{1-t}}{2\sqrt{1-t^2}\sqrt{t}}\right)}$

$$\frac{dy}{dx} = \left(\frac{2\sqrt{(1-y^2)t}}{(2\sqrt{t}-\sqrt{1-t})}\right)$$

5. We have $\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$

$$= \begin{vmatrix} f & g & h \\ xf'+f & xg'+g & xh'+h \\ (x^2f''+4xf') & (x^2g''+4xg') & (x^2h''+4xh') \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ (x^2f''+4xf') & (x^2g''+4xg') & (x^2h''+4xh') \end{vmatrix}$$

$(R_2 \to R_2 - R_1)$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix} \quad (R_3 \to R_3 - R_1)$$

$$= x^3 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

Thus, Δ'

$$= 3x^2 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} + x^3 \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f''' & g''' & h''' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ 3x^2f'' & 3x^2g'' & 3x^2h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f''' & x^3g''' & x^3h''' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^2f''') & (x^2g''') & (x^2h''') \end{vmatrix}$$

Hence, the result.

6. Given $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$

Put $x^3 = \sin\theta$, $y^3 = \sin\varphi$

$\cos\theta + \cos\varphi = a^3(\sin\theta - \sin\varphi)$

$$2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right) = a^3.2\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$$

$$\cot\left(\frac{\theta-\varphi}{2}\right) = a^3$$

$$\left(\frac{\theta-\varphi}{2}\right) = \cot^{-1}(a^3)$$

$$\theta - \varphi = 2\cot^{-1}(a^3)$$

$$\sin^{-1}(x^3) - \sin^{-1}(y^3) = 2\cot^{-1}(a^3)$$

$$\frac{3x^2}{\sqrt{1-x^6}} - \frac{3y^2}{\sqrt{1-y^6}}\frac{dy}{dx} = 0$$

$$\frac{3y^2}{\sqrt{1-y^6}}\frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$$

$$\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$$

7. Let $u = \log_{(1-\sqrt{x})}(\sin^{-1}(1-\sqrt{x}))$

and $v = 2^{2(1-\sqrt{x})}$

$u = \log_t(\sin^{-1}t)$ and $v = 2^{2t}$

where $t = (1-\sqrt{x})$

$$\frac{du}{dt} = \frac{\log t \cdot \dfrac{1}{\sin^{-1}t}\cdot\dfrac{1}{\sqrt{1-t^2}} - \dfrac{1}{t}\log(\sin^{-1}t)}{(\log t)^2}$$

$$\frac{du}{dt} = \frac{t\log t - \log(\sin^{-1}t)\sqrt{1-t^2}\sin^{-1}t}{t\sin^{-1}t(\log t)^2.\sqrt{1-t^2}},$$

$$\frac{dv}{dt} = 2^{2t+1}\log 2 \text{ and } \frac{dt}{dx} = -\frac{1}{2\sqrt{x}}$$

Thus, $\dfrac{du}{dv} = \dfrac{du}{dt}\Big/\dfrac{dv}{dt}$

$$= \frac{t\log t - \log(\sin^{-1}t)\sqrt{1-t^2}\sin^{-1}t}{t\sin^{-1}t(\log t)^2\cdot\sqrt{1-t^2}\cdot 2^{2t+1}\log^2}\cdot -\frac{1}{2\sqrt{x}}$$

where $t = (1-\sqrt{x})$

8. Given $f(x) = \log_x(\sin(x^2)) + (\sin x^2)^{\log_e x}$

and let $g(x) = \sqrt{x+1}$

To find $\dfrac{f'(x)}{g'(x)}$

Now, $f(x) = \log_x(\sin(x^2)) + (\sin x^2)^{\log_e x}$

$$= \frac{\log(\sin(x^2))}{\log x} + e^{\log x \log(\sin x^2)}$$

$$\Rightarrow \quad f'(x) = \frac{\log(x)\cdot\cot(x^2)\cdot 2x - \dfrac{\log(\sin(x^2))}{x}}{(\log x)^2}$$

$$+ (\sin(x^2))^{\log x}\cdot\left(\frac{\log(\sin(x^2))}{x} + 2x\log x.\cot(x^2)\right)$$

Also, $g'(x) = \dfrac{1}{2\sqrt{x+1}}$

Thus, $\dfrac{f'(x)}{g'(x)}$

$$= \frac{1}{2\sqrt{x+1}}\left[\frac{\log(x).\cot(x^2).2x - \dfrac{\log(\sin(x^2))}{x}}{(\log x)^2}\right.$$

$$+ (\sin(x^2))^{\log x}.\left(\frac{\log(\sin(x^2))}{x} + 2x\log x.\cot(x^2)\right)\Bigg]$$

Questions asked in Past IIT-JEE Exams

2. $f'(1) = \lim_{x\to 1}\left(\frac{f(x) - f(1)}{x - 1}\right)$

$$= \lim_{x\to 1}\left(\frac{\frac{x - 1}{2x^2 - 7x + 5} + \frac{1}{3}}{x - 1}\right)$$

$$= \lim_{x\to 1}\left(\frac{3x - 3 + 2x^2 - 7x + 5}{3(x - 1)(2x^2 - 7x - 5)}\right)$$

$$= \lim_{x\to 1}\left(\frac{2}{3} \times \frac{x^2 - 2x + 1}{(x - 1)(2x^2 - 7x + 5)}\right)$$

$$= \lim_{x\to 1}\left(\frac{2}{3} \times \frac{(x - 1)^2}{(x - 1)(2x^2 - 7x + 5)}\right)$$

$$= \lim_{x\to 1}\left(\frac{2}{3} \times \frac{(x - 1)}{(2x - 5)(x - 1)}\right)$$

$$= \lim_{x\to 1}\left(\frac{2}{3} \times \frac{1}{(2x - 5)}\right)$$

$$= -\frac{2}{9}$$

3. Given $y = \frac{5x}{\sqrt[3]{(1 - x)^2}} + \cos^2(2x + 1)$.

$$\frac{dy}{dx} = \frac{(1- x)^{\frac{2}{3}}.5 - \frac{2}{3}(1 - x)^{\frac{1}{3}}}{(1 - x)^{\frac{4}{3}}} - 2\sin(4x + 2)$$

$$\frac{dy}{dx} = \frac{15(1 - x) - 2}{3(1 - x)^{\frac{5}{3}}} - 2\sin(4x + 2)$$

$$\frac{dy}{dx} = \frac{(13 - 15x)}{3(1 - x)^{\frac{5}{3}}} - 2\sin(4x + 2))$$

4. We have $y = e^{x\sin x^3} + (\tan x)^x$

$$= e^{\log(x\sin(x^2))} + e^{x\log(\tan x)}$$

$$\frac{dy}{dx} = e^{\log(x\sin(x^2))}\left(\frac{1}{x\sin(x^2)}.(\cos(x^2) + 2x^2\cos(x^2))\right)$$

$$+ e^{x\log(\tan x)}\left(\log(\tan x) + \frac{x}{\tan x}.\sec^2 x\right)$$

$$\frac{dy}{dx} = (x\sin(x^2))\left(\frac{1}{x}.(\cot(x^2) + 2x^2\cos(x^2))\right)$$

$$+ \left(\tan x)^x(\log(\tan x) + \frac{x}{\tan x}.\sec^2 x\right)$$

5. We have, $h(x) = [f(x)]^2 + [g(x)]^2$

$\Rightarrow \quad h'(x) = 2f(x)f'(x) + 2g(x)g'(x)$

$\Rightarrow \quad h'(x) = 2f(x)g(x) + 2g(x)g'(x)$

$\Rightarrow \quad h'(x) = -2f''(x)g(x) + 2g(x)f''(x) = 0$

Thus, $h(x)$ is a constant function.

But $h(5) = 11$

$\Rightarrow \quad h(x) = 11$

$\Rightarrow \quad h(10) = 11$

6. Given $y = f\left(\frac{2x - 1}{x^2 + 1}\right)$

$$\frac{dy}{dx} = f'\left(\frac{2x - 1}{x^2 + 1}\right)\left(\frac{(x^2 + 1).2 - (2x - 1).2x}{(x^2 + 1)^2}\right)$$

$$\frac{dy}{dx} = f'\left(\frac{2x - 1}{x^2 + 1}\right)\left(\frac{2x^2 + 2 - 4x^2 + 2x}{(x^2 + 1)^2}\right)$$

$$\frac{dy}{dx} = f'\left(\frac{2x - 1}{x^2 + 1}\right)\left(\frac{2 + 2x + 2x^2}{(x^2 + 1)^2}\right)$$

$$\frac{dy}{dx} = \sin\left(\left(\frac{2x - 1}{x^2 + 1}\right)^2\right)\left(\frac{2 + 2x + 2x^2}{(x^2 + 1)^2}\right)$$

7. We have $(a + bx)e^{y/x} = x$

$$\Rightarrow \quad e^{\frac{y}{x}} = \left(\frac{x}{a + bx}\right)$$

$$\Rightarrow \quad \frac{y}{x} = \log\left(\frac{x}{a + bx}\right)$$

$$\Rightarrow \quad \frac{y}{x} = \log x - \log(a + bx)$$

$$\Rightarrow \quad \frac{x.\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{b}{a + bx}$$

$$\Rightarrow \quad \frac{x.\frac{dy}{dx} - y}{x^2} = \frac{a + bx - bx}{x(a + bx)}$$

$$\Rightarrow \quad x.\frac{dy}{dx} - y = \frac{ax}{(a + bx)}$$

$$\Rightarrow \quad x.\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a + bx)a - ax.b}{(a + bx)^2}$$

$$\Rightarrow \quad x.\frac{d^2y}{dx^2} = \frac{a^2}{(a + bx)^2}$$

$$\Rightarrow x^3.\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a + bx)^2} = \left(\frac{ax}{a + bx}\right)^2 = \left(x\frac{dy}{dx} - y\right)^2$$

Hence, the result.

8. Let
$y = (\log_{\sin x}\cos x)(\log_{\cos x}\sin x)^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$\Rightarrow \quad y = \frac{\log(\cos x)}{\log(\sin x)} \times \frac{\log(\cos x)}{\log(\sin x)} + 2\tan^{-1} x$$

$$\Rightarrow \quad y = \frac{\{\log(\cos x)\}^2}{\{\log(\sin x)\}^2} + 2\tan^{-1} x$$

$$\frac{dy}{dx} = \frac{\{\log(\sin x)\}^2(-2\log(\cos x).\tan x) - \{\log(\cos x)\}^2(2\log(\sin x).\cot x)}{\{\log(\sin x)\}^4} + \frac{2}{1+x^2}$$

$$\Rightarrow \quad \left(\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}}\right)$$

$$= \frac{\left\{\log\left(\frac{1}{\sqrt{2}}\right)\right\}^2\left(-2\log\left(\frac{1}{\sqrt{2}}\right)\right) - \left\{\log\left(\frac{1}{\sqrt{2}}\right)\right\}^2\left(2\log\left(\frac{1}{\sqrt{2}}\right)\right)}{\left\{\log\left(\frac{1}{\sqrt{2}}\right)\right\}^4} + 2\left(\frac{1}{1+\frac{\pi^2}{16}}\right)$$

$$= \frac{-4\log\left(\frac{1}{\sqrt{2}}\right)}{\left(\log\left(\frac{1}{\sqrt{2}}\right)\right)^2} + \left(\frac{32}{\pi^2+16}\right)$$

$$= -\frac{4}{(\log 1 - \log(\sqrt{2}))} + \left(\frac{32}{\pi^2+16}\right)$$

$$= \frac{8}{(\log 2)} + \left(\frac{32}{\pi^2+16}\right)$$

9. We have $f(x) = \log x(\ln x)$

$$= \frac{\log(\log x)}{\log x}$$

$$\Rightarrow \quad f'(x) = \frac{\log x . \frac{1}{\log x} . \frac{1}{x} - \log(\log x) . \frac{1}{x}}{(\log x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{\frac{1}{x}(1 - \log(\log x))}{(\log x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{(1 - \log(\log x))}{x(\log x)^2}$$

$$\Rightarrow \quad (f'(x))_{x=e} = \frac{(1-1)}{e^2} = 0$$

10. Let $u = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ and $v = \sqrt{1-x^2}$

Now, $u = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

Put $x = \cos\theta$

So, $u = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$

$$\Rightarrow \quad u = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}(x)$$

$$\Rightarrow \quad \frac{du}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

Also, $v = \sqrt{1-x^2}$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{1 \times -2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

Thus, $\frac{du}{dv} = \dfrac{-\frac{2}{\sqrt{1-x^2}}}{-\frac{x}{\sqrt{1-x^2}}} = \frac{2}{x}$

Now, $\left(\frac{du}{dx}\right)_{x=\frac{1}{2}} = \dfrac{2}{\frac{1}{2}} = 4$

11. No questions asked in 1987.

12. Given $y^2 = P(x)$

$$2y\frac{dy}{dx} = P'(x)$$

$$\Rightarrow \quad 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = P''(x)$$

$$\Rightarrow \quad 2y^3\frac{d^2y}{dx^2} + 2y^2\left(\frac{dy}{dx}\right)^2 = y^2P''(x)$$

$$\Rightarrow \quad 2y^3\frac{d^2y}{dx^2} = y^2P''(x) - 2y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad 2y^3\frac{d^2y}{dx^2} = y^2P''(x) - \frac{1}{2}\left(2y\left(\frac{dy}{dx}\right)\right)^2$$

$$\Rightarrow \quad 2y^3\frac{d^2y}{dx^2} = y^2P''(x) - \frac{1}{2}(P'(x))^2$$

$$\Rightarrow \quad 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$$

$$= 2y\frac{dy}{dx}P''(x) + y^2P'''(x) - \frac{1}{2}.2.P'(x).P''(x)$$

$$= 2y\frac{dy}{dx}P''(x) + y^2P'''(x) - P'(x).P''(x)$$

$$= 2y\frac{dy}{dx}P''(x) + y^2P'''(x) - P'(x).P''(x)$$

$= P'(x)P''(x) + y^2P'''(x) - P'(x).P''(x)$
$= y^2P'''(x)$
$= P(x).P''(x)$

13. We have $x = \sec\theta - \cos\theta$
and $y = \sec^n\theta - \cos^n\theta$

$\Rightarrow \quad \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta$

$\Rightarrow \quad \frac{dx}{d\theta} = \tan\theta(\sec\theta + \cos\theta)$

Also,

$\Rightarrow \quad \frac{dy}{d\theta} = n\sec^{n-1}\theta.\sec\theta.\tan\theta + n\cos^{n-1}\theta.\sin\theta$

$\Rightarrow \quad \frac{dy}{d\theta} = n\sec^n\theta.\tan\theta + n\cos^n\theta.\tan\theta$

$\Rightarrow \quad \frac{dy}{d\theta} = n\tan\theta(\sec^n\theta + \cos^n\theta.)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{n\cdot\tan\theta(\sec^n\theta + \cos^n\theta.)}{\tan\theta(\sec\theta + \cos\theta)}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{n^2(\sec^n\theta - \cos^n\theta)^2 + 4}{(\sec\theta - \cos\theta)^2 + 4}$

$\left(\frac{dy}{dx}\right)^2 = \frac{n^2(y^2 + 4)}{(x^2 + 4)}.$

14. We have $f(x) = |x - 2|$

$$= \begin{cases} (x-2) & : x \ge 2 \\ -(x-2) & : x < 2 \end{cases}$$

Now, $g(x) = f(f(x))$
$= f(x - 2)$
$= (x - 2) - 2$
$= x - 4,\ x \ge 4$

Thus, $g'(x) = 1,\ x > 20$

15. Let $f(x) = ax^2 + bx + c$, $a > 0$ (since $f(x)$ is + ve for all real values of (x)
$f'(x) = 2ax + b$
$f''(x)\ 2a$
Now, $g(x) = f(x) + f'(x) + f''(x)$
$= (ax^2 + bx + c) + (2ax + b) + 2a$
$= ax^2 + (2a + b)x + (2a + b + c)$
Now, $D = (2a + b)^2 - 4a(2a + b + c)$
$= 4a^2 + 4ab + b^2 - 8a^2 - 4ab - 4ac$
$= -4a^2 - 4ac + b^2$
$= -4a^2 + (b^2 - 4ac)$
< 0 for all x in R.
Thus, $g(x) > 0,\ \forall\ x \in R$

16. Find $\frac{dy}{dx}$ at $x = -1$, when
Given
$(\sin y)^{\sin(\pi x/2)} + 2^x\tan(\ln(x + 2)) + \frac{\sqrt{3}}{2}\sec^{-1}(2x) = 0$

$(\sin y)^{\sin(\pi x/2)}\left\{\sin\left(\frac{\pi x}{2}\right)\cot y\frac{dy}{dx}\right.$
$\left. + \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)\log(\sin y)\right\} + 2^x\log 2\tan\log(x + 2)$

$+ \frac{2^x\sec^2(\ln(x + 2))}{(x + 2)} + \frac{\sqrt{3}}{2} \times \frac{2x}{2x\sqrt{4x^2 - 1}} = 0$

Put $x = -1$, we get,

$(\sin y)^{-1}\left(-\cot y\frac{dy}{dx}\right) + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} + \frac{1}{2} = 0$

$\left(-\frac{\cot y}{\sin y}\frac{dy}{dx}\right) + 1 = 0$

$\frac{\cos y}{\sin^2 y}\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{\sin^2 y}{\cos y} = \sec y\ \tan y$

17. No questions asked in 1992.
18. Yes, it is true, the derivative of every odd function is an even function.
19. Ans. (a)
Given $y = (\sin x)^{\tan x}$
$y = e^{\tan x \log\sin x}$
$\frac{dy}{dx} = e^{\tan x\log\sin x}(\tan x.\cot x + \sec^2 x\log\sin x)$
$\frac{dy}{dx} = (\sin x)^{\tan x}(1 + \sec^2 x\log\sin x)$
20. No questions asked in 1995.
21. We have $xe^{xy} = y + \sin^2 x$

$e^{xy}.1 + xe^{xy}\left(x\frac{dy}{dx} + y.1\right) = \frac{dy}{dx} + \sin 2x$

$\Rightarrow \quad e^{xy} + \left(x^2e^{xy}\frac{dy}{dx} + y.1\right) = \frac{dy}{dx} + \sin 2x$

when $x = 0$, then $y = 0$
Thus,
$\Rightarrow \quad 1 + (0 + 0) = \frac{dy}{dx} + 0$

$\Rightarrow \quad \frac{dy}{dx} = 1$

22. Ans. (d)

We have

$$f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 0 & p^2 & p^3 \end{vmatrix}$$

$$\frac{d}{dx}(f(x)) = \begin{vmatrix} 3x^3 & \cos x & -\sin x \\ 6 & -1 & 0 \\ 0 & p^2 & p^3 \end{vmatrix}$$

$$\frac{d^2}{dx^2}(f(x)) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\frac{d^3}{dx^3}(f(x)) = \begin{vmatrix} 6 & -\cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

Now, $\left(\frac{d^3}{dx^3}(f(x))\right)_{x=0}$

$$= \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$= 0$

Which is independent of p.

23. We have

$$y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$$

$$\Rightarrow y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c+x-c}{x-c}$$

$$\Rightarrow y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{x}{x-c}$$

$$\Rightarrow y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx + x^2 - bx}{(x-b)(x-c)}$$

$$\Rightarrow y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x^2}{(x-b)(x-c)}$$

$$\Rightarrow y = \frac{ax^2 + x^3 - ax^2}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow y = \frac{x^3}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow \log y = \log\left(\frac{x^3}{(x-a)(x-b)(x-c)}\right)$$

$\Rightarrow \log y = \log(x^3) - \log((x-a)(x-b)(x-c))$

$\Rightarrow \log y = 3\log x - \log((x-a)(x-b)(x-c))$

$\Rightarrow \log y = 3\log x - \log(x-a) - \log(x-b) - \log(x-c)$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \frac{1}{(x-a)} - \frac{1}{(x-b)} - \frac{1}{(x-c)}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{x} - \frac{1}{(x-a)}\right) + \left(\frac{1}{x} - \frac{1}{(x-b)}\right) + \left(\frac{1}{x} - \frac{1}{(x-c)}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left(\frac{x-a-x}{x(x-a)}\right) + \left(\frac{x-b-x}{(x-b)}\right) + \left(\frac{x-c-x}{(x-c)}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\left(\left(\frac{a}{a-x}\right) + \left(\frac{b}{b-x}\right) + \left(\frac{c}{c-x}\right)\right)$$

Hence, the result.

24. No questions asked in 1999.

25. Ans. (b)

We have $x^2 + y^2 = 1$

$\Rightarrow \quad 2x + 2yy' = 0$

$\Rightarrow \quad x + yy' = 0$

$\Rightarrow \quad 1 + (y')2 + y.y'' = 0$

26. Ans. (c)

We have $F(x) = \int_0^x f(t)dt$

$\Rightarrow \quad F'(x) = f(x)$

Given $F(x^2) = x^2 + x^3$

$\Rightarrow \quad F'(x^2).2x = 2x + 3x^2$

$\Rightarrow \quad F'(x^2) = \frac{2x + 3x^2}{2x} = 1 + \frac{3}{2}x$

$\Rightarrow \quad f(x^2) = 1 + \frac{3}{2}x$

Put $x = 2$

$\Rightarrow \quad f(4) = 1 + \frac{3}{2}\cdot 2 = 1 + 3 = 4$

27. No questions asked in between 2002 -2003.

28. Ans. (a)

We have $\log(x + y) = 2xy$

$$\Rightarrow \quad \frac{1}{(x+y)}\left(1 + \frac{dy}{dx}\right) = 2\left(x\cdot\frac{dy}{dx} + y\cdot 1\right)$$

$$\Rightarrow \quad \frac{1}{(x+y)} + \frac{1}{(x+y)}\cdot\frac{dy}{dx} = 2x\cdot\frac{dy}{dx} + 2y$$

$$\Rightarrow \quad \left(\frac{1}{(x+y)} - 2x\right)\cdot\frac{dy}{dx} = \left(2y - \frac{1}{(x+y)}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\left(2y - \frac{1}{(x+y)}\right)}{\left(\frac{1}{(x+y)} - 2x\right)} = \left(\frac{2y(x+y)-1}{1-2x(x+y)}\right)$$

when $x = 0$, $y = 1$.

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1,\, y=1} = \frac{(2-1)}{(1-0)} = 1$$

29. No questions asked in between 2005-2006.

30. Ans. (d)

We have $\frac{d^2x}{dy^2}$

$$= \frac{d}{dy}\left(\frac{dx}{dy}\right)$$

$$= \frac{d}{dx}\left(\frac{dx}{dy}\right)\cdot\frac{dx}{dy}$$

$$= \frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)\cdot\frac{dx}{dy}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{dx}{dy}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)^3}\cdot\frac{d^2y}{dx^2}$$

$$= -\left(\frac{dy}{dx}\right)^{-3}\cdot\frac{d^2y}{dx^2}$$

31. We have $f(x) = 2 + \cos x,\ \forall\ x \in R$

$f'(x) = -\sin x,\ \forall\ x \in R$

clearly, $f(x)$ is continuous and differentiable so, there exist a point $c \in (t, t + \pi)$ such that $f'(c) = 0$

Thus, statement-I is true,

Also, $f(t + 2\pi) = 2 + \cos(t + 2\pi)$

$= 2 + \cos t$

$= f(t)$

Thus, statement-II is also true

But statement-II is not a correct explanation for statement-I

32. We have $f(x) = g(x)\sin x$

$$\Rightarrow \quad f'(x) = g(x)\cos x + g'(x)\sin x$$

$$\Rightarrow \quad f''(x) = g'(x)\cos x - g(x)\sin x + g'(x)\cos x + g''(x)\sin x$$

Now, $f'(0) = g(0) + g'(0)\cdot 0 = g(0)$

So Statement-II is true and $f''(0) = 2g'(0)$

Now, $\lim_{x\to 0} (g(x)\cot x - g(0)\operatorname{cosec} x)$

$$= \lim_{x\to 0}\left(\frac{g(x)\cos x - g(0)}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{g'(x)\cos x - g(x)\sin x}{\cos x}\right)$$

$$= \left(\frac{g'(0)\cos 0 - g(0)\sin 0}{\cos 0}\right)$$

$= g'(0)$

$= 0$

$= f''(0)$

Thus, statement-I is also true.

But statement-II is not a correct explanation for statement-I

33. Ans. (a)

We have $g(x) = \log(f(x))$

Now, $g(x + 1) - g(x)$

$= \log(f(x + 1)) - \log(f(x))$

$$= \log\left(\frac{f(x+1)}{f(x)}\right)$$

$$= \log\left(\frac{xf(x)}{f(x)}\right)$$

$= \log x$

$$g'(x+1) - g'(x) = \frac{1}{x}$$

$$g''(x+1) - g''(x) = -\frac{1}{x^2}$$

Put $x = k - \frac{1}{2}$

$$g''\left(k + \frac{1}{2}\right) - g''\left(k - \frac{1}{2}\right) = -\frac{4}{(2k-1)^2}$$

Now, $g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right)$

$$= \sum_{k=1}^{N}\left(g''\left(k + \frac{1}{2}\right) - g''\left(k - \frac{1}{2}\right)\right)$$

$$= \sum_{k=1}^{N}\left(-\frac{4}{(2k-1)^2}\right)$$

$$= -4\left(1 + \frac{1}{9} + \frac{1}{25} + \dots + \frac{1}{(2N-1)^2}\right)$$

34. No questions asked in between 2009-2010.

35. (a)

Given $f(x) = \dfrac{b - x}{1 - bx}$

Let $y = \dfrac{b - x}{1 - bx} \Rightarrow x = \dfrac{b - y}{1 - by}$

$$0 < x < 1 \Rightarrow 0 < \frac{b - y}{1 - by} < 1$$

$$\Rightarrow \quad \frac{b - y}{1 - by} > 0 \Rightarrow y > b,\ y < \frac{1}{b}$$

$$\Rightarrow \quad \frac{b - y}{1 - by} - 1 < 0 \Rightarrow -1 < y < \frac{1}{b}$$

$$\Rightarrow \quad -1 < y < b$$

36. We have $f(\theta) = \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

$$= \sin\left(\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\sin^2\theta + \cos 2\theta}}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\frac{\sin\theta}{\sqrt{\sin^2\theta + \cos^2\theta - \sin^2\theta}}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)\right)$$

$$= \sin(\sin^{-1}(\tan\theta))$$

$$= \tan\theta$$

$$\frac{d}{d(\tan\theta)}(f(\theta))$$

$$= \frac{d}{d(\tan\theta)}(\tan\theta)$$

$$= 1.$$

37. No questions asked in between 2012-2014.

CHAPTER 6

Rolle's Theorem and Lagrange's Mean Value Theorem

CONCEPT BOOSTER

1. Rolle's Theorem

1.1 Introduction

In calculus, Rolle's theorem essentially states that any real-valued differentiable function that attains equal values at two distinct points must have a stationary point somewhere between them—that is, a point where the first derivative (the slope of the tangent line to the graph of the function) is zero.

1.2 Rolle's Theorem Statement

Let $f(x)$ be a real function and $a, b \in R$.

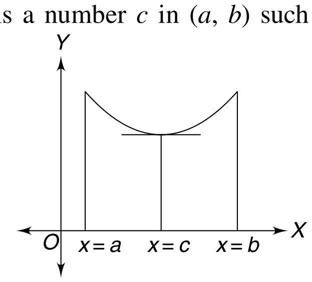
If

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

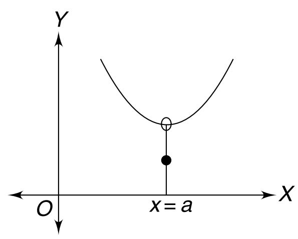
(iii) $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.



1.3 Applicability of Rolle's Theorem

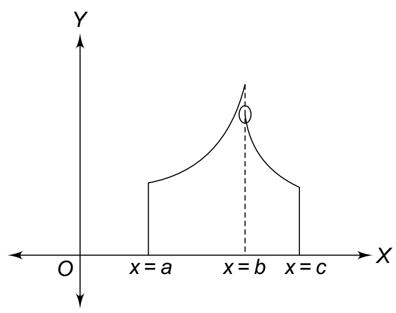
(i)



Here, $f(x)$ is not continuous at $x = a$

So, Rolle's theorem is not applicable here.

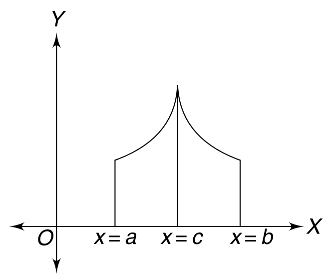
(ii)



Here, $f(x)$ is not continuous at $x = b$

So, Rolle's theorem is not applicable here.

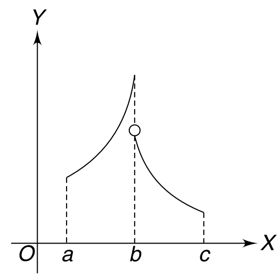
(iii)



Here, $f(x)$ is not differentiable at $x = c$

So, Rolle's theorem is not applicable here.

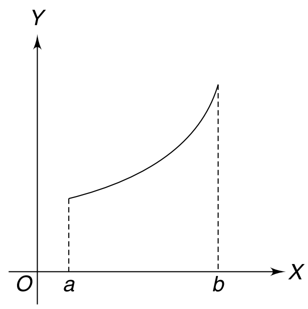
(iii)



Here, $f(x)$ is not continuous at $x = b$

So, Rolle's theorem is not applicable here.

(iv)



Here, $f(a)$ is not equal to $f(b)$.

So, Rolle's thoerem is not applicable here.

Here, $f(a)$ is not equal to $f(b)$.

So, Rolle's thoerem is not applicable here.

1.4 Algebraical Significance of Rolle's Theorem

Between any two roots of a polynomial, there is atleast one root of its derivative.

1.5 Applications of Rolle's Theorem

(i) If $y = f(x)$ satisfies the Rolle's theorem in $[a, b]$, then $f'(x) = 0$ for some x in (a, b).

(ii) Let $x = a$ and $x = b$ be the roots of $f(x) = 0$ and $y = f(x)$ satisfies the condition of Rolle's Theorem on $[a, b]$

Here, $f(a) = f(b) = 0$. Hence we can say that between two roots of $f(x) = 0$ at least one root of $f'(x) = 0$ would lie.

(iii) If $y = f(x)$ be a polynomial functions of degree n and $f(x) = 0$ has real roots, then all the roots of $f'(x) = 0$ are also real.

For if $f(x)$ is of degree n, $f'(x)$ is of degree $(n - 1)$ and a root of $f'(x) = 0$ exists in each of the $(n - 1)$ intervals between the n roots of $f(x) = 0$.

(iv) If all the roots of $f(x) = 0$ are real, so also are those of $f'(x) = 0$, $f''(x) = 0$, $f'''(x) = 0$, ..., and the roots of any one of those equations separate those of the preceeding equation. This follows from (iii).

(v) Not more than two roots of $f(x) = 0$ can

(a) lie between two consecutive roots of $f'(x) = 0$ or

(b) be less than the least of these or

(c) be greater than the greatest of these.

(vi) If $f'(x) = 0$ has n real roots, then $f(x) = 0$ can not have more than $(n + 1)$ real roots.

(vii) If $f^n(x)$ is the nth derivative of $f(x)$ and the equation $f^n(x) = 0$ has some imaginary roots, then $f(x) = 0$ has at least as many imaginary roots.

2. Lagranges Mean Value Theorem

2.1 Introduction

In mathematics, the mean value theorem states, roughly: that given a planar arc between two endpoints, there is at least one point at which the tangent to the arc is parallel to the secant through its endpoints.

The theorem is used to prove global statements about a function on an interval starting from local hypotheses about derivatives at points of the interval.

2.2 Statement

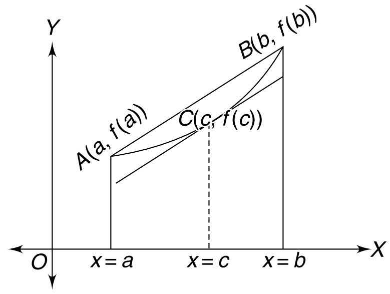
Let $f(x)$ be a real function and $a, b \in R$. If

(i) $f(x)$ is continuous in $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



3. Constant Function Theorem

If f is a continuous function in $[a, b]$ and is differentiable in (a, b), with $f'(x) = 0$ for all x in (a, b), then the function $f(x)$ is constant on $[a, b]$.

4. Constant Difference Theorem

Let f and g be two continuous functions on $[a, b]$ and are differentiable on (a, b) such that $f'(x) = g'(x)$ for all x in (a, b). Then there exists a constant C such that $f(x) = g(x) + C$ for all x in $[a, b]$.

5. Applications of L.M.V. Theorem

(i) If f is continuous on $[a, b]$ and $m \le f'(x) \le M$ for all x in (a, b), then $m(b - a) \le f(b) - f(a) \le M(b - a)$.

(ii) If f is continuous on $[a, b]$ and $|f'(x)| \le M$ for all x in (a, b), then $|f(b) - f(a)| \le M|b - a|$

(iii) If f is continuous on $[a, b]$ and $|f'(x)| \ge m$ for all x in (a, b), then $|f(b) - f(a)| \ge m|b - a|$

(iv) Rolle's Theorem is a particular case of L.M.V. Theorem.

6. Cauchy's Mean Value Theorem

Let f and g be two real functions defined on $[a, b]$ such that

(i) f and g both are continuous on $[a, b]$

(ii) f and g both are differentiable on (a, b) and

(iii) $g'(x)$ does not vanish at any point of (a, b)

then there exists a real number c in (a, b) such that
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

7. Generalised Mean Value Theorem

If f, g and h are continuous on $[a, b]$ and differentiable on (a, b), then there exist a number c in (a, b) such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0\cdot$$

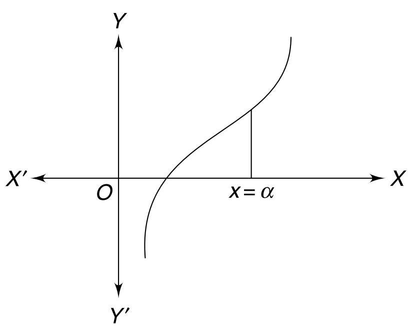
8. Nature of the Cubic Polynomial

Let $f(x) = ax^3 + bx^2 + cx + d$

Then $f'(x) = 3ax^2 + 2bx + c$

(i) when $f'(x) = 0$ has no real root.

Then $f(x)$ always increases as x increases and the equation $f(x) = 0$ has one real root.



Example 1. Let $f(x) = x^3 - x^2 + 4x + 10$

Then $f'(x) = 3x^2 - 2x + 4$

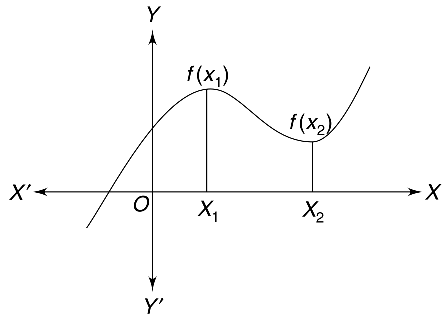
Therefore, $f'(x) > 0$ since $D = 4 - 48 < 0$

Hence, f is monotonic and it has one real root.

(ii) When $f'(x) = 0$ has two distinct roots say x_1 and x_2 $(x_1 < x_2)$

If $f(x_1) \cdot f(x_2) > 0$, then the equation

$f(x) = 0$ has one real root.



Example 2. Let $f(x) = 2x^3 - 9x^2 + 12x + 10$

Then $f'(x) = 6x^2 - 18x + 12$

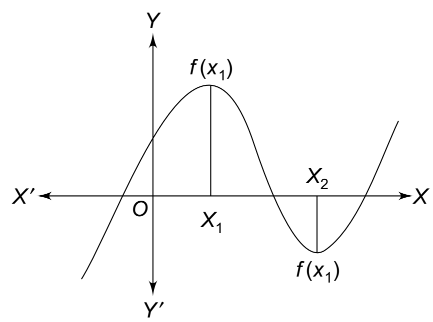
Thus $f'(x) = 0$ has two real roots 1 and 2.

Now $f(1)\cdot f(2) > 0$.

Thus, $f(x)$ has one real root.

(iii) The equation $f'(x) = 0$ has two distinct roots say x_1 and x_2.

If $f(x_1)f(x_2) < 0$, then the eqution $f(x) = 0$ has three distinct real roots.



Example 3. Let $f(x) = x^3 - 3x + 1$

Then $f'(x) = 3x^2 - 3$

Now $f'(x) = 0$ has two real roots 1 and –1

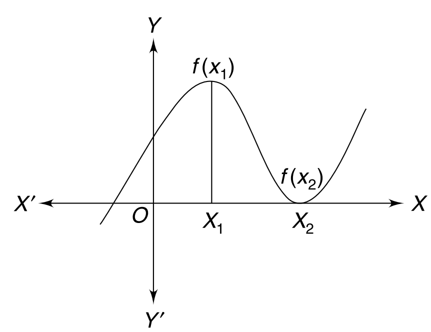
Now, $f(-1)f(1) = (-1 + 3 + 1)(1 - 3 + 1)$

$= -4 < 0$

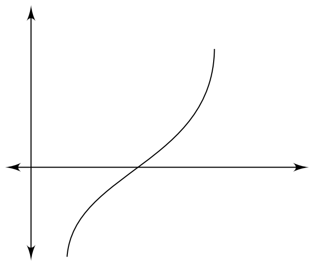
Thus $f(x) = 0$ has three real roots.

(iv) The eqution $f'(x) = 0$ has two distinct roots, say x_1 and x_2

If $f(x_2) = 0$, then $f(x)$ has two real root.



(v) The equation $f'(x) = 0$ has two equal roots, say α. If $f(\alpha) = 0$, then the equation $f(x) = 0$ has one reapeated (treble) root.



Example 4. Let $f(x) = (x - 3)^3$

Then $f'(x) = 3(x - 3)^2$

Thus $f'(x) = 0$ has one repeated root $x = 3$

Also $f(3) = 0$.

Thus, $f(x) = 0$ has one repeated root.

EXERCISES

Level 1

(Problems Based on Fundamentals)

1. Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x + 5$ on $[0, 2]$..
2. Verify Rolle's theorem for the function $f(x) = (x - a)^m (x - b)^n$ on $[a, b]$, $m, n \in I^+$
3. Verify Rolle's theorem for the function $f(x) = \log\left\{\frac{x^2 + ab}{x(a + b)}\right\}$ on $[a, b]$, where $0 < a < b$.
4. Verify Rolle's theorem for the function $f(x) = \sin^4 x + \cos^4 x$ on $\left[0, \frac{\pi}{2}\right]$
5. Verify Rolle's theorem for the function $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$
6. Verify Rolle's theorem for the function $f(x) = \sin x + \cos x - 1$ on $\left[0, \frac{\pi}{2}\right]$
7. Verify Rolle's theorem for the function $f(x) = e^x(\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
8. Suppose that $f(x) = x^{1/2} - x^{3/2}$ on $[0, 1]$. Find a number c that satisfies the conditions of Rolle's theorem.
9. If the value of c prescribed in the Rolle's theorem for the function $f(x) = 2x(x - 3)^n$, $n \in N$ on $[0, 3]$ is $3/4$, then find the value of n.
10. If the function $f(x) = x^3 - 6x^2 + ax + b$ is defined on $[1, 3]$ satisfies the hypothesis of Rolle's theorem, then find the values of a and b.
11. If the function $f(x) = x^3 - px^2 + qx$ is defined on $[1, 3]$ satisfies the hypothesis of Rolle's theorem such that $c = 5/4$, then find the values of $(p + 4q + 17)$.
12. At what points on the curve $y = 12(x + 1)(x - 2)$ on $[-1, 2]$, is the tangent parallel to x–axis?.

Algebraic Meaning of Rolle's Theorem

13. Let $f(x) = x^3 - x^2 - x + 1$. Prove that there is a root of its derivative on $(-1, 1)$.
14. Prove that the equation $x\cos x = \sin x$ has a root between π and 2π.
15. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.
16. Let $f(x) = \sin^5 x + \cos^5 x - 1$ on, $\left[0, \frac{\pi}{2}\right]$

    Then prove that the equation $f(x) = 0$ has two roots in $\left[0, \frac{\pi}{2}\right]$
17. If $2a + 3b + 6c = 0$, then prove that the equation $ax^2 + bx + c = 0$ has at least one root in $(0, 1)$.
18. If $a + b + c = 0$, then show that the equation $3ax^2 + 2bx + c = 0$ has atleast one root in $(0, 1)$.
19. If $3a + 4b + 6c + 12d = 0$, then prove that the equation $ax^3 + bx^2 + cx + d$, where $a, b, c, d \in R$ has atleast one root in $(0, 1)$.
20. If $f(x) = x^2(1 - x)^3$, then prove that the equation $f'(x) = 0$ has atleast one root in $(0, 1)$.
21. If $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$, then find the number of real roots of $f'(x) = 0$

    and indicate the intervals in which they would lie.
22. Find the number of real roots of
    (i) $x^3 - 6x^2 + 15x + 3 = 0$
    (ii) $4x^2 - 21x^2 + 18x + 20 = 0$
    (iii) $3x^4 - 8x^3 - 6x^2 + 24x + 1 = 0$
    (iv) $x^4 - 4x - 2 = 0$

23. Discuss the applicability of Rolle's theorem for each of the following functions on the indicated intervals:
    (i) $f(x) = |x - 2|$ in $[0, 3]$.
    (ii) $f(x) = 3 + (x - 2)^{2/3}$ in $[1, 3]$
    (iii) $f(x) = \sin\left(\frac{1}{x}\right)$ in $[-1, 1]$
    (iv) $f(x) = \begin{cases} -4x + 5, 0 \le x \le 1 \\ 2x - 3, 1 < x \le 2 \end{cases}$.
    (v) $f(x) = [x]$ in $[-1, 1]$, where [,] = G.I.F.
    (vi) $f(x) = \sin|x|$ in $[-2, 2]$
    (vii) $f(x) = e^{-|x|}$ in $[-3, 3]$
    (viii) $f(x) = \left|e^{-|x|} - \frac{1}{2}\right|$ in $[-2, 2]$
    (ix) $f(x) = \left|\frac{1}{x} - 1\right|$ in $[-1, 1]$
    (x) $f(x) = \sin x + |\sin x|$ in $[-\pi, \pi]$

Lagranges Mean Value Theorem

24. Verify Lagranges Mean Value Theorem for the function $f(x) = x^3 - x^2 - x + 1$ on $[0, 2]$.
25. Find the value of c for which the function $f(x) = (x - 1)(x - 2)(x - 3)$ on $[0, 4]$ is applicable on L.M.V. theorem
26. Find the point on the curve $y = 2x^2 - 5x + 3$ where the tangent is parallel to the chord joining the points $A(1, 0)$ and $B(2, 1)$.
27. Find a point on the parabola $y = (x - 4)^2$, where the tangent is parallel to the chord joining (4, 0) and (5, 1).
28. Suppose $f(x)$ is twice differentiable function such that $f(1) = 1, f(2) = 4, f(3) = 9$, then prove that there exist at least one root in (1, 3) such that $f''(x) = 2$, $x \in R$
29. If $f(x)$ satisfies the property of L.M.V. theorem in $[0, 2]$ such that $f(0) = 0$ and $f'(x) \le \frac{1}{2}$ for all x in $[0, 2]$, prove that $f(x) \le 1$.
30. Let $f(x)$ and $g(x)$ be differentiable functions for $x \in [0, 1]$ such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$ Let there exist a real number c in (0, 1) such that $f'(c) = 2g'(c)$, then find the value of $g(1)$.
31. If $a, b, c \in \left(0, \frac{\pi}{2}\right)$ and $a < c < b$, then prove that $\frac{\cos a - \cos b}{\sin a - \sin b} = -\tan c$
32. Discuss the applicability of L.M.V. theorem for each of the following functions to the indicated intervals
    (i) $f(x) = \frac{1}{x}$ on $[-1, 1]$
    (ii) $f(x) = \sin|x|$ on $[-2\pi, 2\pi]$
    (iii) $f(x) = |\sin x|$ on $[0, 2\pi]$
    (iv) $f(x) = \log|x|$ on $[0, 5]$
    (v) $f(x) = |\log|x||$ on $[-1, 1]$
33. Using L.M.V. theorem, prove that
$$\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a},$$
where $0 < a < b$.
34. Using L.M.V. theorem, prove that $(b - a)\sec^2 a < \tan b - \tan a < (b - a)\sec^2 b$, where $0 < a < b < \frac{\pi}{2}$
35. Using L.M.V. theorem, prove that
$$\frac{\beta - \alpha}{1 + \beta^2} < \tan^{-1}\beta - \tan^{-1}\alpha < \frac{\beta - \alpha}{1 + \alpha^2}, \; 0 < \alpha < \beta$$
36. Using L.M.V. theorem, prove that $|\sin x - \sin y| \le |x - y|$
37. Using L.M.V. theorem, prove that $|\tan^{-1} x| \le |x|$ for all x in R.
38. Using L.M.V. theorem, prove that $|\tan x - \tan y| \ge |x - y|$ for all x, y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
39. Let f be a differentiable function for all x in R. If $f(1) = -2$ and $f'(x) \ge 2$ for all x in [1, 6] such that the least value of $f(6)$ is m, then find the value of $2(m + 2)^3 + 17$
40. If f is continuous on [0, 2] and differentiable on (0, 2) such that $f(0) = 2$, $f(2) = 8$ and $f'(x) \le 3$ for all x in (0, 2), then find the value of $f(1)$.

Level II

(Mixed Problems)

Choose the Most Appropriate One (s).

1. The value of c in L.M.V. theorem for the function $f(x) = x^2$ in the interval $[-1, 1]$ is
   (a) 0 (b) $\frac{1}{2}$
   (c) $-\frac{1}{2}$ (d) non existent
2. The value of c of the L.M.V. theorem for which the function $\sqrt{25 - x^2}$ in [0, 5], is
   (a) 5 (b) 3
   (c) 4 (d) None
3. The value of c, of the L.M.V. theorem for which the function $f(x) = x(x - 1)(x - 2)$ in $\left[0, \frac{1}{2}\right]$ is
   (a) $\frac{1}{2}$ (b) 0.3
   (c) 0.24 (d) None

4. The equation $3x^2 + 4ax + b = 0$ has atleast on root in (0, 1) if
(a) $4a + b + 3 = 0$ (b) $2a + b + 1 = 0$
(c) $b = 0,\ a = -\frac{4}{3}$ (d) none

5. If Rolle's theorem holds for the function $f(x) = x^3 + bx^2 + ax + 5$ on [1, 3] with $c = 2 + \frac{1}{\sqrt{3}}$ then (a, b) is/are
(a) (3, –5) (b) (6, –11)
(c) (11, –6) (d) (11, 6)

6. The point on the curve $y = (\cos x - 1)$ in $[0, 2\pi]$ where the tangent is parallel to x–axis is
(a) (0, 0) (b) $(\pi, -2)$
(c) $(2\pi, 0)$ (d) $(\pi/2, 2\pi)$

7. Find out the function in which Rolle's theorem is satisfied.
(a) $f(x) = x^2$ in [1, 2]
(b) $f(x) = x^{2/3}$ in [–1, 1]
(c) $f(x) = |x|$ in [–1, 1]
(d) $f(x) = \sqrt{4 - x^2}$ in [–2, 2]

8. Let $f(x) = \sin^2 x - \sin 2x$ in $[0, \pi]$, then a number c in the Rolle's theorem is
(a) 32°32' (b) 126°23'
(c) 48°48' (d) 148°32'

9. Let $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$, then a number c is the Rolle's theorem is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{6}$ (d) $\frac{3\pi}{4}$

10. Let $f(x) = \sin x + \cos x + c$ is $[0, \pi]$, then a real number c in the Rolle's theorem is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) $\frac{2\pi}{5}$

11. Let $f(x) = (x - 1)(x - 2)^2$ in [1, 2], then a real number c is the Rolle's throrem is
(a) $\frac{4}{3}$ (b) $\frac{3}{4}$
(c) $\frac{5}{4}$ (d) $\frac{4}{5}$

12. If $2a + 3b + 6c = 0$, then atleast one root of the equation $ax^2 + bx + c = 0$ lies in the interval
(a) (0, 1) (b) (1, 2)
(c) (2, 3) (d) None

13. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for all $x \in [1, 6]$ then
(a) $f(6) < 8$ (b) $f(6) > 8$
(c) $f(6) > 5$ (d) None

14. If a, b, c be non-zero real numbers such that
$$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)\,dx$$
$$= \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)\,dx$$
then the equation $ax^2 + bx + c = 0$ will have a root between
(a) (1, 3) (b) (1, 2)
(c) (2, 3) (d) (3, 1)

15. The value of $|\cos a - \cos b|$ is
(a) $\leq |a - b|$ (b) $\geq |a - b|$
(c) 0 (d) $|a + b|$.

16. The length of the longest interval in which Rolle's theorem can be applied for the function $f(x) = |x^2 - a^2|$ is $(a > 0)$
(a) $2a$ (b) $4a^2$
(c) $a\sqrt{2}$ (d) a

17. The number of roots of the equation $\sin x + 2\sin 2x + 3\sin 3x = 0$ is the interval $[0, \pi]$ is
(a) 1 (b) 2
(c) 3 (d) More than 3

18. If $\int_0^1 e^x(x - \alpha)\,dx = 0$, then
(a) $1 < \alpha < 2$ (b) $\alpha < 2$
(c) $0 < \alpha < 1$ (d) $\alpha = 0$

19. $f(x) = x^\alpha \sin\left(\frac{1}{x}\right),\ x \neq 0,\ f(0) = 0$ satisfies conditions of Rolle's theorem on $\left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$ for α equals
(a) –1 (b) 0
(c) 7/2 (d) 5/3

20. If $f(x) = ax + b,\ x \in [-2, 2]$, then the point $c \in (-2, 2)$ where
$$f(c) = \frac{f(2) - f(-2)}{4}$$
(a) does not exist
(b) can be any $c \in (-2, 2)$
(c) can be only 1
(d) can be only –1.

21. Mean value theorem is applicable to $f(x) = x + 2x - \pi| - 2|\cos x|$ equation in the interval
(a) [–2, 2] (b) $[0, \pi]$

(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left[-\frac{\pi}{3}, \frac{2\pi}{3}\right]$.

22. For the function $f(x) = \left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] - [2x]$, Rolle's theorem is applicable on the interval

(a) $\left[0, \frac{1}{4}\right]$ (b) $\left[\frac{1}{4}, 1\right]$

(c) $\left[0, \frac{1}{2}\right]$ (d) $[0, 1]$

23. Rolle's theorem is applicable for the function $f(x) = (x-1)|x| + |x-1|$ in the interval

(a) $[0, 1]$ (b) $\left[\frac{1}{4}, \frac{3}{4}\right]$

(c) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (d) $\left[\frac{1}{5}, \frac{6}{7}\right]$

24. If $a, b, c \in \left(0, \frac{\pi}{2}\right)$ and $a < c < b$, then $\frac{\cos a - \cos b}{\sin a - \sin b}$ may equal to which of the following

(a) $\tan c$ (b) $\cot c$

(c) $-\tan c$ (d) $-\cot c$

Level III

(Problems for JEE-Advanced)

1. Let $f: [0, 4] \to R$ be a differentiable function for some $a, b \in (0, 4)$, show that

$$f^2(4) - f^2(0) = 8f(a)f'(b)$$

2. Let $f: [2, 7] \to [0, \infty)$ be a continuous and differentiable function, show that

$$(f(7) - f(2))\,\frac{\{(f(7))^2 + (f(7))^2 + f(2)\cdot f(7)\}}{3}$$

$= 5f^2(c)\cdot f'(c)$, for all $c \in (2, 7)$

3. If f and g are differentiable on $[0, 1]$, $f(0) = 2, g(0) = 0, f(1) = 6$ and $g(1) = 2, \forall c \in (0, 1)$, then show that

$$f'(c) = 2g'(c)$$

4. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for all $x \in [1, 6]$ then show that

$$f(6) \geq 8$$

5. If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), show that there exists some $c \in (a, b)$

such that $\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$

6. Let f be a continuous in $[a, b]$ and differentiable in (a, b). If $f(a) = a$ & $f(b) = b$, then show that there exists distinct $c_1, c_2 \in (a, b)$ such that

$$f'(c_1) + 'f(c_2) = 2$$

7. Let $f(x)$ be a twice differentiable function such that $f''(x) < 0$ in $[0, 2]$. Then prove that

(i) $f(0) + f(2) = 2f(c), c \in (0, 2)$

(ii) $f(0) + f(2) < f(1)$

8. Let $f(x)$ is continuous in $[0, 2]$ and differentiable in $(0, 2)$. If $f(0) = 2, f(2) = 8$ and $f'(x) \leq 3$ for all x in $(0, 2)$, find the value of $f(1)$.

9. If $a, b \in \left[0, \frac{\pi}{2}\right)$, prove that

$|\cos a - \cos b| \leq |a - b|$

10. Find the value of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in $(0, 1)$.

11. Find the number of distinct real roots of

$x^4 - 4x^3 + 12x^2 + x - 1 = 0$.

12. Find the number of real roots of

$x^3 - 6x^2 + 15x + 3 = 0$

13. Find the number of real roots of

$4x^3 - 21x^2 + 18x + 20 = 0$.

14 Find the number of real roots of

$3x^4 - 8x^3 - 6x^2 + 24x + 1 = 0$

15. Find the number of real roots of

$x^4 - 4x - 2 = 0$

16. Find the value of k for which the equation $x^4 - 14x^2 + 24x - k = 0$ has four real and unequal roots.

17. Find the value of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in $(0, 1)$.

18. Let $x^3 - 2kx^2 - 4kx + k^2 = 0$. If one root is less than 1 and the other is in $(1, 4)$ and the third root is greater than 4 such that $k \in (m + \sqrt{n}, n(m + \sqrt{n}))$, then find the value of $2(m + n - 1)^3 + 17$.

19. Find the value of 'a' for which the equation $x^3 - 3x + a = 0$ would have two distinct roots in $(-1, 1)$.

20. Find the value of k such that Rolle's theorem is applicable to the function $f(x) = x^3 - 2x^2 - 4x + 7$ on the interval $[0, k]$

Level IV

(Tougher Problems for JEE-Advanced)

1. If the equation $x^3 - 3x + 1 = 0$ has three real roots α, β, γ such that $\alpha < \beta < \gamma$, then find the value of $\{\alpha\} + \{\beta\} + \{\gamma\}$, where $\{,\}$ = F.P.F
2. Find the smallest natural number c for which the equation $e^x = cx^2$ has exactly three real and distinct solutions.
3. Let $f(x) = x^4 + 8x^3 + 18x^2 + 8x + b$
   If the equation $f(x) = 0$ has four real and distinct roots, then find b.
4. Let $f: [0, 4] \to R$ be a differentiable function
   (i) for some $a, b \in (0, 4)$, show that,
   $$f^2(4) - f^2(0) = 8f(a)f(b)$$
   (ii) Show that
   $$\int_0^4 f(t)\,dt = 2\{\alpha f(\alpha^2) + \beta f(\beta^2)\}$$
   for some $0 < \alpha, \beta < 2$
5. Prove that $2a_0^2 < 15a$ then all roots of $x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d = 0$ can not be real. It is given that $a_0, a, b, c, d \in R$.
6. For all $x \in [0, 1]$, $f''(x)$ exists and such that $|f''(x)| < 1$. If $f(0) = f(1)$, then show that $|f'(x)| < 1$ for all x in $[0, 1]$.
7. Using L.M.V. theorem, prove that
   $$\frac{x}{1+x} < \log(1+x) < x, \ \forall\ x > 0$$

Integer Type Questions

1. Let $f(x) = x^{1/2} - x^{3/2}$ on [0, 1] such that a number c satisfies the hypothesis of Rolle's theorem, then the value of $3c + 4$ is
2. If the equation $ax^2 + bx + c = 0$ has at-leas one root in (0, 1) such that $2a + 3b + 6c = m$ where $m \in W$, then find the value of $(m + 3)$
3. If m is the number of real roots of $x^4 - 4x - 2 = 0$ and n is the number of real solutions of $2^x = x^2 + 1$, then find the value of $(m + n)$.
4. If p is the number of real roots of $x^4 - 14x^2 + 24x + k = 0$ and q is the number of non differentiable points of $f(x) = |x| + |x^2 - 1|$, then the value of $(p + q + 2)$ is
5. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for all x in [1, 4], then the least value of $f(4)$.
6. Let $f(x) = 2x(x - 3)^n$, $n \in N$ on [0, 3]. If $f(x)$ satisfies the Rolle's theorem for which c is 3/4, then find the value of n.

Comprehensive Link Passages

Passage I

Let $f(x)$ be a real valued function and $a, b \in R$. If f is continuous in $[a, b]$ and differentiable in (a, b), then there exist atleast one real number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

On the basis of the above statement, answers the following questions.

1. The point on the curve $y = x^3$ such that the tangent at that point is parallel to the chord joining the point $A(1, 1)$ and $B(4, 64)$ is
   (a) $(7, \sqrt{7})$ (b) $(5, 5\sqrt{5})$
   (c) $(7, 7\sqrt{7})$ (d) None
2. If $\log(1 + x) < x < e^x$ for all $x > 0$, then which one is most appropriate?
   (a) $0 < \log\left(\frac{e^x - 1}{x}\right) < x$
   (b) $x < \log\left(\frac{e^x - 1}{x}\right) < e^x$
   (c) $\frac{x+1}{x} < \frac{1}{\log(1+x)} < \frac{1}{x}$
   (d) All of the above.
3. If f and g are differentiable on [0, 1], $f(0) = 2$, $g(0) = 0$, $f(1) = 6$, $g(1) = 2$ for all c in (0, 1), then
   (a) $f'(c) = g'(c)$
   (b) $f'(c) = 2g'(c)$
   (c) $f'(c) + g'(c) = 0$
   (d) $2f'(c) - g'(c) = 0.72$.

Passage II

Between any two roots of a polynomial $f(x)$, there is always a root of its derivative.

On the basis of the above information answer the following questions.

1. If a, b, c in R such that the equation $3ax^2 + 2bx + c = 0$ has atleast one root in (0, 1), then $a + b + c$ is
   (a) 1 (b) 0
   (c) 2 (d) –1
2. If $x^4 - 14x^2 + 24x - k = 0$ has four real and unequal roots, then the value of k lies in
   (a) (8, 11) (b) (2, 5)
   (c) (0, 2) (d) (1, 3)
3. The number of real roots of $4x^3 - 21x^2 + 18x + 20 = 0$ is
   (a) 1 (b) 2
   (c) 3 (d) 4

4. The number of real roots $x^4 - 4x - 1 = 0$ is
   (a) 1 (b) 2
   (c) 3 (d) 4
5. The number of real roots of $e^x = x + 1$ is
   (a) 1 (b) 0
   (c) 2 (d) None
6. The number of real roots of $2^x = x^2 + 1$ is
   (a) 1 (b) 2
   (c) 0 (d) None

Passage III

Let $f(x) = x^3 - 3x + k$

1. The number of real roots of $f(x) = 0$ is
   (a) 2 (b) 1
   (c) 3 (d) 0
2. If $f(x) = 0$ has three distinct real root then k lies in
   (a) (–1, 1) (b) (–2, 2)
   (c) (1, 2) (d) (2, 3)
3. The number of real roots of $f(x) = f(x + 1)$ is
   (a) 1 (b) 2
   (c) 3 (d) 0

Passage IV

Let $f(x) = x^4 - 14x^2 + 24x - k$

1. The number of real roots of $f(x) = 0$ is
   (a) 3 (b) 2
   (c) 4 (d) 1
2. If $f(x)$ has four real and unequal roots, then k lies in
   (a) (5, 8) (b) (8, 11)
   (c) (11, 13) (d) (13, 19)
3. The number of imaginary roots of $f(x) = 0$ is
   (a) 2 (b) 1
   (c) 0 (d) 3

Matrix Match
(For JEE-Advanced Exam Only)

1. **Match the following columns:**
   If each of the follwing functions are satisfied by Rolle's theorem, then the value of c is

Column I		Column II	
(A)	$f(x) = e^x \sin x$, $[0, \pi]$	(P)	$\frac{\pi}{3}$
(B)	$f(x) = x + \sin x$, $[0, \pi]$	(Q)	π
(C)	$f(x) = \cos\left(2x - \frac{\pi}{2}\right)$, $\left[0, \frac{\pi}{2}\right]$	(R)	$\frac{\pi}{2}$
(D)	$f(x) = \sin^2 x$, $[0, \pi]$	(S)	$\frac{\pi}{4}$

2. **Match the following columns:**
   If the function $f(x) = x^3 - px^2 + qx$ is defined on [1, 3] satisfies the Rolle's theorem with $c = 5/2$, then

Column I		Column II	
(A)	$4p - q$	(P)	3
(B)	$4p - 2q$	(Q)	13
(C)	p	(R)	23/4
(D)	q	(S)	10

3. **Match the following columns:**

$$\text{Let} \quad f(x) = \begin{cases} ax^2 + b & : |x| < 1 \\ 1 & : |x| = 1 \\ \frac{c}{|x|} & : |x| > 1 \end{cases}$$

   If Rolle's theorem is applicable for the function f in [–3, 3], then

Column I		Column II	
(A)	$|a| + |b| + |c|$	(P)	8
(B)	$2a + 3b + 4c$	(Q)	3
(C)	$a + b$	(R)	1
(D)	c	(S)	2

4. **Match the following columns:**
   Let $f(x) = x^3 + bx^2 + ax$, $\forall\, x \in [1, 3]$ If $f(x)$ satisfies Rolle's theorem such that

$$c = 2 + \frac{1}{\sqrt{3}}, \text{ then}$$

Column I		Column II	
(A)	the value of a is	(P)	17
(B)	the value of b is	(Q)	5
(C)	the value of $a + b$ is	(R)	–6
(D)	the value of $a - b$ is	(S)	11

5. **Match the following columns:**

	Column I	Column II	
(A)	Let $f(x) = 2x(x-3)^n$ on [0, 3] If $f(x)$ satisfies Rolle's theorem on [0, 3] for which $c = 3/4$, then the value of n is	(P)	3
(B)	If $f(x) = x^{3/2} - x^{3/2}$ on [0, 1] satisfies Rolle's theorem, then the value of 'c' is	(Q)	1/3
(C)	If $f(x) = x^4 - 2x^2 + 1$ on [−2, 2] satisfies Rolle's theorem, then the value of 'c' is	(R)	−2
(D)	If $f(x) = x^{\alpha}$log and $f(0) = 0$, then the value of α for which Rolle's theorem can be applied in [0, 1] is	(S)	−1

6. **Match the following columns:**

	Column I	Column II	
(A)	The number of real roots of $4x^3 - 21x^2 + 18x + 20 = 0$ is	(P)	4
(B)	The number of real roots of $x^4 - 4x - 1 = 0$ is	(Q)	3
(C)	The number of real roots of $3x^4 - 8x^3 - 6x^2 + 24x + 1 = 0$ is	(R)	2
(D)	The number of real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is	(S)	1

Qestions asked in Past IIT-JEE Exams

1. If $a + b + c = 0$, then the equation

$3ax^2 + 2bx + c = 0$ has

(a) atleast one root in [0, 1]
(b) one root in [2, 3] and the other in [−2, −1]
(c) imaginary roots.
(d) None **[IIT-JEE, 1983]**

2. In [0, 1] Lagranges Mean Value Theorem is not applicable to

(a) $f(x) = \begin{cases} \left(\frac{1}{2} - x\right) & : x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & : x \geq \frac{1}{2} \end{cases}$

(b) $f(x) = \begin{cases} \frac{\sin x}{x} & : x \neq 0 \\ 1 & : x = 0 \end{cases}$

(c) $f(x) = x|x|$

(d) $f(x) = |x|$

[IIT-JEE, 2003]

3. If $f(x) = x^a \log x$ and $f(0) = 0$, then the value of α for which Rolle's theorem can be applied to [0, 1] is

(a) −2 (b) −1
(c) 0 (d) 1/2.

[IIT-JEE, 2004]

4. Using Rolle's theorem, prove that there is at least one root in $\left(45^{\frac{1}{100}}, 46\right)$ of the polynomial

$P(x) = 51x^{101} - 2323x^{100} - 45x + 1035.$

[IIT-JEE, 2004]

5. If $f(x)$ is a twice differentiable such that

$f(a) = 0, f(b) = 2, f(c) = -1,$

$f(d) = 2, f(e) = 0$ where $a < b < c < d < e$

then find the minimum number of zeroes of $g(x) = (f'(x))^2 + g'(x)f''(x)$ in the interval [a.e]

[IIT-JEE, 2006]

6. The number of distinct real roots of

$x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is ...

[IIT-JEE, 2011]

7. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)$ for all $x > 0$ then

(a) f has a local maximum at $x = 2$
(b) f is decreasing on (2, 3)
(c) there exist some $c \in (0, \infty)$ such that $f''(c) = 0$
(d) f has a local minimum at $x = 3$.

[IIT-JEE, 2012]

ANSWERS

LEVEL-II

1. (d) 2. (d) 3. (c) 4. (b) 5. (d)
6. (a, b, c) 7. (d) 8. (a, b) 9. (a) 10. (b)
11. (a) 12. (a) 13. (b) 14. (b) 15. (a)
16. (a) 17. (d) 18. (c) 19. (d) 20. (b)
21. (b, c, d) 22. (a) 23. (a, b, d) 24. (c)

INTEGER TYPE QUESTIONS

1. (5) 2. (3) 3. (4) 4. (9) 5. (4)
6. (3)

COMPREHENSIVE LINK PASSAGES

Passage I :	1. (c)	2. (a)	3. (b)
Passage II :	1. (b)	2. (a)	3. (c)
	4. (b)	5. (a)	6. (b)
Passage III :	1. (c)	2. (a)	3. (b)
Passage IV :	1. (c)	2. (b)	3. (c)

MATRIX MATCH

1. (A)→(R), (B)→(S), (C)→(P), (D)→(Q)
2. (A)→(Q), (B)→(P), (C)→(R), (D)→(S)
3. (A)→(Q), (B)→(P), (C)→(R), (D)→(R)
4. (A)→(S), (B)→(R), (C)→(Q), (D)→(P)
5. (A)→(P), (B)→(Q), (C)→(S), (D)→(S)
6. (A)→(Q), (B)→(R), (C)→(R), (D)→(R)

HINTS AND SOLUTIONS

Level I

(Problems based on Fundaments)

1. Given $f(x) = x^3 - 3x^2 + 2x + 5$

As we know that, every polynomial function is continuous as well as differentiable.

So, $f(x)$ is continuous and differentiable on the indicated interval.

Also, $f(0) = 5$ and $f(2) = 8 - 12 + 4 + 5 = 5$

i.e. $f(0) = 5 = f(2)$

Thus, all the conditions of Rolle's theorem are satisfied.

Now, we have to show that there exist a point c in (0, 2) such that $f'(c) = 0$

We have $f'(c) = 3c^2 - 6c + 2 = 0 = 0$ gives

We have $f'(c) = 3c^2 - 6c + 2 = 0$ gives

$$\Rightarrow \quad c = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{6 \pm 2\sqrt{3}}{6} = 1 \pm \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$$

Hence, Rolle's theorem is verified.

2. Given $f(x) = (x - a)^m (x - b)^n$

As we know that every polynomial function is continuous and differentiable everywhere.

So, $f(x)$ is continuous and differentiable on the given indicated interval.

Also, $f(a) = 0 = f(b)$.

Thus, all the conditions of Rolle's theorem are satisfied.

Now, we have to show that there exist a point c in (a, b) such that $f'(c) = 0$

So,

$$f'(x) = m(x - a)^{m-1}(x - b)^n + n(x - a)^m (x - b)^{n-1}$$

$$f'(x) = (x - a)^{m-1}(x - b)^{n-1}\,(m(x - b) + n(x - a))$$

Now, $f'(c)$ gives $c = a$, $c = b$ and

$$\Rightarrow \quad (m(c - b) + n(c - a)) = 0$$

$$\Rightarrow \quad c = \frac{mb + na}{m + n} \in (a, b)$$

Hence, Rolle's theorem is verified.

3. Given $f(x) = \log\left\{\dfrac{x^2 + ab}{x(a + b)}\right\}$

As we know that every logarithmic function is continuous and differentiable in Positive real numbers

So it is continuous in $[a, b]$ and differentiable in (a, b)

Also, $f(a) = \log\left(\dfrac{a^2 + ab}{a(a + b)}\right) = \log 1 = 0$

and $f(b) = \log\left(\dfrac{b^2 + ab}{b(a + b)}\right) = \log 1 = 0$

Thus, all the conditions of Rolle's theorem are satisfied.

Now we have to show that there exist a point c in (a, b) such that $f'(c) = 0$

We have $f(x) = \log\left(\dfrac{x^2 + ab}{x(a + b)}\right)$

$$= \log(x^2 + ab) - \log(x(a + b))$$

$$\Rightarrow \quad f'(x) = \frac{2x}{x^2 + ab} - \frac{1}{x}$$

$$\Rightarrow \quad f'(x) = \frac{2x^2 - x^2 - ab}{x(x^2 + ab)} = \frac{x^2 - ab}{x(x^2 + ab)}$$

Now, $f'(c) = 0$ gives $c^2 - ab = 0$

$$\Rightarrow \quad c = \sqrt{ab} \in (a, b)$$

Hence, Rolle's theorem is verified.

4. Given $f(x) = \sin^4 x + \cos^4 x$

As we know that every sine and co-sine functions is continuous and differentiable everywhere.

So, it is continuous on $\left[0, \frac{\pi}{2}\right]$ and

differentiable on $\left(0, \frac{\pi}{2}\right)$

Also, $f(0) = 1 = f\left(\frac{\pi}{2}\right)$

Thus, all the conditions of Rolle's theorem are satisfied.

Now, we have to show that there exist a point c in $\left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$

Therefore,

$$\begin{aligned} f'(x) &= 4\sin^3 x\cos x - 4\cos^3 x\sin x \\ &= 4\sin x\cos x(\sin^2 x - \cos^2 x) \\ &= -2.(2\sin x\cos x)(\cos^2 x - \sin^2 x) \\ &= -2.\sin 2x\cdot\cos 2x \\ &= -\sin 4x \end{aligned}$$

Now, $f'(c) = 0$ gives $\sin(4c) = 0$

$\Rightarrow \quad \sin(4c) = \sin(\pi)$

$\Rightarrow \quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right).$

5. Clearly, $f(x)$ is continuous in $[0, \pi]$ and differentiable in $(0, \pi)$

$f(0) = 0, f(\pi) = 0$

$\Rightarrow \quad f(0) = 0 = f(\pi)$

Thus, all the conditions of Rolle's theorem are satisfied.

Now, we have to show that, there exists a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now, $f'(c) = 0$ gives

$\Rightarrow \quad 2\cos c + 2\cos 2c = 0$

$\Rightarrow \quad \cos c + \cos 2c = 0$

$\Rightarrow \quad \cos c + 2\cos^2 c - 1 = 0$

$\Rightarrow \quad 2\cos^2 c + \cos c - 1 = 0$

$\Rightarrow \quad 2\cos^2 c + 2\cos c - 1 = 0$

$\Rightarrow \quad 2\cos c(\cos c + 1) - (\cos c + 1) = 0$

$\Rightarrow \quad (2\cos c - 1)(\cos c + 1) = 0$

$\Rightarrow \quad \cos c = \frac{1}{2}, -1$

$\Rightarrow \quad c = \frac{\pi}{3}, \pi$

$\Rightarrow \quad c = \frac{\pi}{3} \in [0, \pi]$

Thus, the Rolle's theorem is verified.

6. As we know that, every sine and cosine function is differentiable. So, $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$ and differentiable in $\left(0, \frac{\pi}{2}\right)$

$f(0) = 0 + 1 - 1 = 0$

$f\left(\frac{\pi}{2}\right) = 1 + 0 - 1 = 0$

So, $f(0) = 0 = f\left(\frac{\pi}{2}\right)$

Thus, all the conditions of Rolle's theorem are satisfied.

Now, we have to show that, there exists a point

$c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

$\Rightarrow \quad \cos c - \sin c = 0$

$\Rightarrow \quad \cos c = \sin c$

$\Rightarrow \quad \tan c = 1$

$\Rightarrow \quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

Thus, the Rolle's theorem is verified.

7. As we know that, every exponential, sine and cosine functions are differentiable everywhere.

So, it is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

and differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Now, $f\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$

and $f\left(\frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$

So, $f\left(\frac{\pi}{4}\right) = 0 = f\left(\frac{5\pi}{4}\right)$

Thus, all the conditions of Rolle's theorem are satisfied.

Now, we have to show that, there exists a point

$c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$.

$\Rightarrow \quad e^c(\sin c - \cos c) + e^c(\cos c + \sin c) = 0$

$\Rightarrow \quad (\sin c - \cos c) + (\cos c + \sin c) = 0$

$\Rightarrow \quad 2\sin c = 0$

$\Rightarrow \quad \sin c = 0$

$\Rightarrow \quad c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Thus, all the conditions of Rolle's theorem are verified.

8. Given $f(x) = x^{1/2} - x^{3/2}$ on $[0, 1]$

Since $f(x)$ satisfies all the conditions of Rolle's theorem, so there exists a point $c \in (0, 1)$ such that

$$f'(c) = 0$$
$$\Rightarrow \quad \frac{1}{2}c^{-1/2} - \frac{3}{2}c^{1/2} = 0$$
$$\Rightarrow \quad \frac{1}{c^{1/2}} - 3c^{1/2} = 0$$
$$\Rightarrow \quad 1 - 3c = 0$$
$$\Rightarrow \quad c = \frac{1}{3} \in (0, 1)$$

9. Given $f(x) = 2x(x-3)^n$

$$\Rightarrow \quad f'(x) = 2(x-3)^n + 2nx(x-3)^{n-1}$$

Now, $f'(c) = 0$ gives

$$\Rightarrow \quad 2(c-3)^n + 2nc(c-3)^{n-1} = 0$$
$$\Rightarrow \quad 2(c-3)^n = -2nc(c-3)^{n-1}$$
$$\Rightarrow \quad (c-3)^n = -nc(c-3)^{n-1}$$
$$\Rightarrow \quad (c-3) = -nc$$
$$\Rightarrow \quad c(1+n) = 3$$
$$\Rightarrow \quad (1+n) = \frac{3}{c} = \frac{3}{3/4} = 4$$
$$\Rightarrow \quad n = 3$$

Hence, the value of n is 3.

10. Given $f(x) = x^3 - 6x^2 + ax + b$

$$f(1) = f(3)$$
$$\Rightarrow \quad 1 - 6 + a + b = 27 - 54 + 3a + b$$
$$\Rightarrow \quad a + b - 5 = 3a + b - 27$$
$$\Rightarrow \quad 2a = 22$$
$$\Rightarrow \quad a = 11, b \in R$$

11. Given $f(x) = x^3 - px^2 + qx$

$$f(1) = f(3)$$
$$\Rightarrow \quad 1 - p + q = 27 - 9p + q$$
$$\Rightarrow \quad 1 - p = 27 - 9p$$
$$\Rightarrow \quad 8p = 26$$
$$\Rightarrow \quad p = \frac{13}{4}$$

Also, $f'(c) = 0$

$$\Rightarrow \quad 3c^2 - 2pc + q = 0.$$
$$\Rightarrow \quad 3\left(\frac{5}{4}\right)^2 - 2p\left(\frac{5}{4}\right) + q = 0$$
$$\Rightarrow \quad q = 2p\left(\frac{5}{4}\right) - 3\left(\frac{5}{4}\right)^2$$
$$\Rightarrow \quad q = 2\left(\frac{13}{4} \times \frac{5}{4}\right) - \frac{75}{16} = \frac{130 - 75}{16} = \frac{55}{16}$$

Hence, the value of $(p + 4q + 17)$

$$= \frac{13}{4} + \frac{55}{4} + 17$$
$$= \frac{68}{4} + 17 = 34$$

12. Given curve is $y = 12(x+1)(x-2)$

Since the tangent is parallel to x–axis, so

so, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 12\{(x+1) + (x-2)\} = 0$$
$$\Rightarrow \quad 2x - 1 = 0$$
$$\Rightarrow \quad x = \frac{1}{2}$$

when $x = 1/2$, then $y = 12\left(1 + \frac{1}{2}\right)\left(\frac{1}{2} - 2\right)$

$$\Rightarrow \quad y = 12 \times \frac{3}{2} \times -\frac{3}{2} = -27$$

Hence, the point is $\left(\frac{1}{2}, -27\right)$.

13. Given $f(x) = x^3 - x^2 - x + 1$

$$\Rightarrow \quad f'(x) = 3x^2 - 2x - 1$$
$$f(x) = 0 \text{ gives } 3x^2 - 2x - 1 = 0$$
$$\Rightarrow \quad 3x^2 - 3x + x - 1 = 0$$
$$\Rightarrow \quad 3x(x-1) + 1(x-1) = 0$$
$$\Rightarrow \quad (3x+1)(x-1) = 0$$
$$\Rightarrow \quad x = 1, -\frac{1}{3}$$
$$\Rightarrow \quad x = -\frac{1}{3} \in (-1, 1)$$

Thus, the derivative of the function has a root in $(-1, 1)$.

14. Here, we shall show that the equation $x\cos x = \sin x$ has a root in π and 2π.

Let $f(x) = \frac{\sin x}{x}$

Clearly $f(x) = 0$, at $x = \pi, 2\pi$

Now, $f'(x) = \frac{x\cos x - \sin x}{x^2}$

So, by Rolle's theorem, $f'(x)$ must vanish for some value of x in$(\pi, 2\pi)$.

Hence, the equation $x\cos x - \sin x = 0$ has a root in $(\pi, 2\pi)$.

15. Let $f(x) = x^3 + x - 1$

Now, $f(0) = 0 + 0 - 1 = -1 < 0$

$f(1) = 1 + 1 - 1 = 1 > 0$

So, $f(x)$ has a root in between 0 and 1.

16. Given $f(x) = \sin^5 x + \cos^5 x - 1$

$$f'(x) = 5\sin^4 x\cos x - 5\cos^4 x\sin x$$
$$= 5\sin x\cos x(\sin^3 x - \cos^3 x)$$

Now, $f'(x) = 0$ gives

$$\Rightarrow \quad \sin x\cos x(\sin^3 x - \cos^3 x) = 0$$

$\Rightarrow \quad \tan^3 x = 1$

$\Rightarrow \quad x = \dfrac{\pi}{4}$

Since $f'(x)$ has a real root, so $f(x)$ must have two roots in $\left[0, \dfrac{\pi}{2}\right]$

17. Let $f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx$

Now $f(0) = 0$

and $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c = \dfrac{2a + 3b + 6c}{6} = \dfrac{0}{6} = 0$

Thus, $ax^2 + bx + c = 0$ has at-least one root in (0, 1).

18. Let $f(x) = ax^3 + bx^2 + cx$

Now, $f(0) = 0$

and $f(1) = a + b + c = 0.$

Thus, by Rolle's Theorem, there is atleast one root in its derivative.

19. Let $f(x) = \dfrac{ax^4}{4} + \dfrac{bx^3}{3} + \dfrac{cx^2}{2} + dx$

Now, $f(0) = 0$

and $f(1) = \dfrac{a}{4} + \dfrac{b}{3} + \dfrac{c}{2} + d$

$= \dfrac{3a + 4b + 6c + 12d}{12} = 0$

So, by Rolle's theorem, between any two roots of a polynomial, there is atleast one root of its derivative.

Hence, the result.

20. Given $f(x) = x^2(1 - x)^3$

Clearly, $f(0) = 0 = f(1)$

Thus, $f(x)$ has two roots.

So, by Rolle's theorem, between any two roots of a polynomial, there is atleast one root of its derivative.

Hence, the result.

21. Given $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$

$f'(x) = (x - 1)(x - 2)(x - 3) + (x - 1)(x - 3)(x - 4) + (x - 1)(x - 2)(x - 4)$

$+ (x - 2)(x - 3)(x - 4)$

$= (x^3 - 6x^2 + 11x - 6) + (x^3 - 8x^2 + 19x - 12)$

$+ (x^3 - 7x^2 + 14x - 8) + (x^3 - 9x^2 + 26x - 24)$

$= 4x^3 - 30x^2 + 70x - 50$

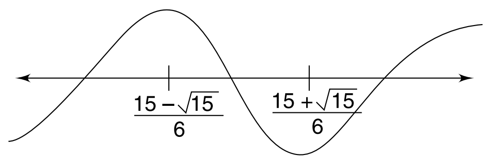
$f''(x) = 12x^2 - 60x + 70$

Clearly, its $D = 3600 - 3360 = 240 > 0$

So, $f''(x)$ has two real roots.

Hence, $f'(x) = 0$ has three real roots.

Now, $f''(x) = 0$ gives $x = \dfrac{15 \pm \sqrt{15}}{6}$



Thus, $f'(x) = 0$ has three real roots

$\left(-\infty\ \dfrac{15 - \sqrt{15}}{6}\right), \left(\dfrac{15 - \sqrt{15}}{6}, \dfrac{15 + \sqrt{15}}{6}\right)$

and $\left(\dfrac{15 + \sqrt{15}}{6}, \infty\right)$

22.

(i) Let $f(x) = x^3 - 6x^2 + 15x + 3$

$f'(x) = 3x^2 - 12x + 15$

$= 3(x^2 - 4x + 5)$

Now, $f'(x) = 0$ has no real roots

So, $f(x) = 0$ has only one real root.

(ii) Let $f(x) = 4x^3 - 21x^2 + 18x + 20$

$\Rightarrow f'(x) = 12x^2 - 42x + 18$

$\Rightarrow f'(x) = 6(2x^2 - 7x + 3)$

$\Rightarrow f'(x) = 6(2x - 1)(x - 3)$

Clearly, $f'(x) = 0$ has two real roots.

Thus, $f(x) = 0$ has three real roots.

(iii) Let $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 1$

$f'(x) = 12x^3 - 24x^2 - 12x + 24$

$= 12(x^3 - 2x^2 - x + 2)$

$= 12\{x^2(x - 2) - 1(x - 2)\}$

$= 12(x - 2)(x^2 - 2)$

Clearly, $f'(x) = 0$ has three real roots.

So, $f(x) = 0$ has 4 real roots.

(iv) Let $f(x) = x^4 - 4x - 2$

$f'(x) = 4x^3 - 4$

$= 4(x^3 - 1)$

$= 4(x - 1)(x^2 + x + 1)$

So, it has only one real root

Thus, $f(x) = 0$ has 2 real roots.

23.

(i) $f(x) = |x - 2|$ in [0, 3].

Clearly, $f(x)$ is not differentiable at $x = 2$

So Rolle's theorem is not applicable.

(ii) $f(x) = 3 + (x - 2)^{2/3}$ in $[1, 3]$

Clearly, $f(x)$ is not differentiable at $x = 2$ So Rolle's theorem is not applicable.

(iii) $f(x) = \sin\left(\frac{1}{x}\right)$ in $[-1, 1]$

Clearly, $f(x)$ is not continuous at $x = 0$ So Rolle's theorem is not applicable.

(iv) $f(x) = \begin{cases} -4x + 5, 0 \le x \le 1 \\ 2x - 3, 1 < x \le 2 \end{cases}$.

Clearly, $f(x)$ is not continuous at $x = 1$ So Rolle's theorem is not applicable.

(v) $f(x) = [x]$ in $[-1, 1]$, where [,] = G.I.F.

Clearly, $f(x)$ is not continuous at $x = -1, 0, 1$ So Rolle's theorem is not applicable.

(vi) $f(x) = \sin|x|$ in $[-2, 2]$

Clearly, $f(x)$ is not differentiable at $x = 0$

So Rolle's theorem is not applicable.

(vii) $f(x) = e^{-|x|}$ in $[-3, 3]$

Clearly, $f(x)$ is not differentiable at $x = 0$

So Rolle's theorem is not applicable.

(viii) $f(x) = \left|e^{-|x|} - \frac{1}{2}\right|$ in $[-2, 2]$

Clearly, $f(x)$ is not differentiable at $x = 0$

So Rolle's theorem is not applicable.

(ix) $f(x) = \left|\frac{1}{x} - 1\right|$ in $[-1, 1]$

Clearly, $f(x)$ is not continuous at $x = 0$

So Rolle's theorem is not applicable.

(x) $f(x) = \sin x + |\sin x|$ in $[-\pi, \pi]$

Clearly, $f(x)$ is not differentiable at $x = 0$

So Rolle's theorem is not applicable.

Lagranges Mean Value Theorem

24. Given $f(x) = x^3 - x^2 - x + 1$

As we know that, every polynomial function is continuous and differentiable everywhere.

So, it is continuous in [0, 2] and differentiable in (0, 2)

Thus, all the conditions of L.M.V. theorem are satisfied

Now, we have to show that, there exists a point c in (0, 2) such that $f'(c) = \frac{f(2) - f(0)}{2 - 0)}$

$\Rightarrow \quad f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2} = 1$

$\Rightarrow \quad 3c^2 - 2c - 1 = 1$

$\Rightarrow \quad 3c^2 - 2c - 2 = 0$

$\Rightarrow \quad c = \frac{2 \pm \sqrt{4 + 24}}{6} = \frac{2 \pm 2\sqrt{7}}{6}$

$\Rightarrow \quad c = \frac{1 \pm \sqrt{7}}{3}$

$\Rightarrow \quad c = \frac{1 + \sqrt{7}}{3} \in (0, 2)$

Hence, L.M.V. theorem is verified.

25. Given $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$

As we know that every polynomial function is continuous and differentiable everywhere.

So, it is continuous in [0, 4] and differentiable in (0, 4)

Thus, all the conditions of L.M.V. theorem are satisfied.

Now we have to show that there exists a point $c \in (0, 4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{24 - 0}{4} = 6$$

$\Rightarrow \quad 4c^3 - 30c^2 + 70c - 50 = -6$

$\Rightarrow \quad 4c^3 - 30c^2 + 70c - 44 = 0$

$\Rightarrow \quad 2(c - 1)(2c^2 - 13c + 11) = 0$

$\Rightarrow \quad 2(c - 1)^2(2c - 11) = 0$

$\Rightarrow \quad c = 1, \frac{11}{2}$

$\Rightarrow \quad c = 1 \in (0, 4)$

Hence, Lagranges Mean Value Theorem is verified.

26. Given curve is $y = 2x^2 - 5x + 3$

The arc AB is continuous in [1, 2] and differentiable in (1, 2)

So, by L.M.V. Theorem,

$$f'(x) = \frac{f(2)f(1)}{2 - 1} = f(2) - f(1)$$

$\Rightarrow \quad 4x - 5 = 1 - 0$

$\Rightarrow \quad 4x = 6$

$\Rightarrow \quad x = \frac{3}{2}$

when $x = \frac{3}{2}$, then $y = 0$

Thus, the point is $\left(\frac{3}{2}, 0\right)$

27. Since the tangent is parallel to the chord joining (4, 0) and (5, 1), so we get,

$$2(x - 4) = \frac{1 - 0}{5 - 4} = 1$$

$$(x-4) = \frac{1}{2}$$

$$x = 4 + \frac{1}{2} = \frac{9}{2}$$

when $x = \frac{9}{2}$, then $y = \left(\frac{9}{2} - 4\right)^2 = \frac{1}{4}$

Hence, the point is $\left(\frac{9}{2}, \frac{1}{4}\right)$

28. Let $g(x) = f(x) - x^2$

$\Rightarrow$ $g(x)$ has atleast 3 real roots which are $x = 1, 2$ and 3

$\Rightarrow$ $g'(x)$ has at least 2 real roots in $x \in (1, 3)$

$\Rightarrow$ $g''(x)$ has at least 1 real root in $x \in (1, 3)$

$\Rightarrow$ $f''(x) = 2$ for at least one root in $x \in (1, 3)$

Hence, the result.

29. Applying L.M.V. theorem, we have,

$$f'(x) = \frac{f(x) - f(0)}{2 - 0} = \frac{f(x)}{2}$$

$$\Rightarrow \quad \frac{f(x)}{2} = f'(x) \le \frac{1}{2}$$

$$\Rightarrow \quad f(x) \le 1$$

Hence, the result.

30. For the function $f(x)$, applying L.M.V. Theorem,

we get, $f'(c) = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0)$ ...(i)

Also, for the function $g(x)$, we have,

$$g'(c) = \frac{g(1) - g(0)}{1 - 0} = g(1) - g(0) \quad \text{...(ii)}$$

Dividing (i) and (ii), we get,

$$\frac{f'(c)}{g'(c)} = \frac{f(1) - f(0)}{g(1) - g(0)}$$

$$\Rightarrow \quad 2 = \frac{6-2}{g(1) - 0} = \frac{4}{g(1)}$$

$$\Rightarrow \quad g(1) = 2$$

31. For the function $f(x) = \cos x$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad -\sin c = \frac{\cos b - \cos a}{b - a} \quad \text{...(i)}$$

For the function $f(x) = \sin x$,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \quad f'(c) = \frac{\sin b - \sin a}{b - a} \quad \text{...(ii)}$$

Dividing (i) and (ii), we get,

$$\frac{\cos b - \cos a}{\sin b - \sin a} = -\frac{\sin c}{\cos c} = -\tan c$$

32.

(i) $f(x) = \frac{1}{x}$ on $[-1, 1]$

Clearly, $f(x)$ is not continuous at $x = 0$

So, L.M.V. theorems is not applicable.

(ii) $f(x) = \sin |x|$ on $[-2\pi, 2\pi]$

Clearly, $f(x)$ is not differentiable at $x = 0$

So, L.M.V theorems is not applicable.

(iii) $f(x) = |\sin x|$ on $[0, 2\pi]$

Clearly, $f(x)$ is not continuous at $x = \pi$

So, L.M.V. theorems is not applicable.

(iv) $f(x) = \log|x|$ on $[0, 5]$

Clearly, $f(x)$ is not continuous at $x = 0$

So, L.M.V. theorems is not applicable.

(v) $f(x) = ||\log|x||$ on $[-1, 1]$

Clearly, $f(x)$ is not continuous at $x = -1, 0, 1$

So, L.M.V. theorems is not applicable.

33. Let $f(x) = \log x$, $0 < a < b$

Clearly, $f(x)$ is continuous and differentiable in (a, b), where $0 < a < b$

$$f'(x) = \frac{1}{x}$$

$$f'(c) = \frac{1}{c}, \; a < c < b$$

By L.M.V, theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\log\left(\frac{b}{a}\right)}{b - a}$$

Now, $a < c < b$

$$\Rightarrow \quad \frac{1}{b} < \frac{1}{c} < \frac{1}{a}$$

$$\Rightarrow \quad \frac{1}{b} < \frac{\log\left(\frac{b}{a}\right)}{b - a} < \frac{1}{a}$$

$$\Rightarrow \quad \frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$$

Hence, the result.

34. Let $f(x) = \tan x$ in (a, b), where $0 < a < b < \frac{\pi}{2}$

Clearly, $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b)

Now, $f'(x) = \sec^2 x$

$\Rightarrow \quad f'(c) = \sec^2 c$

By L.M.V Theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{\tan b - \tan a}{b - a}$$

Here, $c \in (a, b)$

$\Rightarrow \quad a < c < b$

$\Rightarrow \quad \sec^2 a < \sec^2 c < \sec^2 b$

$$\Rightarrow \quad \sec^2 a < \frac{f(b) - f(a)}{b - a} < \sec^2 b$$

$$\Rightarrow \quad \sec^2 a < \frac{\tan b - \tan a}{b - a} < \sec^2 b$$

$\Rightarrow \quad (b - a)\sec^2 a < (\tan b - \tan a) < (b - a)\sec^2 b$

Hence, the result.

35. Let $f(x) = \tan^{-1} x$ in (a, b), $0 < \alpha < \beta$

Clearly $f(x)$ is continuous and differentiable in (α, β)

$$f'(x) = \frac{1}{1 + x^2}$$

$$\Rightarrow \quad f'(c) = \frac{1}{1 + c^2}$$

Now, $\alpha < c < \beta$

$\Rightarrow \quad \alpha^2 < c^2 < \beta^2$

$\Rightarrow \quad 1 + a^2 < 1 + c^2 < 1 + \beta^2$

$$\Rightarrow \quad \frac{1}{1 + \beta^2} < \frac{1}{1 + c^2} < \frac{1}{1 + \alpha^2}$$

$$\Rightarrow \quad \frac{1}{1 + \beta^2} < f'(c) < \frac{1}{1 + \alpha^2}$$

$$\Rightarrow \quad \frac{1}{1 + \beta^2} < \frac{f(\beta) - f(\alpha)}{\beta - \alpha} < \frac{1}{1 + \alpha^2}$$

$$\Rightarrow \quad \frac{\beta - \alpha}{1 + \beta^2} < f(\beta) - f(\alpha) < \frac{\beta - \alpha}{1 + \alpha^2}$$

$$\Rightarrow \quad \frac{\beta - \alpha}{1 + \beta^2} < \tan^{-1}\beta - \tan^{-1}\alpha < \frac{\beta - \alpha}{1 + \alpha^2}$$

Hence, the result.

36. Let $f(x) = \sin x$

Clearly, $f(x)$ is continuous and differentiable So L.M.V. theorem is applicable.

$$f'(x) = \cos x$$

$\Rightarrow \quad f'(c) = \cos c$

$$\Rightarrow \quad \frac{f(y) - f(x)}{y - x} = \cos c$$

$$\Rightarrow \quad \frac{f(x) - f(y)}{x - y} = \cos c$$

$$\Rightarrow \quad \left|\frac{f(x) - f(y)}{x - y}\right| = |\cos c|$$

$$\Rightarrow \quad \left|\frac{\sin x - \sin y}{x - y}\right| \le 1$$

$\Rightarrow \quad |\sin x - \sin y| \le |x - y|$

Hence, the result.

37. Let $f(x) = \tan^{-1} x - x$ in $[0, x]$

Clearly, $f(x)$ is continuous and differentiable. So L.M.V. theorem is applicable.

$$f'(x) = \frac{1}{1 + x^2} - 1 = -\frac{x^2}{1 + x^2}$$

$$\Rightarrow \quad f'(c) = -\frac{c^2}{1 + c^2}$$

$$\Rightarrow \quad \frac{f(x) - f(0)}{x - 0} = -\frac{c^2}{1 + c^2}$$

$$\Rightarrow \quad \frac{f(x)}{x} = -\frac{c^2}{1 + c^2}$$

$$\Rightarrow \quad \frac{f(x)}{x} = -\frac{c^2}{1 + c^2} < 0$$

$$\Rightarrow \quad \frac{\tan^{-1} x - x}{x} \le 0$$

$\Rightarrow \quad \tan^{-1} x - x \le 0$

$\Rightarrow \quad \tan^{-1} x \le x$

$\Rightarrow \quad |\tan^{-1} x| \le |x|$

Hence, the result.

38. Let $f(x) = \tan x$

Clearly, $f(x)$ is continuous and differentiable for all x, y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So, L.M.V. theorems is applicable.

$\Rightarrow \quad f'(x) = \sec^2 x$

$\Rightarrow \quad f'(c) = \sec^2 c$

$$\Rightarrow \quad \frac{f(x) - f(y)}{x - y} = \sec^2 c$$

$$\Rightarrow \quad \frac{\tan x - \tan y}{x - y} = \sec^2 c$$

$\Rightarrow \quad \left|\dfrac{\tan x - \tan y}{x - y}\right| = |\sec^2 c|$

$\Rightarrow \quad \left|\dfrac{\tan x - \tan y}{x - y}\right| \geq 1$

$\Rightarrow \quad |\tan x - \tan y| \geq |x - y|$

Hence, the result.

39. Given that f is differentiable in (1, 6) So by L.M.V Theorems

$$\frac{f(6) - f(1)}{6 - 1} = f'(x)$$

$\Rightarrow \quad \dfrac{f(6) - f(1)}{5} \geq 2$

$\Rightarrow \quad f(6) - f(1) \geq 10$

$\Rightarrow \quad f(6) \geq f(1) + 10$

$\Rightarrow \quad f(6) \geq -2 + 10 = 8$

Thus, $\quad m = 8$

Now, the value of $2(m + 2)^3 + 12$

$= 2(8 + 2)^3 + 17$

$= 2017.$

40. When f is continuous in [0, 1] and differentiable in (0, 1)

So by L.M.V theorem, we get,

$$f'(x) = \frac{f(1) - f(0)}{1 - 0}$$

$\Rightarrow \quad f(1) - f(0) = f'(x)$

$\Rightarrow \quad f(1) - 2 \leq 3$

$f(1) \leq 5$ ...(i)

When f is continuous in [1, 2] and differentiable in (1, 2).

So, by L.M.V. theorem, we get,

$$f'(x) = \frac{f(2) - f(1)}{2 - 1}$$

$\Rightarrow \quad f(2) - f(1) = f'(x)$

$\Rightarrow \quad f(2) - f(1) \leq 3$

$\Rightarrow \quad 8 - f(1) \leq 3$

$\Rightarrow \quad f(1) \geq 5$ ...(ii)

From (i) and (ii), we get,

$f(1) = 5$

Level III

1. As we know that every differentiable function is continuous, so it is continuous in [0, 4] and differentiable in (0, 4)

Thus, by L.M.V. theorem, we have

$$f'(a) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - f(0)}{4} \quad \text{...(i)}$$

Also, by intermediate value theorem, there exists a point b in (0, 4) such that

$$f(b) = \frac{f(4) + f(0)}{2} \quad \text{...(ii)}$$

From (i) and (ii), we get,

$\Rightarrow \quad f'(a)f(b) = \dfrac{(f(4) - f(0))(f(4) + f(0))}{8}$

$\Rightarrow \quad 8f'(a)f(b) = (f^2(4) - f^2(0))$

Hence, the result.

2. We have to show that

$$(f(7) - f(2))\frac{\{(f(7))^2 + (f(2))^2 + f(2).f(7)\}}{3}$$

$= 5f^2(c)\cdot f(c)$

i.e. $\quad \dfrac{f^3(7) - f^3(2)}{7 - 2} = 3f^2(c)\cdot f'(c)$

Let $\quad g(x) = (f(x))^3$

Clearly, $g(x)$ is continuous in [2, 7] and differentiable in (2, 7)

So, by L.M.V. theorem, there exists a point c in (2, 7) such that

$$g'(c) = \frac{g(7) - g(2)}{7 - 2}$$

$\Rightarrow \quad 3f^2(c)f'(c) = \dfrac{f^3(7) - f^3(2)}{7 - 2}$

$\Rightarrow \quad \dfrac{f^3(7) - f^3(2)}{5} = 3f^2(c)f'(c)$

$\Rightarrow \quad \dfrac{f^3(7) - f^3(2)}{3} = 5f^2(c)f'(c)$

Hence, the result.

3. As we know that, every differentiable function is also continuous. So f and g are continuous in [0, 1]

Thus, by L.M.V. theorem, there exist a point c in (0, 1) such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(1) - f(0)}{g(1) - g(0)}$

$$\Rightarrow \quad \frac{f'(c)}{g'(c)} = \frac{6-2}{2-0} = \frac{4}{2} = 2$$

$$\Rightarrow \quad f'(c) = 2g'(c)$$

4. As we know that every differentiable function is continuous. So it is continuous in [1, 6]

Thus, by L.M.V. theorem, there exists a point $c \in (1, 6)$ such that

$$f'(c) = \frac{f(6) - f(1)}{6 - 1} = \frac{f(6) + 2}{5}$$

$$\Rightarrow \quad \frac{f(6) + 2}{5} = f'(c) \geq 2$$

$$\Rightarrow \quad f(6) \geq 10 - 2 = 8.$$

5. Let $\quad h(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix}$

Since $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), so $h(x)$ is continuous in $[a, b]$ and differentiable in (a, b).

Thus, by L.M.V. Theorem, there some c in (a, b)

such that $h'(c) = \dfrac{h(b) - h(a)}{b - a}$

$$\Rightarrow \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix} = \frac{h(b) - h(a)}{b - a}$$

$$\Rightarrow h(b) - h(a) = (b - a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} - \begin{vmatrix} f(a) & f(a) \\ g(a) & g(a) \end{vmatrix} = (b - a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a)\begin{vmatrix} f(a) & f'(c) \\ g(a) & g'(c) \end{vmatrix}$$

Hence, the result.

6. We have by L.M.V. theorem,

$$f'(c_1) = \frac{f(b) - f(a)}{b - a}, \ \forall c_1 \in (a, b)$$

$$\Rightarrow \quad f'(c_1) = \frac{b - a}{b - a} = 1$$

Similarly, $f'(c_2) = 1$

Thus, $\quad f'(c_1) + f'(c_2) = 1 + 1 = 2.$

7. By Intermediate Value Theorem,

$$f(c) = \frac{f(0) + f(2)}{2}, \ 0 < c < 2$$

For some $c_1 \in (0, 1)$ & $c_2 \in (1, 2)$, by L.M.V. Theorem, $f'(c_1) = f(1) - f(0)$

and $\quad f'(c_2) = f(2) - f(1).$

Thus, $\quad f'(c_2) - f'(c_1) = f(0) + f(2) - 2f(1)$

$$\Rightarrow \quad (c_2 - c_1)f'''(c) = f(0) + f(2) - 2f(1), c_1 < c < c_2$$

$$\Rightarrow \quad f(0) + f(2) - 2f(1) = (c_2 - c_1)f'''(c)$$

$$\Rightarrow \quad f(0) + f(2) - 2f(1) < 0$$

$$\Rightarrow \quad f(0) + f(2) < 2f(1)$$

Hence, the result.

8. For some $c_1 \in (0, 1)$ and $c_2 \in (1, 2)$, by L.M.V.

theorem, $f'(c_1) = \dfrac{f(1) - f(0)}{1 - 0}$

and $\quad f'(c_2) = \dfrac{f(2) - f(1)}{2 - 1}$

Now, $\quad f'(c_1) = \dfrac{f(1) - f(0)}{1 - 0} = f(1) - 2$

$$\Rightarrow \quad f(1) - 2 = f'(c_1) \leq 3$$

$$\Rightarrow \quad f(1) \leq 5$$

Also, $\quad f(c_2) = \dfrac{f(2) - f(1)}{2 - 1} = 8 - f(1)$

$$\Rightarrow \quad 8 - f(1) = f'(c_2) \leq 3$$

$$\Rightarrow \quad f(1) \geq 5$$

From (i) and (ii), we get, $f(1) = 5$

9. Let $f(x) = \sin x$

Clearly $f(x)$ is continuous in $\left[0, \dfrac{\pi}{2}\right)$ and differentiable in $\left(0, \dfrac{\pi}{2}\right)$

Thus, by L.M.V. theorem, there exists a point c such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$$\Rightarrow \quad \frac{\cos b - \cos a}{b - a} = \cos c$$

$$\Rightarrow \quad \left|\frac{\cos b - \cos a}{b - a}\right| = |\cos c|$$

$$\Rightarrow \quad \left|\frac{\cos a - \cos b}{a - b}\right| = |\cos c|$$

$$\Rightarrow \quad \left|\frac{\cos a - \cos b}{a - b}\right| = |\cos c| \leq 1$$

$$\Rightarrow \quad |\cos a - \cos b| \leq |a - b|$$

Hence, the result.

10. Let $f(x) = x^3 - 3x + k$

Now, $f(x) = 3x^2 - 3 = 3(x^2 - 1)$

$= 3(x - 1)(x + 1)$

Thus $f'(x) = 0$ has two distinct roots.

So, $f(-1)f(1) < 0$

$\Rightarrow (1 - 3 + k)(-1 + 3 + k) < 0$

$\Rightarrow (k - 2)(k + 2) < 0$

$\Rightarrow -2 < k < 2$

11. Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$\Rightarrow f'(x) = 4x^3 - 12x^2 + 24x + 1$

$\Rightarrow f'(x) = 12x^2 - 24x + 24$

$\Rightarrow f''(x) = 12(x^2 - 2x + 2)$

$\Rightarrow f''(x) = 12(x - 1)^2 + 1) > 0, \forall x \in R$

So $f'(x) = 0$ has only one real root and $f(x) = 0$ has two real roots.

Further more, $f(-1) = 15 > 0, f(0) = -1 < 0$ and $f(1) = 9 > 0$.

Thus, $f(x) = 0$ has one real root in $(-1, 0)$ and another real root in $(0, 1)$.

12. Let $f(x) = x^3 - 6x^2 + 15x + 3$

Now, $f'(x) = 3x^2 - 12x + 5$

$= 3(x^2 - 4x + 3)$

$= 3(x - 1)(x - 3)$

Thus, $f'(x)$ has 2 real roots.
So, $f(x)$ has 3 real roots.

13. Let $f(x) = 4x^3 - 21x^2 + 18x + 20$

Then $f'(x) = 12x^2 - 42x + 18$

$= 6(2x^2 - 7x + 3)$

$= 6(2x - 1)(x - 3)$

Thus, $f'(x) = 0$ has 2 real roots
So, $f(x) = 0$ has 3 real roots.

14. Let $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 1$

Now, $f'(x) = 12x^3 - 24x^2 - 12x + 24$

$= 12(x^3 - 2x^2 - x + 2)$

$= 12(x^2(x - 2) - (x - 2))$

$= 12(x - 2)(x^2 - 1)$

$= 12(x - 2)(x + 1)(x - 1)$

Thus, $f'(x) = 0$ has 3 real roots.
So, $f(x) = 0$ has 4 real roots.

15. Let $f(x) = x^4 - 4x - 2$

Then $f'(x) = 4x^3 - 4 = 4(x^3 - 1)$

$= 4(x - 1)(x - \omega)(x - \omega^2)$

Thus, $f'(x) = 0$ has one real root and 2 imaginary roots.
So, $f(x) = 0$ has 2 real and 2 imaginary roots.

16. Let $f(x) = x^4 - 14x^2 + 24x - k$

Then $f'(x) = 4x^3 - 28x + 24$

$= 4(x^3 - 7x + 6)$

$= 4(x - 1)(x - 2)(x + 3)$

So, $f(x) = 0$ has four unequal roots.

Now, $f(-3) = -117 - k$,

$f(1) = 11 - k$ and $f(2) = 8 - k$

By the sign scheme,

$f(1) < 0, f(2) > 0$ & $f(-3) < 0$

$\Rightarrow (8 - k) < 0, 11 - k > 0$ & $-117 - k < 0$

$\Rightarrow k > 8, k < 11$ and $k > -117$

$\Rightarrow 8 < k < 11$

Therfore, $8 < k < 11$

17. Let $f(x) = x^3 - 3x + k$

$\Rightarrow f'(x) = 3x^2 - 3$

Now, $3(x^2 - 1) = 0$ gives $x = \pm 1$

Since it has two distinct roots in $(0, 1)$

so, $f(0)f(1) < 0$

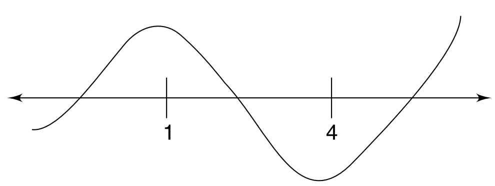
$\Rightarrow k(k - 2) < 0$

$\Rightarrow 0 < k < 2$

Hence, the value of k is $(0, 2)$.

18. Let $f(x) = x^3 - 2kx^2 - 4kx + k^2$

where $f(0) = k^2 \geq 0$



Clearly, $f(1) > 0$

$\Rightarrow k^2 - 6k + 1 > 0$

$\Rightarrow (k - 3)^2 - (2\sqrt{2})^2 > 0$

$\Rightarrow (k - 3)^2 > (2\sqrt{2})^2$

$\Rightarrow |(k - 3)| > (2\sqrt{2})$

$\Rightarrow (k - 3) > (2\sqrt{2}), (k - 3) < -2\sqrt{2}$

$\Rightarrow \quad k > (3 + 2\sqrt{2}),\ k < 3 - 2\sqrt{2} \quad$ ...(i)

Also, $\quad f(4) < 0$

$64 - 32k - 16k + k^2 > 0$

$\Rightarrow \quad k^2 - 48k + 64 > 0$

$\Rightarrow \quad (k - 24)^2 - 512 > 0$

$\Rightarrow \quad (k - 24)^2 - (16\sqrt{2})^2 > 0$

$\Rightarrow \quad (k - 24)^2 > (16\sqrt{2})^2$

$\Rightarrow \quad |(k - 24)| > (16\sqrt{2})$

$\Rightarrow \quad (k - 24) > (16\sqrt{2}),\ (k - 24) < -16\sqrt{2}$

$\Rightarrow \quad k > (24 + 16\sqrt{2}),\ k < 24 - 16\sqrt{2} \quad$ ...(ii)

From (i) and (ii), we get,

$3 + 2\sqrt{2} < k < 24 + 16\sqrt{2}$

$\Rightarrow \quad 3 + \sqrt{8} < k < 8(3 + \sqrt{8})$

Thus $\quad m = 3,\ n = 8$

Hence, the value of $2(m + n - 1)^3 + 17$

$= 2(3 + 8 - 1)^3 + 17$

$= 2017.$

19. Let $\quad f(x) = x^3 - 3x + a$

$\Rightarrow \quad f'(x) = 3x^2 - 3$

Now, $f'(x) = 0$ gives $x = \pm 1$

Since it has two distinct roots in (–1, 1), so

$f(-1)f(1) < 0$

$\Rightarrow \quad (-1 + 3 + a)(1 - 3 + a) < 0$

$\Rightarrow \quad (a + 2)(a - 2) < 0$

$\Rightarrow \quad -2 < a < 2$

Hence, the value of a is (–2, 2).

20. Given $\quad f(x) = x^3 - 2x^2 - 4x + 7$

$\Rightarrow \quad f'(x) = 3x^2 - 4x - 4$

Since $f(x)$ is applicable of Rolle's theorem, so

$f'(k) = 0$

$\Rightarrow \quad 3k^2 - 4k - 4 = 0$

$\Rightarrow \quad 3k^2 - 6k + 2k - 4 = 0$

$\Rightarrow \quad 3k(k - 2) + 2(k - 2) = 0$

$\Rightarrow \quad (3k + 2)(k - 2) = 0$

$\Rightarrow \quad k = 2, -\frac{2}{3}$

Hence, the value of k is 2.

Level IV

1. Given $\quad f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$= 3(x + 1)(x - 1)$

Clearly, $f(x)$ incresing in $(-\infty, -1) \cup (1, \infty)$ and decreasing in $(-1, 1)$

Now, $\quad f(-2) = -8 + 6 + 1 = -1$

$f(-1) = -1 + 3 + 1 = 3$

$f(0) = 1$

$f(1) = 1 - 3 + 1 = -1$

$f(2) = 8 - 6 + 1 = 3$

Now, $\quad f(-2)f(-1) < 0$

Thus, there is a root lies in (–2, –1)

So, $\quad [\alpha] = -2$

Also, $\quad f(0)f(1) < 0$

Thus, there is a root lies in (0, 1)

So, $\quad [\beta] = 0$

Again, $\quad f(1)f(2) < 0$

Thus, there is a root lies in (1, 2)

So, $\quad [\gamma] = 1$

Since α, β, γ are the roots of $x^3 - 3x + 1 = 0$

so, $\quad \alpha + \beta + \gamma = 0$

Now, $\quad \{\alpha\} + \{\beta\} + \{\gamma\}$

$= \alpha - [\alpha] + \beta - [\beta] + \gamma - [\gamma]$

$= (\alpha + \beta + \gamma) - ([\alpha] + [\beta] + [\gamma])$

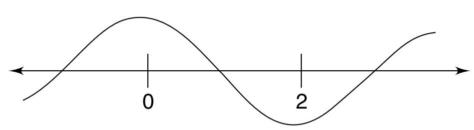
$= 0 - (-2 + 0 + 1)$

$= 1.$

2. Let $\quad f(x) = \frac{e^x}{x^2}$

$$f'(x) = \frac{x^2 \cdot e^x - 2x \cdot e^x}{x^4}$$

$$f'(x) = \frac{xe^x - 2e^x}{x^3} = \frac{(x - 2)e^x}{x^3}$$



Clearly, it has three real and distinct roots.

At the neighbourhood of $x = 2$, $f(x)$ will provide us the minimum value

Hence, the minimum value of c is 2.

3. We have $f(x) = x^4 + 8x^3 + 18x^2 + 8x + b$

$$= \{(x^2 + 4x)^2 + 2(x^2 + 4x) + 1\} + b - 1$$

$$= (x^2 + 4x + 1)^2 + (b - 1)$$

Now, $f(x) = 0$ gives

$\Rightarrow \quad (x^2 + 4x + 1)^2 + (b - 1) = 0$

$\Rightarrow \quad (x^2 + 4x + 1)^2 = 1 - b$

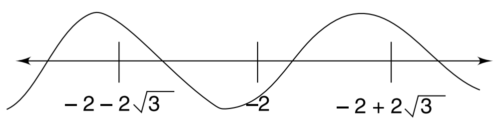
Let $\quad g(x) = (x^2 + 4x + 1)^2$

$$g'(x) = 2(x^2 + 4x + 1)(2x + 4)$$

$$= 4(x^2 + 4x + 1)(x + 2)$$

$$g'(x) = 0 \text{ gives } x = -2 \pm 2\sqrt{3}, 2$$

Clearly, the range of $g(x)$ is (0, 9)



For four distict solutions, $(1 - b) \in (0, 9)$

$\Rightarrow \quad 0 < 1 - b < 9$

$\Rightarrow \quad -1 < -b < 8$

$\Rightarrow \quad -8 < b < 1$

$\Rightarrow \quad b \in (-8, 1)$

4.

(i) By Lagranges Mean Value theorem

$$\frac{f(4) - f(0)}{4 - 0} = f'(a),\ a \in (0, 4) \ldots\text{(i)}$$

Also, from intermediate value theorem

$$\frac{f(4) + f(0)}{2} = f(b),\ b \in (0, 4) \ \ldots\text{(ii)}$$

From (i) and (ii), we get,

$$\Rightarrow \left(\frac{f(4) - f(0)}{4}\right)\left(\frac{f(4) + f(0)}{2}\right) = f'(a)f(b)$$

for $a, b \in (0, 4)$

$\Rightarrow (f(4))^2 - (f(0))^2 = 8f'(a)f(b)$, for $a, b \in (0, 4)$

Hence, the result.

(ii) Replacing t by z^2, we get,

$$\int_0^4 f(t)\,dt = \int_0^2 2zf(z^2)\,dz$$

From Lagranges Mean Value theorem

$$\frac{\int_0^2 2zf(z^2)\,dz - \int_0^0 2zf(z^2)\,dz}{2 - 0} = 2\gamma f(\gamma^2)$$

where $\gamma \in (0, 2)$

For $0 < \alpha < \gamma < \beta < 2$, using intermediate mean value theorem

$$\int_0^2 2zf(z^2)\,dz - 2(2\gamma f(\gamma^2))$$

$$= 2\left(\frac{2\,gaf(\alpha^2) + 2\,\beta\beta(\beta^2)}{2}\right)$$

$$\Rightarrow \int_0^4 f(t)\,dt = 2[\alpha f(\alpha^2) + \beta f(\beta^2)], \text{ for } 0 < \alpha, \beta < 2$$

Hence, the result.

5. Let $\quad f(x) = x^5 - a_0x^4 + 3ax^3 + bx^2 + cx + d$

$$f'(x) = 5x^4 - 4a_0x^3 + 9ax^2 + 2bx + c$$

$$f''(x) = 20x^3 - 12a_0x^2 + 18ax + 2b$$

$$f'''(x) = 60x^2 - 24a_0x + 18a$$

$$= 6(10x^2 - 4a_0x + 3a)$$

Now, its discriminant is $16a_o^2 - 120a$

$$= 8(2a_o^2 - 15a) < 0$$

Thus, $f'''(x)$ has no real roots

Therefore, all the roots of $f(x) = 0$ can not be real.

6. Since $f''(x)$ exists for all $x \in [0, 1]$

So, $f(x)$, $f'(x)$ are differentiable and continuous in [0, 1]

Now, $f(x)$ is continuous in [0, 1] and differentiable in (0, 1) and $f(0) = f(1)$

Thus, by Rolle's theorem there exists atleast one c in (0, 1) such that $f'(c) = 0$.

Now, let $x \in [0, 1]$

Case-I: When $x < c$

since f is differentiable in [0, 1]

So, $f'(x)$ will be continuous and differentiable in (0, 1)

By L.M.V. theorem

$$\frac{f'(x) - f'(c)}{x - c} = f''(\alpha),\ \forall\ \alpha \in (c, x)$$

$$\frac{f'(x) - 0}{x - c} = f''(\alpha)$$

$$f'(x) = (x - c)f''(\alpha)$$

$$|f(x)| = |(x - c)|f''(\alpha) \qquad \ldots\text{(i)}$$

As $\quad |f''(x)| \le 1\ \forall\ x \in [0, 1]$

$$|f''(\alpha)| \le 1,\ \forall\ x \in [0, 1]$$

$$|f'(x)| < 1,\ \forall\ x \in [0, 1]$$

Case-II: When $x > c$

By L.M.V. theorem,

$$\frac{f'(c) - f'(x)}{c - x} = f''(\alpha)$$

$$\frac{f'(x) - f'(c)}{x - c} = f''(\alpha)$$

$$f'(x) = (x - c)f''(\alpha)$$

$$|f'(x)| = |(x - c)|f''(\alpha)$$

$$|f'(x)| < 1 \text{ for all } x \text{ in } [0, 1]$$

Case-III: When $x = c$

$$f'(x) = f'(c) = 0$$

$$|f'(x)| = 0 < 1$$

$$|f'(x)| < 1, \text{ for all } x \in [0, 1]$$

From all the cases, we get,

$$|f'(x)| < 1, \text{ for all } x \in [0, 1]$$

Hence, the result.

7. Let $f(x) = \log(1 + x)$ in $[0, x]$

Since $f(x)$ satisfies the condition of L.M.V. theorem in $[0, x]$, there exists θ $(0 < \theta < 1)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\theta x)$$

$$\Rightarrow \quad \frac{\log(1 + x)}{x} = \frac{1}{1 + \theta x}$$

Now, $0 < \theta < 1, x > 0 \Rightarrow \theta x < x$

$$\Rightarrow \quad 1 + \theta x < 1 + x$$

$$\Rightarrow \quad \frac{1}{1 + \theta x} > \frac{1}{1 + x}$$

$$\Rightarrow \quad \frac{x}{1 + \theta x} > \frac{x}{1 + x} \quad \text{...(i)}$$

Again $0 < \theta < 1, x > 0$

$$\Rightarrow \quad \theta x > 0$$

$$\Rightarrow \quad 1 + \theta x > 1$$

$$\Rightarrow \quad \frac{1}{1 + \theta x} < 1$$

$$\Rightarrow \quad \frac{x}{1 + \theta x} < x \quad \text{.....(ii)}$$

From (i) and (ii), we get,

$$\frac{x}{1 + x} < \log(1 + x) < x$$

Hence, the result.

Questions asked in IIT-JEE Exams

1. *Ans.* (a)

Let $f(x) = ax^3 + bx^2 + cx$

$$f(0) = a \cdot 0 + b \cdot 0 + c \cdot 0 = 0$$

Also, $f(1) = a + b + c = 0$

Thus, from Rolle's theorem, there exists atleast a root of its derivative.

So, $f'(x)$ has a root in (0, 1).

2. *Ans.* (a)

(a) We have $f(x) = \begin{cases} \left(\frac{1}{2} - x\right) & : x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2 & : x \geq \frac{1}{2} \end{cases}$

$$f'(x) = \begin{cases} -1 & : x < \frac{1}{2} \\ -2\left(\frac{1}{2} - x\right) & : x \geq \frac{1}{2} \end{cases}$$

Clearly $f(x)$ is not differentiable at $x = \frac{1}{2}$

Thus, Lagranges Mean Value Theorem is not applicable.

(b) We have $f(x) = \begin{cases} \frac{\sin x}{x} & : x \neq 0 \\ 1 & : x = 0 \end{cases}$

Clearly $f(x)$ is continuous in [0, 1] and differentiable in (0, 1)

Thus, Lagranges Mean Value Theorem is applicable.

(c) We have $f(x) = x|x| = x^2$ in [0, 1]

As we know that every polynomial function is continuous and differentiable everywhere.

So it is continuous in [0, 1] and differentiable in (0, 1).

Thus, Lagranges Mean Value Theorem is applicable.

(d) Also, $f(x) = |x| = x$

Clearly it is continuous in [0, 1] and differentiable in (0, 1).

Thus, Lagranges Mean Value Theorem is applicable.

3. Ans. (d)

Given $f(x) = x^{\alpha} \log x$

Since it is applicable for Rolle's theorem in [0, 1], so it must be continuous in [0, 1]

Now, we should check it at the right end of $x = 0$

i.e. $\lim_{x \to 0^+} f(x) = f(0) = 0$

Now, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^{\alpha} \log x)$

$$= \lim_{x \to 0^+} \left(\frac{\log x}{x^{-\alpha}} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{\frac{1}{x}}{-\alpha x^{-\alpha - 1}} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{1}{-\alpha x^{-\alpha}} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{x^{\alpha}}{-\alpha} \right)$$

It will be zero only when $\alpha > 0$

4. Let $f(x) = \int P(x)\, dx$

$$f(x) = \int (51 x^{101} - 2323 x^{100} - 45x + 1035)\, dx$$

$$f(x) = \frac{51}{102} x^{102} - \frac{2323}{101} x^{101} - \frac{45}{2} x^2 + 1035x$$

$$f(x) = \frac{1}{2} x^{102} - 23x^{101} - \frac{45}{2} x^2 + 1035x$$

Clearly $f(x)$ is continuous and differentiable in

$$\left(45^{\frac{1}{100}}, 46\right)$$

Also, $f\left(45^{\frac{1}{100}}\right)$

$$= \frac{1}{2}\left(45^{\frac{102}{101}}\right) - 23\left(45^{\frac{101}{100}}\right) - \frac{45}{2} (45)^{\frac{2}{100}} + 1035 (45)^{\frac{1}{100}}$$

$$= \frac{1}{2} \cdot 45 \cdot 45^{\frac{2}{101}} - 23 \cdot 45 \cdot 45^{\frac{1}{100}} - \frac{1}{2}\, 45 \cdot 45^{\frac{2}{100}} + 23 \cdot 45^{\frac{1}{100}}$$

$$= 0$$

Also, $f(46)$

$$= \frac{1}{2}(46)^{102} - 23\,(46)^{101} - \frac{45}{2}(46)^2 + 1035\,(46)$$

$$= \frac{1}{2}(23.2)^{102} - 23\,(23.2)^{101} - \frac{45}{2} \cdot 46 \cdot 46 + 1035\,(46)$$

$$= 0$$

Thus, all the conditions of Rolle's theorem are satisfied.

So there exist a point $c \in \left(45^{\frac{1}{100}}, 46\right)$ such that $f'(c) = 0$.

Hence, the result.

5. We have $g(x) = (f'(x))^2 + g'(x) f''(x)$

$$g(x) = \frac{d}{dx}(f(x) \cdot f'(x))$$

By the Rolle's theorem, between any two roots of a polynomial, there is a root of its derivative.

Now, $f(x) \cdot f'(x) = 0$

Either $f(x) = 0$ or $f'(x) = 0$

Thus, $f(x)$ has four zeroes at $x = a$, between b and c, between c and d and at e

So, $f'(x)$ has atleast 3 zeroes.

Thus, $f(x) f'(x)$ has atleast 7 zeroes.

Therefore, $g(x)$ has atleast 6 zeroes.

6. We have $x^4 - 4x^3 + 12x^2 + x - 1 = 0$

Now, $f(-1) = 1 + 4 + 12 - 1 - 1 = 15 > 0$

$f(0) = -1 < 0$

and $f(1) = 1 - 4 + 12 + 1 - 1 = 9 > 0$

So $f(x) = 0$ has a root in (–1, 0) and a root in (0, 1).

Thus, $f(x) = 0$ has atleast two distinct real roots.

Also, $f'(x) = 4x^3 - 12x^2 + 24x + 1$

and $f''(x) = 12x^2 - 24x + 24$

$$= 12(x^2 - 2x + 1) + 1$$

$$= 12(x - 1)^2 + 1 > 0, \forall x \in R$$

$\Rightarrow$ $f'(x)$ increases on R.

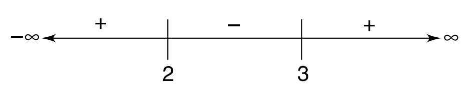
$\Rightarrow$ $y = f'(x)$ intersects the x-axis exactly once.

$\Rightarrow$ $y = f(x)$ has exactly three real roots.

7. *Ans.* (a, b, c, d)

We have $f(x) = \int_0^x e^{t^2} (t - 2)(t - 3)\, dt$

$$f'(x) = ex^2 (x - 2)(x - 3)$$



Clearly, has maximum at $x = 2$ and minimum at $x = 3$ and $f(x)$ decreasing in (2, 3)

So, by Rolle's Theorem $f'(x) = 0$ for

$x = 2$ and $x = 3$.

Thus, there exists a point $c \in (2, 3)$ such that $f''(c) = 0$.

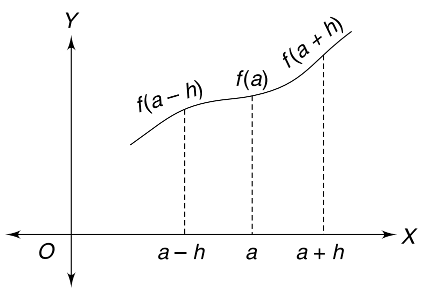
CHAPTER 7

Monotonocity

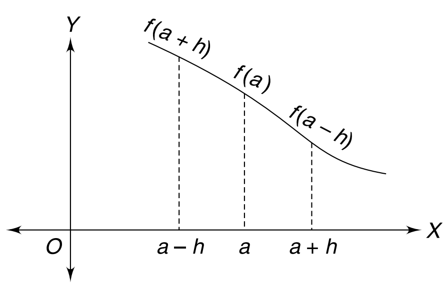
CONCEPT BOOSTER

1. Definitions

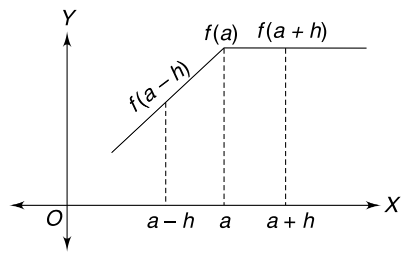
1. A function $f(x)$ is said to be strictly increasing about the point $x = a$ if $f(a - h) < f(a) < f(a + h)$, where h is a very small positive arbitrary number.



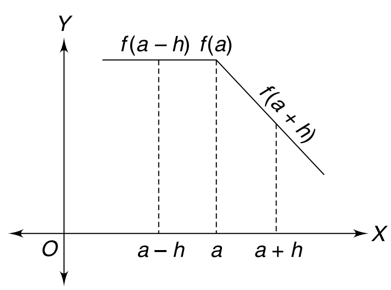
2. A function $f(x)$ is said to be strictly decreasing about the point $x = a$ if $f(a + h) < f(a) < f(a - h)$, where h is a very small positive arbitrary number.



3. A function $f(x)$ is said to be non-increasing about the point $x = a$ if $f(a - h) \le f(a) \le (a + h)$, where h is a very small positive arbitrary number.



4. A function $f(x)$ is said to be non-decreasing about the point $x = a$ if $f(a + h) \le f(a) \le (a - h)$, where h is a very small positive arbitrary number.



5. Let a function f be differentiable at $x = a$
   - (i) If $f'(a) > 0$, then $f(x)$ is strictly increasing at $x = a$.
   - (ii) If $f'(a) < 0$, then $f(x)$ is strictly decreasing at $x = a$.
   - (iii) If $f'(a) = 0$, then we need to examine the signs of $f'(a - h)$ and $f'(a + h)$
   - (a) If $f'(a - h) > 0$ and $f'(a + h) > 0$, then $f(x)$ is strictly increasing at $x = a$.
   - (b) If $f'(a - h) < 0$ and $f'(a + h) < 0$, then $f(x)$ is strictly decreasing at $x = a$.
   - (c) If $f'(a - h)$ and $f'(a + h)$ have opposite signs, then $f(x)$ is neither increasing nor decreasing (non-monotonic) at $x = a$.

2. Test for Finding the Monotonicity at an End Point

Let the function f be differentiable at $x = a$.

(i) If $x = a$ is the left end point, then we check as follows

(a) If $f'(a^+) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(b) If $f'(a^+) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

(c) If $f'(a^+) = 0$, but $f'(a + h) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(d) If $f'(a^+) = 0$, but $f'(a + h) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

(ii) If $x = a$ is the right end point, then we check as follows

(a) If $f'(a^-) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(b) If $f'(a^-) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

(c) If $f'(a^-) = 0$, but $f'(a - h) > 0$, then $f(x)$ is strictly increasing at $x = a$.

(d) If $f'(a^-) = 0$, but $f'(a - h) < 0$, then $f(x)$ is strictly decreasing at $x = a$.

3. Necessary Condition for Monotonicity

Let $f(x)$ be a differentiable function

(i) A function $f(x)$ is said to be increase in an interval if $f'(x) \geq 0$.

(ii) A function $f(x)$ is said to be decrease in an interval if $f'(x) \leq 0$.

(iii) If $f(x)$ does not vary in an interval, then $f'(x) = 0$.

4. Sufficient Condition for Monotonicity

(i) If $f'(x) > 0$ in (a, b), then $f(x)$ is increasing in (a, b).

(ii) If $f'(x) < 0$ in (a, b), then $f(x)$ is decreasing in (a, b).

(iii) If $f'(x) = 0$, then $f(x)$ does not vary in (a, b).

Monotonicity at Points where $f(x)$ does not Exist

Let $f(x)$ be a continuous function whose derivative $f'(x)$ does not exist at $x = c$ but exist in the nbd of $x = c$.

(i) If $f'(c^-) > 0$ and $f'(c^+) > 0$, then $f(x)$ is strictly increasing at $x = c$.

(ii) If $f'(c - h) > 0, f'(c^-) \geq 0$, and $f'(c^-) \geq 0, f'(c + h) > 0$, then $f(x)$ is strictly increasing at $x = c$.

(iii) If $f'(c^-) < 0, f'(c^+) < 0$, then $f(x)$ is strictly decreasing at $x = c$.

(iv) If $f'(c - h) < 0, f'(c^-) \leq 0$ and $f'(c + h) < 0, f'(c^+) \leq 0$, then $f(x)$ is stictly decreasing at $x = c$.

Note:

Discrete point

1. If $f'(x) = 0$ at discrete points, we mean that the points where $f'(x)$ becomes 0 do not form an interval. i.e. they are separated from each other.
2. If $f'(x) \geq 0$ for all $x \in (a, b)$ where $f'(x) = 0$ at discrete points in (a, b), then $f(x)$ is also increasing in (a, b)

5. Critical Point

A critical point of a function f' is a number 'a' in the domain of f such that either $f'(a) = 0$ or $f'(a)$ does not exist.

Sometimes we will want to distinguish critical numbers at which $f'(x) = 0$ from those at which f is not differentiable.

We will call a point on the graph of f at which $f'(x) = 0$, a **stationary point** of f.

The **stationary point** of f are the x-intercepts of the graph of f.

If $x = a$ is a critical point of f, then it is also a critical point of the function $g(x) = f(x) + k, \forall\ k = R$.

For example, $x = 0$ is a critical point of $f(x) = x^2$, then $x = 2$ is a critical point of $f(x) = (x - 2)^2$.

Remember the Step to Find the Intervals of Increasing and Decreasing

(i) First we find $f'(x)$, and then find where $f'(x) = 0$ or does not exist. These points are the critical points for the function $f(x)$.

(ii) Plot the critical points on the number line and put + and – in a alternative way.

(iii) On a certain interval, if $f'(x)$ is +ve, then $f(x)$ is strictly increasing and is $f'(x)$ is –ve then $f(x)$ is strictly decreasing.

(iv) On a certain interval, if $f'(x) \geq 0$, then $f(x)$ is increasing and if $f'(x) \leq 0$ then $f(x)$ is decreasing.

6. Application of Monotonocity in Isolations of Roots

Let $f(x)$ be a real function and a, b be two real numbers such that $a < b$. If

(i) f is continuous on $[a, b]$ and differentiable on (a, b).

(ii) $f(a)$ and $f(b)$ have opposite signs.

(iii) either $f'(x) > 0$ or $f'(x) < 0$ on (a, b)

Then $f(x)$ has an exactly one root between a and b.

7. Algebra of Monotonous Functions

Let increasing = I and decreasing = D

(where both the functions take +ve values) and can not say anything = Ω

(i) **Addition**

(a) $I + I = I$ (b) $D + D = D$

(c) $I + D = \Omega$ (d) $D + I = \Omega$

(ii) **Negativity**

(a) $-I = D$ (b) $-D = I$

(iii) **Difference**

(a) $I - I = \Omega$ (b) $D - D = \Omega$

(c) $I - D = I$ (d) $D - I = D.$

(iv) **Product**

(a) $I \times I = I$ (b) $I \times D = \Omega$

(c) $D \times D = D$ (d) $D \times I = \Omega.$

(v) **Reciprocality**

(a) $1/I = D$ (b) $1/D = I$

(vi) **Division**

(a) $I/I = D$ (b) $I/D = I$

(c) $D/D = D$ (d) $D/I = D$

(vii) **Composition**

(a) $I(I) = I$ (b) $I(D) = D$

(c) $D(I) = D$ (d) $D(D) = I.$

8. General Approach to Proving Inequalities

Some basic idea to proving inequalities

(i) If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), where $f'(x) > 0$ for all x in (a, b) and $f(a) \ge 0$, then $f(x) > 0$, $\forall\ x \in (a, b)$.

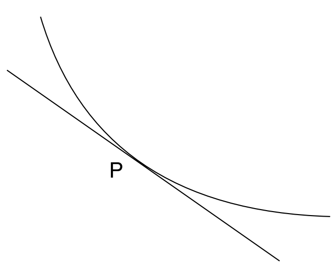
(ii) If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), where $f'(x) < 0$ for all x in (a, b) and $f(a) \le 0$, then $f(x) < 0$, $\forall\ x \in (a, b)$.

(iii) If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), where $f'(x) > 0$ for all x in (a, b) and $f(a) \le 0$, then $f(x) < 0$, $\forall\ x \in [a, b)$.

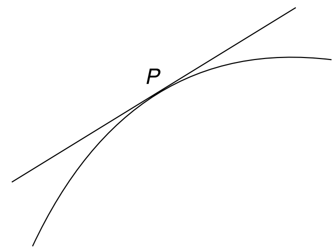
(iv) If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), where $f'(x) < 0$, for all x in (a, b) and $f(b) \ge 0$, then $f(x) > 0$, $\forall\ x \in [a, b)$.

9. Concave up and Concave Down

(i) **Concave Up:** If the second derivative $f''(x)$ is everywhere is positive within an interval, then the arc of the curve $y = f(x)$ corresponding to that interval is Concave Up.



(ii) **Concave Down:** If the second derivative $f''(x)$ is everywhere is negative within an interval, then the arc of the curve $y = f(x)$ corresponding to that interval is Concave Down.



(iii) **Hyper-Critical Point:** A hyper-critical point of a function f is a number c in the domain of f such that either $f''(c) = 0$ or $f''(c)$ does not exist.

Rule to find out the Intervals of Concavity

(i) First we find $f''(x)$ and then we should find the hyper-critical points i.e. points at which $f''(x)$ equals zero or does not exist.

(ii) Plot the hyper-critical points on the number line, and then put + and – in a alternative way.

(iii) On a certain interval, if $f''(x) > 0$, then the function $f(x)$ is Concave Up and if $f''(x) < 0$, then the function $f(x)$ is Concave Down.

10. Inflection Point

Let us consider the graph of a function has a tangent line (possibly vertical) at the point $P(c, f(c))$ and that the graph is concave up on one side of P and Concave down on the other side of P. Then P is called an inflection point of the graph.

Some Basic Idea on Inflection Points

(i) A continuous function f need not have an inflection point where $f''(x) = 0$ For example:

1. Let $f(x) = x^4$. Then $f''(0) = 0$, but the graph is always Concave Up.
2. Let us take another function $f(x) = x^5 + 5x^4$.

   Then $\quad f''(x) = 20x^2(x + 3).$

   At the nbd of $x = -3$, $f''(x)$ changes it sign from negative to positive. So $x = -3$ is the inflection point. But at the *nbd* of $x = 0$, $f''(x)$ does not change its sign. So the graph of the given function is concave up on both sides of the origin.
3. If $x = b$ is a point of inflection of a curve $y = f(x)$ and $f''(b)$ is exist, then $f''(b)$ must be equal to zero.
4. The point $(-1, 0)$ in $y = (x - 1)^3$, being both a critical point and a point of inflection, is a point of horizontal inflection.
5. If a function f is such that $f'''(x)$ is continuous and $f''(b) = 0$, while $f''(b) \neq 0$, then the curve $y = f(x)$ has a point of inflection for $x = b$.

6. It shold be noted that a point separating a Concave Up arc of a curve from a Concave Down, one may be such that the tangent at that point is perpendicular to the x-axis, that is, either a vertical tangent or a tangent does not exist.

Let us take $f(x) = \sqrt[3]{x}$ in the vicinity of the origin. In such a case, we speak of a point of inflection with vertical tangent.

7. A number b such that $f''(b)$ is not defined and the concavity of f changes at b will correspond to an inflection point if and only if $f(b)$ is defined.

EXERCISES

Level I

(Problems Based on Fundamentals)

1. Find the interval of the monotonicity of the function $f(x) = 2x^3 - 12x^2 + 18x + 5$.
2. Find the interval of the monotonicity of the function $f(x) = 5 + 36x + 3x^2 - 2x^3$.
3. Find the interval of the monotonicity of the function $f(x) = (x - 1)^3(x - 2)^2$.
4. Find the interval of the monotonicity of the function $f(x) = 2x^3 - 3x^2 + 6x + 10$
5. Find the interval of the monotonicity of the function $f(x) = 2x^3 + 3x^2 + 12x + 20$.
6. Find the interval of the monotonicity of the function $f(x) = \frac{x}{2} + \frac{2}{x}$.
7. Find the interval of the monotonicity of the function $f(x) = 5x^{3/2} - 3x^{5/2}$, $x > 0$.
8. Find the interval of the monotonicity of the function $f(x) = \log\left(x + \sqrt{1 + x^2}\right)$.
9. Find the interval of the monotonicity of the function $f(x) = \frac{x}{\log x}$.
10. Find the interval of the monotonicity of the function $f(x) = x - \cot^{-1}x - \log\left(x + \sqrt{x^2 + 1}\right)$.
11. Find the least value of m for which the function $f(x) = -x^2 + mx + 1$ is strictly increasing in [1, 2]
12. For what values of b, the function $f(x) = \sin x - bx + c$ is strictly decreasing for all x in R.
13. Find all possible values of 'a' for which the function $f(x) = e^{2x} - (a + 1)e^x + 2x$ is strictly increasing for all x in R.
14. For what values of a is the function $f(x) = \left(\frac{a^2 - 1}{3}\right)x^3 + (a - 1)x^2 + 2x + 1$ strictly increasing?
15. For what values of a, the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ is strictly decreasing for all x in R.

Critical Points

16. Find the critical points of $f(x) = \frac{e^x}{x - 1}$
17. Find the critical points of $f(x) = \frac{5x^2 - 18x + 45}{x^2 - 9}$
18. Find the critical points of the function $f(x) = x^{4/5}(x - 4)^2$.
19. Find the critical points of the function $f(x) = x + \cos^{-1}x + 1$.
20. Find the critical points of the function $f(x) = \sqrt{x^2 - 6x + 15}$

Increasing and Decreasing Functions

21. Find the interval of increasing and decreasing of a function $f(x) = 2x^2 - \ln|x|$.
22. Find the intervals for the function $f(x) = \frac{|x - 1|}{x^2}$ is increasing and decreasing.
23. Find the intervals for the function $f(x) = x^2 e^{\frac{-x^2}{a^2}}$, $a > 0$ is increases. Isolation points.
24. Show that the equation $x^3 = 3x + 1$ has a real root in [–1, 1].
25. Show that the equation $e^x = 1 + x + \frac{x^2}{2}$ has a real root in [–1, 1].

Algebra of Monotonic Functions

26. Find the interval where the function $f(x) = \tan^{-1}(e^x)$ is strictly increasing.
27. Find the interval in which $f(x) = \tan^{-1}(\log_{1/3}x)$ is strictly decreasing.
28. Find the interval in which $f(x) = \cot^{-1}(\log_4 x)$ is strictly decreasing.
29. Find the interval in which $f(x) = \cot^{-1}(\log_{1/10}x)$ is strictly increasing.
30. Find the interval of the monotonocity of the function $f(x) = \sqrt{3x - x^2}$.
31. Find the interval of the monotonicity of the function $f(x) = \tan^{-1}(\sin x + \cos x)$ in $(0, 2\pi)$.

32. Find the interval of the monotonocity of the function $f(x) = \log\left(\frac{\log x}{x}\right)$.
33. Find the interval in which the function $f(x) = \sin(\log x) + \cos(\log x)$ is decreases.
34. Find the interval of the monotonocity of the function $f(x) = \log_e(\cos x)$ for all $x \in (0, \pi)$.
35. Find the interval of the monotonocity of the function $f(x) = \sin(\sin x) + \cos(\sin x)$ in $(0, \pi)$.

Inequality

36. Prove the inequality, $\log(1 + x) > x - \frac{x^2}{2}$ for all x in R^+
37. Prove the inequality $\log(1 + x) > \frac{x}{1 + x}$ for $x > 0$
38. Prove that $(e^x - 1) > (1 + x)\log(1 + x)$, if $x > 0$
39. Prove that $2x\tan^{-1} > \log(1 + x^2)$ for all x in R^+.
40. Prove that $1 + x\log\left(x + \sqrt{x^2 + 1}\right) \geq \sqrt{1 + x^2}$ for all $x \geq 0$.
41. Prove that $\cos(\sin x) > \sin(\cos x)$, if $x \in \left(0, \frac{\pi}{2}\right)$.
42. Find the smallest positive constant B such that $x \leq Bx^2$ for all $x > 0$.
43. If $x^2 + \frac{b}{x} \geq c$, $\forall\ x \in R^+$, where a, b, c are +ve constants, prove that $27ab^2 \geq 4c^3$

Concavity

44. Find the interval of the concavity of the function $f(x) = x^5 + 5x - 6$.
45. Find the interval of the concavity for the function $f(x) = x^4 - 5x^3 - 15x^2 + 30$.
46. Find the interval of the concavity for the function $f(x) = (\sin x + \cos x)e^x$ in $(0, 2\pi.)$
47. Show that the curve $y = f(x) = Ax^2 + Bx + c$ is concave up if $A > 0$ and concave down if $A < 0$.

Point of Inflection

48. Find the inflection point of the function $f(x) = x^4 - 4x^3 + x - 10$
49. Find the point of inflection of the curve $y = f(x) = (x - 2)^{2/3} + 10$
50. Find the point of inflections of the curve $f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$
51. Find the point of inflection of the curve $y = f(x) = x^2 - \frac{1}{6x^3}$
52. Find the inflection point of the curve $y = f(x) = e^{-x^2}$

Level II

(Mixed Problems)

1. The function $f(x) = \frac{1}{x}$ on its domain is
   (a) increasing (b) decreasing
   (c) constant (d) none
2. Let $y = x^2 e^{-x}$, then the interval in which y increases with respect to x is
   (a) $(-\infty, \infty)$ (b) $(-2, 0)$
   (c) $(2, \infty)$ (d) $(0, 2)$
3. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is
   (a) inc. on $\left[-\frac{1}{2}, 1\right]$ (b) dec. on R
   (c) Inc. on R (d) dec. on $\left[-\frac{1}{2}, 1\right]$
4. If $f(x) = \int_{x^2}^{x^2+1} e^{t^2} dt$, then the interval in which $f(x)$ is inc. is
   (a) $(0, \infty)$ (b) $(-\infty, 0)$
   (c) $[-2, 2]$ (d) no where
5. If the domain $f(x) = \frac{a\sin x + 2\cos x}{\sin x + \cos x}$ is inc. for all values of x, then
   (a) $a < 1$ (b) $a > 1$
   (c) $a < 2$ (d) $a > 2$
6. The interval of increases of the function $f(x) = x - e^x + \tan\left(\frac{2\pi}{7}\right)$ is equal to
   (a) $(0, \infty)$ (b) $(-\infty, 0)$
   (c) $(1, \infty)$ (d) $(-\infty, -1)$
7. The function $f(x) = \cot^{-1}x + x$ increases in the interval
   (a) $(1, \infty)$ (b) $(-1, \infty)$
   (c) $(-\infty, \infty)$ (d) (∞, ∞)
8. The function $f(x) = x^x$ decreases on the interval
   (a) $(0, e)$ (b) $(0, 1)$
   (c) $\left(0, \frac{1}{e}\right)$ (d) none
9. The function $f(x) = \frac{x}{\log x}$ increases on the interval
   (a) $(0, \infty)$ (b) $(0, e)$
   (c) (e, ∞) (d) none
10. The function $f(x) = \frac{\log x}{x}$ is increasing in the interval
    (a) $(1, 2e)$ (b) $(0, e)$
    (c) $(2, 2e)$ (d) $\left(\frac{1}{e}, 2e\right)$.

11. The function $f(x) = \frac{x}{4 + x^2}$ decreases on the interval
(a) $(-\infty, -1)$ (b) $(-\infty, 0)$
(c) $(-\infty, -2) \cup (2, \infty)$ (d) $(-2, 2)$

12. The function $f(x) = \tan x - x$
(a) always increases (b) always decreases
(c) never decreases (d) sometimes increases

13. If $a < 0$, the function $f(x) = e^{ax} + e^{-ax}$ is a monotonic dec. function for all values of x, where
(a) $x > 0$ (b) $x < 0$
(c) $x > 1$ (d) $x < 1$

14. The value of b for which the function $f(x) = \sin x - bx + c$ is dec. in $(-\infty, \infty)$ is given by
(a) $b < 1$ (b) $b \geq 1$
(c) $b > 1$ (d) $b \leq 1$

15. The value of a for which the function $f(x) = \sin x - \cos x - ax + b$ decreasing for all real values of x is given by
(a) $a \geq \sqrt{2}$ (b) $a \geq 1$
(c) $a < \sqrt{2}$ (d) $a < 1$

16. $y = \sin x - a \sin 2x - \frac{1}{3} \sin 3x + 2ax$, then y increases for all values of x, if
(a) $a = 1$ (b) $a > 1$
(c) $a > 0$ (d) $0 < a < 1$

17. The set of values of a for which the function $f(x) = x^2 + ax + 1$ is an inc. function on $[1, 2]$ is
(a) $(-\infty, -4)$ (b) $[-4, \infty)$
(c) $[4, \infty)$ (d) $(-\infty, 4]$

18. The length of a longest interval in which the function $3\sin x - 4\sin^3 x$ is increasing, is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{2}$
(c) $\frac{3\pi}{2}$ (d) π

19. If $f(x) = 2x + \cot^{-1} x + \left(\sqrt{1 + x^2} + 1\right)$, then $f(x)$
(a) inc. on $[0, \infty)$
(b) dec. on $[0, \infty)$
(c) neither inc. nor dec. on $[0, \infty)$
(d) inc. on $(-\infty, \infty)$

20. Let $f(x) = \int e^x (x - 1)(x - 2)\, dx$, then f decreases in the interval
(a) $(-\infty, -2)$ (b) $(-2, -1)$
(c) $(1, 2)$ (d) $(2, \infty)$

21. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for every real number x then
(a) h is inc. whenever f is inc.
(b) h is inc. whenever f is decreasing
(c) h is dec. whenever f is inc.
(d) nothing can be said in general.

22. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$, $f(x)$ is
(a) increasing
(b) decreasing
(c) has local max. or min.
(d) bounded

23. The function $f(x) = 2\log(x - 2) - x^2 + 4x + 1$ increases on the interval
(a) $(1, 2)$ (b) $(2, 3)$
(c) $\left(\frac{5}{2}, 3\right)$ (d) $(2, 4)$

24. Let $y = x^2 (x - 3)^2$ increases for all values of x lying in the interval
(a) $0 < x < \frac{3}{2}$ (b) $0 < x < \infty$
(c) $-\infty < x < 0$ (d) $1 < x < 3$

25. Let $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an inc. function on the set R. Then a and b satisfy
(a) $a^2 - 3b - 15 > 0$ (b) $a^2 - 3b + 15 > 0$
(c) $a^2 - 3b + 15 < 0$ (d) $a < 0$ and $b > 0$

26. The function $f(x) = \sin^4 x + \cos^4 x$ increases if
(a) $0 < x < \frac{\pi}{8}$ (b) $\frac{\pi}{4} < x < \frac{3\pi}{8}$
(c) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (d) $\frac{5\pi}{8} < x < \frac{3\pi}{4}$

27. The set of all x for which $\log(1 + x) \leq x$ is
(a) $(0, \infty)$ (b) $(-1, \infty)$
(c) $(-1, 0)$ (d) none

28. For all $x \in (0, 1)$,
(a) $e^x < 1 + x$ (b) $\log(1 + x) < x$
(c) $\sin x > x$ (d) $\log_e x > x$

29. If $f(x) = \left(\frac{\lambda^2 - 1}{\lambda^2 + 1}\right)x^3 - 3x + 5$ is a dec. function of in R then the set of all possible values of λ (independent of x) is
(a) $[-1, 1]$ (b) $(-\infty, -1)$
(c) $(1, \infty)$ (d) none

30. Let $f(x) = x^3 + 6x^2 + px + 2$. If the largest possible interval in which $f(x)$ is dec. function in $(-3, -1)$ then p is

(a) 2 (b) 6
(c) 8 (d) 9

31. If $f(x) = \sin^2 x - 3\cos^2 x + 2ax - 4$ is increasing for all $x \geq 0$, then the value of a lies in
(a) $[-2, 0)$ (b) $(-\infty, 2]$
(c) $[2, \infty)$ (d) $(-\infty, 2]$.

32. Let $f(x) = \sin^2 x - (2a + 1)\sin x + (a - c)$. If $f(x) \leq 0$ for all x in $\left[0, \frac{\pi}{2}\right]$, then the range of a is
(a) $[-3, 0]$ (b) $[3, \infty)$
(c) $[-3, 3]$ (d) $(-\infty, 3]$.

33. The set of values of a for which the function
$f(x) = (4a - 3)(x + 5) + 2(a - 7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)$. does not posses any critical point is given by
(a) $\left(-\infty, -\frac{4}{3}\right)$ (b) $(-\infty, -1)$
(c) $\left(-\frac{4}{3}, 2\right)$ (d) $\left(-\infty, -\frac{4}{3}\right) \cup (2, \infty)$

34. The number of roots of the cubic $f(x) = x^3 + 3x + k$, $\forall\ k \in R$ is
(a) 0 (b) 1
(c) 2 (d) 3

35. The number of inflection points on the curve represented by the equations $x = t^2$, $y = 3t + t^3$ is
(a) 0 (b) 1
(c) 2 (d) 3

36. If $f'(x^2 - 4x + 3) > 0\ \forall\ x \in (2, 3)$, then $f(\sin x)$ is increasing on
(a) $\left(n\pi, \frac{n\pi}{2}\right)$, $n \in I$
(b) $\left((2n + 1)\pi, (4n + 3)\frac{\pi}{2}\right)$ $n \in I$
(c) $\left((4n - 1)\frac{\pi}{2}, 2n\pi\right)$, $n \in I$
(d) None.

37. Let $f'(x) = |x| - \{x\}$, $\{,\}$ = F.P.F, then $f(x)$ is decreasing on
(a) $\left(-\frac{1}{2}, 0\right)$ (b) $\left(-\frac{1}{2}, 2\right)$
(c) $(0, 2)$ (d) $\left(\frac{1}{2}, \infty\right)$.

38. Let f be a function such that $f'(x) = \log_{1/3}(\sin x + a)$. If f is decreasing for all real values of x, then the range of a is
(a) $(1, 4)$ (b) $(4, \infty)$
(c) $(2, 3)$ (d) $(2, \infty)$

39. The length of the largest interval in which the function $f(x) = 4x - \tan 2x$ is monotonic is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{8}$ (d) $\frac{\pi}{16}$

40. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$, in $[0, 2\pi]$ is
(a) 1 (b) 2
(c) 3 (d) 0

41. The number of solutions of the equation $x^3 + 2x^2 + 5x + 2\sin x = 0$ in $[0, \pi]$ is
(a) 3 (b) 2
(c) 1 (d) 0

Level III

(Problems for JEE-Advanced)

1. Find the length of the largest continuous interval in which the function $f(x) = 4x - \tan 2x$ is monotonic increasing.
2. Find the number of solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$
3. If $f(x) = (ab - b^2 - 2)x + \int_0^x (\cos^4\theta + \sin^4\theta)\,d\theta$ is a decreasing function of x for all x, b in R, find a.
4. If $f(x) = \cos x + a^2 x + b$ is an increasing function for all values of x, find a.
5. If $f(x) = 2e^x - a e^{-x} + (2a + 1)x - 3$ is increasing for all x, find a.
6. If $y = f(x)$ be given by $x = \frac{1}{1 + t^2}$, $y = \frac{1}{1 + t^2}$ $t > 0$. If f is increasing in $(0, a)$, find the greatest value of a.
7. Find the range values of 'b' so that for all real x, $f(x) = \int_0^x (bt^2 + t\sin t)\,dt$ is monotonic.
8. Find the interval of strictly increases for the function $f(x) = \sin^4 x + \cos^4 x$.
9. Find the interval of increasing and decreasing for the function $f(x) = x e^{x(1 - x)}$.
10. Find the interval of increasing and decreasing for the function $f(x) = \frac{\log(\pi + x)}{\log(e + x)}$.
11. Find the interval for which the function $f(x) = e^{-ax} + e^{ax}$, $a > 0$ is monotonically decreasing.
12. Find the set of values of a for which the function $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$ is invertible.
13. If the function $f(x) = 2x^2 - kx + 5$ is increasing on $[1, 2]$. then find k.

14. Find the interval of the increases or decreases for the function $f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\,dt$

15. Let $f(x) = \begin{cases} xe^{ax} & : x \le 0 \\ x + ax^2 - x^3 & : x > 0 \end{cases}$,

where a is a +ve const. Find the interval in which $f'(x)$ is inc.

16. Let $f(x) = \int_{0}^{x} |\log_2(\log_3(\log_4(\cos t + a)))|\,dt$ be increasing for all real values of x, find a.

17. Let $f(x) = x^3 - 3x + a$. If the equation $f(x) = 0$ has exactly one root, find the value of a.

18. Let $f(x) = x^3 - 3x + a$. If the equation $f(x) = 0$ has three real and distinct roots, find the value of a.

19. Let $f(x) = \int_{0}^{x} \{(a-1)(t^2 + t + 1)^2 - (a+1)(t^4 + t^2 + 1)\}\,dt$

Then find the set of values of a for which $f'(x) = 0$ has two distinct real roots.

20. If $y = f(x) = \dfrac{ae^x + be^{-x}}{ce^x + de^{-x}}$ is an increasing function of x, then find a relation in a, b, c and d.

21. If a, b, c are real numbers, find the intervals in which

$$f(x) = \begin{vmatrix} x + a^2 & ab & ac \\ ab & x + b^2 & bc \\ ac & bc & x + c^2 \end{vmatrix}$$

is increasing or decreasing.

Level III

(Tougher Problems for JEE-Advanced)

1. Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality $1 - f(x) - f^3(x) > f(1 - 5x)$.

2. A function $f(x)$ is given by the equation $x^2 f'(x) + 2xf(x) - x + 1 = 0$, where $x \ne 0$.

If $f(1) = 0$, then find the interval of the monotonocity of the function f.

3. $f(x)$ be a differentiable function such that $f'(x) = \log_{1/3}(\log_3(\sin x + a))$. If $f(x)$ is decreasing for all real values of x, find a.

4. $f(x)$ be a differentiable function such that $f'(x) = \dfrac{1}{\log_3(\log_{1/4}(\cos x + b))}$. If $f(x)$ be an increasing function, find b.

5. Let $f''(x) > 0$ for all x in R and $g(x) = f(2 - x) + f(4 + x)$. Find the interval in which $g(x)$ is increasing.

6. Find the set of all real values of a for which $f(x) = \left(1 - \dfrac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^3 + 5x + \log_e 17$ increases for all real x in R.

7. Find the set of all real values of 'a' for which $f(x) = \left(\dfrac{\sqrt{a + 4}}{1 - a} - 1\right)x^5 - 3x + \log_e 5$ decreases for all values of x in R.

8. If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ is monotonically increases for all x in R, then the set of all values of a.

9. Find the interval in which $f(x) = \int_{0}^{x} 2\sqrt{2}\sin^2 t + (2 - \sqrt{2})\sin t - 1)\,dt$ where $0 < x < 2\pi$ is increasing or decreasing.

10. Let $f(x) = \{-b^2 + (a - 1)b - 2\}x + \int (\sin^2 x + \cos^4 x)\,dx$, $a, b \in R$

If $f(x)$ be an increasing function, find all the permissible values of a.

11. Let $g(x) = 2f\left(\dfrac{x^2}{2}\right) + f(6 - x^2)$ for all x in R and $f''(x) > 0$, $\forall\, x \in R$. Find the intervals of increase and decrease of $g(x)$.

12. Let $f(x) = \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x + 1}{\sqrt{3}}\right) - \log(x^2 + x + 1) + (b^2 - 5b + 3)x + 4$. If $f(x)$ is a decreasing function for all real values of x in R, find the permissible values of b.

13. If $h(x) = f(x) + 2f\left(1 - \dfrac{x}{2}\right)$, $0 < x < 1$ and $f'(x) > 0$, $\forall\, x \in (0, 1)$. Find the interval where $h(x)$ is increasing or decreasing where $f(x)$ and $h(x)$ are two differentiable function.

14. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$ for all x in $\left(0, \dfrac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then find the interval where $g(x)$ increases or decreases.

15. Let $f'(x) < 0$ for all x in R such that $f'(3) = 0$.

If $g(x) = f(\tan^2 x - 2\tan x + 4)$ for all x in $\left(0, \dfrac{\pi}{2}\right)$, find the interval where $g(x)$ is increasing.

Integer Type Questions

1. The number of zeroes of the curve $f(x) = x^3 + 3x + m$, $m \in R$ is...

2. The number of real roots of the function $f(x) = x^3 - 3x + m$, $m \in R$ is...

3. If $f(x) = \sin^2 x - 3\cos^2 x + 2ax - 4$ is increasing for all $x \geq 0$, then the least value of a is...
4. If the number of real solutions of the equation $x^3 + 2x^2 + 5x + 2\cos x = 0$ in $[0, 2\pi]$ is m, then the value of $(m + 3)$ is...
5. The number of critical points of the function $f(x) = |x^2 - 2x|$ is...
6. If the function $f(x) = \frac{x^3}{3} + (m-1)x^2 + (m+5)x + 7$ is increasing for all $x > 0$, then find the integral values of m.
7. Let $f(x) = |x| + |x^2 - 1|$. Then the number of real solutions of $6f(x) = 7$ is...
8. If m is the number of real roots of $ke^x = 5 + x + \frac{x^2}{2}$, $k \in R^+$ then the value of $(m + 3)$ is...
9. If $f(x) = e^{2x} - (a+1)e^x + 2x$ is increasing for all real values of x in R, then find the maximum value of a.
10. If m is the number of solution of the equation $3\tan x + x^3 = 2$, in $\left(0, \frac{\pi}{4}\right)$ and n is the number of inflection points of $f(x) = 3x^4 - 4x^3$, then the value of $(m + n + 2)$ is...

Comprehensive Link Passages (For JEE-Advanced Exam)

Passage I

If f is strictly increasing in (a, b) and g is strictly increasing in $(f(a), f(b))$, then $(g_0 f)$ is strictly increasing in (a, b).

1. $f(x) = \tan^{-1}(\sin x + \cos x)^3$ is strictly inc. in

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(c) $\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ (d) $(\pi, 2\pi)$

2. $f(x) = e^{\sin x + \cos x}$ is strictly increasing in

(a) $\left(0, \frac{\pi}{4}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(c) $\left(\frac{\pi}{2}, \pi\right)$ (d) $\left(0, \frac{\pi}{2}\right)$

3. $f(x) = \tan^{-1}(\log_2 x)$ is strictly increasing in

(a) $(-\infty, 0)$ (b) $\left(-\frac{\pi}{2}, 0\right)$
(c) $(0, \infty)$ (d) $(-\infty, \infty)$

Passage II

If f is strictly decreasing in (a, b) and g is strictly decreasing in $(f(a), f(b))$, then $(g_0 f)$ is strictly decreasing in (a, b).

1. $f(x) = \cot^{-1}\left(\log_{\frac{1}{2}} x\right)$ is strictly increasing in

(a) $(-\infty, 0)$ (b) $(-\infty, \infty)$
(c) $(0, \infty)$ (d) None

2. $f(x) = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$ is strictly increasing in

(a) $(0, \infty)$ (b) $(-\infty, \infty)$
(c) $(-2, 2)$ (d) $(-\infty, 0)$

3. $f(x) = e^{-(x - x^2)}$ is strictly increasing in

(a) $(1, \infty)$ (b) $(-\infty, 0)$
(c) $(0, 1)$ (d) $(-2, 0)$

Passage III

A critical point is a point where either $f'(x) = 0$ or $f'(x)$ does not exist, whereas a stationary point is a point where f is differentiable and $f'(x) = 0$.

1. The critical points of the function $f(x) = x^{2/3}(2 - x)$ is/are

(a) 1 (b) 2
(c) 4/5 (d) 0

2. The critical points of the function $f(x) = x + \cos^{-1} x + 1$ is/are

(a) 1 (b) 2
(c) –1 (d) 0

3. The critical points of the function $f(x) = \max\{\sin x, \cos x\}$, $\forall\ x \in (0, 2\pi)$ is/are

(a) 0 (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) $\frac{5\pi}{2}$

Passage IV

Let $f(x)$ be a real function and $a, b \in R$. If

(i) $f(x)$ is continuous in $[a, b]$
(ii) $f(x)$ is differentiable in (a, b)
(iii) $f(a)$ and $f(b)$ are of opposite signs
(iv) either $f'(x) > 0$ on (a, b) or $f'(x) < 0$ on (a, b)

then the function $f(x)$ has exactly one root in $[a, b]$

1. The equation $x^5 - 3x - 1 = 0$ has a unique root in

(a) $[0, 1]$ (b) $[1, 2]$
(c) $[-1, 0]$ (d) $[2, 3]$

2. The equation $x^4 + 2x^2 - 2 = 0$ has a unique root in

(a) $[0, 1]$ (b) $[-1, 1]$
(c) $[1, 2]$ (d) $[-2, -1]$

3. The equation $xe^x - 2 = 0$ has exactly one root in

(a) $[-1, 0]$ (b) $[1, 2]$
(c) $[0, 1]$ (d) $\left[0, \frac{1}{2}\right]$

Passage V

A point c in the domain of a function $f(x)$ is said to be a hyper-critical point at which $f''(c)$ = or $f''(c)$ does not exist.

1. The hyper-critical point of the function $f(x) = x^5 + 5x - 6$ is
   (a) 1 (b) 2
   (c) 0 (d) –1
2. The hyper-critical point of the function $f(x) = \sin x - \cos x$ in $[0, 2\pi]$ is
   (a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$
   (c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{4}$
3. The hyper-critical point of the function $f(x) = 2 - |x^5 - 1|$ is
   (a) 1 (b) 0
   (c) –1 (d) 2

Passage VI

An inflection point is point on a curve at which the curve changes from concave down-ward to concave upward and vice-versa.

1. The inflection point of the function $f(x) = 4x^4 - x^3 + 2$ is
   (a) (0, 1) (b) (0, 1)
   (c) (–1, 1) (d) (–2, 0)
2. The inflection point of the function $f(x) = -x^3 + 3x^2 + 1$ is
   (a) (1, 3) (b) (0, 1)
   (c) (1, 2) (d) (–1, 0)
3. The inflection point of the curve $f(x) = x^4 - 6x^2 + 8x + 10$ is
   (a) (0, 1) (b) (1, 2)
   (c) (–1, –3) (d) (2, 3)

Passage VII

If $y = f'(x)$ is increasing, then $y = f(x)$ is concave up and if $y = f'(x)$ is decreasing then $y = f(x)$ is concave down.

1. The function $f(x) = x^{1/3} + 2$ is concave up in
   (a) $(0, \infty)$ (b) (0, 1)
   (c) $(-\infty, 0)$ (d) (1, 2)
2. The function $f(x) = x^4 - 6x^2 + x - 3$ is concave down in
   (a) $(-\infty, 0)$ (b) (–1, 1)
   (c) $(1, \infty)$ (d) (1, 2)
3. The function $f(x) = \sin x$, $\forall\ x \in [0, 2\pi]$ is concave up in
   (a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2}, \pi\right)$
   (c) $(\pi, 2\pi)$ (d) $(0, \pi)$

Matrix Match (For JEE-Advanced Exam)

1. **Match the following columns**

   The number of critical points of the function

Column I		Column II	
(A)	$f(x) = \|x\|e^{-x}$ is	(P)	2
(B)	$f(x) = (ax^2 + bx + c)\|x\|$ where a c < 0, is	(Q)	3
(C)	$f(x) = (1 - x)\|x - 2\|$ is	(R)	1
(D)	$f(x) = x^2 - 2\|x\|$ is	(S)	0

2. **Match the following columns**

   The function $f(x)$ is increasing in

Column I		Column II	
(A)	$f(x) = x - e^x$	(P)	$(-\infty, 1)$
(B)	$f(x) = 2x^2 - \log x$	(Q)	$\left(\frac{1}{2}, \infty\right)$
(C)	$f(x) = \log\left(x + \sqrt{x^2 + 1}\right)$	(R)	$(-\infty, 0)$
(D)	$f(x) = 5x^{3/2} - 3x^{5/2}, x > 0$	(S)	$(-\infty, \infty)$

3. **Match the following columns**

   Let $f(x) = |x + 1|\{|x| + |x - 1|\}$. Then

Column I		Column II	
(A)	$f(x)$ is increasing in	(P)	(–2, –1)
(B)	$f(x)$ is decreasing in	(Q)	$(-\infty, \infty)$
(C)	$f(x)$ is continuous in	(R)	(0, 1)
(D)	The number of non-differentiable points is	(S)	3
		(T)	4

4. **Match the following columns**

   Let $f(x) = \max\left\{\text{sgn}(x), -\sqrt{9 - x^2}, x^3\right\}$ Then

Column I		Column II	
(A)	$f(x)$ is increasing in	(P)	1
(B)	The number of discontinuous points is	(Q)	4
(C)	The number of non-differentiable points is	(R)	$(1, \infty)$
(D)	$f(x)$ is decreasing in	(S)	$(-3, -2\sqrt{2})$
		(T)	(–1, 0)

5. **Match the following columns**

Let $f(x) = \begin{cases} x^3 - x^2 + x + 1 & : 0 \le x \le 1 \\ 3 - x & : 1 < x \le 2 \end{cases}$

Then

Column I		Column II	
(A)	$f(x)$ is increasing in	(P)	$(1, 2)$
(B)	$f(x)$ is decreasing in	(Q)	$(0, 1)$
(C)	$f(x)$ is continuous in	(R)	$(0, 1) \cup (1, 2)$
(D)	$f(x)$ is differentiable in	(S)	$[0, 2]$

6. **Match the following columns**

Let $f(x) = \sin(|x|) + |x|$ and

$g(x) = \cos(|x|) + |x|$. Then

Column I		Column II	
(A)	$f(x)$ is increasing in	(P)	$(-\infty, 0)$
(B)	$f(x)$ is decreasing in	(Q)	$(0, \infty)$
(C)	The number of disc. Points of $f(x)$ and $g(x)$ is	(R)	2
(D)	The number of non-differentiable points of $f(x)$ is	(S)	0
		(T)	1

7. **Match the following columns**

Let $f(x) = \{1, \cos x, 1 - \sin x\}$, $x \in \left[-\frac{\pi}{2}, \pi\right]$

Then

Column I		Column II	
(A)	$f(x)$ is increasing in	(P)	$[0, \pi]$
(B)	$f(x)$ is decreasing in	(Q)	0
(C)	The number of non-differentiable points is	(R)	$\left[-\frac{\pi}{2}, 0\right]$
(D)	The number of discontinuous points is	(S)	2

8. **Match the following columns**

Let $f(x) = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$

and $g(x) = \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$. Then

Column I		Column II	
(A)	$f(x)$ is increasing in	(P)	$(0, \infty)$
(B)	$g(x)$ is increasing in	(Q)	$(-\infty, 0)$
(C)	$f(x)$ is decreasing in	(R)	$(-1, 1)$
(D)	$g(x)$ is decreasing in	(S)	$(1, \infty)$

9. **Match the following columns**

Let $f(x) = |x - 1| + |x - 2| + |x - 3| + |x - 4|$

and $a, b, c, d \in I^+$. Then

Column I		Column II	
(A)	If $f(x)$ is increasing in $[a. b]$ then $(a + b)$ is	(P)	3
(B)	If $f(x)$ is decreasing in $[c, d]$ then $(c + d)$ is	(Q)	0
(C)	The number of discontinuous points is	(R)	7
(D)	The number of non-differentiable points is	(S)	4

10. **Match the following columns**

Let $f(x) = \frac{x}{x^2 + 1}$, $g(x) = \tan^{-1} x - x$. Then

Column I		Column II	
(A)	$f(x)$ is increasing in	(P)	$(-\infty, -1]$
(B)	$g(x)$ is increasing in	(Q)	$[-1, 1]$
(C)	$f(x)$ is decreasing in	(R)	$[1, \infty)$
(D)	$g(x)$ is decreasing in	(S)	$[0, \infty)$

Question asked in Previous Years' JEE-Advanced Examinations

1. Use the function $f(x) = x^{1/x}$, $x > 0$ to determine the bigger of the two numbers e^{π} and π^{c}. **[IIT-JEE, 1981]**

2. If $ax^2 + \frac{b}{x} \ge c$ for all +ve x where $a > 0$ and $b > 0$, show that $27ab^2 \ge 4c^3$ **[IIT-JEE, 1982]**

3. Show that $1 + x\log\left(x + \sqrt{x^2 + 1}\right) \ge \sqrt{1 + x^2}$ for all $x \ge 0$ **[IIT-JEE, 1983]**

4. The larger of $\cos(\ln\theta)$ and $\ln(\cos\theta)$ if $e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$... **[IIT-JEE, 1983]**

5. The function $y = 2x^2 - \ln|x|$ is monotonically increasing for values of $x \ne 0$ satisfying the inequalities ... and

monotonically decreasing for values of x satisfying the inequalities ... **[IIT-JEE, 1983]**

6. No questions asked in between 1984 to 1986.

7. Let f and g be increasing and decreasing functions respectively from $[0, \infty)$ to $[0, \infty)$. Let $h(x) = f(g(x))$. If $h(0) = 0$, then $h(x) - h(1)$ is
   (a) always zero (b) always –ve
   (c) always +ve (d) strictly inc.
   [IIT-JEE, 1987]

8. The set of all x for which $(1 + x) \le x$ is equal to ... **[IIT-JEE, 1987]**

9. No questions asked in between 1988 to 1992.

10. Show that $2\sin x + \tan x \ge 3x$, where $0 \le x \le \frac{\pi}{2}$
    [IIT-JEE, 1993]

11. If $f(x) = \begin{cases} 3x^3 - 12x - 1 & : -1 \le x \le 2 \\ 37 - x & : 2 < x \le 3 \end{cases}$
    then $f(x)$ is
    (a) increasing in $[-1, 2]$
    (b) continuous in $[-1, 3]$
    (c) greatest at $x = 2$
    (d) all
    [IIT-JEE, 1993]

12. Let $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for all x in R. Then
    (a) h is inc. whenever f is inc.
    (b) h is inc. whenever f is dec.
    (c) h is dec. whenever f is dec.
    (d) nothing can be said in general.
    [IIT-JEE, 1993]

13. The function f is defined by $f(x) = (x + 2)e^{-x}$ is
    (a) decreasing for all x
    (b) decreasing in $(-\infty, -1)$ and increasing in $(-1, \infty)$
    (c) increasing for all x.
    (d) decreasing in $(-1, \infty)$ and increasing in $(-\infty, -1)$
    [IIT-JEE, 1994]

14. The function $f(x) = \frac{\ln(x + \pi)}{\ln(x + e)}$ is
    (a) increasing on $[0, \infty)$
    (b) decreasing on $[0, \infty)$
    (c) increasing on $\left(0, \frac{\pi}{e}\right)$ and decreasing on $\left(\frac{\pi}{e}, \infty\right)$
    (d) decreasing on $\left(0, \frac{\pi}{e}\right)$ and increasing on $\left(\frac{\pi}{e}, \infty\right)$
    [IIT-JEE, 1995]

15. Let $f(x) = \begin{cases} xe^{ax} & : x \le 0 \\ x + ax^2 - x^3 & : x > 0 \end{cases}$
    where a is a +ve constant. Find the interval in which $f'(x)$ is increasing.
    [IIT-JEE, 1996]

16. If $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$ where $0 < x \le 1$, then
    (a) both $f(x)$ and $g(x)$ are inc. functions
    (b) both $f(x)$ and $g(x)$ are dec. functions
    (c) $f(x)$ is an inc. function.
    (d) $g(x)$ is an inc. function.
    [IIT-JEE, 1997]

17. No questions asked in 1998.

18. The function $f(x) = \sin^4 x + \cos^4 x$ increases if
    (a) $0 < x < \frac{\pi}{8}$ (b) $\frac{\pi}{4} < x < \frac{3\pi}{8}$
    (c) $\frac{3\pi}{8} < x < \frac{5\pi}{8}$ (d) $\frac{5\pi}{8} < x < \frac{3\pi}{5}$
    [IIT-JEE, 1999]

19. Consider the following statements S and R.
    S: Both $\sin x$ and $\cos x$ are decreasing functions in the interval $\left(\frac{\pi}{2}, \pi\right)$
    R: If a differentiable function decreases in an interval (a, b), then its derivative also decrease in (a, b)
    Which of the following is true.
    (a) both S and R are wrong.
    (b) both S and R are correct but R is not the correct explanation for S.
    (c) S is correct and R is the correct explanation for S.
    (d) S is correct and R is wrong.
    [IIT-JEE, 2000]

20. Let $f(x) = \int e^x (x - 1)(x - 2)\,dx$. Then f decrease in the interval
    (a) $(-\infty, -2)$ (b) $(-2, -1)$
    (c) $(1, 2)$ (d) $(2, \infty)$
    [IIT-JEE, 2000]

21. If $f(x) = xe^{x(1 - x)}$, then $f(x)$ is
    (a) increasing on $[-1/2, 1]$
    (b) decreasing on R.
    (c) increasing on R.
    (d) decreasing on $[-1/2, 1]$
    [IIT-JEE, 2001]

22. The length of the longest interval in which the function $f(x) = 3\sin x - 4\sin^3 x$ is increasing is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) π

[IIT-JEE, 2002]

23. Using the relation $2(1 - \cos x) < x^2$, $x \neq 0$ or otherwise, prove that $\sin(\tan x) > x$, $\forall\ x \in \left(0, \frac{\pi}{4}\right)$

[IIT-JEE, 2003]

24. If $P(1) = 0$ and $\frac{dp(x)}{dx} > P(x)$ for all $x \geq 1$, then prove that $P(x) > 0$ for all $x > 1$

[IIT-JEE, 2003]

25. Prove that for $x \in \left[0, \frac{\pi}{2}\right]$, $\sin x + 2x \geq \left(\frac{3x(x+1)}{\pi}\right)$. Explain the identity if any used in the proof.

[IIT-JEE, 2003]

26. No questions asked in between 2004 to 2007.

27. Let the function $f: (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then g is

(a) even and is strictly inc. in $(0, \infty)$
(b) odd and is strictly dec. in $(-\infty, \infty)$
(c) odd and is strictly inc. in $(-\infty, \infty)$
(d) neither even nor odd, but is strictly inc. in $(-\infty, \infty)$.

[IIT-JEE, 2008]

28. For function $f(x) = x\cos\left(\frac{1}{x}\right)$ $x \geq 1$

(a) for atleast one x in interval $[1, \infty)$, $f(x + 2) - f(x) < 2$

(b) $\lim_{x\to\infty} (f(x)) = 1$

(c) for all x in interval $[1, \infty)$, $f(x + 2) - f(x) > 2$

(d) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$.

[IIT-JEE, 2009]

29. Consider the function $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

(i) The real number s lies in the interval

(a) $\left(-\frac{1}{4}, 0\right)$

(b) $\left(-1, -\frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$

(d) $\left(0, \frac{1}{4}\right)$

(ii) The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$ lies in the interval

(a) $\left(\frac{3}{4}, 3\right)$

(b) $\left(\frac{21}{64}, \frac{11}{16}\right)$

(c) $(9, 10)$

(d) $\left(0, \frac{21}{64}\right)$

(iii) The function $f'(x)$ is

(a) inc. in $\left(-t, \frac{1}{4}\right)$ and dec. in $\left(-\frac{1}{4}, t\right)$

(b) dec. in $\left(-t, -\frac{1}{4}\right)$ and inc. in $\left(-\frac{1}{4}, t\right)$

(c) inc. in $(-t, t)$

(d) dec. in $(-t, t)$

[IIT-JEE, 2010]

30. If $f(x) = x^{3/2}(3x - 10)$, $x \geq 0$, then $f(x)$ is increasing in...

[IIT-JEE, 2011]

31. Let $f(x) = (1 - x)^2\sin^2 x + x^2$ for all x in R and $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)\,dt$ for all $x > 1$. which of the following is true?

(a) g is inc. on $(1, \infty)$
(b) g is dec. on $(1, \infty)$
(c) g is inc. on $(1, 2)$ and dec. on $(2, \infty)$
(d) g is dec. on $(1, 2)$ and inc. on $(2, \infty)$.

[IIT-JEE, 2012]

32. No questions asked in 2013.

33. Let $f : (0, \infty) \to R$ be given by $f(x) = \int_{1/x}^{x} e^{-\left(t + \frac{1}{t}\right)} \frac{dt}{t}$, then

(a) $f(x)$ is monotonic inc. on $[1, \infty)$.

(b) $f(x)$ is monotonic dec. on $(0, 1)$.

(c) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$.

(d) $f(2^x)$ is an odd function of x on R.

[IIT-JEE, 2014]

ANSWERS

LEVEL II

1. (b)	2. (d)	3. (a)	4. (b)	5. (d)
6. (b)	7. (c)	8. (c)	9. (c)	10. (b)
11. (c)	12. (c)	13. (b)	14. (c)	15. (a)
16. (b)	17. (b)	18. (a)	19. (a, d)	20. (c)
21. (a, c)	22. (a)	23. (b, c)	24. (a)	25. (c)
26. (b)	27. (b)	28. (b)	29. (a)	30. (d)
31. (c)	32. (c)	33. (c)	34. (b)	35. (c)
36. (d)	37. (a)	38. (b)	39. (b)	40. (d)
41. (d)				

INTEGER TYPE QUESTIONS

1. 1	2. 3	3. 2	4. 3	5. 3
6. 4	7. 6	8. 4	9. 3	10. 5

COMPREHENSIVE LINK PASSAGES

Passage I :	1. (a)	2. (b)	3. (c)
Passage II:	1. (c)	2. (d)	3. (c)
Passage III:	1. (c, d)	2. (a, c, d)	3. (b, c, d)
Passage IV:	1. (b)	2. (a)	3. (c)
Passage V:	1. (c)	2. (a)	3. (b)
Passage VI:	1. (b)	2. (a)	3. (c)
Passage VII:	1. (c)	2. (b)	3. (c)

MATRIX MATCH

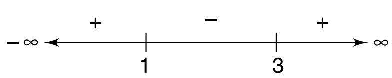
1. (A)→(P), (B)→(Q), (C)→(P), (D)→(Q)
2. (A)→(R), (B)→(Q), (C)→(Q), (D)→(P)
3. (A)→(R), (B)→(P), (C)→(Q), (D)→(S)
4. (A)→(R,T), (B)→(P), (C)→(Q), (D)→(S)
5. (A)→(Q), (B)→(P), (C)→(S), (D)→(R)
6. (A)→(Q), (B)→(P), (C)→(S), (D)→(T)
7. (A)→(R), (B)→(P), (C)→(S), (D)→(Q)
8. (A)→(R), (B)→(P), (C)→(S), (D)→(Q)
9. (A)→(R), (B)→(P), (C)→(Q), (D)→(S)
10. (A)→(Q), (B)→(T), (C)→(P, R), (D)→(P, Q, R, S)

HINTS AND SOLUTIONS

Level I

1. Given $f(x) = 2x^3 - 12x^2 + 18x + 5$

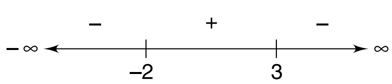
$$= 6x^2 - 24x + 18$$
$$= 6(x^2 - 4x + 3)$$
$$= 6(x^2 - 4x + 3)$$
$$= 6(x - 1)(x - 3)$$



By the sign scheme, $f(x)$ is strictly increases in $(-\infty, 1) \cup (3, \infty)$ and strictly decreases in $(1, 3)$

2. Given $f(x) = 5 + 36x + 3x^2 - 2x^3$

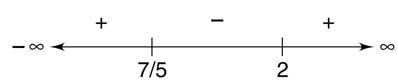
$$\Rightarrow \quad f'(x) = 36 + 6x - 6x^2$$
$$= -6(x^2 - x - 6)$$
$$= -6(x - 3)(x + 2)$$



By the sign scheme, we can say that, $f(x)$ is strictly increases in $(-2, 3)$ and strictly decreases in $(-\infty, -2) \cup (3, \infty)$.

3. Given $f(x) = (x - 1)^3 (x - 2)^2$

$$\Rightarrow \quad f'(x) = 3(x - 1)^2 (x - 2)^2 + 2(x - 1)^3 (x - 2)$$
$$= (x - 1)^2 (x - 2)(3(x - 2) + 2(x - 1))$$
$$= (x - 1)^2 (x - 2)(5x - 7)$$



By the sign scheme, we can say that, $f(x)$ is strictly increase in $\left(-\infty, \frac{7}{5}\right) \cup (2, \infty)$ and strictly decreases in $\left(\frac{7}{5}, 2\right)$

4. Given $f(x) = 2x^3 - 3x^2 + 6x + 10$

$$\Rightarrow \quad f'(x) = 6x^2 - 6x + 6$$
$$= 6(x^2 - x + 2) > 0, \text{ for all } x \text{ in } R$$

Thus, the function $f(x)$ is strictly increases for all x in R.

5. Given $f(x) = 2x^3 + 3x^2 + 12x + 20$

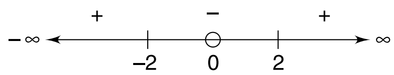
$$\Rightarrow \quad f'(x) = 6x^2 + 6x + 12$$
$$= 6(x^2 + x + 2) > 0 \text{ for all } x \text{ in } R$$

Thus, $f(x)$ is strictly increases in $(-\infty, \infty)$

6. Given $f(x) = \frac{x}{2} + \frac{2}{x}$

$\Rightarrow \quad f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}$

$= \frac{(x-2)(x+2)}{2x^2}$



By the sign scheme, we can say that, $f(x)$ is strictly increases in $(-\infty, -2) \cup (2, \infty)$and strictly decreases in $(-2, 0) \cup (0, 2)$

7. Given $f(x) = 5x^{3/2} - 3x^{5/2}$, $x > 0$

$\Rightarrow \quad f'(x) = 15x^{1/2} - 15x^{3/2}$

$= 15\sqrt{x}(1 - x) - 15\sqrt{x}(x - 1)$

By the sign scheme, we can say that, $f(x)$ is strictly increases in (0, 1) and strictly decreases in $(1, \infty)$

8. Given $f(x) = \log\left(x + \sqrt{1 + x^2}\right)$

$\Rightarrow \quad f'(x) = \frac{1}{\left(x + \sqrt{x^2 + 1}\right)}\left(1 + \frac{1 \times 2x}{2\sqrt{x^2 + 1}}\right)$

$\Rightarrow \quad f'(x) = \frac{1}{\left(x + \sqrt{x^2 + 1}\right)}\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right)$

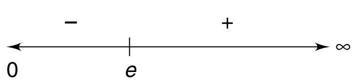
$= \frac{1}{\left(x + \sqrt{x^2 + 1}\right)}\left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right)$

$= \frac{1}{\sqrt{x^2 + 1}} > 0 \ \forall \ x \in R$

Thus, $f(x)$ is strictly increases in $(-\infty, \infty)$

9. Given $f(x) = \frac{x}{\log x}$

$\Rightarrow \quad f'(x) = \frac{\log x - 1}{(\log x)^2}$



By the sign scheme, we can say that, $f(x)$ is strictly increases in (e, ∞) and strictly decreases in $(0, e)$.

10. Given $f(x) = \cot^{-1} x - \log\left(x + \sqrt{x^2 + 1}\right)$

$\Rightarrow \quad f'(x) = 1 + \frac{1}{1 + x^2} - \frac{1}{\left(x + \sqrt{x^2 + 1}\right)}$

$\times \frac{\left(x + \sqrt{x^2 + 1}\right)}{\sqrt{x^2 + 1}}$

$\Rightarrow \quad f'(x) = 1 + \frac{1}{1 + x^2} - \frac{1}{\sqrt{x^2 + 1}} > 0 \ \forall \ x \in R$

Thus, $f(x)$ is strictly increases in $(-\infty, \infty)$

11. Given $f(x) = -x^2 + mx + 1$

$\Rightarrow \quad f'(x) = -2x + m$

Since f is strictly increasing, so $f'(x) > 0$

$\Rightarrow \quad -2x + m > 0$

$\Rightarrow \quad m > 2x$

$\Rightarrow \quad m > 2, \ \forall \ x \in [1, 2]$

Hence, the least value of m is 2.

12. Given $f(x) = \sin x - bx + c$

$\Rightarrow \quad f'(x) = \cos x - b$

Since f is strictly decreasing, so $f < 0$

$\Rightarrow \quad \cos x - b < 0$

$\Rightarrow \quad b > \cos x$

$\Rightarrow \quad b > 1$

Hence, $b \in (1, \infty)$

13. Given $f(x) = e^{2x} - (a + 1)e^x + 2x$

$\Rightarrow \quad f'(x) = 2e^{2x} - (a + 1)e^x + 2$

$= 2(e^x)^2 - (a + 1)e^x + 2$

Since f is strictly increasing, so $f'(x) > 0$

$\Rightarrow \quad 2(e^x)^2 - (a + 1)e^x + 2 > 0$

$\Rightarrow \quad 2e^x - (a + 1)e^x + \frac{2}{e^x} > 0$

$\Rightarrow \quad 2\left(e^x + \frac{1}{e^x}\right) - (a + 1) > 0$

$\Rightarrow \quad (a + 1) < 2\left(e^x + \frac{1}{e^x}\right)$

$\Rightarrow \quad (a + 1) < 2.2 = 4$

$\Rightarrow \quad a < 3$

Hence, the value of a is $\in (-\infty, 3)$

14. Given $f(x) = \left(\frac{a^2 - 1}{3}\right)x^3 + (a - 1)x^2 + 2x + 1$

$\Rightarrow \quad f'(x) = 3\left(\frac{a^2 - 1}{3}\right)x^2 + 2(a - 1)x + 2$

$\Rightarrow \quad f'(x) = (a^2 - 1)x^2 + 2(a - 1)x + 2$

Since f is strictly increasing, so $f'(x) > 0$

$\Rightarrow \quad (a^2 - 1)x^2 + 2(a - 1)x + 2 > 0$

$\Rightarrow \quad (a^2 - 1) > 0 \ \& \ 4(a - 1)^2 - 8(a^2 - 1) < 0$

$\Rightarrow \quad a^2 > 1 \ \& \ (a - 1)^2 - 2(a^2 - 1) < 0$

$\Rightarrow \quad (a + 1)(a - 1) > 0 \ \& \ (a + 3)(a - 1) > 0$

$\Rightarrow \quad a \in (-\infty, -1) \cup (1, \infty)$

and $\quad a \in (-\infty, -3) \cup (1, \infty)$

Hence, the values of a are $a \in (-\infty, -3) \cup (1, \infty)$

15. Given $\quad f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$

$\Rightarrow \quad f'(x) = 3(a + 2)x^2 - 6ax + 9a$

Since f is strictly decreasing for all x in R, so

$f'(x) < 0$

$\Rightarrow \quad 3(a + 2)x^2 - 6ax + 9a < 0$

$\Rightarrow \quad (a + 2)x^2 - 2ax + 3a < 0$

Thus, $\quad (a + 2) < 0$ and $4a^2 - 12a(a + 2) < 0$

$\Rightarrow \quad a < -2$ and $a^2 - 3a(a + 2) < 0$

$\Rightarrow \quad a < -2 \ \& \ a(a + 3) > 0$

$\Rightarrow \quad a \in (-\infty, -2) \ \& \ a \in (-\infty, -3) \cup (0, \infty)$

Thus, $\quad a \in (-\infty, -3)$

16. Given $\quad f(x) = \dfrac{e^x}{x - 1}$

$\Rightarrow \quad f'(x) = \dfrac{(x - 1)e^x - e^x \cdot 1}{(x - 1)^2} = \dfrac{e^x(x - 2)}{(x - 1)^2}$

Also, $\quad D_f = R - \{1\}$

Since $x = 1$ is not an interior point in the domain of f, so, $x = 1$ is not a critical point of f.

Thus, the critical point of f is $x = 2$.

17. Given $f(x) = \dfrac{5x^2 - 18x + 45}{x^2 - 9}$

$\Rightarrow \quad f'(x) = \dfrac{(x^2 - 9)(10x - 18) - (5x^2 - 18x + 45) \cdot 2x}{(x^2 - 9)^2}$

$\Rightarrow \quad f'(x) = \dfrac{18(x^2 - 10 + 9)}{(x^2 - 9)^2}$

$\Rightarrow \quad f'(x) = \dfrac{18(x - 1)(x - 9)}{(x^2 - 9)^2}$

Also, $\quad D_f = R - \{-3, 3\}$

Since $x = -3, 3$ are not an interior point of the domain of f, so $x = -3, 3$ are not the critical point of f.

Thus, the critical points of f are $x = 1$ and 9.

18. Given $\quad f(x) = x^{4/5}(x - 4)^2$

$\Rightarrow \quad f'(x) = 2x^{4/5}(x - 4) + \dfrac{4}{5x^{1/5}}(x - 4)^2$

$\Rightarrow \quad f'(x) = \dfrac{10x(x - 4) + 4(x - 4)^2}{5x^{1/5}}$

$\Rightarrow \quad f'(x) = \dfrac{(x - 4)(10x + 4x - 16)}{5x^{1/5}}$

$\Rightarrow \quad f'(x) = \dfrac{2(x - 4)(7x - 8)}{5x^{1/5}}$

Also, $\quad D_f = R$

Thus, the critical points of f are $x = 0, \dfrac{8}{7}, 4$

19. Given $\quad f(x) = x + \cos^{-1}x + 1$

$\Rightarrow \quad f'(x) = \dfrac{1 - 1}{\sqrt{1 - x^2}}$

Now, $\quad f'(x) = 0 \Rightarrow 1 - \dfrac{1}{\sqrt{1 - x^2}} = 0$

$\Rightarrow \quad \dfrac{1}{\sqrt{1 - x^2}} = 1$

$\Rightarrow \quad x = 0.$

Also, $\quad D_f = [-1, 1]$

Thus, the critical points of 'f' is $x = 0$.

20. Given $\quad f(x) = \sqrt{x^2 - 6x + 15}$

$\Rightarrow \quad f'(x) = \dfrac{1}{2\sqrt{x^2 - 6x + 15}} \times (2x - 6)$

$\Rightarrow \quad f'(x) = \dfrac{1}{2\sqrt{x^2 - 6x + 15}} \times 2(x - 3)$

$\Rightarrow \quad f'(x) = \dfrac{(x - 3)}{\sqrt{x^2 - 6x + 15}}$

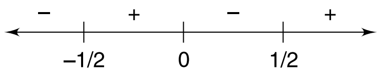
Also $\quad D_f = R$

Thus, the critical points of f is $x = 3$.

21. Given $\quad f(x) = 2x^2 - \ln|x|$

$\Rightarrow \quad f'(x) = 4x - \dfrac{1}{x} = \dfrac{4x^2 - 1}{x}$

$= \dfrac{(2x - 1)(2x + 1)}{x}.$

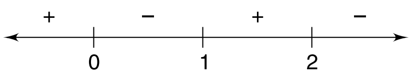


By the sign scheme for $f'(x)$, we have, $f(x)$ is increasing in $\left[-\frac{1}{2}, 0\right) \cup \left[\frac{1}{2}, \infty\right)$ and decreasing in $\left(-\infty, -\frac{1}{2}\right]$ $\cup \left(0, \frac{1}{2}\right]$.

22. Given $f(x) = |x - 1|/x^2$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{x-1}{x^2} = \dfrac{1}{x} - \dfrac{1}{x^2} & : x \ge 1 \\ \dfrac{1-x}{x^2} = \dfrac{1}{x^2} - \dfrac{1}{x} & : x < 1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -\dfrac{1}{x^2} + \dfrac{2}{x^3} = \dfrac{2-x}{x^3} & : x \ge 1 \\ -\dfrac{2}{x^3} + \dfrac{1}{x^2} = \dfrac{x-2}{x^3} & : x < 1 \end{cases}$$



By the sign scheme for the function $f'(x)$, we have $f(x)$ is increasing in $(-\infty, 0) \cup [1, 2]$ and decreases in $(0, 1] \cup [2, \infty)$.

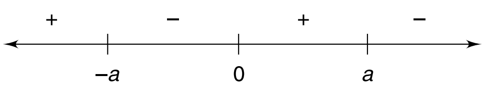
23. Given $f(x) = x^2 e^{-x^2/a^2}$, $a > 0$

$$\Rightarrow \quad f'(x) = 2x e^{-x^2/a^2} + x^2 \cdot e^{-x^2/a^2} \times \left(-\frac{2x}{a^2}\right)$$

$$= 2x e^{-x^2/a^2}\left(1 - \frac{x^2}{a^2}\right)$$

$$= -2x e^{-x^2/a^2}\left(\frac{(x-a)(x+a)}{a^2}\right)$$

$$= \left(-\frac{2}{a^2}\right) e^{-x^2/a^2} \times x(x-a)(x+a)$$



Now, by the sign scheme for the function $f'(x)$, we have $f(x)$ is increases in $(-\infty, -a] \cup [0, a]$

24. Given $x^3 = 3x + 1$

Let $f(x) = x^3 - 3x - 1$

$\Rightarrow \quad f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$\Rightarrow \quad f'(x) = 3(x^2 - 1) < 0, \ \forall \ x \in (-1, 1)$

Thus $f(x)$ is strictly decreases in $(-1, 1)$

Now, $f(-1) = -1 + 3 - 1 = 1 > 0$

and $f(1) = 1 - 3 - 1 = -2 < 0$

Thus, the curve $y = f(x) = x^3 - 3x - 1$ will cut the x-axis exactly one point in $(-1, 1)$

25. Let $f(x) = e^x - 1 - x - \dfrac{x^2}{2}$

$\Rightarrow \quad f'(x) = e^x - 1 - x < 0, \ \forall \ x \in (-1, 1)$

Thus, $f(x)$ is strictly decreases in $(-1, 1)$

Now, $f(-1) = \dfrac{1}{e} - 1 + 1 - \dfrac{1}{2} = \dfrac{1}{e} - \dfrac{1}{2} < 0$

and $f(1) = e - 1 - 1 - \dfrac{1}{2} = e - \dfrac{5}{2} > 0$

Thus, the equation $e^x = 1 + x + \dfrac{x^2}{2}$ has a real root in $(-1, 1)$

26. As we know that $\tan^{-1}x$ & e^x, both are strictly increasing for all x in R.
Therefore $f(x) = \tan^{-1}(e^x)$ is strictly increasing for all x in R.

27. As we know that $\tan^{-1}x$ is strictly increasing for all x in R and $(\log_{1/3}x)$ is strictly decreasing for all $x \in R^+$.
Therefore, $f(x) = \tan^{-1}(\log_{1/3}x)$ is strictly decreasing for all $x \in R$.

28. As we know that $\cot^{-1}x$ is strictly decreasing for all x in R and $(\log_4 x)$ is increasing for all $x \in R^+$.
Therefore, $f(x) = \cot^{-1}(\log_4 x)$ is strictly decreasing for all $x \in R^+$.

29. As we know that $(\log_{1/10}x)$ is strictly decreasing for all $x \in R^+$ and $(\cot^{-1}x)$ is strictly decreasing for all x.
Thus $f(x) = \cot^{-1}(\log_{1/10}x)$ is strictly increasing for all $x > 0$.

30. Let $f(x) = 3x - x^2$ and $g(x) = \sqrt{x}$

Now, $f'(x) = 3 - 2x$

By the sign scheme, f is strictly inc. in $\left(-\infty, \frac{3}{2}\right)$ and strictly decreasing in $\left(\frac{3}{2}, \infty\right)$.

Also, g is srictly increasing in $[0, \infty)$.

Now, $D_f = [0, 3]$

Thus, the function $y = \sqrt{3x - x^2}$ is strictly increasing in $\left(0, \frac{3}{2}\right)$ and strictly decreasing in $\left(\frac{3}{2}, 2\right)$.

31. Let $g(x) = \tan^{-1}$ and $h(x) = (\sin x + \cos x)$

Now, $h'(x) = (\cos x - \sin x)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$$

$h(x)$ is strictly increasing if $h'(x) > 0$

$\Rightarrow \quad \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad 0 < \left(x + \frac{\pi}{4}\right) < \frac{\pi}{2}$ & $\frac{3\pi}{2} < \left(x + \frac{\pi}{4}\right) < 2\pi$

$\Rightarrow \quad -\frac{\pi}{4} < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < \frac{7\pi}{4}$

Thus, the given function $f(x)$ is strictly increases in $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$ and strictly decreasing in $\left(0, \frac{\pi}{4}\right)$ $\cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$.

32. Let $g(x) = \frac{\log x}{x}$ and $h(x) = \log x$

Now, $g'(x) = \frac{1 - \log x}{x^2}$

By the sign scheme, $g(x)$ is strictly increasing in $(0, e)$ and strictly decreasing in (e, ∞)

Also, $h(x)$ is strictly increasing for all $x > 0$.

Also, $D_f = (1, \infty)$

Thus, $f(x)$ is strictly increasing in $(1, e)$ and strictly decreasing in (e, ∞)

33. Let $g(x) = \sin x + \cos x$ and $h(x) = \log x$

Since $h(x)$ is an increasing function, $f(x)$ will be decreases if $g(x)$ decreases.

Now, $g'(x) = \cos x - \sin x$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \sqrt{2}\left(\cos\left(x + \frac{\pi}{4}\right)\right)$$

Since $g(x)$ decreases, so $g'(x) < 0$

$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) < 0$

$\Rightarrow \quad 2n\pi + \frac{\pi}{2} < \left(x + \frac{\pi}{4}\right) < 2n\pi + \frac{3\pi}{2}$

$\Rightarrow \quad 2n\pi + \frac{\pi}{4} < x < 2n\pi + \frac{5\pi}{4}$

$\Rightarrow \quad 2n\pi + \frac{\pi}{4} < \log x < 2n\pi + \frac{5\pi}{4}$

$\Rightarrow \quad e^{\left(2n\pi + \frac{\pi}{4}\right)} < x < e^{\left(2n\pi + \frac{5\pi}{4}\right)}$

34. Let $g(x) = \log_e x$ and $h(x) = \cos x$

Here, $g(x)$ is strictly increases for all $x > 0$

Also, $h(x)$ is strictly decreases in $(0, \pi)$

Again, for the domain of the function, $\cos x > 0$

$\Rightarrow \quad x \in \left(0, \frac{\pi}{2}\right)$

Therefore, the function $f(x)$ is strictly decreases in $\left(0, \frac{\pi}{2}\right)$.

35. Let $g(x) = \sin x + \cos x$ and $h(x) = \sin x$

Now, $g'(x) = \cos x - \sin x$

$$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$$

When $g'(x) > 0$. $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) > 0$

$\Rightarrow \quad -\frac{\pi}{2} < \left(x + \frac{\pi}{4}\right) < \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2} - \frac{\pi}{4} < x < \frac{\pi}{2} - \frac{\pi}{4}$

$\Rightarrow \quad -\frac{3\pi}{2} < x < \frac{\pi}{4}$

When $g'(x) < 0$, $\frac{\pi}{2} < \left(x + \frac{\pi}{4}\right) < \frac{3\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} - \frac{\pi}{4} < x < \frac{3\pi}{2} - \frac{\pi}{4}$

$\Rightarrow \quad \frac{\pi}{4} < x < \frac{5\pi}{4}$

Thus, $f(x)$ is strictly increases in $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing in $\left(\frac{\pi}{4}, \pi\right)$.

36. Let $f(x) = \log(1 + x) - x + \frac{x^2}{2}$

$\Rightarrow \quad f'(x) = \frac{1}{1 + x} - 1 + x$

$\Rightarrow \quad f'(x) = \frac{1}{1 + x} - (1 - x)$

$\Rightarrow \quad f'(x) = \frac{1 - 1 + x^2}{1 + x} = \frac{x^2}{x + 1} > 0 \ \forall \ x \in R^+$

$\Rightarrow$ $f(x)$ is strictly increasing in $(0, \infty)$

Thus $f(x) > f(0)$

$\Rightarrow$ $\log(1 + x) - x + \frac{x^2}{2} > 0$

$\Rightarrow$ $\log(1 + x) > x - \frac{x^2}{2}$

Hence, the result.

37. Consider $f(x) = \log(1 + x) - \frac{x}{x + 1}$

$\Rightarrow$ $f'(x) = \frac{1}{1 + x} - \frac{1}{(x + 1)^2}$

$\Rightarrow$ $f'(x) = \frac{x + 1 - 1}{(x + 1)^2} = \frac{x}{(x + 1)^2}$

$\Rightarrow$ $f'(x) > 0$ for all $x > 0$

Thus, $f(x)$ is strictly increasing

$\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $\log(1 + x) - \frac{x}{x + 1} > 0$

$\Rightarrow$ $\log(1 + x) > \frac{x}{x + 1}$ for all $x > 0$

Hence, the result.

38. Let $f(x) = (1 + x)\log(1 + x) - e^x + 1$

$\Rightarrow$ $f'(x) = \frac{(1 + x)}{(1 + x)} + \log(1 + x)\cdot 1 - e^x$

$\Rightarrow$ $f'(x) = 1 + \log(1 + x) - e^x$

$\Rightarrow$ $f'(x) < 0$ for all $x < 0$

Thus, $f(x)$ is strictly decreasing function

$\Rightarrow$ $f'(x) < f(0)$

$\Rightarrow$ $(1 + x)\log(1 + x) - e^x + 1 < 0$

$\Rightarrow$ $(1 + x)\log(1 + x) < e^x - 1$

Hence, the result.

39. We have $f(x) = 2x\tan^{-1}x - \log(1 + x^2)$

$\Rightarrow$ $f'(x) = 2\tan^{-1}x + \frac{2x}{1 + x^2} - \frac{2x}{1 + x^2}$

$\Rightarrow$ $f'(x) = 2\tan^{-1}x > 0$ for all x in R^+

Thus, $f(x)$ is strictly increasing in R^+

$\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $2x\tan^{-1}x - \log(1 + x^2) > 0$

$\Rightarrow$ $2x\tan^{-1}x > \log(1 + x^2)$

Hence, the result.

40. Let $f(x) = 1 + x\log\left(x + \sqrt{x^2 + 1}\right) - \sqrt{1 + x^2}$

$f'(x) = \log\left(x + \sqrt{x^2 + 1}\right) + \left(\frac{x}{x + \sqrt{x^2 + 1}}\right)$

$\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) - \frac{x}{\sqrt{1 + x^2}}$

$f'(x) = \log\left(x + \sqrt{x^2 + 1}\right) + \frac{x}{\sqrt{1 + x^2}}$

$- \frac{x}{\sqrt{1 + x^2}}$

$f'(x) = \log\left(x + \sqrt{x^2 + 1}\right)$

$f'(x) = \log\left(x + \sqrt{x^2 + 1}\right) \geq 0, \forall x \geq 0$

Thus, $f(x)$ is increasing in $[0, \infty)$

$f(x) \geq f(0)$

$1 + x\log\left(x + \sqrt{x^2 + 1}\right) - \sqrt{x^2 + 1} \geq 0$

$\Rightarrow$ $1 + x\log\left(x + \sqrt{x^2 + 1}\right) \geq \sqrt{x^2 + 1}$

Hence, the result.

41. Let $f(x) = x - \sin x$

$\Rightarrow$ $f'(x) = 1 - \cos x > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$

Thus, $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ $f(x) > f(0)$

$\Rightarrow$ $x - \sin x > 0$

$\Rightarrow$ $x > \sin x$

$\Rightarrow$ $\cos x < \cos(\sin x)$ ...(i)

Also, for all x in $\left(0, \frac{\pi}{2}\right)$, $0 < \cos x < 1$

$\Rightarrow$ $\cos x < 1$

$\Rightarrow$ $\cos x > \sin(\cos x)$ ...(ii)

From (i) and (ii), we get,

$\sin(\cos x) < \cos x < \cos(\sin x)$

42. Given $f(x) = \log x - Bx^2$

$\Rightarrow$ $f'(x) = \frac{1}{x} - 2Bx = \frac{1 - 2Bx^2}{x}$

The critical points of 'f' are

$x = 0, \frac{1}{\sqrt{2B}}, -\frac{1}{\sqrt{2B}}$

Now, $f'(x) > 0, \forall x \in \left(0, \frac{1}{\sqrt{2B}}\right)$

and $\quad f'(x) < 0,\ \forall x \in \left(\frac{1}{\sqrt{2B}}, \infty\right)$

Now, $\quad \log x < Bx^2$ for $x > 0$

It holds good for $x = \frac{1}{\sqrt{2B}}$

Thus $\quad \log\left(\frac{1}{\sqrt{2B}}\right) < B,\ \frac{1}{2B} = \frac{1}{2}$

$\Rightarrow \quad -\log(\sqrt{2B}) < \frac{1}{2}$

$\Rightarrow \quad \log(\sqrt{2B}) < -\frac{1}{2}$

$\Rightarrow \quad \sqrt{2B} < e^{-\frac{1}{2}}$

$\Rightarrow \quad 2B < e^{-1}$

$\Rightarrow \quad B > \frac{1}{2e}$

Thus, the least value of B is $\frac{1}{2e}$

43. Let $\quad f(x) = ax^2 + \frac{b}{x} - c$

$\Rightarrow \quad f'(x) = 2ax - \frac{b}{x^2}$

Now, $\quad f'(x) = 0$ gives $2ax - \frac{b}{x^2} = 0$

$\Rightarrow \quad 2ax^3 = b$

$\Rightarrow \quad x = \left(\frac{b}{2a}\right)^{1/3}$

Thus, the least value of $f(x)$ occurs at $x = \left(\frac{b}{2a}\right)^{1/3}$

we have $a\left\{\frac{b}{2a}\right\}^{2/3} + \frac{b}{\left\{\frac{b}{2a}\right\}^{1/3}} \geq c$

$\Rightarrow \quad a\left(\frac{b}{2a}\right) + b \geq c \cdot \left(\frac{b}{2a}\right)^{1/3}$

$\Rightarrow \quad \left(\frac{3b}{2}\right)^3 \geq \frac{b}{2a} \cdot c^2$

$\Rightarrow \quad \frac{27b^3}{8} \geq \frac{b}{2a} \cdot c^2$

$\Rightarrow \quad 27b^2 a \geq 4c^3$

$\Rightarrow \quad 27ab^2 \geq 4c^3$

Hence, the result.

44. We have $f(x) = x^5 + 5x - 6$

$\Rightarrow \quad f'(x) = 5x^4 + 5$

$\Rightarrow \quad f''(x) = 20x^3$

$\Rightarrow \quad f'''(x) = 60x^2$ is exists for all x

Now, $\quad f''(x) = 0$ gives $x = 0$

By the sign scheme for $f''(x) = 0$, we have, $f(x)$ is concave down in $(-\infty, 0)$ and concave up in $(0, \infty)$.

45. We have $f(x) = x^4 - 5x^3 - 15x^2 + 30$

$\Rightarrow \quad f'(x) = 4x^3 - 15x^2 - 30x$

$\Rightarrow \quad f''(x) = 12x^2 - 30x - 30$

Now, $\quad f''(x) = 0$ gives $12x^2 - 30x - 30 = 0$

$\Rightarrow \quad 6x^2 - 15x - 15 = 0$

$\Rightarrow \quad 2x^2 - 5x - 5 = 0$

$\Rightarrow \quad x = \frac{5 \pm \sqrt{25 + 40}}{4}$

$\Rightarrow \quad x = \frac{5 \pm 8}{4} = \frac{13}{4}, -\frac{3}{4}$

By the sign scheme for the function $f''(x)$, the function $f(x)$ is concave down in $\left(-\frac{3}{4}, \frac{13}{4}\right)$ and concave up in $\left(-\infty, -\frac{3}{4}\right) \cup \left(\frac{13}{4}, \infty\right)$

46. We have $f(x) = (\sin x + \cos x)e^x$

$f'(x) = (\sin x + \cos x)e^x + e^x(\cos x - \sin x)$

$f'(x) = e^x(\sin x + \cos x + \cos x - \sin x)$

$f'(x) = 2e^x \cos x$

$f''(x) = 2(e^x \cos x - e^x \sin x)$

$f''(x) = 2e^x(\cos x - \sin x)$

Now, $\quad f''(x) = 0$ gives $2e^x(\cos x - \sin x) = 0$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$

By the sign scheme for the function $f''(x) = 0$, we have $f(x)$ is concave down in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and concave up in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)$

47. Given curve is $y = f(x) = Ax^2 + Bx + C$

$f'(x) = 2Ax + B$

$f''(x) = 2A$

Thus, the curve concave up if $f''(x) > 0$ and concave down if $f''(x) < 0$ i.e. concave up if $A > 0$ and concave down if $A < 0$.

48. We have $f(x) = x^4 - 4x^3 + x - 10$

$\Rightarrow \quad f'(x) = 4x^3 - 12x^2 + 1$

$\Rightarrow \quad f''(x) = 12x^2 - 24x = 12x(x - 2)$

Now, $f''(x) = 0$ gives $x = 0$ and $x = 2$.

when $x = 0, y = -10$

when $x = 2, y = -24$

Thus, the point of inflection are (0, 10) and (2, –24)

49. We have $y = f(x) = (x - 2)^{2/3} + 10$

$\Rightarrow \quad f'(x) = \frac{2}{3}(x - 2)^{-1/3} = \frac{2}{3(x - 2)^{1/3}}$

$\Rightarrow \quad f''(x) = \frac{2}{9(x - 2)^{4/3}}$

Thus, $f''(x)$ does not exist at $x = 2$.

when $x = 2, y = 0 + 10 = 10$

Thus, the inflection point is (2, 10)

50. We have $f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$

$\Rightarrow \quad f'(x) = 4x^3 - 18x^2 + 24x - 8$

$\Rightarrow \quad f''(x) = 12x^2 - 36x + 24$

$= 12(x^2 - 3x + 2)$

$= 12(x - 1)(x - 2)$

Now, $f''(x) = 0$ gives $x = 1, 2$

when $x = 1, y = 2$

when $x = 2, y = 3$

Thus, the point of inflection are (1, 2) and (2, 3).

51. Given $y = f(x) = x^2 - \frac{1}{6x^3}$

$\Rightarrow \quad f'(x) = 2x + \frac{1}{2x^4}$

$\Rightarrow \quad f''(x) = 2 - \frac{2}{x^5}$

Now, $f''(x) = 0$ gives $2 - \frac{2}{x^5} = 0$

Thus, $x = 1$

when $x = 1, y = \frac{5}{6}$

Thus, the point of inflection is $\left(1, \frac{5}{6}\right)$.

52. We have $y = f(x) = e^{-x^2}$

$\Rightarrow \quad f'(x) = e^{-x^2} \times -2x$

$\Rightarrow \quad f'(x) = -2e^{-x^2} \times x$

$\Rightarrow \quad f''(x) = -2\left(e^{-x^2} \cdot 1 + x \cdot e^{-x^2} \cdot (-2x)\right)$

$\Rightarrow \quad f''(x) = 2e^{-x^2}(2x^2 - 1)$

Now, $f''(x) = 0$ gives $2e^{-x^2}(2x^2 - 1) = 0$

$\Rightarrow \quad x = \pm \frac{1}{\sqrt{2}}$

when $x = \frac{1}{\sqrt{2}}, y = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

when $x = -\frac{1}{\sqrt{2}}, y = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$.

Thus, the point of inflection are

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right) \ \& \ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$$

Level III

1. Given $f(x) = 4x - \tan 2x$

$f'(x) = 4 - 2\sec^2 x$

Now, $f'(x) > 0$ gives $4 - 2\sec^2 2x > 0$

$\Rightarrow \quad \sec^2 2x < 2$

$\Rightarrow \quad (\sec 2x + \sqrt{2})(\sec 2x - \sqrt{2}) < 0$

$\Rightarrow \quad -\sqrt{2} < \sec 2x < \sqrt{2}$

$\Rightarrow \quad -\frac{\pi}{4} < 2x < \frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{8} < x < \frac{\pi}{8}$

Thus, the length of the longest interval

$= \left(\frac{\pi}{8} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$

2. We have $x^3 + 2x^2 + 5x + 2\cos x = 0$

Let $f(x) = x^3 + 2x^2 + 5x + 2\cos x$

$\Rightarrow \quad f'(x) = 3x^2 + 4x + 5 - 2\sin x$

Let $g(x) = 3x^2 + 4x + 5$ and $h(x) = 2\sin x$

Max value of $g(x)$ is $-\frac{16 - 60}{6} = \frac{44}{6} = \frac{22}{3}$ and max value of $h(x)$ is 2.

Thus, $f'(x) > 0$

$\Rightarrow \quad f(x)$ is strictly increasing function

Also, $f(0) = 2 > 0$ and $f(2\pi) > 0$

Therefore $f(x)$ has no real root.

3. We have

$$f(x) = (ab - b^2 - 2)x + \int_0^x (\cos^4\theta + \sin^4\theta)\,d\theta$$

$\Rightarrow \quad f'(x) = (ab - b^2 - 2) + (\cos^4 x + \sin^4 x)$

Since $f(x)$ is a decreasing function, so $f'(x) \le 0$

$\Rightarrow \quad (ab - b^2 - 2) + (\cos^4 x + \sin^4 x) \le 0$

$\Rightarrow \quad (ab - b^2 - 2) + 1 \le 0$

$\Rightarrow \quad (ab - b^2 - 2) \le 0$

$\Rightarrow \quad (b^2 - ab + 1) \ge 0$

$\Rightarrow \quad a^2 - 4 \le 0$

$\Rightarrow \quad (a + 2)(a - 2) \le 0$

$\Rightarrow \quad -2 \le a \le 2$

$\Rightarrow \quad a \in [-2, 2]$

4. Given $\quad f(x) = \cos x + a^2 x + b$

$\Rightarrow \quad f'(x) = -\sin x + a^2$

Since $f(x)$ is increasing, so $f'(x) \ge 0$

$\Rightarrow \quad a^2 - \sin x \ge 0$

$\Rightarrow \quad a^2 \ge \sin x$

$\Rightarrow \quad a^2 \ge 1$

$\Rightarrow \quad a^2 - 1 \ge$

$\Rightarrow \quad (a + 1)(a - 1) \ge 0$

$\Rightarrow \quad a \le -1, a \ge 1$

$\Rightarrow \quad a \in (-\infty, -1] \cup [1, \infty)$

5. Given $\quad f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$

$\Rightarrow \quad f'(x) = 2e^x + ae^{-x} + (2a + 1)$

Since $f(x)$ is increasing, so $f'(x) \ge 0$

$\Rightarrow \quad 2e^x + ae^{-x} + (2a + 1) \ge 0$

$\Rightarrow \quad 2(e^x)^2 + (2a + 1)e^x + a \ge 0$

Therefore, $D \le 0$

$\Rightarrow \quad (2a + 1)^2 - 8a \le 0$

$\Rightarrow \quad 4a^2 + 4a + 1 - 8a \le 0$

$\Rightarrow \quad 4a^2 - 4a + 1 \le 0$

$\Rightarrow \quad (2a + 1)^2 \le 0$

$\Rightarrow \quad (2a - 1)^2 = 0$

$\Rightarrow \quad a = \frac{1}{2}$

6. ***

7. We have $f(x) = \int_0^x (bt^2 + t\sin t)\,dt$

$\Rightarrow \quad f'(x) = bx^2 + x\sin x$

$\Rightarrow \quad f'(x) = x^2\left(b + \frac{\sin x}{x}\right)$

Since $f(x)$ is monotonic, so $f'(x) > 0$ or $f'(x) < 0$

$\Rightarrow \quad x^2\left(b + \frac{\sin x}{x}\right) > 0 \;\&\; x^2\left(b + \frac{\sin x}{x}\right) < 0$

$\Rightarrow \quad \left(b + \frac{\sin x}{x}\right) > 0 \;\&\; \left(b + \frac{\sin x}{x}\right) < 0$

Therefore, $b > 1$

$\Rightarrow \quad b \in (-\infty, 1)$

8. We have $f(x) = \sin^4 x + \cos^4 x$

$$\begin{aligned} f'(x) &= 4\sin^3 x\cdot\cos x - 4\cos^3 x\cdot\sin x \\ &= -4\sin x\cdot\cos x(\cos^2 x - \sin^2 x) \\ &= -2(2\sin x\cdot\cos x)(\cos^2 x - \sin^2 x) \\ &= -(2\sin 2x\cdot\cos 2x) \\ &= -\sin 4x \end{aligned}$$

When $f(x)$ is strictly increasing, then $f'(x) > 0$

$\Rightarrow \quad -\sin 4x > 0$

$\Rightarrow \quad \sin 4x < 0$

$\Rightarrow \quad \pi < 4x < 2\pi$

$\Rightarrow \quad \frac{\pi}{2} < x < \frac{\pi}{2}$

Thus, $\quad x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

9. We have $f(x) = xe^{x(1-x)}$

$\Rightarrow \quad f'(x) = e^{x(1-x)} + xe^{x(1-x)} \times (1 - 2x)$

$\Rightarrow \quad f'(x) = (1 + x(1 - 2x))e^{x(1-x)}$

$\Rightarrow \quad f'(x) = (1 + x - 2x^2)e^{x(1-x)}$

$$\begin{aligned} &= -(2x^2 - x - 1)e^{x(1-x)} \\ &= -(2x + 1)(x - 1)e^{x(1-x)} \end{aligned}$$

By the sign scheme for $f'(x)$, the function $f(x)$ is increasing $\left[-\frac{1}{2}, 1\right]$ and decreasing in $\left(-\infty, -\frac{1}{2}\right)$ $[1, \infty)$

10. Given $\quad f(x) = \frac{\log(\pi + x)}{\log(e + x)}$

$$\Rightarrow \quad f'(x) = \frac{\log(e+x)\frac{1}{(\pi+x)} - \log(\pi+x)\frac{1}{(e+x)}}{\{\log(e+x)\}^2}$$

$$= \frac{(e+x)\log(e+x) - (\pi+x)\log(\pi+x)}{(e+x)(\pi+x)\{\log(e+x)\}^2}$$

Since $1 < e < \pi$

$(1 + x) < (e + x) < (\pi + x)$ ...(i)

Also, $\log(e + x) < \log(\pi + x)$ ...(ii)

Multiplying (i) and (ii), we get,

$(e + x)\log(e + x) < (\pi + x)\log(\pi + x)$

Thus, $f'(x) < 0$

Therefore, $f(x)$ is decreasing in $(0, \infty)$.

11. Given, $f(x) = e^{-ax} + e^{ax}, a > 0$

$\Rightarrow \quad f'(x) = -a e^{-ax} + a e^{ax}$

$\Rightarrow \quad f'(x) = a(e^{ax} - e^{-ax})$

Since $f(x)$ is monotonic increasing, so $f'(x) > 0$

$\Rightarrow \quad a(e^{ax} - e^{-ax}) > 0$

$\Rightarrow \quad a\left(\frac{e^{2ax} - 1}{e^{ax}}\right) > 0$

$\Rightarrow \quad e^{2ax} - 1 > 0$

$\Rightarrow \quad e^{2ax} > e^0$

$\Rightarrow \quad 2ax > 0$

$\Rightarrow \quad x > 0$

Hence, the required interval $(0, \infty)$.

12. We have $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$

$\Rightarrow \quad f'(x) = 3x^2 + 2(a + 2)x + 3a$

Since $f(x)$ is invertible, so $f(x)$ is bijective function
Therefore, f is one-one function
Thus, we can conclude that $f(x)$ is strictly increasing or strictly decreasing.
Here, the co-efficient of x^2 is 3, so $f(x)$ is strictly increasing.

Therefore, $f'(x) > 0$

$\Rightarrow \quad 3x^2 + 2(a + 2)x + 3a > 0$

So, $D < 0$

$\Rightarrow \quad 4(a + 2)^2 - 36a < 0$

$\Rightarrow \quad (a + 2)^2 - 9a < 0$

$\Rightarrow \quad a^2 + 4a + 4 - 9a < 0$

$\Rightarrow \quad a^2 - 5a + 4 < 0$

$\Rightarrow \quad (a - 1)(a - 4) < 0$

$\Rightarrow \quad 1 < a < 4$

Thus, $a \in (1, 4)$

13. Given $f(x) = 2x^2 - kx + 5$

$\Rightarrow \quad f'(x) = 4x - k$

$f''(x) = 4$

Now, $f''(x) = 4 > 0 \ \forall \ x \in [1, 2]$

Thus, $f'(1)$ is the least value of $f(x)$.

Now, $f'(1) > 0$ gives $4 - k > 0$

$\Rightarrow \quad k < 4$

$\Rightarrow \quad k \in (-\infty, 4)$

14. We have $f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\,dt$

$\Rightarrow \quad f'(x) = (x^2 + 2x)(x^2 - 1)$

$\Rightarrow \quad f'(x) = x(x + 2)(x + 1)(x - 1)$

By the sign scheme, we get, $f(x)$ is strictly increasing in $(-\infty, -2) \cup (-1, 0) \cup (1, \infty)$ and strictly decreasing in $(-2, -1) \cup (0, 1)$

15. Given $f(x) = \begin{cases} xe^{ax} & : x \le 0 \\ x + ax^2 - x^3 & : x > 0 \end{cases}$

Clearly f is continuous at $x = 0$

$$\Rightarrow \quad f'(x) = \begin{cases} (ax+1)e^{ax} & : x < 0 \\ 1 & : x = 0 \\ 1 + 2ax - 3x^2 & : x > 0 \end{cases}$$

$$\Rightarrow \quad f''(x) = \begin{cases} a(ax+2)e^{ax} & : x < 0 \\ 2a & : x = 0 \\ 2a - 6x & : x > 0 \end{cases}$$

Now, for $x < 0, f''(x) > 0$

$\Rightarrow \quad ax + 2 > 0$

$\Rightarrow \quad x > -\frac{2}{a}$

$\Rightarrow \quad f'(x)$ increases on $\left[-\frac{2}{a}, 0\right]$

Also, $x > 0, f''(x) > 0$

$\Rightarrow \quad 2a - 6x > 0$

$\Rightarrow \quad x > \frac{a}{3}$

Thus, $f'(x)$ increases on $\left[0, \frac{a}{3}\right]$

Hence, $f'(x)$ increases on $\left[-\frac{2}{a}, \frac{a}{3}\right]$.

16. We have

$$f(x) = \int_0^x |\log_2(\log_3(\log_4(\cos t + a)))|\, dt$$

$$f'(x) = |\log_2(\log_3(\log_4(\cos x + a)))|$$

Since $f(x)$ is increasing, so $f(x)$ is defined for all x.

$$(\log_3(\log_4(\cos x + a))) > 0$$

$$(\log_4(\cos x + a)) > 1$$

$$(\cos x + a) > 4$$

$$a > 4 - \cos x$$

$$a > 5 \text{ for all } x \text{ in } R$$

17. We have $f(x) = x^3 - 3x + a$

$\Rightarrow$ $f'(x) = 3x^2 - 3$

$= 3(x - 1)(x + 1)$

Now, $f(x) = 0$ will have only one root if

$$f(1)f(-1) > 0$$

$\Rightarrow$ $(a - 2)(a + 2) > 0$

$\Rightarrow$ $a < -2, a > 2$

$\Rightarrow$ $a \in (-\infty, -2) \forall (2, \infty)$

18. We have $f(x) = x^3 - 3x + a$

$\Rightarrow$ $f'(x) = 3x^2 - 3$

Now, $f'(x) = 0$ gives $3x^2 - 3 = 0$

$\Rightarrow$ $x^2 - 1 = 0$

$\Rightarrow$ $x^2 = 1$

$\Rightarrow$ $x = \pm 1$

It has three roots only when

$$f(-1)f(1) < 0$$

$\Rightarrow$ $(a + 2)(a - 2) < 0$

$\Rightarrow$ $-2 < a < 2$

19. We have

$$f(x) = \int_0^x \left\{(a-1)(t^2+t+1)^2 - (a+1)(t^4+t^2+1)\right\} dt$$

$\Rightarrow$ $f'(x) = \left\{(a-1)(x^2+x+1)^2 - (a+1)(x^4+x^2+1)\right\}$

Now, $f'(x) = 0$

$\Rightarrow$ $\left\{(a-1)(x^2+x+1)^2 - (a+1)(x^4+x^2+1)\right\} = 0$

$\Rightarrow$ $\left\{(a-1)(x^2+x+1) - (a+1)(x^2-x+1)\right\} = 0$

$\Rightarrow$ $(a-1-a-1)x^2 + (a-1+a+1)x + (a-1-a-1) = 0$

$\Rightarrow$ $-2x^2 + 2ax - 2 = 0$

$\Rightarrow$ $x^2 - ax + 1 = 0$

It has two distinct roots only when, $D \geq 0$

$\Rightarrow$ $a^2 - 4 \geq 0$

$\Rightarrow$ $(a + 2)(a - 2) \geq 0$

$\Rightarrow$ $a \leq -2, a \geq 2$

$\Rightarrow$ $a \in (-\infty, 2] \cup [2, \infty)$

20. We have $y = f(x) = \dfrac{ae^x + be^{-x}}{ce^x + de^{-x}}$

$$= \frac{(ce^x + de^{-x})(ae^x - be^{-x}) - (ce^x - de^{-x})(ae^x + be^{-x})}{(ce^x + de^{-x})^2}$$

$$= \frac{2(ad - bc)}{(ce^x + de^{-x})^2}$$

Since $f(x)$ is increasing function, so $f'(x) \geq 0$

$\Rightarrow$ $\dfrac{2(ad - bc)}{(ce^x + de^{-x})^2} \geq 0$

$\Rightarrow$ $(ad - bc) \geq 0$

$\Rightarrow$ $ad \geq bc$

21. We have

$$f(x) = \begin{vmatrix} x + a^2 & ab & ac \\ ab & x + b^2 & bc \\ ac & bc & x + c^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a(x + a^2) & a^2b & a^2c \\ ab^2 & b(x + b^2) & b^2c \\ ac^2 & bc^2 & c(x + c^2) \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} (x + a^2) & a^2 & a^2 \\ b^2 & (x + b^2) & b^2 \\ c^2 & c^2 & (x + c^2) \end{vmatrix}$$

$$= \begin{vmatrix} (x + a^2) & a^2 & a^2 \\ b^2 & (x + b^2) & b^2 \\ c^2 & c^2 & (x + c^2) \end{vmatrix}$$

$$= \begin{vmatrix} (x+a^2+b^2+c^2) & (x+a^2+b^2+c^2) & (x+a^2+b^2+c^2) \\ b^2 & (x + b^2) & b^2 \\ c^2 & c^2 & (x + c^2) \end{vmatrix}$$

$$= (x + a^2 + b^2 + c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & (x+b^2) & b^2 \\ c^2 & c^2 & (x+c^2) \end{vmatrix}$$

$$= (x + a^2 + b^2 + c^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & x & 0 \\ c^2 & 0 & x \end{vmatrix}$$

$= (x + a^2 + b^2 + c^2)x^2$

$\Rightarrow \quad f(x) = (x + a^2 + b^2 + c^2)x^2$

$\Rightarrow \quad f'(x) = 2(x + a^2 + b^2 + c^2)x + x^2$

$\Rightarrow \quad f'(x) = x\{3x + 2(a^2 + b^2 + c^2)\}$

Now, $\quad f'(x) = 0$ gives

$\Rightarrow \quad x\{3x + 2(a^2 + b^2 + c^2)\} = 0$

$\Rightarrow \quad x = 0, -\dfrac{2(a^2 + b^2 + c^2)}{3}$

Thus, $f(x)$ increasing in

$$\left(-\infty, -\frac{2(a^2 + b^2 + c^2)}{3}\right] \cup [0, \infty)$$

and decreasing in $\left[-\dfrac{2(a^2 + b^2 + c^2}{3}, 0\right]$

Level III (Tougher Problems for JEE-Advanced)

1. We have $f(x) = 1 - x - x^3$

$\Rightarrow \quad f'(x) = -1 - 3x^2 < 0, \forall\ x \in R$

$f(x)$ is strictly decreasing

Now, $\quad f(f(x)) = 1 - f(x) - (f(x))^3$

$\Rightarrow \quad f(f(x)) > f(1 - 5x)$

$\Rightarrow \quad f(x) < (1 - 5x)$

$\Rightarrow \quad 1 - x - x^3 < (1 - 5x)$

$\Rightarrow \quad x^3 - 4x > 0$

$\Rightarrow \quad x(x^2 - 4) > 0$

$\Rightarrow \quad x(x + 2)(x - 2) > 0$

$\Rightarrow \quad x \in (-2, 0) \cup (2, \infty)$

2. We have $x^2 f'(x) + 2xf(x) - x + 1 = 0$

Put $\quad y = f(x)$

Thus, $\quad x^2\dfrac{dy}{dx} + 2xy = x - 1$

$\Rightarrow \quad \dfrac{d}{dx}(x^2 y) = x - 1$

On integration, we get,

$$x^2 y = \frac{x^2}{2} - x + c$$

when $\quad x = 1, y = 0$, then $c = 1/2$

Thus, $\quad x^2 y = \dfrac{x^2}{2} - x + \dfrac{1}{2}$

$\Rightarrow \quad y = \dfrac{1}{2} - \dfrac{1}{x} + \dfrac{1}{2x^2}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{x^2} - \dfrac{1}{x^3} = \dfrac{x-1}{x^3}$

So, by the sign scheme, $f(x)$ is increasing in $x \in (-\infty, 0) \cup (1, \infty)$ and decreasing in $x \in (0, 1)$

3. Since $f(x)$ be a decreasing function, we have

$$\log_{1/3}(\log_3(\sin x + a)) < 0$$

$\Rightarrow \quad (\log_3(\sin x + a)) > 1$

$\Rightarrow \quad (\sin x + a) > 3$

$\Rightarrow \quad a > 3 - \sin x$

$\Rightarrow \quad a > 4$ for all x in R.

4. Since $f(x)$ be an increasing function, we have

$$\log_3(\log_{1/4}(\cos x + b)) > 0$$

$\Rightarrow \quad (\log_{1/4}(\cos x + b)) > 1$

$\Rightarrow \quad 0 < (\cos x + b) < \dfrac{1}{4}$ for all x in R.

$\Rightarrow \quad -\cos x < b < \dfrac{1}{4} - \cos x$

$\Rightarrow \quad 1 < b < \dfrac{1}{4} - 1$

$\Rightarrow \quad 1 < b < -\dfrac{3}{4}$

which is not possible.

So, $\quad b = \varphi$

5. We have $f''(x) > 0$

$\Rightarrow \quad f'(x)$ is increasing for all x in R

Also, $\quad g(x) = f(2 - x) + f(4 + x)$

$\Rightarrow \quad g'(x) = -f'(2 - x) + f'(4 + x)$

Since $g(x)$ is increasing, so $g'(x) > 0$

$\Rightarrow \quad -f'(2 - x) + f'(4 + x) > 0$

$\Rightarrow \quad f'(4 + x) > f'(2 - x)$

$\Rightarrow \quad (4 + x) > (2 - x)$

$\Rightarrow \quad 2x > -2$

$\Rightarrow \quad x > -1$

$\Rightarrow \quad x \in (-1, \infty)$

6. Find the set of all real values of a for which

$$f(x) = \left(1 - \frac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^3 + 5x + \log_e 17$$

inreases for all real x in R.

6. We have

$$f(x) = 1 - \left(\frac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^3 + 5x + \log_e 17$$

$$\Rightarrow \quad f'(x) = 3\left(1 - \frac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^2 + 5$$

Since $f(x)$ is increasing, so $f'(x) \geq 0$

$$\Rightarrow \quad 3\left(1 - \frac{\sqrt{21 - 4a - a^2}}{a + 1}\right)x^2 + 5 \geq 0$$

$$\Rightarrow \quad \left(1 - \frac{\sqrt{21 - 4a - a^2}}{a + 1}\right) \leq -\frac{5}{3x^2}$$

$$\Rightarrow \quad \left(\frac{\sqrt{21 - 4a - a^2}}{a + 1}\right) \leq \frac{5}{3x^2}$$

$$\Rightarrow \quad \left(\frac{\sqrt{21 - 4a - a^2}}{a + 1} - 1\right) \leq 0$$

$$\Rightarrow \quad \left(\frac{\sqrt{21 - 4a - a^2}}{a + 1}\right) \leq 1$$

Case I: when $a + 1 > 0$

$\Rightarrow \quad \sqrt{21 - 4a - a^2} \leq a + 1$

$\Rightarrow \quad 21 - 4a - a^2 \leq (a + 1)^2$

$\Rightarrow \quad 21 - 4a - a^2 \leq a^2 + 2a + 1$

$\Rightarrow \quad 2a^2 + 6a - 20 \geq 0$

$\Rightarrow \quad a^2 + 3a - 10 \geq 0$

$\Rightarrow \quad (a + 5)(a - 2) \geq 0$

$\Rightarrow \quad a \leq -5, a \geq 2$

Also, $\quad 21 - 4a - a^2 \geq 0$

$\Rightarrow \quad a^2 + 4a - 21 \leq 0$

$\Rightarrow \quad (a + 7)(a - 3) \leq 0$

$\Rightarrow \quad -7 \leq a \leq 3$

Thus, $\quad a \in [2, 3]$ ...(i)

Case II: When $a + 1 < 0$

Then $\quad 21 - 4a - a^2 \geq 0$

$\Rightarrow \quad a^2 + 4a - 21 \leq 0$

$\Rightarrow \quad (a + 7)(a - 3) \leq 0$

$\Rightarrow \quad -7 \leq a \leq 3$

Thus, $\quad -7 \leq a < -1$ ...(ii)

From (i) and (ii), we get,

$$a \in [-7, -1) \cup [2, 3]$$

7. We have

$$f(x) = \left(\frac{\sqrt{a + 4}}{1 - a} - 1\right)x^5 - 3x + \log_e 5$$

$$\Rightarrow \quad f'(x) = 5\left(\frac{\sqrt{a + 4}}{1 - a} - 1\right)x^4 - 3$$

Since $f(x)$ decreases for all x in R, so

$f'(x) \leq 0$

$$\Rightarrow \quad 5\left(\frac{\sqrt{a + 4}}{1 - a} - 1\right)x^4 - 3 \leq 0$$

$$\Rightarrow \quad \left(\frac{\sqrt{a + 4}}{1 - a} - 1\right) \leq \frac{3}{5x^4}$$

It is possible only when

$$\left(\frac{\sqrt{a + 4}}{1 - a} - 1\right) \leq 0$$

$$\Rightarrow \quad \frac{\sqrt{a + 4}}{1 - a} \leq 1$$

Case I: when $1 - a > 0$

$\Rightarrow \quad \sqrt{a + 4} \leq 1 - a$

$\Rightarrow \quad (a + 4) \leq (1 - a)^2$

$\Rightarrow \quad (a + 4) \leq 1 - 2a + a^2$

$\Rightarrow \quad a^2 - 3a - 3 \geq 0$

$$\Rightarrow \quad a \leq \frac{3 - \sqrt{21}}{2}, \ a \geq \frac{3 + \sqrt{21}}{2}$$

But $\quad -4 \leq a < 1$

$$\text{Hence, } \quad a \in \left[-4, \frac{3 - \sqrt{21}}{2}\right]$$

Case II: When $a > 1$

Then $\quad \sqrt{a + 4} \geq 1 - a$

which is true for $a > 1$

Hence, the values of a is

$$a \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$$

8. We have

$$f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$$

$$\Rightarrow \quad f'(x) = 2e^x + ae^{-x} + (2a+1)$$

Since $f(x)$ is monotonically increases in R, so

$$\Rightarrow \quad f'(x) \ge 0$$

$$\Rightarrow \quad 2e^x + ae^{-x} + (2a+1) \ge 0$$

$$\Rightarrow \quad 2e^{2x} + (2a+1)e^x + a \ge 0$$

$$\Rightarrow \quad 2(e^x)^2 + (2a+1)e^x + a \ge 0$$

which is true for all $a \ge 0$.

9. We have

$$f(x) = \int_0^x (2\sqrt{2}\sin^2 t + (2-\sqrt{2})\sin t - 1)\,dt$$

$$\Rightarrow \quad f'(x) = (2(\sqrt{2}\sin^2 x + (2-\sqrt{2})\sin x - 1)$$

$$= (2\sin x - 1)(\sqrt{2}\sin x + 1)$$

When $f(x)$ is increasing, so $f'(x) > 0$

$$\Rightarrow \quad (2\sin - 1)(\sqrt{2}\sin x + 1) > 0$$

$$\Rightarrow \quad (\sqrt{2}\sin x + 1) < 0, (2\sin x - 1) > 0$$

$$\Rightarrow \quad \sin x < -\frac{1}{\sqrt{2}}, \sin x > \frac{1}{2}$$

$$\Rightarrow \quad x \in \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right) \text{ and } x \in \left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$$

$$\Rightarrow \quad x \in \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right) \cup \left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$$

when $f(x)$ is decreasing, so

$$\Rightarrow \quad f'(x) < 0$$

$$\Rightarrow \quad (2\sin x - 1)(\sqrt{2}\sin x + 1) < 0$$

$$\Rightarrow \quad \frac{-1}{\sqrt{2}} < \sin x < \frac{1}{2}$$

$$\Rightarrow \quad x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$$

10. We have $f(x) = \{-b^2 + (a-1)b - 2\}x + \int (\sin^2 x + \cos^4 x)\,dx,\ a, b \in R$

$$f'(x) = \{-b^2 + (a-1)b - 2\} + (\sin^2 x + \cos^4 x)$$

Since $f(x)$ is increasing function, so $f'(x) \ge 0$

$$\{-b^2 + (a-1)b - 2\} + \sin^2 x + \cos^4 x) \ge 0$$

$\{-b^2 + (a-1)b - 2\} + \frac{3}{4} \ge 0$, since the minimum value of $\sin^2 x + \cos^4 x$ is $\frac{3}{4}$

$$\{-4b^2 + 4(a-1)b - 8 + 3\} \ge 0$$

$$-4b^2 + 4(a-1)b - 5 \ge 0$$

$$4b^2 - 4(a-1)b + 5 \le 0$$

So its $\quad D < 0$

$$16(a-1)^2 - 80 < 0$$

$$(a-1)^2 - 5 < 0$$

$$(a-1)^2 < 5$$

$$|(a-1)| < \sqrt{5}$$

$$-\sqrt{5} < (a-1) < \sqrt{5}$$

$$1 - \sqrt{5} < a < 1 + \sqrt{5}$$

Hence, the value of a is $(1-\sqrt{5}, 1+\sqrt{5})$.

11. We have $g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$$\Rightarrow \quad g'(x) = 2f'\left(\frac{x^2}{2}\right)\left(\frac{2x}{2}\right) + f'(6-x^2)(-2x)$$

$$= 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right\}$$

when $g(x)$ is increasing, so $g(x) \ge 0$

$$\Rightarrow \quad 2x\left\{f'\left(\frac{x^2}{2}\right) - f'(6-x^2)\right\} \ge 0$$

But $\quad f''(x) > 0 \Rightarrow f'(x)$ is increasing

Case I: when $\frac{x^2}{2} > (6 - x^2)$

$$\Rightarrow \quad x^2 > 4$$

$$\Rightarrow \quad x \in (-\infty, 2) \in (2, \infty)$$

Now, $\quad f'\left(\frac{x^2}{2}\right) > f(6-x^2)$

$$\Rightarrow \quad f'\left(\frac{x^2}{2}\right) - f(6-x^2) > 0$$

Thus, $\quad$ if $x > 0$, $g'(x) > 0$ for $x \in (2, \infty)$

and $\quad g'(x) < 0$ for $x \in (-\infty, -2)$

Case II: when $\frac{x^2}{2} < (6 - x^2)$

$$\Rightarrow \quad x^2 < 4$$

$$\Rightarrow \quad -2 < x < 2$$

Now, $f'\left(\frac{x^2}{2}\right) < f'(6 - x)$

$\Rightarrow$ $f'\left(\frac{x^2}{2}\right) - f'(6 - x) < 0$ for all x in $(-2, 2)$

If $x > 0$, $g'(x) > 0$ for $x \in (-2, 0)$

and $x < 0$, $g'(x) < 0$ for $x \in (0, 2)$

Thus, $g(x)$ is increasing in $(-2, 0) \cup (2, \infty)$ and decreasing in $(-\infty, -2) \cup (0, 2)$

12. We have $f(x) = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \log(x^2 + x + 1)$

$$+(b^2 - 5b + 3)x + 4$$

$\Rightarrow$ $$f'(x) = \frac{2}{\sqrt{3}} \times \frac{2/\sqrt{3}}{1 + \left(\frac{2x+1}{\sqrt{3}}\right)^2}$$

$$-\frac{2x+1}{(x^2+x+1)} + (b^2 - 5b + 3)$$

$$= \frac{4}{3} \times \frac{3}{(4x^2 + 4x + 4)} - \frac{2x+1}{x^2+x+1}$$

$$+ (b^2 - 5b + 3)$$

$$= \frac{1}{(x^2+x+1)} - \frac{2x+1}{x^2+x+1} + (b^2 - 5b + 3)$$

$$= \frac{1 - 2x - 1}{(x^2 + x + 1)} + (b^2 - 5b + 3)$$

$$= (b^2 - 5b + 3) - \frac{2x}{(x^2 + x + 1)}$$

$$= (b^2 - 5b + 3) - \frac{2}{\left(x + \frac{1}{x} + 1\right)}$$

Since $f(x)$ be a decreasing function, so $f'(x) < 0$

$\Rightarrow$ $$(b^2 - 5b + 3) - \frac{2}{\left(x + \frac{1}{x} + 1\right)} < 0$$

$\Rightarrow$ $$(b^2 - 5b + 3) < \frac{2}{\left(x + \frac{1}{x} + 1\right)}$$

$\Rightarrow$ $(b^2 - 5b + 3) < -2$, since the minimum value of

$$\frac{2}{\left(x + \frac{1}{x} + 1\right)} \text{ is } -2$$

$\Rightarrow$ $(b^2 - 5b + 5) < 0$

$\Rightarrow$ $$\left(b - \frac{5}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 < 0$$

$\Rightarrow$ $$\left(b - \frac{5}{2}\right)^2 < \left(\frac{\sqrt{5}}{2}\right)^2$$

$\Rightarrow$ $$\left|\left(b - \frac{5}{2}\right)\right| < \left(\frac{\sqrt{5}}{2}\right)$$

$\Rightarrow$ $$-\frac{\sqrt{5}}{2} < \left(b - \frac{5}{2}\right) < \left(\frac{\sqrt{5}}{2}\right)$$

$\Rightarrow$ $$\frac{5}{2} - \frac{\sqrt{5}}{2} < b < \frac{5}{2} + \frac{\sqrt{5}}{2}$$

$\Rightarrow$ $$\frac{5 - \sqrt{5}}{2} < b < \frac{5 + \sqrt{5}}{2}$$

Hence, the value of b is

$$\left(\frac{5 - \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}\right)$$

13. We have $h(x) = f(x) + 2f\left(1 - \frac{x}{2}\right)$, $0 < x < 1$

$\Rightarrow$ $$h'(x) = f'(x) + 2f'\left(1 - \frac{x}{2}\right) \times -\frac{1}{2}$$

$\Rightarrow$ $$h'(x) = f'(x) - f'\left(1 - \frac{x}{2}\right) \quad \text{...(i)}$$

when $x < \frac{2}{3}$, then $1 - \frac{x}{2} > \frac{2}{3}$

$\Rightarrow$ $f'\left(1 - \frac{x}{2}\right) > f'(x)$, since $f(x)$ is inc.

$\Rightarrow$ $h'(x) < 0$

$\Rightarrow$ $h(x)$ is decreasing in $\left(0, \frac{2}{3}\right)$

when $x > \frac{2}{3}$ then $1 - \frac{x}{2} < \frac{2}{3}$

$\Rightarrow$ $f'(x) > f'\left(1 - \frac{x}{2}\right)$

$\Rightarrow$ $h'(x) > 0$

Thus, $h(x)$ is increasing in $\left(\frac{2}{3}, 1\right)$

14. We have, $g(x) = f(\sin x) + f(\cos x)$

$\Rightarrow$ $g'(x) = f'(\sin x)\cos x + f'(\cos x)(-\sin x)$

$\Rightarrow$ $g'(x) = f'(\sin x)\cos x - f'(\cos x)\sin x$

$\Rightarrow$ $$g''(x) = f''(\sin x)\cos^2 x - \sin x f'(\sin x) + f''(\cos x)\sin x - f'(\cos x)\cos x$$

$\Rightarrow \quad g''(x) < 0$, since $f'(\sin x) < 0, f''(\sin x) > 0$

and $f'(\cos x) > 0, f''(\cos x) < 0$

in the interval $\left[0, \frac{\pi}{2}\right]$

$\Rightarrow \quad g'(x)$ is a decreasing function

Now, $\quad g'\left(\frac{\pi}{4}\right) = 0$

If $\quad x < \frac{\pi}{4}$ than $g'(x) < \left(\frac{\pi}{4}\right)$

$\Rightarrow \quad g'(x) > 0$

$\Rightarrow \quad g'(x)$ is increasing in $\left[0, \frac{\pi}{4}\right]$

If $\quad x > \frac{\pi}{4}$ then $g'(x) < g'\left(\frac{\pi}{4}\right)$

$\Rightarrow \quad g'(x) < 0$

$\Rightarrow \quad g(x)$ is decreasing in $\frac{\pi}{4} < x \le \frac{\pi}{2}$.

15. We have $g(x) = f(\tan^2 x - 2\tan x + 4)$

$\Rightarrow \quad g'(x) = f'(\tan^2 x - 2\tan x + 4)(2\tan x - 2)\sec^2 x$

$\Rightarrow \quad g'(x) = 2f'(\tan^2 x - 2\tan x + 4)(\tan x - 1)\sec^2 x$

Since $\quad f''(x) > 0$, so $f'(x)$ is increasing

Now, $\quad g'(x) = 2f'(\tan^2 x - 2\tan x + 4)(\tan x - 1)\sec^2 x$

$= 2f'((\tan x - 1)^2 + 3)(\tan x - 1)\sec^2 x$

> 0 for all x in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Thus, $g(x)$ is increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Integer Type Questions

1. Given $\quad f(x) = x^3 + 3x + m,\ m \in R$

$\Rightarrow \quad f'(x) = 3x^2 + 3 > 0$

Clearly, $f'(x)$ has no real roots.
Hence, the number of zeroes of the function $f(x)$ is 1.

2. Given $\quad f(x) = x^3 - 3x + m,\ m \in R$

$\Rightarrow \quad f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

Thus, $\quad f'(x) = 0$ gives $x = \pm 1$

Clearly, $f'(x)$ has two real roots.

Hence, the number of real roots of $f(x)$ is 3.

3. Given $\quad f(x) = \sin^2 x - 3\cos^2 x + 2ax - 4$

$\Rightarrow \quad f'(x) = \sin 2x + 3\sin 2x + 2a$

$\Rightarrow \quad f'(x) = 4\sin 2x + 2a$

Since $f(x)$ is increasing for all $x \ge 0$, so

$4\sin 2x + 2a \ge 0$

$\Rightarrow \quad 2a \ge -4\sin 2x$

$\Rightarrow \quad a \ge -2\sin 2x$

$\Rightarrow \quad a \ge 2$

Hence, the least value of a is 2.

4. The given equation is

$\Rightarrow \quad x^3 + 2x^2 + 5x + 2\cos x = 0$

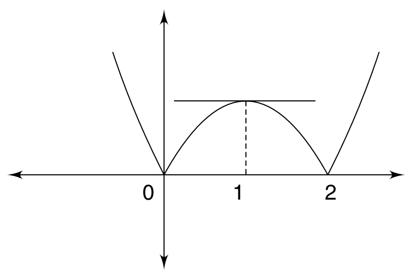
$\Rightarrow \quad x^3 + 2x^2 + 5x = -2\cos x$

$\Rightarrow \quad x(x^2 + 2x + 5) = -2\cos x$

Clearly, it has no solution.
Thus $\quad m = 0$
Hence, the value of $(m + 3)$ is 3.

5. Given function is

$f(x) = |x^2 - 2x|$



Thus, the number of critical points is 3 at $x = 0, 1, 2$.

6. We have

$$f(x) = \frac{x^3}{3} + (m - 1)x^2 + (m + 5)x + 7$$

$\Rightarrow \quad f'(x) = x^2 + 2(m - 1)x + (m + 5)$

Since $f(x)$ is increasing, so $f'(x) > 0$

$\Rightarrow \quad x^2 + 2(m - 1)x + (m + 5) > 0$

Clearly, its $D < 0$

$\Rightarrow \quad 4(m - 1)^2 - 4(m + 5) < 0$

$\Rightarrow \quad (m - 1)^2 - (m + 5) < 0$

$\Rightarrow \quad m^2 - 2m + 1 - m - 5 < 0$

$\Rightarrow \quad m^2 - 3m - 4 < 0$

$\Rightarrow \quad (m - 4)(m + 1) < 0$

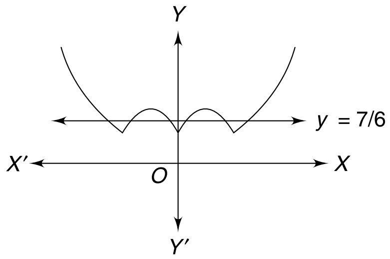
$\Rightarrow \quad -1 < m < 4$

Thus, the integral values of m are 0, 1, 2, 3.

Hence, the number of integral values of m is 4

7. Given $f(x) = |x| + |x^2 - 1|$

$$|x| + |x^2 - 1| = \frac{7}{6}$$



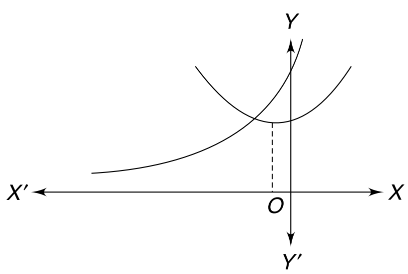
Clearly, the number of solution is 6.

8. Given equation is

$$ke^x = 5 + x + \frac{x^2}{2},\ k \in R^+$$

$$\Rightarrow \quad 2ke^x = 10 + 2x + x^2$$

$$\Rightarrow \quad 2ke^x = (x + 1)^2 + 9$$



Clearly, the number of solution is 1

Thus, $m = 1$

Hence, the value of $(m + 4)$ is 4.

9. Given $f(x) = e^{2x} - (a + 1)e^x + 2x$

$$\Rightarrow \quad f'(x) = 2e^{2x} - (a + 1)e^x + 2$$

Since $f(x)$ is increasing, so

$$2e^{2x} - (a + 1)e^x + 2 \geq 0$$

Thus, its descriminant $D < 0$

$$\Rightarrow \quad (a + 1)^2 - 16 < 0$$

$$\Rightarrow \quad (a + 1)^2 - 4^2 < 0$$

$$\Rightarrow \quad (a + 1 + 4)(a + 1 - 4) < 0$$

$$\Rightarrow \quad (a + 5)(a - 3) < 0$$

$$\Rightarrow \quad -5 < a < 3$$

Hence, the maximum value of a is 3.

10. Clearly, the number of solutions of

$$3\tan x + x^3 = 2, \text{ in } \left(0, \frac{\pi}{4}\right) \text{ is 1.}$$

Thus $m = 1$

Also, $f(x) = 3x^4 - 4x^2$

$$\Rightarrow \quad f'(x) = 12x^3 - 12x^2$$

$$\Rightarrow \quad f''(x) = 36x^2 - 24x = 12x(3x - 2)$$

Now, $f''(x) = 0$ gives $x = 0, \frac{2}{3}$

So, the number of points of inflection is 2

Thus, $n = 2$

Hence, the value of $(m + n + 2)$ is 5.

Questions asked in Past IIT-JEE Exams

1. Given $f(x) = x^{1/x},\ x > 0$

$$\Rightarrow \quad f(x) = e^{\frac{1}{x}\log x}$$

$$\Rightarrow \quad f'(x) = e^{\frac{1}{x}\log x}\left(\frac{1}{x^2} - \frac{1}{x^2}\log x\right)$$

$$\Rightarrow \quad f'(x) = x^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{1}{x^2}\log x\right)$$

$$\Rightarrow \quad f'(x) = \frac{x^{\frac{1}{x}}}{x^2}(1 - \log x)$$

$$\Rightarrow \quad f'(x) < 0\ (\because x > e \Rightarrow \log x > 1)$$

$\Rightarrow$ $f(x)$ is strictly decreasing function.

when $x > e$

$$\Rightarrow \quad f(x) < f(e)$$

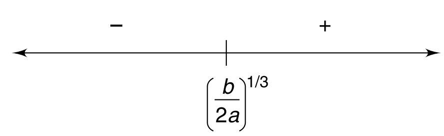
$$\Rightarrow \quad \pi^{\frac{1}{\pi}} < e^{\frac{1}{e}}$$

$$\Rightarrow \quad \left(\pi^{\frac{1}{\pi}}\right)^{\pi e} < \left(e^{\frac{1}{e}}\right)^{\pi e}$$

$$\Rightarrow \quad (\pi)^e < (e)^\pi.$$

2. Let $f(x) = ax^2 + \frac{b}{x} - c$

$$f'(x) = 2ax - \frac{b}{x^2}$$



$$f''(x) = 2a + \frac{2b}{x^3}$$

Now, $f'(x) = 0$ gives $x = \left(\frac{b}{2a}\right)^{1/3}$

which is a positive critical point.

So we shal find the least value of $f(x)$

which occurs at $x = \left(\frac{b}{2a}\right)^{1/3}$

Since $ax^2 + \frac{b}{x} \geq c,\ \forall\ x \in R^+$, we should

have $\quad f\left(\left(\frac{n}{2a}\right)^{1/3}\right) \geq 0$

$$\Rightarrow \quad a\left(\frac{b}{2a}\right)^{2/3} + \frac{b}{\left(\frac{b}{2a}\right)^{1/3}} \geq c$$

$$\Rightarrow \quad a\left(\frac{b}{2a}\right) + b \geq c\left(\frac{b}{2a}\right)^{1/3}$$

$$\Rightarrow \quad \left(\frac{b}{2} + b\right) \geq x\left(\frac{b}{2a}\right)^{1/3}$$

$$\Rightarrow \quad \left(\frac{3b}{2}\right)^3 \geq c^3\left(\frac{b}{2a}\right)$$

$$\Rightarrow \quad 27ab^2 \geq 4c^3$$

3. Let $\quad f(x) = 1 + x\log\left(x + \sqrt{x^2+1}\right) - \sqrt{1+x^2}$

$$f(x) = 1 + x\log(x + \sqrt{x^2+1}) - \sqrt{1+x^2}$$

$$\Rightarrow \quad f'(x) = \log\left(x + \sqrt{x^2+1}\right) + \frac{x}{\left(x + \sqrt{x^2+1}\right)}\left(1 + \frac{2x}{2\sqrt{x^2+1}}\right) - \frac{2x}{2\sqrt{1+x^2}}$$

$$= \log\left(x + \sqrt{x^2+1}\right) + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}$$

$$= \log\left(x + \sqrt{x^2+1}\right) \geq 0$$

Thus, f is an increasing function

So, when $x \geq 0$

$$\Rightarrow \quad f(x) \geq f(0)$$

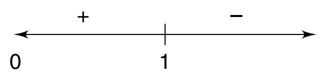
$$\Rightarrow \quad 1 + x\log\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} \geq 0$$

$$\Rightarrow \quad 1 + x\log\left(x + \sqrt{1+x^2}\right) \geq \sqrt{1+x^2}$$

Hence, the result.

4. Let $\quad f(x) = \ln x - x,\ x > 0$

$$\Rightarrow \quad f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}$$



Thus, $f(x)$ is strictly increasing in (0, 1) and strictly decreasing in $(1, \infty)$

So, $f(x)$ has the greatest value at $x = 1$.

$$\Rightarrow \quad f(x) \leq f(1) = -1 < 0$$

$$\Rightarrow \quad f(x) < 0$$

$$\Rightarrow \quad \ln x < x$$

Now, we have $e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$

$$\Rightarrow \quad 0 < \theta < \frac{\pi}{2}$$

$$\Rightarrow \quad 0 < \cos\theta < 1$$

$$\Rightarrow \quad \ln(\cos\theta) < 0 \qquad \text{...(i)}$$

$(\because \ln x < 0\ \forall\ x \in (0, 1))$

Also, we have $e^{-\frac{\pi}{2}} < \theta < \frac{\pi}{2}$

$$\Rightarrow \quad -\frac{\pi}{2} < \ln\theta < \ln\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad -\frac{\pi}{2} < \ln\theta < \left(\frac{\pi}{2}\right)\ (\because \ln x < x)$$

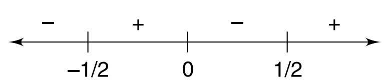
$$\Rightarrow \quad 0 < \cos(\ln\theta < 1 \qquad \text{...(ii)}$$

From (i) and (ii), we conclude that,

$$\ln(\cos\theta) < \cos(\ln\theta)$$

5. Given $\quad y = 2x^2 - \ln|x|$

$$\frac{dy}{dx} = 4x - \frac{1}{x} = \frac{(2x-1)(2x+1)}{x}$$



From the sign scheme for $f'(x)$, $f(x)$ is strictly increasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$ and strictly decreasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

6. No questions asked in between 1984 to 1986.
7. Ans. (a)

Given $h(x) = f(g(x))$

Since f is inc. and g is dec. so h is dec.

$(\because I(D) = D)$

$\Rightarrow \quad h(x) \le h(0) \cdot \forall\, x \ge 0$

$\Rightarrow \quad h(x) \le 0 \cdot \forall \ge 0$ ...(i)

Also, $\quad h(x) \ge 0 \cdot \forall\, x \in [0, \infty]$ ...(ii)

From (i) and (ii), we conclude that,

$h(x) = 0$ for all $x \in [0, \infty)$

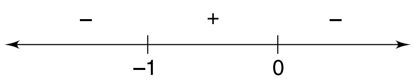
Thus, $\quad h(x) - h(1)$

$= 0 - 0$

$= 0.$ for all $x \in [0, \infty)$

8. Let $\quad f(x) = \ln(1 + x) - x$

$$f'(x) = \frac{1}{1+x} - 1 = \frac{1 - 1 - x}{1 + x} = -\frac{x}{1+x}$$



From the sign scheme, we can say that f is decreasing for $x \ge 0$

when $\quad x \ge 0 \Rightarrow f(x) \le f(0)$

$\Rightarrow \quad \log(1 + x) - x \le 0$

$\Rightarrow \quad \log(1 + x) \le x$

Thus, $\quad \log(1 + x) \le x, \ \forall\, x \ge 0$

9. No questions asked in between 1988 to 1992.

10. Let $\quad f(x) = 2\sin x + \tan x - 3x$

$\Rightarrow \quad f'(x) = 2\cos x + \sec^2 x - 3$

$[g(x) = 2\cos x + \sec^2 x$

$= \cos x + \cos x + \sec^2 x$

> 3, by A.M $\ge$ G.M]

Thus, $\quad f'(x) \ge 0$

So, f is increasing function for $x \ge 0$.

When $x \ge 0 \Rightarrow f(x) \ge f(0)$

$\Rightarrow \quad 2\sin x + \tan x - 3x \ge 0$

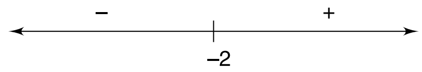
$\Rightarrow \quad 2\sin x + \tan x \ge 3x$

11. Ans.(a, b, c, d)

Given $\quad f(x) = \begin{cases} 3x^2 + 12x - 1 & : -1 \le x \le 2 \\ 37 - x & : 2 < x \le 3 \end{cases}$

$\Rightarrow \quad f'(x) = \begin{cases} 6x + 12 & : -1 \le x \le 2 \\ -1 & : 2 < x \le 3 \end{cases}$

Now, in $\quad [-1, 2], f'(x) = 6x + 12$



Thus, $f(x)$ decreases in $(-\infty, -2)$ and increases in $(-2, \infty)$

Also, $\quad f(2^+) = 37 - 2 = 35$

and $\quad f(2^-) = 12 + 24 - 1 = 35$

Hence, $\quad f(x)$ is continuous at $x = 2$.

Also, $\quad f(2^+) < f(2)$ and $f(2^-) < f(2)$

Thus, $\quad x = 2$ is the point of maxima.

Hence, the result.

12. Ans (a, c)

Given $\quad h(x) = f(x) - (f(x))^2 + (f(x))^3$

$\Rightarrow \quad h'(x) = f'(x) - 2f(x)f'(x) + 3(f(x))^2 f'(x)$

$\Rightarrow \quad h'(x) = (3(f(x))^2 - 2f(x) + 1)f'(x)$

$$\Rightarrow \quad h'(x) = \left(\frac{(3(f(x))^2 - 2f(x) + 1)}{+\text{ve}}\right) f'(x)$$

Thus, $\quad h'(x) > 0$, when $f'(x) > 0$

and $\quad h'(x) < 0$, when $f'(x) < 0$

Therefore h is increasing when f is increasing and h is decreasing when f is increasing.

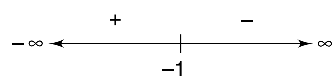
13. Ans. (d)

Given $\quad f(x) = (x + 2)e^{-x}$

$\Rightarrow \quad f'(x) = e^{-x} - (x + 2)e^{-x}$

$f'(x) = (1 - x - 2)e^{-x}$

$f'(x) = -(1 + x)e^{-x}$



Thus, $f(x)$ increasing in $(-\infty, -1)$ and decreasing in $(-1, \infty)$.

14. Ans. (b).

Given $\quad f(x) = \dfrac{\ln(x + \pi)}{\ln(x + e)}$

$$\Rightarrow \quad f'(x) = \frac{\frac{1}{\pi + x}\log(e + x) - \frac{1}{e + x}\log(\pi + x)}{\{\log(e + x)\}^2}$$

$$= \frac{(e + x)\log(e + x) - (\pi + x)\log(\pi + x)}{(\pi + x)(e + x)\{\log(e + x)\}^2}$$

< 0 for $x > 0$

Thus, f decreases on $[0, \infty)$

15. Given $f(x) = \begin{cases} xe^{ax} & : x \le 0 \\ x + ax^2 - x^3 & : x > 0 \end{cases}$

Clearly f is continuous at $x = 0$

$$\Rightarrow \quad f'(x) = \begin{cases} (ax+1)e^{ax} & : x < 0 \\ 1 & : x = 0 \\ 1 + 2ax - 3x^2 & : x > 0 \end{cases}$$

$$\Rightarrow \quad f''(x) = \begin{cases} a(ax+2)e^{ax} & : x < 0 \\ 2a & : x = 0 \\ 2a - 6x & : x > 0 \end{cases}$$

Now, for $x < 0, f''(x) > 0$

$\Rightarrow \quad ax + 2 > 0$

$\Rightarrow \quad x > -\frac{2}{a}$

$\Rightarrow \quad f'(x)$ increases on $\left[-\frac{2}{a}, 0\right]$

Also, $x > 0, f''(x) > 0$

$\Rightarrow \quad 2a - 6x > 0$

$\Rightarrow \quad x > \frac{a}{3}$

Thus, $f'(x)$ increases on $\left[0, \frac{a}{3}\right]$

Hence, $f'(x)$ increases on $\left[-\frac{2}{a}, \frac{a}{3}\right]$.

16. Ans. (b)

Let $F(x) = \begin{cases} \frac{x}{\sin x} & : 0 < x \le 1 \\ 0 & : x = 0 \end{cases}$

Clearly F is continuous on [0, 1] and differentiable on (0, 1)

Now, $$F'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$$

$$= \frac{\cos x(\tan x - x)}{\sin^2 x}$$

$\Rightarrow \quad F'(x) > 0$, for $0 < x < 1$

Thus, $F(x)$ increases on [0, 1]

Hence, $f(x) = \frac{\sin x}{x}$ increases on $0 < x \le 1$

Again, let $G(x) = \begin{cases} \frac{x}{\tan x} & : 0 < x \le 1 \\ 0 & : x = 0 \end{cases}$

Clearly, G is continuous on [0, 1] and differentiable on (0, 1).

Now, $$G'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}$$

Again, let, $H(x) = \tan x - x\sec^2 x$

So H is continuous on [0, 1] and differentiable on (0, 1).

$$H'(x) = \sec^2 x - \sec^2 x - x\sec^2 x \tan x$$

$$H'(x) = -x\sec^2 x \tan x < 0 \text{ for } 0 < x < 1.$$

H is a decreasing function on [0, 1]

Thus, $H(x) < H(0)$

$\Rightarrow \quad H(x) < 0$

$\Rightarrow \quad G'(x) < 0$

Thus, $G(x)$ is a decreasing function on [0, 1]

Therefore, $g(x) = \frac{x}{\tan x}$ is a decreasing function on $0 < x \le 1$.

17. No questions asked in 1998.

18. Ans. (b)

Given $f(x) = \sin^4 x + \cos^4 x$

$$f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$$
$$= -4\sin x\cos x(\cos^2 x - \sin^2 x)$$
$$= -2(2\sin x\cos x)(\cos^2 x - \sin^2 x)$$
$$= -2\sin 2x\cos 2x$$
$$= -\sin 4x$$

f increases when $f'(x) > 0$

$\Rightarrow \quad -\sin 4x > 0$

$\Rightarrow \quad \sin 4x < 0$

$\Rightarrow \quad \pi < 4x < 2\pi$

$\Rightarrow \quad \frac{\pi}{4} < x < \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{4} < x < \frac{3\pi}{8}$

19. Ans. (d)

S is the correct statement.

But R is the wrong statement.

Let $f(x) = \sin x \ \forall \ x \in \left(\pi, \frac{3\pi}{2}\right)$

Clearly, f is decreases in the given interval.

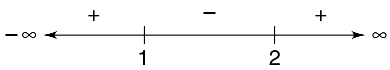
Now, $f'(x) = \cos x$

which is increases in $\left(\pi, \frac{3\pi}{2}\right)$.

20. Ans. (c)

Given $f(x) = \int e^x(x-1)(x-2)\,dx$

$f'(x) = e^x(x-1)(x-2)$



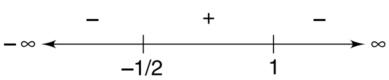
f is decreasing, when $f'(x) \le 0$

Thus f is decreasing in $[1, 2]$

21. Ans. (a)

Given $f(x) = xe^{x(1-x)}$

$$f'(x) = e^{x(1-x)} + xe^{x(1-x)}(1-2x)$$
$$= e^{x(1-x)}(1 + x(1-2x))$$
$$= -e^{x(1-x)}(2x^2 - x - 1)$$
$$= -e^{x(1-x)}(2x^2 - 2x + x - 1)$$
$$= -e^{x(1-x)}(x-1)(2x+1)$$



f is increasing when $f'(x) \ge 0$ and decreasing when $f'(x) \le 0$.

Thus f is increasing in $[-1/2, 1]$.

22. Ans (a)

Given $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

$\Rightarrow \quad f'(x) = 3\cos 3x$

f is increasing when $f'(x) \ge 0$

$\Rightarrow \quad 3\cos 3x \ge 0$

$\Rightarrow \quad -\dfrac{\pi}{2} \le 3x \le \dfrac{\pi}{2}$

$\Rightarrow \quad -\dfrac{\pi}{6} \le x \le \dfrac{\pi}{6}$

Hence, the length of the longest interval

$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$

23. Let $f(x) = \sin(\tan x) - x,\ 0 \le x \le \dfrac{\pi}{4}$

$$\Rightarrow \quad f'(x) = \cos(\tan x)\sec^2 x - 1$$
$$= \cos(\tan x)(1 + \tan^2 x) - 1$$
$$= \cos(\tan x)\tan^2 x - (1 - \cos(\tan x))$$
$$> \cos(\tan x)\tan^2 x - \frac{1}{2}\tan^2 x$$
$$= \tan^2 x(\cos(\tan x) - 1) + \frac{1}{2}\tan^2 x$$
$$> -\tan^2 x\left(\frac{1}{2}\tan^2 x\right) + \frac{1}{2}\tan^2 x$$
$$= \frac{1}{2}\tan^2 x(1 - \tan^2 x)$$
$$> 0 \text{ for all } x \text{ in } \left(0, \frac{\pi}{4}\right)$$

Thus, $f'(x) > 0$ for all x in $\left(0, \dfrac{\pi}{4}\right)$

Therefore f is increasing in $\left[0, \dfrac{\pi}{4}\right]$

So, $\quad x \ge 0 \Rightarrow f(x) \ge f(0)$

$\Rightarrow \quad \sin(\tan x) - x \ge 0$

$\Rightarrow \quad \sin(\tan x) \ge x$ for all x in $\left[0, \dfrac{\pi}{4}\right]$

Hence, the result.

24. Given that $\dfrac{d(P(x))}{dx} > P(x)$ for all $x \ge 1$.

$\Rightarrow \quad \dfrac{d(P(x))}{dx} - P(x) > 0$

$\Rightarrow \quad e^{-x}\dfrac{d(P(x))}{dx} - e^{-x}P(x) > 0$

$\Rightarrow \quad \dfrac{d}{dx}(e^{-x}P(x)) > 0$

$\Rightarrow \quad (e^{-x}P(x))$ is an increasing function.

Thus, $\quad x > 1 \Rightarrow e^{-x}P(x) > e^{-x}P(1) = 0$

$\Rightarrow \quad e^{-x}P(x) > 0$

$\Rightarrow \quad P(x) > 0,\ \forall\ x > 1$

Hence, the result.

25. Let $f(x) = \sin x + 2x - \left(\dfrac{3x(x+1)}{\pi}\right)$

$\Rightarrow \quad f'(x) = \cos x + 2 - \left(\dfrac{3(2x+1)}{\pi}\right)$

$\Rightarrow \quad f''(x) = -\sin x - \dfrac{6}{\pi}$

Thus, $\quad f'(x)$ is a decreasing function.

$\Rightarrow \quad f$ is an increasing function.

So, $\quad x \ge 0$

$\Rightarrow \quad f(x) \ge f(0)$

$\Rightarrow \quad \sin x + 2x - \left(\dfrac{3x(x+1)}{\pi}\right) \ge 0$

$\Rightarrow \quad \sin x + 2x \ge \left(\dfrac{3x(x+1)}{\pi}\right)$

26. No questions asked in between 2004 to 2007.
27. Ans. (c)

Given $g(u) = 2\tan^{-1}(e^u) - \dfrac{\pi}{2}$

$$\Rightarrow \quad g(-u) = 2\tan^{-1}(e^{-u}) - \frac{\pi}{2}$$
$$= 2\tan^{-1}\left(\frac{1}{e^u}\right) - \frac{\pi}{2}$$
$$= 2\cot^{-1}(e^u) - \frac{\pi}{2}$$
$$= 2\left(\frac{\pi}{2} - \tan^{-1}(u)\right) - \frac{\pi}{2}$$
$$= (\pi - 2\tan^{-1}(u)) - \frac{\pi}{2}$$
$$= \left(\frac{\pi}{2} - 2\tan^{-1}(u)\right)$$
$$= -\left(2\tan^{-1}(u) - \frac{\pi}{2}\right)$$
$$= -g(u)$$

Thus $g(u)$ is an odd function.

Also, $g'(u) = \dfrac{2e^u}{1 + e^{2u}} > 0 \ \forall\ u \in R$

Therefore, g is strictly increasing function.
Hence, g is odd as well as strictly inc. on R.

28. Ans. (a, b, c, d)

Given $f(x) = x\cos\left(\dfrac{1}{x}\right),\ x \geq 1$

$$\Rightarrow \quad f'(x) = \cos\left(\frac{1}{x}\right) + \frac{1}{x}\sin\left(\frac{1}{x}\right) > 0,\ \forall\ x > 1$$

$\Rightarrow$ f is strictly increasing on $[1, \infty)$

Also,

$$\Rightarrow \quad f''(x) = \frac{1}{x^2}\sin\left(\frac{1}{x}\right) - \frac{1}{x^2}\sin\left(\frac{1}{x}\right) - \frac{1}{x^3}\cos\left(\frac{1}{x}\right)$$
$$= -\frac{1}{x^3}\cos\left(\frac{1}{x}\right) < 0,\ \forall\ x > 1$$

$\Rightarrow$ f' is strictly decreasing on $[1, \infty)$

Again, $\lim\limits_{x\to\infty}(f'(x))$

$$= \lim_{x\to\infty}\cos\left(\frac{1}{x}\right) + \frac{1}{x}\sin\left(\frac{1}{x}\right)$$
$$= 1 + 0$$
$$= 1$$

Finally, let $x \geq 1$

By the Lagranges Mean Value Theorem, there exists some $c \in (x, x + 2)$ for which

$$\frac{f(x+2) - f(x)}{2} = f'(c)$$

As f' is strictly decreasing on $[1, \infty)$,

so, $f'(c) > \lim\limits_{x\to\infty} f'(x) = 1$

Thus, $\dfrac{f(x+2) - f(x)}{2} > 1$

$\Rightarrow$ $f(x + 2) - f(x) > 2$

Hence, the result.

29. Ans. (i). (c), (ii). (a), (iii). (b).

(i) Ans. (c)

Since $f\left(-\dfrac{1}{2}\right) f\left(-\dfrac{3}{4}\right) < 0$

so, s lies in $\left(-\dfrac{3}{4}, -\dfrac{1}{2}\right)$

(ii) Ans. (a)

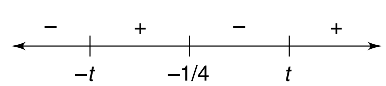
Here, $-\dfrac{3}{4} < s < -\dfrac{1}{2}$

$$\Rightarrow \quad \frac{1}{2} < t < \frac{3}{4}$$
$$\Rightarrow \quad \int_0^{1\backslash 2}(4x^3 + 3x^2 + 2x + 1)\,dx < \text{area}$$
$$< \int_0^{3/4}(4x^3 + 3x^2 + 2x + 1)\,dx$$
$$\Rightarrow \quad (x^4 + x^3 + x^2 + x)\Big|_0^{1/2} < \text{area}$$
$$< (x^4 + x^3 + x^2 + x)\Big|_0^{3/4}$$
$$\Rightarrow \quad \frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} < \text{area} < \frac{81}{256} + \frac{27}{64} + \frac{9}{16} + \frac{3}{4}$$
$$\Rightarrow \quad \frac{15}{16} < \text{area} < \frac{525}{256}$$

(iii) Ans. (b)

Given $f(x) = 1 + 2x + 3x^2 + 4x^3$

$$\Rightarrow \quad f'(x) = 2 + 6x + 12x^2$$
$$\Rightarrow \quad f''(x) = 6 + 24x$$



From the sign scheme of $f''(x)$,

$$f'(x) \text{ dec. in } \left(-t, -\frac{1}{4}\right) \text{ and inc. in } \left(-\frac{1}{4}, t\right)$$

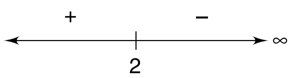
30. Given $f(x) = x^{3/2}(3x - 10),\ x \ge 0$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}}(3x - 10) + x^{\frac{3}{2}} \cdot 3$$

$$= \frac{3}{2}x^{\frac{1}{2}}(3x - 10 + 2x)$$

$$= \frac{3}{2}x^{\frac{1}{2}}(5x - 10)$$

$$= \frac{15}{2}x^{\frac{1}{2}}(x - 2)$$



Thus, $f(x)$ is increasing in $[2, \infty)$

31. Ans. (b)

$$g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right) f(x)$$

For $x > 1,\ f(x) > 0$

Let $h(x) = \left(\dfrac{2(x-1)}{x+1} - \ln x\right)$

$$h'(x) = \left(\frac{4}{(x+1)^2} - \frac{1}{x}\right) = \frac{-(x-1)^2}{x(x+1)^2} < 0$$

Also, $h(1) = 0$

So $h(x) < 0$ for all $x > 1$

Thus, $g(x)$ is decreasing on $[0, \infty)$.

32. No questions asked in 2013.

33. Ans. (a, c, d)

Given $f(x) = \displaystyle\int_{1/x}^{x} e^{-\left(t + \frac{1}{t}\right)} \frac{dt}{t}$

$$f'(x) = e^{-\left(x + \frac{1}{x}\right)} \cdot \frac{1}{x} - e^{-\left(\frac{1}{x} + x\right)}\left(-\frac{1}{x^2}\right) \cdot x$$

$$= \frac{2e^{-\left(x + \frac{1}{x}\right)}}{x} > 0 \text{ for all } x > 1$$

Thus, f is monotonic increasing on $[1, \infty)$

Also, $f(x) + f\left(\dfrac{1}{x}\right)$

$$= \int_{\frac{1}{x}}^{x} e^{-\left(t + \frac{1}{t}\right)} \frac{dt}{t} + \int_{x}^{\frac{1}{x}} e^{-\left(t + \frac{1}{t}\right)} \frac{dt}{t}$$

$$= \int_{\frac{1}{x}}^{x} e^{-\left(t + \frac{1}{t}\right)} \frac{dt}{t} - \int_{\frac{1}{x}}^{x} e^{-\left(t + \frac{1}{t}\right)} \frac{dt}{t}$$

$$= 0.$$

Let $g(x) = f(2^x) = \displaystyle\int_{2^{-x}}^{2^x} \left(\frac{e^{\left(-t + \frac{1}{t}\right)}}{t} dt\right)$

$$g(-x) = f(2^{-x}) = \int_{2^x}^{2^{-x}} \left(\frac{e^{\left(-t + \frac{1}{t}\right)}}{t} dt\right) = -g(x)$$

Thus, $g(x)$ is an odd function.

So, $f(2^x)$ is an odd function.

Hence, the result.

CHAPTER

8 The Tangent and Normal

CONCEPT BOOSTER

1. Introduction

In geometry, the tangent line (or simply tangent) to a plane curve at a given point is the straight line that "just touches" the curve at that point.

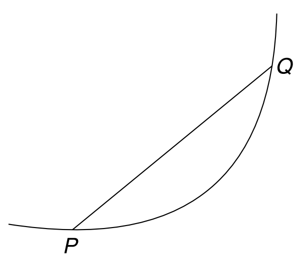
Leibniz defined it as the line through a pair of infinitely close points on the curve. More precisely, a straight line is said to be a tangent of a curve $y = f(x)$ at a point $x = c$ on the curve if the line passes through the point $(c, f(c))$ on the curve and has slope $f'(c)$ where f' is the derivative of f. A similar definition applies to space curves and curves in n-dimensional Euclidean space.

As it passes through the point where the tangent line and the curve meet, called the point of tangency, the tangent line is "going in the same direction" as the curve, and is thus the best straight-line approximation to the curve at that point.

Similarly, the tangent plane to a surface at a given point is the plane that "just touches" the surface at that point. The concept of a tangent is one of the most fundamental notions in differential geometry and has been extensively generalized.

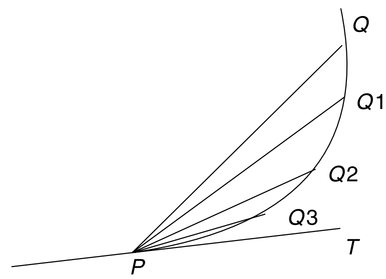
The word tangent comes from the Latin word tangere, to touch.

2. Secant



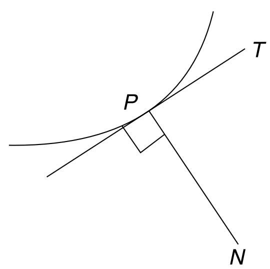
If a line intersects the curve in two distinct points, then it is called a secant. Here, PQ be a secant.

3. Tangent



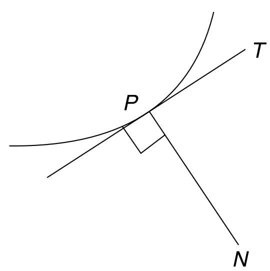
If a line intersects the curve in two coincident points, then it is called a tangent. Here, PT be a tangent.

4. Normal



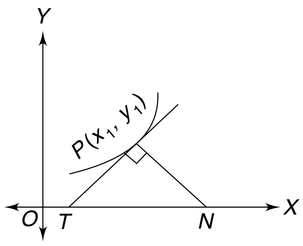
If a line, which is perpendicular to the point of contact to the tangent, then it is called a normal. Here, PN be a normal.

5. Relation between Tangent and Normal



m(Tangent) × m (Normal) = –1

6. Equations of tangent and Normal



(i) **Tangent:** $(y - y_1) = m(x - x_1)$, where

$$m = \left(\frac{dy}{dx}\right)_{at(x_1,\, y_1)}$$

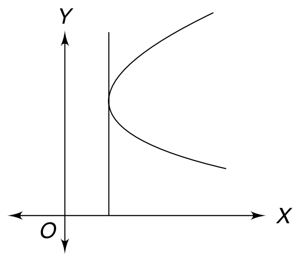
(ii) **Normal:** $(y - y_1) = -\frac{1}{m}(x - x_1)$, where

$$m = \left(\frac{dy}{dx}\right)_{at(x_1,\, y_1)}$$

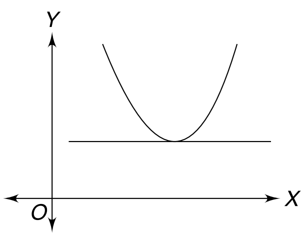
7. Conditions of Different Types of Tangents

(i) **Vertical Tangent**

If a line touches the curve either vertically or parallel to y-axis or perpendicular to x-axis, then $\frac{dx}{dy} = 0$.

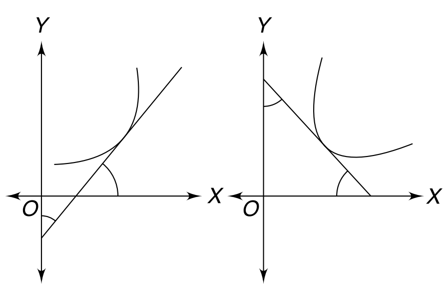


(ii) **Horizontal Tangent**



If a line touches the curve either horizontally or parallel to x-axis or perpendicular to y-axis, then $\frac{dy}{dx} = 0$.

(iii) **Equally Inclined with the Axes**



If a tangent at any point on the curve is equally inclined to both the axes, then $\frac{dy}{dx} = \pm 1$.

(iv) If the tangent at any point on the curve cuts both the axes at equal distances from origin, then $\frac{dy}{dx} = \pm 1$ or it passes through the origin.

(v) If the tangent at any point makes equal intercepts on the co-ordinate axes, then $\frac{dy}{dx} = -1$ or it passes through the origin.

8. Equation of Tangent and Normal to a Second Degree Curve

Equation of tangent to a second degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Equation of normal to a second degree curve is

$$\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}.$$

Note: Whenever we shall write the equation of tangent to any type of second degree curve, we should remember the following replacement.

1. x^2 is replaced by xx_1
2. y^2 is replaced by yy_1
3. xy is replaced by $\frac{xy_1 + x_1y}{2}$
4. x is replaced by $\frac{x + x_1}{2}$
5. y is replaced by $\frac{y + y_1}{2}$
6. constant is no change, c will remain c.

Some Common Parametric Curves

(i) $x^2 + y^2 = a^2$

$\Rightarrow x = a\cos\theta,\ y = a\sin\theta$

(ii) $y^2 = 4ax$

$\Rightarrow x = \text{at}^2,\ y = 2\text{at}$

(iii) $x^2 = 4ay$

$\Rightarrow x = 2\text{at},\ y = \text{at}^2$

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow x = a\cos\theta,\ y = b\sin\theta$

(v) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow x = a\sec\theta,\ y = b\tan\theta$

(vi) $x^2 - y^2 = a^2$

$\Rightarrow x = a\sec\theta,\ y = a\tan\theta$

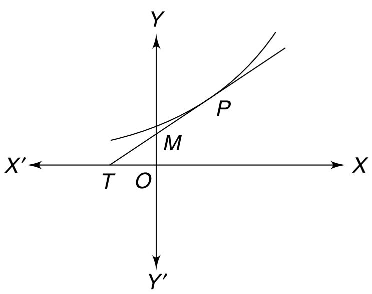
(vii) $x^{2/3} + y^{2/3} = a^{2/3}$

$\Rightarrow x = a\cos^3\theta,\ y = a\sin^3\theta$

(viii) $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$\Rightarrow x = a\cos^4\theta,\ y = a\sin^4\theta$

9. Length of Intercepts of the Tangents by the Axes



(i) The equation of tangent to a curve at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{at(x_1,\ y_1)} (x - x_1)$$

(ii) The length of the x-intercept is:

$$= \text{OT} = x_1 - y_1\left(\frac{dx}{dy}\right)_{at(x_1,\ y_1)}$$

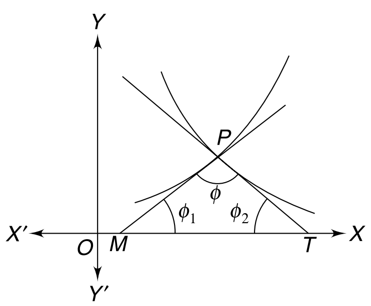
(iii) The length of the y-intercept is:

$$= \text{OM} = x - y\left(\frac{dx}{dy}\right)_{at(x_1,\ y_1)}$$

10. Tangents at the Origin or Through Origin

If a curve passing through the origin given by a rational integral algebraic equation, the equation of tangent or tangents at the origin is obtained by equating to zero the terms of the lowest degree in the given equation.

11. Angle Between two Curves



Let two curves be $y = f(x)$ and $y = g(x)$ and $p(x_1, y_1)$ be the point of intersection.

Let $m_1 = \left(\dfrac{d(f(x))}{dx}\right)_{\text{at } P}$ and

$m_2 = \left(\dfrac{d(g(x))}{dx}\right)_{\text{at } P}$ and θ be the angle between them.

Then $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$

(i) **Condition of Parallelism**

when $\theta = 0$ or π, then $m_1 = m_2$

(ii) **Condition of Perpendicularity**

when $\theta = \dfrac{\pi}{2}$, then $m_1 \times m_2 = -1$

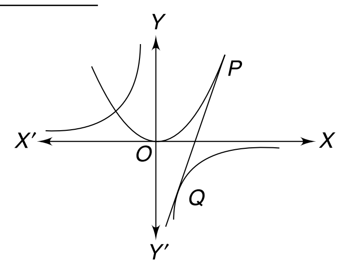
(iii) **Condition for two Curves Touch Each Other**

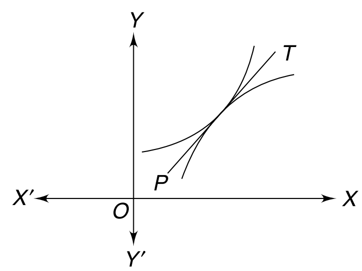
when $\theta = 0$, then $m_1 = m_2$.

12. Shortest Distance

The shortest distance between two non inter secting curves is found along the common normal to the two curves. In fact, if two curves also have the largest distance between them, then it is also found along the common normal to the two curves.

13. Common Tangent

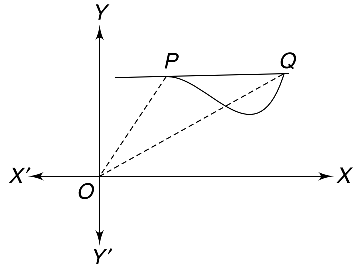




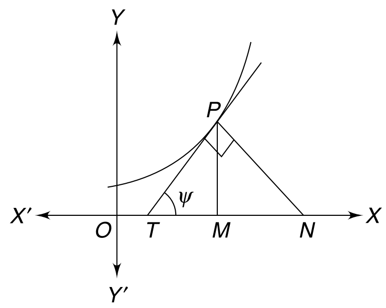
A line which touches two given curves is called a common tangent to the curves. From the above figures, we can say that, the point of contact P is the common point of both the curves. Here, the curves touch each other. In this case, the slope of the common tangent is equal to the slope of the curves at their point of contact.

14. Tangent to the curve at point P intersects the curve again at Q

Tangent to the curve at a point P intersects the curve again at Q.



15. Lengths of Tangent, Sub-Tangent, Normal and Sub-Normal to the Curve at a Point



Let the curve be $y = f(x)$ and the point be $P(x, y)$. Then $\tan(\psi) = \frac{dy}{dx}$.

(i) Length of the Tangent (PT)

$$\sin\psi = \frac{y}{PT}$$

$$\Rightarrow \quad PT = y\,\text{cosec}\,\psi$$

$$\Rightarrow \quad PT = y\sqrt{1 + \cot^2\psi}$$

$$\Rightarrow \quad PT = y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

(ii) Length of the sub-tangent (TM)

$$\tan\psi = \frac{y}{TM}$$

$$\Rightarrow \quad TM = y\cot\psi$$

$$\Rightarrow \quad TM = y\cdot\left(\frac{dx}{dy}\right)$$

(iii) Length of the normal (PN)

$$\cos\psi = \frac{y}{PN}$$

$$\Rightarrow \quad PN = y\sec\psi$$

$$\Rightarrow \quad PN = y\sqrt{1 + \tan^2\psi}$$

$$\Rightarrow \quad PN = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(iv) Length of the sub-normal (MN)

$$\tan\psi = \frac{y}{MN}$$

$$\Rightarrow \quad MN = y\left(\frac{dy}{dx}\right).$$

Exercises

Level I

(Problems Based on Fundamentals)

ABC of Tangents and Normals

1. Find the slope of the tangent to the curve $y = x^3 + 3x^2 + 3x - 10$ at $x = 2$.
2. Find the slope of the normal to the curve $y = x^x + 1$ at $x = 2$
3. If the slope of the curve $y = \frac{ax}{b - x}$ at the point (1, 1) be 2, find the value of $a + b + 10$.
4. Find the equation of the tangent to the curve $y = e^{2x}$ at $x = 0$
5. Find the equation of the tangent to the curve $y = \sqrt[3]{x - 1}$ at $x = 1$.
6. Find the equation of the tangent to the curve $y = be^{-x/a}$ at the point where it crosses the y-axis.
7. Find the equation of the tangent to the curve $y = \frac{4}{x^2 + 2}$ at $x = 0$.
8. Find the equation of the normal to the curve $x^3 + y^3 = 6xy$ at (3, 3).
9. Find the equation of the normal to the curve $y = 3x^2 + 2\sin x + 4\cos x + 10$ at $x = 0$.
10. Find the equation of the normal to the curve $x + y = x^y$, where it cuts the x-axis.
11. Find the equation of the normal to the curve $y = |x^2 - |x||$ at $x = -2$.
12. Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from point (1, 2).
13. Find the equation of the normal to the curve $x^2 = 4y$, which passes through the point (1, 2)

Vertical Tangent

14. Show that the curve $y = x - e^{xy}$ has a vertical tangent at (1, 0).
15. The curve $x + y - \log(x + y) = 2x + 5$ has a vertical tangent at the point (p, q), then find the value of $p + q + 10$.

Horizontal Tangent

16. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$, where the tangent is horizontal.
17. Find the points on the curve $y = x^3 - x^2 - x + 3$, where the tangents are parallel to x-axis.

Equally Inclined with the Axes

18. At what points on the curve $y = \frac{x^3}{3} + \frac{x^2}{2}$, the tangents make equal angles with the co-ordinate axes?

Equation of Tangent and Normal to a 2nd Degree Curve

19. Find the equation of the tangent to the curve $y = 4ax$ at 't'.
20. Find the equation of the tangent to the curve $x^2 + y^2 + x + y = 0$ at $(1, -1)$
21. Find the equation of the normal to the curve $x^2 + y^2 = 10$ at $(3, 1)$.
22. Find the equation of the normal to the curve $x^2 + y^2 + 4x + 6y + 9 = 0$ at $(-4, -3)$.
23. Find the equation of the normal to the curve $\frac{x^2}{9} + \frac{y^2}{4} = 2$ at $(3, 2)$.
24. Find the number of tangents to the curve $y^2 - 2x^2 - 4y + 8 = 0$, which pass through the point $(1, 2)$.
25. Any tangent at a point $P(x, y)$ to the ellipse $\frac{x^2}{8} + \frac{y^2}{18} = 1$ meets the co-ordinate axes in the points A and B such that the area of the triangle OAB is least, then find the point P.

Length of Intercepts of the Tangents by the Axes

26. If the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at any point on it cuts the axes OX, OY respectively, then prove that OP + OQ = a.
27. Show that the x-intercept of the tangent at an arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to the cube of the abscissa of the point of tangency.

Tangent to the Curve at the Origin

28. Find the tangent to the curve $x^3 + 3xy + y^3 + x^2y = 0$ at the origin.
29. Find the tangent to the curve $ax^2 + 2hxy + by^2 + ax + by = 0$ at the origin.
30. Find the tangent to the curve $(x^4 + y^4)^2 = 2013(x^2 - y^2)$ at the origin.
31. Find the tangent to the curve $x^5 + y^5 + 2010x^2 - 2011y^2 + 2012x - 2013y = 0$ at the origin.

Angle between two Curves

32. Find the angle of intersection of the curves $x^2 = y$ and $y^2 = x$.
33. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.
34. Find the angle between the curves $y^2 = 4x$ and $y = e^{-x/2}$.
35. Find the acute angle between the curves $y = \sin x$ and $y = \cos x$.
36 Find the angle between the curves $2y^2 = x^3$ and $y^2 = 32x$.
37. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6y + 1 = 0$ touch each other at the point $(1, 2)$.
38. Prove that the curves $y = 6 - x + x^2$ and $y = (x - 1)(x + 2)$ touch each other at $(2, 4)$.
39. Prove that the curves $x = y$ and $xy = k$ cut at right angles if $8k^2 = 1$.
40. Find the values of a if the curves $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ cut each other orthogonally.
41. Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.
42. Find the angle of intersection of the curves $y = [|\sin x| + |\cos x|]$, where [,] = G.I.F and $x^2 + y^2 = 5$.

Shortest Distance between two Curves

43. Find the shortest distance between the line $y = x - 2$ and the curve $y = x^2 + 3x + 2$
44. Find the shortest distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$.
45. Find the shortest distance between the curves $y^2 = x^3$ and $9x^2 + 9y^2 - 30y + 16 = 0$.
46. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is minimum.
47. Find the least distance between any two points of the curves $y = 3^x$ and $\log_3 x$.

Common Tangent between two Curves

48. Find the common tangent to the curves $y = x^2 + x + 1$ and $y = x^2 - 5x + 6$.
49. Find the equation of the common tangent to the curves $y = 3x^2$ and $y = 2x^3 + 1$.
50. Find the equation of the common tangent to the curves $y = 6 - x - x^2$ and $xy = x + 3$.

Tangent at a Point Intersect the Curve Again 2nd Point

51. If the tangent at $P(1, 1)$ to the curve $y^2 = x(2 - x)^2$ meets the curve again at Q, then find the co-ordinates of Q.
52. If the tangent at P to the curve $y^2 = x^3$ intersects the curve again at Q and the straight line OP, OQ makes

angles α, β with the x-axis, where O is origin, then find the value of $\left(\frac{\tan\alpha}{\tan\beta} + 2013\right)$.

53. If the tangent at a variable point P on the curve $y = x^2 - x^3$ meets it again at Q. Show that the locus of the middle point of PQ is $y = 1 - 9x + 28x^2 - 28x^3$.
54. A curve is given by the equations $x = \sec^2\theta$ and $y = \cot\theta$. If the tangent at P, where $\theta = \frac{\pi}{4}$ meets the curve again at Q. Find the length of PQ.

Lengths of Tangent, Normal, Sub-tangent and Sub-normal

55. Find the lengths of the tangent, sub-tangent, normal and sub-normal to the curve $y^2 = 4ax$ at the point $P(at^2, 2at)$.
56. Prove that the length of the sub-tangent at any point to the curve $y = be^{x/a}$ is always constant.
57. Find the length of the tangent and normal to the curves $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at the point, where $\theta = \frac{\pi}{2}$.
58. Find the length of the sub-normal to the curve $y^2 = x^3$ at (4, 8).
59. Find the length of the sub-tangent to the curve $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$ at any point (x, y).

Level II

(Mixed Problems)

1. For the curve $x + t^2 - 1$, $y = t^2 - t$, the tangent line is perpendicular to x-axis, where
   (a) $t = 0$ (b) $t = \infty$
   (c) $t = \frac{1}{\sqrt{3}}$ (d) $t - \frac{1}{\sqrt{3}}$
2. The slope of the tangent to the curve $x = t^2 + 3t - 8y = 2t^2 - 2t - 5$ at the point (2, –1) is
   (a) $\frac{22}{7}$ (b) $\frac{6}{7}$
   (c) –6 (d) None
3. Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is
   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$
   (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$
4. The equation of the tangent at the point $P(t)$, where t is any parameter, to the parabola $y^2 = 5ax$ is
   (a) $yt = x + at^2$ (b) $y = xt + at^2$
   (c) $y = tx$ (d) $y = x + \frac{a}{t}$
5. The values of a for which $y = x^2 + ax + 25$ touches the axis of x are
   (a) ±5 (b) ±10
   (c) ±15 (d) None
6. The point on the curve $y^2 = x$, the tangent at which values an angle of 45° with x-axis will be given by
   (a) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$
   (c) (2, 4) (d) $\left(\frac{1}{4}, \frac{1}{2}\right)$
7. If the tangent to the curve $x + y = e^{xy}$ be parallel to the y-axis, then the point of contact is
   (a) (1, 0) (b) (0, 1)
   (c) (1, 1) (d) None
8. If the parametric equation of curve is given by $x = e^t \cos t$, $y = e^t \sin t$, then the tangent to the curve at the point $t = \frac{\pi}{4}$ values with the axis of the angle is
   (a) 0 (b) $\frac{\pi}{4}$
   (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
9. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point
   (a) (1, 1) (b) at no point
   (c) (0, 1) (d) (1, 0)
10. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets the x-axis of
    (a) $(0, a)$ (b) (2, 0)
    (c) $\left(-\frac{1}{2}, 0\right)$ (d) None
11. The line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = b^{-x/a}$ at the point
    (a) $\left(a, \frac{b}{a}\right)$ (b) $\left(-a, \frac{b}{a}\right)$
    (c) $\left(a, \frac{a}{b}\right)$ (d) $(0, b)$
12. The tangent to the curve $y = x^2 + 3x$ will pass through the point (0, –9) if it is at the point
    (a) (3, 18) (b) (1, 4)
    (c) (–4, 4) (d) (–3, 0)
13. If the tangent at $P(1, 1)$ on $y^2 = x(2 - x)^2$ meets the curve again at Q then the point Q is
    (a) (–1, 2) (b) $\left(\frac{9}{4}, \frac{3}{8}\right)$
    (c) (4, 4) (d) None
14. The co-ordinates of the point on the curves $y = x^2 + 3x + 4$ the tangent of at which passes through the origin is equal to

(a) (2, 14) (–2, 2) (b) (2, 14) (–2, –2)
(c) (2, 14) (2, 2) (d) None

15. If the tangent (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then P is
(a) (–1, 2) (b) (4, 4)
(c) $\left(\frac{9}{4}, \frac{3}{8}\right)$ (d) None

16. The number of points on the curve $x^{3/2} + y^{3/2} = a^{3/2}$ where the tangents are equally inclined to the axes is
(a) 1 (b) 2
(c) 4 (d) None

17. The point on the curve $\sqrt{x} + \sqrt{y} = 2a^2$ at which the tangent is equally inclined to the axes is
(a) $(4a^4, 0)$ (b) $(0, 4a^4)$
(c) (a^4, a^4) (d) None

18. The area of the triangle formed by the tangent to the curve $y = \frac{8}{4 + x^2}$ at $x = 2$ and the co-ordinate axes is
(a) 2 sq. units (b) 4 sq. units
(c) 8 sq. units (d) $\frac{7}{2}$ sq. units

19. Any tangent at a point $P(x, y)$ to the ellipse $\frac{x^2}{8} + \frac{y^2}{18} = 1$ meets the co-ordinate axes in the points A and B such that the area of the triangle OAB is least, then the point P is
(a) $(\sqrt{8}, 0)$ (b) $(0, \sqrt{18})$
(c) (2, 3) (d) None

20. The co-ordinates of the point P on the curve $y^2 = 2x^3$, the tangent at which is perpendicular to the line $4x - 3y + 2 = 0$ are given by
(a) (2, 4) (b) (0, 0)
(c) $\left(\frac{1}{8}, -\frac{1}{16}\right)$ (d) None.

21. If $y = 4x - 5$ is a tangent to the curve $y^2 = ax^3 + b$ at (2, 3) then
(a) $a = 2, b = -7$ (b) $a = -2, b = 7$
(c) $a = -2, b = -7$ (d) $a = 2, b = 7$

22. If the tangent to the curve $xy + ax + by = 0$ at (1, 1) is inclined at an angle $\tan^{-1} 2$ to axis of x then (a, b) is
(a) (–1, –2) (b) (–1, 2)
(c) (1, –2) (d) (1, 2)

23. A function $y = f(x)$ has a 2nd order derivative $f'(x) = 6x - 1$. If the graph passes through the point (2, 1) and at this point tangent to the graph is $y = 3x - 1$, then the function is
(a) $(x - 1)^3$ (b) $(x - 1)^2$
(c) $(x + 1)^3$ (d) $(x + 1)^2$

24. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then
(a) $a > 0, b > 0$ (b) $a > b, b < 0$
(c) $a < 0, b > 0$ (d) $a < 0, b < 0$.

25. The equation to the normal to the curve $y = \sin x$ at (0, 0) is
(a) $x = 0$ (b) $y = 0$
(c) $x + y = 0$ (d) $x - y = 0$

26. The normal to the curve $x = a(1 + \cos\theta)$, if $a\sin\theta$ at θ always passes through the fixed point
(a) (a, a) (b) $(a, 0)$
(c) $(0, a)$ (d) None

27. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the $+ve$ x-axis $f'(3)$ is
(a) –1 (b) $-\frac{3}{4}$
(c) $\frac{4}{3}$ (d) 1

28. The point on the curve where the normal to the curve $9y^2 = x^3$ makes equal intercepts with the axes is
(a) $\left(4, \frac{8}{3}\right)$ (b) $\left(-4, \frac{8}{3}\right)$
(c) $\left(4, -\frac{8}{3}\right)$ (d) None

29. The normal at any point $P\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve at $Q\left(ct_1, \frac{c}{t_1}\right)$ then t_1 is
(a) $-t$ (b) $\frac{1}{t^2}$
(c) $-\frac{1}{t^3}$ (d) None

30. The values of parameter 'a' so that the line $(3 - a)x + ay + a^2 - 1 = 0$ is a normal to the curve $xy = 1$ is/are
(a) $(3, \infty)$ (b) $(-\infty, 0)$
(c) (0, 3) (d) None

31. The angle between the curves $y^2 = x$ and $x^2 = y$ at (1, 1) is
(a) $\tan^{-1}$ (4/5) (b) $\tan^{-1}$ (3/4)
(c) 45° (d) 90°

32. Angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$ is
(a) $\frac{\pi}{2}$ (b) $\tan^{-1}\left(\frac{4}{3}\right)$
(c) $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ (d) None

33. The curve $x^3 - 3xy^2 + 2 = 0$ and $3x^2 y - y^3 - 2 = 0$ cut at an angle of
(a) 45° (b) 60°
(c) 90° (d) 30°

34. If the curve $y^2 = 16x$ and $9x^2 + by^2 = 16$ cut each other at right angles, then the values of b is
(a) 2 (b) 4
(c) $\frac{9}{2}$ (d) None

35. If $x + y = 9$ is a normal to the parabola $y^2 - 2y - 8x + 17 = 0$, then the point is
(a) (5, 4) (b) (4, 5)
(c) (4, –5) (d) (–4, 5)

36. A line L is perpendicular to the curve $y = \frac{x^2}{4} - 2$ at its point P and passes through (10, –1). Then the co-ordinates of point P is
(a) (2, –1) (b) (6, 7)
(c) (0, –2) (d) (4, 2).

37. If the line joining the points (0, 3) and (5, –2) is a tangent to the curve $y = \frac{ax}{x+1}$, then the value of a is/are
(a) $a = 1 \pm \sqrt{3}$ (b) $a = 2 \pm 2\sqrt{3}$
(c) $a = -1 \pm 1\sqrt{3}$ (d) $a = 2 - \sqrt{3}$

38. Minimum distance between two points P and Q, where P lies on the parabola $y^2 - x + 2 = 0$ and Q lies on the parabola $x^2 - y + 2 = 0$ is
(a) $7\sqrt{2}$ (b) 4
(c) $\frac{7}{2\sqrt{2}}$ (d) None

39. If normal drawn at any point P of the parabola $y^2 = 4x$ meets the curve again at Q, then the least distance of Q from the origin is
(a) $6\sqrt{3}$ (b) $4\sqrt{6}$
(c) $9\sqrt{6}$ (d) None

40. The equation of the common tangent to the curve $y = 6 - x - x^2$ and $xy = x + 3$ is
(a) $3x + y = 7$ (b) $3x - y = 7$
(c) $3x + y + 7 = 0$ (d) None

41. The number of tangents to the curve $y^2 - 2x^2 - 4y + 8 = 0$ that pass through (1, 2) is
(a) 3 (b) 1
(c) 2 (d) 6

42. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at (1, 1) and the co-ordinate axes, lies in the first quadrant. If its area is 2, then the value of b is
(a) –1 (b) 3
(c) –3 (d) 1

43. The minimum slope of the curve $y = x^3 + 3x^2 + 3x - 1$ is
(a) 0 (b) –2
(c) –1 (d) None

44. The length of sub-normal to the parabola $y^2 = 4ax$ at any point is
(a) $a\sqrt{2}$ (b) $a \cdot 2\sqrt{2}$
(c) $\frac{a}{\sqrt{2}}$ (d) $2a$

45. The length of sub-tangent to the curve $\sqrt{x} + \sqrt{y} = 3$ at (4, 1) is
(a) 2 (b) $\frac{1}{2}$
(c) –3 (d) 4

46. The length of normal to the curve
$x = a(t + \sin t)$, $y = a\left(1 - \cos t \text{ at } t = \frac{\pi}{2}\right)$ is
(a) $2a$ (b) $a\sqrt{2}$
(c) $\frac{a}{2}$ (d) $\frac{a}{\sqrt{2}}$

47. The length of the normal to the curve at (x, y)
$y = a\left(\frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}\right)$ at any point varies as
(a) x (b) x^2
(c) y (d) y^2

48. The value of n for which the length of the sub-normal to the curve $xy^n = a^{n+1}$ is constant, is
(a) 1 (b) –1
(c) 2 (d) –2

49. For the parabola $y^2 = 4ax$, the ratio of the sub-tangent to the abscissa is
(a) 1 : 1 (b) 2 : 1
(c) $x : y$ (d) $x^2 : y$

50. The length of the normal at θ on the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ is
(a) $a\sin^2\theta$ (b) $a\sin 2\theta + \tan\theta$

51. Tangent at $P(2, 8)$ on the curve $y = x^3$ meets the curve again at $Q(a, b)$. Then
(a) $a = -4$ (b) –64
(c) $a - b = 60$ (d) $a - b = -60$

52. The points on the curve $y = x^2 + x + 3$ the tangent at which passes through the origin is/are
(a) (2, –2) (b) (–2, 2)
(c) (2, 14) (d) (14, 2)

53. Let $y = |x^2 - |x||$ at $x = -2$. Then
(a) Point of contact is (–2, 2)
(b) slope of tangent is 1/3

(c) Equation of normal is $3y = x + 8$

(d) Equation of normal is $y = x + 8$

54. Let two curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other at two points. Then

(a) the points of intersections are (2, 2) and (–2, –2)

(b) the points of intersections are (3, 2) and (–3, –2)

(c) equations of common tangents are $x + y \pm 4 = 0$

(d) equations of common tangents are $x + y \pm 9 = 0$

55. Let the line be $y = x - 2$ and the parabola be $y = x^2 + 3x + 2$. Then

(a) the nearest point between the curves is (–1, 0)

(b) the nearest point between the curves is (–2, 0)

(c) the shortest distance between the curves is $\frac{3}{\sqrt{2}}$

(d) the shortest distance between the curves is $\frac{5}{\sqrt{2}}$

Level III

(Problems for JEE-Advanced)

1. Does the straight line $\frac{x}{a} + \frac{y}{b} = 2$ touch the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 2$? If it touches, find the co-ordinates of the point of contact. **[Roorkee JEE, 1984]**

2. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on $x^2y^2 = x^2 - y^2$. **[Roorkee JEE, 1986]**

3. In the curve $x^a y^b = k^{a+b}$, prove that the portion of the tangent intercepted between the co-ordinate axes is divided at its point of contact into segments which are in a constant ratio. **[Roorkee JEE, 1988]**

4. Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2). **[Roorkee JEE, 1990]**

5. Show that the normal to the curve $5x^5 - 10x^3 + x + 2y + 6 = 0$ at $P(0, -3)$ meets the curve again at two points . Find the equations of the tangents to the curve at those points. **[Roorkee JEE, 1992]**

6. Find the points on the curve $9y^2 = x^3$ where normal to the curve makes equal intercepts with the axes. **[Roorkee JEE, 1993]**

7. Find the acute angles between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection. **[Roorkee JEE, 1998]**

8. Find the equations of the common tangents of the circle $x^2 + y^2 - 6y + 4 = 0$ and the parabola $y^2 = x$ **[Roorkee JEE, 1999]**

9. Find the equation of the straight line which has tangent at one point and normal at another point of the curve $x = 3t^2$, $y = 2t^2$ **[Roorkee JEE, 2000]**

10. Find the equation of the tangent to the curve $y = (2x - 1)e^{2(1-x)}$ at the point of its maximum.

11. Let the parabola $y = x^2 + ax + b$ and $y = x(c - x)$ touch each other at the point (1, 0), then find the value of $(a + b + c + 4)$.

12. Find all points at which the tangents to the curves $y = x^3 - x + 1$ are $y = 3x^2 - 4x + 1$ are parallel.

13. Find the equation of the staright line which is tangent at one point and normal at another point of the curve $x = 3t^2$, $y = 2t^3$.

14. Find the points on the curve $9y^2 = 3x^3$ where normal to the curve makes equal intercepts with the axes.

15. The traingle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the co-ordinate axes, lies in the first quadrant. If the area is 2, then find the value of b.

16. Find the equation of the normal to the curve $x^3 + y^3 = 8xy$ at the point where it meets the curve $y^2 = 4x$ other than origin.

17. Find the value of a, if $1 - \cos x = \frac{\sqrt{3}}{2}|x| + a$ where $x \in \left(0, \frac{\pi}{2}\right)$ has exactly one solution.

18. Find the equation of the common tangent to the curves $y = x^2 - 5x + 4$ and $y = x^2 + x + 1$.

19. If the curves $ay + x^2 = 7$ and $y = x^3$ cut each other orthogonally at a point, find a.

20. If the curves $y = 1 - ax^2$ and $y = x^2$ are orthogonal, then find a.

21. Let $y = f(x)$ be a curve whose parametric equation is $x = t^2 + t + 1$, $y = t^2 - t + 1$, where $t > 0$. Find the number of tangents that can be drawn to this curve from (1, 1).

22. If $y = e^x$ and $y = kx^2$ touches each other, then find k.

24. Find all the lines that pass through the point (1, 1) and are tangent to the curve represented parametrically as $x = 2t - t^2$, $y = t + t^2$ provided $t \neq 1$.

25. Find all the normals to the curve $x = a\cos t + at\sin t$, $y = a\sin t - a\,t\cos t$ and also find its distance from the origin.

Level IV

(Tougher Problems for JEE-Advanced)

1. If three normals can be drawn to the curve $y^2 = x$ from the point $(c, 0)$, then find c.

2. If the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ intersect orthogonally, then prove that

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$$

3. Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2).
4. Find the equation of the straight line which is a tangent at one point and normal at another point to the curve $y = 8t^3 - 1$, $x = 4t^2 + 3$.
5. Three normals are drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$. Find the co-ordinates of the feet of the normals.
6. If the equation of the normal of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the positive direction of x-axis, prove that the equation of the normal $x\sin\phi - y\cos\phi + a\cos 2\phi = 0$.
7. Find all the tangents to the curve $y = \cos(x + y)$, $[-2\pi, 2\pi]$ that are parallel to the line $x + 2y = 0$.
8. A curve is given by the equations $x = \sec^2\theta$ and $y = \cot\theta$. If the tangent at P where $\theta = \frac{\pi}{4}$ meets the curve again at Q. Find PQ.
9. If p_1 and p_2 be the lengths of perpendiculars from the origin on the tangent and normal respectively at any point (x, y) on the curve $x^{2/3} + y^{2/3} = a^{2/3}$, then show that $4p_1^2 + p_2^2 = a^2$.
10. If the tangent at a variable point P on the curve $y = x^2 - x^3$ meets it again at Q, then prove that the locus of the middle point of PQ is
$y = 1 - 9x + 28x^2 - 28x^3$.

Integer Type Questions

1. Find the number of tangents to the curve $x^{3/2} + y^{3/2} = a^{3/2}$, $a > 0$ which are equally inclined to the axes.
2. Find the number of tangents to the curve
$y^2 - 2x^2 - 4y + 8 = 0$ that pass through (1, 2).
3. Find the maximum slope of the curve
$y = -x^3 + 3x^2 + 3x - 1$.
4. The number of values of c such that the straight line $3x + 4y = c$ touches the curve
$\frac{x^4}{2} = x + y$.
5. If $y = 4x - 5$ is a tangent to the curve $y^2 = px^3 + q$ at (2, 3), find the value of $(p - q - 4)$.
6. Find the value of c such that the line joining the points (0, 3) & (5 – 2) becomes tangent to the curve $y = \frac{c}{x+1}$.
7. Find the slope of the tangent to the curve $y = x^2 - x$ at the point where the line $y = 2$ cuts the curve in the first quadrant.
8. If the tangent at any point $P(4m^2, 8m^3)$ of $x^3 - y^2 = 0$ is a normal also the curve $x^3 - y^2 = 0$ then find the value of $(9m^2 + 2)$.
9. Find the sum of the intercepts made on the axes of co-ordinates by any tangent to the curve $\sqrt{x} + \sqrt{y} = 2$.
10. If m is the length of the subnormal to the curve $y^2 = x^3$ at the point (4, 8), then find the value of $\sqrt{m} + 1$.
11. The curve $(x + y) - \ln(x + y) = 2x + 5$ has a vertical tangent at the point (α, β), then the value of $(|\alpha + \beta| + 4)$.
12. If $x + y = a$ be a tangent to the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$, then find the value of $(|a| + 1)$.
13. If m is the number of points on the curve $xy = 2$ where the tangent makes an acute angle with the x-axis, then find the value of $(m + 4)$.
14. If the co-ordinates of the point $y^2 = 8x$ which is nearest to the circle $x^2 + (y + 6)^2 = 1$ is (α, β) then find the value of $(|\alpha + \beta| + 3)$.
15. If m is the slope of the common tangents $y = x^2 - x + 1$ and $y = x^2 - 3x + 1$, then find the value of $(|m| + 3)$.

Comprehensive Link Passages (For JEE-Advanced Exam Only)

Passage I

If the equation of the curve be given in the parametric form, say, $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$. The equation of tangent at any point 't' on the curve is $y - g(t) = \frac{g'(t)}{f'(t)}(x - f(t))$ and the equation of the normal at any point 't' is $y - g(t) = -\frac{f'(t)}{g'(t)}(x - f(t))$.

On the basis of the above information, answers the following questions.

1. The equation of tangent to the curve
$x = \theta + \sin\theta$, $y = 1 + \cos\theta$ at $\theta = \frac{\pi}{4}$ is
(a) $y = (\sqrt{2} - 1)x + 2$
(b) $y = -(\sqrt{2} - 1)x + (\sqrt{2} - 1)\frac{\pi}{4} + 2$
(c) $y = -(\sqrt{2} - 1)x - (\sqrt{2} - 1)\frac{\pi}{4} + 2$
(d) $y = -(\sqrt{2} - 1)x - (\sqrt{2} + 1)\frac{\pi}{4} + 2$
2. The equation of the tangent to the curve
$x = 2t - t^2$ & $y = t + t^2$ at (1, 1) is
(a) $5x - 4y = 1$ (b) $4x - 5y = 1$
(c) $3x - 4y = 1$ (d) $4x - 3y = 1$

3. The equation of the normal to the curve
$x = a(\cos\theta + \theta\sin\theta)$,
$y = a(\sin\theta - \theta\cos\theta)$ at any point θ is
(a) $x\cos\theta + y\sin\theta = a$ (b) $x\cos\theta - y\sin\theta = a$
(c) $-x\cos\theta + y\sin\theta = a$ (d) $x\sin\theta + y\cos\theta = a$

Passage II

The angle of intersection between two curves is the angle between their tangents at their point of intersection.

Let the equation of the curves be

$$f(x, y) = 0 \quad ...(i)$$

and

$$g(x, y) = 0 \quad ...(ii)$$

Suppose $p(x_1, y_1)$ be the point of intersection of (i) and (ii).

Let $m_1 = \left(\frac{dy}{dx}\right)_1$ at $p(x_1, y_1)$,

$m_2 = \left(\frac{dy}{dx}\right)_2$ at $p(x_1, y_1)$ and θ be the angle between the curves.

Then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

On the basis of the above information answer the following questions.

1. The angle between the curves $y = x^2$ and $x = y^2$ is
(a) $\tan^{-1}\left(\frac{1}{3}\right)$ (b) $\tan^{-1}\left(\frac{2}{3}\right)$
(c) $\tan^{-1}\left(\frac{3}{4}\right)$ (d) None.
2. The angle between the curves $y = \sin 2x$ and $y = \cos 2x$ is
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$
3. The angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their intersection is
(a) $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$ (b) $\tan^{-1}\left(\frac{2\sqrt{2}}{7}\right)$
(c) $\tan^{-1}\left(\frac{5\sqrt{2}}{8}\right)$ (d) $\tan^{-1}\left(\frac{7\sqrt{2}}{8}\right)$
4. The angle between the curves $2y^2 = x^3$ and $y^2 = 32x$ is
(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}\left(\frac{1}{3}\right)$
(c) $\tan^{-1}\left(\frac{1}{4}\right)$ (d) $\tan^{-1}\left(\frac{1}{2}\right)$

Passage III

A line which touches two given curves is called a common tangent to the curves.

1. The equation of the common tangent of the curves $xy = 4$ and $x^2 + y^2 = 8$ is
(a) $x + y = 4$ (b) $x - y = 4$
(c) $-x + y = 4$ (d) $x + y + 5 = 0$
2. A tangent drawn to the curve C_1: $y = x^2 + 4x + 8$ at its point P touches the curve C_2 : $y = x^2 + 8x + 4$ at its point Q. Then the co-ordinates of P and Q are
(a) $P(2, 10)$, $Q(0, 4)$ (b) $P(2, 20)$, $Q(0, 4)$
(c) $P(10, 2)$, $Q(0, 4)$ (d) $P(20, 2)$, $Q(4, 0)$.
3. The common tangent to the curves $y = x^2 - 5x + 6$ and $y = x^2 + x + 1$ is
(a) $3x + 9y = 5$ (b) $9x + 3y = 5$
(c) $5x + 9y = 3$ (d) $9x + 5y = 3$.

Passage IV

The shortest (largest) distance between two non-intersecting curves is found along the common normal to the two curves.

1. The shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$ is
(a) $\frac{3}{\sqrt{2}}$ (b) $\frac{5}{\sqrt{2}}$
(c) $\frac{7}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$
2. The minimum distance between the curves $y^2 = 4x$ and $x^2 + y^2 - 12x + 31 = 0$ is
(a) $\sqrt{21}$ (b) $\sqrt{21} - \sqrt{5}$
(c) $\sqrt{26} - \sqrt{5}$ (d) $\sqrt{26} + \sqrt{5}$
3. The point on the curve $x^2 + y^2 = 6$ whose distance from the line $x + y = 7$ is minimum is
(a) (1, 2) (b) (2, 1)
(c) (1, 3) (d) (3, 1).

Passage V

The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$ and cuts the y-axis at Q, where its gradient is 3. Then

1. The value of a is
(a) $-1/2$ (b) $-1/4$
(c) $-3/4$ (d) $-5/4$
2. The value of b is
(a) $-1/2$ (b) $-1/4$
(c) $-3/4$ (d) $-5/4$
3. The value of $2a + 4b + c$ is
(a) -1 (b) -2
(c) -3 (d) -4.

Passage VI

A function $y = f(x)$ has a 2nd order derivative $f''(x) = 6(x - 1)$. If the curve passes through the point (2, 1) and at this point tangent to the curve is $y = 3x - 1$. Then

1. The slope of the curve $y = f(x)$ is
(a) 2 (b) 3
(c) 4 (d) –1
2. The equation of the given curve is
(a) $(x + 1)^3$ (b) $(x - 2)^3$
(c) $(x - 1)^3$ (d) $(x + 2)^3$
3. The equation of the normal to the curve $y = f(x)$ at (2, 3) is
(a) $x + 3y = 11$ (b) $x - 3y = 12$
(c) $3x - y = 11$ (d) $3x - y = 12$

Matrix Match (For JEE-Advanced Exam Only)

1. **Match the Following Columns**

	Column I		Column II
(A)	The equation of the tangent to the curve $y = e^x$ at $x = 0$ is	(P)	$y = x - 1$
(B)	The equation of the normal to the curve $x + y = x^y$, where it cuts the x-axis is	(Q)	$y = x + 1$
(C)	The equation of the normal to the curve $y = \lvert x^2 - \lvert x \rvert \rvert$ at $x = -2$ is	(R)	$y = 2x + 2$
(D)	The equation of the tangent to the curve $y = x^4 + 2e^x$ at (0, 2) is	(S)	$3y = x + 8$

2. **Match the following columns**
The tangent $y = ax^2 + bx + 10$ at (1, 2) is parallel to the normal at the point (2, 3) on the curve $y = x^2 + 6x + 20$. Then

	Column I		Column II
(A)	The value of a is	(P)	–79/10
(B)	The value of b is	(Q)	–159/10
(C)	The value of $2a + b$ is	(R)	–8
(D)	The value of $a + b$ is	(S)	–1/10

3. **Match the following columns**
Let the equation of the curve is $y = x^3 + 3x + 4x - 1$ at $x = 0$

	Column I		Column II
(A)	The length of the tangent is	(P)	4
(B)	The length of the normal is	(Q)	1/4
(C)	The length of the sub-tangent is	(R)	$\sqrt{17}$
(D)	The length of the sub-normal is	(S)	$\sqrt{17}/4$

4. **Match the following columns**

	Column I		Column II
(A)	If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $3\pi/4$ with the positive x-axis, then $f'(3)$ is	(P)	3
(B)	If m is the number of horizontal tangents and n is the number of vertical tangents to the curve $y^2 - 3xy + 2 = 0$, then $m + n$ is	(Q)	1
(C)	If the tangent to the curve $xy + ax + by = 0$ at (1, 1) is inclined at an angle of $\tan^{-1}2$ with x-axis, then $2a + b$ is	(R)	–3
(D)	The minimum slope of the curve $y = x^3 + 3x^2 + 3x - 1$ is	(S)	0

5. **Match the following columns**

	Column I		Column II
(A)	The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$. If the curve passes through the point (1, 2), then the area bounded by the curve, the x-axis and the line $x = 1$ is $m/6$ sq. u, then the value of m is	(P)	3
(B)	If the normal to the curve $y = f(x)$ at (3, 4) makes an angle $3\pi/4$ with the $+ve$ x-axis, then $f'(3)$ is	(Q)	2
(C)	The ordinate of the point on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical is	(R)	1
(D)	The angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is $\frac{\pi}{\lambda}$, then the value of λ is	(S)	5

6. **Match the following columns**
Let the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$. Then

	Column I		Column II
(A)	the equation of the tangent is $y = 2x + \lambda$, where λ is	(P)	5
(B)	if the tangent touches the given circle at P, then the x-co-ordinate of P is	(Q)	–6

(C)	the equation of the normal to the curve $y = x^2 + 6$ at (1, 7) is $x + 2y = 5k$, where k is	(R)	3
(D)	the value of 'c/13' is	(S)	9

7. **Match the following columns**

Let $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2$ for all $x_1, x_2 \in R$. Then

Column I		Column II	
(A)	the equation of the tangent to the curve $y = f(x)$ at (1, 2) is	(P)	$y - 2 = 0$
(B)	the equation of the normal to the curve $y = f(x)$ at (2, 3) is	(Q)	$x - 2 = 0$
(C)	the area bounded by the curve $y = f(x)$, tangent at (1, 2) and the lines $x = 0$, $x = 2$ is m sq. u, where m is	(R)	4
(D)	the area bounded by the curve $y = f(x)$, tangent at (1, 2), normal at (2, 3) and the axes is n sq. u, where n is	(S)	5

Questions asked in Previous Years' IIT-JEE Exams

1. Find the shortest distance of the point (0, c) from the parabola $y = x^2$ where $0 \le c \le 5$ **[IIT-JEE, 1982]**
2. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point θ is such that
   (a) it makes a constant angle with the x-axis
   (b) it passes through the origin.
   (c) it is at a constant distance from the origin
   (d) None of these.
   [IIT-JEE, 1983]
3. No questions asked in 1984.
4. Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ that are parallel to the line $x + 2y = 0$ **[IIT-JEE, 1985]**
5. If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then
   (a) $a > 0, b > 0$ (b) $a > 0, b < 0$
   (c) $a < 0, b > 0$ (d) $a < 0, b < 0$
   [IIT-JEE, 1986]
6. What normal to the curve $y = x^2$ forms the shortest chord? **[IIT-JEE, 1992]**
7. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point
   (a) (1, 1) (b) at no point
   (c) (0, 1) (d) (1, 0)
   [IIT-JEE, 1992]
8. Find the equations of the normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$. **[IIT-JEE, 1993]**
9. Tangent at a point P_1 (other than (0, 0) on the curve $y = x^3$ meets the curve at P_2 and tangent at P_2 meets the curve at P_3 and so on. Show that the abcissae of $P_1, P_2, ..., P_n$ form a G.P. Also, find the ratio of
   $\left(\dfrac{Ar(\Delta P_1P_2P_3)}{Ar(\Delta P_2P_3P_4)}\right)$ **[IIT-JEE, 1993]**
10. If $y = 4x - 5$ is a tangent to the curve $y^2 = px^3 + q$ at (2, 3), then
    (a) $p = 2, q = 7$ (b) $p = -2, q = 7$
    (c) $p = -2, q = -7$ (d) None
    [IIT-JEE, 1994]
11. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$ and cuts the y-axis at Q where its gradient is 3. Find a, b, c. **[IIT-JEE, 1994]**
12. Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of the point on the curve C where the tangent is vertical, then H = ... and V = ...
13. If the normal to the curve $y = f(x)$ at the point (3, 4) makes an angle $\dfrac{3\pi}{4}$ with the $+ve$ x-axis, then the value of $f'(3)$ is
    (a) –1 (b) –3/4
    (c) 4/3 (d) 1
    [IIT-JEE, 2000]
14. The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the co-ordinate axes, lies in the first quadrant. If its area is 2 sq. unit, then the value of b is
    (a) –1 (b) 3
    (c) –3 (d) 1
    [IIT-JEE, 2001]
15. The points on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical, is (are)
    (a) $\left(\pm\dfrac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\sqrt{\dfrac{11}{3}}, 1\right)$
    (c) (0, 0) (d) $\left(\pm\dfrac{4}{\sqrt{3}}, 2\right)$
    [IIT-JEE, 2002]

16. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is minimum.
[IIT-JEE, 2003]

17. Tangent to the curve $y = x^2 + 6$ at a point $P(1, 7)$ touches the curve $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the co-ordinates of Q are

(a) $(-9, -13)$ (b) $(-10, -15)$
(c) $(-6, -7)$ (d) $(6, -7)$
[IIT-JEE, 2005]

18. If a function $f(x)$ satisfies the condition $|f(x) - f(y)| \le (x - y)^2$, $\forall x, y \in R$. Find an equation of tangent to the curve $y = f(x)$ at the point (1, 2).
[IIT-JEE, 2005]

19. No questions asked in 2006.

20. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$, $(c + 1, e^{c+1})$

(a) on the left of $x = c$
(b) on the right of $x = c$
(c) at no point
(d) at all points.
[IIT-JEE, 2007]

21. No questions asked in between 2008 to 2010.

22. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to the straight line $2x - y = 1$, The points of contact to the tangent and the hyperbola are

(a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
(c) $(3\sqrt{3}, -2\sqrt{2})$ (d) $(-3\sqrt{3}, 2\sqrt{2})$
[IIT-JEE, 2011]

23. No questions asked in between 2012 to 2013.

24. The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1, 3) is...
[IIT-JEE, 2014]

ANSWERS

LEVEL II

1. (a)	2. (b)	3. (a)	4. (a)	5. (b)
6. (a)	7. (a)	8. (d)	9. (d)	10. (c)
11. (d)	12. (a)	13. (b)	14. (a)	15. (c)
16. (a)	17. (c)	18. (b)	19. (c)	20. (c)
21. (a)	22. (c)	23. (a)	24. (b)	25. (c)
26. (b)	27. (d)	28. (a)	29. (c)	30. (a)
31. (b)	32. (c)	33. (c)	34. (c)	35. (b)
36. (d)	37. (b,d)	38. (c)	39. (b)	40. (a)
41. (c)	42. (d)	43. (d)	44. (a)	45. (b)
46. (d)	47. (d)	48. (b)	49. (b)	50. (c)
51. (a,b,c)	52. (c,c)	53. (a,b,c)	54. (a,c)	55. (a,c)

INTEGER TYPE QUESTIONS

1. 1	2. 2	3. 3	4. 1	5. 5
6. 4	7. 2	8. 4	9. 4	10. 5
11. 5	12. 6	13. 4	14. 5	15. 5

COMPREHENSIVE LINK PASSAGES

Passage I:	1. (b),	2. (a),	3. (a)
Passage II:	1. (c),	2. (c),	3. (a)
Passage III:	1. (a),	2. (b),	3. (a)
Passage IV:	1. (c),	2. (b),	3. (b)
Passage V:	1. (a),	2. (c),	3. (a)
Passage VI:	1. (b),	2. (a),	3. (a)

MATRIX MATCH

1. (A)→(Q), (B)→(P), (C)→(S), (D)→(R)
2. (A)→(P), (B)→(Q), (C)→(S), (D)→(R).
3. (A)→(S), (B)→(R), (C)→(Q), (D)→(P).
4. (A)→(Q), (B)→(Q), (C)→(R), (D)→(S).
5. (A)→(S), (B)→(R), (C)→(Q), (D)→(P).
6. (A)→(P), (B)→(Q), (C)→(R), (D)→(S).
7. (A)→(P), (B)→(Q), (C)→(R), (D)→(R).

HINTS AND SOLUTIONS

Level I

1. The given curve is

$$y = x^3 + 3x^2 + 3x - 10$$

$$\frac{dy}{dx} = 3x^3 + 6x + 3$$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=2} = 3.4 + 6.2 + 3 = 27$

Hence, the slope of the tangent is 27.

2. The given curve is $y = x^x + 1$

$$\frac{dy}{dx} = x^x(\log x + 1)$$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=2} = 2^2(\log 2 + 1) = 4(\log 2 + 1)$

Hence, the slope of the normal is $= -\frac{1}{4(\log 2 + 1)}$

3. The given curve is $y = \frac{ax}{b - x}$

Since the point (1, 1) lies on the curve,

so $1 = \frac{a}{b-1}$

$\Rightarrow \quad a = b - 1$

Also, $\frac{dy}{dx} = \frac{ab}{(b-x)^2}$

Now, $\left(\frac{dy}{dx}\right)_{(1,1)} = 2$

$\Rightarrow \quad \frac{ab}{(b-1)^2} = 2$

$\Rightarrow \quad a(a+1) = 2a^2$

$\Rightarrow \quad a^2 + a = 2a^2$

$\Rightarrow \quad a^2 = a$

$\Rightarrow \quad a = 0, 1$

when $a = 0, b = 1$

Then $a + b + 10 = 0 + 1 + 10 = 11$

when $a = 1, b = 2$

Then $a + b + 10 = 1 + 2 + 10 = 13.$

4. when $x = 0, y = 1$

Thus, the point is (0, 1)

Now, $\frac{dy}{dx} = 2.e^{2x}$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=0} = 2.1 = 2$

Hence, the equation of the tangent is

$y - 1 = 2(x - 0)$

$\Rightarrow \quad y = 2x + 1$

5. when $x = 1, y = 0$

Thus, the point is (1, 0)

Now, $\frac{dy}{dx} = \frac{1}{3}(x-1)^{-\frac{2}{3}}$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=1} = \infty$

Hence, the equation of the tangent to the curve is

$y - 0 = \infty(x - 1)$

$\Rightarrow \quad (x-1) = \frac{y}{\infty} = 0$

$\Rightarrow \quad x = 1$

6. The equation of the given curve is $y = be^{-x/a}$

put $x = 0$, then $y = b$

So, the point is $(0, b)$

Now, $\frac{dy}{dx} = -\frac{b}{a}e^{-x/a}$

Thus, $m = \left(\frac{dy}{dx}\right)_{(0,b)} = -\frac{b}{a}$

Hence, the equation of the tangent is

$y - b = -\frac{b}{a}(x - 0)$

$\Rightarrow \quad ay - ab = -bx$

$\Rightarrow \quad bx + ay = ab$

$\Rightarrow \quad \frac{bx}{ab} + \frac{ay}{ab} = \frac{ab}{ab}$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$

7. when $x = 0, y = 2$

So, the point is (0, 2)

Now, $\frac{dy}{dx} = -\frac{8x}{(x^2+2)^2}$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=0} = -\frac{8}{4} = -2$

Hence, the equation of the tangent is

$y - 2 = -2(x - 0)$

$\Rightarrow \quad y = 2 - 2x.$

8. The given curve is $x^3 + y^3 = 3xy$

$\Rightarrow \quad 3x^2 + 3y^2\frac{dy}{dx} = 6\left(x\frac{dy}{dx} + y \cdot 1\right)$

$\Rightarrow \quad (y^2 - 2x)\frac{dy}{dx} = (2y - x^2)$

$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{2y - x^2}{y^2 - 2x}\right)$

Thus, $m = \left(\frac{dy}{dx}\right)_{(3,3)} = \frac{6-9}{9-6} = -1$

Hence, the equation of the normal is

$y - 3 = 1(x - 3)$

$\Rightarrow \quad x - y = 0$

9. when $x = 0, y = 14$

Hence, the point is (0, 14)

Now, $\frac{dy}{dx} = 6x + 2\cos x - 4\sin x$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=0} = 2$

Hence, the equation of the normal is

$y - 14 = -\frac{1}{2}(x - 0)$

$\Rightarrow \quad 2y - 28 = -x$

$\Rightarrow \quad x + 2y = 28.$

10. The equation of the given curve is $x + y = x^y$ put $y = 0$, then $x = 1$.

So, the point is (1, 0)

Now, $x + y = x^y$

$\Rightarrow \log(x + y) = y \log x$

$\Rightarrow \frac{1}{(x + y)}\left(1 + \frac{dy}{dx}\right) = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dy}$

put $x = 1, y = 0, \left(1 + \frac{dy}{dx}\right) = 0$

$\Rightarrow \left(\frac{dy}{dx}\right) = -1$

$\Rightarrow$ Slope of normal = 1

Hence, the equation of the normal is

$y - 0 = 1\ (x - 1)$

$\Rightarrow y = x - 1.$

11. The equation of the given curve is

$y = |x^2 - |x||$

$\Rightarrow y = |x^2 - (-x)|$ (since at $x = -2$, $|x| = -x$)

$\Rightarrow y = |x^2 + x|$

$\Rightarrow y = x^2 + x$ (at $x = -2$, $x^2 + x > 0$)

when $x = -2, y = 2$

So, the point is (–2, 2)

Now, $\frac{dy}{dx} = 2x + 1$

Thus, $m = \left(\frac{dy}{dx}\right)_{x=-2} = -3$

So, slope of the normal = 1/3

Hence, the equation of the normal is

$y - 2 = \frac{1}{3}(x + 2)$

$\Rightarrow 3y - 6 = x + 2$

$\Rightarrow 3y = x + 8$

12. The equation of the given curve is

$y^2 - 2x^3 - 4y + 8 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{y - 2}$

Let the point (α, β) lies on the curve

Thus, $m = \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = \frac{3\alpha^2}{\beta - 2}$

Therefore, the equation of the tangent at

(α, β) is $y - \beta = \left(\frac{3\alpha^2}{\beta - 2}\right)(x - \alpha)$ ...(1)

which is passing through (1, 2)

So, $(2 - \beta) = \left(\frac{3\alpha^2}{\beta - 2}\right)(1 - \alpha)$ ...(2)

Also, the point (α, β) lies on the curve

$y^2 - 2x^3 - 4y + 8 = 0$

So, $\beta^2 - 2\alpha^2 - 4\beta + 8 = 0$ ...(3)

From (2) and (3), we get,

$2(\alpha^3 - 2) = 3\alpha^2(\alpha - 1)$

$\Rightarrow \alpha^3 - 3\alpha^2 + 4 = 0$

$\Rightarrow \alpha = 2$

when $\alpha = 2, \beta = \pm 2\sqrt{3}$

Hence, the equation of the tangents are

$(y - (2 \pm 2\sqrt{3})) = \pm 2\sqrt{3}(x - 2)$

13. The equation of the given curve is $x^2 = 4y$

$\Rightarrow 2x = 4\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{2}$

Let the point on the given curve be (α, β)

Now, $m = \left(\frac{dy}{dx}\right)_{(\alpha,\ \beta)} = \frac{\alpha}{2}$

Therefore the equation of normal at (α, β) is

$y - \beta = -\frac{2}{\alpha}(x - \alpha)$ ...(1)

which is passing through (1, 2).

So, $2 - \beta = -\frac{2}{\alpha}(1 - \alpha)$ ...(2)

Also, the point (α, β) lies on the curve

$\alpha^2 = 4\beta$ ...(3)

Solving (2) and (3), we get, $\alpha = 2, \beta = 1$

Hence, the equation of the normal is

$y - 1 = -\frac{2}{2}(x - 2)$

$\Rightarrow y - 1 = -x + 2$

$\Rightarrow x + y = 3.$

14. The equation of the given curve is $y = x - e^{xy}$

$\Rightarrow \frac{dy}{dx} = 1 - e^{xy}\left(y \cdot 1 + x \cdot \frac{dy}{dx}\right)$

$\Rightarrow (1 - xe^{xy})\frac{dy}{dx} = (ye^{xy} - 1)$

$\Rightarrow \frac{dy}{dx} = \frac{ye^{xy} - 1}{1 - xe^{xy}}$

For a vertical tangent, $\frac{dx}{dy} = 0$

$\Rightarrow 1 - xe^{xy} = 0$

$\Rightarrow x = 1, y = 0$

Therefore, the curve $y = x - e^{xy}$ has a vertical tangent at (1, 0).

15. The given curve is $x + y - \log(x + y) = 2x + 5$

$\Rightarrow 1 + \frac{dy}{dx} - \frac{1}{x + y}\left(1 + \frac{dy}{dx}\right) = 2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x + y + 1}{x + y - 1}$

Now, $\left(\frac{dy}{dx}\right)_{(p,q)} = \frac{p + q + 1}{p + q - 1}$

Since, the tangent is vertical, so $\frac{dx}{dy} = 0$

$\Rightarrow \quad p + q = 1$

The value of $p + q + 10 = 11$.

16. Since the curve has horizontal tangent, so $\frac{dx}{dy} = 0$

$\Rightarrow \quad 6x^2 + 6x - 12 = 0$

$\Rightarrow \quad x^2 + x - 2 = 0$

$\Rightarrow \quad (x + 2)(x - 1) = 0$

$\Rightarrow \quad x = 1, -2$

When $x = 1, y = 2 + 3 - 12 + 1 = -6$

So, the point is $(1, -6)$

when $x = -2, y = 16 + 12 - 26 + 1 = 3$

So, the point is $(-2, 3)$

Hence, the points are $(1, -6)$ and $(-2, 3)$.

17. Since the tangents are parallel to x-axis, so $\frac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 - 2x - 1 = 0$

$\Rightarrow \quad (3x + 1)(x - 1) = 0$

$\Rightarrow \quad x = 1, -1/3$

when $x = 1, y = 1 - 1 - 1 + 3 = 2$

when $x = -1/3, y = 70/27$

Hence, the points are $(1, 2)$ and $(-1/3, 70/27)$

18. The given curve is $y = \frac{x^3}{3} + \frac{x^2}{2}$

$\Rightarrow \quad \frac{dy}{dx} = 2x^2 + x$

Since the tangents make equal angles with the axes, so $\frac{dy}{dx} = \pm 1$

$\Rightarrow \quad 2x^2 + x = \pm 1$

$\Rightarrow \quad 2x^2 + x - 1 = 0, 2x^2 + x + 1 = 0$

$\Rightarrow \quad (2x - 1)(x + 1) = 0$

$\Rightarrow \quad x = 1/2, -1$

when $x = 1/2, y = 5/24$

and when $x = -1, y = 1/6$

Hence, the points are $(1/2, 5/24)$ & $(-1, 1/6)$

19. The equation of the tangent to the curve

$y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$

$\Rightarrow \quad y \cdot 2at = 2a(x + at^2)$

$\Rightarrow \quad yt = x + at^2$

20. The equation of the tangent to the curve $x^2 + y^2 + x + y = 0$ is

$xx_1 + yy_1 + \left(\frac{x + x_1}{2}\right) + \left(\frac{y + y_1}{2}\right) = 0$

$\Rightarrow \quad x \cdot 1 - y \cdot 1 + \left(\frac{x + 1}{2}\right) + \left(\frac{y - 1}{2}\right) = 0$

$\Rightarrow \quad 2x - 2y + x + 1 + y - 1 = 0$

$\Rightarrow \quad 3x - y = 0$

21. Equation of the normal to the curve

$x^2 + y^2 = 10$ is $\frac{x}{x_1} = \frac{y}{y_1}$

$\Rightarrow \quad \frac{x}{3} = \frac{y}{1}$

$\Rightarrow \quad x - 3y = 0$

22. The equation of the normal to the curve $x^2 + y^2 + 4x + 6y + 9 = 0$ at (x_1, y_1) is

$\frac{x - x_1}{x_1 + 2} = \frac{y - y_1}{y_1 + 3}$

$\Rightarrow \quad \frac{x + 4}{-4 + 2} = \frac{y + 3}{-3 + 3}$

$\Rightarrow \quad y + 3 = 0.$

23. Equation of the tangent to the curve is

$\frac{xx_1}{9} + \frac{yy_1}{4} = 2$

$\Rightarrow \quad \frac{3x}{9} + \frac{2y}{4} = 2$

$\Rightarrow \quad \frac{x}{3} + \frac{y}{2} = 2$

$\Rightarrow \quad 2x + 3y = 6$

Thus, slope of the tangent is $= -2/3$

Therefore, the slope of the normal is $= 3/2$

Equation of the normal to the curve at $(3, 2)$ is

$(y - 2) = \frac{3}{2}(x - 3)$

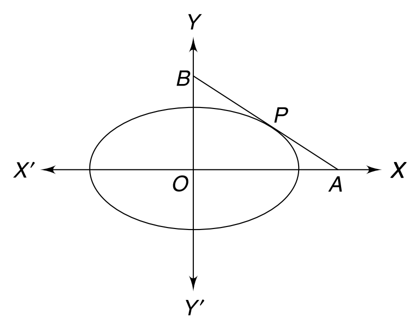
$\Rightarrow \quad 2y - 4 = 3x - 9$

$\Rightarrow \quad 3x - 2y = 5.$

24. Clearly, the point $(1, 2)$ lies outside of the curve $y^2 - 2x^2 - 4y + 8 = 0$. (as $4 - 2 - 4 + 8 = 12 - 6 = 6 > 0$)

Since the point lies outside of the given curve, so the number of tangent will be 2.

25.



Let the point P be $(2\sqrt{2}\cos\theta, 3\sqrt{2}\sin\theta)$

Equation of the tangent at P is

$$\frac{xx_1}{8} + \frac{yy_1}{18} = 1$$

$$\Rightarrow \quad \frac{x \cdot 2\sqrt{2}\cos\theta}{8} + \frac{y \cdot 3\sqrt{2}\sin\theta}{18} = 1$$

$$\Rightarrow \quad \frac{x}{2\sqrt{2}\sec\theta} + \frac{y}{3\sqrt{2}\operatorname{cosec}\theta} = 1$$

Thus, the co-ordinates of A and B are

$(2\sqrt{2}\sec\theta, 0)$ and $(0, 3\sqrt{2}\operatorname{cosec}\theta)$

Now, $ar(\Delta OAB) = \frac{1}{2} \times OA \times OB$

$$= \frac{1}{2} \times 2\sqrt{2}\sec\theta \times 3\sqrt{2}\operatorname{cosec}\theta$$

$$= \frac{12}{2\sin\theta\cos\theta}$$

$$= \frac{12}{\sin 2\theta}$$

The area of the triangle is maximum, when $\sin 2\theta = 1$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

Hence, the point P is (2, 3).

26. Let any point on the curve be $M(x_1, y_1)$

The equation of the given curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_M = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$$

Now, x-intercept $= OP = x_1 - \frac{y_1}{dy/dx}$

$$= x_1 - \frac{y_1}{\left(-\frac{\sqrt{y_1}}{\sqrt{x_1}}\right)} = x_1 + \sqrt{x_1 y_1}$$

y-intercept $= y_1 - x_1 \frac{dy}{dx}$

$$= y_1 - x_1\left(-\frac{\sqrt{y_1}}{\sqrt{x_1}}\right) = y_1 + \sqrt{x_1 y_1}$$

Thus, $OP + OQ$

$$= x_1 + \sqrt{x_1 y_1} + y_1 + \sqrt{x_1 y_1}$$

$$= x_1 + y_1 + 2\sqrt{x_1 y_1}$$

$$= (\sqrt{x_1} + \sqrt{y_1})^2$$

$$= (\sqrt{a})^2 = a$$

27. Let the point on the curve be $P(\alpha, \beta)$

The equation of the given curve is $\frac{a}{x^2} + \frac{b}{y^2} = 1$

$$\Rightarrow \quad -\frac{2a}{x^3} - \frac{2b}{y^3} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{ay^3}{bx^3}$$

Now, $\left(\frac{dy}{dx}\right)_p = -\frac{a\beta^3}{b\alpha^3}$

Therefore, x-intercept

$$= x - \frac{y}{dy/dx}$$

$$= \alpha - \frac{\beta}{\left(-\frac{a\beta^3}{b\alpha^3}\right)} = \alpha + \frac{b\alpha^3}{a\beta^2}$$

$$= \alpha + \frac{\alpha^3}{a} \times \frac{b}{\beta^2}$$

$$= \alpha + \frac{\alpha^3}{a} \times \left(1 - \frac{a}{\alpha^2}\right)$$

$$= \alpha + \frac{\alpha^3}{a} - \alpha$$

$$= \frac{\alpha^3}{a}$$

$\Rightarrow$ x-intercept is proportional to α^3.

28. The equation of the tangent to the origin is

$xy = 0.$

$\Rightarrow \quad x = 0$ and $y = 0$

29. The equation of the tangent at the origin is

$ax + by = 0.$

30. The equation of the tangent to the curve at the origin is

$x^2 - y^2 = 0$

$\Rightarrow \quad x + y = 0$ and $x - y = 0$.

31. The equation of the tangent to the curve at the origin is $2012x - 2013y = 0$.

32. The given curves are $x^2 = y$ and $y^2 = x$

We have, $x = x^4$

$\Rightarrow \quad x(1 - x^3) = 0$

$\Rightarrow \quad x = 0, 1$

when $x = 0$, $y = 0$ and when $x = 1$, $y = 1$

So, the point of intersections are (0, 0), (1, 1)

At the point of intersection (0, 0)

Now, $y = x^2$

$$\Rightarrow \quad \frac{dy}{dx} = 1$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(0,0)} = 1$$

Also, $y^2 = x$

$\Rightarrow \quad 2y \dfrac{dy}{dx} = 1$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{2y}$

$$m_2 = \left(\frac{dy}{dx}\right)_{(0,0)} = \frac{1}{0} = \infty$$

Let θ be the angle between them

Then, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \infty$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$

Now, consider the point of intersection (1, 1)

$$m_1 = \left(\frac{dy}{dx}\right)_{(1,1)} \text{ and } m_2 = \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{1}{2}$$

Let ϕ be the angle between them

Then $\tan(\varphi) = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{1 - \frac{1}{2}}{1 + 1\cdot\frac{1}{2}}\right| = \dfrac{1}{3}$

$\Rightarrow \quad \varphi = \tan^{-1}\left(\dfrac{1}{3}\right).$

33. The given curves are $y = 4 - x^2$ and $y = x^2$

On solving, we get, $x^2 = 4 - x^2$

$\Rightarrow \quad x = \pm\sqrt{2}$

when $x = \pm\sqrt{2}$, $y = 2$

So, the points of intersections are

$(\sqrt{2}, 2)$ & $(-\sqrt{2}, 2)$

Now, $y = 4 - x^2$

$\Rightarrow \quad \dfrac{dy}{dx} = -2x$

$$m_1 = \left(\frac{dy}{dx}\right)_{(\sqrt{2},\, 2)} = -2\sqrt{2}$$

Also, $y = x^2$

$\Rightarrow \quad \dfrac{dy}{dx} = 2x$

$$m_2 = \left(\frac{dy}{dx}\right)_{(\sqrt{2},\, 2)} = 2\sqrt{2}$$

Let θ be the angle between them

Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 - 8}\right| = \left(\frac{4\sqrt{2}}{7}\right)$$

$\Rightarrow \quad \theta = \tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$

Similarly, we can find, $\varphi = \tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$

34. The given curves are $y^2 = 4x$ and $y = e^{-x/2}$

Let the point of intersection be (x_1, y_1)

Now, $y^2 = 4x$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{2}{y}$

$\Rightarrow \quad m_1 = \left(\dfrac{dy}{dx}\right)_{(x_1,\, y_1)} = \dfrac{2}{y_1}$

Also, $y = e^{-x/2}$

$$\frac{dy}{dx} = e^{-x/2} \times -\frac{1}{2}$$

$\Rightarrow \quad m_2 = \left(\dfrac{dy}{dx}\right)_{(x_1,\, y_1)} = -\dfrac{1}{2} \times e^{\frac{-x_1}{2}} = -\dfrac{1}{2} y_1$

Let θ be the angle between them

Then $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$

$\Rightarrow \quad \tan\theta = \left|\dfrac{-\frac{y_1}{2} - \frac{2}{y_1}}{1 + \left(\frac{-y_1}{2} \times \frac{2}{y_1}\right)}\right| = \infty$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$

35. The given curves are $y = \sin x$ and $y = \cos x$.

On solving, we get, the point of intersection is

$\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right).$

Now, $y = \sin x$

$\Rightarrow \quad \dfrac{dy}{dx} = \cos x$

$\Rightarrow \quad m_1 = \left(\dfrac{dy}{dx}\right)_{\left(\frac{\pi}{4},\, \frac{1}{\sqrt{2}}\right)} = \dfrac{1}{\sqrt{2}}$

Also, $y = \cos x$

$\Rightarrow \quad \dfrac{dy}{dx} = -\sin x$

$\Rightarrow \quad m_2 = \left(\dfrac{dy}{dx}\right)_{\left(\frac{\pi}{4},\, \frac{1}{\sqrt{2}}\right)} = -\dfrac{1}{\sqrt{2}}$

Let θ be the angle between them

Then $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1, m_2}\right|$

$\Rightarrow \quad \tan\theta = \left|\dfrac{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{1 - \frac{1}{2}}\right| = 2\sqrt{2}$

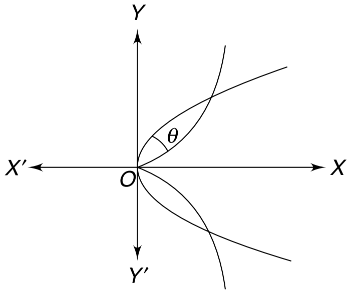
$\Rightarrow \quad \theta = \tan^{-1}(2\sqrt{2})$

36. The given curves are $2y^2 = x^3$ and $y^2 = 32x$

On solving we get, the point of intersections are $O(0, 0)$, $P(8, 16)$ and $Q(8, -16)$.

From the diagram, it is clear that, the angle of intersection at (0, 0) is $\frac{\pi}{2}$.

Now, $\quad 2y^2 = x^3$



From the diagram, it is clear that, the angle of intersection at (0, 0) is $\frac{\pi}{2}$.

Now, $\quad 2y^2 = x^3$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{4y}$

$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = \frac{3 \times 64}{4 \times 16} = 3$

Also, $\quad y^2 = 32x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{32}{2y} = \frac{16}{y}$

$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_P = \frac{16}{16} = 1$

Let θ be the angle between them

Then, $\quad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

$\Rightarrow \quad \tan\theta = \left|\frac{1-3}{1+1.3}\right| = \frac{1}{2}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{1}{2}\right)$

Thus, the angle of intersection at P and Q is $\tan^{-1}\left(\frac{1}{2}\right)$.

37. The given curves are $y^2 = 4x$ and

$x^2 + y^2 - 6x + 1 = 0$

Now, $\quad y^2 = 4x$

$\Rightarrow \quad 2y\frac{dy}{dx} = 4$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{y}$

$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{2}{2} = 1$

Also, $\quad x^2 + y^2 - 6x + 1 = 0$

$\Rightarrow \quad 2x + 2y\cdot\frac{dy}{dx} - 6 = 0$

$x + y\frac{dy}{dx} - 3 = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3-x}{y}$

$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{3-1}{2} = 1$

Let θ be the angle between them

Then $\quad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1\cdot m_2}\right| = \left|\frac{1-1}{1+1.1}\right| = 0$

$\Rightarrow \quad \theta = 0$

Hence, the curves touch each other.

38. The given curves are $y = 6 - x + x^2$

and $\quad y(x-1) = x + 2.$

Now, $\quad y = 6 - x + x^2$

$\frac{dy}{dx} = -1 + 2x$

$m_1 = \left(\frac{dy}{dx}\right)_{(2,4)} = -1 + 4 = 3$

Also, $\quad y = (x-1)(x+2)$

$\Rightarrow \quad \frac{dy}{dx} = x + 2 + x - 1 = 2x + 1$

$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_{(2,4)} = 3$

Let θ be the angle between them

Then, $\quad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1\cdot m_2}\right|$

$\Rightarrow \quad \tan = \theta = \left|\frac{3-3}{1+9}\right| = 0$

$\Rightarrow \quad \theta = 0$

Hence, the curves touch each other

39. The given curves are $x = y^2$ and $x\,y = k$

On solving we get, $y = k^{1/3}$, $x = k^{2/3}$

So, the point of intersection is $P\,(k^{2/3}, k^{1/3})$

Now, $\quad y^2 = x$

$\Rightarrow \quad 2y\frac{dy}{dx} = 1$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$

$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = \frac{1}{2k^{1/3}}$

Also, $\quad xy = k \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{k}{x^2}$

$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_P = \frac{k}{k^{4/3}} = -\frac{1}{k^{1/3}}$

since the given curves cut at right angles, so

$m_1 \times m_2 = -1$

$\Rightarrow \quad \frac{1}{2k^{1/3}} \times -\frac{1}{k^{1/3}} = -1$

$\Rightarrow \quad \frac{1}{2k^{2/3}} = 1$

$\Rightarrow \quad \frac{1}{8k^2} = 1$

$\Rightarrow \quad 8k^2 = 1$

40. The given curves are $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$

Let the point of intersection be (x_1, y_1)

Now, $\quad \frac{x^2}{a^2} + \frac{y^2}{4} = 1$

$\Rightarrow \quad \frac{2x}{a^2} + \frac{2y}{4}\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{x}{a^2} + \frac{y}{2}\cdot\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a^2y}$

$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = -\frac{2x_1}{a^2y_1}$

Also, $\quad y^3 = 16x$

$\Rightarrow \quad 3y^2\frac{dy}{dx} = 16$

$\Rightarrow \quad \frac{dy}{dx} = \frac{16}{3y^2}$

$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_P = \frac{16}{3y_1^2}$

Since two curves are orthogonal, so

$m_1 \times m_2 = -1$

$\Rightarrow \quad -\frac{2x_1}{a^2y_1} \times \frac{16}{3y_1^2} = -1$

$\Rightarrow \quad \frac{32x_1}{3a^2y_1^3} = 1$

$\Rightarrow \quad \frac{32x_1}{3a^2 \cdot 16x_1} = 1$

$\Rightarrow \quad a^2 = \frac{2}{3}$

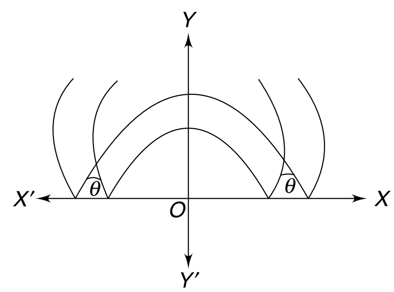
$\Rightarrow \quad a = \pm\sqrt{\frac{2}{3}}$

41. The given curves are $y = |x^2 - 1|$ and $y = |x^2 - 3|$

On solving, we get, $x^2 - 1 = -x^2 + 3$

$\Rightarrow \quad 2x^2 = 4$

$\Rightarrow \quad x^2 = 2$



$\Rightarrow \quad x = \pm\sqrt{2}$

when $\quad x = \pm\sqrt{2}$, then $y = 2 - 1 = 1$

so, the point of intersection is $(\pm\sqrt{2}, 1)$

Now, consider the point of intersection is $P(\sqrt{2}, 1)$

Now, $\quad y = |x^2 - 1|$

$\Rightarrow \quad y = x^2 - 1$

$\Rightarrow \quad \frac{dy}{dx} = 2x$

$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = 2\sqrt{2}$

Also, $\quad y = |x^2 - 3|$

$\Rightarrow \quad y = -x^2 + 3$

$\Rightarrow \quad \frac{dy}{dx} = -2x$

$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_P = -2\sqrt{2}$

Let θ be the angle between them

Then $\quad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-2\sqrt{2} - 2\sqrt{2}}{1 + 2\sqrt{2}\cdot(-2\sqrt{2})}\right|$

$\Rightarrow \quad \tan\theta = \frac{4\sqrt{2}}{7}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right).$

42. We have, $y = [|\sin x| + |\cos x|]$

$\Rightarrow \quad y = 1$, since $1 \le |\sin x| + |\cos x| \le \sqrt{2}$

when $\quad y = 1, x = \pm 2$

So, the point of intersection are (2, 1) and (–2, 1)

Now, $\quad y = 1 \Rightarrow \dfrac{dy}{dx} = 0$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_p = 0$$

Also, $\quad x^2 + y^2 = 5$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

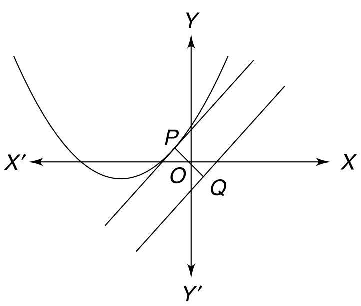
$$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_p = -2$$

Let θ be the angle between them

Then $\quad \tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 \cdot m_2}\right| = \left|\dfrac{-2 - 0}{1 + 0}\right| - 2$

$$\Rightarrow \quad \theta = \tan^{-1}(2)$$

43.



The given curves are $y = x^2 + 3x + 2$ and

$$y = x - 2$$

$$\Rightarrow \quad \frac{dy}{dx} = 2x + 3 \text{ and } \frac{dy}{dx} = 1$$

Since the tangents are parallel, so their slopes are same.

Therefore, $2x + 3 = 1$

$$\Rightarrow \quad x = 1$$

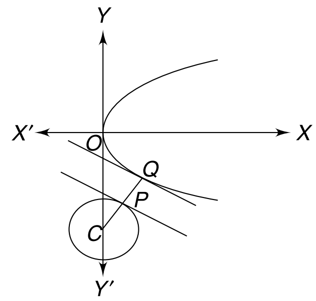
when $x = 1$, then y is 6.

Thus, the point lies on the curve is (1, 6)

Hence, the length of the shortest distance

$$= \left|\frac{1 - 6 + 2}{\sqrt{1^2 + 1^2}}\right| = \frac{3}{\sqrt{2}}$$

44.



The given curves are $y^2 = 4x$ and

$$x^2 + y^2 - 12x + 31 = 0.$$

In this case, tangent at P on the parabola is parallel to the tangent at Q on the circle.

So their slopes are same.

Thus, $\quad \dfrac{4}{2y} = \dfrac{6 - x}{y}$

$$\Rightarrow \quad x = 4$$

when $\quad x = 4$, then $y = -4$

So, the point on the parabola is (4, –4)

Now shortest distance $= PQ$

$= CP - CQ$

$= 2\sqrt{5} - \sqrt{5}$

$= \sqrt{5}$

45. The given curves are $y^2 = x^3$ and

$$9x^2 + 9y^2 - 30y + 16 = 0.$$

In this case, tangent at P on the curve $y^2 = x^3$

So their slopes are same.

Now, $\quad \dfrac{3x^2}{2y} = -\dfrac{3x}{3y - 5}$

$$\Rightarrow \quad y = \frac{5x}{3x + 2}$$

Put the value of y in $y^2 = x^3$, we get,

$$\frac{25x^2}{(3x + 2)^2} = x^3$$

$$\Rightarrow \quad 9x^3 + 12x^2 + 4x - 25 = 0$$

$$\Rightarrow \quad x = 1.$$

when $\quad x = 1$, then $y = 1$

So, the point is (1, 1)

Now, the equation of the normal to the curve

$y^2 = x^3$ at (1, 1) is $y - 1 = -\dfrac{2}{3}(x - 1)$

$$\Rightarrow \quad 2x + 3y = 9$$

On solving, we get, the point on the ellipse is

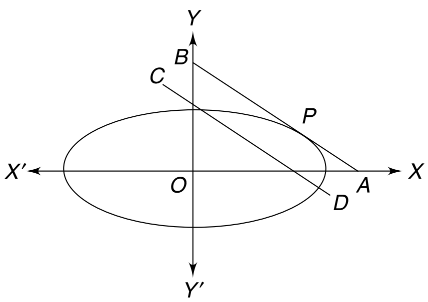
$$\left(\frac{3}{\sqrt{13}}, \frac{5\sqrt{13}-6}{3\sqrt{13}}\right)$$

Hence, the required shortest distance

$$= \sqrt{\left(\frac{3}{\sqrt{13}}-1\right)^2 + \left(\frac{5\sqrt{13}-6}{3\sqrt{13}}\right)^2}$$

$$= \sqrt{\frac{1}{13}(110 - 30\sqrt{3})}.$$

46.



The given curve is $x^2 + 2y^2 = 6$

$\Rightarrow \quad \frac{x^2}{6} + \frac{y^2}{3} = 1$

since the tangent at P to the ellipse is parallel to the given line, so their slopes are same.

Now, $\quad -\frac{x}{2y} = -1$

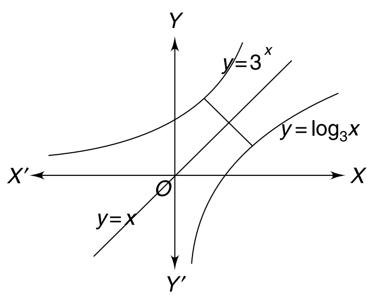
On solving, we get, $6y^2 = 6$

$\Rightarrow \quad y^2 = 1$

$\Rightarrow \quad y = \pm 1$

So, the point can be either (2, 1) or (–2, –1).

47.



The given curves are $y = 3^x$ and $y = \log_3 x$.

Clearly, $y = \log_3 x$ is the image of the curve $y = 3^x$ with respect to the line $y = x$.

Therefore, $3^x \cdot \log 3 = 1$

$\Rightarrow \quad 3^x = \frac{1}{\log 3} = (\log 3)^{-1}$

$\Rightarrow \quad x = \log_3 (\log 3)^{-1} = -\log_3 (\log 3)$

when $\quad x = -\log_3 (\log 3)$, then $y = \frac{1}{\log 3}$

Thus, the point $\left(-\log_3(\log 3), \frac{1}{\log 3}\right)$ lies on the curve $y = 3^x$.

Since the curve $y = \log_3 x$ is the image of the curve $y = 3^x$ with respect to the line $y = x$, so the point on the curve $y = \log_3 x$ is

$$\left(\frac{1}{\log 3}, -\log_3(\log 3)\right)$$

Hence, the shortest distance

$$= \sqrt{\left(\frac{1}{\log 3} + \log_3(\log 3)\right)^2 + \left(\log_3(\log 3) + \frac{1}{\log 3}\right)^2}$$

$$= \sqrt{2}\left(\frac{1}{\log 3} + \log_3(\log 3)\right)$$

$$= \sqrt{2}\left(\frac{1 + \log(\log 3)}{\log 3}\right)$$

$$= \sqrt{\left(\frac{1}{\log 3} + \log_3(\log 3)\right)^2 + \left(\log_3(\log 3) + \frac{1}{\log 3}\right)^2}$$

$$= \sqrt{2}\left(\frac{1}{\log 3} + \log_3(\log 3)\right)$$

$$= \sqrt{2}\left(\frac{1 + \log(\log 3)}{\log 3}\right)$$

48. The given curves are $y = x^2 + x + 1$ and

$$y = x^2 - 5x + 6$$

Let the common tangent be $y = ax + b$.

On solving with both the given curves, we have,

$ax + b = x^2 + x + 1$ and $ax + b = x^2 - 5x + 6$

$\Rightarrow \quad x^2 + (1 - a)x + (1 - b) = 0$

and $\quad x^2 + (5 + a)x + (6 - b) = 0$

Since they have a common tangent, so the given equations have equal roots.

Thus, $\quad D = 0$

$\Rightarrow \quad (1 - a)^2 - 4(1 - b) = 0$

and $\quad (5 + a)^2 - 4(6 - b) = 0$

$\Rightarrow \quad a^2 - 2a + 4b - 3 = 0$

and $\quad a^2 + 10a + 4b + 1 = 0$

$\Rightarrow \quad a = -1/3$ and $b = 5/9$.

Hence, the equation of the common tangent is

$$3x + 9y = 5.$$

49. The given curves are $y = 3x^2$ and $y = 2x^3 + 1$

$$\frac{dy}{dx} = 6x \text{ and } \frac{dy}{dx} = 6x^2$$

Since the given curves have common tangent, so their slopes are same.

Thus, $6x = 6x^2$.

$\Rightarrow \quad x = 0$ and 1.

when $x = 0$, $y = 0$ and when $x = 1$, $y = 3$.

So, the points are (0, 0) and (1, 3).

Hence, the equations of the common tangents are

$$y = 0 \text{ and } y - 1 = 6(x - 1) = 6x - 6$$

$\Rightarrow \quad y = 0$ and $y = 6x - 5$.

50. The given curves are $y = 6 - x - x^2$ and $y = 1 + \frac{3}{x}$

$\Rightarrow \quad \frac{dy}{dx} = -1 - 2x$ and $\frac{dy}{dx} = -\frac{3}{x^2}$

Since the given curves have their common tangent,

so $\quad -\frac{3}{x^2} = -1 - 2x$

$\Rightarrow \quad 2x^3 + x^2 - 3 = 0$

$\Rightarrow \quad x = 1$

when $\quad x = 1$, $y = 6 - 1 - 1 = 4$

So, the point is (1, 4)

Hence, the equation of the common tangent is

$$y - 4 = -3(x - 1)$$

$\Rightarrow \quad y - 4 = -3x + 3$

$\Rightarrow \quad 3x + y = 7$.

51. The given curve is $y^2 = x(2 - x)^2$

$\Rightarrow \quad 2y\frac{dy}{dx} = (2 - x)^2 - 2x(2 - x)$

$\Rightarrow \quad 2y\frac{dy}{dx} = 4 - 4x + x^2 - 4x + 2x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2 - 8x + 4}{2y}$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{3 - 8 + 4}{2} = -\frac{1}{2}$

Hence, the equation of the tangent is

$$y - 1 = -\frac{1}{2}(x - 1)$$

$\Rightarrow \quad 2y - 2 = -x + 1$

$\Rightarrow \quad x + 2y = 3$

On solving, the given curve and the tangent, we get,

$$y^2 = (3 - 2y)(2 - 3 + 2y)^2$$

$\Rightarrow \quad y^2 = (3 - 2y)(2y - 1)^2$

$\Rightarrow \quad 8y^3 - 19y^2 + 14y - 3 = 0$

$\Rightarrow \quad (y - 1)(8y^2 - 11y + 3) = 0$

$\Rightarrow \quad (y - 1)^2(8y - 3) = 0$

$\Rightarrow \quad y = 1, 3/8$

when $y = 3/8$, $x = 3 - 2y = 3 - \frac{3}{4} = \frac{9}{4}$

Hence, the point is $\left(\frac{9}{4}, \frac{3}{8}\right)$.

52. 2015.

53.

54. Given $\quad x = \sec^2\theta$ and $y = \cot\theta$

$$\frac{dx}{d\theta} = 2\sec^2\theta\tan\theta \text{ and } \frac{dy}{d\theta} = -\text{cosec}^2\theta$$

Thus, $\quad \frac{dy}{dx} = -\frac{\text{cosec}^2\theta}{2\sec^2\theta\tan\theta} = -\frac{1}{2}\cot^3\theta$

Now, $\quad \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -\frac{1}{2}$

So, the point P is (2, 1)

Equation of the tangent at P is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$\Rightarrow \quad 2y - 2 = -(x - 2) = -x + 2$

$\Rightarrow \quad x + 2y = 4 \quad$ ...(i)

Eliminating 'θ' between $x = \sec^2\theta$ & $y = \cot\theta$

we get, $x - \frac{1}{y^2} = 1 \quad$ ...(ii)

On solving (i) and (ii), we get,

$\Rightarrow \quad 4 - 2y - \frac{1}{y^2} = 1$

$\Rightarrow \quad 4y^2 - 2y^3 - 1 = y^2$

$\Rightarrow \quad 3y^2 - 2y^3 - 1 = 0$

$\Rightarrow \quad 2y^3 - 3y^2 + 1 = 0$

$\Rightarrow \quad 2y^3 - 2y^2 - y^2 + y - y + 1 = 0$

$\Rightarrow \quad 2y^2(y - 1) - y(y - 1) - (y - 1) = 0$

$\Rightarrow \quad (y - 1)(2y^2 - y - 1) = 0$

$\Rightarrow \quad (y - 1) = 0, (2y^2 - y - 1) = 0$

$\Rightarrow \quad (y - 1) = 0, (2y^2 - 2y + y - 1) = 0$

$\Rightarrow \quad (y - 1) = 0, (2y(y - 1) + (y - 1)) = 0$

$\Rightarrow \quad (y - 1) = 0, (y - 1)(2y + 1) = 0$

$\Rightarrow \quad y = 1, -\frac{1}{2}$

$\Rightarrow \quad y = -\frac{1}{2}$

when $y = -\frac{1}{2}$, then $x = 4 - 2y = 4 + 2\cdot\frac{1}{2} = 5$

Thus, the point Q is $\left(5, -\frac{1}{2}\right)$

Now, the length of PQ

$$= \sqrt{(2-5)^2 + \left(1 + \frac{1}{2}\right)^2}$$

$$= \sqrt{9 + \frac{9}{4}}$$

$$= \sqrt{\frac{45}{4}}$$

$$= \frac{3\sqrt{5}}{2}$$

55. The given curve is $y^2 = 4ax$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

Now, $\left(\frac{dy}{dx}\right)_P = \frac{2a}{2at} = \frac{1}{t}$

(i) The length of the tangent

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= 2at\sqrt{1 + t^2}$$

(ii) The length of the sub-tangent

$$= y \cdot \frac{dx}{dy}$$

$$= 2at \times t = 2at^2$$

(iii) The length of the normal

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= 2at \times \sqrt{1 + \frac{1}{t^2}}$$

$$= 2a\sqrt{t^2 + 1}$$

(iv) The length of the sub-normal

$$= y \cdot \frac{dy}{dx}$$

$$= 2at \times \frac{1}{t}$$

$$= 2a.$$

56. The given curve is $y = be^{x/a}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{b}{a}e^{x/a}$$

Let the point be (x, y)

Now, the length of the sub-tangent

$$= y \cdot \frac{dx}{dy}$$

$$= be^{x/a} \times \frac{a}{be^{x/a}}$$

$$= a$$

$$= \text{constant.}$$

57. The given curves are $x = a(\theta - \sin\theta)$,

$$y = a(1 - \cos\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = a(1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \cot\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \cos\left(\frac{\pi}{4}\right) = 1$$

when $\theta = \frac{\pi}{4}$, $y = a\left(\frac{1-1}{\sqrt{2}}\right)$

The length of the tangent

$$= y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$= a\left(1 - \frac{1}{\sqrt{2}}\right) \times \sqrt{2}$$

$$= a(\sqrt{2} - 1)$$

The length of the normal

$$= y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= a\left(1 - \frac{1}{\sqrt{2}}\right) \times \sqrt{2}$$

$$= a(\sqrt{2} - 1)$$

58. The given curve is $y^2 = x^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{2y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(4,8)} = \frac{3.16}{2.8} = 3$$

Hence, the length of the sub-normal

$$= y \cdot \frac{dy}{dx} = 8.3 = 24$$

59. The given curve is $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{c}{2}\left(\frac{1}{c}e^{\frac{x}{c}} - \frac{1}{c}e^{-\frac{x}{c}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$$

Hence, the length of the sub-tangent

$$= y \cdot \frac{dx}{dy}$$

$$= \frac{\frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)}{\frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)}$$

$$= \frac{c}{2}\left(\frac{e^{\frac{2x}{c}} + 1}{e^{\frac{2x}{c}} - 1}\right).$$

Level III

1. The straight line $\frac{x}{a} + \frac{y}{b} = 2$ ...(i)

touches the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 2$ ...(ii)

if the intersection of (i) and (ii) is a unique point

From (i) and (ii), we get,

$$\left(2 - \frac{y}{b}\right)^2 + \left(\frac{y}{b}\right)^2 = 2$$

$$\Rightarrow \quad 4 - \frac{4y}{b} + \frac{y^2}{b^2} + \left(\frac{y}{b}\right)^2 = 2$$

$$\Rightarrow \quad \frac{2y^2}{b^2} - \frac{4y}{b} + 2 = 0$$

$$\Rightarrow \quad \frac{y^2}{b^2} - \frac{2y}{b} + 1 = 0$$

$$\Rightarrow \quad \left(\frac{y}{b} - 1\right)^2 = 0$$

$$\Rightarrow \quad \left(\frac{y}{b} - 1\right) = 0$$

$$\Rightarrow \quad y = b$$

Thus, the straight line (i) is a tangent to the curve (ii)

Also, the point of contact is (a, b).

2. Let the tangent from the origin to the curve $y = \sin x$ meet the curve again at (x_1, y_1)

Equation of tangent at (x_1, y_1) is

$$y - y_1 = \cos(x_1)(x - x_1)$$

since it passes through the origin, so

$$y_1 = x_1 \cos(x_1) \quad \text{...(i)}$$

Also, the point (x_1, y_1) lies on the curve, so

$$y_1 = \sin(x_1) \quad \text{...(ii)}$$

From (i) and (ii), we get,

$$\sin(x_1) = x_1 \cos(x_1)$$

$$\Rightarrow \quad x_1 = \tan(x_1)$$

$$\Rightarrow \quad x_1^2 = \tan^2(x_1)$$

$$\Rightarrow \quad x_1^2 = \sec^2(x_1) - 1$$

$$\Rightarrow \quad x_1^2 = \left(\frac{{x_1}^2}{{y_1}^2}\right) - 1$$

$$\Rightarrow \quad x_1^2 y_1^2 = x_1^2 - y_1^2$$

Hence, the locus of (x_1, y_1) is

$$x^2y^2 = x^2 - y^2$$

3. Let $P(x_1, y_1)$ be the point of contact of the tangent.

Given $\quad x^a y^b = k^{a+b}$

$$\Rightarrow \quad a\log x + b\log y = (a + b)\log k$$

$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{ay}{bx}$$

Equation of the tangent at P is

$$y - y_1 = \left(-\frac{ay_1}{bx_1}\right)(x - x_1) \quad \text{...(i)}$$

Put $\quad y = 0$ in (i), we get, $x = \left(\frac{a + b}{a}\right)x_1$

Thus, $\quad A = \left(\left(\frac{a + b}{a}\right)x_1, 0\right)$

put $\quad x = 0$ in (i), we get, $y = \left(\frac{a + b}{b}\right)y_1$

Thus, $\quad B = \left(0, \left(\frac{a + b}{b}\right)y_1\right)$

Let P divide AB in the ratio $\lambda : 1$

Thus,

$$P = \left(\frac{\lambda.0 + 1\cdot\left(\frac{a + b}{a}\right)x_1}{\lambda + 1}, \frac{\lambda.\left(\frac{a + b}{b}\right)y_1 + 1.0}{\lambda + 1}\right)$$

$$= \left(\frac{\left(\frac{a + b}{a}\right)x_1}{\lambda + 1}, \frac{\lambda.\left(\frac{a + b}{b}\right)y_1}{\lambda + 1}\right)$$

Thus, $\quad x_1 = \frac{\left(\frac{a + b}{a}\right)x_1}{\lambda + 1}, \; y_1 = \frac{\lambda.\left(\frac{a + b}{b}\right)y_1}{\lambda + 1}$

$$\Rightarrow \quad \lambda + 1 = \left(\frac{a + b}{a}\right), \; \lambda + 1 = \left(\frac{a + b}{b}\right)$$

$$\Rightarrow \quad \lambda = \frac{b}{a} \text{ or } \frac{a}{b}$$

Therefore P divides AB in the ratio $a : b$.

4. Given curve is $y^2 - 2x^3 - 4y + 8 = 0$

$$\Rightarrow \quad 2y\frac{dy}{dx} - 6x^2 - 4\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} - 3x^2 - 2\frac{dy}{dx} = 0$$

$\Rightarrow \quad (y-2)\frac{dy}{dx} = 3x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{y-2}$

Let the point of contact be (h, k)
Equation of tangent at (h, k) is

$$y - k = \left(\frac{3h^2}{k-2}\right)(x-h)$$

which is also passing through (1, 2), so

$\Rightarrow \quad 2 - k = \left(\frac{3h^2}{k-2}\right)(1-h)$

$\Rightarrow \quad 3h^2 - 3h^3 = -(k^2 - 4k + 4)$

$\Rightarrow \quad 3h^2 - 3h^3 + (k^2 - 4k + 4) = 0$ ...(i)

Also, (h, k) lies on the curve, so

$$k^2 - 2h^3 - 4k + 8 = 0$$

$\Rightarrow \quad k^2 - 4k + 4 = 2h^3 - 4$ ...(ii)

From (i) and (ii), we get,

$$3h^2 - 3h^3 + 2h^3 - 4 = 0$$

$\Rightarrow \quad -h^3 + 3h^2 - 4 = 0$

$\Rightarrow \quad h^3 - 3h^2 + 4 = 0$

$\Rightarrow \quad (h+1)(h-2)^2 = 0$

$\Rightarrow \quad h = -1, 2$

when $h = -1$, then $k^2 - 4k + 10 = 0$

Thus k is imaginary
So $h = -1$ is rejected.
When $h = 2$, then $k^2 - 4k - 8 = 0$

Thus, $k = 2 \pm \sqrt{3}$
Therefore, the points of contact are

$$(2, 2 + \sqrt{3}) \ \& \ (2, 2 - \sqrt{3})$$

Hence, the equations of tangents are

$$y - 2\sqrt{3} = \frac{3.4}{2 - \sqrt{3}}(x-2)$$

or

$$y + 2\sqrt{3} = \frac{3.4}{2 + \sqrt{3}}(x-2)$$

$\Rightarrow \quad y + 2\sqrt{3}\,x - 2(1 + \sqrt{3}) = 0$

or

$$y - 2\sqrt{3}\,x - 2(1 - \sqrt{3}) = 0$$

5. Given curve is $5x^5 - 10x^3 + x + 2y + 6 = 0$. ...(i)

$\Rightarrow \quad 25x^4 - 30x^2 + 1 + 2\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{25x^4 - 30x^2 + 1}{2}\right)$

Slope of the tangent to the curve at (0, –3) is

$$= -\frac{1}{2}$$

Equation of normal to the curve at (0, –3) is

$$y + 3 = 2(x - 0) = 2x$$

$\Rightarrow \quad 2x - y = 3$ ...(ii)

On solving (i) and (ii), we get,

$$5x^5 - 10x^3 + 5x = 0$$

$\Rightarrow \quad 5x(x^4 - 2x^2 + 1) = 0$

$\Rightarrow \quad 5x(x^2 - 1)^2 = 0$

$\Rightarrow \quad x = 0, -1, 1$

when $x = 0$, then $y = -3$, so *pt* is (0, –3)
when $x = 1$, then $y = -1$, so the point is (1, –1)
when $x = -1$, then $y = -5$,
so the point is (–1, –5)
Thus, the points are (1, –1) and (–1, –5).
Therefore, the equation of tangents at (1, –1) and (–1, –5) are

$$y + 1 = 2(x - 1) \text{ and } y + 5 = 2(x + 1)$$

$\Rightarrow \quad 2x - y - 3 = 0$ and $2x - y + 5 = 0$.

6. Let the point be (h, k)
Given curve is $9y^2 = x^3$

$\Rightarrow \quad 18y\frac{dy}{dx} = 3x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{18y} = \frac{x^2}{6y}$

Equation of normal at (h, k) is

$$y - k = -\frac{6k}{h^2}(x - h)$$

put $y = 0$, then $x = \frac{h(h+6)}{6}$

put $x = 0$, then $y = k\left(\frac{6+h}{h}\right)$

Given, $h\left(\frac{h+6}{6}\right) = k\left(\frac{6+h}{h}\right)$

$\Rightarrow \quad h^2 = 6k$ ...(i)

Since the point (h, k) lies on the curve,
so $9k^2 = h^3$

$\Rightarrow \quad 9\left(\frac{h^2}{6}\right)^2 = h^3$

$\Rightarrow \quad 9h^4 = 36h^3$

$\Rightarrow \quad h = 4$

when $h = 4$, then $k = \pm\frac{8}{3}$

Hence, the points are $\left(4, \frac{8}{3}\right)$ & $\left(4, -\frac{8}{3}\right)$

7. Given curves are $y = |x^2 - 1|$ and $y = |x^2 - 3|$
The points of intersection are $(\pm\sqrt{2}, 1)$.
Since the curves are symmetrical about y-axis,
so the angle of intersection at $(\sqrt{2}, 1)$
= the angle of intersection at $(-\sqrt{2}, 1)$

At $(\sqrt{2}, 1)$, $m_1 = 2x = 2\sqrt{2}$
and $m_2 = -2x = -2\sqrt{2}$
Let θ be the angle between them

Then $\tan\theta = \left|\frac{4\sqrt{2}}{1-8}\right| = \frac{4\sqrt{2}}{7}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

8. Given curves are $x^2 + y^2 - 6y + 4 = 0$ and $y^2 = x$

$\Rightarrow x^2 + (y-3)^2 = 5$ & $y^2 = x$

$\Rightarrow 2x + 2(y-3)\frac{dy}{dx} = 0$ & $2y\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y-3}$ & $\frac{dy}{dx} = \frac{1}{2y}$

Thus, $\frac{1}{2y} = -\frac{x}{y-3}$

$\Rightarrow x = \frac{3-y}{2y}$

$\Rightarrow y^2 = \frac{3-y}{2y}$

$\Rightarrow 2y^3 + y - 3 = 0$

$\Rightarrow (y-1)(2y^2 + 2y + 3) = 0$

$\Rightarrow y = 1$

when $y = 1$, then $x = 1$

So, the point is (1, 1).

Hence, the equation of the common tangent is

$$y - 1 = \frac{1}{2}(x - 1)$$

$\Rightarrow 2y - 2 = x - 1$

$\Rightarrow x - 2y + 1 = 0.$

9. Given curves are $x = 3t^2$, $y = 2t^3$
Eliminating 't', we get, $27y^2 = 4x^3$

Let the tangent at $P(3t_1^2, 2t_1^3)$ meets the curve again at $Q(3t_2^2, 2t_2^3)$

Equation of the tangent at $P(3t_1^2, 2t_1^3)$ is

$$\frac{y - 2t_1^3}{x - 3t_1^2} = t_1\left(\text{since}\frac{dy}{dx} = t\right)$$

$\Rightarrow y - t_1x + t_1^3 = 0$ ...(i)

Equation of normal at $Q(3t_2^2, 2t_2^3)$ is

$$\frac{y - 2t_2^3}{x - 3t_2^3} = -\frac{1}{t^2}$$

$\Rightarrow t_2y + x - 2t_2^4 - 3t_2^2 = 0$ ...(ii)

Equations (i) and (ii) are identical

$$t_2 = -\frac{1}{t_1} = -\frac{(2t_2^4 + 3t_2^2)}{t_1^2}$$

$\Rightarrow t_1 = -\frac{1}{t_2}, t_1^2 = (2t_2^4 + 3t_2^2)$

$\Rightarrow \left(\frac{1}{t_2}\right)^2 = (2t_2^4 + 3t_2^2)$

$\Rightarrow (2t_2^6 + 3t_2^4 - 1) = 0$

$\Rightarrow 2p^3 + 3p^2 - 1 = 0, p = t_2^2$

$\Rightarrow (p+1)(2p^2 + p - 1) = 0$

$\Rightarrow (p+1)^2(2p - 1) = 0$

$\Rightarrow p = \frac{1}{2}, p = -1$

$\Rightarrow = \frac{1}{2}$

$\Rightarrow t_2^2 = \frac{1}{2}$

$\Rightarrow t_2 = \pm\frac{1}{\sqrt{2}}$ and $t_1 = \mp\frac{1}{\sqrt{2}}$

Hence, the equation of the line PQ is

$$y \pm \sqrt{2}x \mp 2\sqrt{2} = 0$$

10. Given curve is

$$y = (2x-1)e^{2(1-x)}$$

$\Rightarrow \frac{dy}{dx} = 2e^{2(1-x)} + (2x-1)e^{2(1-x)} + (-2)$

For max or min, $\frac{dy}{dx} = 0$

$\Rightarrow 2e^{2(1-x)} + (2x-1)e^{2(1-x)}(-2) = 0$

$\Rightarrow 1 - (2x - 1) = 0$

$\Rightarrow 2 - 2x = 0$

$\Rightarrow x = 1$

Clearly, slope = m = 0

when $x = 1$, then $y = 1$

Hence, the equation of the tangent is

$$y - 1 = m(x - 1)$$

$$\Rightarrow \quad y - 1 = 0.(x - 1)$$

$$\Rightarrow \quad y - 1 = 0$$

11. Given curves are $y = x^2 + ax + b$ and $y = x(c - x)$ since the point (1, 0) lie on the both curves, so

$$a + b + 1 = 0, c - 1 = 0$$

$$a + b = -1, c = 1. \quad ...(i)$$

Again, both the curves touch each other at (1, 0)

So, $(2x + a)_{(1, 0)} = (c - 2x)_{(1, 0)}$

$$2 + a = c - 2$$

$$a = c - 4 \quad ...(ii)$$

From (i) and (ii), we get,

$$a = -3, b = 2, c = 1$$

Hence, the value of $(a + b + c + 4)$ is 4

12. Given curves are

$$y = x^3 - x + 1 \text{ and } y = 3x^2 - 4x + 1$$

Since the tangents are parallel, so their slopes are same

Let the points (α, β) and (γ, δ) on the two given curves where the tangents are parallel

Thus, $3\alpha^2 - 1 = 6\gamma - 4$

$$\Rightarrow \quad 3\alpha^2 - 6\gamma + 3 = 0$$

$$\Rightarrow \quad \alpha^2 - 2\gamma + 1 = 0$$

which is possible for infinite number of ordered pairs

Hence, the number of solutions is infinite.

13. Given curve is $x = 3t^2$, $y = 2t^3$

Let the tangent at $P(3t_1^{\ 2}, 2t_1^{\ 3})$ and the normal at $Q(3t_2^{\ 2}, 2t_2^{\ 3})$

Slope of the tangent at $P = t_1$

Slope of the normal at $Q = t_2$

But, slope of the tangent × slope of the normal = –1

$$t_1 \cdot t_2 = -1$$

$$\Rightarrow \quad t_2 = -\frac{1}{t_1}$$

Equation of tangent at P is

$$y - 2t_1^3 = t_1(x - 3t_1^2)$$

which is passing through $Q(3t_2^2, 2t_2^3)$

So, $2(t_2^3 - t_1^3) = t_1(3t_2^2 - 3t_1^2)$

$$\Rightarrow \quad 2(t_2^2 + t_1^2 + t_1 t_2) = 3t_1(t_2 + t_1)$$

$$\Rightarrow \quad 2\left(t_1^2 + \frac{1}{t_2} - 1\right) = 3t_1^2 - 1$$

$$\Rightarrow \quad -t_1^2 + \frac{2}{t_1^2} + 1 = 0$$

$$\Rightarrow \quad t_1^4 - t_1^2 - 2 = 0$$

$$\Rightarrow \quad (t_1^2 - 2)(t_1^2 + 1) = 0$$

$$\Rightarrow \quad (t_1^2 - 2) = 0$$

$$\Rightarrow \quad t_1 = \pm\sqrt{2}$$

Thus, $t_2 = \mp\dfrac{1}{\sqrt{2}}$

Hence, the equation of the required lines are

$$y - (\pm 4\sqrt{2}) = \pm\sqrt{2}(x - 6)$$

14. Given curve is $9y^2 = 3x^3$ ...(i)

Slope of the tangent $= \dfrac{x^2}{2y}$

Slope of the normal $= -\dfrac{2y}{x^2}$

Since the normal makes equal intersepts with the axes

So, $-\dfrac{2y}{x^2} = \pm 1$

$$\Rightarrow \quad 2y = \mp x^2 \quad ...(ii)$$

Solving (i) and (ii), we get,

$$x = 0, \frac{4}{3}$$

when $x = 0$, then $y = 0$

when $x = 4/3$, then $y = -\dfrac{8}{9}, \dfrac{9}{9}$

Hence, the points are

$$(0, 0), \left(\frac{4}{3}, -\frac{8}{9}\right), \left(\frac{4}{3}, \frac{8}{9}\right)$$

15. Given curves are

$$y = 2^x \log_e x \text{ and } y = x^{2x} - 1$$

Clearly, (1, 0) is the point of intersection of the curves

Now, $y = 2^x \log_e x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2^x}{x} + \log_e x . 2^x \log 2$$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_{(1, 0)} = 2$$

Also, $y = x^{2x} - 1 = e^{2x \log x} - 1$

$$\Rightarrow \quad \frac{dy}{dx} = e^{2x \log x}\left(\frac{2x}{x} + 2\log x\right)$$

$\Rightarrow \quad \frac{dy}{dx} = e^{2x\log x}(2 + 2\log x)$

$\Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_{(1,0)} = 2$

Let θ be the angle between them.

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = 0$$

$\Rightarrow \quad \tan\theta = 0$

$\Rightarrow \quad \theta = 0$

Now, $(2\cos\theta + 3\sin\theta)$

$= 2\cos(0) + 3\sin(0)$

$= 2 + 0$

$= 2.$

16. Given curves are

$x^3 + y^3 = 8xy$ and $y^2 = 4x$

On solving, we get, $x = 0, 4$

when $x = 0$, then $y = 0$

when $x = 4$, then $y = \pm 4$

Thus, the points are (0, 0), (4, –4), (4, 4)

But (4, –4) does not satisfy the equations

Hence, the required point is (4, 4)

Now, $x^3 + y^3 = 8xy$

$\Rightarrow \quad 3x^2 + 3y^2\frac{dy}{dx} = 8y + 8x\frac{dy}{dx}$

$\Rightarrow \quad (3y^2 - 8x)\frac{dy}{dx} = (8y - 3x^2)$

$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{8y - 3x^2}{3y^2 - 8x}\right)$

Thus, $m = \left(\frac{dy}{dx}\right)_{(4,4)} = \frac{32 - 48}{48 - 32} = -\frac{16}{16} = -1$

Hence, the equation of the normal is

$y - 4 = 1(x - 4)$

$\Rightarrow \quad x - y = 0$

17. Let the given curves can be written as

$y = 1 - \cos x$ and $y = \frac{\sqrt{3}}{2}|x| + a$

Since it has exactly one solution, so both the curves touch each other

Thus, the slope of the first curve is equal to the slope of the second curve

when $x > 0$, so their slopes are $\sin x$ and $\frac{\sqrt{3}}{2}$

According to the condition, we have

$\sin x = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad x = \frac{\pi}{3}$

Hence, the point of contact is $\left(\frac{\pi}{3}, \frac{1}{2}\right)$

when the point is $\left(\frac{\pi}{3}, \frac{1}{2}\right)$, then $a = \frac{1}{2} - \frac{\pi}{2\sqrt{3}}$

18. Let the common tangent be $y = ax + b$

On solving with both the curves

$ax + b = x^2 - 5x + 4$

$ax + b = x^2 + x + 1$

$\Rightarrow \quad x^2 - (5 + a)x + 4 - b = 0$

$x^2 + (1 - a)x + 1 - b = 0$

Putting $D = 0$, we get,

$(a + 5)^2 - 4(4 - b) = 0$

$(1 - a)^2 - 4(1 - b) = 0$

$\Rightarrow \quad a^2 + 10a + 4b + 9 = 0$

$a^2 - 2a + 4b - 3 = 0$

On solving, we get, $a = -1, b = 0$

Hence, the equation of the common tangent be

$x + y = 0$

19. Given curves are $ay + x^2 = 7$ and $y = x^3$ ...(i)

$\Rightarrow \quad a\frac{dy}{dx} + 2x = 0, \frac{dy}{dx} = 3x^2$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a}, \frac{dy}{dx} = 3x^2$

Since the curves cut orthogonally, so

$\Rightarrow \quad -\frac{2x}{a} \times 3x^2 = -1$

$\Rightarrow \quad a = 6x^3$ ...(ii)

On solving (i) and (ii), we get,

$6x^6 + x^2 = 7$

$\Rightarrow \quad x = 1$

Hence, the value of a is 6 .

20. Given curves are

$y = 1 - ax^2$ and $y = x^2$ ...(i)

$\Rightarrow \quad \frac{dy}{dx} = -2ax; \frac{dy}{dx} = 2x$

since the curves cut orthogonally, so

$\Rightarrow \quad -2ax \times 2x = -1$

$\Rightarrow \quad ax^2 = \frac{1}{4}$ ...(ii)

On solving (i) and (ii), we get,

$y = \frac{5}{4} = x^2$

Now, from (ii), we get,

$$a \times \frac{5}{4} = \frac{1}{4}$$

$\Rightarrow \quad a = \frac{1}{5}.$

21. Given parametric equations are

$$x = t^2 + t + 1 = t^2 - t + 1$$

Thus, the equation of the curve is

$$2(x + y) = (x - y)^2 + 4$$

Since the point (1, 1) lies on the curve, so the number of tangents can be drawn is 1.

22. Given curves are $y = e^x$ and $y = kx^2$ ...(i)

$$\frac{dy}{dx} = e^x; \frac{dy}{dx} = 2kx$$

Since both the curves touch each other, so

$$e^x = 2kx \quad \text{...(ii)}$$

On solving (i) and (ii), we get,

$$x = 2$$

Hence, the value of k is $\frac{e^2}{4}$

24. Let the tangent at (α, β)

Now, $\frac{dy}{dx} = \frac{2t + 1}{2 - 2t}$

$\Rightarrow \quad \frac{\beta - 1}{\alpha - 1} = \frac{2t + 1}{2 - 2t}$

$\Rightarrow \quad \frac{t + t^2 - 1}{2t - t^2 - 1} = \frac{2t + 1}{2 - 2t}$

$\Rightarrow \quad \frac{t^2 + t - 1}{t^2 - 2t + 1} = \frac{2y + 1}{2(t - 1)}$

$\Rightarrow \quad \frac{t^2 + t - 1}{(t - 1)^2} = \frac{2t + 1}{2(t - 1)}$

$\Rightarrow \quad \frac{t^2 + t - 1}{(t - 1)} = \frac{2t + 1}{2}$

$\Rightarrow \quad 2t^2 + 2t - 2 = 2y^2 - t - 1$

$\Rightarrow \quad 2t - 2 = - t - 1$

$\Rightarrow \quad 3t = 1$

$\Rightarrow \quad t = \frac{1}{3}$

Now, $\frac{dy}{dx} = \frac{2t + 1}{2 - 2t} = \frac{2 + 3}{6 - 2} = \frac{5}{4}$

Thus, the equation of the tangent is

$$y - 1 = \frac{5}{4}(x - 1)$$

$\Rightarrow \quad 4y - 4 = 5(x - 1)$

$\Rightarrow \quad 5x - 4y = 1$

25. Given curve is

$x = a\cos t + at\sin t,\ y = a\sin t - at\cos t$

Now, $\frac{dy}{dx} = \frac{at\sin t}{at\cos t} = \tan t$

Hence, the equation of the normal is

$$y - (a\sin t - at\cos t)$$

$$= -\frac{1}{\tan t}\{x - (a\cos t + at\sin t)\}$$

$\Rightarrow \quad x\cos t + y\sin t = a$

Hence, the distance of the normal from the origin is

$$= \left| \frac{0 + 0 - a}{\sqrt{\cos^2 t + \sin^2 t}} \right|$$

= a unit.

Level IV

(Tougher Problems for JEE-Advanced)

1. We have $y^2 = x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$

Let the point $P(t^2, t)$ lies on the curve $y^2 = x$

Equation of the normal to the curve at P is

$$y - t = -2t(x - t^2)$$

which passes through the point $(c, 0)$

So, $\quad -t = -2t(c - t^2)$

$\Rightarrow \quad (c - t^2) = \frac{1}{2}$

$\Rightarrow \quad c - \frac{1}{2} = t^2$

$\Rightarrow \quad c - \frac{1}{2} > 0$

$\Rightarrow \quad c > \frac{1}{2}$

Hence, the value of c is $c \in \left(\frac{1}{2}, \infty\right)$

2. Given curves are

$$ax^2 + by^2 = 1 \text{ and } a'x^2 + b'y^2 = 1 \quad \text{...(i)}$$

$\Rightarrow \quad 2ax + 2by\frac{dy}{dx} = 0;\ 2a'x + 2b'y\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{ax}{by};\ \frac{dy}{dx} = -\frac{a'x}{b'y}$

Since both the curves intersect orthogonally, so, $-\frac{ax}{by} \times -\frac{a'x}{b'y} = -1$

$\Rightarrow \quad \frac{aa'x^2}{bb'y^2} = -1$

$\Rightarrow \quad aa'x^2 = -bb'y^2$ ...(ii)

From (i), we get,

$(a - a')x^2 + (b - b')y^2 = 0$ ...(iii)

Dividing (iii), by (ii), we get,

$$\frac{(a - a')}{aa'} = \frac{(b - b')}{bb'}$$

$\Rightarrow \quad \frac{1}{a'} \frac{-1}{a} = \frac{1}{b'}, \frac{-1}{b}$

$\Rightarrow \quad \frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$

Hence, the result.

3. Let $P(\alpha, \beta)$ be any point on the curve

$$y^2 - 2x^3 - 4y + 8 = 0$$

Now, $2y\frac{dy}{dx} - 6x^2 - 4\frac{dy}{dx} = 0$

$\Rightarrow \quad (2y - 4)\frac{dy}{dx} - 6x^2$

$\Rightarrow \quad \frac{dy}{dx} = \frac{6x^2}{(2y - 4)}$

Thus, slope $= m = \left(\frac{dy}{dx}\right)_P = \frac{6\alpha^2}{(2\beta - 4)}$

Now, the equation of the tangent at P is

$$y - \beta = \frac{6\alpha^2}{(2\beta - 4)}(x - \alpha)$$

which is passing through (1, 2)

$\Rightarrow \quad 2 - \beta = \frac{6\alpha^2}{(2\beta - 4)}(1 - \alpha)$

$\Rightarrow \quad 2 - \beta = \frac{3\alpha^2}{(\beta - 2)}(1 - \alpha)$

$\Rightarrow \quad -(\beta - 2)^2 = 3\alpha^2(1 - \alpha)$

$\Rightarrow \quad 3\alpha^3 - 3\alpha^2 - \beta^2 + 4\beta - 4 = 0$ ...(i)

Also, $P(\alpha, \beta)$ lies on $y^2 - 2x^3 - 4y + 8 = 0$

So, $\beta^2 - 2\alpha^3 - 4\beta + 8 = 0$ ...(ii)

From (i) and (ii), we get,

$\Rightarrow \quad -\alpha^3 + 3\alpha^2 - 4 = 0$

$\Rightarrow \quad \alpha^3 - 3\alpha^2 + 4 = 0$

$\Rightarrow \quad \alpha^3 + \alpha^2 - 4\alpha^2 - 4\alpha + 4\alpha + 4 = 0$

$\Rightarrow \quad \alpha^2(\alpha + 1) - 4\alpha(\alpha + 1) + 4(\alpha + 1) = 0$

$\Rightarrow \quad (\alpha + 1)(\alpha^2 - 4\alpha + 4) = 0$

$\Rightarrow \quad (\alpha + 1)(\alpha - 2)^2 = 0$

$\Rightarrow \quad \alpha = -1, 2$

when $\alpha = -1$, then $\beta^2 + 2 - 4\beta + 8 = 0$

$\Rightarrow \quad \beta^2 - 4\beta + 10 = 0$

Thus, β is imaginary.

when $\alpha = 2$, then $\beta^2 - 16 - 4\beta + 8 = 0$

$\Rightarrow \quad \beta^2 - 4\beta - 8 = 0$

$\Rightarrow \quad (\beta - 2)^2 = 12 = (2\sqrt{3})^2$

$\Rightarrow \quad (\beta - 2) = \pm (2\sqrt{3})$

$\Rightarrow \quad \beta = 2 \pm 2\sqrt{3}$

Hence, the point of contacts are

$(2, 2 + 2\sqrt{3})$ and $(2, 2 - 2\sqrt{3})$

Slope of the tangents are $2\sqrt{3}, -2\sqrt{3}$

Hence, the equations of tangents are

$$y - (2 + 2\sqrt{3}) = 2\sqrt{3}(x - 2)$$

and

$$y - (2 - 2\sqrt{3}) = -2\sqrt{3}(x - 2)$$

4. We have $y = 8t^3 - 1, x = 4t^2 + 3$

$\Rightarrow \quad \frac{dy}{dx} = \frac{24t^2}{8t} = 3t$

The equation of the tangent at $P(4t_1^2 + 3, 8t_1^3 - 1)$

is $y - (8t_1^3 - 1) = 3t_1(x - (4t_1^2 + 3))$ ...(i)

Let another point on the given curve is

$Q(4t_2^2 + 3, 8t_2^3 - 1)$

Since the equation (i), passes through Q, so

$(8t_2^3 - 1) - (8t_1^3 - 1) = 3t_1((4t_2^2 + 3) - (4t_1^2 + 3))$

$\Rightarrow \quad (8t_2^3 - 8t_1^3) = 3t_1(4t_2^2 - 4t_1^2)$

$\Rightarrow \quad 2(t_2^3 - t_1^3) = 3t_1(t_2^2 - t_1^2)$

$\Rightarrow \quad 2(t_2^2 + t_1^2 + t_1t_2) = 3t_1(t_2 + t_1)$

$\Rightarrow \quad 2t_2^2 - t_1t_2 - t_1^2 = 0$

$\Rightarrow \quad (2t_2 + t_1)(t_2 - t_1) = 0$

$\Rightarrow \quad (2t_2 + t_1) = 0 \quad (\because t_2 \neq t_1)$

$\Rightarrow \quad t_2 = -\frac{t_1}{2}$

Now, the point Q becomes $Q(t_1^2 + 3, -t_1^3 - 1)$

Slope of the tangent at $Q = -\frac{3t_1}{2}$

Thus, the equation of the normal at $Q(t_1^2 + 3, -t_1^3 - 1)$ is

$$y + (t_1^3 + 1) = \frac{2}{3t_1}(x - t_1^2 - 3) \quad ...(ii)$$

Equations (i) and (ii) are identical

$$\frac{1}{1} = \frac{3t_1}{\frac{2}{3}t_1} = \frac{8t_1^3 - 1 - 12t_1^3 - 9t_1}{-t_1^3 - 1 - \frac{2t_1}{3} - \frac{2}{t_1}}$$

$\Rightarrow \quad 3t_1^3 + 9t_1 - \frac{2}{3}t_1 - \frac{2}{t_1} = 0$

$\Rightarrow \quad 3t_1\left(t_1^2 - \frac{2}{9}\right) + \frac{9}{t_1}\left(t_1^2 - \frac{2}{9}\right) = 0$

$\Rightarrow \quad \left(t_1^2 - \frac{2}{9}\right)\left(3t_1 + \frac{9}{t_1}\right) = 0$

$\Rightarrow \quad \left(t_1^2 - \frac{2}{9}\right) = 0$

$\Rightarrow \quad t_1 = \pm\frac{\sqrt{2}}{3}$

Hence, the equations of the straight lines are

$$27\sqrt{2}x - 27y - (89\sqrt{2} + 27) = 0$$

and

$$27\sqrt{2}x + 27y - (89\sqrt{2} - 27) = 0$$

5. The given curve is $y^2 - 16x - 8y = 0$.

$\Rightarrow \quad y^2 - 8y = 16x$

$\Rightarrow \quad (y - 4)^2 = 16(x + 1) \quad ...(i)$

Consider any point on (i) as $P(4t^2 - 1, 8t + 4)$

Now, $\quad \left(\frac{dy}{dx}\right)_P = \frac{1}{t}$

Thus, the equation of normal at point P is

$$y - (8t + 4) = -t(x - (4t^2 - 1))$$

which is passing through (14, 7)

So, $\quad 7 - (8t + 4) = -t(14 - (4t^2 - 1)$

$\Rightarrow \quad 3 - 8t = -15t + 4t^3$

$\Rightarrow \quad 4t^3 - 7t - 3 = 0$

$\Rightarrow \quad 4t^3 + 4t^2 - 4t^2 - 4t - 3t - 3 = 0$

$\Rightarrow \quad 4t^2(t + 1) - 4t(t + 1) - 3(t + 1) = 0$

$\Rightarrow \quad (4t^2 - 4t - 3)(t + 1) = 0$

$\Rightarrow \quad (4t^2 - 6t + 2t - 3)(t + 1) = 0$

$\Rightarrow \quad \{2t(2t - 3) + 1(2t - 3)\}(t + 1) = 0$

$\Rightarrow \quad (2t + 1)(2t - 3)(t + 1) = 0$

$\Rightarrow \quad t = -1, -\frac{1}{2}, \frac{3}{2}$

Hence, the foot of normals are

(3, – 4), (8, 16) and (0, 0).

6. Given curve is $x^{2/3} + y^{2/3} = a^{2/3} \quad ...(i)$

Differentiating w.r.t x, we get,

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}$

Let $P(\alpha, \beta)$ be any point on the curve (i)

Slope of the normal at P is $m = \left(\frac{\alpha}{\beta}\right)^{1/3}$

It is given that, the slope of the normal = $\tan\varphi$

Thus, $\quad \left(\frac{\alpha}{\beta}\right)^{1/3} = \tan\varphi$

$\Rightarrow \quad \left(\frac{\alpha}{\beta}\right)^{1/3} = \tan\varphi = \frac{\sin\varphi}{\cos\varphi}$

$\Rightarrow \quad \frac{\alpha^{1/3}}{\sin\varphi} = \frac{\beta^{1/3}}{\cos\varphi} = \frac{\sqrt{\alpha^{2/3} + \beta^{2/3}}}{\sqrt{\sin^2\varphi + \cos^2\varphi}} = \frac{a^{1/3}}{1}$

$\Rightarrow \quad \alpha = a\sin^3\varphi, \beta = a\cos^3\varphi$

Hence, the equation of the normal is

$$y - a\cos^3\varphi = \tan\varphi(x - a\sin^3\varphi)$$

$\Rightarrow \quad y\cos\varphi - a\cos^4\varphi = x\sin\varphi - a\sin^4\varphi$

$\Rightarrow \quad x\sin\varphi - y\cos\varphi = a\sin^4\varphi - a\cos^4\varphi$

$\Rightarrow \quad y\cos\varphi - x\sin\varphi = a\cos^4\varphi - a\sin^4\varphi$

$\Rightarrow \quad y\cos\varphi - x\sin\varphi = a(\cos^2\varphi - \sin^4\varphi)$

$\Rightarrow \quad y\cos\varphi - x\sin\varphi = a\cos 2\varphi$

7. Given curve is $y = \cos(x + y)$

$\Rightarrow \quad \frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow \quad \frac{dy}{dx} = -\sin(x + y) - \sin(x + y)\frac{dy}{dx}$

$\Rightarrow \quad (1 + \sin(x + y))\frac{dy}{dx} = -\sin(x + y)$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sin(x + y)}{1 + \sin(x + y)}$

Now, slope of the line $x + 2y = 0$ is –1/2.

Since the tangents are parallel, so

$$-\frac{\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2}$$

$$\Rightarrow \quad \frac{\sin(x+y)}{1+\sin(x+y)} = \frac{1}{2}$$

$$\Rightarrow \quad 2\sin(x+y) = 1 + \sin(x+y)$$

$$\Rightarrow \quad \sin(x+y) = 1 \quad \text{...(ii)}$$

Squaring (i) and (ii) and adding, we get,

$$y^2 + 1 = \sin^2(x+y) + \cos^2(x+y)$$

$$\Rightarrow \quad y^2 + 1 = 1$$

$$\Rightarrow \quad y = 0$$

when $y = 0$, then $\cos x = 0$

$$\Rightarrow \quad x = (2n+1)\frac{\pi}{2},\ n \in I$$

$$\Rightarrow \quad x = -\frac{3\pi}{2}, \frac{\pi}{2}$$

Hence, the points are $\left(-\frac{3\pi}{2}\right), \left(\frac{\pi}{2}, 0\right)$

Therefore, the equation of tangents are

$$y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$$

and $$y - 0 = -\frac{1}{2}\left(x - \frac{3\pi}{2}\right)$$

$$2x + 4y + 3\pi = 0$$

and $$2x + 4y - \pi = 0$$

8. Given curve is $x = \sec^2\theta$ and $y = \cot\theta$

Eliminating θ, we get,

$$\sec^2\theta - \tan^2\theta = 1$$

$$\Rightarrow \quad x - \frac{1}{y^2} = 1 \quad \text{...(i)}$$

Clearly, the point P is (2, 1)

Now, $$1 + \frac{2}{y^3}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y^3}{2}$$

Slope of the tangent at $P = \left(\frac{dy}{dx}\right)_P = -\frac{1}{2}$

Equation of the tangent at P is

$$y - 1 = -\frac{1}{2}(x - 2)$$

$$\Rightarrow \quad x + 2y = 4 \quad \text{...(ii)}$$

On solving (i) and (ii), we get,

$$y = -\frac{1}{2},\ x = 5$$

Thus, the point Q is $\left(5, -\frac{1}{2}\right)$

Hence, the length of PQ

$$= \sqrt{(5-2)^2 + \left(1 + \frac{1}{2}\right)^2}$$

$$= \sqrt{9 + \frac{9}{4}} = \sqrt{\frac{45}{4}} = \frac{3\sqrt{5}}{2}$$

9. Given curve is $x^{2/3} + y^{2/3} = a^{2/3}$ ...(i)

Any point on (i) can be considered as

$$(a\cos^3\theta, a\sin^3\theta)$$

Now, slope of the tangent $= -\tan\theta$

Equation of the tangent is

$$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$$

$$\Rightarrow \quad x\sin\theta + y\cos\theta = a\sin\theta\cos\theta$$

$$\Rightarrow \quad x\sin\theta + y\cos\theta = \frac{a}{2}\sin 2\theta \quad \text{...(ii)}$$

Slope of the normal $= \cot\theta$

Equation of the normal is

$$y - a\sin^3\theta = \cot\theta(x - a\cos^3\theta)$$

$$\Rightarrow \quad x\cos\theta - y\sin\theta = a\cos 2\theta \quad \text{...(ii)}$$

Now, $$p_1 = \left|-\frac{a\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \cos^2\theta}}\right| = \frac{a}{2}\sin 2\theta$$

$$\Rightarrow \quad 2p_1 = a\sin 2\theta \quad \text{...(iii)}$$

and $$p_2 = \left|-\frac{a\cos 2\theta}{\sqrt{\sin^2\theta + \cos^2\theta}}\right| = a\cos 2\theta$$

$$\Rightarrow \quad p_2 = a\cos 2\theta \quad \text{...(iv)}$$

Squaring (iii) and (iv), we get,

$$4p_1^2 + p_2^2 = a^2$$

Hence, the result.

10. Given curve is $y = x^2 - x^3$

$$\Rightarrow \quad \frac{dy}{dx} = 2x - 3x^2$$

Let the point P and Q be $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$

slope of the tangent at $P = m = (2\alpha_1 - 3\alpha_1^2)$

Equation of the tangent at P is

$$y - \beta_1 = (2\alpha_1 - 3\alpha_1^2)(x - \alpha_1)$$

which is passing through Q

So, $$\beta_2 - \beta_1 = (2\alpha_1 - 3\alpha_1^2)(\alpha_2 - \alpha_1) \quad \text{...(i)}$$

Since the point P and Q lie on the given curve, so

$$\beta_1 = \alpha_1^2 - \alpha_1^3,\ \beta_2 = \alpha_2^2 - \alpha_2^3 \quad \text{...(ii)}$$

From (i) and (ii), we get,

$(\alpha_2^2 - \alpha_2^3) - (\alpha_1^2 - \alpha_1^3) = (2\alpha_1 - 3\alpha_1^2)(\alpha_2 - \alpha_1)$

$\Rightarrow (\alpha_2^2 - \alpha_1^2) + (\alpha_1^3 - \alpha_2^3) = (2\alpha_1 - 3\alpha_1^2)(\alpha_2 - \alpha_1)$

$\Rightarrow (\alpha_2 + \alpha_1) - (\alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2) = (2\alpha_1 - 3\alpha_1^2)$

$\Rightarrow (2\alpha_1^2 - \alpha_2^2 - \alpha_1\alpha_2 - \alpha_1 + \alpha_2) = 0$

$\Rightarrow (2\alpha_1^2 - \alpha_1\alpha_2 - \alpha_2^2 - \alpha_1 + \alpha_2) = 0$

$\Rightarrow (\alpha_1 - \alpha_2)(2\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) = 0$

$\Rightarrow (\alpha_1 - \alpha_2)(2\alpha_1 + \alpha_2 - 1) = 0$

$\Rightarrow (2\alpha_1 + \alpha_2 - 1) = 0 \; (\because \alpha_1 \neq \alpha_2)$

$\Rightarrow 2\alpha_1 + \alpha_2 = 1$ ...(iii)

Let $M(h, k)$ be the mid-point of P and Q

Thus, $h = \dfrac{\alpha_1 + \alpha_2}{2}, \; k = \dfrac{\beta_1 + \beta_2}{2}$

On solving, we get,

$\alpha_1 = 1 - 2h, \; \alpha_2 = 4h - 1$

Now, $k = \dfrac{\beta_1 + \beta_2}{2}$

$= \dfrac{\alpha_1^2 - \alpha_1^3 + \alpha_2^2 - \alpha_2^3}{2}$

$= \dfrac{(\alpha_1^2 + \alpha_2^2) - (\alpha_1^3 + \alpha_2^3)}{2}$

$= \dfrac{(\alpha_1 + \alpha_2)^2 - 2\alpha_1\alpha_2 - (\alpha_1 + \alpha_2)^3 + 3\alpha_1\alpha_2(\alpha_1 + \alpha_2)}{2}$

$= \dfrac{4h^2 - 2\alpha_1\alpha_2 - 8h^3 + 6h\alpha_1\alpha_2}{2}$

$= 2h^2 - \alpha_1\alpha_2 - 4h^3 + 3h\alpha_1\alpha_2$

$= 2h^2 - 4h^3 + \alpha_1\alpha_2(3h - 1)$

$= 2h^2 - 4h^3 + (1 - 2h)(4h - 1)(3h - 1)$

$= 2h^2 - 4h^3 + (1 - 2h)(12h^2 - 7h + 1)$

$= 2h^2 - 4h^3 + (12h^2 - 7h + 1 - 24h^3 + 14h^2 - 2h)$

$k = 1 - 9h + 28h^2 - 28h^3$

Hence, the locus of the mid-point $M(h, k)$ is

$y = 1 - 9x + 28x^2 - 28x^3.$

Hence, the result.

Integer Type Questions

1. Given curve is $x^{3/2} + y^{3/2} = a^{3/2}, \; a > 0$

Since $a > 0$, so the curve lies in the 1st quadrant

Thus, the number of tangents which are equally inclined with the axes is 1.

2. Given curve is $y^2 - 2x^2 - 4y + 8 = 0$

Now, $4 - 2.1 - 8 + 8$

$= 4 - 2 = 2 > 0$

So, the point lies outside of the curve.

Thus, the number of tangents to the curve is 2.

3. Given curve is $y = -x^3 + 3x^2 + 3x - 1$

$\Rightarrow \dfrac{dy}{dx} = -3x^2 + 6x + 3$

Thus, $m = -3x^2 + 6x + 3$

$\Rightarrow \dfrac{dm}{dx} = -6x + 6$

$\Rightarrow \dfrac{d^2m}{dx^2} = -6 < 0$

So slope is maximum

For max or min, $\dfrac{dm}{dx} = 0$ gives

$-6x + 6 = 0$

$\Rightarrow x = 1$

Hence, the maximum slope is $-3 + 6 + 3 = 6$.

4. We have $\dfrac{x^4}{2} = x + y$ and the line is $3x + 4y = c$

Since both the curves touch each other, so

$\Rightarrow \dfrac{4x^3}{2} - 1 = -\dfrac{3}{4}$

$\Rightarrow 2x^3 = 1 - \dfrac{3}{4} = \dfrac{1}{4}$

$\Rightarrow x^3 = \dfrac{1}{8}$

$\Rightarrow x = \dfrac{1}{2}$

when $x = \dfrac{1}{2}, \; y = -\dfrac{5}{32}$

Thus, $c = 3x + 4y = \dfrac{3}{2} - \dfrac{5}{8} = \dfrac{12 - 5}{8} = \dfrac{7}{8}$

Hence, the number of values of c is 1.

5. Given curve is $y^2 = px^3 + q$

$\Rightarrow 2y\dfrac{dy}{dx} = 3px^2$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3px^2}{2y}$

$\Rightarrow m = \left(\dfrac{dy}{dx}\right)_{(2,\,3)} = \dfrac{3px^3}{2y} = \dfrac{12p}{6} = 2p$

Hence, the equation of the tangent at (2, 3) is

$$y - 3 = 2p(x - 2)$$

which is identical with $y = 4x - 5$

Thus, $2p = 4 \Rightarrow p = 2$

Since the point (2, 3) lies on the curve, so

$$8p + q = 9$$

$$\Rightarrow 16 + q = 9$$

$$\Rightarrow q = 9 - 16 = -7$$

Hence, the value of $(p - q - 4)$

$$= 2 + 7 - 4$$

$$= 9 - 4 = 5.$$

6. Equation of the line joining the given points is $x + y = 3$.

and the given curve is $y = \dfrac{c}{x + 1}$

Clearly, $3 - x = \dfrac{c}{x + 1}$

$$\Rightarrow (3 - x)(x + 1) = c$$

$$\Rightarrow x^2 - 2x + (c - 3) = 0$$

For tangency, its D is zero

So, $4 - 4(c - 3) = 0$

$$\Rightarrow 1 - (c - 3) = 0$$

$$\Rightarrow c = 4$$

7. Given curve is $y = x^2 - x$

The given curve cut the line $y = 2$ at the point (2, 2)

Now, $\dfrac{dy}{dx} = 2x - 1$

Thus, $m = \left(\dfrac{dy}{dx}\right)_{(2,2)} = 2.2 - 1 = 3$

8. Given curve is $x^3 - y^2 = 0$ ...(i)

$$\Rightarrow 3x^2 - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$$

Slope of the tangent

$$= \left(\frac{dy}{dx}\right)_{(4m^2,\ 8m^3)} = \frac{3 \times 16m^4}{2 \times 8m^2} = 3m$$

Hence, the equation of the tangent is

$$y - 8m^2 = 3m(x - 4m^2)$$

$$\Rightarrow y = 3mx - 4m^3 \quad \text{...(ii)}$$

On solving (i) and (ii), we get,

$$x = 4m^2, m^2$$

Thus, the new point is $Q(m^2, -m^3)$

Slope of the tangent at $Q(m^2, -m^3)$ is $= -\dfrac{3}{2}m$

Slope of the normal at $Q(m^2, -m^2)$ is $= \dfrac{2}{3m}$

Since tangent at $P(4m^2, 8m^3)$ and normal at $Q(m^2, -m^3)$ coincide, we get,

$$\frac{2}{3m} = 3m$$

$$\Rightarrow 9m^2 = 2$$

$$\Rightarrow 9m^2 + 2 = 2 + 2 = 4$$

9. Given curve is $\sqrt{x} + \sqrt{y} = 0$

Let $P(\alpha, \beta)$ be a variable point on the given curve.

Equation of tangent at $P(\alpha, \beta)$ is

$$y - \beta = -\frac{\sqrt{\beta}}{\sqrt{\alpha}}(x - \alpha)$$

$$\Rightarrow \frac{y - \beta}{\sqrt{\beta}} = -\frac{(x - \alpha)}{\sqrt{\alpha}}$$

$$\Rightarrow \frac{y}{\sqrt{\beta}} - \sqrt{\beta} = -\frac{x}{\sqrt{\alpha}} + \sqrt{\alpha}$$

$$\Rightarrow \frac{x}{\sqrt{\alpha}} + \frac{y}{\sqrt{\beta}} = \sqrt{\alpha} + \sqrt{\beta}$$

$$\Rightarrow \frac{x}{\sqrt{\alpha}} + \frac{y}{\sqrt{\beta}} = 2$$

Thus, the sum of the intercepts

$$= 2(\sqrt{\alpha} + \sqrt{\beta})$$

$$= 2 \times 2$$

$$= 4.$$

10. Given curve is $y^2 = x^3$

$$\Rightarrow 2y\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$$

Now, $\left(\dfrac{dy}{dx}\right)_{(4,8)} = \dfrac{3 \times 16}{16} = 3$

Hence, the length of the sub-normal is

$$= y\frac{dy}{dx} = 8 \times 3 = 24$$

Thus, $m = 24$

Hence, the value of $\sqrt{m + 1}$ is 5.

11. The given curve is

$$(x + y) - \ln(x + y) = 2x + 5$$

$$\Rightarrow 1 + \frac{dy}{dx} - \frac{1}{(x + y)}\left(1 + \frac{dy}{dx}\right) = 2$$

$$\Rightarrow \quad \left(1+\frac{dy}{dx}\right)-\frac{1}{(x+y)}\left(1+\frac{dy}{dx}\right)=2$$

$$\Rightarrow \quad \left(1+\frac{dy}{dx}\right)\left(1-\frac{1}{(x+y)}\right)=2$$

$$\Rightarrow \quad \left(1+\frac{dy}{dx}\right)\left(\frac{x+y-1}{(x+y)}\right)=2$$

$$\Rightarrow \quad \left(1+\frac{dy}{dx}\right)=\frac{2(x+y)}{x+y-1}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2(x+y)}{x+y-1}-1=\frac{2x+2y-x-y+1}{x+y-1}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x+y+1}{x+y-1}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{x+y-1}{x+y-1}$$

Since it has a vertical tangent, so $\frac{dy}{dx}=0$

$$\Rightarrow \quad \frac{x+y-1}{x+y+1}=0$$

$$\Rightarrow \quad x+y-1=0$$

$$\Rightarrow \quad x+y=1$$

Clearly, the point is (1, 0)

Thus, $\alpha = 1, \beta = 0$

Hence, the value of $(|\alpha + \beta| + 4)$ is 5.

12. The line $y = mx + c$, will be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $c^2 = a^2m^2 + b^2$

Here, $c = a, m = -1, a^2 = 9, b^2 = 16$

Thus, $a^2 = 16 + 9 = 25$

$$\Rightarrow \quad a = \pm 5$$

$$\Rightarrow \quad (|a| + 1) = |\pm 5| + 1 = 6$$

13. We have $xy = 2$

$$y=\frac{2}{x}$$

$$\frac{dy}{dx}=-\frac{2}{x^2}<0$$

So, the tangent makes obtuse angle with the x-axis

Thus, the number of points is zero. i.e. $m = 0$

Hence, the value of $(m + 4)$ is 4.

14. Given curves are

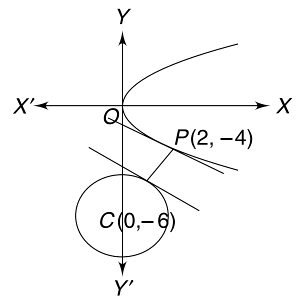
$$y^2 = 8x \text{ and } x^2 + (y+6)^2 = 1$$

$$\Rightarrow \quad 2y\frac{dy}{dx}=8;\ 2x+2(y+6)\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{4}{y};\ \frac{dy}{dx}=-\frac{x}{(y+6)}$$

Since the tangents are parallel, so their slopes are same

$$\frac{4}{y}=-\frac{x}{(y+6)}$$



$$\Rightarrow \quad 4y + 24 + xy = 0$$

$$\Rightarrow \quad 4y + 24 + y\cdot\frac{y^2}{8} = 0$$

$$\Rightarrow \quad y^3 + 32y + 192 = 0$$

$$\Rightarrow \quad y^3 + 4y^2 - 4y^2 - 16y + 48y + 192 = 0$$

$$\Rightarrow \quad y^2(y+4) - 4y(y+4) + 48(y+4) = 0$$

$$\Rightarrow \quad (y^2 - 4y + 48)(y+4) = 0$$

$$\Rightarrow \quad (y+4) = 0$$

$$\Rightarrow \quad y = -4$$

when $y = -4$, then $8x = 16$

$x = 2$

Hence, the nearest point is (2, – 4)

Thus, $(\alpha, \beta) = (2, -4)$

Hence, the value of $(|\alpha + \beta|)$ is 2 + 3 = 5.

15. Let the common tangent be $y = ax + b$

On solving with both the curves

$$x^2 - x + 1 = ax + b$$

$$x^2 - 3x + 1 = ax + b$$

$$\Rightarrow \quad x^2 - (1+a)x + (1-b) = 0$$

$$x^2 - (a+3)x + (1-b) = 0$$

Putting $D = 0$, we get,

$$(a+1)^2 - 4(1-b) = 0$$

$$(a+3)^2 - 4(1-b) = 0$$

$$\Rightarrow \quad a^2 + 2a + 4b - 3 = 0$$

$$a^2 + 6a + 4b + 5 = 0$$

On solving, we get, $a = -2$

Thus, slope of the common tangent = $m = -2$

Hence, the value of $(|m| + 3)$ is 5.

Questions asked in Past IIT-JEE Exams

1. Given curve is $y = x^2$

   Any point on the given parabola is (x, x^2)

   Distance between (x, x^2) and $(0, c)$ is

   $z = \sqrt{(x^2) + (x^2 - c)^2}$

$\Rightarrow z^2 = (x^2) + (x^2 - c)^2$

$= x^4 - (2c - 1)x^2 + c^2$

$= \left(x^2 - \left(\frac{2c-1}{2}\right)\right)^2 + c^2 - \left(\frac{2c-1}{2}\right)^2$

$= \left(x^2 - \left(\frac{2c-1}{2}\right)\right)^2 + \left(c^2 - \left(\frac{4c^2 - 4c + 1}{4}\right)\right)$

$= \left(x^2 - \left(\frac{2c-1}{2}\right)\right)^2 + \left(c - \frac{1}{4}\right)$

Which is minimum when $x^2 = \left(\frac{2c-1}{2}\right)$

and the minimum value is $= z_{\min} = \sqrt{\left(c - \frac{1}{4}\right)}$

2. *Ans.* (c)

   Given $\quad x = a(\cos\theta + \theta\sin\theta)$

   $\frac{dx}{d\theta} = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta$

   Also, $\quad y = a(\sin\theta - \theta\cos\theta)$

   $\frac{dy}{d\theta} = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta$

   $\frac{dy}{dx} = \tan\theta$

   Slope of normal $= -\cot\theta$

   Equation of normal at θ is

   $y - a(a(\cos\theta - \theta\cos\theta) = -\cot\theta(x - a(\cos\theta + \theta\sin\theta))$ $(\cos\theta)x + (\sin\theta)y = a$

   Length of perpendicular from the origin is

   $$= \left|\frac{a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right| = a = \text{costant}$$

4. Given $\quad y = \cos(x + y)$

$\Rightarrow \quad \frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow \quad \frac{dy}{dx} = -\sin(x + y) - \sin(x + y)\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx}(1 + \sin(x + y)) = -\sin(x + y)$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sin(x + y)}{1 + \sin(x + y)}$

Since the tangent to the curve is parallel to the line $x + 2y = 0$

so, $\quad -\frac{\sin(x + y)}{1 + \sin(x + y)} = -\frac{1}{2}$

$\Rightarrow \quad \frac{\sin(x + y)}{1 + \sin(x + y)} = \frac{1}{2}$

$\Rightarrow \quad 2\sin(x + y) = 1 + \sin(x + y)$

$\Rightarrow \quad \sin(x + y) = 1$

$\Rightarrow \quad (x + y) = (4n + 1)\frac{\pi}{2}, n \in I$

When $\quad (x + y) = (4n + 1)\frac{\pi}{2}, n \in I$ then

$y = \cos(x + y) = 0$

Thus, $\quad x = (4n + 1)\frac{\pi}{2} = -\frac{3\pi}{2}, \frac{\pi}{2}$

Therefore, the points are $\left(-\frac{3\pi}{2}, 0\right)$ & $\left(\frac{\pi}{2}, 0\right)$

Hence, the equations of the tangents at these points are

$$y = -\frac{1}{2}\left(x - \frac{\pi}{2}\right) \;\&\; y = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$$

$\Rightarrow \quad x + 2y = \frac{\pi}{2}$ & $x + 2y = -\frac{3\pi}{2}$

5. *Ans.* (b)

   Given curve is $xy = 1$

   Let the point be (h, k)

   Now, $\quad y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$

   Sople of normal $= \left(\frac{dy}{dx}\right)_{(h,k)} = h^2$

   Equation of normal at (h, k) is

   $y - k = h^2(x - h)$

$\Rightarrow \quad h^2x - y + (k - h) = 0$

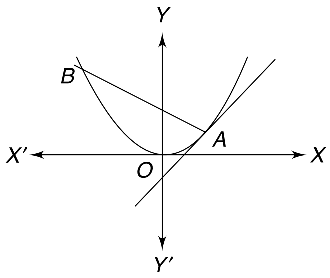
Given normal is $ax + by + c = 0$

Comparing the co-efficients of x and y, we get,

$$\frac{a}{h^2} = \frac{b}{-1} = \frac{c}{(k - h)}$$

Thus, $\quad a > 0$ and $b < 0$.

6. Given curve is $y = x^2$



Let $A = (t, t^2)$

Equation of the tangent at $A = (t, t^2)$ is

$$y - t^2 = \left(-\frac{1}{2t}\right)(x - t)$$

Let the normal intersects the curve again at B

where $B = (t_1, t_1^2)$

Thus, $(t_1^2 - t^2) = \left(-\frac{1}{2t}\right)(t_1 - t)$

$\Rightarrow$ $(t_1 + t) = \left(-\frac{1}{2t}\right)$

$\Rightarrow$ $t_1 = -\left(t + \frac{1}{2t}\right)$

Now, $AB = \sqrt{(t - t_1)^2 + (t^2 - t_1^2)^2}$

$\Rightarrow$ $AB = (t - t_1)^2 + (t^2 - t_1^2)^2$

$\Rightarrow$ $AB = (t - t_1)^2(1 + (t - t_1)^2)$

$\Rightarrow$ $L = \left(2t + \frac{1}{2t}\right)^2\left(1 + \frac{1}{4t^2}\right)$

$\Rightarrow$ $L = 4t^2\left(1 + \frac{1}{4t^2}\right)^3$

$$\frac{dL}{dt} = 8t\left(1 + \frac{1}{4t^2}\right)^3 - 12t^2\left(1 + \frac{1}{4t^2}\right)^2\left(\frac{2}{4t^3}\right)$$

$$\frac{dL}{dt} = 4\left(1 + \frac{1}{4t^2}\right)^2\left(2t - \frac{1}{t}\right)$$

For max/min, $\frac{dL}{dt} = 0$

$\Rightarrow$ $4\left(1 + \frac{1}{4t^2}\right)^2\left(2t - \frac{1}{t}\right) = 0$

$\Rightarrow$ $t^2 = \frac{1}{2}$

$\Rightarrow$ $t = \pm\frac{1}{\sqrt{2}}$

$$\left(\frac{d^2L}{dt^2}\right)_{t=\pm\frac{1}{\sqrt{2}}} = 16 \times \frac{9}{4} = 35 > 0$$

So L is minimum.

When $t = \frac{1}{\sqrt{2}}$ then

$$A = \left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \text{ \& } B = (-\sqrt{2}, 2) \text{ and } AB = \frac{27}{4}$$

Thus, the equation of AB is

$$\frac{y - 2}{\frac{1}{2} - 2} = \frac{x + 2}{\frac{1}{\sqrt{2}} + \sqrt{2}}$$

$\Rightarrow$ $\frac{y - 2}{\frac{1}{\sqrt{2}} - \sqrt{2}} = \frac{x + 2}{1}$

$\Rightarrow$ $\sqrt{2}x + 2y - 2 = 0$

When $t = -\frac{1}{\sqrt{2}}$, then

$$A = \left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \text{ \& } B = (\sqrt{2}, 2) \text{ and } AB = \frac{27}{4}$$

Thus, the equation of AB is

$$\frac{y - 2}{\frac{1}{2} - 2} = \frac{x - \sqrt{2}}{-\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)}$$

$\Rightarrow$ $\frac{y - 2}{\frac{1}{\sqrt{2}} - \sqrt{2}} = \frac{x - \sqrt{2}}{-1}$

$\Rightarrow$ $\sqrt{2}x - 2y + 2 = 0$

7. *Ans.* (d)

Given curve is $y - e^{xy} + x = 0$

$\Rightarrow$ $\frac{dy}{dx} - e^{xy}\left(x\frac{dy}{dx} + y.1\right) + 1 = 0$

$\Rightarrow$ $\frac{dy}{dx}(1 - xe^{xy}) = (ye^{xy} - 1)$

$\Rightarrow$ $\frac{dy}{dx} = \left(\frac{ye^{xy} - 1}{1 - xe^{xy}}\right)$

Since it has a vertical tangent, so $\frac{dy}{dx} = 0$

$\Rightarrow$ $\left(\frac{1 - xe^{xy}}{ye^{xy} - 1}\right) = 0$

$\Rightarrow$ $1 - xe^{xy} = 0$

when $y = 0$, then $x = 1$.

So, the point is (1, 0)

8. Given curve is $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$

$$y = e^{y\log(1+x)} + \sin^{-1}(\sin^2 x)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{y\log(1+x)}\left(\frac{y}{1+x} + \log(1+x)\frac{dy}{dx}\right)$$

$$+ \frac{\sin 2x}{\sqrt{1-\sin^4 x}}$$

when $x = 0$, then $y = 1$.

So, the point is (0, 1)

$$\left(\frac{dy}{dx}\right)_{(0,1)} = 1$$

So, slope of the normal = –1

Hence, the equation of the normal is

$$y - 1 = -1(x - 0) = -x$$

$$\Rightarrow \quad x + y = 1$$

9. Let $P_1(x_1, y_1)$ be a point on a curve $y = x^3$

Now, $\frac{dy}{dx} = 3x^2$

Slope of the tangent at $P_1 = m_1 = 3x_1^2$

Equation of the tangent at P_1 is

$$y - y_1 = 3x_1^2(x - x_1)$$

$$\Rightarrow \quad x^3 - x_1^3 = 3x_1^2(x - x_1)$$

$$\Rightarrow \quad x^3 - 3x_1^2 x - 2x_1^3 = 0$$

$$\Rightarrow \quad (x - x_1)(x^2 + xx_1 - 2x_1^2) = 0$$

$$\Rightarrow \quad (x - x_1)(x - x_1)(x + 2x_1) = 0$$

$$\Rightarrow \quad x = x_1, -2x_1$$

$$\Rightarrow \quad x = -2x_1$$

Thus, $x_2 = -2x_1$, $y_2 = x_2^3 = -8x_1^3$

Therefore, $P_2 = (x_2, y_2) = (-2x_1, -8x_1^3)$

Now, we find P_3.

Equation of tangent at P_2 is

$$\Rightarrow \quad y - y_2 = 3x_2^2(x - x_2)$$

$$\Rightarrow \quad x^3 - x_2^3 = 3x_2^2(x - x_2)$$

$$\Rightarrow \quad (x - x_2)(x - x_2)(x + 2x_2) = 0$$

$$\Rightarrow \quad x = -2x_2$$

Thus, $x_3 = -2x_2 = -2.\ -2x_1 = 4x_1$

and $y_3 = x_2^3 = (4x_1)^3 = 64x_1^3$

Therefore, $P_3 = (x_3, y_3) = (4x_1, 64x_1^3)$

The abscissae of $P_1, P_2, P_3, \ldots$ are

$$x_1, -2x_1, 4x_1, -8x_1$$

Which are in G.P with a common ratio -2.

Now, $ar(\Delta P_1 P_2 P_3)$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & x_1^3 & 1 \\ -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \end{vmatrix}$$

$$= \frac{x_1^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

Similarly, $ar(\Delta P_2 P_3 P_4)$

$$= \frac{1}{2}\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \\ -8x_1 & -512x_1^3 & 1 \end{vmatrix}$$

$$= \frac{x_1^4}{16}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

Therefore, $\frac{ar(\Delta P_1 P_2 P_3)}{ar(\Delta P_2 P_3 P_4)} = \frac{1}{16}$

10. *Ans.* (d)

Given curve is $y^2 = px^3 + q$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 3px^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3px^2}{2y}$$

Slope of the tangent

$$= m = \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3p.4}{2.3} = 2p$$

Equation of the tangent at (2, 3) is

$$y - 3 = 2p(x - 2)$$

$$\Rightarrow \quad y - 3 = 2px - 4p$$

$$\Rightarrow \quad y = 2px + (3 - 4p)$$

Given tangent is $y = 4x - 5$

Comparing the co-efficients of x and the constant term, we get, $p = 2$

Also, the point (2, 3) lies on the curve, so

$$9 = 16 + q \quad \Rightarrow \quad q = -7$$

Hence, the values of p and q are

$$p = 2 \text{ and } q = -7.$$

11. Given curve is $y = ax^3 + bx^2 + cx + 5$ ...(i)

Since the curve touches the x-axis at $(-2, 0)$

So equation of tangent is $y = 0$

$\Rightarrow \quad \frac{dy}{dx} = 3ax^2 + 2bx + c$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(-2,0)} = 0$

$12a - 4b + c = 0$ ...(ii)

Also, the curve intersects the y-axis at $(0, 5)$

So, $\left(\frac{dy}{dx}\right)_{(0,5)} = 3$

$\Rightarrow \quad 3a \cdot 0 + 2b \cdot 0 + c = 3$

$\Rightarrow \quad c = 3$ ...(iii)

Again the curve (i) passes through the point

$(-2, 0)$, so $-8a + 4b - 2c + 5 = 0$

$\Rightarrow \quad -8a + 4b - 6 + 5 = 0$

$\Rightarrow \quad -8a + 4b = 1$ ...(iv)

From (ii), we get, $12a - 4b = -3$ ...(v)

Solving (iv) and (v), we get,

$$a = -1/2,\ b = -3/4$$

Hence, the values are

$$a = -1/2,\ b = -3/4 \text{ and } c = 3.$$

12. Given curve is $y^3 - 3xy + 2 = 0$

$\Rightarrow \quad 3y^2\frac{dy}{dx} - 3\left(x\frac{dy}{dx} + y\right) = 0$

$\Rightarrow \quad y^2\frac{dy}{dx} - \left(x\frac{dy}{dx} + y\right) = 0$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{(y^2 - x)}$

Since the tangent is horizontal, so $\frac{dy}{dx} = 0$

$\Rightarrow \quad y = 0$

which is not possible, since the point $y = 0$ does not lie on the curve.

Thus, $H = \varphi =$ null set

When the tangent is vertical, so $\frac{dx}{dy} = 0$

$\Rightarrow \quad \frac{y^2 - x}{y} = 0$

$\Rightarrow \quad x = y^2$

when $x = y^2$, then $y^3 - 3y^3 + 2 = 0$

$\Rightarrow \quad 2y^3 = 2$

$\Rightarrow \quad y^3 = 1$

$\Rightarrow \quad y = 1$

when $y = 1$, then $x = 1$

Hence, $V = \{(1, 1)\}$

13. *Ans.* (d)

Slope the tangent to the curve

$$= \left(\frac{dy}{dx}\right)_{(3,4)} = f'(3)$$

Slope of the normal $= -\frac{1}{f'(3)}$

Thus, $-\frac{1}{f'(3)} = \tan\left(\frac{3\pi}{4}\right) = -1$

$\Rightarrow \quad f'(3) = 1$

14. *Ans.* (c)

Given curve is $f(x) = x^2 + bx - b$

$\Rightarrow \quad f'(x) = 2x + b$

Slope of the tangent to the curve at $(1, 1)$

$$= m = 2 + b.$$

Equation of the tangent to the curve at $(1, 1)$ is

$$y - 1 = (b + 2)(x - 1) \quad \text{...(i)}$$

Let the tangent (i) intersect the x-axis at A and y-axis at B respectively.

Put $x = 0$ in (i), we get, $y - 1 = -(b + 2)$

$y = -(1 + b)$

Thus, $B = (0, -(1 + b))$

Also, put $y = 0$ in (i), we get,

$$x = 1 - \frac{1}{b+2} = \frac{b+1}{b+2}$$

Thus, $A = \left(\frac{b+1}{b+2}, 0\right)$

Again, it is given that, the area of $\Delta = 2$

$\Rightarrow \quad \frac{1}{2} \times OA \times PB = \pm 2$

$\Rightarrow \quad \frac{1}{2} \times \left(\frac{b+1}{b+2}\right) \times (1 + b) = \pm 2$

$\Rightarrow \quad \left(\frac{(b+1)^2}{b+2}\right) = \pm 4$

$\Rightarrow \quad b^2 + 2b + 1 = 4b + 8, -4b - 8$

$\Rightarrow \quad b^2 - 2b - 7 = 0,\ b^2 + 6b + 9 = 0$

$\Rightarrow \quad b^2 + 6b + 9 = 0$

$\Rightarrow \quad (b + 3)^2 = 0$

$\Rightarrow \quad b = -3.$

Hence, the value of b is –3.

15. *Ans.* (a, d)

Given curve is $y^3 + 3x^2 = 12y$

$\Rightarrow \quad 3y^2\dfrac{dy}{dx} + 6x = 12\dfrac{dy}{dx}$

$\Rightarrow \quad y^2\dfrac{dy}{dx} + 2x = 4\dfrac{dy}{dx}$

$\Rightarrow \quad \dfrac{dy}{dx} = \left(\dfrac{2x}{y^2 - 4}\right)$

Since, the tangent is vertical, so $\dfrac{dy}{dx} = 0$

$\Rightarrow \quad \left(\dfrac{y^2 - 4}{2x}\right) = 0$

$\Rightarrow \quad y = \pm 2$

when $y = \pm 2$, then

$\Rightarrow \quad 3x^2 = 12y - y^3 = (24 - 8)$

$\Rightarrow \quad x^2 = \dfrac{16}{3}$

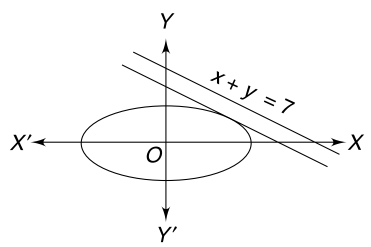
$\Rightarrow \quad x = \pm \dfrac{4}{\sqrt{3}}$

Thus, the points are

$$\left(\pm\frac{4}{\sqrt{3}}, 2\right) \text{ and } \left(\pm\frac{4}{\sqrt{3}}, -2\right)$$

16 Given curve is $x^2 + 2y^2 = 6$

$$\frac{x^2}{6} + \frac{y^2}{3} = 1$$



Since the tangent is parallel to the line, so

$$\frac{dy}{dx} = -1$$

$\Rightarrow \quad -\dfrac{x}{2y} = -1$

$\Rightarrow \quad x = 2y$

when $\quad x = 2y$, then $4y^2 + 2y^2 = 6$

$\Rightarrow \quad 6y^2 = 6$

$\Rightarrow \quad y^2 = 1$

$\Rightarrow \quad y = \pm 1$

when $y = \pm 1$, then $x = \pm 2$

So, the point on the curve is (2, 1) or (–2, –1).

17. *Ans.* (d)

Given parabola is $y = x^2 + 6$

$\Rightarrow \quad \dfrac{dy}{dx} = 2x$

Slope of the tangent = m = 2 at (1, 7)

Equation of the tangent at $P(1, 7)$ is

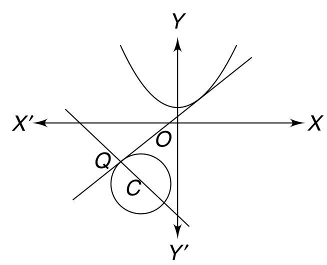
$$y - 7 = 2(x - 1)$$

$\Rightarrow \quad 2x - y + 5 = 0 \quad$ ...(i)

Given circle is $x^2 + y^2 + 16x + 12y + c = 0$

$$(x + 8)^2 + (y + 6)^2 = r^2$$

where $\quad r = \sqrt{100 - c}$



Here, CQ is perpendicular to the tangent.

Thus, slope of $CQ = -1/2$

Equation of CQ is

$$y + 6 = -\frac{1}{2}(x + 8)$$

$\Rightarrow \quad x + 2y = -20 \quad$ ...(ii)

Solving (i) and (ii), we get,

$$x = -6,\ y = -7$$

Hence, the point Q is (–6, –7).

18. Given condition is $|f(x) - f(y)| \le (x - y)^2$

$\Rightarrow \quad \left|\dfrac{f(x) - f(y)}{x - y}\right| \le (x - y)$

$\Rightarrow \quad \lim\limits_{y\to x}\left|\dfrac{f(x) - f(y)}{x - y}\right| \le \lim\limits_{y\to x}(x - y)$

$\Rightarrow \quad \lim\limits_{y\to x}\left|\dfrac{f(y) - f(x)}{y - x}\right| \le \lim\limits_{y\to x}(x - y)$

$\Rightarrow \quad f'(x) \le 0$

$\Rightarrow \quad f'(x) = 0$

$\Rightarrow \quad f(x) = \text{costant} = c$

Thus, $y = f(x) = c$

which is passing through (1, 2).

Thus, $c = 2$

Hence, the equation of the tangent to the curve at (1, 2) is $y = 2$.

19. No questions asked in 2006.

20. *Ans.* (a)

Given curve is $y = e^x$

$$\Rightarrow \quad \frac{dy}{dx} = e^x$$

Equation of the tangent to the curve at (c, e^c) is

$$y - e^c = e^c(x - c) \quad \text{...(i)}$$

Equation of the line joining

$$(c - 1.\ e^{c-1}),\ (c + 1,\ e^{c+1}) \text{ is}$$

$$y - e^{c+1} = \frac{1}{2}(e^{c+1} - e^{c-1})(x - c - 1) \quad \text{...(ii)}$$

(ii)-(i), we get,

$$e^c(e - 1) = e^c\left[(x - c)\left\{1 - \frac{1}{2}(e - e^{-1})\right\} + \frac{1}{2}(e + e^{-1})\right]$$

$$\Rightarrow (e - 1) = \left[(x - c)\left\{1 - \frac{1}{2}(e - e^{-1})\right\} + \frac{1}{2}(e + e^{-1})\right]$$

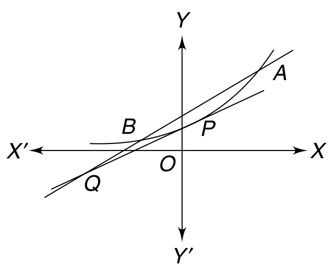
$$\Rightarrow \frac{1}{2}(e + e^{-1} - 2) = -\frac{1}{2}(x - c)(e - e^{-1} - 2)$$

$$\Rightarrow (c - x) = \frac{(e + e^{-1} - 2)}{(e - e^{-1} - 2)}$$

$$\Rightarrow x = c - \frac{(e + e^{-1} - 2)}{(e - e^{-1} - 2)}$$

$$\Rightarrow x < c$$

Thus, two lines meet on the left of $x = c$.



22. *Ans.* (a, b)

Given curve is $\frac{x^2}{9} - \frac{y^2}{4} = 1$

$$\Rightarrow \quad \frac{2x}{9} = \frac{2y}{4}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4x}{9y}$$

Since, the tangent is parallel to the straight line

$$2x - y = 1, \text{ so } \frac{4x}{9y} = 2$$

$$\Rightarrow \quad x = \frac{9y}{2}$$

$$\text{when} \quad x = \frac{9y}{2}, \text{ then } \frac{\left(\frac{9y}{2}\right)^2}{9} - \frac{y^2}{4} = 1$$

$$\Rightarrow \quad \frac{9y^2}{4} - \frac{y^2}{4} = 1$$

$$\Rightarrow \quad \frac{8y^2}{4} = 1$$

$$\Rightarrow \quad y^2 = \frac{1}{2}$$

$$\Rightarrow \quad y = \pm\frac{1}{\sqrt{2}}$$

$$\text{when} \quad y = \pm\frac{1}{\sqrt{2}}, \text{ then } x = \pm\frac{9}{2\sqrt{2}}$$

Hence, the points of contact are

$$\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \ \& \ \left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

24. Given curve is $(y - x^5)^2 = x(1 + x^2)^2$

$$\Rightarrow 2(y - x^5)\left(\frac{d}{dx} - 5x^4\right) = (1 + x^2)^2 + 4x(1 + x^2)$$

$$\Rightarrow 2(y - x^5)\left(\frac{dy}{dx} - 5x^4\right) = (1 + x^2)(1 + 5x^2)$$

when $x = 1$, $y = 3$, then

$$2(3 - 1)\left(\frac{dy}{dx} - 5\right) = (1 + 1)(1 + 5) = 12$$

$$\Rightarrow \quad \left(\frac{dy}{dx} - 5\right) = 3$$

$$\Rightarrow \quad \frac{dy}{dx} = 5 + 3 = 8.$$

Hence, the slope of the tangent = 8.

CHAPTER

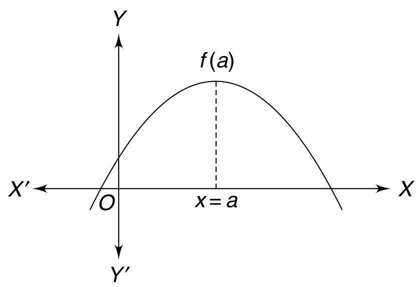
9

The Maxima and Minima

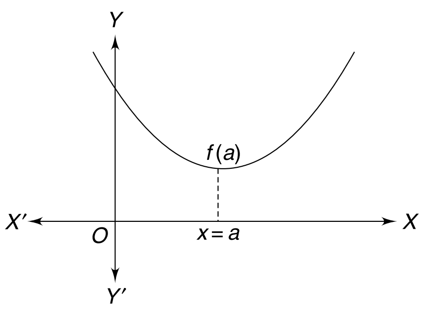
CONCEPT BOOSTER

9.1 Definitions

1.



Let $y = f(x)$ be a real function defined at $x = a$. Then the function $f(x)$ is said to have a maximum value at $x = a$, if $f(x) \le f(a)\ \forall\ a \in R$

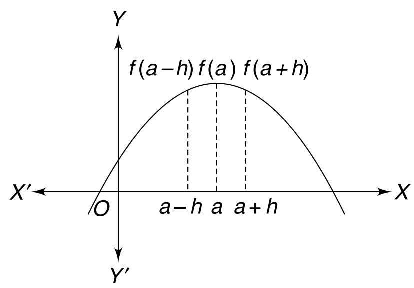


And also the function $f(x)$ is said to have a minimum value at $x = a$, if $f(x) \ge f(a)\ \forall\ a \in R$.

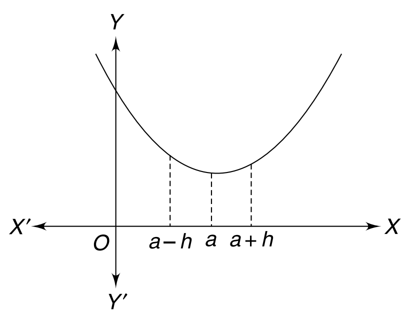
> ***Note:*** If the range of a function is R, then the function has no extremum.
>
> Let $f(x) = x^3 + 1$
>
> Clearly, $R_f = R$.
>
> So, the function has no extrema.

9.2 Concept of Local Maxima and Local Minima

(i) A Function $f(x)$ is said to have a local maximum at $x = a$, if it is maximum at the neighbourhood of $x = a$. that is, $f(a - h) < f(a) > f(a + h)$., where h is very very small arbitrary positive number.

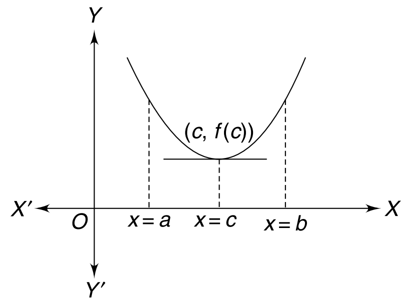


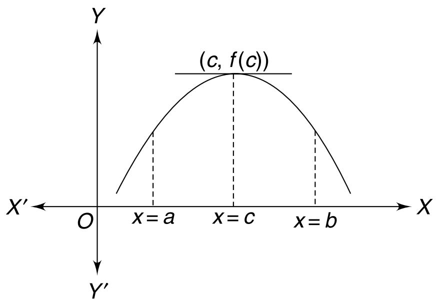
(ii) A function $f(x)$ is said to have a local minimum at $x = a$, if it is minimum at the neighbourhood of $x = a$. that is, $f(a - h) > f(a)$ and also $f(a + h) > f(a)$



(iii) **Fermat Theorem**

Let $f(x)$ be a real function and a, b be two real numbers such that $a < b$. If $c \in (a, b)$ such that $f'(a)$ exists and $(c, f(c))$ is a point of Local extremum (either a max or a min), then $f'(c) = 0$

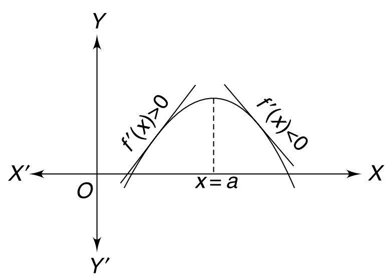




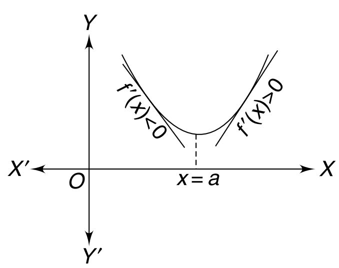
9.3 The First Order derivative Test

Let $y = f(x)$ be a differentiable function and $x = a$ be a critical point of $y = f(x)$.

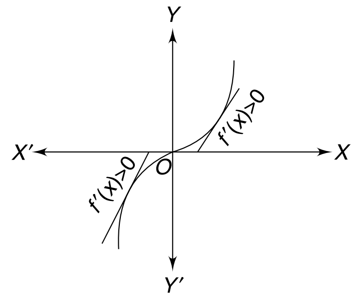
(i) If $f'(x)$ changes from positive to negitive at $x = a$, then f has a Local Maximum at $x = a$

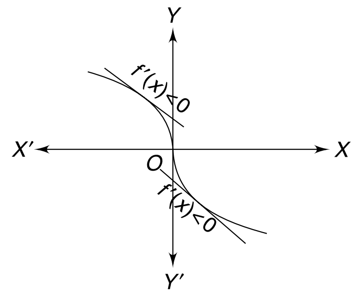


(ii) If $f'(x)$ changes from negitive to positive at $x = a$, then f has a Local Minimum at $x = a$.



(iii) If $f'(x)$ does not change any sign at $x = a$, then f has neither Local maxima nor Local minimum at $x = a$. That time, we shall say that $x = a$ is the point of inflection.





(iv) **Critical points**

Critical points are interior points in the domain of the function $f(x)$ where $f'(x) = 0$ or $f'(x)$ does not exist (but $f(x)$ is defined)

Example 1. Let $y = f(x) = \dfrac{x^2}{x-2}$

Then $\quad \dfrac{dy}{dx} = \dfrac{(x-2).2x - x^2.1}{(x-2)^2}$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x(x-4)}{(x-2)^2}$

Now, $\quad \dfrac{dy}{dx} = 0$ gives $x = 0$ and 4

At $x = 2$, $\dfrac{dy}{dx}$ is undefined and y is also undefined

Thus, the critical points are $x = 0$ and $x = 4$.

Example 2. Let $y = f(x) = |x^2 - 2x|$

Then $$y = f(x) = \begin{cases} (x^2 - 2x) & : x \le 0, x \ge 2 \\ -(x^2 - 2x) & : 0 < x < 2 \end{cases}$$

$$\frac{dy}{dx} = f'(x) = \begin{cases} (2x - 2) & : x \le 0, x \ge 2 \\ -(2x - 2) & : 0 < x < 2 \end{cases}$$

Now, $\quad f'(x) = 0$ gives $x = 1$

and $f'(x)$ does not exist at $x = 0$ and 2.

Thus, the critial points are $x = 0, 1, 2$.

9.4 Test for Local maxima/Local Minima, when $f(x)$ is not Differentiable at $x = a$

(i) When $f(x)$ is continuous at $x = a$ and $f'(a^-)$ and $f'(a^+)$ both are exist and non-zero.

If $f'(a^-) > 0$ and $f'(a^+) < 0$, then $x = a$ is a point of Local Maxima.

If $f'(a^-) < 0$ and $f'(a^+) > 0$, then $x = a$ is a point of Local Minima.

(ii) When $f(x)$ is continuous and one or both of $f'(a^-)$ and $f'(a^+)$ are either zero or does not exist. Then we should check the sign of $f'(a - h)$ and $f'(a + h)$.

If $f'(a - h) > 0$ and $f'(a + h) < 0$, then $x = a$ is a point of Local Maximum.

If $f'(a - h) < 0$ and $f'(a + h) > 0$, then $x = a$ is a point of Local Minimum.

(iii) If $f(x)$ is discontinuous at $x = a$, then we should compare the values of $f(x)$ at the neighbourhood of $x = a$ to check the Local Maxima or Local Minima.

9.5 Extremum at End-Points

Let $f(x)$ be a differentiable function and $a, b \in R$ such that $a < b$.

At a left end point of $x = a$, If $f'(a + h) < 0$ for $x > a$, then $f(x)$ has a Local Maximum at $x = a$.

If $f'(a + h) > 0$ for $x > a$, then $f(x)$ has a Local Minimum at $x = a$.

At a right end at $x = b$, If $f'(b - h) < 0$ for $x < b$, then $f(x)$ has a Local Minimum at $x = b$.

If $f'(b - h) > 0$ for $x > b$, then $f(x)$ has a Local Minimum at $x = b$.

9.6 More Idea on Local Maximum and Local Minimum

(i) If $y = f(x)$ has a local maximum at $x = b$, then $y = -f(x)$ has a local minimum at $x = b$.

(ii) If $y = f(x)$ has a local minimum at $x = b$, then $y = -f(x)$ has a local maximum at $x = b$.

(iii) If $f(x)$ and $g(x)$ both have a local maximum at $x = b$, then the function $y = f(x) + g(x)$ also has a local maximum at $x = b$.

(iv) If $f(x)$ and $g(x)$ both have a local minimum at $x = b$, then the function $y = f(x) + g(x)$ also has a local minimum at $x = b$.

(v) Let $f(x)$ and $g(x)$ be two differentiable function such that both have a local maximum at $x = b$. If $f(b) > 0$ and $g(b) > 0$, then the function $y = f(x) . g(x)$ has a local maximum at $x = b$.

(vi) Let $f(x)$ and $g(x)$ be two differentiable function and $b \in R$

(a) If $f(x)$ has a local maximum at $x = b$ and $g(x)$ has a local maximum at $x = f(b)$, then the function $(g_0 f)(x)$ also has a local maximum at $x = b$.

(b) If $f(x)$ has a local minimum at $x = b$ and $g(x)$ has a local minimum at $x = f(b)$, then the function $(g_0 f)(x)$ also has a local minimum at $x = b$ the function $(g_0 f)(x)$ has a local minimum at $x = b$.

(c) If $f(x)$ has a local maximum at $x = b$ and $g(x)$ is increasing at $x = f(b)$, then the function $(g_0 f)(x)$ also has a local maximum at $x = b$.

(d) If $f(x)$ has a local minimum at $x = b$ and $g(x)$ is increasing at $x = f(b)$, then the function $(g_0 f)(x)$ also has a local minimum at $x = b$.

(e) If $f(x)$ has a local maximum at $x = b$ and $g(x)$ is decreasing at $x = f(b)$, then the function $(g_0 f)(x)$ also has a local maximum at $x = b$

(f) If $f(x)$ has a local minimum at $x = b$ and $g(x)$ is decreasing at $x = f(b)$, then the function $(g_0 f)(x)$ has a local maximum at $x = b$.

(vii) (a) An even function has an extremum at $x = 0$. It is noted that the odd function does not have any extremum at $x = 0$.

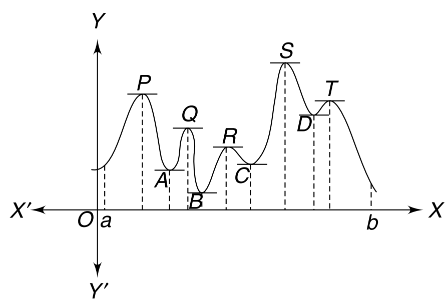
(b) If an even function $f(x)$ has a local maximum at $x = b$, then it also has a local maximum at $x = -b$.

(c) If an even function $f(x)$ has a local minimum at $x = b$, then it also has a local minimum at $x = -b$.

9.7 Global Maximum/Global Minimum

Let $f(x)$ be a real function on $[a, b]$.

First of all, we shall find out all the critical points of $f(x)$ on (a, b).



Let $c_1, c_2, \ldots c_n$ are the critical points of $f(x)$ then we shall find the value of the function for all those critical points.

Let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at critical points.

Let $M_1 = \max\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$ and.

$$M_2 = \min\{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$$

Then M_1 is called the Global Maximum or Absolute Maximum or the Greatest value of the function and M_2 is called the Global Minimum or Absolute Minimum or the Least value of the function.

9.8 Algebra of Global Extrema

(i) If two functions f and g attain their greatest (least) value at $x = a$, then $y = f(x) + g(x)$ also attains its greatest (least) value at $x = a$.

Let us take $y = \cos x + \left(\frac{1}{x^2 + 1}\right)$.

The greatest value of y is 2 at $x = 0$.

Since both $\cos x$ and $\left(\frac{1}{x^2 + 1}\right)$ attain their greatest value at $x = 0$.

(ii) If f and g are non-negative functions, in which, both attains their greatest (least) value at $x = a$, then $y = f(x) \cdot g(x)$ also attains its greatest (least) value at $x = a$.

Let us take, $y = (1 + \sin x)\sin^2 x$ attains its maximum value 2 at $x = \frac{\pi}{2}$.

(iii) If f is non-negative and g is positive, so that f attains its greatest (least) value at $x = a$ and g attains its least (greatest) value at $x = a$, then $y = \frac{f(x)}{g(x)}$ attains its greatest (least) value at $x = a$.

Let us take, $y = \frac{e^{-x} + e^{x}}{(3 + 2\cos \pi x)}$ attains its minimum value at $x = 0$, is $\frac{1}{2}$.

9.9 The Second Order derivative Test

Let $f(x)$ be a differentiable function and $x = b$ be a stationary point of $f(x)$, that is, $f'(b) = 0$.

(i) First we find $f'(x)$, $f''(x)$

(ii) $f'(x) = 0$ gives, say $x = b$.

(iii) If $f''(b) < 0$, then $f(x)$ has a maximum value at $x = b$.

(iv) If $f''(b) > 0$, then $f(x)$ has a minimum value at $x = b$.

(v) If $f''(b) = 0$, then we find $f'''(b)$, $f^{iv}(b)$, ,..., $f^{n}(b)$

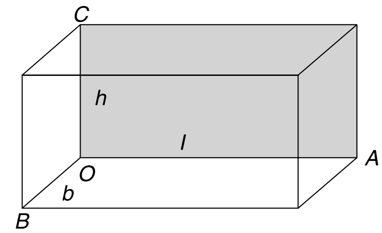
(vi) If the first non-vanishing derivative $f^{n+1}(b)$ is a derivative of even order, then the function $f(x)$ has a Local Maximum at $x = b$, if $f^{n+1}(b) > 0$ and has a Local Minimum at $x = b$, if $f^{n+1}(b) < 0$

But if the first non-vanishing derivative $f^{n+1}(b)$ is a derivative of odd order, then the function $f(x)$ has neither maximum nor minimum, that is, $f(x)$ increases if $f^{n+1}(b) > 0$ and $f(x)$ decreases if $f^{n+1}(b) < 0$.

Important Formulae to Remember for Solid Geometrical Problems

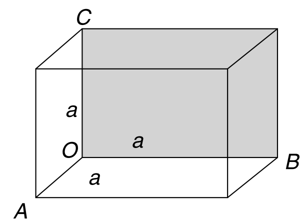
(i) Volume of a cuboid = $l.b.h$

(ii) Surface area of Cuboid = $2(lb + bh + hl)$



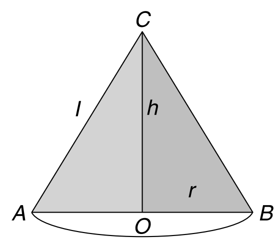
(iii) Volume of a cube = a^3

(iv) Surface area of Cube = $6a^2$



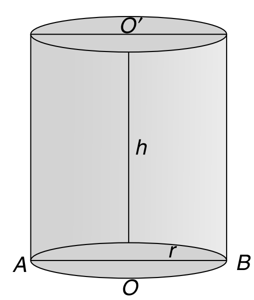
(v) Volume of a cone = $\frac{1}{3} \times \pi r^2 h$

(vi) Curved surface area of a Cone = $\pi r l$ (where l = slant height)



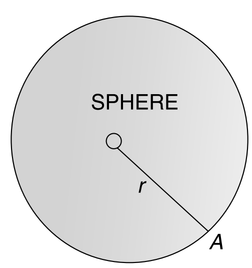
(vii) Curved surface area of a Cylinder = $2\pi r h$

(viii) Total surface area of a cylinder = $2\pi r h + 2\pi r^2$

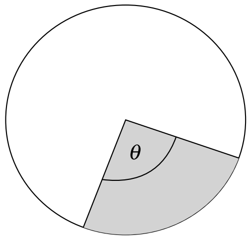


(ix) Volume of a Sphere = $\frac{4}{3}\pi r^3$

(x) Surface area of a sphere = $4\pi r^2$



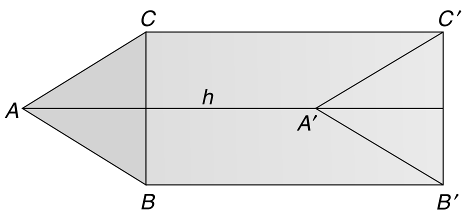
(xi) Area of a circular sector $= \frac{1}{2}r^2\theta.$, where θ is in radians. = (area of the base) × (height)



(xii) Lateral surface of a Prism = (perimeter of the base) × (height)

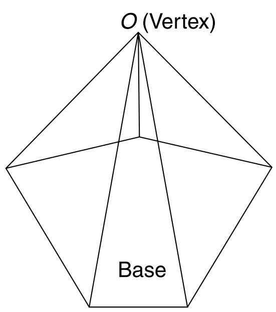
(xiii) Total surface of a Prism = (Lateral Surface) × 2 (area of the base)

(It is note that, lateral surfaces of a prism are all rectangle.)

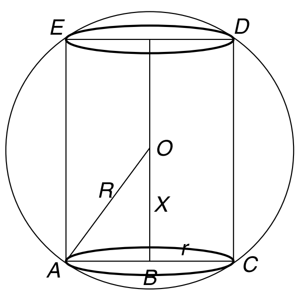


(xiv) Volume of a Pyramid $= \frac{1}{3}$ × (area of the base) × (height)

(xv) Curved Surface of a Pyramid $= \frac{1}{2}$ × (perimeter of the base) × (slant height).



(xvi) **Right circular cylinder in a sphere**



Here, $AB = \sqrt{R^2 - x^2}$

Volume of the cylinder $= V = \pi r^2 h$

$= \pi(R^2 - x^2)\cdot 2x$

$= 2\pi x(R^2 - x^2)$

Curved surface area of the cylinder

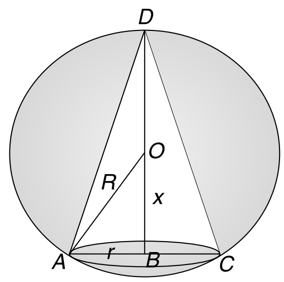
$= S = 2\pi rh = 2\pi r 2x = 4\pi x\sqrt{R^2 - x^2}$

Total Surface area of the cylinder

$= 2\pi rh + 2\pi r^2$

$= 4\pi x\sqrt{R^2 - x^2} + 2\pi(R^2 - x^2)$

(xvii) **Right circular cone in a sphere**



Here, $AB = \sqrt{R^2 - x^2}$

Volume of the cone

$= V = \frac{1}{3}\pi r^2 h$

$= \frac{2\pi x}{3}(R^2 - x^2)$

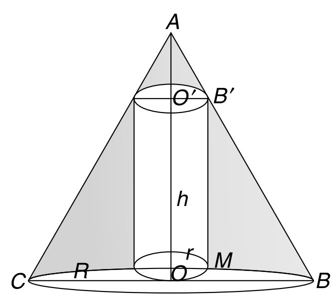
Curved surface area of the cone

$= S = \pi rl$

Total surface area of the cone

$= \pi rl + \pi r^2$

(xviii) **Right circular cylinder in a cone**



Here $OA = H$, $OC = R$, $OM = r$.

Δ's AOB $AO'B'$ are similar.

Thus, $\frac{AO'}{AO} = \frac{O'B'}{OB} = \frac{H-h}{H} = \frac{r}{R}$

$h = H\left(1 - \frac{r}{R}\right)$

Volume of the cylinder

$= V = \pi r^2 h = \pi r^2 \times H\left(1 - \frac{r}{R}\right)$

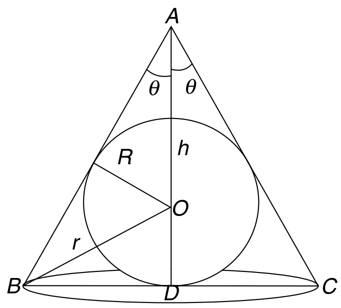
Curved surface area of the cylinder

$= S = 2\pi r h = \pi r \times H\left(1 - \frac{r}{R}\right)$

Total surface area of the cylinder

$= 2\pi r h + 2\pi r^2 = \pi r \times H\left(1 - \frac{r}{R}\right) + 2\pi r^2$

(xix) **Cone around sphere**



Here, $h = OD + OA = R + R \operatorname{cosec}\theta$

and $r = h \tan\theta = R(1 + \operatorname{cosec}\theta)\tan\theta$

Volume of the cone = V

$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi R^3(1 + \operatorname{cosec}\theta)^3 \tan^2\theta$

Curved surface area of the cone

$= S = \pi r l = \pi r \cdot r \operatorname{cosec}\theta$

$= \pi R^2(1 + \operatorname{cosec}\theta)^2 \operatorname{cosec}\theta$

Total surface area of the cone

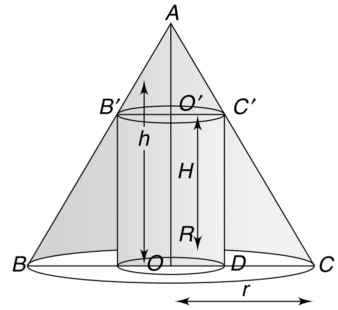
$= \pi r l + \pi r^2$

$= \pi R^2(1 + \operatorname{cosec}\theta)^2 \operatorname{cosec}\theta$

$+ \pi R^2(1 + \operatorname{cosec}\theta)^2$

$= \pi R^2(1 + \operatorname{cosec}\theta)^3.$

(xx) **Cone around cylinder**



$\Delta's\ AOC, AO'C'$ are similar

Thus, $\frac{AO'}{AO} = \frac{O'C'}{OC}$

$\Rightarrow \quad \frac{h - H}{h} = \frac{R}{r}$

$\Rightarrow \quad h = \frac{rH}{r - R}$

Volume of the cone

$= V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\frac{\pi r^2 H}{(r - R)}$

Curved surface area of the cone

$= S = \pi r l = \pi r\sqrt{r^2 + h^2}$

$= \pi r\sqrt{r^2 + \frac{r^2 H^2}{(r - R)^2}}$

$= \frac{\pi r^2}{(r - R)}\sqrt{(r - R)^2 + H^2}$

EXERCISES

Level I

(Problems Based on Fundamentals)

1. Find the maximum and the minimum values of each of the following functions (without using derivative)

(i) $f(x) = x^2 - 4x + 10$

(ii) $f(x) = -x^2 + 6x - 15$

(iii) $f(x) = (x - 3)^2 + 5$

(iv) $f(x) = -(x - 1)^2 + 2$

(v) $f(x) = |x + 2|$

(vi) $f(x) = -|x - 1| + 5$

(vii) $f(x) = |x - 1| + |x - 3|$

(viii) $f(x) = |x + 2| + |x - 3|$

(ix) $f(x) = x^3 - 1$

(x) $f(x) = 3\sin x + 4\cos x + 10$

(xi) $f(x) = \sin x + \cos x$

(xii) $f(x) = \sin x - \cos x$

(xiii) $f(x) = \sin(2x) + 5$

(xiv) $f(x)\ 3 - 2\sin x$

(xv) $f(x) = 2\cos x + 3$

(xvi) $f(x) = 3 - 4\cos x$

(xvii) $f(x) = 2\sin^2 x + 4$

(xviii) $f(x) = 5 - 3\sin^2 x$

(xix) $f(x) = 3\sin^2 x + 2\cos^2 x$

(xx) $f(x) = 4\sin^2 x + 5\cos^2 x + 6\sin x\cos x + 10$

2. Find the maximum and the minimum values of each of the following functions without using derivative

(i) $f(x) = \sin(\sin x)$

(ii) $f(x) = \cos(\cos x)$

(iii) $f(x) = \sin(\sin x) + \cos(\sin x)$
(iv) $f(x) = \sin^2 x + \cos^4 x$
(v) $f(x) = \sin^4 x + \cos^2 x$
(vi) $f(x) = \sin^4 x + \cos^4 x$
(vii) $f(x) = \sin^6 x + \cos^6 x$
(viii) $f(x) = \sin^2(\sin x) + \cos^2(\cos x)$
(ix) $f(x) = |\sin x| + |\cos x|$
(x) $f(x) = \sin x + \operatorname{cosec} x$ for all x in $\left(0, \frac{\pi}{2}\right)$

3. Find the min value of $f(x) = |x - a| + |x - b|$ where $0 < a < b$.
4. Find the min value of $f(x) = |x - a| + |x - b| + |x - c|$ where $0 < a < b < c$.
5. Let $f(x) = |x - a| + |x - b| + |x - c| + |x - d|$ where $a < b < c < d$, find the minimum value of $f(x)$.

Local Max or Local Min

6. Find the points of extremum of the function $f(x) = (x - 1)^2(x - 2)^3$
7. Find the points of extremum of the function $f(x) = (x - 2)x^{2/3}$
8. Find the points of extremum of the function $f(x) = x^3 - 6x^2 + 12x - 8$.
9. Find the points of extremum of the function $f(x)$ for which $f'(x) = (x - 1)(x - 2)^2(x - 3)^3(x - 4)^4$
10. Find the points of extremum of the function $f(x) = \sqrt{2x^2 - x + 2}$.
11. Find the points of extremum of the function $f(x) = x - \ln(1 + x^2)$.
12. Discuss the extremum of
$$f(x) = \begin{cases} 1 + \sin x & : x < 0 \\ x^2 - x + 1 & : x \geq 0 \end{cases}$$
13. Find the points of extremum of the function $f(x) = 1 + (x - 1)^{2/3}$
14. Find the points of extremum of the function $f(x) = 1 + (x - 2)^{4/5}$
15. Find the points of extremum of the function
$$f(x) = \begin{cases} x^2 & : x \leq 0 \\ 2\sin x & : x > 0 \end{cases}$$
16. Find the points of extremum of the function $f(x) = |x| + |x^2 - 1|$.
17. Find the points of extremum of the function $f(x) = 2\log(x - 2) - x^2 + 4x + 1$.

Max or Min at the end points

18. Find the absolute max and min of $f(x) = x^3 - 3x^2 + 1$, for all x in $\left[-\frac{1}{2}, 4\right]$
19. Find the absolute max or min of $f(x) = x^{2/3}(5 - 2x)$ in $[-1, 2]$
20. Find the absolute max or min of $f(x) = x + 2\sqrt{x}$, $\forall x \in [0, 4]$
21. Find the absolute max or min of $f(x) = x^2 \ln x$, $\forall x \in [1, e]$
22. Find the absolute max or min of $f(x) = x + \sin 2x$, $\forall x \in [0, 2\pi]$
23. Find the absolute max or min of $f(x) = \sin x + \frac{1}{2}\cos 2x$, $\forall x \in \left[0, \frac{\pi}{2}\right]$
24. Find the max or min values of $f(x) = \tan^{-1} x - \frac{1}{2}\ln x$, $\forall x \in \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$
25. Find the max or min values of $f(x) = \{(1 - x^2)(2x^2 + 1)\}^{1/2}$, $\forall x \in [-1, 1]$
26. Find the greatest and least values of $f(x) = \dfrac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$ in $\left[0, \frac{\pi}{2}\right]$
27. Find the greatest and least values of $f(x) = \cos x + \cos(\sqrt{2}x)$
28. Find the greatest and least value of the function $f(x) = \dfrac{1}{\sin x + 4} + \dfrac{1}{\cos x - 4}$ for all x in R.
29. Find the global max or global min of $f(x) = \dfrac{14}{x^4 - 8x^2 + 2}$ for all x in R.
30. Find the global max or global min of $f(x) = (x - 1)^2\sqrt{x^2 - 2x + 3}$ in $[0, 3]$
31. Find the max and min values of the function $y = \dfrac{x^2 - 7x + 6}{x - 10}$
32. Find the max or min values of $f(x) = 2\sin x + \cos 2x$
33. Find the max or min values of $f(x) = 3x^4 - 2x^3 - 3x^2 + 10$
34. Find the max or min values of $f(x) = \ln(x^4 - 2x^2 + 3)$
35. Find the max or min values of $f(x) = \sqrt{3x^2 - 2x^3}$

36. Find the max or min values of
$$f(x) = \frac{10}{3x^4 + 4x^3 - 12x^2 + 11}$$
37. Find the max or min values of
$$f(x) = x^5 + 5x^4 + 5x^3 - 1$$

Mensuration Problems

38. Show that of all the rectangles with a given perimeter, the square has the largest area.
39. Show that of all the rectangles inscribed in a given circle, the square has the maximum area.
40. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is max when the angle between them is $\frac{\pi}{3}$.
41. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.
42. Show that the semi vertical angle of a cone of max volume and of given slant height is $\tan^{-1}(\sqrt{2})$.
43. Show that the semi vertical angle of a right circular cone of given surface area and max volume is $\sin^{-1}\left(\frac{1}{3}\right)$.
44. Show that the height of the cylinder of max volume that can be inscribed in a sphere of radius a is $\frac{2a}{\sqrt{3}}$.
45. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.
46. Show that the volume of the largest cylinder which can be inscribed in a cone of height h and semi vertical angle α is $\frac{4}{27} \times \pi h^3 \tan^2\alpha$.

Co-ordinate Geometrical Problems

47. Find the point on the curve $y^2 = 4x$ which is closest to the point (2, 1).
48. A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). Find the shortest distance between the soldier and the jet.
49. Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $0 \le c \le 5$.
50. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point (0, –2).
51. What normal to the curve $y = x^2$ forms the shortest chord?
52. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is minimum.
53. Find the shortest distance between the curves $x^2 + y^2 = 2$ and $xy = 9$.
54. Find the shortest distance between the curves $y^2 = 4x$ and $x^2 + (y + 12)^2 = 1$

Miscellaneous Problems

55. Find the least value of
$$f(x) = ax + \frac{b}{x}, \ a, b, x > 0$$
56. Find the least value of
$$f(x) = x^2 + \frac{1}{x^2 + 1}$$
57. Find the least value of
$$f(x) = \frac{x^3 + x + 2}{x}, \ x > 0$$
58. Find the least value of
$$f(x) = 2\cos x + \sec^2 x, \ x \in \left[0, \frac{\pi}{2}\right)$$
59. Find the least value of
$$f(x) = 2\log_{10}x - \log_x(0.01), \ x > 1$$
60. Find the min values of
$$f(x) = \frac{a^2}{\sin^2 x} + \frac{b^2}{\cos^2 x}$$
61. Find the min values of
$$f(x) = 2^x + 3^x + 5^x + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{5^x}, \ x > 0$$
62. Find the Min values of
$$f(a) = a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10}, \ a > 0$$
63. Find the min values of
$$f(x) = x^{10} + x^7 + \frac{2}{x^3} + \frac{4}{x^2} + \frac{3}{x}, \ x > 0$$
64. Find the min value of
$$f(a, b, c, d) = \frac{(a^2 + 1)(b^2 + 1)(c^2 + 1)(d^2 + 1)}{abcd}$$
where $a, b, c, d > 0$.
65. Find the max value of x^2y^3, where $2x + 3y = 5$.
66. Find the max value of x^3y^2z, where $3x + 2y + z = 14$.
67. Find the max value of x y z, if
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
68. Find the max value of
$$y = \frac{x}{ax^2 + b}, \ a, b, x > 0$$
69. Find the least value of
$$f(x) = x^2 + 1 + \frac{4}{x^2 + 3}$$
70. Find the least value of
$$f(x, y, z) = \frac{(x^3 + 2)(x^3 + 2)(x^3 + 2)}{xyz}$$
where $x, y, z > 0$

Level II

(Mixed Problems)

1. If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has local max. and min. at $x = p$ and $x = q$ resp., then (p, q) is...
   (a) (0, 1) (b) (1, 3)
   (c) (1, 0) (d) None

2. The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at $x =$
   (a) –2 (b) 0
   (c) 1 (d) 2

3. The real number x when added to its inverse gives the min. value of the sum at x equal to
   (a) –2 (b) –1
   (c) 1 (d) 2

4. The max. or min. value of $\frac{x^2 + x + 1}{x^2 - x + 1}$ are
   (a) (2, 1) (b) $\left(3, \frac{1}{3}\right)$
   (c) (1, 0) (d) (3, 1)

5. The max. value of $\frac{\log x}{x}$ is
   (a) 1 (b) $\frac{2}{e}$
   (c) e (d) $\frac{1}{e}$

6. x^x has a stationary point at $x =$
   (a) e (b) $\frac{1}{e}$
   (c) 1 (d) $\sqrt{e}$

7. Greatest value of $\left(\frac{1}{x}\right)^x$ is
   (a) e (b) $e^{1/e}$
   (c) $\left(\frac{1}{e}\right)^e$ (d) None

8. The max. value of $\left(\frac{1}{x}\right)^{2x^2}$ is
   (a) 1 (b) e
   (c) $e^{1/e}$ (d) None

9. The min. value of $x^{1/x}$ at $x = e (x > 0)$ is
   (a) $e^{\pi} > \pi^e$ (b) $e^{\pi} < \pi^e$
   (c) $e^{\pi} = \pi^e$ (d) $e^{\pi} \le \pi^e$

10. The min. value of $ax + by$ when $xy = r^2$ is
   (a) $2r\sqrt{ab}$ (c) $2ab\sqrt{r}$
   (c) $-2r\sqrt{ab}$ (d) None

11. On the interval [0, 1], the function $f(x) = x^{25} (1 - x)^{75}$ touches the max. value or the point
   (a) $\frac{1}{2}$ (c) $\frac{1}{3}$
   (c) $\frac{1}{4}$ (d) 0

12. If $y = a \log|x| + bx^2 + x$ has its extreme values of $x = -1$ and $x = 2$, then
   (a) $a = 2, b = -1$ (b) $a = 2, b = -\frac{1}{2}$
   (c) $a = -2, b = \frac{1}{2}$ (d) None

13. If $f(x) = 2x^3 - 3x^2 - 12x + 5$ on [–2, 4], then relative maximum occurs at $x =$
   (a) –2 (b) –1
   (c) 2 (d) 4

14. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$, where $a > 0$, attains its max. and min. at p and q respectively such that $p^2 = q$ then a is
   (a) $\frac{1}{2}$ (b) 1
   (c) 2 (d) 3

15. The function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1 = 0$ has a local max. at $x = \alpha$ and a local minimum at $x = \beta$ such that $\alpha^2 = \beta$ then a is
   (a) 0 (b) 1/4
   (c) 2 (d) either 0 or 2

16. If min. value of $f(x) = x^2 + 2bx + 2c^2$ is greater than max. value of $g(x) = -x^2 - 2cx + b^2$ then for x is real
   (a) $|e| > |b|\sqrt{2}$ (b) $|c| > \frac{b}{\sqrt{2}}$
   (c) $0 < c < b\sqrt{2}$ (d) None

17. The points on the curve $5x^2 - xy + 5y^2 = 4$ where distance from the origin is max. or min. are
   (a) $(\sqrt{2}, \sqrt{2})$ (b) $(-\sqrt{2}, -\sqrt{2})$
   (c) $\left(\frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}\right)$ (d) $\left(-\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}\right)$

18. The longest distance of the point $(a, 0)$ from the curve $2x^2 + y^2 - 2x = 0$ is given by
   (a) $\sqrt{1 + 2a + 2a^2}$ (b) $\sqrt{1 + 2a - a^2}$
   (c) $\sqrt{1 - 2a + 2a^2}$ (d) $\sqrt{1 - 2a + a^2}$

19. The point on the curve $y = \frac{x}{1 + x^2}$ where the tangent to the curve has greatest slope is
   (a) (0, 0) (b) $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$
   (c) $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$ (d) None

20. The point (0, 5) is closest to the curve $x^2 = 2y$ is
   (a) $(2\sqrt{2}, 0)$ (b) (0, 0)
   (c) (2, 2) (d) None

21. Let $f(x) = \frac{9}{\cos^2 x} + \frac{4}{\sin^2 x}$, $x \in \left(0, \frac{\pi}{2}\right)$. Then the min value of $f(x)$ is
(a) 25 (b) 36
(c) 20 (d) 10

22. The co-ordinates of a point on the parabola $y^2 = 8x$ where distance from the circle $x^2 + (y + 6)^2 = 1$ is minimum is
(a) (2, 4) (b) (2, −4)
(c) (18, −12) (d) (8, 8)

23. If a function $f(x)$ has $f'(a) = 0$ and $f''(a) = 0$ then
(a) $x = a$ is a max. for $f(x)$
(b) $x = a$ is a min. for $f(x)$
(c) It is difficult to say (a) and (b)
(d) $f(x)$ is necessarily a const. function

24. The largest value of $f(x) = 2x^3 - 3x^2 - 12x + 5$ for $[-2, 4]$ occurs at $x =$
(a) −2 (b) −1
(c) 2 (d) 4

25. The difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is
(a) π (b) $\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)$
(c) $\left(\frac{\sqrt{3} + \sqrt{2}}{2} + \frac{\pi}{6}\right)$ (d) $\left(\frac{\sqrt{2} + \sqrt{3}}{2}\right)$

26. The difference between the greatest and the least values of the function
$f(x) = \int_0^x (at^2 + 1 + \cos t)\,dt$, $a > 0$ $x \in [2, 3]$ is
(a) $\frac{19}{3}a + 1 + \sin 3 - \sin 2$
(b) $\frac{18}{3}a + 1 + 2\sin 3$
(c) $\frac{18}{3}a - 1 + 2\sin 3$
(d) None

27. The function $\int_1^x ((2t-1)(t-2)^3 + 3(t-1)^2(t-2)^2)\,dt$ attains its max. at $x =$
(a) 1/3 (b) 2
(c) 3 (d) 4

28. The function $f(x) = \int_1^x (t(e^t - 1)(t-1)(t-2)^3(t-3)^5)\,dt$ has a local minimum at $x =$
(a) 0 (b) 1
(c) 2 (d) 3

29. The points of extreme of the function $\int_1^x \left(e^{\frac{-t^2}{2}}(1 - t^2)\right)dt$ are
(a) $x = 0$ (b) $x = 1$
(c) $x = \frac{1}{2}$ (d) $x = -1$

30. The minimum value of xy is, if $a^2x^4 + b^2y^4 = c^6$
(a) $\frac{c^3}{\sqrt{ab}}$ (b) $\frac{c^3}{\sqrt{2ab}}$
(c) $\frac{c^3}{ab}$ (d) $\frac{c^3}{2ab}$

31. The min. value of $f(x) = |3 - x| + |2 + x| + |5 - x|$ is
(a) 7 (b) 10
(c) 8 (d) 0

32. A cubic $f(x)$ vanishes at $x = -2$ and has relative min./max. at $x = -1$, $x = \frac{1}{3}$ such that $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$. Then $f(x)$ is
(a) $x^3 + x^2 - x$ (b) $x^3 + x^2 - x + 1$
(c) $x^3 + x^2 - x + 2$ (d) $x^3 + x + x + 10$

33. The minimum value of $2^{(x^2-3)^3 + 27}$ is
(a) 2^{27} (b) 2
(c) 1 (d) 0

34. Let $f(x) = \begin{cases} |x|, 0 < |x| \le 2 \\ 1, x = 0 \end{cases}$
Then at $x = 0$, $f(x)$ has
(a) a local max. (b) no local max.
(c) a local min. (d) No extreme

35. A rod of fixed length l slides along the co-ordinate axes. If it meets the axes at $A(a, 0)$ and $B(0, b)$, then the minimum value of $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2$ is
(a) $l^2 - 4 + \frac{4}{x^2}$ (b) $l^2 + 4 + \frac{4}{x^2}$
(c) $l^2 - 4 - \frac{4}{x^2}$ (d) $l^2 - 2 - \frac{2}{x^2}$

36. If $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 2$, has positive point of max. then a varies over an interval of length
(a) $\frac{8}{7}$ (b) $\frac{6}{7}$
(c) $\frac{4}{7}$ (d) $\frac{3}{7}$

37. The interval into which the function $y = \frac{x - 1}{x^2 - 3x + 3}$ transforms into the extremum is

(a) $\left[\frac{1}{3}, 2\right]$ (b) $\left[-\frac{1}{3}, 1\right]$

(c) $\left[-\frac{1}{3}, 2\right]$ (d) None

38. Let (h, k) be a fixed point, where $h > 0$, $k > 0$. A straight line passing through this points cuts the positive direction of the co-ordinate axes at the point P and Q.

Which of the following is the min. area of the ΔOPQ, O being the origin

(a) hk (b) $2hk$

(c) $\frac{1}{2}hk$ (d) None

39. If $A > 0$, $B > 0$ and $A + B = \pi/3$, then the max. value of $\tan A . \tan B$ is

(a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{3}$

(c) 3 (d) $\sqrt{3}$

40. Let $f(x) = 1 + 2\sin x + 3\cos^2 x\left(0 \le x \le \frac{2\pi}{3}\right)$ is

(a) Min. at $x = 90°$

(b) Max. of $x = \sin^{-1}\left(\frac{1}{\sqrt{8}}\right)$

(c) Min. of $x = 30°$

(d) Max. of $x = \sin^{-1}\left(\frac{1}{3}\right)$

41. The min. value of $f(x) = \frac{a^2}{\cos^2 x} + \frac{b^2}{\sin^2 x}$ is

(a) $(a - b)^2$ (b) $a^2 + b^2$

(c) $(a + b)^2$ (d) $a^2 - b^2$

42. A tangent is drawn at the point $(3\sqrt{3}\cos\theta, \sin\theta)$, $0 < \theta < \frac{\pi}{2}$ of an ellipse $\frac{x^2}{27} + \frac{y^2}{1} = 1$, the least value of the sum of the intercepts on the co-ordinates axes by this tangent is attained at $\theta =$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

43. The greatest value of the function $f(x) = \frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$ on the interval $\left[0, \frac{\pi}{2}\right]$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) 1 (d) $-\sqrt{2}$

44. The least value of a for which the equation $\frac{4}{\sin x} + \frac{1}{1 - \sin x} = a$ has at least one solution on the interval $\left(0, \frac{\pi}{2}\right)$ is

(a) 9 (b) 4

(c) 8 (d) 1

45. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(a) $\sqrt{ab}$ (b) $\frac{a}{b}$

(c) $2ab$ (d) ab

46. Min. area of the triangle formed by any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the co-ordinate axes is

(a) $\frac{a^2 + b^2}{2}$ (b) $\frac{(a + b)^2}{2}$

(c) ab (d) $\frac{(a - b)^2}{2}$

47. Rectangle of max. area that can be inscribed in an equilateral Δ of side a will have area =

(a) $\frac{a^2\sqrt{3}}{2}$ (b) $\frac{a^2\sqrt{3}}{4}$

(c) $\frac{a^2\sqrt{3}}{8}$ (d) None

48. The minimum radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ is of length

(a) $(a - b)$ (b) $(a + b)$

(c) $2(a + b)$ (d) $2(a - b)$

49. The value of x for which the slope of $y = x^6 + 5x^5 + 4x^4 + ... + 1$ is maximum is

(a) $x = 0$ (b) $x = 1$

(c) $x = 2$ (d) None.

50. Let $f(x) = (x - 2)^6 (x - 3)^5$. Then $f(x)$ has minima at

(a) 5/11 (b) 28/11

(c) 3/10 (d) 27/10

Level III

(Problems for JEE-Advanced)

1. If $a > b > 0$, then find the positive max. value of

$$f(x) = \frac{ab(a^2 - b^2)\sin x \cos x}{a^2\sin^2 x + b^2\cos^2 x}, x \in \left[0, \frac{\pi}{2}\right]$$

2. If $a > b > 0$ and $f(\theta) = \frac{(a^2 - b^2)\cos\theta}{a - b\sin\theta}$, find the max. value of $f(\theta)$.

3. Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$

The set of values of b for which $f(x)$ has greatest value of $x = 1$ is given by

4. Find the max. and the min values of $f(x) = x^3 - 3x$ subject to condition $x^4 + 36 \leq 13x^2$.
5. Find the co-ordinates of the point on the curve $(x^2 + 1)(y - 3) = x$ where a tengent to the curve has the greatest slope.
6. Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$
7. Find the maximum value of $f(x) = (\sqrt{-3 + 4x - x^2} + 4)^2 + (x - 5)^2$ where $1 \leq x \leq 3$.
8. Let $f(x, y) = x^2 + y^2 - xy$ where x and y are connected to the relation $x^2 + 4y^2 = 4$. Find the greatest value of $f(x, y)$.
9. If $x^2 + y^2 = 1$, find the least value of $f(x, y) = x^2 + y^2 + xy + 3$.
10. Find the least value of a for which the equation $\frac{4}{\sin x} + \frac{1}{1 - \sin x} = a$ has at-least one solution in the interval $\left(0, \frac{\pi}{2}\right)$.
11. Let $y = f(x) = \frac{x^4 - x^2}{x^6 + 2x^3 - 1}$, $x > 1$

    If the maximum value of y is $\frac{a}{b}$, where $a, b \in N$, find the value of $(a + b + 2)$.
12. Find the greatest and the least value of $f(x) = \sqrt{4 - 4x + x^2} - \sqrt{4 + 4x + x^2}$, $x \in R$
13. Find the greatest and the least value of the function $f(x) = \frac{1}{\sin x + 4} - \frac{1}{\cos x - 4}$, $x \in R$
14. Let $x + y = 1$, where $0 < x, y < 1$. Find the minimum value of $f(x, y) = x^x + y^y$.
15. Find the shortest distance between the curves $9x^2 + 9y^2 - 30y + 16 = 0$ and $y^2 = x^3$.
16. Let $f(x)$ be a polynomial of degree 3 having local maximum at $x = -1$ and $f'(x)$ has local minimum at $x = 1$. If $f(1) = -6$, $f(-1) = 10$ then find the distance bewteen the local maximum and local minimum of the curve.
17. Let $f: R \to R$ be defined as $f(x) = |x| - |x^2 - 1|$, find the number of points at which f attains either local maximum and local minimum.
18. Let $f: R \to \mathrm{R}$ be defined as $f(x) = |x| - |x^3 - 1|$ If f has local maximum at two points, find the distance between them.
19. Let $f: R \to R$ be defined as $f(x) = |3x - 1| - x^3$, find the points of extremum of $f(x)$.
20. Find the values of a for which all roots of the equation $3x^4 + 4x^3 - 12x^2 + a = 0$ are real and distinct.
21. Let $f(x)$ be a polynomial of degree 3 satisfying $f(-1) = 4$, $f(0) = 3$ and $f(x)$ has local maximum at $x = -1$ and $f'(x)$ has local minimum at $x = 1$. Find the function $f(x)$.
22. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, -2$ and $\lim_{x \to 0} \left(1 + \frac{p(x)}{x^2}\right) = 4$. Find the polynomial $p(x)$ and $p'(2)$.
23. Find the max value of $y = \sqrt{x - 2} + \sqrt{4 - x}$.
24. Find the max value of $y = \sqrt{x^2 - 1} + \sqrt{4 - x^2}$.
25. Find the max or min values of $f(x, y) = x^2 + y^2 - xy$, where $x^2 + 4y^2 = 4$
26. Find the min value of $f(x, y) = 1 - x^2 + 2y^2$, where $x + y = 1$
27. Find the min value of $f(x, y) = x^2 + y^2$, where $x^2(xy - 1) = y^2(1 + xy)$
28. Find the min value of $x + y$, where $x^2 + y^2 = 1$
29. Find the least value of $\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)$ where $x^2 + y^2 + z^2 = 1$
30. Find the absolute maximum value of $f(x) = \frac{1}{|x - 4| + 1} + \frac{1}{|x + 8| + 1}$
31. For what values of a do the points of extremum of the function $y = x^3 - 3ax^2 + 3(a^2 - 1)x + 1$ lie in the interval $(-2, 4)$?
32. Find all the values of the parameter a for which the point of minimum of the function $f(x) = 1 + a^2x - x^3$ satisfies the inequality $\frac{x^2 + x + 2}{x^2 + 5x + 6} < 0$.
33. Find the greatest and the least value of the function $f(x) = \frac{a^2}{x} + \frac{b^2}{(1 - x)}$ in $(0, 1)$, $a, b > 0$.
34. A polynomial $f(x)$ of degree 4 has a relative maximum and minimum at $x = -1, 0, 1$ and $f(0) = 3$, $f(1) = 4$ find $f(x)$.
35. A polynomial $f(x)$ of degree 4 with leading co-efficient unity and also has a local max or min at $x = 1, 2, 3$ respectively. If $f(0) = 2$, find the polynomial $f(x)$.
36. Find the greatest value of the function $f(x) = \frac{(x + 1)^4}{x^4 - x^3 + x^2 - x + 1}$

37. Let $f(x) = \dfrac{4x(x^2 + 1)}{x^2 + (x^2 + 1)^2}$, $x \geq 0$

Then find the greatest and the least values of $f(x)$.

38. Find the number of points of extremum of

$$f(x) = |x^2 - 1| + |x^3 - 1| + |x^5 - 1|$$

39. Let $f: R \to R$ be defined as $f(x) = |x^4 - 1| + |x^6 - 1|$. Then find the total number of points at which f attains either a local maximum or local minimum.

40. Let $f: R \to R$ be defined as

$$f(x) = |x^2 - 1| + |x| + |x^6 - 1|$$

Then find the total number of points at which f attains either a local maximum or local minimum.

41. Let $f: R \to R$ be defined as

$$f(x) = |x^2 - 1| + |x| + |x^3 - 1|$$

Then find the total number of points at which f attains either a local maximum or local minimum.

42. If m is the number of points of extremum of $f(x) = |x| + |x^3 - 1|$ and n is the least value of $g(x) = \dfrac{1}{|x - 2| + 1} + |x - 3|$, find the value of $(m + 2n + 2)$.

Level IV

(Tougher Problems for JEE-Advanced)

Mensuration Related Problems

1. A box of max volume with top open is to be made out of a square tin sheet of sides 6 ft length by cutting out small equal squares from four corners of the sheet. Find the height of the box.
2. How should a wire 20 cm long be divided into two parts, if one part is to be bent into a circle, the other part is to be bent into a square and the two plane figures are to have areas the sum of which is max.
3. A window is in the shape of a rectangle surmounted by a semi-circle. If its perimeter is 20 ft, then find the dimensions of the window so that it may admitmaximum light.
4. Find the greatest area of the rectangular plot which can be made out within a triangle of base 36 ft and altitude 12 ft.
5. Rectangles are inscribed in a semi-circle of radius r. Find the rectangle with max area.
6. A sheet of area 40 sq m is used to make an open tank with square base. Find the dimensions of the base such that the volume of this tank is max.
7. Find the height of a cone of the greatest volume, where slant height is given.
8. Find the height of a cone of least volume that can be drawn around a hemisphere of radius R (where the centre of the base of the cone falls on the centre of the sphere).

Co-ordinate Geometrical Problems

9. Let $A(p^2, -p)$, $B(q^2, -q)$, $C(r^2, -r)$ be the vertices of a triangle ABC. A parallelogram $AFDE$ is drawn with vertices D, E and F on the line segments BC, CA and AB respectively. Using calculus, show that the maximum area of such a parallelogram is $\frac{1}{4}(p + q)(p + r)(q + r)$.
10. The circle $x^2 + y^2 = 1$ cuts the x-axis at P and Q Another circle with centre at Q and variable radius intersect the first circle at R above the x-axis and line segment PQ at S. Find the maximum area of the triangle QSR.
11. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then find the maximum value of A.
12. Let (h, k) be a fixed point, where $h > 0$, $k > 0$. A straight line passing through this point cuts the positive direction of the co-ordinate axies at the points P and Q. Find the minimum area of the triangle OPQ, O being the origin.
13. Find the point on the curve $ax^2 + 2bxy + ay^2 = c$, $0 < a < b < c$, whose distance from the origin is minimum.
14. Find the co-ordinate of all the points P on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N is the foot of perpendicular from O to the tangent at P.
15. Find the normals of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ which are farthest from its centre.
16. A straight line L with positive slope passes through the point (8, 2) and cuts the positive co-ordinate axies at points P and Q. Find the absolute minimum value of $OP + OQ$ as L varies, where O is the origin.
17. Let $x^2 + y^2 = r^2$ be a circle of variable radius from a point $P(6, 8)$ tangents are drawn to the circle. Find r such that the triangle formed by chord of contact and two tangents is maximum.

Integer Type Questions

1. Let $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$, $x \in R$. If the minimum value of f is m, find the value of $(m^2 + m + 2)$.
2. Find the number of points of extremum of the function $f(x) = \dfrac{x}{x^2 + 1}$.

3. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and $m(b)$ is the minimum value of $f(x)$. Find the maximum value of $m(b)$.

4. Let $f(x) = \dfrac{x^2 + 3x + 4}{x + 3}$, $x > 0$. Then the minimum value of $f(x)$.

5. Let $f(x) = \dfrac{x^4 + x^2 + 4}{x^2 + 1}$, $x > 0$, find the least value of $f(x)$.

6. Find the maximum value of xyz for which subject to condition is $\dfrac{x^2}{3} + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 1$.

7. Find the maximum value of y for which
$2x^2 - 4xy + 3y^2 - 8x + 8y - 1 = 0$

8. Let $f(x) = x^3 - 3x$. If m is minimum value and n is maximum value of $f(x)$, find the value of
$(m + n + 4)$

9. Let $f(x) = \sqrt{9 - x^2} + \sqrt{x^2 - 4}$. If the range of f is $\lfloor\sqrt{a}, \sqrt{b}\rfloor$, where $a, b \in N$, find the value of
$\left(\dfrac{b}{a} + 3\right)$

10. Let $f(x) = x^4 - 4x^3 + 10$. If m is the number of point of inflection and n is the number of the points of extremum, find the value of $(m + n + 1)$.

Comprehensive Link Passages

Passage I

Let $f(x) = \cos^{-1}(4x^3 - 3x)$. Then

1. $f(x)$ is increasing in
   (a) $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ (b) $\left(-1, -\dfrac{1}{2}\right)$
   (c) $\left(\dfrac{1}{2}, 1\right)$ (d) None

2. The number of points of extremum is
   (a) 0 (b) 1
   (c) 2 (d) 3

3. The point of maximum of f is
   (a) 0 (b) 1/2
   (c) 1 (d) –1/2.

Passage II

Let $f(x) = \dfrac{x^3}{3} - \dfrac{3}{2}x^2 + 2x + 6$. Then

1. f is decreasing in
   (a) (1, 2) (b) (0, 1)
   (c) (–1, 0) (d) (–1, 1)

2. The point of local maximum is
   (a) $x = 0$ (b) $x = 1$
   (c) $x = 2$ (d) $x = -1$

3. The greatest value of $f(x)$ is
   (a) $\dfrac{41}{6}$ (b) $\dfrac{41}{3}$
   (c) $\dfrac{20}{3}$ (d) $\dfrac{22}{3}$

Passage III

Let $f(x) = 2.3^{3x} - 4.3^{2x} + 2.3^x$, $x \in [-1, 1]$. Then

1. $f(x)$ is strictly increasing in
   (a) (–1, 0) (b) (0, 1)
   (c) $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ (d) (–1, 1)

2. The point of maximum of $f(x)$ is
   (a) $x = 1$ (b) $x = -1$
   (c) $x = 0$ (d) $x = 1/2$

3. The greatest value of $f(x)$ is
   (a) 21 (b) 22
   (c) 23 (d) 24.

Passage IV

Let $f(x) = \dfrac{1}{x^2 + 1}$, $g(x) = \dfrac{x}{x^2 + 1}$ and

$h(x) = \dfrac{x^2}{1 + x^2}$ for all x in R. Then

1. $g(x)$ is strictly increasing in
   (a) $(-\infty, -1)$ (b) $(-1, 1)$
   (c) $(0, 1)$ (d) $(1, \infty)$.

2. If m, n and p are the number of points of extremum of f, g and h, then the value of $(m + n + p)$ is
   (a) 2 (b) 3
   (c) 4 (d) 1

3. If a, b are the greatest values of f and g, then the value of $(a + 2b + 2)$ is
   (a) 3 (b) 4
   (c) 1 (d) 2

Passage V

Let $f(x) = 2\sin x + \cos 2x$, $x \in [0, 2\pi]$. Then

1. The number of points of extremum of $f(x)$ is
   (a) 2 (c) 3
   (c) 4 (d) 1

2. The greatest value of $f(x)$ is
   (a) 3/2 (b) 2/3
   (c) 1/2 (d) 2

3. The least value of $f(x)$ is
   (a) -2 (b) -3
   (c) 2 (d) 3

Passage VI

Let $f(x) = \begin{cases} \sin\left(\dfrac{\pi x}{2}\right) & ; \ x < 1 \\ 3 - 2x & : \ x \geq 1 \end{cases}$. Then

1. The number of discontinuous points of f is
   (a) 0 (b) 1
   (c) 2 (d) None
2. The function $f(x)$ is strictly increasing in
   (a) $(-1, 1)$ (b) $(0, 1)$
   (c) $(1, 2)$ (d) $(-2, 1)$
3. The point of local maximum in $(-2, \infty)$ is
   (a) 1 (b) 2
   (c) 0 (d) 3
4. The local maximum value of $f(x)$ is
   (a) 1 (b) 2
   (c) 0 (d) -1

Passage VII

Let $f(x) = \dfrac{x^4 - x^2}{x^6 + 2x^3 - 1}$, $x > 1$

If the maximum value of y is $\dfrac{a}{b}$, where a and b are in their lowest form and extremum occurs at $x = \dfrac{m + \sqrt{n}}{p}$ where m, n and p are pairwise relatively prime numbers, then

1. The maximum value of y is
   (a) 1/6 (b) 1/5
   (c) 1/4 (d) 1/3
2. The value of $(a + b + 3)$ is
   (a) 7 (b) 10
   (c) 8 (d) 9
3 The value of $(m + n + p)$ is
   (a) 7 (b) 6
   (c) 8 (d) 9.

Passage VIII

Let $f(x) = |x^2 - 5|x| + 6|$. Then

1. The number of non-differentiable points of $f(x)$ is
   (a) 4 (b) 3
   (c) 5 (d) 6
2. The number of points of extremum is
   (a) 4 (b) 5
   (c) 6 (d) 7
3. The function $f(x)$ is strictly increasing in
   (a) $(-2, 0)$ (b) $(-3, -2)$
   (c) $(0, 2)$ (d) $(2, 3)$
4. The number of solutions of $4f(x) = 1$ is
   (a) 5 (b) 7
   (c) 8 (d) 9
5. The number of solutions of $8f(x) + 1 = 0$ is
   (a) 5 (b) 7
   (c) 8 (d) 0

Passage IX

Let $f(x)$ be a polynomial of degree 3 satisfying $f(-1) = 4$, $f(0) = 3$ and $f(x)$ has local maximum at $x = -1$ and $f'(x)$ has local minimum at $x = 1$. Then

1. The function $f(x)$ is
   (a) $f(x) = \dfrac{1}{5}(x^3 - 3x^2 - 9x + 15)$
   (b) $f(x) = \dfrac{1}{5}(x^3 + 3x^2 - 9x + 15)$
   (c) $f(x) = \dfrac{1}{5}(x^3 + 3x^2 + 9x - 15)$
   (d) $f(x) = \dfrac{1}{5}(-x^3 + 3x^2 + 9x - 15)$
2. The function $f(x)$ is strictly decreasing in
   (a) $(3, \infty)$ (b) $(-\infty, \infty)$
   (c) $(-1, 3)$ (d) $(3, 10)$
3. The point of local minima occurs at
   (a) $x = -1$ (b) $x = 1$
   (c) $x = 3$ (d) $x = 0$

Passage X

Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, -2$ and $\lim_{x \to 0}\left(1 + \dfrac{p(x)}{x^2}\right) = 4$. Then

1. The polynomial $p(x)$ is
   (a) $p(x) = -\dfrac{3}{4}x^4 - x^3 + 3x^2$
   (b) $p(x) = \dfrac{3}{4}(x^4 + 4x^3 + 4x^2 + 1)$
   (c) $p(x) = \dfrac{3}{4}(x^4 - 4x^3 + 4x^2 + 1)$
   (d) $p(x) = \dfrac{3}{4}(x^4 - 4x^3 + 4x^2 - 1)$
2. The value of $p(0) + p(1) + p(2)$ is
   (a) 0 (b) 1
   (c) 3/4 (d) None
3. The value of $p'(0) + p'(1) + p'(2)$ is
   (a) 0 (b) 1
   (c) 3/4 (d) None.

Matrix Match

1. **Match the following columns**

	Column I		Column II
(A)	The min value of $y = x^2 + 6x + 11$ is	(P)	-2
(B)	The max value of $y = -x^2 + 4x - 10$	(Q)	non exist
(C)	The max value of $y = x^3 - 6x^2 + 11x - 6$ is	(R)	2
(D)	The max value of $y = x + \frac{1}{x}$ is	(S)	6

2. **Match the following columns**

	Column I		Column II
(A)	The min value of $y = \lvert x\rvert + \lvert x - 2\rvert$ is	(P)	1
(B)	The max value of $y = \lvert x - 2\rvert - \lvert x - 4\rvert$ is	(Q)	4
(C)	The min value of $y = \lvert x + 2\rvert + \lvert x\rvert + \lvert x - 2\rvert$ is	(R)	2
(D)	The min value of $y = \lvert x\rvert + \lvert x^2 - 1\rvert$ is	(S)	6

3. **Match the following columns**

	Column I		Column II
(A)	The least value of $y = 2\log_{10} x - \log_x(0.01)$ when $x > 1$, is	(P)	3
(B)	The least value of $y = 3\cos x + \sec^3 x$ for all x in $\left(0, \frac{\pi}{2}\right]$, is	(Q)	4
(C)	The least value of $y = \frac{x^2y^2 + x^2 + y^2 + 1}{xy}$ for all x, $y > 0$, is	(R)	9
(D)	The least value of $y = \frac{(x^3 + 2)(y^3 + 2)}{xy}$ for all x, $y > 0$, is	(S)	2

4. **Match the following columns**

	Column I		Column II
(A)	The least value of $y = \sqrt{x - 2} + \sqrt{4 - x}$ is	(P)	$\sqrt{3}$
(B)	The least value of $y = \frac{2}{x^2 + 2}$ is	(Q)	$\sqrt{2}$
(C)	The least value of $y = \sqrt{4 - x^2} + \sqrt{x^2 - 1}$ is	(R)	1
(D)	The least value of $f(x, y) = 1 + x^2 - 2y^2$ when $x + y = 1$, is	(S)	0

5. **Match the following columns**

	Column I		Column II
(A)	The greatest value of $y = \frac{x}{x^2 + 4}$, $x > 0$, is	(P)	
(B)	The greatest value of $y = \frac{x}{x^2 + x + 4}$, $x > 0$, is	(Q)	1
(C)	The greatest value of x^2y^2, where $2x + 3y = 5$ and $x, y > 0$, is	(R)	1/5
(D)	The greatest value of $x^4y^3z^2$, where $4x + 3y + 2z = 9$ and $x, y, z > 0$, is	(S)	1/4

6. **Match the following columns**

The number of points at f attains either a local max or local min is

	Column I		Column II
(A)	$f(x) = \lvert x\rvert + \lvert x^2 - 1\rvert$	(P)	2
(B)	$f(x) = x(x - 1)^2$	(Q)	4
(C)	$f(x) = \int_0^x e^{t^2}(t - 2)(t - 3)dt$	(R)	5
(D)	$f(x) = \begin{cases} (x+2)^3 & : -3 < x \le -1 \\ x^{2/3} & : -1 < x < 2 \end{cases}$	(S)	3

7. **Match the following columns**

	Column I		Column II
(A)	The absolute max of $f(x) = x^3 - 3x^2 + 1$ in $[-1/2, 4]$ is	(P)	0
(B)	The absolute max of $f(x) = \frac{x^3}{3} - \frac{5x^2}{2}$ in $[1, 4]$ is	(Q)	23/6

(C)	The absolute min of $f(x) = x^{2/3}(5 - 2x)$ in $[-1, 2]$ is	(R)	1
(D)	The absolute min of $f(x) = x + 2\sqrt{x}$ in $[0, 4]$	(S)	17

8. **Match the following columns**
Investigate the following functions for extrema.

Column I		Column II	
(A)	$f(x) = 1 - (x - 2)^{2/3}$	(P)	local max at $x = 1$
(B)	$f(x) = (x - 2)^{4/5}(2x + 1)$	(Q)	local max at $x = 2$
(C)	$f(x) = \begin{cases} x^2 & : x \le 0 \\ 2\sin x & : x > 0 \end{cases}$	(R)	local min at $x = 0$
(D)	$f(x) = 3 \cdot \sqrt[3]{x^2} - x^2$	(S)	local min at $x = 3$

9. **Match the following columns**
Let $f(x) = 2x^3 - 9x^2 + 12a^2x + 1$, where a is positive. If f attains its max at p and min at q such that $p^2 = q$. Then

Column I		Column II	
(A)	p is	(P)	$2a$
(B)	q is	(Q)	a
(C)	the value of a is	(R)	3
(D)	the value of $p + q - 3$ is	(S)	2

10. **Match the following columns**

Column I		Column II	
(A)	The point on the curve $y = -x^3 + 3x^2 + 2x + 27$ where the tangent has the max slope is	(P)	(0, 0)
(B)	The point on the curve $y = \dfrac{x}{x^2 + 1}$ has the max slope is	(Q)	(1, 31)
(C)	The point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is min is	(R)	(1, 2)
(D)	The point on the curve $y^2 = 4x$ which is nearest to the point (2, 1) is	(S)	(2, 1)

Question asked in Previous Years' IIT-JEE Examinations

1. Prove that the minimum value of $\dfrac{(a + x)(b + x)}{(c + x)}$ is $(\sqrt{a - c} + \sqrt{b - c})^2$, where $a, b > c, x > -c$.
[IIT-JEE, 1979]

2. Let x and y be two real variable $x > 0$ and $xy = 1$ Find the minimum value of $x + y$.
[IIT-JEE, 1981]

3. For all x in [0, 1], let the 2nd derivative of a function exist and satisfy $|f''(x)| < 1$. If $f(0) = f(1)$, then show that $|f'(x)| < 1$ for all x in [0, 1]
[IIT-JEE, 1981]

4. Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $0 \le c \le 5$
[IIT-JEE, 1982]

5. If $y = a\ln|x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then
(a) $a = 2, b = -1$ (b) $a = 2, b = -\frac{1}{2}$
(c) $a = -2, b = \frac{1}{2}$ (d) None.
[IIT-JEE, 1983]

6. AB is a diameter of a circle and C is any point on the circumference of the circle. Then
(a) the area of ΔABC is maximum when it is isosceles.
(b) the area of ΔABC is minimum when it is isosceles.
(c) the perimeter of ΔABC is minimum when it is isosceles.
(d) None.
[IIT-JEE, 1983]

7. For $0 < a < x$, the minimum value of the function $\log_a x + \log_x a$ is 2. Is it true or false ?
[IIT-JEE, 1984]

8. Find the co-ordinates of the point on the curve $y = \dfrac{x}{1 + x^2}$ where the tangent to the curve has the greatest slope.
[IIT-JEE, 1984]

9. Let $f(x) = \sin^3 x + \lambda \sin^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ Find the interval in which λ should lies in order that $f(x)$ has exactly one minimum and one maximum.
[IIT-JEE, 1985]

10. Let $A(p^2, -p)$, $B(q^2, -q)$, $C(r^2, -r)$ be the vertices of a triangle ABC. A parallelogram $AFDE$ is drawn with vertices D, E and F on the line segments BC, CA and AB respectively.

Using calculus, show that the maximum area of such a parallelogram is $\frac{1}{4}(p + q)(p + r)(q + r)$

[IIT-JEE, 1986]

11. Let $P(x) = a_0 + a_1x^2 + a_2x^4 + \dots + a_nx^{2n}$ be a polynomial in real variable x with $0 < a_0 < a_1 < a_2 < \dots < a_n$. The function $P(x)$ has
   (a) Neither a max nor a min
   (b) Only one maximum
   (c) Only one minimum
   (d) Only one maximum and only one minimum.
   (e) None

[IIT-JEE, 1986]

12. Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$

[IIT-JEE, 1987]

13. Investigate the maxima and minima for the function $f(x) = \int_1^x (2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2)\,dt$

[IIT-JEE, 1988]

14. Find all maxima and minima of the function $y = x(x - 1)^2$, $0 \le x \le 2$.

[IIT-JEE, 1989]

15. Show that $2\sin x + \tan x \ge 3x$, where $0 \le x < \frac{\pi}{2}$

[IIT-JEE, 1990]

16. *A* point *P* is given on the circumference of the circle of radius *r*, chord *QR* is parallel to the tangent at *P*. Determine the maximum possible area of triangle *PQR*.

[IIT-JEE, 1990]

17. A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semi-circular portion is fitted with coloured glass while the rectangular part is fitted with clear glass. The clear glass transmits three times as much light per square meter as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits the maximum light?

[IIT-JEE, 1991]

18. A cubic $f(x)$ vanishes at $x = -2$ and has relative maximum/minimum at $x = -1$ and $x = \frac{1}{3}$. If $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$, find the cubic $f(x)$.

[IIT-JEE, 1992]

19. Let $f(x) = \begin{cases} -x^3 + \left(\dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}\right) & : 0 \le x \le 1 \\ 2x - 3 & : 1 \le x \le 3 \end{cases}$

Find all possible real values of b such that $f(x)$ has the smallest value at $x = 1$.

[IIT-JEE, 1993]

20. Find the values of x for which the function $f(x) = 1 + 2\sin x + 3\cos^2 x$, $0 \le x \le \frac{2\pi}{3}$ is a maximum or minimum. Also find these values of the function.

[IIT-JEE, 1993]

21. If $A > 0$, $B > 0$ and $A + B = \frac{\pi}{3}$, then find the maximum value of $\tan A \tan B$.

[IIT-JEE, 1993]

22. The circle $x^2 + y^2 = 1$ cuts the x-axis at *P* and *Q* Another circle with centre at *Q* and variable radius intersect the first circle at *R* above the *x*-axis and line segment *PQ* at *S*. Find the maximum area of the triangle *QSR*.

[IIT-JEE, 1994]

23. Let *P* be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If *A* is the area of the triangle PF_1F_2, then the maximum value of *A* is...

[IIT-JEE, 1994]

24. Let (h, k) be a fixed point, where $h > 0$, $k > 0$. A straight line passing through this point cuts the + ve direction of the co-ordinate axies at the points *P* and *Q*. Find the minimum area of the triangle *OPQ*, *O* being the origin.

[IIT-JEE, 1995]

25. On the interval [0, 1], the function $f(x) = x^{25}(1 - x)^{75}$ takes its maximum value at the point
   (a) 0 (b) 1/4
   (c) 1/2 (d) 1/3

[IIT-JEE, 1995]

26. Determine the points of maxima and minima of the function $f(x) = \frac{1}{8}\log x - bx + x^2$, $x > 0$ where b (≥ 0) is a constant.

[IIT-JEE, 1996]

27. No questions asked in 1997.

28. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$ for every real x, then the min value of f
   (a) does not exist because f is unbounded.
   (b) is not attained even though f is bounded.
   (c) is equal to 1
   (d) is equal to –1

[IIT-JEE, 1998]

29. The number of values of x where the function $f(x) = \cos x + \cos(x\sqrt{2})$ attains its maximum is

(a) 0 (b) 1

(c) 2 (d) infinite.

[IIT-JEE, 1998]

30. Suppose $f(x)$ is a function satisfying the following conditions

(a) $f(0) = 2, f(1) = 1$

(b) f has minimum value at $x = 1/2$

(c) for all x

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

where a and b are some constants. Dtermine the constants a and b and the function $f(x)$.

[IIT-JEE, 1998]

31. Find the point on the curve $ax^2 + 2bxy + ay^2 = c$, $0 < a < b < c$, whose distance from the origin is minimum.

[IIT-JEE, 1998]

32. Find the co-ordinate of all the points P on the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ for which the area of the triangle PON is maximum, where O denotes the origin and N is the foot of perpendicular from O to the tangent at P.

[IIT-JEE, 1999]

33. Find the normals of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ which are farthest from its centre.

[IIT-JEE, 1999]

34. The function

$f(x) = \int_{-1}^{x} t(e^t - 1)(t-1)(t-2)^3(t-3)^5\,dt$ has a local minimum at $x =$

(a) 0 (b) 1

(c) 2 (d) 3

[IIT-JEE, 1999]

35. Let $-1 \le p \le 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in [1/2, 1] and identify it.

[IIT-JEE, 2000]

36. Let $f(x) = \begin{cases} |x| & : 0 < |x| \le 2 \\ 1 & : x = 0 \end{cases}$.

Then at $x = 0$, f has

(a) a local min

(b) no local maximum

(c) a local max

(d) no extremum.

[IIT-JEE, 2000]

37. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies the range of $m(b)$ is...

(a) [0, 1] (b) $\left(0, \frac{1}{2}\right]$

(c) $\left[\frac{1}{2}, 1\right]$ (d) (0, 1]

[IIT-JEE, 2001]

38. A straight line L with –ve slope passes through the point (8, 2) and cuts the +ve co-ordinate axies at points P and Q. Find the absolute minimum value of $OP + OQ$ as L varies, where O is the origin.

[IIT-JEE, 2002]

39. A tangent is drawn at the point $(3\sqrt{3}\cos\theta, \sin\theta)$ $0 < \theta < \frac{\pi}{2}$ of an ellipse $\frac{x^2}{27} + \frac{y^2}{1} = 1$, the least value of the sum of the intercepts on the co-ordinates axes by this tangent is attained at $\theta =$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

[IIT-JEE, 2003]

40. If x be real then minimum value of $f(x) = x^2 + 2bx + 2c^2$ is greater than the maximum value of $g(x) = -x^2 - 2cx + b^2$ for...

[IIT-JEE, 2003]

41. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$ is minimum.

[IIT-JEE, 2003]

42. Let $x^2 + y^2 = r^2$ be a circle of variable radius from a point $P(6, 8)$ tangents are drawn to the circle. Find r such that the triangle formed by chord of contact and two tangents is maximum.

[IIT-JEE, 2003]

43. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$

(a) $f(x)$ is strictly increasing function

(b) $f(x)$ has a local maxima

(c) $f(x)$ is strictly decreasing function

(d) $f(x)$ is bounded.

[IIT-JEE, 2004]

44. Minimum area of the triangle formed by any tagent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the co-ordinate axes is...

[IIT-JEE, 2005]

45. Find the cubic polynomial $P(x)$ such that $P(x)$ has maximum at $x = -1$ and $P'(x)$ has minimum at $x = 1$. $P(-1) = 10$, $P(1) = -6$. Also find the distance between

the points of local maximum and local minimum of the curve.

[IIT-JEE, 2005]

46. If $f(x) = \begin{cases} e^x, & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \text{ and} \\ x - e, & 2 < x \le 3 \end{cases}$

$g(x) = \int_0^x f(t)\,dt,\ x \in [1, 3]$ then $g(x)$ has

(a) local maxima at $x = 1 + \log 2$

(b) local maxima at $x = 1$ and local minima at $x = 2$

(c) local maxima at $x = e$

(d) local minima at $x = e$.

[IIT-JEE, 2006]

47. Let $f(x)$ is a cubic polynomial which has a local max at $x = -1$. If $f(2) = 18$, $f(-1) = -1$ and $f'(x)$ has local min at $x = 0$, then...

(a) the distance between $(-1, 2)$ and $(a, f(a))$ where $x = a$ is the point of local minima is $2\sqrt{5}$

(b) $f(x)$ is increasing for $x \in \lfloor 1, 2\sqrt{5} \rfloor$

(c) $f(x)$ has local minima at $x = 1$

(d) the value of $f(0)$ is 5.

[IIT-JEE, 2006]

48. For a twice differentiable function $f(x)$, $g(x)$ is defined as $g(x) = (f'(x))^2 + f''(x) f(x)$ on $[a, e]$. If $a < b < c < d < e$, $f(a) = 0$,

$f(b) = 2$, $f(c) = -1$, $f(d) = 2$, $f(e) = 0$, then find the maximum number of zeroes of $g(x)$.

[IIT-JEE, 2006]

49. No questions asked in 2007.

50. The total number of local max and local min of the function $f(x) = \begin{cases} (2+x)^3 & : -3 < x \le -1 \\ x^{2/3} & : -1 < x < 2 \end{cases}$ is...

(a) 0 (b) 1

(c) 2 (d) 3

[IIT-JEE, 2008]

51. Consider the function $f: R \to R$ defined by

$f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}$, where $0 < a < 2$

(i) which of the following is true ?

(a) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

(b) $(2 - a)^2 f''(1) - (2 - a)^2 f''(-1) = 0$

(c) $f'(1)f'(-1) = (2 - a)^2$

(d) $f'(1)f'(-1) = (2 + a)^2$

(ii) which of the following is true?

(a) $f(x)$ is a decreasing on $(-1, 1)$ and has a local minimum at $x = 1$.

(b) $f(x)$ is a increasing on $(-1, 1)$ and has a local maximum at $x = 1$.

(c) $f(x)$ is increasing on $(-1, 1)$ and has neither local maximum nor a local minimum at $x = 1$

(d) $f(x)$ is decreasing on $(-1, 1)$ and has neither local maximu nor a local minimum at $x = 1$.

(iii) Let $g(x) = \int_0^{e^x} \left(\dfrac{f'(t)}{1 + t^2}\right) dt$. Which of the following is true?

(a) $g'(x)$ is +ve on $(-\infty, 0)$ and –ve on $(0, \infty)$

(b) $g'(x)$ is –ve on $(-\infty, 0)$ and +ve on $(0, \infty)$

(c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$

(d) $g'(x)$ does not change sign on $(-\infty, \infty)$

[IIT-JEE, 2008]

52. The max value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x : x^2 + 20 \le 9x\}$ is...

[IIT-JEE, 2009]

53. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x \to 0} \left(1 + \dfrac{p(x)}{x^2}\right) = 2$. Then the value of $p(2)$ is...

[IIT-JEE, 2009]

54. Let f be a function defined on R (the set of all real numbers) such that

$f'(x) = 2010(x - 2009)(x - 2010)2.$

$(x - 2011)^3 (x - 2012)^4$ for all x in R. If g is a function on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$ for all x in R then the number of points in R at which g has a local minimum is...

[IIT-JEE, 2010]

55. Let f, g and h be real valued functions defined on the interval $[0, 1]$ by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2 e^{x^2} + e^{-x^2}$.

If a, b and c denote respectively the absolute maximum on $[0, 1]$ then

(a) $a = b$, $c \ne b$ (b) $a = c$, $a \ne b$

(c) $a \ne b$, $c \ne a$ (d) $a = b = c$

[IIT-JEE, 2010]

56. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is...

[IIT-JEE, 2011]

57. Let $f : R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or local minimum is...

[IIT-JEE, 2012]

58. Let $p(x)$ be a polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then find the value of $p'(0)$.

[IIT-JEE, 2012]

59. If $f(x) = \int_0^x e^{t^2}(t - 2)(t - 3)$ for all $x > 0$ then

(a) f has a local maximum at $x = 2$

(b) f is decreasing on (2, 3)

(c) there exist some $c \in (0, \infty)$ such that $f''(c) = 0$

(d) f has a local minimum at $x = 3$.

[IIT-JEE, 2012]

60. The number of points in $(-\infty, \infty)$ for which $x^2 - x \sin x - \cos x = 0$ is...

(a) 6 (b) 4

(c) 2 (d) 0

[IIT-JEE, 2013]

61. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are

(a) 24 (b) 32

(c) 45 (d) 60

[IIT-JEE, 2013]

62. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum and local maximum at $x =$

(a) –2 (b) $-\frac{2}{3}$

(c) 2 (d) $\frac{2}{3}$

[IIT-JEE, 2013]

63. Let $f : [0, 1] \to R$ be a function. Suppose the function f is twice differentiable $f(0) = 0 = f(1)$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$

(i) which of the following is true for $0 < x < 1$?

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

(ii) If the function $e^{-x}f(x)$ assumes its minimum in the interval [0, 1] at $x = 1/4$. Which of the following is true?

(a) $f'(x) < f(x)$, $\frac{1}{4} < x < \frac{3}{4}$

(b) $f'(x) > f(x)$, $0 < x < \frac{1}{4}$

(c) $f'(x) < f(x)$, $0 < x < \frac{1}{4}$

(d) $f'(x) < f(x)$, $\frac{3}{4} < x < 1$

[IIT-JEE, 2013]

63. A line $L : y = mx + 3$ meets y-axis at $E(0, 3)$ and the arc of the parabola $y^2 = 16x$, $0 \leq y \leq 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum.

List-I	List-II
(P) $m =$	1. 1/2
(Q) Max area of	2. 4 triangle EFG is
(R) $y_0 =$	3. 2
(S) $y_1 =$	4. 1

Codes:

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

[IIT-JEE, 2013]

64. Let $a \in R$ and $f : R \to R$ be given by $f(x) = x^5 - 5x + a$, then

(a) $f(x)$ has three real roots if $a > 4$

(b) $f(x)$ has only one real root if $a > 4$

(c) $f(x)$ has three real roots if $a < -4$

(d) $f(x)$ has three real roots if $-4 < a < 4$.

[IIT-JEE, 2014]

65. The number of points in the interval $\lfloor -\sqrt{13}, \sqrt{13} \rfloor$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value is...

[IIT-JEE, 2014]

ANSWERS

LEVEL II

1. (b)	2. (d)	3. (c)	4. (b)	5. (d)
6. (b)	7. (b)	8. (c)	9. (a)	10. (a)
11. (c)	12. (b)	13. (d)	14. (c)	15. (c)
16. (a)	17. (a,b,c,d)		18. (c)	19. (a)
20. (d)	21. (a)	22. (b)	23. (c)	24. (d)

25. (a)	26. (a)	27. (a)	28. (b,d)	29. (b,d)
30. (b)	31. (a)	32. (a)	33. (c)	34. (d)
35. (b)	36. (a)	37. (b)	38. (b)	39. (b)
40. (a,d)	41. (c)	42. (a)	43. (c)	44. (a)
45. (c)	46. (c)	47. (d)	48. (b)	49. (d)
50. (b)				

INTEGER TYPE QUESTIONS

1. 2	2. 2	3. 1	4. 1	5. 3
6. 2	7. 5	8. 4	9. 5	10. 4

COMPREHENSIVE LINK PASSAGES

Passage I	1. (a)	2. (c)	3. (b)		
Passage II	1. (a)	2. (b)	3. (a)		
Passage III	1. (a,b,c,d)	2. (a)	3. (d)		
Passage IV	1. (b,c)	2. (c)	3. (b)		
Passage V	1. (a)	2. (a)	3. (b)		
Passage VI	1. (a)	2. (a,b)	3. (a)	4. (a)	
Passage VII	1. (a)	2. (b)	3. (c)		
Passage VIII	1. (c)	2. (d)	3. (a)	4. (c)	5. (d)
Passage IX	1. (a)	2. (c)	3. (c)		
Passage X	1. (a)	2. (d)	3. (d)		

MATRIX MATCH

1. (A)→R, (B)→S, (C)→Q, (D)→P.
2. (A)→R, (B)→R, (C)→Q, (D)→P.
3. (A)→Q, (B)→Q, (C)→Q, (D)→R.
4. (A)→Q, (B)→S, (C)→P, (D)→R.
5. (A)→S, (B)→R, (C)→Q, (D)→Q.
6. (A)→R, (B)→P, (C)→P, (D)→P.
7. (A)→S, (B)→Q, (C)→P, (D)→P.
8. (A)→Q, (B)→P, (C)→R, (D)→P, R
9. (A)→Q, (B)→P, (C)→S, (D)→R
10. (A)→Q, (B)→P, (C)→S, (D)→R

HINTS AND SOLUTIONS

Level I

1. (i) We have $f(x) = x^2 - 4x + 10$

$$= (x-2)^2 + 6 \geq 6$$

Thus, the min value of $f(x)$ is 6 at $x = 2$.

(ii) $f(x) = -x^2 + 6x - 15$

We have $f(x) = -x^2 + 6x - 15$

$$= -(x^2 - 6x + 9) - 6$$
$$= -(x-3)^2 - 6 \leq -6$$

Thus, the maximum value of $f(x)$ is -6 at $x = 3$.

(iii) We have

$$f(x) = (x-3)^2 + \geq 5$$

Hence, the min value of $f(x)$ is 5.

(iv) We have $f(x) = -(x-1)^2 + 2 \leq 2$

Hence, the maximum value of $f(x)$ is 2.

(v) We have $f(x) = |x + 2|$ clearly, $|x + 2| \geq 0$

$\Rightarrow \quad f(x) \geq 0$

Thus, the min value of $f(x)$ is 0 at $x = -2$

(vi) We have $f(x) = -|x - 1| + 5$

Now, $|x - 1| \geq 0$

$\Rightarrow \quad -|x - 1| \leq 0$

$\Rightarrow \quad -|x - 1| + 5 \leq 5$

$\Rightarrow \quad f(x) \leq 5$

Thus, the maximum value of $f(x)$ is 5 at $x = 1$

(vii) We have $f(x) = |x - 1| + |x - 3|$

$$= \begin{cases} -(x-1)-(x-3) & : x<1 \\ (x-1)-(x-3) & : 1 \leq x < 3 \\ (x-1)+(x-3) & : x \geq 3 \end{cases}$$

$$= \begin{cases} -2x+4 & : x<1 \\ 2 & : 1 \leq x < 3 \\ 2x-4 & : x \geq 3 \end{cases}$$

Clearly, the min value of $f(x)$ is 2 at x in [1, 3]

(viii) We have $f(x) = |x + 2| + |x - 3|$

$$= \begin{cases} -2x+1 & : x<-2 \\ 5 & : -2 \leq x < 3 \\ 2x-1 & : x \geq 3 \end{cases}$$

Clearly, the min value of $f(x)$ is 5 at x in [–2, 3]

(ix) We have $f(x) = x^3 - 1$

Since the range of $f(x)$ is R, so it has no extrema.

(x) We have $f(x) = 3\sin x + 4\cos x + 10$

Max value of $f(x) = \sqrt{3^2 + 4^2} + 10 = 5 + 10 = 15$

Min value of $f(x) = -\sqrt{3^2 + 4^2} + 10 = -5 + 10 = 5$

(xi) We have $f(x) = \sin x + \cos x$

Max value of $f(x) = \sqrt{1^2 + 1^2} = \sqrt{2}$

Min value of $f(x) = -\sqrt{1^2 + 1^2} = -\sqrt{2}$

(xii) We have $f(x) = \sin x - \cos x$

Max value of $f(x) = \sqrt{1^2 + 1^2} = \sqrt{2}$

Min value of $f(x) = -\sqrt{1^2 + 1^2} = -\sqrt{2}$

(xiii) We have $f(x) = \sin(2x) + 5$

Max value of $f(x) = 1 + 5 = 6$

Min value of $f(x) = -1 + 5 = 4$

(xiv) We have $f(x) = 3 - 2\sin x = 3 + 2(-\sin x)$

Max value of $f(x) = 3 + 2.1 = 5$

Min value of $f(x) = 3 + 2(-1) = 3 - 2 = 1$

(xv) We have $f(x) = 2\cos x + 3$

Max value of $f(x) = 2.1 + 3 = 5$

Min value of $f(x) = 2.(-1) + 3 = -2 + 3 = 1$

(xvi) We have $f(x) = 3 - 4\cos x = 3 + 4(-\cos x)$

Max value of $f(x) = 3 + 4.1 = 7$

Min value of $f(x) = 3 + 4.(-1) = 3 - 4 = -1$

(xvii) We have $f(x) = 2\sin^2 x + 4$

Max value of $f(x) = 2.1 + 4 = 6$

Min value of $f(x) = 2.0 + 4 = 4$

(xviii) We have $f(x) = 5 - 3\sin^2 x = 5 + 3(-\sin^2 x)$

Max value of $f(x) = 5 + 3.0 = 5$

Min value of $f(x) = 5 + 3(-1) = 5 - 3 = 2.$

(xix) We have $f(x) = 3\sin^2 x + 2\cos^2 x$

$= 2(\sin^2 x + \cos^2 x) + \sin^2 x$

$= 2 + \sin^2 x$

Max value of $f(x) = 2 + 1 = 3$

Min value of $f(x) = 2 + 0 = 2$

(xx) We have

$f(x) = 4\sin^2 x + 5\cos^2 x + 6\sin x\cos x + 10$

$= 4(\sin^2 x + \cos^2 x) + \cos^2 x + 3\sin 2x + 10$

$= 4 + \cos^2 x + 3\sin 2x + 10$

$= \cos^2 x + 3\sin 2x + 14$

$= \frac{1}{2}(1 + \cos 2x) + 3\sin 2x + 14$

$= \frac{1}{2}\cos 2x + 3\sin 2x + \frac{29}{2}$

Max value of $f(x) = \frac{\sqrt{37}}{2} + \frac{29}{2}$

Min value of $f(x) = -\frac{\sqrt{37}}{2} + \frac{29}{2}$

2. (i) We have $f(x) = \sin(\sin x)$

As we know that $-1 \le \sin x \le 1$

$\Rightarrow \sin(-1) \le \sin(\sin x) \le \sin(1)$

$\Rightarrow -\sin(1) \le \sin(\sin x) \le \sin(1)$

$\Rightarrow -\sin(1) \le f(x) \le \sin(1)$

Min value of $f(x)$ is $\sin(1)$ and the max value of $f(x)$ is $-\sin(1)$

(ii) We have $f(x) = \cos(\cos x)$

As we know that $-1 \le \cos x \le 1$

$\Rightarrow \cos(1) \le \cos(\cos x) \le 1$

Thus, the max value of $f(x)$ is 1

and the min value of $f(x)$ is cos 1

(iii) We have $f(x) = \sin(\sin x) + \cos(\sin x)$

As we know that,

$-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$

$\Rightarrow -\sqrt{2} \le \sin(\sin x) + \cos x(\sin x) \le \sqrt{2}$

Thus, the max value of $f(x)$ is $\sqrt{2}$

and the min value of $f(x)$ is $-\sqrt{2}$

(iv) We have $f(x) = \sin^2 x + \cos^4 x$

$= \frac{1}{2}(2\cos^2 x) + \frac{1}{4}(2\sin^2 x)^2$

$= \frac{1}{2}(1 + \cos 2x) + \frac{1}{4}(1 - \cos 2x)^2$

$= \frac{1}{2}(1 + \cos 2x) + \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$= \frac{1}{2} + \frac{1}{4}(1 + \cos^2 2x)$

$= \frac{3}{4} + \frac{1}{4}\cos^2 2x$

Max value of $f(x) = \frac{3}{4} + \frac{1}{4}\cdot 1 = \frac{3}{4} + \frac{1}{4} = 1$

Min value of $f(x) = \frac{3}{4} + \frac{1}{4}(-1) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

(v) We have $f(x) = \sin^4 x + \cos^2 x$

$= \frac{1}{4}(2\sin^2 x)^2 + \frac{1}{2}(2\cos^2 x)$

$= \frac{1}{4}(1 - \cos 2x)^2 + \frac{1}{2}(1 + \cos 2x)$

$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) + \frac{1}{2}(1 + \cos 2x)$

$= \frac{1}{4}(1 + \cos^2 2x) + \frac{1}{2}$

$= \frac{3}{4} + \frac{1}{4}\cos^2 2x$

Max value of $f(x) = \frac{3}{4} + \frac{1}{4}\cdot 1 = \frac{3}{4} + \frac{1}{4} = 1$

Min value of $f(x) = \frac{3}{4} + \frac{1}{4}(-1) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

(vi) We have $f(x) = \sin^4 x + \cos^4 x$

$= (\sin^2 x)^2 + (\cos^2 x)^2$

$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$= 1 - 2\sin^2 x\cos^2 x$

$= 1 - \frac{1}{2}(4\sin^2 x\cos^2 x)$

$= 1 - \frac{1}{2}(2\sin x\cos x)^2$

$= 1 - \frac{\sin^2 2x}{2}$

$= 1 + \frac{1}{2}(-\sin x^2 2x)$

Max value of $f(x) = 1 + \frac{1}{2}\cdot 0 = 1$

Min value of $f(x) = 1 + \frac{1}{2}\cdot(-1) = 1 - \frac{1}{2} = \frac{1}{2}$

(vii) We have $f(x) = \sin^6 x + \cos^6 x$

$= 1 - 3\sin^2 x\cos^2 x$

$= 1 - \frac{3}{4}\sin^2 2x$

$= 1 + \frac{3}{4}(-\sin^2 2x)$

Max value of $f(x) = 1 + \frac{3}{4}\cdot 0 = 1$

Min value of $f(x) = 1 + \frac{3}{4}\cdot(-1) = 1 - \frac{3}{4} = \frac{1}{4}$

(viii) We have $f(x) = \sin^2(\sin x) + \cos^2(\cos x)$

$= \sin^2(\cos x) + \cos^2(\cos x) + \sin^2(\sin x)$
$- \sin^2(\cos x)$

$= 1 + \sin^2(\sin x) - \sin^2(\cos x)$

Max value of $f(x) = 1 + \sin^2(1)$

Min value of $f(x) = 1 - \sin^2(1)$

(ix) We have $f(x) = |\sin x| + |\cos x|$

Max value of $f(x) = 1$

Min value of $f(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

(x) We have $f(x) = \sin x + \operatorname{cosec} x$ for all x in $\left(0, \frac{\pi}{2}\right)$

As we know that, A.M $\geq$ G.M

$\Rightarrow \left(\frac{\sin x + \operatorname{cosec} x}{2}\right) \geq \sqrt{\sin x\cdot\operatorname{cosec} x}$

$\Rightarrow \left(\frac{\sin x + \operatorname{cosec} x}{2}\right) \geq 1$

$\Rightarrow \sin x + \operatorname{cosec} x \geq 2$

Hence, the min value of $f(x)$ is 2.

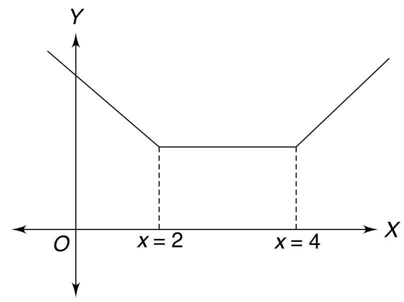
3. We have $f(x) = |x - a| + |x - b|$

$$= \begin{cases} -(x-a)-(x-a) & : x < a \\ (x-a)-(x-b) & : a \leq x < b \\ (x-a)+(x-b) & : x \geq b \end{cases}$$

$$= \begin{cases} -2x+a+b & : x < a \\ b-a & : a \leq x < b \\ 2x-a-b & : x \geq b \end{cases}$$

Thus, the function has a min value $a - b$ at $x \in [a, b]$

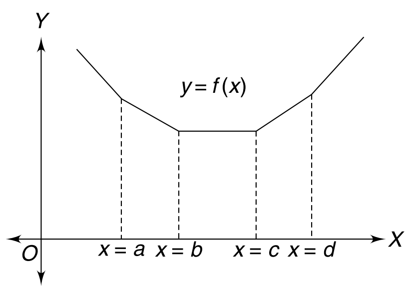
4. We have $f(x) = |x - 2| + |x - 4|$



Thus, the function has a min value 2 at $x \in [2, 4]$

5. We have

$$f(x) = \begin{cases} -4x+(a+b+c+d) & : x < a \\ 2x-a+(b+c+d) & : a \leq x < b \\ c+d-a-b & : b \leq x < c \\ 2x-(a+b+c)-d & : c \leq x < d \\ 4x-(a+b+c+d) & : x \geq d \end{cases}$$



From the graph, it is clear that, $f(x)$ has a min value

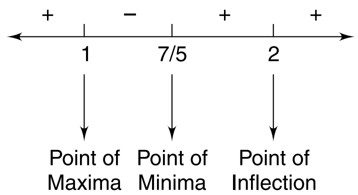
$= c + d - a - b$ at $x \in [b, c]$

6. We have $f(x) = (x - 1)^2(x - 2)^3$

$\Rightarrow f'(x) = 2(x - 1)(x - 2)^3 + 3(x - 1)^2(x - 2)^2$

$\Rightarrow f'(x) = (x - 1)(x - 2)^2(2(x - 2) + 3(x - 1))$

$\Rightarrow f'(x) = (x - 1)(x - 2)^2(5x - 7)$



Thus, $f(x)$ has a local max value at $x = 1$ and local min value at $x = 7/5$.

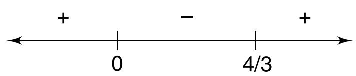
7. We have, $f(x) = (x - 2)x^{2/3}$

$$\Rightarrow \quad f'(x) = x^{2/3} + (x - 2)\cdot\frac{2}{3}\cdot x^{-1/3}$$

$$\Rightarrow \quad f'(x) = x^{2/3} + \frac{2}{3}\frac{(x - 2)}{x^{1/3}}$$

$$\Rightarrow \quad f'(x) = \frac{x + 2x - 4}{3\sqrt[3]{x}}$$

$$\Rightarrow \quad f'(x) = \frac{3x - 4}{3\sqrt[3]{x}}$$



Thus, $f(x)$ has a local max value at $x = 0$ and local min value at $x = 4/3$.

8. Given $\quad f(x) = x^3 - 6x^2 + 12x - 8$

$$\Rightarrow \quad f'(x) = 3x^2 - 12x + 12$$

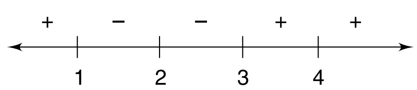
$$\Rightarrow \quad f'(x) = 3(x^2 - 4x + 4) = 3(x - 2)^2$$

So, $f'(x)$ does not changes it sign at the *nbd* of $x = 2$. Thus, $f(x)$ has no point of extremum.

Therefore, $x = 2$ is the point of inflection.

9. We have

$$f'(x) = (x - 1)(x - 2)^2(x - 3)^3(x - 4)^4$$



Thus, $x = 1$ is the point of local max and $x = 3$ is the point of local min and the points $x = 2, 4$ are the point of inflections.

10. We have $f(x) = \sqrt{2x^2 - x + 2}$

$$\Rightarrow \quad f'(x) = \frac{1}{2\sqrt{2x^2 - x + 2}} \times (4x - 1)$$

$$\Rightarrow \quad f'(x) = \frac{(4x - 1)}{2\sqrt{2x^2 - x + 2}}$$

So, $f'(x)$ changes it sign at $x = 1/4$ from negative to positive, so $f(x)$ has a point of local min.

11. We have $f'(x) = 1 - \dfrac{2x}{(1 + x^2)}$

$$\Rightarrow \quad f'(x) = \frac{x^2 + 1 - 2x}{(1 + x^2)} = \frac{(x - 1)^2}{(x^2 + 1)}$$

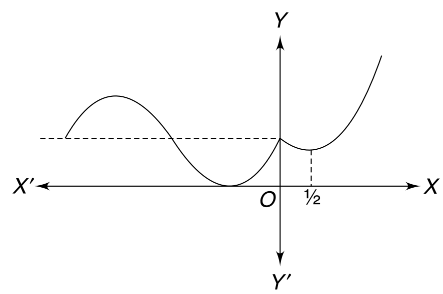
So, $f'(x)$ does not changes it sign at the *nbd* of $x = 1$.

Thus, $f(x)$ has no point of local max or min.

12. We have $f(x) = \begin{cases} 1 + \sin x & : x < 0 \\ x^2 - x + 1 & : x \geq 0 \end{cases}$

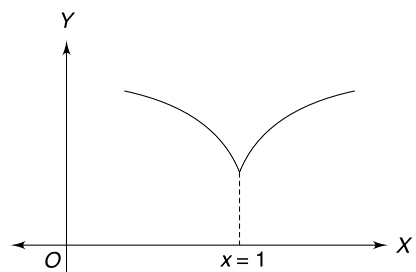
$$f'(x) = \begin{cases} \cos x & : x < 0 \\ 2x - 1 & : x \geq 0 \end{cases}$$

Thus, $\quad f'(0^+) = 1 = f'(0^-)$.



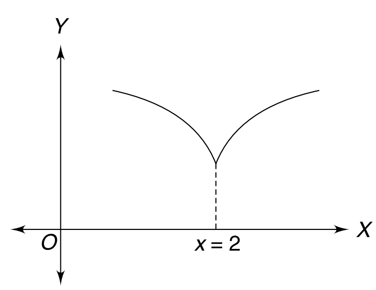
So, $f(x)$ has a local max value at $x = 0$

13. Given $\quad f(x) = 1 + (x - 1)^{2/3}$



From the graph, it is clear that, $f'(x)$ changes its sign from negative to positive, so $x = 1$ is the point of minimum.

14. We have, $f(x) = 1 + (x - 2)^{4/5}$



So, $x = 2$ is the point of minimum.

15. We have $f(x) = \begin{cases} x^2 & : x \le 0 \\ 2\sin x & : x > 0 \end{cases}$

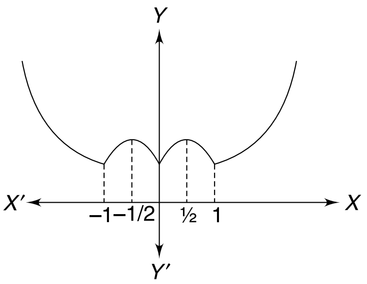
$\Rightarrow \quad f'(x) = \begin{cases} 2x & : x \le 0 \\ 2\cos x & : x > 0 \end{cases}$

Now, $\quad f'(0^+) = 2$ & $f'(0^-) = 0$

So, $f(x)$ is not differentiable at $x = 0$.

At the neighbourhood of $x = 0$, $f'(x)$ changes its sign from negative to positive, so $x = 0$ is the point of maximum.

16. Given $\quad f(x) = |x| + |x^2 - 1|$



Thus, $f(x)$ has local minimum values at $x = -1, 0, 1$ and local maximum values at $x = -1/2$ and $1/2$.

17. We have $f(x) = 2\log(x - 2) - x^2 + 4x + 1$

$\Rightarrow \quad f'(x) = \dfrac{2}{x - 2} - 2x + 4$

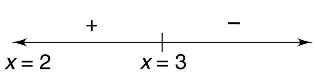
$= \dfrac{2 - 2(x - 2)^2}{x - 2}$

$= \dfrac{2(1 - (x - 2)^2)}{x - 2}$

$= \dfrac{2(1 - (x - 2))(1 + (x - 2))}{x - 2}$

$= \dfrac{2(3 - x)(x - 1)}{(x - 2)}$

$= -\dfrac{2(x - 1)(x - 3)}{(x - 2)}$



Here, $x = 2$ is not a critical point and function is not defined at $x < 2$.

Thus, the given function $f(x)$ has local max value at $x = 3$.

18. Given $\quad f(x) = x^3 - 3x^2 + 1$

$\Rightarrow \quad f'(x) = 3x^2 - 6x = 3x(x - 2)$

Now, $\quad f'(x) = 0$ gives $x = 0, 2$

We have,

$f(0) = 1, f(2) = -3, f(4) = 17, f\left(-\dfrac{1}{2}\right) = \dfrac{3}{8}$

Therefore, the max value is 17 at $x = 4$ and min value is -3 at $x = 2$.

19. Given $\quad f(x) = x^{2/3}(5 - 2x)$

$\Rightarrow \quad f'(x) = \dfrac{2}{3} \cdot \dfrac{1}{x^{1/3}} \cdot (5 - 2x) - 2x^{2/3}$

Now, $\quad f'(x) = 0$ gives

$\Rightarrow \quad \dfrac{2}{3} \cdot \dfrac{1}{x^{1/3}} \cdot (5 - 2x) - 2x^{2/3} = 0$

$\Rightarrow \quad (5 - 2x) - 3x = 0$

$\Rightarrow \quad x = 1$

Thus, $\quad f(1) = 3, f(-1) = 7, f(2) = 2^{2/3}$

Therefore, the absolute max = 7 at $x = -1$ and absolute min = $2^{2/3}$ at $x = 2$.

20. Given $\quad f(x) = x + 2\sqrt{x}, \ \forall x \in [0, 4]$

$\Rightarrow \quad f'(x) = 1 + \dfrac{1}{\sqrt{x}}$

So, it has no critical point.

Now, $\quad f(0) = 0, f(4) = 4 + 2 \cdot \sqrt{4} = 8$

Thus, the absolute max = 8 at $x = 4$ and absolute min = 0 at $x = 0$.

21. Given $\quad f(x) = x^2 \ln x, \ \forall x \in [1, e]$

$\Rightarrow \quad f'(x) = x^2 \cdot \dfrac{1}{x} + 2x \cdot \ln x = x(1 + 2\ln x)$

Now, $\quad f'(x) = 0$ gives $x(1 + 2\ln x) = 0$

$x = 0, x = \dfrac{1}{\sqrt{e}}$

Now, $\quad f(1) = 0, f(e) = e^2$,

$f\left(\dfrac{1}{\sqrt{e}}\right) = \dfrac{1}{e} \cdot \ln\left(\dfrac{1}{\sqrt{e}}\right) = -\dfrac{1}{2e}$

Therefore, the absolute max = e^2 at $x = e$ and absolute min = $-\dfrac{1}{2e}$ at $x = \dfrac{1}{\sqrt{e}}$.

22. Given $\quad f(x) = x + \sin 2x, \ \forall x \in [0, 2\pi]$

$\Rightarrow \quad f'(x) = 1 + 2\cos 2x$

Now, $\quad f'(x) = 0$ gives $1 + 2\cos 2x = 0$

$\Rightarrow \quad \cos 2x = -\dfrac{1}{2}$

$\Rightarrow \quad 2x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

$\Rightarrow \quad x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

Now, $f(0) = 0, f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2},$

$$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}, f(2\pi) = 2\pi$$

Therefore, the abolute max value = 2π and absolute min value = 0

23. Given $f(x) = \sin x + \frac{1}{2}\cos 2x$

$\Rightarrow$ $f'(x) = \cos x - \sin 2x$

Now, $f'(x) = 0$ gives $\cos x - \sin 2x = 0$

$\Rightarrow$ $\cos x(1 - 2\sin x) = 0$

$\Rightarrow$ $\cos x = 0, (1 - 2\sin x) = 0$

$\Rightarrow$ $\cos x = 0, \sin x = \frac{1}{2}$

$\Rightarrow$ $x = \frac{\pi}{6}, \frac{\pi}{2}$

Now, $f(0) = \frac{1}{2}, f\left(\frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$

$$f\left(\frac{\pi}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore, the absolute max value = 3/4 and absolute min = 1/2.

24. Given $f(x) = \tan^{-1} x - \frac{1}{2}\ln x$

$\Rightarrow$ $f'(x) = \frac{1}{1 + x^2} - \frac{1}{2x} = -\frac{x^2 - 2x + 1}{2x(x^2 + 1)}$

Now, $f'(x) = 0$ gives $x = 1$

Thus, $f(1) = \frac{\pi}{4}, f(\sqrt{3}) = \frac{\pi}{3} - \frac{1}{4}\log 3,$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{4}\log 3$$

Therefore, the max value = $\frac{\pi}{6} + \frac{1}{4}\log 3$ and min value = $\frac{\pi}{3} - \frac{1}{4}\log 3$.

25. Given $f(x) = \{(1 - x^2)(2x^2 + 1)\}^{1/2}$

$\Rightarrow$ $f'(x) = \frac{(1 - x^2)\cdot 4x + (2x^2 + 1)\cdot(2x)}{2(\sqrt{1 - x^2)(2x^2 + 1)}}$

$\Rightarrow$ $f'(x) = \frac{2x(2x^2 + 1 + 2 - 2x^2)}{2\sqrt{(1 - x^2)(2x^2 + 1)}}$

$\Rightarrow$ $f'(x) = \frac{6x}{2\sqrt{(1 - x^2)(2x^2 + 1)}}$

Now, $f'(x) = 0$ gives $x = 0$.

Thus, $f(0) = 1, f(-1) = 0, f(1) = 0$

Therefore, the max value = 1 and min value = 0.

26. Given $f(x) = \frac{\sin 2x}{\sin\left(x + \frac{\pi}{4}\right)}$

Max value occurs at $x = \frac{\pi}{4}$

So, Max value = 1

Min value occurs at $x = 0$.

Min value = 0

27. Given $f(x) = \cos x + \cos(\sqrt{2}x)$

The maximum value of $f(x)$ occurs when

$\Rightarrow$ $\cos x = 1$ & $\cos(x\sqrt{2}) = 1$

$\Rightarrow$ $x = 2n\pi$ & $x\sqrt{2} = 2m\pi, m, n \in I$

$\Rightarrow$ $x = 2n\pi$ & $x = \sqrt{2}m\pi, m, n \in I$

$\Rightarrow$ $2n\pi = \sqrt{2}m\pi$

$\Rightarrow$ $m = \sqrt{2}n$

It is possible only when $x = 0$.

Thus, the function will provide us the maximum value at $x = 0$ and the maximum value is 2.

28. We have $f(x) = \frac{1}{\sin x + 4} + \frac{1}{\cos x - 4}$

$\Rightarrow$ $f'(x) = \frac{\cos x}{(\sin x + 4)^2} - \frac{\sin}{(\cos x - 4)^2}$

Now $f'(x) = 0$ gives

$\Rightarrow$ $\frac{\cos x}{(\sin x + 4)^2} - \frac{\sin x}{(\cos x - 4)^2} = 0$

$\Rightarrow$ $\frac{\cos x}{(\sin x + 4)^2} = \frac{\sin x}{(\cos x - 4)^2}$

$\Rightarrow$ $\cos x(\cos x - 4)^2 = \sin x(\sin x + 4)^2$

$\Rightarrow$ $\cos x(\cos^2 x - 8\cos x + 16)$

$= \sin x(\sin^2 x + 8\sin x + 16)$

$\Rightarrow$ $(\cos^3 x - \sin^3 x) - 8(\cos^2 x - \sin^2 x)$

$+ 16(\cos x - \sin x) = 0$

$\Rightarrow$ $(\cos x - \sin x) \times \{(1 - \sin x\cos x)$

$- 8(\cos x + \sin x) + 16\} = 0$

$\Rightarrow$ $\tan x = \frac{\pi\cdot}{4}$ and put $\sin x + \cos x = t$

Hence, the min value of the function

$$= \frac{1}{\frac{1}{\sqrt{2}} + 4} + \frac{1}{\frac{1}{\sqrt{2}} - 4}$$

$$= \frac{4 - \frac{1}{\sqrt{2}}}{16 - \frac{1}{2}} - \frac{4 + \frac{1}{\sqrt{2}}}{16 - \frac{1}{2}}$$

$= \frac{2}{31}\left(4 - \frac{1}{\sqrt{2}} - 4 - \frac{1}{\sqrt{2}}\right)$

$= \frac{2}{31} \times \frac{-2}{\sqrt{2}}$

$= -\frac{4}{31\sqrt{2}}$

29. Given $f(x) = \frac{14}{x^4 - 8x^2 + 2}$

$= \frac{14}{(x^2 - 4)^2 - 14}$

Hence, the global max value is –1 at $x = \pm 2$ and global min value is 7 at $x = 0$.

30. Given $f(x) = (x - 1)^2\sqrt{x^2 - 2x + 3}$

$\Rightarrow f'(x) = 2(x - 1)\sqrt{x^2 - 2x + 3} + (x - 1)^2 \times \frac{x - 1}{\sqrt{x^2 - 2x + 3}}$

$\Rightarrow f'(x) = 2(x - 1)\sqrt{x^2 - 2x + 3} + \frac{(x - 1)^3}{\sqrt{x^2 - 2x + 3}}$

$\Rightarrow f'(x) = \frac{2(x - 1)(x^2 - 2x + 3) + (x - 1)^3}{\sqrt{x^2 - 2x + 3}}$

$f'(x) = 0$ gives

$\Rightarrow \frac{2(x - 1)(x^2 - 2x + 3) + (x - 1)^3}{\sqrt{x^2 - 2x + 3}} = 0$

$\Rightarrow (x - 1)(3x^2 - 6x + 7) = 0$

$\Rightarrow x = 1$

Now, $f(1) = 0, f(0) = \sqrt{3}, f(3) = 4\sqrt{6}$

Thus, the global max value $= 4\sqrt{6}$ and global min is 0.

31. Given $y = \frac{x^2 - 7x + 6}{x - 10}$

$\Rightarrow \frac{dy}{dx} = \frac{(x - 10)(2x - 7) - (x^2 - 7x + 6)\cdot 1}{(x - 10)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 20x + 64}{(x - 10)^2} = \frac{(x - 10)^2 - 36}{(x - 10)^2}$

$\Rightarrow \frac{dy}{dx} = 1 - \frac{36}{(x - 10)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{72}{(x - 10)^2}$

Now, $\frac{dy}{dx} = 0$ gives $x = 4, 16$

At $x = 4, \frac{d^2y}{dx^2} = -\frac{1}{3} < 0$

Thus, y is max at $x = 4$ and the max value = 1

At $x = 16, \frac{d^2y}{dx^2} = \frac{1}{3} > 0$

Thus, y is min at $x = 16$ and the min value = 25.

32. Given $f(x) = 2\sin x + \cos 2x$

$\Rightarrow f'(x) = 2\cos x - 2\sin 2x$

$\Rightarrow f''(x) = -2\sin x - 4\cos 2x$

Now $f'(x) = 0$ gives

$\Rightarrow 2\cos x - 2\sin 2x = 0$

$\Rightarrow 2\cos x - 4\sin x\cos x = 0$

$\Rightarrow 2\cos x(1 - 2\sin x) = 0$

$\Rightarrow 2\cos x = 0, (1 - 2\sin x) = 0$

$\Rightarrow \cos x = 0, \sin x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

At $x = \frac{\pi}{6}, f''(x) = -1 - \sqrt{3} < 0$

Thus, f is max at $x = \frac{\pi}{6}$ and the max value is

$= 2\cdot\frac{1}{2} + \frac{1}{2} = \frac{3}{2}$

At $x = \frac{\pi}{2}, f''(x) = -2 + 4 = 2 > 0$

Thus, f is min at $x = \frac{\pi}{2}$ and the min value is $= 2 - 1 = 1$.

Also, at $x = \frac{5\pi}{6}, f''(x) = -2\cdot\frac{1}{2} - 4\cdot\frac{1}{2} = -3$

Thus, f is max at $x = \frac{5\pi}{6}$ and the max value is = 3/2

Hence, the max value = 3/2 and min value = 1.

33. Given $f(x) = 3x^4 - 2x^3 - 3x^2 + 10$

$\Rightarrow f'(x) = 12x^3 - 6x^2 - 6x$

$\Rightarrow f'(x) = 6(2x^3 - x^2 - x)$

$\Rightarrow f''(x) = 6(6x^2 - 2x - 1)$

Now, $f'(x) = 0$ gives $x(2x^2 - x - 1) = 0$

$\Rightarrow x(x - 1)(2x + 1) = 0$

$\Rightarrow x = 0, 1, -1/2$

At $x = 0, f''(x) = -6 < 0,$

Thus f is max and the max value $= f(0) = 10$

At $x = 1$, $f''(x) = 18 > 0$, so f is min and the min value $= f(1) = 8$.

At $x = -1/2, f''(x) = 6\left(\frac{6}{4} + 2\cdot\frac{1}{2} - 1\right) = 9 > 0$

so f is min and the min value

$$= f\left(-\frac{1}{2}\right) = \frac{3}{16} + \frac{2}{8} - \frac{3}{4} + 10$$

$$= 10 - \frac{5}{16} = \frac{155}{16} = 9\frac{11}{16}$$

Hence, the max value = 10 and the min value = 9.

34. Given $f(x) = \ln(x^4 - 2x^2 + 3)$

Let $g(x) = (x^4 - 2x^2 + 3)$

So g is max or min, then f is max or min.

Now, $g'(x) = 4x^3 - 4x$

$\Rightarrow g''(x) = 12x^2 - 4$

$\Rightarrow g''(x) = 4(3x^2 - 1)$

Now, $g'(x) = 0$ gives $x^3 - x = 0$

$\Rightarrow x(x + 1)(x - 1) = 0$

$\Rightarrow x = 0, -1, 1.$

At $x = 0, g''(x) = -4 < 0$, so g is max

Thus f is max at $x = 0$ and the max value $= \ln 3$

At $x = -1$, $g''(x) = 8 > 0$, so g is min

Thus f is min at $x = -1$ and the min value $= \ln 2$

At $x = -1$, $g''(x) = 8 > 0$, so g is min.

Thus f is min at $x = 1$ and the min value $= \ln 2$.

Hence the min value $= \ln 2$ and max value $= \ln 3$.

35. Given $f(x) = \sqrt{3x^2 - 2x^3}$

Let $g(x) = 3x^2 - 2x^3$

So g is max or min, then f is max or min.

$\Rightarrow g'(x) = 6x - 6x^2$

$\Rightarrow g''(x) = 6 - 12x$

Now, $g'(x) = 0$ gives $x - x^2 = 0$

$\Rightarrow x = 0, 1$

At $x = 0$, $g''(x) = 6 > 0$, g is min

Thus, f is min and the min value = 0

At $x = 1$, $g''(x) = 6 - 12 = -6 < 0$, g is max

Thus f is max and the max value = 1.

36. Given $f(x) = \dfrac{10}{3x^4 + 4x^3 - 12x^2 + 11}$

Let $g(x) = 3x^4 + 4x^3 - 12x^2 + 11$

So g is max or min, then f is min or max.

Now, $g'(x) = 12x^3 + 12x^2 - 24x$

$\Rightarrow g'(x) = 12(x^3 + x^2 - 2x)$

$\Rightarrow g''(x) = 12(3x^2 + 2x - 2)$

Now, $g'(x) = 0$ gives $x(x^2 + x - 2) = 0$

$\Rightarrow x(x + 2)(x - 1) = 0$

$\Rightarrow x = -2, 0, 1.$

At $x = -2$, $g''(x) = 72 > 0$, so g is min

Thus, f is max and the max value $= -\frac{1}{21}$

At $x = 0$, $g''(x) = -24 < 0$, g is max

Thus f is min and the min value is $= \frac{1}{11}$

At $x = 1$, $g''(x) = 36 > 0$, so g is min

Thus, f is max and the max value $= \frac{1}{6}$

Hence, the max value $= \frac{1}{6}$ and min value $= \frac{1}{11}$.

37. Given $f(x) = x^5 + 5x^4 + 5x^3 - 1$

$\Rightarrow f'(x) = 5x^4 + 20x^3 + 15x^2$

$\Rightarrow f'(x) = 5x^2(x + 1)(x + 3)$

$\Rightarrow f''(x) = 5(4x^3 + 12x^2 + 6x)$

$\Rightarrow f'''(x) = 5(12x^2 + 24x + 6)$

Now, $f'(x) = 0$ gives $x = -3, -1, 0$

At $x = -3, f''(x) = -90 < 0$

So, f is max and the max value = 26

At $x = -1, f''(x) = 10 > 0$

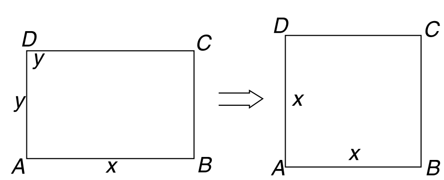
So f is min and the min value = -2

At $x = 0, f''(x) = 0$

Now, $f'''(0) = 30$.

So f has neither max nor min at $x = 0$.

38.



Let x and y be the sides of a rectangle.

Given $P = 2(x + y)$

$\Rightarrow y = \left(\frac{p}{2} - x\right)$

Now, $A = xy = x\left(\frac{p}{2} - x\right)$

$$\frac{dA}{dx} = \left(\frac{p}{2} - x\right) \cdot x + x(-1) = \left(\frac{p}{2} - 2x\right)$$

$$\frac{d^2A}{dx^2} = -2 < 0$$

So area is maximum.

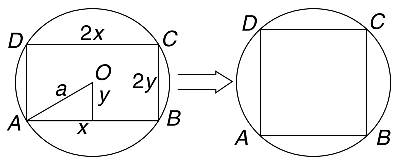
For max or min, $\frac{dA}{dx} = 0$ gives $x = \frac{p}{4}$

when $x = \frac{p}{4}$, then $y = \frac{p}{2} - \frac{p}{4} = \frac{p}{4}$

$\Rightarrow x = y$

$\Rightarrow$ Rectangles becomes square.

39.



Let the lengths of the sides be $2x$ and $2y$ respectively

and a be the radius of the circle.

Now, $x^2 + y^2 = a^2$

$\Rightarrow \quad y = \sqrt{a^2 - x^2}$

We have, $A = 4xy = 4x\sqrt{a^2 - x^2}$

$\Rightarrow \quad A^2 = 16x^2(a^2 - x^2)$

$\Rightarrow \quad B = 16(a^2x^2 - x^4)$

$\Rightarrow \quad \frac{dB}{dx} = 16(2a^2x - x^3)$

$\Rightarrow \quad \frac{d^2B}{dx^2} = 16(2a^2 - 12x^2)$

For max or min, $\frac{dB}{dx} = 0$ gives

$\Rightarrow \quad 2a^2x - 4x^3 = 0$

$\Rightarrow \quad a^2 - 2x^2 = 0$

$\Rightarrow \quad x = \frac{a}{\sqrt{2}}$

At $\quad x = \frac{a}{\sqrt{2}}, \frac{d^2B}{dx^2} = 16(2a^2 - 6a^2) = -64a^2 < 0$

So, B is maximum

Thus, area is maximum.

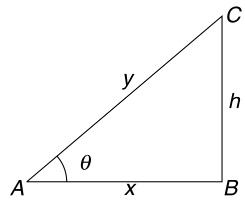
when $\quad x = \frac{a}{\sqrt{2}}$, then $y = \sqrt{a^2 - \frac{a^2}{2}} = \frac{a}{\sqrt{2}}$

$\Rightarrow \quad x = y$

$\Rightarrow \quad 2x = 2y$

$\Rightarrow$ Rectangles become squares

40. Let base = x and hypotenuse = y resp and $x + y = k$



Here, $\quad x^2 + h^2 = y^2$

$\Rightarrow \quad h^2 = y^2 - x^2$

$\Rightarrow \quad h = \sqrt{y^2 - x^2}$

Area of the triangle

$$= A = \frac{1}{2} \times h = \frac{1}{2}\sqrt{y^2 - x^2}$$

$\Rightarrow \quad A = \frac{x}{2}\sqrt{(k - x)^2 - x^2}$

$\Rightarrow \quad A = \frac{x}{2}\sqrt{k^2 - 2kx}$

$\Rightarrow \quad B = A^2 = \frac{x^2}{4}(k^2 - 2kx)$

$\Rightarrow \quad B = \frac{1}{4}(k^2x^2 - 2kx^3)$

$\Rightarrow \quad \frac{dB}{dx} = \frac{1}{4}(2k^2x - 6kx^2)$

$\Rightarrow \quad \frac{d^2B}{dx^2} = \frac{1}{4}(2k^2 - 12kx)$

For max or min, $\frac{dB}{dx} = 0$ gives $x = \frac{k}{3}$

At $\quad x = \frac{k}{3}, \frac{d^2B}{dx^2} = \frac{1}{4}(2k^2 - 4k^2) = -\frac{k^2}{2} < 0$

So, B is maximum

Thus, area is maximum

when $\quad x = \frac{k}{3}$, then $y = k - x = \frac{2k}{3}$

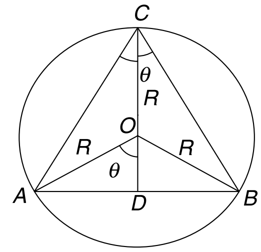
Let θ be the angle between them

$\Rightarrow \quad \cos\theta = \frac{x}{y} = \frac{k/3}{2k/3} = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}$

Hence, the result.

41.



Here, $\cos\theta = \frac{OD}{R}$ and $\sin\theta = \frac{AD}{R}$

$\Rightarrow \quad OD = R\cos\theta,\ AD = R\sin\theta$

We have, $A = \frac{1}{2}\cdot AB\cdot CD$

$\Rightarrow \quad A = \frac{1}{2}\cdot(2R\sin\theta)\cdot(R + R\cos\theta)$

$\Rightarrow \quad A = R^2\cdot(\sin\theta + \sin\theta\cos\theta)$

$\Rightarrow \quad \frac{dA}{d\theta} = R^2\cdot(\cos\theta + \cos 2\theta)$

$\Rightarrow \quad \frac{d^2A}{d\theta} = R^2(-\sin\theta - 2\sin 2\theta)$

For max or min, $\frac{dA}{d\theta} = 0$ gives

$\Rightarrow \quad \cos\theta + \cos 2\theta = 0$

$\Rightarrow \quad 2\cos^2\theta - 1 + \cos\theta = 0$

$\Rightarrow \quad 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \quad 2\cos^2\theta + 2\cos\theta - \cos\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(1 + \cos\theta) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}, -1$

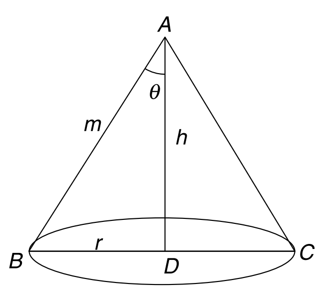
$\Rightarrow \quad \theta = \frac{\pi}{3}$

At $\quad \theta = \frac{\pi}{3}, \frac{d^2A}{d\theta^2} < 0$

So area is maximum.

Thus, the triangle is equilateral, where $\theta = \frac{\pi}{3}$.

42.



Let m = slant height, radius = r of the cone

Also, $\quad \sin\theta = \frac{r}{m}, \cos\theta = \frac{h}{m}$

$\Rightarrow \quad r = m\sin\theta,\ h = m\cos\theta$

Volume of the cone = V

$$= \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(m\sin\theta)^2(m\cos\theta)$$

$$= \frac{m^3\pi}{3}(\sin^2\theta\cos\theta)$$

$$= \frac{m^3\pi}{3}(\cos\theta - \cos^3\theta)$$

$\Rightarrow \quad \frac{dV}{d\theta} = \frac{m^3\pi}{3}(-\sin\theta + 3\cos^2\theta\cdot\sin\theta)$

$\Rightarrow \quad \frac{dV}{d\theta} = \frac{m^3\pi}{3}(2\sin\theta - \sin^3\theta)$

$\Rightarrow \quad \frac{d^2V}{d\theta^2} = \frac{m^3\pi}{3}(2\cos\theta - 9\sin^2\theta\cos\theta)$

For max or min, $\frac{dV}{d\theta} = 0$ gives $2\sin\theta - 3\sin^3\theta = 0$

$\Rightarrow \quad \sin\theta = \sqrt{\frac{2}{3}}$

$\Rightarrow \quad \tan\theta = \sqrt{2}$

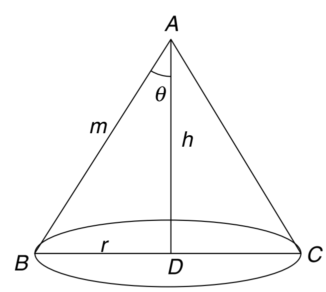
$\Rightarrow \quad \theta = \tan^{-1}(\sqrt{2})$

At $\quad \sin\theta = \sqrt{\frac{3}{2}}, \frac{d^2V}{d\theta^2} < 0$

So, the volume is maximum.

Hence, the semi vertical angle of a right circular cone $= \tan^{-1}(\sqrt{2})$.

43.



Let r be the radius, m be the slant height and h be the height of the right circular cone.

Given surface area = constant

$\Rightarrow \quad \pi r^2 + \pi r m = k$

$$m = \frac{k - \pi r^2}{\pi r}$$

Volume of the cone

$$= V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi r^2\sqrt{m^2 - r^2}$$

$\Rightarrow \quad V^2 = \frac{\pi^2}{9}\times r^4(m^2 - r^2)$

$\Rightarrow \quad z = \frac{\pi^2}{9}\times(m^2r^4 - r^6)$

$\Rightarrow \quad z = \frac{\pi^2}{9} \times \left(r^4\left(\frac{k - \pi r^2}{\pi r}\right)^2 - r^6\right)$

$\Rightarrow \quad z = \frac{\pi^2}{9} \times \left(r^4\left(\frac{k^2 - 2k\pi r^2 + \pi^2 r^4}{\pi^2 r^2}\right) - r^6\right)$

$\Rightarrow \quad z = \frac{\pi^2}{9} \times \frac{1}{\pi^2 r^2} \times (r^4(k^2 - 2k\pi r^2 + \pi^2 r^4) - \pi^2 r^8)$

$\Rightarrow \quad z = \frac{1}{9} \times (r^2(k^2 - 2k\pi r^2))$

$\Rightarrow \quad z = \frac{1}{9} \times (k^2 r^2 - 2k\pi r^4)$

$\Rightarrow \quad \frac{dz}{dr} = \frac{1}{9} \times (2k^2 r - 8k\pi r^3)$

$\Rightarrow \quad \frac{d^2z}{dr^2} = \frac{1}{9} \times (2k^2 - 24k\pi r^2)$

For max or min, $\frac{dz}{dr} = 0$ gives

$2k^2 r - 8k\pi r^3 = 0$

$\Rightarrow \quad k - 4\pi r^2 = 0$

$\Rightarrow \quad r = \sqrt{\frac{k}{4\pi}}$

At $\quad r = \sqrt{\frac{k}{4\pi}}, \frac{d^2z}{dr^2} < 0$

So, z is maximum

Thus, volume is maximum.

when $r = \sqrt{\frac{k}{4\pi}}$, then $m = \frac{3}{2}\sqrt{\frac{k}{\pi}}$

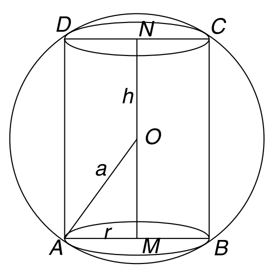
Let θ be the semi vertical angle of the cone.

Then $\quad \sin\theta = \frac{r}{m} = \frac{\frac{1}{2}\sqrt{\frac{k}{\pi}}}{\frac{3}{2}\sqrt{\frac{k}{\pi}}} = \frac{1}{3}$

$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{3}\right)$

Hence, the result.

44.



Here, a = radius of the sphere

r = radius and h = height of the cylinder resp.

Now, $\quad r^2 + \frac{h^2}{4} = a^2$

$\Rightarrow \quad h = 2\sqrt{a^2 - r^2}$

Volume of the cylinder

$= V = \pi r^2 h = \pi r^2 2\sqrt{a^2 - r^2}$

$\Rightarrow \quad V = 2\pi r^2 \sqrt{a^2 - r^2}$

$\Rightarrow \quad z = V^2 = 4\pi^2 r^4(a^2 - r^2)$

$\Rightarrow \quad z = 4\pi^2(a^2 r^4 - r^6)$

$\Rightarrow \quad \frac{dz}{dr} = 4\pi^2(4a^2 r^3 - 6r^5)$

$\Rightarrow \quad \frac{d^2z}{dr^2} = 4\pi^2(12a^2 r^2 - 30r^4)$

For max or min, $\frac{dz}{dr} = 0$ gives

$\Rightarrow \quad 4a^2 r^3 - 6r^5 = 0$

$\Rightarrow \quad 2a^2 - 3r^2 = 0$

$\Rightarrow \quad r = \sqrt{\frac{2}{3}}a$

When $\quad r = \sqrt{\frac{2}{3}}a$, then

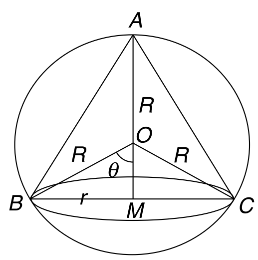
$\Rightarrow \quad h = 2\sqrt{a^2 - \frac{2}{3}a^2} = \frac{2a}{\sqrt{3}}$

At $\quad r = \sqrt{\frac{2}{3}}a, \frac{d^2z}{dr^2} < 0$

So z is maximum

Thus, Volume is maximum.

45.



Let R be the radius of the sphere and r be the radius of the cone.

$\sin\theta = \frac{r}{R}$ and $\cos\theta = \frac{OM}{R}$

Let V_1 be the volume of the sphere and V_2 be the volume of the cone

To prove, $V_2 = \frac{8}{27} \times V_1$

Now, $\quad V_1 = \frac{4}{3}\pi R^3$

and $V_2 = \pi r^2 h$

$\Rightarrow \quad V_2 = \pi r^2 (OA + OM)$

$\Rightarrow \quad V_2 = \pi R^2 \sin^2\theta \cdot (R + R\cos\theta)$

$\Rightarrow \quad V_2 = \pi R^3 \cdot (\sin^2\theta + \sin^2\theta\cos\theta)$

$\Rightarrow \quad \frac{dV_2}{d\theta} = \pi R^3 (\sin 2\theta + \sin 2\theta\cos\theta - \sin^3\theta)$

$\Rightarrow \quad \frac{d^2V_2}{d\theta^2} = \pi R^3 [2\cos 2\theta + 2\cos 2\theta\cos\theta - \sin 2\theta\sin\theta - 3\sin^2\theta\cos\theta]$

For max or min, $\frac{dV_2}{d\theta} = 0$ gives

$\sin 2\theta + \sin 2\theta\cos\theta - \sin^3\theta = 0$

$\Rightarrow \quad 2\cos\theta + 2\cos^2\theta - \sin^2\theta = 0$

$\Rightarrow \quad 3\cos^2\theta + 2\cos\theta - 1 = 0$

$\Rightarrow \quad (3\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{3}$

Thus, $\sin\theta = \frac{2\sqrt{2}}{3}$

At $\cos\theta = \frac{1}{3}$ and $\sin\theta = \frac{2\sqrt{2}}{3}$,

$\Rightarrow \quad \frac{d^2V_2}{d\theta^2} < 0$

So, the volume of the cone is maximum.

Thus, $V_2 = \pi r^2 h = \pi(R\sin\theta)^2(R + R\cos\theta)$

$= \pi R^3 \cdot \sin^2\theta \cdot (1 + \cos\theta)$

$= \pi R^3 \left(\frac{2\sqrt{2}}{3}\right)^2 \cdot \frac{4}{3}$

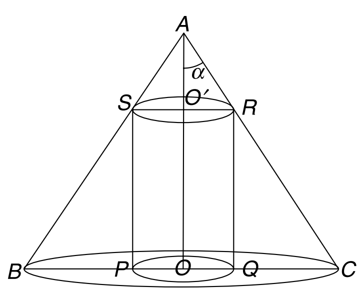
$= \pi R^3 \cdot \frac{8}{9} \cdot \frac{4}{3}$

$= \frac{8}{27} \times \left(\frac{4}{3}\pi R^3\right)$

$= \frac{8}{27} \times V_1$

Hence, the result.

46.



Let r be the radius of the cylinder

Now, $O'A = r\cot\alpha$

Height of the cylinder $= OO' = h - r\cot\alpha$

Volume of the cylinder

$= V = \pi r^2 (h - r\cot\alpha)$

$\Rightarrow \quad V = \pi(r^2 h - r^3 \cot\alpha)$

$\Rightarrow \quad \frac{dV}{dr} = \pi(2rh - 3r^2\cot\alpha)$

$\Rightarrow \quad \frac{d^2V}{dr^2} = \pi(2h - 6r\cot\alpha)$

For max or min, $\frac{dV}{dr} = 0$ gives

$2rh - 3r^2\cot\alpha = 0$

$\Rightarrow \quad r = \frac{2h}{3\cot\alpha}$

$\Rightarrow \quad r = \frac{2h}{3\tan\alpha}$

At $r = \frac{2h}{3}\tan\alpha, \frac{d^2V}{dh^2} = -2\pi h < 0$

So, volume is maximum

when $r = \frac{2h}{3}\tan\alpha$, then

height $= h - r\cot\alpha = h - \frac{2}{3}h = \frac{1}{3}h$

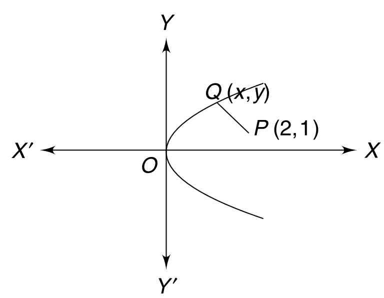
Thus, Volume of the cylinder

$= \pi\left(\frac{2h}{3}\tan\alpha\right)^2 \cdot \frac{h}{3}$

$= \left(\frac{4}{27}\right) \times (\pi h^3 \tan^2\alpha)$

Hence, the result.

47.



Let $P = (2, 1)$ and $Q = (x, y)$

$PQ = \sqrt{(x - 2)^2 + (y - 1)}^2$

$\Rightarrow \quad PQ^2 = (x - 2)^2 + (y - 1)^2$

$\Rightarrow \quad z = PQ^2 = \left(\frac{y^2}{4} - 2\right)^2 + (y - 1)^2$

$\Rightarrow \quad \frac{dz}{dy} = 2\left(\frac{y^2}{4} - 2\right)\left(\frac{2y}{4}\right) + 2(y - 1)^2$

$\Rightarrow \quad \frac{dz}{dy} = \left(\frac{y^3}{4} - 2y\right) + 2(y - 1)$

$\Rightarrow \quad \frac{dz}{dy} = \left(\frac{y^3}{4} - 2\right)$

$\Rightarrow \quad \frac{d^2z}{dy^2} = \frac{3y^2}{4}$

For max or min, $\frac{dz}{dy} = 0$

$\Rightarrow \quad \left(\frac{y^3}{4} - 2\right) = 0$

$\Rightarrow \quad y = 2$

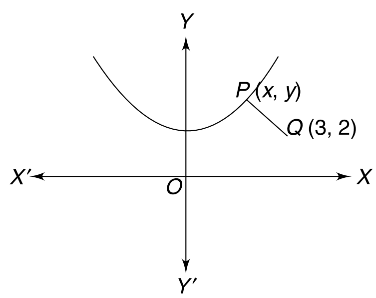
At $\quad y = 2, \frac{d^2z}{dy^2} = 3 > 0$

So, z is minimum

Thus, distance is minimum when $y = 2$, then $x = 1$

Hence, the point $Q = (1, 2)$.

48. Given curve is $y = x^2 + 2$



Let the closest point of Q be $P(x, y)$

Now, $\quad PQ = \sqrt{(x - 3)^2 + (y - 2)^2}$

$\Rightarrow \quad PQ^2 = (x - 3)^2 + (y - 2)^2$

$\Rightarrow \quad z = PQ^2 = (x - 3)^2 + x^4$

$\Rightarrow \quad \frac{dz}{dx} = 2(x - 3) + 4x^3$

$\Rightarrow \quad \frac{d^2z}{dx^2} = 2 + 12x^2$

For max or min, $\frac{dz}{dx} = 0$ gives

$\Rightarrow \quad 2(x - 3) + 4x^3 = 0$

$\Rightarrow \quad (x - 3) + 2x^3 = 0$

$\Rightarrow \quad 2x^3 + x - 3 = 0$

$\Rightarrow \quad x = 1$

At $\quad x = 1, \frac{d^2z}{dx^2} = 14 > 0$

So, z is minimum Thus, the distance is minimum. Hence, the minimum distance

$= \sqrt{4 + 1} = \sqrt{5}$

49. Given curve is $y = x^2$

Any point on the given parabola is (x, x^2)

Distance between (x, x^2) and $(0, c)$ is

$z = \sqrt{(x^2) + (x^2 - c)^2}$

$\Rightarrow z^2 = (x^2) + (x^2 - c)^2 = x^4 - (2c - 1)x^2 + c^2$

$= \left(x^2 - \left(\frac{2c - 1}{2}\right)\right)^2 + \left(c^2 - \left(\frac{2c - 1}{2}\right)\right)^2$

$= \left(x^2 - \left(\frac{2c - 1}{2}\right)\right)^2 + \left(c^2 - \left(\frac{4c^2 - 4c + 1}{4}\right)\right)$

$= \left(x^2 - \left(\frac{2c - 1}{2}\right)\right)^2 + \left(c - \frac{1}{4}\right)$

Which is minimum when $x^2 = \left(\frac{2c - 1}{2}\right)$ and the minimum value is

$= z_{\min} = \sqrt{\left(c - \frac{1}{4}\right)}$

50. Given curve is $4x^2 + a^2y^2 = 4a^2$

$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{4} = 1 \quad$ ...(i)

Any point of (i) is $P(a\cos\theta, 2\sin\theta)$ and given point is $(0, -2)$

Let $z = PQ^2 = (a\cos\theta)^2 + (2\sin\theta + 2)^2$

$\Rightarrow \quad \frac{dz}{d\theta} = 2(a\cos\theta)(-a\sin\theta) + 2(2\sin\theta + 2)(2\cos\theta)$

$\Rightarrow \quad \frac{dz}{d\theta} = 2\cos\theta(4(1 + \sin\theta) - a^2\sin\theta)$

$\Rightarrow \quad \frac{dz}{d\theta} = 2\cos\theta(4 + (4 - a^2)\sin\theta)$

$\Rightarrow \quad \frac{d^2z}{d\theta} = 2\cos^2\theta(4 - a^2) - 2(4 - a^2)\sin\theta)\sin\theta$

For max or min, $\frac{dz}{d\theta} = 0$

$\Rightarrow \quad 2\cos\theta(4 + (4 - a^2)\sin\theta) = 0$

$\Rightarrow \quad 2\cos\theta = 0, (4 + (4 - a^2)\sin\theta) = 0$

$\Rightarrow \quad \cos\theta = 0,\ \sin\theta = \dfrac{4}{4 - a^2}$

$\Rightarrow \quad \theta = \dfrac{\pi}{2},\ \sin\theta = \dfrac{4}{4 - a^2}$

$\Rightarrow \quad \theta = \dfrac{\pi}{2},\ \sin\theta = \dfrac{4}{4 - a^2} > 1$

$\Rightarrow \quad \theta = \dfrac{\pi}{2}$

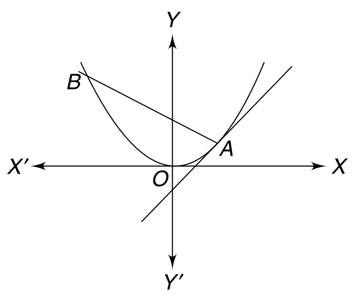
$\Rightarrow \quad \left(\dfrac{d^2z}{d\theta}\right)_{\theta=\frac{\pi}{2}} = 2(a^2 - 8) < 0$

So z is maximum

Thus, distance is maximum.

Hence, the co-ordinates of the point P be (0, 2)

51. Given curve is $y = x^2$



Let $\quad A = (t, t^2)$

Equation of the tangent at $A = (t, t^2)$ is

$$y - t^2 = \left(-\frac{1}{2t}\right)(x - t)$$

Let the normal intersects the curve again at B

where $\quad B = (t_1, t_1^2)$

Thus, $\quad \left(t_1^2 - t^2\right) = \left(-\dfrac{1}{2t}\right)(t_1 - t)$

$\Rightarrow \quad (t_1 + t) = \left(-\dfrac{1}{2t}\right)$

$\Rightarrow \quad t_1 = -\left(t + \dfrac{1}{2t}\right)$

Now, $\quad AB = \sqrt{(t - t_1)^2 + (t^2 - t_1^2)^2}$

$\Rightarrow \quad AB^2 = (t - t_1)^2 + (t^2 - t_1^2)^2$

$\Rightarrow \quad AB^2 = (t - t_1)^2(1 + (t + t_1)^2)$

$\Rightarrow \quad L = \left(2t + \dfrac{1}{2t}\right)^2\left(1 + \dfrac{1}{4t^2}\right)$

$\Rightarrow \quad L = 4t^2\left(1 + \dfrac{1}{4t^2}\right)^3$

$$\frac{dL}{dt} = 8t\left(1 + \frac{1}{4t^2}\right)^3 - 12t^2\left(1 + \frac{1}{4t^2}\right)^2\left(\frac{2}{4t^3}\right)$$

$$\frac{dL}{dt} = 4\left(1 + \frac{1}{4t^2}\right)^2\left(2t - \frac{1}{t}\right)$$

For max/min, $\dfrac{dL}{dt} = 0$

$\Rightarrow \quad 4\left(1 + \dfrac{1}{4t^2}\right)^2\left(2t - \dfrac{1}{t}\right) = 0$

$\Rightarrow \quad t^2 = \dfrac{1}{2}$

$\Rightarrow \quad t = \pm\dfrac{1}{\sqrt{2}}$

$$\left(\frac{d^2L}{dt^2}\right)_{t = \pm\frac{1}{\sqrt{2}}} = 16 \times \frac{9}{4} = 36 > 0$$

So L is minimum.

When $\quad t = \dfrac{1}{\sqrt{2}}$ then

$A = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$ & $B = (-\sqrt{2}, 2)$ and $AB = \dfrac{27}{4}$

Thus, the equation of AB is

$$\frac{y - 2}{\frac{1}{2} - 2} = \frac{x + 2}{\frac{1}{\sqrt{2}} + \sqrt{2}}$$

$\Rightarrow \quad \dfrac{y - 2}{\frac{1}{\sqrt{2}} - \sqrt{2}} = \dfrac{x + 2}{1}$

$\Rightarrow \quad \sqrt{2x} + 2y - 2 = 0$

When $\quad t = -\dfrac{1}{\sqrt{2}}$, then

$A = \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$ & $B = (\sqrt{2}, 2)$ and $AB = \dfrac{27}{4}$

Thus, the equation of AB is

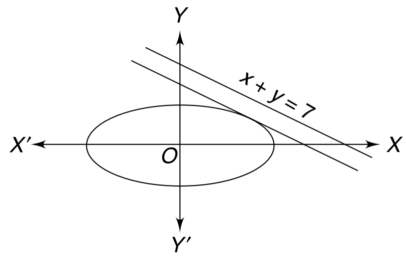
$$\frac{y - 2}{\frac{1}{2} - 2} = x - \frac{\sqrt{2}}{-\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)}$$

$\Rightarrow \quad \dfrac{y - 2}{\frac{1}{\sqrt{2}} - \sqrt{2}} = \dfrac{x - \sqrt{2}}{-1}$

$\Rightarrow \quad \sqrt{2}x - 2y + 2 = 0$

52. Given curve is $x^2 + 2y^2 = 6$

$\Rightarrow \quad \dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$



Since the tangent is parallel to the line, so their slopes are same

$$\frac{dy}{dx} = -1$$

$\Rightarrow \quad \frac{-x}{2y} = -1$

$\Rightarrow \quad x = 2y$

when $\quad x = 2y$, then $4y^2 + 2y^2 = 6$

$\Rightarrow \quad 6y^2 = 6$

$\Rightarrow \quad y^2 = 1$

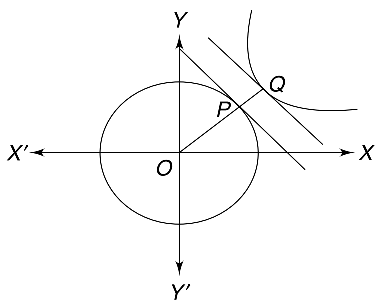
$\Rightarrow \quad y = \pm 1$

when $\quad x = \pm 1$, then $x = \pm 2$

So, the point on the curve is

(2, 1) or (–2, –1).

53.



Given curves are $x^2 + y^2 = 2$ and $xy = 9$

Since both the tangents at P and Q are parallel, so their slopes are same

$$-\frac{x}{y} = -\frac{9}{x^2}$$

$\Rightarrow \quad x^3 = 9y$

$\Rightarrow \quad x^3 = 9 \cdot \frac{9}{x} = \frac{81}{x}$

$\Rightarrow \quad x^4 = 81$

$\Rightarrow \quad x = 3$

when $x = 3$, then $y = 3$

Thus, the point $Q = (3, 3)$

Therefore, the shortest distance

$= PQ$

$= OQ - OP$

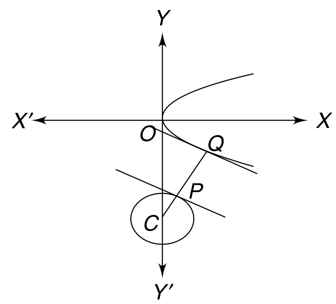
$= \sqrt{9 + 9} - \sqrt{2}$

$= 3\sqrt{2} - \sqrt{2}$

$= 2\sqrt{2}$

54. Given curves are

$$y^2 = 4x \text{ and } x^2 + (y + 12)^2 = 1$$



Since both the tangents at P and Q are parallel. So their slopes are same.

$$-\frac{2}{y} = -\frac{x}{(y + 12)}$$

$\Rightarrow \quad \frac{2}{y} = \frac{x}{(y + 12)}$

$\Rightarrow \quad 2y + 24 = xy$

$\Rightarrow \quad 2y + 24 = y \cdot \frac{y^2}{4}$

$\Rightarrow \quad y^3 - 8y - 96 = 0$

$\Rightarrow \quad y = 4$

when $y = 4$, then $x = 4$

So the point Q is (4, 4)

Here, $\quad C = (0, -12)$

Thus, Shortest distance

$= PQ$

$= CQ - CP$

$= \sqrt{16 + 64} - 1$

$= \sqrt{80} - 1$

$= 2\sqrt{5} - 1$

55. Given $\quad f(x) = ax + \frac{b}{x}$, $a, b, x > 0$

Applying A.M ≥ G.M, we get,

$\Rightarrow \quad \frac{ax + \frac{b}{x}}{2} \geq \sqrt{ax \cdot \frac{b}{x}} = \sqrt{ab}$

$\Rightarrow \quad ax + \frac{b}{x} \geq 2\sqrt{ab}$

Thus, the min value of $f(x)$ is $2\sqrt{ab}$

56. We have, $f(x) = x^2 + \frac{1}{x^2+1}$

$\Rightarrow \quad f(x) = \left(x^2 + 1 + \frac{1}{x^2+1}\right) - 1$

$\Rightarrow \quad f(x) \geq 2 - 1 = 1$

Hence, the least value of $f(x)$ is 1.

57. We have, $f(x) = \frac{x^3 + x + 2}{x}, x > 0$

$\Rightarrow \quad f(x) = x^2 + \frac{2}{x} + 1$

$\Rightarrow \quad f(x) = \left(x^2 + \frac{1}{x} + \frac{1}{x}\right)$

$\Rightarrow \quad f(x) \geq 3 + 1 = 4$

Hence, the min value of $f(x)$ is 4.

58. We have $f(x) = 2\cos x + \sec^2 x$

$\Rightarrow \quad f(x) = \cos x + \cos x + \sec^2 x$

$\Rightarrow \quad f(x) = \cos x + \cos x + \sec^2 x \geq 3$

Hence, the min value of $f(x)$ is 3.

59. We have $f(x) = 2\log_{10} x - \log_x(0.01), x > 1$

$\Rightarrow \quad f(x) = 2\log_{10} x - \log_x(10)^{-2}$

$\Rightarrow \quad f(x) = 2\log_{10} x + 2\log_x(10)$

$\Rightarrow \quad f(x) = 2(\log_{10} x + \log_x(10))$

$\Rightarrow \quad f(x) = 2(\log_{10} x + \log_x(10)) \geq 4$

Hence, the min value of $f(x)$ is 4.

60. We have, $f(x) = \frac{a^2}{\sin^2 x} + \frac{b^2}{\cos^2 x}$

$\Rightarrow \quad f(x) = \frac{a^2}{\sin^2 x} + \frac{b^2}{\cos^2 x}$

$\Rightarrow \quad f(x) = a^2 \operatorname{cosec}^2 x + b^2 \sec^2 x$

$\Rightarrow \quad f(x) = a^2(1 + \cos^2 x) + b^2(1 + \tan^2 x)$

$\Rightarrow \quad f(x) = a^2 + b^2 + (a^2\cot^2 x + b^2\tan^2 x)$

$\Rightarrow \quad f(x) \geq a^2 + b^2 + 2ab = (a+b)^2$

Hence, the min value of $f(x)$ is $(a+b)^2$

61. We have

$f(x) = 2^x + 3^x + 5^x + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{5^x}, x > 0$

$\Rightarrow \quad f(x) = \left(2^x + \frac{1}{2^x}\right) + \left(3^x + \frac{1}{3^x}\right) + \left(5^x + \frac{1}{5^x}\right)$

$\Rightarrow \quad f(x) \geq 2 + 2 + 2 = 6$

Hence, the min value of $f(x)$ is 6.

62. We have

$f(a) = a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10}, a > 0$

$\Rightarrow f(a) = a^{10} + a^8 + 1 + \frac{1}{a^5} + \frac{1}{a^4} + \frac{1}{a^3} + \frac{1}{a^3} + \frac{1}{a^3}$

$\Rightarrow f(a) = \left(a^{10} + a^8 + \frac{1}{a^5} + \frac{1}{a^4} + \frac{1}{a^3} + \frac{1}{a^3} + \frac{1}{a^3}\right) + 1$

$\Rightarrow f(x) \geq 7 + 1 = 8$

Hence, the min value of $f(a)$ is 8.

63. Given $\quad f(x) = x^{10} + x^7 + \frac{2}{x^3} + \frac{4}{x^2} + \frac{3}{x}, x > 0$

$\Rightarrow \quad f(x) = x^{10} + x^7 + \left(\frac{1}{x^3} + \frac{1}{x^3}\right)$

$+ \left(\frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{x^2}\right) + \left(\frac{1}{x} + \frac{1}{x} + \frac{1}{x}\right)$

$\Rightarrow \quad f(x) \geq 11$

Min value of $f(x)$ is 11.

64. Given

$f(a, b, c, d) = \frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd}$

$= \left(\frac{a^2+1}{a}\right)\left(\frac{b^2+1}{b}\right)\left(\frac{c^2+1}{c}\right)\left(\frac{d^2+1}{d}\right)$

$= \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right)\left(d + \frac{1}{d}\right)$

$\geq 2.2.2.2 = 16$

Hence, the min value of $f(a, b, c, d)$ is 16.

65. We have $2x + 3y = 5$

$\Rightarrow \quad (x + x + y + y + y) = 5$

Now, $\quad \left(\frac{x + x + y + y + y}{5}\right) \geq \sqrt{x^2 \cdot y^3}$

$\Rightarrow \quad \left(\frac{5}{3}\right) \geq \sqrt{x^2 \cdot y^3}$

$\Rightarrow \quad x^2 \cdot y^3 \leq 1$

Hence, the max value of $x^2 \cdot y^3$ is 1.

66. We have, $3x + 2y + z = 14$

Now, $\quad \left(\frac{x + x + x + y + y + z}{6}\right) \geq \sqrt[6]{x^3 y^2 z}$

$\Rightarrow \quad \left(\frac{3x + 2y + z}{6}\right) \geq \sqrt[6]{x^3 y^2 z}$

$\Rightarrow \quad \left(\frac{14}{6}\right) \geq \sqrt[6]{x^3 y^2 z}$

$\Rightarrow \quad x^3y^2z \leq \left(\frac{14}{6}\right)^6 = \left(\frac{7}{3}\right)^6$

Hence, the max value of x^3y^2z is $\left(\frac{7}{3}\right)^6$

67. We have

$$\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}{3} \geq \sqrt[3]{\left(\frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{c^2}\right)}$$

$\Rightarrow \quad \frac{1}{3} \geq \sqrt[3]{\left(\frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{c^2}\right)}$

$\Rightarrow \quad \left(\frac{1}{3}\right)^3 \geq \left(\frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{c^2}\right)$

$\Rightarrow \quad \left(\frac{1}{3}\right)^3 \geq \left(\frac{xyz}{abc}\right)^2$

$\Rightarrow \quad (xyz) \leq \left(\frac{abc}{3^{3/2}}\right)$

Hence, the max value of xyz is $\left(\frac{abc}{3^{3/2}}\right)$

68. We have $y = \frac{x}{ax^2+b}$, $a, b, x > 0$

$\Rightarrow \quad y = \frac{1}{ax + \frac{b}{x}}$

$\Rightarrow \quad y \leq \frac{1}{2\sqrt{ab}}$

Hence, the max value of y is $\frac{1}{2\sqrt{ab}}$

69. We have

$$f(x) = x^2 + 1 + \frac{4}{x^2+3}$$

$$= (x^2+3) + \frac{4}{(x^2+3)} - 2 \geq 4 - 2 = 2$$

Hence, the minimum value of $f(x)$ is 2

70. We have

$$f(x, y, z) = \frac{(x^3+2)(x^3+2)(x^3+2)}{xyz}$$

$$= \left(\frac{x^3+2}{x}\right)\left(\frac{y^3+2}{y}\right)\left(\frac{z^3+2}{z}\right)$$

$$= \left(x^2+\frac{2}{x}\right)\left(y^2+\frac{2}{y}\right)\left(z^2+\frac{2}{z}\right)$$

$$\geq 3.3.3 = 27$$

Hence, the minimum value of $f(x)$ is 27.

Level III

1. We have

$$f(x) = \frac{ab(a^2-b^2)\sin x\cos x}{a^2\sin^2 x + b^2\cos^2 x},\ x \in \left(0, \frac{\pi}{2}\right)$$

$$= \frac{ab(a^2-b^2)}{a^2\tan x + b^2\cot x}$$

As we know that, A.M ≥ G.M

$\Rightarrow \quad \frac{a^2\tan x + b^2\cot x}{2} \geq \sqrt{a^2b^2}$

$\Rightarrow \quad a^2\tan x + b^2\cot x \geq 2ab$

$\Rightarrow \quad \frac{1}{a^2\tan x + b^2\cot x} \leq \frac{1}{2ab}$

$\Rightarrow \quad \frac{ab(a^2-b^2)}{a^2\tan x + b^2\cot x} \leq \frac{ab(a^2-b^2)}{2ab}$

$\Rightarrow \quad \frac{ab(a^2-b^2)}{a^2\tan x + b^2\cot x} \leq \frac{(a^2-b^2)}{2}$

$\Rightarrow \quad f(x) \leq \frac{(a^2-b^2)}{2}$

Thus, the maximum value of $f(x)$ is $\frac{(a^2-b^2)}{2}$

2. We have $f(\theta) = \frac{(a^2-b^2)\cos\theta}{a - b\sin\theta}$

$$= \frac{(a^2-b^2)}{a\sec\theta - b\tan\theta}$$

$$= \frac{(a^2-b^2)}{g(\theta)},\ g(\theta) = a\sin\theta - b\tan\theta$$

$f(\theta)$ is max or min according as $g(\theta)$ is min and maximum

Now, $g(\theta) = a\sec\theta - b\tan\theta$

$\Rightarrow \quad g'(\theta) = a\sec\theta\tan\theta - b\sec^2\theta$

$\Rightarrow \quad g''(\theta) = a\sec^3\theta + a\sec\theta\tan^2\theta - 2b\sec^2\theta\tan\theta$

$= a\sec^3\theta + a\sec\theta(\sec^2\theta - 1) - 2b\sec^2\theta\tan\theta$

$= 2a\sec^3\theta - a\sec\theta - 2b\sec^2\theta\tan\theta$

$= \frac{a + a\sin^2\theta - 2b\sin\theta}{\cos^3\theta}$

For max or min, $g'(\theta)$ gives

$\Rightarrow \quad a\sec\theta\tan\theta - b\sec^2\theta = 0$

$\Rightarrow \quad a\sin\theta - b = 0$

$\Rightarrow \quad \sin\theta = b/a.$

Now, $(h''(\theta))_{\sin\theta = \frac{b}{a}} = \frac{a^2-b^2}{\cos^3\theta} > 0$

$\Rightarrow \quad h(\theta)$ will provide us the min value

$\Rightarrow$ $f(\theta)$ will provide us the maximum value

Now, the maximum value of $f(\theta)$

$$= \frac{(a^2 - b^2)\dfrac{\sqrt{a^2 - b^2}}{a}}{a - b\left(\dfrac{b}{a}\right)}$$

$$= \frac{(a^2 - b^2)\sqrt{a^2 - b^2}}{(a^2 - b^2)}$$

$$= \sqrt{a^2 - b^2}$$

3. We have

$$f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \le 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} 3x^3 - 2x + 10 & : x \le 1 \\ -2 & : x > 1 \end{cases}$$

Now, $f(1) \ge f(1^+)$

$\Rightarrow$ $1 - 1 + 10 - 5 \ge -2 + \log_2(b^2 - 2)$

$\Rightarrow$ $\log_2(b^2 - 2) - 2 \le 5$

$\Rightarrow$ $\log_2(b^2 - 2) \le 7$

$\Rightarrow$ $(b^2 - 2) \le 2^7 = 128$ and $(b^2 - 2) > 0$

$\Rightarrow$ $2 < b^2 \le 130$

$\Rightarrow$ $\sqrt{2} < |b| \le \sqrt{130}$

$\Rightarrow$ $b \in \lfloor -\sqrt{130}, -\sqrt{2}) \cup (\sqrt{2}, 130\rfloor$

4. Given condition is

$x^4 + 36 - 13x^2 \le 0$

$\Rightarrow$ $x^4 - 13x^2 + 36 \le 0$

$\Rightarrow$ $(x^2 - 9)(x^2 - 4) \le 0$

$\Rightarrow$ $(x + 3)(x - 3)(x + 2)(x - 2) \le 0$

$\Rightarrow$ $x \in [-3, -2] \cup [2, 3]$

Now, $f(x) = x^3 - 3x$

$\Rightarrow$ $f'(x) = 3x^2 - 3$

For max or min $f'(x) = 0$ gives

$\Rightarrow$ $x^2 - 1 = 0$

$\Rightarrow$ $x = \pm 1$

Now, $f(1) = 1 - 3 = -2$

$\Rightarrow$ $f(-1) = -1 + 3 = 2$

$\Rightarrow$ $f(2) = 8 - 6 = 2$

$\Rightarrow$ $f(-2) = -8 + 6 = -2$

$\Rightarrow$ $f(3) = 27 - 9 = 18$

$\Rightarrow$ $f(-3) = -27 + 9 = -18$

Thus, the maximum value of $f(x)$ is 18 at $x = 3$ and the minimum value of $f(x)$ is -18 at $x = -3$.

5. The given curve is

$(x^2 + 1)(y - 3) = x$

$$\Rightarrow \quad (y - 3) = \frac{x}{(x^2 + 1)}$$

$$\Rightarrow \quad y = 3 + \frac{x}{(x^2 + 1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x^2 + 1)1 - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$\Rightarrow \quad m = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$\Rightarrow \quad \frac{dm}{dx} = \frac{(x^2 + 1)^2(-2x) - 2(1 - x^2)2x(x^2 + 1)}{(x^2 + 1)^4}$$

$$\Rightarrow \quad \frac{dm}{dx} = \frac{(x^2 + 1)(-2x) - 2(1 - x^2)2x}{(x^2 + 1)^3}$$

$$\Rightarrow \quad \frac{dm}{dx} = \frac{2x(x^2 + 1 + 2 - 2x^2)}{(x^2 + 1)^3}$$

$$\Rightarrow \quad \frac{dm}{dx} = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$$

For max or min, $\frac{dm}{dx} = 0$ gives

$$\Rightarrow \quad \frac{2x(x^2 - 3)}{(x^2 + 1)^3} = 0$$

$\Rightarrow$ $2x(x^2 - 3) = 0$

$\Rightarrow$ $x = -\sqrt{3}, 0, \sqrt{3}$

By the sign scheme, the point of local max of m is $x = 0$

when $x = 0$, then $y = 3$

Hence, the point of a tangent to the curve has the greatest slope is (0, 3).

6. We have $x^2 + xy + y^2 = 12$

Put $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow r^2\cos^2\theta + r^2\sin\theta\cos\theta + r^2\sin^2\theta = 12$

$\Rightarrow r^2 + r^2\sin\theta\cos\theta = 12$

$\Rightarrow r^2(1 + \sin\theta\cos\theta) = 12$

$$\Rightarrow r^2 = \frac{12}{(1 + \sin\theta\cos\theta)}$$

$$\Rightarrow r^2 = \frac{12}{\left(1 + \frac{1}{2}\sin 2\theta\right)} = \frac{24}{(2 + \sin 2\theta)}$$

Max value of r^2 is at $\theta = -\frac{\pi}{4}$

i.e. $r = 2\sqrt{6}$

and $(x, y) = (r\cos\theta, r\sin\theta)$

$= (2\sqrt{3}, -2\sqrt{3})$

Min value of r^2 is $\frac{24}{3} = 8$

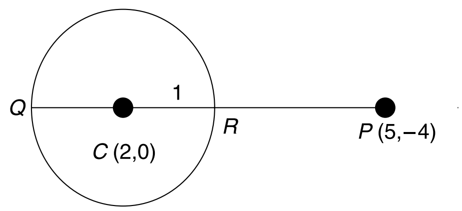
i.e $r = 2\sqrt{2}$ and $(x, y) = (2, -2)$

7. The function represents the distance between the point P(–4, 5) and the circle

$$y = \sqrt{-3 + 4x - x^2}$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 3 = 0$$

$$\Rightarrow \quad (x - 2)^2 + y^2 = 1$$



Maximum distance is

$$= PQ^2$$
$$= (PC + CQ)^2$$
$$= (\sqrt{(5-2)^2 + (4-0)^2} + 1)^2$$
$$= (\sqrt{9 + 16} + 1)^2$$
$$= (5 + 1)^2$$
$$= 36.$$

8. Given condition is $x^2 + 4y^2 = 4$

$$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{1} = 1$$

Put $x = 2\cos\theta$ and $y = \sin\theta$

Now, $f(x, y) = x^2 + y^2 - xy$

$$= 4\cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta$$
$$= 3\cos^2\theta - 2\cos\theta\sin\theta + 1$$
$$= \frac{3}{2}(2\cos^2\theta) - \sin 2\theta + 1$$
$$= \frac{3}{2}(1 + \cos 2\theta) - \sin 2\theta + 1$$
$$= \frac{3}{2}\cos 2\theta - \sin 2\theta + \frac{5}{2}$$

Max value $= \sqrt{\frac{9}{4} + 1} + \frac{5}{4} = \frac{\sqrt{13} + 5}{2}$

Min Value $= -\sqrt{\frac{9}{4} + 1} + \frac{5}{2} = \frac{5 - \sqrt{13}}{2}$

Thus, the greatest value of $f(x, y)$ is $\frac{5 + \sqrt{13}}{2}$.

9. Given $x^2 + y^2 = 1$

Put $x = \cos\theta$, $y = \sin\theta$

Now, $f(\theta) = \cos^2\theta + \sin^2\theta + \sin\theta\cos\theta + 3$

$$= 1 + \frac{1}{2}\sin 2\theta + 3$$
$$= \frac{1}{2}\sin 2\theta + 4$$

Max value $= \frac{1}{2}\cdot 1 + 4 = \frac{9}{2}$

Min value $= \frac{1}{2}\cdot(-1) + 4 = \frac{7}{2}$

Hence, the least value of $f(x, y)$ is $\frac{7}{2}$

10. The given equation is

$$\frac{4}{\sin x} + \frac{1}{1 - \sin x} = a \quad \text{...(i)}$$

Let $f(x) = \frac{4}{\sin x} + \frac{1}{1 - \sin x}$

$$\Rightarrow \quad f'(x) = -\frac{4\cos x}{\sin^2 x} + \frac{\cos x}{(1 - \sin x)^2}$$

For max or min, $f'(x) = 0$ gives

$$-\frac{4\cos x}{\sin^2 x} + \frac{\cos x}{(1 - \sin x)^2} = 0$$

$$\Rightarrow \quad \frac{\cos x}{(1 - \sin x)^2} = \frac{4\cos x}{\sin^2 x}$$

$$\Rightarrow \quad \frac{1}{(1 - \sin x)^2} = \frac{4}{\sin^2 x}$$

$$\Rightarrow \quad \frac{1}{(1 - \sin x)} = \frac{2}{\sin x}$$

$$\Rightarrow \quad \sin x = 2 - 2\sin x$$

$$\Rightarrow \quad 3\sin x = 2$$

$$\Rightarrow \quad \sin x = \frac{2}{3}$$

Put the value of $\sin x = \frac{2}{3}$ in (i), we get,

$$a = \frac{4}{\frac{2}{3}} + \frac{1}{1 - \frac{2}{3}} = 6 + 3 = 9$$

11. We have

$$y = f(x) = \frac{x^4 - x^2}{x^6 + 2x^3 - 1}, \; x > 1$$

$$= \frac{x^3\left(x - \frac{1}{x}\right)}{x^3\left(x^3 - \frac{1}{x^3} + 2\right)}$$

$$= \frac{\left(x - \frac{1}{x}\right)}{\left(x^3 - \frac{1}{x^3} + 2\right)}$$

$$= \frac{\left(x - \frac{1}{x}\right)}{\left(x - \frac{1}{x}\right)^3 + 3\left(x - \frac{1}{x}\right) - 2}$$

$$= \frac{t}{t^3 + 3t - 2}, \text{ where } \left(x - \frac{1}{x}\right) = t$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{(t^3 + 3t - 2)\cdot 1 - t(3t^2 + 3)}{(t^3 + 3t - 2)^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{(t^3 + 3t - 2 - 3t^3 - 3t)}{(t^3 + 3t - 2)^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{(-2t^3 - 2)}{(t^3 + 3t - 2)^2}$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{2(t^3 + 1)}{(t^3 + 3t - 2)^2}$$

For max or min, $\frac{dy}{dt} = 0$ gives

$$\Rightarrow \quad -\frac{2(t^3 + 1)}{(t^3 + 3t - 2)^2} = 0$$

$$\Rightarrow \quad (t^3 + 1) = 0$$

$$\Rightarrow \quad t = -1$$

Hence, the maximum value of $f(x)$ is $\frac{1}{6}$

at $\quad x - \frac{1}{x} = -1$

$$\Rightarrow \quad x^2 + x - 1 = 0$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$\Rightarrow \quad x = \frac{-1 + \sqrt{5}}{2}, \text{ since } x > 1.$$

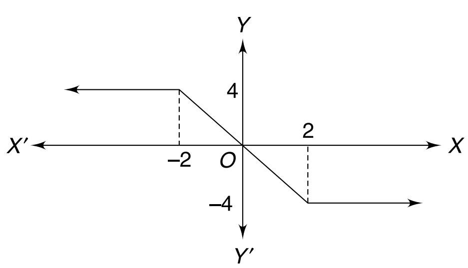
Therefore, the value of $(a + b + 2)$ is 9.

12. We have

$$f(x) = \sqrt{4 - 4x + x^2} - \sqrt{4 + 4x + x^2}, \; x \in R$$

$$= \sqrt{(x - 2)^2} - \sqrt{(x + 2)^2}$$

$$= |x - 2| - |x + 2|$$



Hence, the greatest value of $f(x)$ is 4 at $x = -2$ and the least value of $f(x)$ is -4 at $x = 2$.

13. We have

$$f(x) = \frac{1}{\sin x + 4} - \frac{1}{\cos x - 4}, \; x \in R$$

$$\Rightarrow \quad f'(x) = -\frac{\cos x}{(\sin x + 4)^2} - \frac{\sin x}{(\cos x - 4)^2}$$

For max or min $f'(x) = 0$ gives

$$-\frac{\cos x}{(\sin x + 4)^2} - \frac{\sin x}{(\cos x - 4)^2} = 0$$

$$\Rightarrow \quad -\frac{\cos x}{(\sin x + 4)^2} = \frac{\sin x}{(\cos x - 4)^2}$$

$$\Rightarrow \quad \frac{\cos x}{(\sin x + 4)^2} + \frac{\sin x}{(\cos x - 4)^2} = 0$$

$$\Rightarrow \quad \frac{\cos x}{(\cos x - 4)^2} + \frac{\sin x}{(\sin x + 4)^2} = 0$$

$$\Rightarrow \quad \cos x(\cos x - 4)^2 + \sin x(\sin x + 4)^2 = 0$$

$$\Rightarrow \quad \cos x(\cos^2 x - 8\cos x + 16) + \sin x(\sin^2 x + 8\sin x + 16) = 0$$

$$\Rightarrow \quad (\sin^3 x + \cos^3 x) + 8(\sin^2 x - \cos^2 x) + 16(\sin x + \cos x) = 0$$

$$\Rightarrow \quad (\sin x + \cos x) = 0 \text{ and}$$
$$\{(1 - \sin x \cos x) + 8(\sin x - \cos x) + 16\} = 0$$

Now

$$8(\sin x - \cos x) - \sin x \cos x + 17 = 0$$

$$\Rightarrow \quad 8t - \left(\frac{1 - t^2}{2}\right) + 17 = 0, \; t = \sin x - \cos x$$

$$\Rightarrow \quad 16 - 1 + t^2 + 34 = 0$$

$$\Rightarrow \quad t^2 + 16t + 33 = 0$$

$$\Rightarrow \quad (t + 8)^2 = 64 - 33 = 31$$

$$\Rightarrow \quad (t + 8) = \pm\sqrt{31}$$

$$\Rightarrow \quad t = -8 \pm \sqrt{31}$$

It is not possible

Thus $\quad \sin x + \cos x = 0$

$$\Rightarrow \quad \tan x = -1$$

$$\Rightarrow \quad x = (4n - 1)\frac{\pi}{4}, \; n \in I$$

Thus, the max value of $f(x)$

$$= \frac{1}{4 - \frac{1}{\sqrt{2}}} - \frac{1}{\frac{1}{\sqrt{2}} - 4}$$

$$= \frac{2\sqrt{2}}{4\sqrt{2} - 1}$$

$$= \frac{4}{8 - \sqrt{2}}$$

and the least value of $f(x)$

$$= \frac{1}{\frac{1}{\sqrt{2}} + 4} - \frac{1}{-\frac{1}{\sqrt{2}} - 4}$$

$$= \frac{1}{\frac{1}{\sqrt{2}} + 4} + \frac{1}{\frac{1}{\sqrt{2}} + 4}$$

$$= \frac{2\sqrt{2}}{4\sqrt{2} + 1}$$

$$= \frac{4}{8 + \sqrt{2}}$$

14. Let $z = x^x + y^x$

$\Rightarrow \quad z = x^x + (1 - x)^x$

$\Rightarrow \quad z = f(x) + f(1 - x), f(x) = x^x$

$\Rightarrow \quad \frac{dz}{dx} = f'(x) - f'(1 - x)$

For max or min, $\frac{dz}{dx} = 0$, we get,

$f'(x) - f'(1 - x) = 0$

$\Rightarrow \quad f'(x) = f'(1 - x)$

$\Rightarrow \quad x = 1 - x$

$\Rightarrow \quad x = \frac{1}{2}$

Now, $\quad \frac{d^2z}{dx^2} = f''(x) + f''(1 - x)$

$\Rightarrow \quad \left(\frac{d^2z}{dx^2}\right)_{x=\frac{1}{2}} = f''\left(\frac{1}{2}\right) + f''\left(1 - \frac{1}{2}\right)$

$\Rightarrow \quad \left(\frac{d^2z}{dx^2}\right)_{x=\frac{1}{2}} = f''\left(\frac{1}{2}\right) + f''\left(\frac{1}{2}\right) = 2f''\left(\frac{1}{2}\right)$

$\Rightarrow \quad \left(\frac{d^2z}{dx^2}\right)_{x=\frac{1}{2}} = 2f''\left(\frac{1}{2}\right) > 0$

Thus, z will provide us the minimum value

Thus, the minimum value of z is

$$= f\left(\frac{1}{2}\right) + f\left(1 - \frac{1}{2}\right)$$

$$= f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right)$$

$$= 2f\left(\frac{1}{2}\right)$$

$$= 2\left(\frac{1}{2}\right)^{1/2} = \frac{2}{\sqrt{2}} + \sqrt{2}$$

15. Given curve is

$$9x^2 + 9y^2 - 30y + 16 = 0$$

$\Rightarrow \quad x^2 + y^2 - \frac{30}{9}y + \frac{16}{9} = 0$

$\Rightarrow \quad x^2 + y^2 - \frac{10}{3}y + \frac{16}{9} = 0$

$\Rightarrow \quad x^2 + \left(y - \frac{5}{3}\right)^2 = 1$

Any point on the curve $y^2 = x^3$ can be written as (t^2, t^3).

Let d be the distance between the centre of the circle and the point (t^2, t^3).

Let $\quad z = d^2 = t^4 + \left(t^3 - \frac{5}{3}\right)^2$

$\Rightarrow \quad dz/dt = 4t^3 + 2\left(t^3 - \frac{5}{3}\right)\cdot 3t^2$

For max or min, $\frac{dz}{dt} = 0$ gives

$\Rightarrow \quad 4t^3 + 2\left(t^3 - \frac{5}{3}\right)\cdot 3t^2 = 0$

$\Rightarrow \quad 4t^3 + 6t^5 - 10t^2 = 0$

$\Rightarrow \quad 2t^2(2t + 3t^3 - 5) = 0$

$\Rightarrow \quad t = 0, 1$

Also, $\quad \frac{d^2z}{dt^2} = 12t^2 + 30t^4 - 20t$

$\Rightarrow \quad \left(\frac{d^2z}{dt^2}\right)_{t=1} = 12 + 30 - 20 = 12 > 0$

$\Rightarrow \quad z$ is min at $t = 1$

At $t = 0$, there is neither max nor min.

Hence, the shortest distance

$$= \sqrt{t^4 + \left(t^2 - \frac{5}{3}\right)^2} - 1$$

$$= \sqrt{1 + \left(1 - \frac{5}{3}\right)^2} - 1$$

$$= \sqrt{1 + \frac{4}{9}} - 1$$

$$= \left(\frac{\sqrt{13}}{3} - 1\right)$$

16. Let $f''(x) = a(x - 1)$

$\Rightarrow \quad f'(x) = \frac{a}{2}(x - 1)^2 + b$

$\Rightarrow \quad f(x) = \frac{a}{6}(x - 1)^3 + bx + c \quad$ ...(i)

Now, $f'(-1) = 0$ gives

$\Rightarrow \quad \frac{a}{2}(-1 - 1)^2 + b = 0$

$\Rightarrow \quad 2a + b = 0 \quad$ ...(ii)

Also, $f(1) = -6$

$\Rightarrow \quad b + c = -6 \quad$ ...(iii)

Again $f(-1) = 0$ gives

$\Rightarrow \quad -\frac{4}{3}a - b + c = 10 \quad$ ...(iv)

On solving (ii), (iii) and (iv), we get,

$$a = 6, b = -12, c = 6$$

Thus, the function is

$$y = f(x) = (x - 1)^3 - 12x + 6$$
$$= x^3 - 3x^2 - 9x + 5.$$

Thus, the point of local max is $x = -1$ and the point of local min is $x = 3$

when $x = -1$, then $y = -6$

when $x = 3$, then $y = -22$

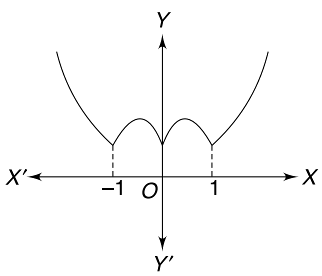
Thus, the two points are

$$P(-1, -6) \text{ and } Q(3, -22)$$

Hence, the distance between P and Q is

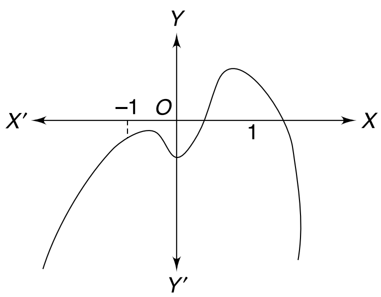
$$= \sqrt{(3 + 1)^2 + (-22 + 6)^2}$$
$$= \sqrt{16 + 256} = \sqrt{272} = 4\sqrt{17}.$$

17. We have $f(x) = |x| - |x^2 - 1|$



Clearly, the number of points of extremum is 5 at $x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$

18. We have $f(x) = |x| - |x^3 - 1|$



$$= \begin{cases} -x + (x^3 - 1) & : x < 0 \\ x + x^3 - 1 & : 0 \le x < 1 \\ x - x^3 + 1 & : x \ge 1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -1 + (3x^2 - 1) & : x < 0 \\ 1 + 3x^2 - 1 & : 0 \le x < 1 \\ 1 - 3x^2 + 1 & : x \ge 1 \end{cases}$$

Now, for max or min, $f'(x) = 0$ gives

$$x = -\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{2}{3}}$$

Thus, the function provide us the point of local max are $x = -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}$

when $x = -\sqrt{\frac{2}{3}}$, then $y = \frac{1}{3}\sqrt{\frac{2}{3}} - 1$

when $x = \sqrt{\frac{2}{3}}$, then $y = \frac{1}{3}\sqrt{\frac{2}{3}} + 1$

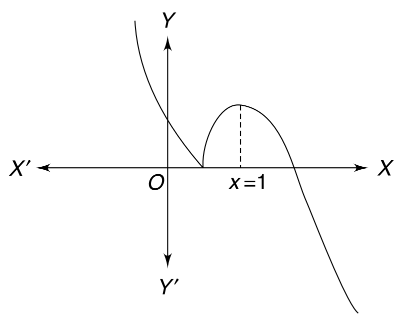
Thus, the points of local maximum are

$P\left(-\sqrt{\frac{2}{3}}, \frac{1}{3}\sqrt{\frac{2}{3}} - 1\right)$ and $Q\left(\sqrt{\frac{2}{3}}, \frac{1}{3}\sqrt{\frac{2}{3}} + 1\right)$

Hence, the distance between P and Q is

$$= \sqrt{\left(\sqrt{\frac{2}{3}} + \sqrt{\frac{2}{3}}\right)^2 + (1 + 1)^2}$$
$$= \sqrt{\frac{8}{3} + 4} = \sqrt{\frac{20}{3}} = 2\sqrt{\frac{5}{3}}$$

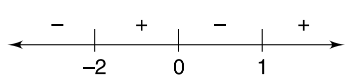
19. We have $f(x) = |3x - 1| - x^3$



Thus, the number of points of extremum of $f(x)$ is 2 at $x = 1/3, 1$

20. Let $f(x) = 3x^4 + 4x^3 - 12x^2 + a$

$$\Rightarrow \quad f'(x) = 12x^3 + 12x^2 - 24x$$
$$= 12x(x^2 + x - 2)$$
$$= 12x(x - 1)(x + 2)$$



The function $f(x)$ will provide us four real and distinct roots if $f(-2) < 0, f(0) > 0, f(1) < 0$

$\Rightarrow \quad 48 - 32 - 48 + a < 0, a > 0, 3 + 4 - 12 + a < 0$

$\Rightarrow \quad a < 32, a > 0, a < 5$

$\Rightarrow \quad 0 < a < 5$

$\Rightarrow \quad a \in (0, 5)$

21. Let $\quad f''(x) = a(x - 1)$

$\Rightarrow \quad f'(x) = \frac{a}{2}(x - 1)^2 + b$

$\Rightarrow \quad f(x) = \frac{a}{6}(x - 1)^3 + bx + c \quad$ ...(i)

Now, $f(0) = 3$ gives $c - \frac{a}{6} = 3$

$\Rightarrow \quad 6c - a = 18 \quad$ ...(ii)

Also, $f(-1) = 4$ gives

$\Rightarrow \quad -\frac{8a}{6} - b + c = 4$

$\Rightarrow \quad -8a - 6b + 6c = 24 \quad$ ...(iii)

Again, $f'(-1) = 0$

$\Rightarrow \quad f'(-1) = \frac{a}{2}(-1-1)^2 + b = 0$

$\Rightarrow \quad b + 2a = 0$

$\Rightarrow \quad b = -2a \quad$ ...(iv)

From (ii), (iii) and (iv), we get,

$-8a - 6b + a + 18 = 24$

$\Rightarrow \quad -7a - 6b = 6$

$\Rightarrow \quad -7a + 12a = 6$

$\Rightarrow \quad 5a = 6$

$\Rightarrow \quad a = \frac{6}{5}$

Now, $\quad b = -2a = -\frac{12}{5}$

and $\quad c = \frac{a}{6} + 3 = \frac{1}{5} + 3 = \frac{16}{5}$

Put the values of a, b and c, we get,

$$f(x) = \frac{1}{5}(x-1)^3 - \frac{12}{5}x + \frac{16}{5}$$
$$= \frac{1}{5}[(x-1)^3 - 12x + 16]$$
$$= \frac{1}{5}[x^3 - 3x^2 + 3x - 1 - 12x + 16]$$
$$= \frac{1}{5}(x^3 - 3x^2 - 9x + 15)$$

22. Let $p(x) = ax^4 + bx^3 + cx^3 \quad$ ...(i)

Now, $\quad \lim_{x\to 0}\left(1 + \frac{p(x)}{x^2}\right) = 4$

$\Rightarrow \quad \lim_{x\to 0}\left\{1 + \left(\frac{ax^4 + bx^3 + cx^2}{x^2}\right)\right\} = 4$

$\lim_{x\to 0}\{1 + (ax^2 + bx + c)\} = 4$

$\Rightarrow \quad c + 1 = 4$

$\Rightarrow \quad c = 3 \quad$ ...(ii)

Also, $\quad p'(x) = 4ax^3 + 3bx^2 + 2cx$

$\Rightarrow \quad p'(1) = 0$ gives

$\Rightarrow \quad 4a + 3b + 2c = 0 \quad$ ...(iii)

And, $\quad p'(-2) = 0$ gives

$\Rightarrow \quad -32a + 12b - 4c = 0$

$\Rightarrow \quad 8a - 3b + c = 0 \quad$ ...(iv)

On solving (ii), (iii) and (iv), we get,

$$a = -\frac{3}{4},\ b = -1,\ c = 3$$

Thus, $\quad p(x) = -\frac{3}{4}x^4 - x^3 + 3x^2$

and $\quad p'(2) = -24$

23. Given $y = \sqrt{x-2} + \sqrt{4-x}$

The domain of the given function is [2, 4]
At $x = 2$, the value of y is $\sqrt{2}$ and at $x = 4$, the value of y is also $\sqrt{2}$.
So minimum value of y is $\sqrt{2}$
But at $x = 1$, the value of y is 2
Hence, the max value of y is 2.

24. Given $y = \sqrt{x^2 - 1} + \sqrt{4 - x^2}$

$$D_f = [-2, -1] \cup [1, 2]$$

Put $\quad x^2 = 4\cos^2\theta + \sin^2\theta$

Then $\quad (4 - x^2) = 3\sin^2\theta,\ (x^2 - 1) = 3\cos^2\theta$

Then the given function reduces to

$$y = \sqrt{3}|\sin\theta| + \sqrt{3}|\cos\theta|$$

$\Rightarrow \quad y_{\min} = \sqrt{3}$ and $y_{\max} = \sqrt{3}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \sqrt{6}$

25. We have $x^2 + 4y^2 = 4$

$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{1} = 1$

Put $\quad x = 2\cos^2\theta$ and $y = \sin\theta$

Then $f(x, y) = x^2 + y^2 - xy$

$$= 4\cos^2\theta + \sin^2\theta - \sin 2\theta$$
$$= 2(2\cos^2\theta) + \frac{1}{2}(2\sin^2\theta) - \sin 2\theta$$
$$= 2(1 + \cos 2\theta) + \frac{1}{2}(1 - \cos 2\theta) - \sin 2\theta$$
$$= \frac{3}{4}\cos 2\theta - \sin 2\theta + \frac{5}{2}$$

Max value $= \sqrt{\frac{9}{4} + 1} + \frac{5}{2} = \frac{5 + \sqrt{13}}{2}$

Min value $= -\sqrt{\frac{9}{4} + 1} + \frac{5}{2} = \frac{5 - \sqrt{13}}{2}$

26. Given $f(x, y) = 1 - x^2 + 2y^2$

$$= 1 - x^2 + 2(1 - x)^2$$
$$= 1 - x^2 + 2 - 4x + 2x^2$$

$= x^2 - 4x + 3$

$= (x - 2)^2 - 1$

Hence, the min value is –1.

27. Put $x = r\cos\theta$, $y = r\sin\theta$

We have $x^2(xy - 1) = y^2(1 + xy)$

$\Rightarrow \quad xy(x^2 - y^2) = x^2 + y^2$

$\Rightarrow \quad r^4\sin\theta\cdot\cos\theta\cdot\cos 2\theta = r^2$

$\Rightarrow \quad r^2(2\sin\theta\cdot\cos\theta)\cdot\cos 2\theta = 2$

$\Rightarrow \quad r^2(2\sin 2\theta\cos 2\theta) = 4$

$\Rightarrow \quad r^2(\sin 4\theta) = 4$

$\Rightarrow \quad r^2 = \dfrac{4}{\sin 4\theta} = 4\operatorname{cosec} 4\theta$

$\Rightarrow \quad f(x, y) = x^2 + y^2$

Thus, $f(x, y) = x^2 + y^2$

$= r^2$

$= 4\operatorname{cosec} 4\theta$

Min value is 4.

28. Given $x^2 + y^2 = 1$

$\Rightarrow \quad \left(\dfrac{x^2 + y^2}{2}\right) \le \left(\dfrac{x + y}{2}\right)^2$

$\Rightarrow \quad \left(x + \dfrac{y}{2}\right)^2 \ge \left(\dfrac{1}{2}\right)$

$\Rightarrow \quad \left(\dfrac{x + y}{2}\right) \ge \left(\dfrac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad (x + y) \ge \left(\dfrac{2}{\sqrt{2}}\right)$

$\Rightarrow \quad (x + y) \ge \sqrt{2}$

Hence, the min value of $x + y$ is $\sqrt{2}$

29. By mth power theorem, we get,

$\Rightarrow \quad \left(\dfrac{x^{-2} + y^{-2} + z^{-2}}{3}\right) \ge \left(\dfrac{x^2 + y^2 + z^2}{3}\right)^{-1}$

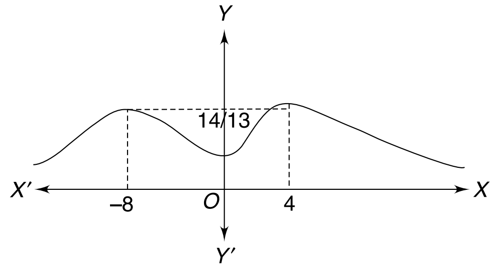
$\Rightarrow \quad \left(\dfrac{x^{-2} + y^{-2} + z^{-2}}{3}\right) \ge \left(\dfrac{3}{x^2 + y^2 + z^2}\right)$

$\Rightarrow \quad x^{-2} + y^{-2} + z^{-2} \ge \left(\dfrac{9}{1}\right) = 9$

$\Rightarrow \quad \dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} \ge 9$

30. We have

$f(x) = \dfrac{1}{|x - 4| + 1} + \dfrac{1}{|x + 8| + 1}$



Now $\quad \forall x \in (-8, 4)$

$f(x) = \dfrac{1}{x + 9} + \dfrac{1}{5 - x}$

$= \dfrac{5 - x + x + 9}{(x + 9)(5 - x)} = \dfrac{14}{(x + 9)(5 - x)}$

Which will have maximum either at $x = 4$ or $x = -8$ and minima of $f(x)$ does not exist.

$f(4) = 1 + \dfrac{1}{13} = \dfrac{14}{13} = f(-8)$

Hence, the absolute maximum value of $f(x)$ is $\dfrac{14}{13}$.

31. We have

$y = x^3 - 3ax^2 + 3(a^2 - 1)\,x + 1$

$\Rightarrow \quad \dfrac{dy}{dx} = 3x^2 - 6ax + 3(a^2 - 1)$

For max or min, $\dfrac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 - 6ax + 3(a^2 - 1) = 0$

$\Rightarrow \quad x^2 - 2ax + (a^2 - 1) = 0$

Let $\quad g(x) = x^2 - 2ax + (a^2 - 1)$

Clearly, $g(-2) > 0$ and $g(4) > 0$

Now, $g(-2) > 0$ gives

$\Rightarrow \quad 4 + 4a + a^2 - 1 > 0$

$\Rightarrow \quad a^2 + 4a + 3 > 0$

$\Rightarrow \quad (a + 1)(a + 3) > 0$

$\Rightarrow \quad a < -3,\ a > -1$ ...(i)

Also, $g(4) > 0$ gives

$16 - 8a + a^2 - 1 > 0$

$\Rightarrow \quad a^2 - 8a + 15 > 0$

$\Rightarrow \quad (a - 3)(a - 5) > 0$

$\Rightarrow \quad a < 3,\ a > 5$ ...(ii)

Again, $\quad -2 < a < 4$ ...(iii)

From (i), (ii) and (iii), we get,

$-1 < a < 3$

$\Rightarrow \quad a \in (-1, 3)$

32. We have

$$\frac{x^2 + x + 2}{x^2 + 5x + 6} < 0$$

$$\Rightarrow \quad \frac{x^2 + x + 2}{(x + 2)(x + 3)} < 0$$

$$\Rightarrow \quad \frac{1}{(x + 2)(x + 3)} < 0$$

($\because x^2 + x + 2 > 0$ for all x in R

$$\Rightarrow \quad \frac{1}{(x + 2)(x + 3)} < 0$$

$$\Rightarrow \quad (x + 2)(x + 3) < 0$$

$$\Rightarrow \quad -3 < x < -2 \quad ...(i)$$

Again, since $f(x) = 1 + a^2x - x^3$

$$\Rightarrow \quad f'(x) = a^2 - 3x^2$$

$$\Rightarrow \quad f''(x) = -6x > 0, \ \forall x \in (-3, -2)$$

Thus, $f(x)$ will provide us the min value.

Now, for max or min, $f'(x) = 0$ gives

$$\Rightarrow \quad a^2 - 3x^2 = 0$$

$$\Rightarrow \quad x = \pm \frac{a}{\sqrt{3}}$$

Case-I: When $a > 0$

Then $x = -\frac{a}{\sqrt{3}} < 0$

$$\Rightarrow \quad f''\left(\frac{-a}{\sqrt{3}}\right) > 0$$

$$\Rightarrow \quad -3 < -\frac{a}{\sqrt{3}} < -2$$

$$\Rightarrow \quad 2 < \frac{a}{\sqrt{3}} < 3$$

$$\Rightarrow \quad 2\sqrt{3} < a < 3\sqrt{3} \quad ...(ii)$$

Case-II: When $a < 0$

Then $x = \frac{a}{\sqrt{3}} < 0$

$$\Rightarrow \quad f''\left(\frac{a}{\sqrt{3}}\right) < 0$$

$$\Rightarrow \quad -3 < \frac{a}{\sqrt{3}} < -2$$

$$\Rightarrow \quad -3\sqrt{3} < a < -2\sqrt{3} \quad ...(iii)$$

From (ii) and (iii), we get,

$$a \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$$

33. We have $f(x) = \frac{a^2}{x} + \frac{b^2}{(1 - x)}$

$$\Rightarrow \quad f'(x) = -\frac{a^2}{x^2} + \frac{b^2}{(1 - x)^2}$$

For max or min, $f'(x) = 0$ gives

$$\Rightarrow \quad -\frac{a^2}{x^2} + \frac{b^2}{(1 - x)^2} = 0$$

$$\Rightarrow \quad \frac{a^2}{x^2} = \frac{b^2}{(1 - x)^2}$$

$$\Rightarrow \quad \frac{a}{x} = \pm \frac{b}{(1 - x)}$$

$$\Rightarrow \quad x = \frac{a}{a + b}, \frac{a}{a - b}$$

Thus, the maximum value of $f(x)$ is

$$= \frac{a^2}{\frac{a}{a + b}} + \frac{b^2}{1 - \frac{a}{a + b}}$$

$$= \frac{a^2(a + b)}{a} + \frac{b^2(a + b)}{b}$$

$$= a(a + b) + b(a + b)$$

$$= a^2 + ab + ab + b^2$$

$$= a^2 + 2ab + b^2$$

$$= (a + b)^2$$

and the least value of $f(x)$ is

$$= \frac{a^2}{\frac{a}{a - b}} + \frac{b^2}{1 - \frac{a}{a - b}}$$

$$= \frac{a^2(a - b)}{a} - \frac{b^2(a - b)}{b}$$

$$= a(a - b) - b(a - b)$$

$$= a^2 - ab - ab + b^2$$

$$= a^2 - 2ab + b^2$$

$$= (a - b)^2$$

34. Let $f'(x) = ax(x + 1)(x - 1)$

$$\Rightarrow \quad f'(x) = ax(x^2 - 1)$$

$$\Rightarrow \quad f'(x) = a(x^3 - x)$$

$$\Rightarrow \quad f(x) = a\left(\frac{x^4}{4} - \frac{x^2}{2}\right) + b$$

Now, $f(0) = 3$ gives $b = 3$

$$\Rightarrow \quad f(1) = 4 \text{ gives } a\left(\frac{1}{4} - \frac{1}{2}\right) + b = 4$$

$$\Rightarrow \quad a\left(\frac{1 - 2}{4}\right) + 3 = 4$$

$$\Rightarrow \quad a\left(\frac{1 - 2}{4}\right) = 4 - 3 = 1$$

$$\Rightarrow \quad a = -4$$

Thus, the function $f(x)$ is $= -4\left(\frac{x^4}{4} - \frac{x^2}{2}\right) + 3$

$\Rightarrow \quad f(x) = -4x^4 + 2x^2 + 3$

35. Let $\quad f'(x) = (x-1)(x-2)(x-3)$

$\Rightarrow \quad f'(x) = x^3 - 6x^2 + 11x - 6$

$\Rightarrow \quad f(x) = \frac{x^4}{4} - 6\frac{x^3}{3} + 11\frac{x^2}{2} - 6x + b$

$\Rightarrow \quad f(x) = \frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x + b$

$f(0) = 2$ gives $b = 2$

Hence, the polynomial function is

$$f(x) = \frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x + 2$$

$\Rightarrow \quad f(x) = \frac{1}{4}(x^4 - 8x^3 + 44x^2 - 24x + 8)$

36. We have

$$f(x) = \frac{(x+1)^4}{x^4 - x^3 + x^2 - x + 1}$$

$$= \frac{x^4 + 4x^3 + 6x^2 + 4x + 1}{x^4 - x^3 + x^2 - x + 1}$$

$$= \frac{\left(x^2 + \frac{1}{x^2}\right) + 4\left(x + \frac{1}{x}\right) + 6}{\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) + 1}$$

Let $\quad g(x) = \left(x + \frac{1}{x}\right)$

$\Rightarrow \quad g'(x) = \left(1 - \frac{1}{x^2}\right)$

$\Rightarrow \quad g'(x) = 0$ gives $\left(1 - \frac{1}{x^2}\right) = 0$

$\Rightarrow \quad x = \pm 1$

Hence, the greatest value of $f(x)$ is

$$= f(1) = \frac{2 + 8 + 6}{2 - 2 + 1} = 16$$

37. We have

$$f(x) = \frac{4x(x^2 + 1)}{x^2 + (x^2 + 1)^2}, \; x \geq 0$$

$$= \frac{4(x^3 + x)}{x^4 + 3x^2 + 1}$$

$$= \frac{4\left(x + \frac{1}{x}\right)}{x^2 + \frac{1}{x^2} + 3}$$

$$= \frac{4\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right) + 1}$$

Let $\quad g(x) = \left(x + \frac{1}{x}\right)$

$\Rightarrow \quad g'(x) = \left(1 - \frac{1}{x^2}\right)$

$\Rightarrow \quad g'(x) = 0$ gives $\left(1 - \frac{1}{x^2}\right) = 0$

$\Rightarrow \quad x = \pm 1$

Thus, the greatest value of $f(x)$ is

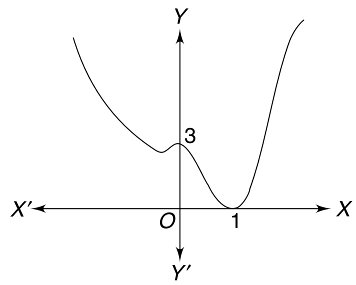
$$= f(-1) = \frac{4 \cdot (-2)}{-2 + 1} = 8$$

and the least value of $f(x)$ is

$$= f(1) = \frac{4 \cdot 2}{2 + 1} = \frac{8}{3}$$

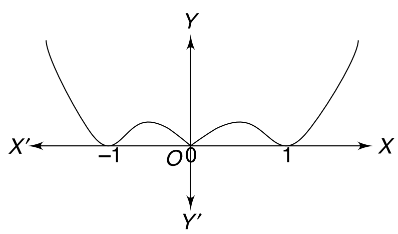
38. We have

$f(x) = |x^2 - 1| + |x^3 - 1| + |x^5 - 1|$



Clearly, the number of points of extremum is 3 at $x = 1/2, 0, 1$

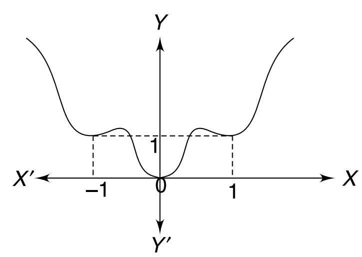
39. We have $f(x) = |x^4 - 1| + |x^6 - 1|$



Clearly, the number of extremum is 5

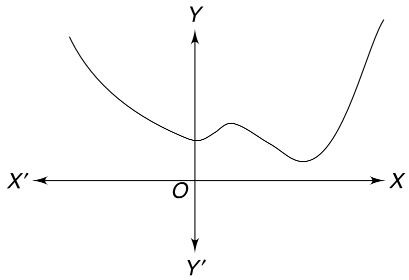
at $x = -1, -\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{2}{3}}, 1$

40. We have $f(x) = |x^2 - 1| + |x| + |x^6 - 1|$



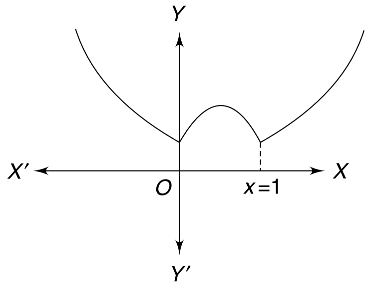
Thus, the number of points of extremum is 5 at $x = -1, -0.85, 0, 0.85, 1$

41. We have $f(x) = |x^2 - 1| + |x| + |x^3 - 1|$



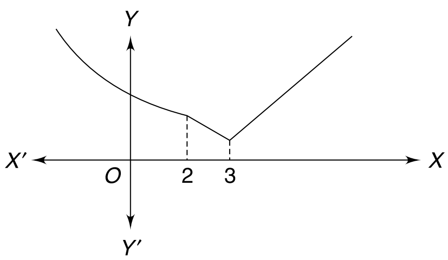
Thus, the number of points of extremum is 3

42. We have $f(x) = |x| + |x^3 - 1|$



Thus, the number of points of extremum is 3

$$g(x) = \frac{1}{|x-2|+1} + |x-3|$$



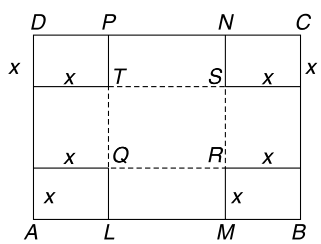
Thus, the minimum value of $g(x)$ is 1/2 at $x = 3$.

So, $n = 1/2$

Hence, the value of $(m + 2n + 2) = 3 + 1 + 2 = 6$.

Level II

(Tougher Problems for JEE-Advance)

1. 

Let x be the length of each side of the square cut out

So height of the box $= x$

Here, $QR = 6 - 2x$

Volume of the box $= V = x(6 - 2x)^2$

$$\Rightarrow \quad \frac{dV}{dx} = (6-2x)^2 + x \cdot 2(6-2x) \cdot (-2)$$

$$\Rightarrow \quad \frac{dV}{dx} = (6-2x)(6-2x-4x)$$

$$\Rightarrow \quad \frac{dV}{dx} = 12(3-x)(1-x)$$

$$\Rightarrow \quad \frac{d^2V}{dx^2} = -24(2-x)$$

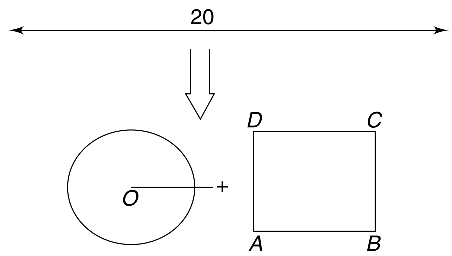
For max or min, $\frac{dV}{dx} = 0$ gives

$$\Rightarrow \quad x = 3, 1$$

At $x = 1$, $\frac{d^2V}{dx^2} = -24 < 0$

So the volume is maximum.

Hence, the height of the box = 1 ft.

2. 

Let r be the radius of the circle and x be the side of a square.

Total area $= A = x^2 + \pi r^2$

and $\quad 4x + 2\pi r = 20$

$$\Rightarrow \quad r = \frac{10-2x}{\pi}$$

We have, $A = x^2 + \pi r^2 = x^2 + \pi\left(\frac{10-2x}{\pi}\right)^2$

$$\Rightarrow \quad \frac{dA}{dx} = 2x + \frac{2}{\pi}(10-2x)\cdot(-2)$$

$$\Rightarrow \quad \frac{dA}{dx} = 2x - \frac{8}{\pi}(5-x)$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = 2 + \frac{8}{\pi}$$

For max or min, $\frac{dA}{dx} = 0$ gives

$$2x - \frac{8}{\pi}(5-x) = 0$$

$$\Rightarrow \quad x = \frac{4}{\pi}(5-x)$$

$$\Rightarrow \quad \left(1 + \frac{4}{\pi}\right)x = \frac{20}{\pi}$$

$\Rightarrow \quad x = \dfrac{20}{4 + \pi}$

Here, $\quad 0 \le x \le 20$

Area is minimum at $x = \dfrac{20}{4 + \pi}$ and maximum at $x = 0$ and 20

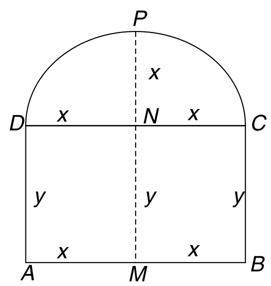
when $x = 0$, then $A = \dfrac{100}{\pi}$

when $x = 20$, then $A = 400 + \dfrac{900}{\pi}$

Thus, the max area is occured at $x = 20$.

Therefore, the area will be max when the whole wire should be bent into a circle.

3.



Let $AB = 2x$ and $BC = y$ and radius of the circle $= x$.

Given $2x + 2y + \pi x = 20$

$\Rightarrow \quad y = \dfrac{20 - (\pi + 2)x}{2}$

Total area $= A = 2xy + \dfrac{\pi x^2}{2}$

Clearly, window will admit max light if its area is maximum

Thus, $\quad A = 2x\left(\dfrac{20 - (2 + \pi)x}{2}\right) + \dfrac{\pi x^2}{2}$

$\Rightarrow \quad A = (20x - (2 + \pi)x^2) + \dfrac{\pi x^2}{2}$

$\Rightarrow \quad \dfrac{dA}{dx} = (20 - 2(2 + \pi)x) + \pi x$

$\Rightarrow \quad \dfrac{dA}{dx} = 20 - (4 + \pi)x$

$\Rightarrow \quad \dfrac{d^2A}{dx^2} = -(4 + \pi) < 0$

So, the area is maximum

For max or min, $x = \dfrac{dA}{dx} = 0$

$\Rightarrow \quad x = \dfrac{20}{\pi + 4}$

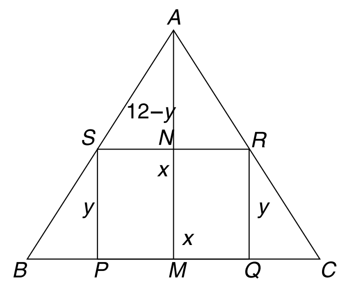
when $x = \dfrac{20}{\pi + 4}$, then

$$y = \frac{20 - (2 + \pi)\cdot\left(\dfrac{20}{\pi + 4}\right)}{2} = \frac{20}{\pi + 4}$$

Thus, the radius of the semi-circle $= \dfrac{20}{\pi + 4}$ and the sides of the rectangle are

$$2x = \frac{40}{\pi + 4} \text{ and } y = \frac{20}{\pi + 4}$$

4.



Let the sides of a rectangle be x and y respectively

Here $AM = 12$, $BC = 36$, $AN = 12 - y$

$PQ = x$, $QR = y$

Also, $\quad \dfrac{36}{x} = \dfrac{12}{12 - y}$

$\Rightarrow \quad x = 36 - 3y$

$\Rightarrow \quad y = \dfrac{(36 - x)}{3}$

Now, $A = xy = x\left(\dfrac{36 - x}{3}\right)$

$\Rightarrow \quad A = \dfrac{1}{3}(36x - x^2)$

$\Rightarrow \quad \dfrac{dA}{dx} = \dfrac{1}{3}(36 - 2x)$

$\Rightarrow \quad \dfrac{d^2A}{dx^2} = -\dfrac{2}{3} < 0$

So, area is maximum

For max or min, $\dfrac{dA}{dx} = 0$

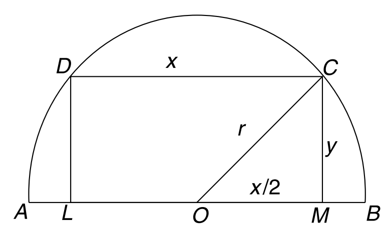
$\Rightarrow \quad \dfrac{1}{3}(36 - 2x) = 0$

$\Rightarrow \quad x = 18$

when $x = 18$, then $y = 6$.

Maximum area $= A = 18.6 = 108$ sq ft.

5.



Let the sides of the rectangle be x and y respectively

Here, $\quad \dfrac{x^2}{4} + y^2 = r^2$

$\Rightarrow \quad y = \sqrt{r^2 - \frac{x^2}{4}}$

Now, $A = xy = x\sqrt{r^2 - \frac{x^2}{4}}$

$\Rightarrow \quad A^2 = x^2\left(r^2 - \frac{x^2}{4}\right) = r^2x^2 - \frac{x^4}{4}$

$\Rightarrow \quad B = r^2x^2 - \frac{x^4}{4}$

$\Rightarrow \quad \frac{dB}{dx} = 2xr^2 - x^3$

$\Rightarrow \quad \frac{d^2B}{dx^2} = 2r^2 - 3x^2$

Now, $\frac{dB}{dx} = 0$ gives $x = r\sqrt{2}$

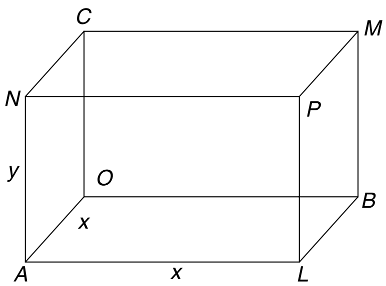
At $x = r\sqrt{2}$, $\frac{d^2B}{dx^2} = 2r^2 - 6r^2 = -4r^2 < 0$

So, area is max.

Thus the sides of a rectangle are

$x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$

6.



Let length = x and height = y

Given $\quad x^2 + 4xy = 40$

$\Rightarrow \quad y = \frac{40 - x^2}{4x}$

Now, Volume = $V = x^2y$

$= x^2\left(\frac{40 - x^2}{4x}\right)$

$= x(40 - x^2)$

$= 40x - x^3$

$\Rightarrow \quad \frac{dV}{dx} = 40 - 3x^2$

$\Rightarrow \quad \frac{d^2V}{dx^2} = -6x$

For max or min, $\frac{dV}{dx} = 0$ gives $x = \sqrt{\frac{40}{3}}$

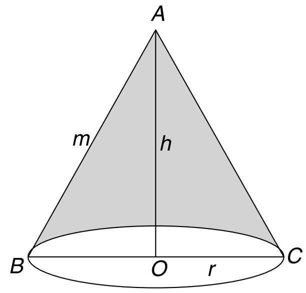
So, at $x = \sqrt{\frac{40}{3}}$, $\frac{d^2V}{dx^2} < 0$

Thus, volume is maximum.

Therefore, length = $x = \sqrt{\frac{40}{3}}$

and height = $y = \sqrt{\frac{10}{3}}$

7.



Let h = height of the cone, r = radius of the cone m = slant height

$r^2 + h^2 = m^2$

$r = \sqrt{m^2 - h^2}$

We have, $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow \quad V = \frac{1}{3}\pi(m^2 - h^2)h$

$\Rightarrow \quad V = \frac{1}{3}\pi(m^2h - h^3)$

$\Rightarrow \quad \frac{dV}{dh} = \frac{1}{3}\pi(m^2 - 3h^2)$

$\Rightarrow \quad \frac{d^2V}{dh^2} = \frac{\pi}{3}(-6h) = -2\pi h$

Now, $\quad \frac{dV}{dh} = 0$ gives $m^2 - 3h^2 = 0$

$\Rightarrow \quad m = h\sqrt{3}$

$\Rightarrow \quad h = \frac{m}{\sqrt{3}}$

At $h = \frac{m}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -2\pi\frac{m}{\sqrt{3}} < 0$

So volume is max

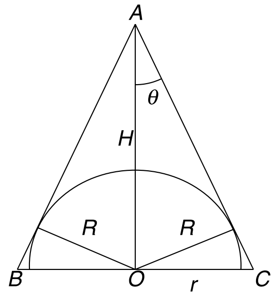
Hence, the greatest volume

$= V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{2}{3}m^2\right)\cdot\frac{m}{\sqrt{3}}$

$= \left(\frac{2\pi}{9\sqrt{3}}\right)m^3$

8.



Let H = height of the cone, r = radius of the cone and R = radius of the sphere.

Now, $\tan\theta = \frac{r}{H}$, $\sin\theta = \frac{R}{H}$

$\Rightarrow \quad r = \frac{R}{\sin\theta}, H = \frac{R}{\cos\theta}$

Thus, $V = \frac{1}{3}\pi r^2 H = \frac{1}{3}\pi\left(\frac{R}{\cos\theta}\right)^2\left(\frac{R}{\sin\theta}\right)$

$\Rightarrow \quad V = \frac{\pi R^3}{3}\cdot\frac{1}{\cos^2\theta}\cdot\frac{1}{\sin\theta}$

$\Rightarrow \quad V = \frac{\pi R^3}{3}\cdot\frac{1}{\sin\theta - \sin^3\theta}$

$\Rightarrow \quad \frac{dV}{d\theta} = \frac{\pi R^3\cos\theta}{(\sin\theta - \sin^3\theta)^2}\left(\sin\theta + \frac{1}{\sqrt{3}}\right)\left(\sin\theta - \frac{1}{\sqrt{3}}\right)$

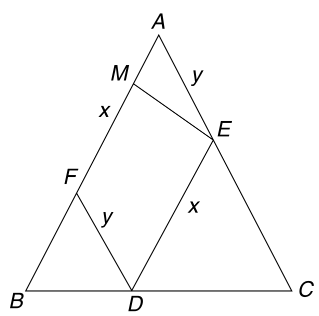
Now, $\frac{dV}{d\theta} = 0$ gives $\left(\sin\theta - \frac{1}{\sqrt{3}}\right) = 0$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ to be the point of maximum

Hence, the height = $H = \frac{R}{\sin\theta} = R\sqrt{3}$

9. Let $AF = x = DE$ and $AE = y = DF$ and $AB = c$, $BC = a$ and $CA = b$ respectively.



Since the triangles CAB and CED are similar,

so, $\quad \frac{CE}{CA} = \frac{DE}{AB}$

$\Rightarrow \quad \frac{b - y}{b} = \frac{x}{c}$

Now, area of the parallelogram $AFDE = P = AF\cdot EM$ $= xy\sin A$

$\Rightarrow \quad P = xy\sin A$

$\Rightarrow \quad p = x\cdot b\left(1 - \frac{x}{c}\right)\sin A$

$\Rightarrow \quad \frac{dP}{dx} = \frac{b}{c}(c - 2x)\sin A$

For extremum, $\frac{dP}{dx} = 0$

$\Rightarrow \quad \frac{b}{c}(c - 2x)\sin A = 0$

$\Rightarrow \quad x = \frac{c}{2}$

Also, $\quad \left(\frac{d^2P}{dx^2}\right)_{x=\frac{c}{2}} = -\frac{2b}{c} < 0$

So P is maximum, when $x = \frac{c}{2}$

Thus, $P_{\max} = \frac{1}{4}bc\sin A = \frac{1}{2}\left(\frac{1}{2}bc\sin A\right)$

$= \frac{1}{2}(\Delta ABC)$

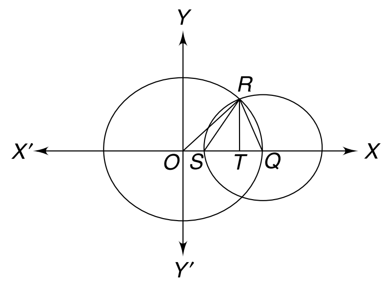
$= \frac{1}{4}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & -q & 1 \\ r^2 & -r & 1 \end{vmatrix}$

$= \frac{1}{4}(p + q)(q + r)(r + p)$

10. Given circle is $x^2 + y^2 = 1$ ...(i)

Centre = (0, 0) and radius = $OP = OQ = 1$

so, $Q = (1, 0)$



Equation of a circle with centre at Q and variable radius is $(x - 1)^2 + y^2 = r^2$ ...(ii)

From (i) and (ii), we get,

$2x - 1 = 1 - r^2$

$\Rightarrow \quad x = 1 - \frac{r^2}{2}$

Now, $\quad RT = \sqrt{OR^2 - OT^2}$

$= \sqrt{1 - \left(1 - \frac{r^2}{2}\right)^2}$

$= \sqrt{r^2 - \frac{r^4}{4}}$

Now, the area of ΔQSR

$\Rightarrow \quad A = \frac{1}{2}\cdot QS\cdot RT$

$\Rightarrow \quad A^2 = \frac{1}{4}\cdot(QS)^2(RT)^2$

$\Rightarrow \quad B = \frac{1}{4}\cdot r^2\left(r^2 - \frac{r^4}{4}\right)$

$\Rightarrow \quad B = \frac{1}{16}(4r^4 - r^6)$

$\Rightarrow \quad \frac{dB}{dr} = \frac{1}{16}(16r^3 - 6r^5)$

$\Rightarrow \quad \frac{d^2B}{dr^2} = \frac{1}{16}(48r^2 - 30r^4)$

For extremum, $\frac{dB}{dr} = 0$ gives

$\Rightarrow \quad \frac{1}{16}(16r^3 - 6r^5) = 0$

$\Rightarrow \quad (16r^3 - 6r^5) = 0$

$\Rightarrow \quad 3r^2 = 8$

$\Rightarrow \quad r = 2\sqrt{\frac{2}{3}}$

At $\quad r = 2\sqrt{\frac{2}{3}}, \frac{d^2B}{dr^2} < 0$

So, B is maximum

Thus, area is maximum

Hence, the maximum area $= \frac{4}{3\sqrt{3}}$ sq.u.

11. Given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let $P(x, y)$ be any point and the foci are $F_1(-ae, 0)$ and $F_2(ae, 0)$ respectively.

Then the area of the triangle PF_1F_2

$$= A = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -ae & 0 & 1 \\ ae & 0 & 1 \end{vmatrix}$$

$\Rightarrow \quad A = \frac{1}{2}(2aey) = aye = aeb\sqrt{1 - \frac{x^2}{a^2}}$

$\Rightarrow \quad A = eb\sqrt{a^2 - x^2}$

So A is maximum, when $x = 0$ and the maximum area $= aeb = b\sqrt{a^2 - b^2}$

12. Equation of any line passing through (h, k) is $(y - h) = m(x - k)$

Put $y = 0$, then $x = \left(k - \frac{h}{m}\right)$

So, the point P is $\left(\left(k - \frac{h}{m}\right), 0\right)$

Put $x = 0$, then $y = h - mk$

so, the point Q is $(0, (h - mk))$

Hence, the area of the triangle OPQ

$$= A = \frac{1}{2}\cdot OP \cdot OQ$$

$\Rightarrow A = \frac{1}{2}\cdot\left(k - \frac{h}{m}\right)\cdot(h - mk)$

$\Rightarrow \frac{dA}{dm} = \frac{1}{2}\left[(h - mk)\left(\frac{h}{m^2}\right) + \left(k - \frac{h}{m}\right)\cdot(-k)\right].$

For extrema, $\frac{dA}{dm} = 0$ gives

$\Rightarrow \frac{1}{2}\left[(h - mk)\left(\frac{h}{m^2}\right) + \left(k - \frac{h}{m}\right)\cdot(-k)\right] = 0$

$\Rightarrow \left[(h - mk)\left(\frac{h}{m^2}\right) - \left(k - \frac{h}{m}\right)\cdot k\right] = 0$

$\Rightarrow (h - mk)\left(\frac{h}{m^2}\right) = \left(k - \frac{h}{m}\right)\cdot k$

$\Rightarrow (h - mk)\left(\frac{h}{m^2}\right) = -\left(\frac{h - km}{m}\right)\cdot k$

$\Rightarrow (h - mk)\left(\frac{h}{m^2} + \frac{k}{m}\right) = 0$

$\Rightarrow (h - mk)(h + km) = 0$

$\Rightarrow m = \frac{h}{k}, -\frac{h}{k}$

Since $h > 0$, $k > 0$ and $m < 0$, so the value of m is m $= -\frac{h}{k}$

Hence, the min area of the triangle OPQ

$$= \frac{1}{2}\left(k + \frac{h}{\frac{h}{k}}\right)\left(h + \frac{h}{k}\cdot k\right)$$

$$= \frac{1}{2}\cdot 2k\cdot 2h = 2hk$$

13. Let the point P be $(r\cos\theta, r\sin\theta)$

Given curve is $ax^2 + 2bxy + ay^2 = c$ ...(i)

(i) reduces to $ar^2\cos^2\theta + r^2 b\sin 2\theta + ar^2\sin^2\theta = c$

$\Rightarrow \quad r^2 = \frac{c}{(a\cos^2\theta + b\sin 2\theta + a\sin^2\theta)}$

$\Rightarrow \quad r^2 = \frac{c}{a + b\sin 2\theta}$

$\Rightarrow r^2$ will be minimum when $(a + b\sin 2\theta)$ is maximum

$$\sin 2\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

Therefore, $r^2 = \frac{c}{a + b}$

Thus, the points are

$= (r\cos\theta, r\sin\theta)$

$= \left(r\cos\left(\frac{\pi}{4}\right), r\sin\left(\frac{\pi}{4}\right)\right), \left(r\cos\left(\frac{5\pi}{4}\right), r\sin\left(\frac{5\pi}{4}\right)\right)$

$= \left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right), \left(-\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}\right)$

14. Given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of any point $P(a\cos\theta, b\sin\theta)$ is

$$\frac{a\cos\theta x}{a^2} + \frac{b\sin\theta y}{b^2} = 1$$

$$\Rightarrow \quad \frac{x}{a\sec\theta} + \frac{y}{b\operatorname{cosec}\theta} = 1 \quad \text{...(ii)}$$

ON is perpendicular to the tangent

Thus, equation of ON is

$$\frac{x}{b\operatorname{cosec}\theta} - \frac{y}{a\sec\theta} = 0$$

$$\Rightarrow \quad \frac{x}{b\operatorname{cosec}\theta} = \frac{y}{a\sec\theta}$$

$$\Rightarrow \quad y = \frac{ax}{b}\tan\theta \quad \text{...(iii)}$$

On solving (ii) and (iii), we get,

$$x = \frac{ab^2\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

and $$y = \frac{a^2\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

Thus, the co-ordinates of N

$$= \left(\frac{ab^2\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}, \frac{a^2b\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta}\right)$$

Therefore, the area of the triangle $OPN = A$

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\cos\theta & b\sin\theta & 1 \\ \frac{ab^2\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta} & \frac{a^2b\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times \frac{a\cos\theta \cdot b\sin\theta}{a^2\sin\theta + b^2\cos^2\theta}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ b^2 & a^2 & 1 \end{vmatrix}$$

$$= \frac{1}{4} \times \frac{ab\sin 2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}(a^2 - b^2)$$

$$= \frac{(a^2 - b^2)ab}{4} \times \frac{2}{a^2\tan\theta + b^2\cot\theta}$$

$$\leq \frac{(a^2 - b^2)ab}{2} \times \frac{1}{2ab} = \left(\frac{a^2 - b^2}{4}\right)$$

Hence, the co-ordinates of the point P

$$= \left(\frac{ab}{\sqrt{a^2 + b^2}}, \frac{ab}{\sqrt{a^2 + b^2}}\right)$$

15. Any normal to the ellipse at $(a\cos\theta, b\cos\theta)$ is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

Here, $a = 3$ and $b = 2$, so the normal reduces to $3x\sec\theta - 2y\operatorname{cosec}\theta = 3^2 - 2^2 = 5$

If p be the length of the perpendicular from the origin, so $p = \frac{5}{\sqrt{9\sec^2\theta + 4\operatorname{cosec}^2\theta}}$

$$\Rightarrow \quad p = \frac{5}{\sqrt{9(1 + \tan^2\theta) + 4(1 + \cot^2\theta)}}$$

$$\Rightarrow \quad p = \frac{5}{\sqrt{13 + (9\tan^2\theta + 4\cot^2\theta)}}$$

$$\Rightarrow \quad f(\theta) = (9\tan^2\theta + 4\cot^2\theta)$$

$$\Rightarrow \quad f(\theta) = (3\tan\theta)^2 + (2\cot\theta)^2$$

$$\Rightarrow \quad f(\theta) = ((3\tan\theta) - (2\cot\theta))^2 + 12\tan\theta \cdot \cot\theta$$

$$\Rightarrow \quad f(\theta) = ((3\tan\theta) - (2\cot))^2 + 12$$

$f(\theta)$ is minimum, when

$$\Rightarrow \quad ((3\tan\theta) - (2\cot\theta))^2 = 0$$

$$\Rightarrow \quad (3\tan\theta) = (2\cot\theta)$$

$$\Rightarrow \quad \tan^2\theta = \frac{2}{3}$$

$$\Rightarrow \quad \tan\theta = \sqrt{\frac{2}{3}}$$

Thus, p is maximum, when $f(\theta)$ is minimum

where $\tan\theta = \sqrt{\frac{2}{3}}$

So, $\sec\theta = \sqrt{\frac{5}{3}}, \operatorname{cosec}\theta = \sqrt{\frac{5}{2}}$

Hence, the equation of the normal is

$$3 \cdot \sqrt{\frac{5}{3}} \cdot x - 2 \cdot \sqrt{\frac{5}{2}} \cdot y = 5$$

$$\Rightarrow \quad x \cdot \sqrt{3} - y \cdot \sqrt{2} = \sqrt{5}.$$

16. Equation of any line passing through (8, 2) is

$y - 2 = m(x - 8)$

Put $y = 0$, then $x = 8 - \frac{2}{m}$

So, $OP = \left(8 - \frac{2}{m}\right)$

Put $x = 0$, then $y = 2 - 8m$

So, $OQ = (2 - 8m)$.

Let $S = OP + OQ$

$\Rightarrow \quad S = \left(8 - \frac{2}{m}\right) + (2 - 8m)$

$\Rightarrow \quad \frac{dS}{dm} = \frac{2}{m^2} - 8$

$\Rightarrow \quad \frac{d^2S}{dm^2} = -\frac{4}{m^3}$

Now, $\frac{dS}{dm} = 0$ gives $\frac{2}{m^2} - 8 = 0$

$\Rightarrow \quad 8m^2 = 2$

$\Rightarrow \quad 4m^2 = 1$

$m = \pm\frac{1}{2}$

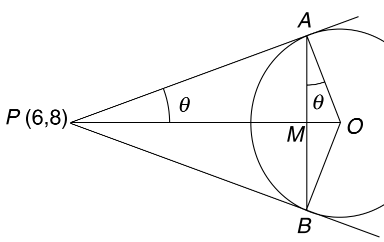
Since slope is positive, so $m = -1/2$

when $m = -1/2$, $\frac{d^2S}{dm^2} > 0$.

So, the function will provide us the minimum value

Thus, the minimum value = (8 + 4) + (2 + 4) = 18.

17. Given circle is $x^2 + y^2 = r^2$



We have $OP = \sqrt{6^2 + 8^2} = 10$

$BM = r\cos\theta$, $OM = r\sin\theta$

where $0 < \theta < \frac{\pi}{2}$

Also, $\sin\theta = \frac{r}{10}$

If A denote the area of the triangle PAB, then

$\Rightarrow \quad A = 2ar(\Delta PBM)$

$\Rightarrow \quad A = 2 \times \frac{1}{2} \times PM \times BM$

$\Rightarrow \quad A = PM \times BM$

$= (OP - OM)\cdot BM$

$= (10 - r\sin\theta)\cdot r\cos\theta$

$= (10 - 10\sin\theta\cdot\sin\theta)(10\sin\theta\cdot\cos\theta)$

$= 100\sin\theta\cos^3\theta$

$\frac{dA}{d\theta} = 100\cos\theta\cos^3\theta - 300\sin^2\theta\cos^2\theta$

$= 300\cos^4\theta\left(\frac{1}{\sqrt{3}} - \tan\theta\right)\left(\frac{1}{\sqrt{3}} + \tan\theta\right)$

For maxima and minima, $\frac{dA}{d\theta} = 0$ gives

$\Rightarrow \quad \frac{1}{\sqrt{3}} - \tan\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{6}$

Thus, $\frac{dA}{d\theta} = \begin{cases} > 0 : 0 < \theta < \frac{\pi}{6} \\ < 0 : \frac{\pi}{6} < \theta < \frac{\pi}{2} \end{cases}$

Therefore A is maximum, when $\theta = \frac{\pi}{6}$

and $r = 10\cdot\sin\left(\frac{\pi}{6}\right) = 5$.

Integer Type Questions

1. We have $f(x) = \frac{x^2 - 1}{x^2 + 1}$, $x \in$ R

$= \frac{(x^2 + 1) - 2}{x^2 + 1}$

$= 1 - \frac{2}{x^2 + 1}$

Clearly, the min value of $f(x)$ is –1 at $x = 0$

Thus, $m = -1$

Hence, the value of $(m^2 + m + 2)$ is 2.

2. We have $f(x) = \frac{x}{x^2 + 1}$

$\Rightarrow \quad f'(x) = \frac{(x^2 + 1)\cdot 1 - x\cdot 2x}{(x^2 + 1)^2}$

$\Rightarrow \quad f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$

Now, $f'(x) = 0$ gives $1 - x^2 = 0$

$\Rightarrow \quad x = \pm 1$

Thus, the number of points of extremum is 2.

3. We have $f(x) = (1 + b^2)x^2 + 2bx + 1$

$\Rightarrow \quad f'(x) = 2(1 + b^2)x + 2b$

For max or min, $f'(x) = 0$ gives

$\Rightarrow \quad 2(1 + b^2)x + 2b = 0$

$\Rightarrow \quad x = -\frac{b}{1 + b^2}$

Thus, the min value of $f(x) = m(b)$

$= (1 + b^2)\left(\frac{b^2}{(1 + b^2)^2}\right) + 2b\left(-\frac{b}{1 + b^2}\right) + 1$

$= \frac{b^2}{(1 + b^2)} - \frac{2b^2}{1 + b^2} + 1$

$$= \frac{b^2 - 2b^2 + 1 + b^2}{(1 + b^2)}$$

$$= \frac{1}{(1 + b^2)}$$

Hence, the maximum value of $m(b)$ is 1.

4. We have $f(x) = \frac{x^2 + 3x + 4}{x + 3}$, $x > 0$

$$= \frac{x(x + 3) + 4}{(x + 3)}$$

$$= x + \frac{4}{(x + 3)}$$

$$= (x + 3) + \frac{4}{(x + 3)} - 3$$

$$\geq 2\sqrt{(x + 3) \cdot \frac{4}{(x + 3)}} - 3$$

$$= 2.2 - 3$$

$$= 4 - 3$$

$$= 1$$

Hence, the minimum value of $f(x)$ is 1.

5. We have $f(x) = \frac{x^4 + x^2 + 4}{x^2 + 1}$, $x > 0$

$$= \frac{x^2(x^2 + 1) + 4}{(x^4 + 1)}$$

$$= x^2 + \frac{4}{(x^2 + 1)}$$

$$= (x^2 + 1) + \frac{4}{(x^2 + 1)} - 1$$

$$\geq 2 \times \sqrt{(x^2 + 1) \cdot \frac{4}{(x^2 + 1)}} - 1$$

$$= 4 - 1 = 3$$

Hence, the min value of $f(x)$ is 3.

6. Given $\frac{x^2}{3} + \frac{y^2}{4} + \frac{z^2}{9} = 1$

Applying, A.M $\geq$ G.M

$$\Rightarrow \quad \frac{\frac{x^2}{3} + \frac{y^2}{4} + \frac{z^2}{9}}{3} \geq \sqrt[3]{\frac{x^2}{3} \cdot \frac{y^2}{4} \cdot \frac{z^2}{9}}$$

$$\Rightarrow \quad \frac{1}{3} \geq \sqrt[3]{\frac{x^2y^2z^2}{108}}$$

$$\Rightarrow \quad \frac{1}{27} \geq \frac{x^2y^2z^2}{108}$$

$$\Rightarrow \quad \frac{x^2y^2z^2}{108} \leq \frac{1}{27}$$

$$\Rightarrow \quad x^2y^2z^2 \leq \frac{108}{27} = 4$$

$$\Rightarrow \quad (xyz) \leq 2$$

Hence, the max value of xyz is 2.

7. We have

$$2x^2 - 4xy + 3y^2 - 8x + 8y - 1 = 0 \quad \text{...(i)}$$

$$\Rightarrow \quad 4x - 4x\frac{dy}{dx} - 4y + 6y\frac{dy}{dx} - 8 + 8\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (8 - 4x + 6y)\frac{dy}{dx} = (8 + 4y - 4x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(8 + 4y - 4x)}{(8 - 4x + 6y)}$$

For max or min, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{(8 + 4y - 4x)}{(8 - 4x + 6y)} = 0$$

$$\Rightarrow \quad (8 + 4y - 4x) = 0$$

$$\Rightarrow \quad y = x - 2 \quad \text{...(ii)}$$

On solving (i) and (ii), we get,

$$2x^2 - 4x(x - 2) + 3(x - 2)^2 - 8x + 8(x - 2) = 1$$

$\Rightarrow x^2 - 4x - 5 = 0$

$\Rightarrow (x - 5)(x + 1) = 0$

$\Rightarrow x = -1, 5$

when $x = 5$, then $y = 3$

when $x = -1$, then $y = -3$.

Hence, the maximum value of y is 3.

8. We have $f(x) = x^3 - 3x$

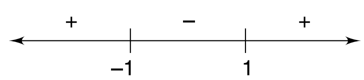
$$\Rightarrow \quad f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

For max or min, $f'(x) = 0$

$$\Rightarrow \quad 3(x^2 - 1) = 0$$

$$\Rightarrow \quad (x^2 - 1) = 0$$

$$\Rightarrow \quad x = \pm 1$$



Clearly, the function $f(x)$ have a point of max at $x = -1$ and the point of minima at $x = 1$

Thus m = min value of $f(x) = 1 - 3 = -2$

and n = max value of $f(x) = -1 + 3 = 2$

Hence, the value of $(m + n + 4)$

$$= -2 + 2 + 4$$

$$= 4.$$

9. We have $f(x) = \sqrt{9 - x^2} + \sqrt{x^2 - 4}$

Put $x = 9\cos^2\theta + 4\sin^2\theta$

Then $f(x) = \sqrt{5\sin^2\theta} + \sqrt{5\cos^2\theta}$

$= \sqrt{5}|\sin\theta| + \sqrt{5}|\cos\theta|$

$= \sqrt{5}(|\sin\theta| + |\cos\theta|)$

Max value of $f(x) = \sqrt{5}$ and the min value

of $f(x) = \sqrt{5}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \sqrt{5} \times \sqrt{2} = \sqrt{10}$

Thus, $R_f = \lfloor\sqrt{5}, \sqrt{10}\rfloor$

So, $a = 5$ and $b = 10$

Hence, the min value of $\left(\frac{b}{a} + 3\right)$

$= \left(\frac{10}{5} + 3\right) = 2 + 3 = 5$

10. We have $f(x) = x^4 - 4x^3 + 10$

$\Rightarrow \quad f'(x) = 4x^3 - 12x^2$

$\Rightarrow \quad f''(x) = 12x^2 - 24x$

Now, $f''(x) = 0$ gives $12x^2 - 24x = 0$

$\Rightarrow \quad 12x(x - 2) = 0$

$\Rightarrow \quad x = 0, 2$

Thus, the number of point of inflection is 2

So, $\quad m = 2$

Also, $f'(x) = 0$ gives $4x^3 - 12x^2 = 0$

$\Rightarrow \quad 4x^2(x - 3) = 0$

$\Rightarrow \quad x = 0, 3$

Thus, the number of point of extrema is 1 at $x = 3$, since $x = 0$ is the point of inflection

So, $n = 1$

Hence, the value of $(m + n + 1)$

$= 2 + 1 + 1$

$= 4$

Questions asked in Past IIT-JEE Exams

1. Let $y = \frac{(a + x)(b + x)}{(c + x)}$

$\frac{dy}{dx} = \frac{(c + x).(a + b + 2x) - (a + x)(b + x)}{(c + x)^2}$

Now, $\frac{dy}{dx} = 0$ gives

$\Rightarrow \frac{(c + x).(a + b + 2x) - (a + x)(b + x)}{(c + x)^2} = 0$

$\Rightarrow (c + x).(a + b + 2x) - (a + x)(b + x) = 0$

$\Rightarrow (c + x).(a + b + 2x) = (a + x)(b + x)$

$\Rightarrow ac + ax + bc + bx + 2cx + 2x^2$

$= ab + ax + bx + x^2$

$\Rightarrow x^2 + 2cx + ac + bc - ab = 0$

$\Rightarrow (x + c)^2 = ab - bc - ac + c^2$

$\Rightarrow (x + c)^2 = (b - c)(a - c)$

$\Rightarrow \quad x = \sqrt{(b - c)(a - c)} - c$

Hence, the minimum value is

$= \frac{(a + \sqrt{(b - c)(a - c)} - c)(b + \sqrt{(b - c)(a - c)} - c)}{\sqrt{(b - a)(a - c)}}$

$= \frac{(a - c + \sqrt{(b - c)(a - c)})(b - c + \sqrt{(b - c)(a - c)})}{\sqrt{(b - a)(a - c)}}$

$= (\sqrt{(a - c)} + \sqrt{(b - c)})(\sqrt{(b - c)} + \sqrt{(a - c)})$

$= (\sqrt{(a - c)} + \sqrt{(b - c)})^2$

2. As we know that A.M $\geq$ G.M

Thus, $\quad \left(\frac{x + y}{2}\right) \geq \sqrt{xy}$

$(x + y) \geq 2\sqrt{xy} = 2.1 = 2$

Therefore, the min value of $x + y$ is 2.

3. As $f(0) = f(1)$, by the Rolle's theorem, there exist $\alpha \in (0, 1)$ such that $f'(\alpha) = 0$

Let $x \in [0, 1]$ and $x \neq \alpha$ Applying Lagranges mean value theorem to $[x, \alpha]$ if $0 \leq x < \alpha$ or $[\alpha, x]$ if $\alpha < x \leq 1$, we get there exists a point β lying between α and x such that

$\Rightarrow \quad \frac{f'(x) - f'(\alpha)}{x - \alpha} = f''(\beta)$

$\Rightarrow \quad \frac{f'(x)}{x - \alpha} = f''(\beta) \quad (\because f'(\alpha) = 0)$

$\Rightarrow \quad f'(x) = (x - \alpha) f''(\beta)$

$\Rightarrow \quad |f'(x)| = |(x - \alpha)||f''(\beta)| \leq |(x - \alpha)| < 1$

$(\because x, \alpha \in [0, 1], x \neq \alpha)$

$\Rightarrow \quad |f'(x)| < 1, \forall x \in [0, 1]$

4. Given curve is $y = x^2$

Any point on the given parabola is (x, x^2) Distance between (x, x^2) and $(0, c)$ is

$z = \sqrt{(x^2) + (x^2 - c)^2}$

$\Rightarrow \quad z^2 = (x^2) + (x^2 - c)^2 = x^4 - (2c - 1)x^2 + c^2$

$= \left(x^2 - \left(\frac{2c - 1}{2}\right)\right)^2 + c^2 - \left(\frac{2c - 1}{2}\right)^2$

$= \left(x^2 - \left(\frac{2c - 1}{2}\right)\right)^2 + \left(c^2 - \left(\frac{4c^2 - 4c + 1}{4}\right)\right)$

$= \left(x^2 - \left(\frac{2c - 1}{2}\right)\right)^2 + \left(c - \frac{1}{4}\right)$

Which is minimum when $x^2 = \left(\frac{2c - 1}{2}\right)$

and the minimum value is

$= z_{\min} = \sqrt{\left(c - \frac{1}{4}\right)}$

5. (b)

Given $y = a\ln|x| + bx^2 + x$

$\Rightarrow \quad \frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

Now, $\left(\frac{dy}{dx}\right)_{x=-1} = 0$

$\Rightarrow \quad -a - 2b + 1 = 0$

$\Rightarrow \quad a + 2b = 1 \quad$ ...(i)

Also, $\left(\frac{dy}{dx}\right)_{x=2} = 0$

$\Rightarrow \quad \frac{a}{2} + 4b + 1 = 0$

$\Rightarrow \quad a + 8b + 2 = 0$

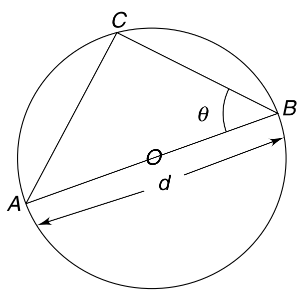
$\Rightarrow \quad a + 8b = -2 \quad$ ...(ii)

On solving (i) and (ii), we get,

$\Rightarrow \quad 6b = -3$

$\Rightarrow \quad b = -1/2$ and $a = 2$.

6. Ans. (a)



Area of the triangle ABC

$= \frac{1}{2} \times AC \times BC$

$= \frac{1}{2} \times d\sin\theta \times d\cos\theta$

$= \frac{d^2}{4} \times \sin 2\theta$

which is maximum, when $\sin(2\theta) = 1 \Rightarrow \frac{\pi}{4}$

Thus, AC = BC.

So, the triangle is isosceles.

7. Yes, it is true.

Applying, A.M. ≥ G.M.,

we get, $\frac{\log_a x + \log_x a}{2} \geq \sqrt{\log_a x \cdot \log_x a}$

$\Rightarrow \quad \frac{\log_a x + \log_x a}{2} \geq 1$

$\Rightarrow \quad (\log_a x + \log_x a) \geq 2$

Thus, the minimum value of $(\log_a x + \log_x a)$ is 2.

8 Given curve is $y = \frac{x}{1 + x^2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{(1 + x^2)\cdot 1 - x\cdot 2x}{(1 + x^2)^2}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{(1 - x^2)}{(1 + x^2)^2}$

$\Rightarrow \quad m = \frac{(1 - x^2)}{(1 + x^2)^2}$

$\Rightarrow \quad \frac{dm}{dx} = \frac{(1 + x^2)^2 \times (-2x) - 4x(1 - x^2)(1 + x^2)}{(1 + x^2)^4}$

Now, $\frac{dm}{dx} = 0$ gives

$\Rightarrow \quad \frac{(1 + x^2)^2 \times (-2x) - 4x(1 - x^2)(1 + x^2)}{(1 + x^2)^4} = 0$

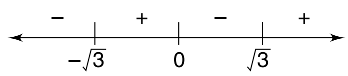
$\Rightarrow \quad (1 + x^2)^2 \times (-2x) - 4x(1 - x^2)(1 + x^2) = 0$

$\Rightarrow \quad (1 + x^2)^2 \times (-2x) = 4x(1 - x^2)(1 + x^2)$

$\Rightarrow \quad -2x(1 - x^2) = 4x(1 + x^2)$

$\Rightarrow \quad x(6 - 2x^2) = 0$

$\Rightarrow \quad x = 0, \pm\sqrt{3}$



At the *nbd* of $x = 0$, $f(x)$ will provide us the max slope.

Hence, the maximum slope is = 1

9. Given $f(x) = \sin^3 x + \lambda\sin^2 x$

$\Rightarrow \quad f'(x) = 3\sin^2 x\cos x + \lambda\sin 2x$

$\Rightarrow \quad f'(x) = \sin x\cos x(3\sin x + 2\lambda)$

$\Rightarrow \quad f'(x) = 0$ gives

$\Rightarrow \quad \sin x\cos x(3\sin x + 2\lambda) = 0$

$\Rightarrow \quad \sin x = 0, \cos x = 0, (3\sin x + 2\lambda) = 0$

$\Rightarrow \quad x = 0, x = \frac{\pi}{2}, x = \sin^{-1}\left(-\frac{2\lambda}{3}\right)$

$\Rightarrow \quad x = 0, x = \sin^{-1}\left(-\frac{2\lambda}{3}\right)$, since $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Also, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\Rightarrow \quad \sin\left(-\frac{\pi}{2}\right) < \sin x < \sin\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad 1 < \frac{-2\lambda}{3} < 1$

$\Rightarrow \quad -\frac{3}{2} < \lambda < \frac{3}{2}$

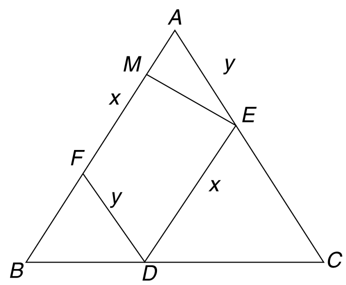
But if $\lambda = 0$, then $\sin x = 0$ has only one solution.

Thus, $\lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

For these value of λ, there are two distinct solutions.

Since $f(x)$ is continuous function, these solutions give one maxima and one minima. Beacuse for a continuous function, local max and min occur alternate way.

10. Let AF = x = DE and AE = y = DF and AB = c, BC = a and CA = b respectively.



Since the triangles CAB and CED are similar, so,

$$\frac{CE}{CA} = \frac{DE}{AB}$$

$$\Rightarrow \quad \frac{b-y}{b} = \frac{x}{c}$$

Now, area of the parallelogram AFDE = P = AF.EM = $x\, y \sin A$

$$\Rightarrow \quad P = xy \sin A$$

$$\Rightarrow \quad P = x \cdot b\left(1 - \frac{x}{c}\right)\sin A$$

$$\Rightarrow \quad \frac{dP}{dx} = \frac{b}{c}(c - 2x)\sin A$$

For extremum, $\frac{dP}{dx} = 0$

$$\Rightarrow \quad \frac{b}{c}(c - 2x)\sin A = 0$$

$$\Rightarrow \quad x = \frac{c}{2}$$

Also, $\left(\frac{d^2P}{dx^2}\right)_{x=\frac{c}{2}} = -\frac{2b}{c} < 0$

So P is maximum, when $x = c/2$

Thus, $P_{\max} = \frac{1}{4} bc \sin A = \frac{1}{2}\left(\frac{1}{2} bc \sin A\right)$

$$= \frac{1}{2}(\Delta ABC)$$

$$= \frac{1}{4}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & -q & 1 \\ r^2 & -r & 1 \end{vmatrix}$$

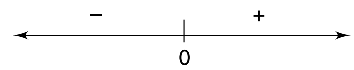
$$= \frac{1}{4}(p+q)(q+r)(r+p)$$

11. Ans. (c)

Given $P(x) = a_0 + a_1x^2 + a_2x^4 + \ldots + a_nx^{2n}$

$$\Rightarrow \quad P'(x) = 2a_1x + 4a_2x^3 + \ldots + 2na_nx^{2n-1}$$

$$\Rightarrow \quad P'(x) = x(2a_1 + 4a_2x^2 + \ldots + 2na_nx^{2n-2}$$



At the *nbd* of $x = 0$, $f'(x)$ changes from negative to positive, so $x = 0$ is the point of minima. Minimum value = $P(0) = a_0$, when $x = 0$

12. Given curve is $4x^2 + a^2y^2 = 4a^2$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{4} = 1 \qquad \ldots(i)$$

Any point of (i) is $P(a\cos\theta, 2\sin\theta)$ and given point is $(0, -2)$

Let $z = PQ^2 = (a\cos\theta)^2 + (2\sin\theta + 2)^2$

$$\Rightarrow \frac{dz}{d\theta} = 2(a\cos\theta)(-a\sin\theta) + 2(2\sin\theta + 2)(2\cos\theta)$$

$$\Rightarrow \frac{dz}{d\theta} = 2\cos\theta(4(1 + \sin\theta) - a^2\sin\theta)$$

$$\Rightarrow \frac{dz}{d\theta} = 2\cos\theta(4 + (4 - a^2)\sin\theta)$$

$$\Rightarrow \frac{d^2z}{d\theta^2} = 2\cos^2\theta(4 - a^2) - 2(4 + (4 - a^2)\sin\theta)\sin\theta)$$

For max or min, $\frac{dz}{d\theta} = 0$

$$\Rightarrow \quad 2\cos\theta(4 + (4 - a^2)\sin\theta) = 0$$

$$\Rightarrow \quad 2\cos\theta = 0, \ (4 + (4 - a^2)\sin\theta) = 0$$

$$\Rightarrow \quad \cos\theta = 0, \ \sin\theta = \frac{4}{4 - a^2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}, \ \sin\theta = \frac{4}{4 - a^2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}, \ \sin\theta = \frac{4}{4 - a^2} > 1$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}$$

$$\Rightarrow \quad \left(\frac{d^2z}{d\theta^2}\right)_{\theta=\frac{\pi}{2}} = 2(a^2 - 8) < 0$$

So z is maximum. Thus, distance is maximum.

Hence, the co-ordinates of the point P be (0, 2)

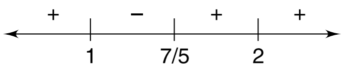
13. Given

$$f(x) = \int_1^x (2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2)dt$$

$$\Rightarrow f'(x) = 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2$$

$= (x-1)(x-2)^2(2(x-2)+3(x-1))$

$= (x-1)(x-2)^2(5x-7)$

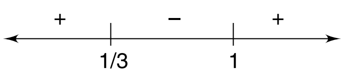


Here, $x = 1$ is the point of maxima, $x = 7/5$ is the point of minima and $x = 2$ is the point of inflection.

14. Given $y = x(x-1)^2$

$$\Rightarrow \quad \frac{dy}{dx} = (x-1)^2 + 2x(x-1)$$

$$\Rightarrow \quad \frac{dy}{dx} = (x-1)(3x-1)$$



Here, $x = 1/3$ is the point of maxima and $x = 1$ is the point of minima. But the maximum value is 2 at $x = 2$ and the minum value is 0 at $x = 1, 0$

15. Let $f(x) = 2\sin x + \tan x - 3x$

$$f'(x) = 2\cos x + \sec^2 x - 3$$

$$\Rightarrow \quad f'(x) = (\cos x + \cos x + \sec^2 x) - 3$$

$$\Rightarrow \quad f'(x) \geq 3 - 3 = 0 \quad (\because \text{A.M} \geq \text{G.M})$$

$$\Rightarrow \quad f'(x) \geq 0$$

$f(x)$ is strictly increasing function

Thus, $x \geq 0 \Rightarrow f(x) \geq f(0)$

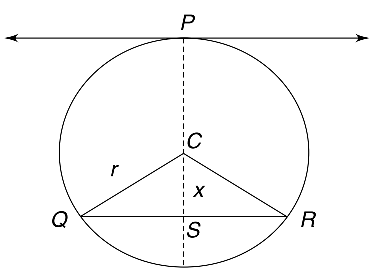
$$\Rightarrow \quad 2\sin x + \tan x - 3x \geq 0$$

$$\Rightarrow \quad 2\sin x + \tan x \geq 3x$$

Hence, the result.

16. Let S be the mid point of QR.

Then PS is perpendicular to QR and PS passes through the centre C.



Let x be the distance of the chord QR from the centre C of the circle of radius r. Here, S is the mid-point of QR, then QS $= \sqrt{r^2 - x^2}$ and PS $= r + x$. Therefore,

$$f(x) = Ar(\Delta PQR) = (x+r)\sqrt{r^2 - x^2}$$

$$\Rightarrow \quad g(x) = (x+r)^2(r^2 - x^2)$$

$$\Rightarrow \quad g'(x) = 2(x+r)(r^2 - x^2) - 2x(x+r)^2$$

For extrema, $g'(x) = 0$ gives

$$2(x+r)(r^2 - x^2) - 2x(x+r)^2 = 0$$

$$\Rightarrow \quad 2(x+r)(r^2 - x^2) = 2x(x+r)^2$$

$$\Rightarrow \quad (r^2 - x^2) = 2x(x+r)$$

$$\Rightarrow \quad (r - x) = x$$

$$\Rightarrow \quad x = \frac{r}{2}$$

when $g(x)$ is maximum, so $f(x)$ is maximum. Hence, $f(x)$ is maximum at $x = \frac{r}{2}$ and the maximum area of the triangle PQR

$$= \left(r + \frac{r}{2}\right)\sqrt{r^2 - \frac{r^2}{4}}$$

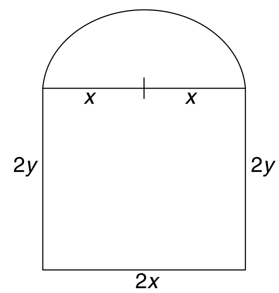
$$= \frac{3r}{2} \cdot \frac{r\sqrt{3}}{2} = \frac{3\sqrt{3}}{4} \cdot r^2$$

17. Let the sides of the rectangle be $2x$ and $2y$ respectively. Suppose semi-circle is on the side $2x$ and the fixed perimeter be P.

Thus, $\quad P = 2(2x + 2y) + \pi x$

$$P = (4 + \pi)x + 4y \qquad \text{...(i)}$$

Light transmitted by the semi-circle is $\frac{1}{2}.\pi.x^2$ and the light trasmitted by the rectangular part is $3.2x.2y = 12xy$



Let $\quad L = \frac{1}{2}\cdot \pi x^2 + 12xy$

$$L = \frac{1}{2}\cdot \pi x^2 + 12x\left(\frac{P - (4+\pi)x}{4}\right)$$

$$L = \frac{1}{2}\cdot \pi x^2 + 3x\,P - 3(4+\pi)x^2$$

$$L = 3xP - \left(12 + \frac{5\pi}{2}\right)x^2$$

$$\frac{dL}{dx} = 3P - (24 + 5\pi)x$$

$$\frac{d^2L}{dx^2} = -(24 + 5x) < 0$$

So, L is maximum.

For max or min, $\frac{dL}{dx} = 0$ gives $x = \frac{3P}{24 + 5\pi}$

From (i), we have, $4 + \pi + 4\left(\frac{y}{x}\right) = \frac{p}{x}$

$$\left(\frac{y}{x}\right) = \frac{1}{4}\left(\frac{p}{x} - 4 - \pi\right)$$

$$\left(\frac{y}{x}\right) = \frac{1}{4}\left(\frac{P(24 + 5\pi)}{3P} - 4 - \pi\right)$$

$$\left(\frac{y}{x}\right) = \frac{1}{12}((24 + 5\pi) - 12 - 3\pi)$$

$$\left(\frac{y}{x}\right) = \frac{1}{12}(12 + 2\pi) = \frac{6 + \pi}{6}$$

Hence, $y : x = (6 + \pi) : 6$

18. Let $f(x) = ax^3 + bx^2 + cx + d$

$\Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$

$\Rightarrow \quad f'(-1) = 0 \Rightarrow 3a - 2b + c = 0$

$\Rightarrow \quad f'\left(\frac{1}{3}\right) = 0 \Rightarrow 3a \cdot \frac{1}{9} + \frac{2b}{3} + c = 0$

$\Rightarrow \quad a + 2b + 3c = 0$

Also, $f(-2) = 0 \Rightarrow -8a + 4b - 2x + d = 0$

Again, $\int_{-1}^{1} (ax^3 + bx^2 + cx + d)dx = \frac{14}{3}$

$\Rightarrow \quad \int_{-1}^{1} (bx^2 + d)dx = \frac{14}{3}$

$\Rightarrow \quad \left(b\frac{x^3}{3} + dx\right)_{-1}^{1} = \frac{14}{3}$

$\Rightarrow \quad \left(\left(\frac{b}{3} + d\right) - \left(-\frac{b}{3} - d\right)\right) = \frac{14}{3}$

$\Rightarrow \quad \left(\frac{b}{3} + d\right) = \frac{7}{3}$

$\Rightarrow \quad b + 3d = 7$

On solving, we get, $a = 1$, $b = 1$, $c = -1$ and $d = 2$.
Thus, the given curve is

$$f(x) = x^3 + x^2 - x + 2$$

19. Obviously, f is decreasing in $[0, 1)$ and increasing in $[1, 3]$ Also, $f(1) = -1$.

Since $f(x)$ has the smallest value at $x = 1$

so, $\quad f(1 - h) \geq f(1)$

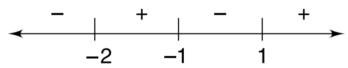
$\Rightarrow \quad \lim_{h \to 0} f(1 - h) \geq f(1)$

$\Rightarrow \quad \lim_{h \to 0} \left(-(1 - h)^3 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)}\right) \geq f(1)$

$\Rightarrow \quad \lim_{h \to 0} \left(-(1 - h)^3 + \frac{(b - 1)(b^2 + 1)}{(b + 1)(b + 2)}\right) \geq -1$

$\Rightarrow \quad \left(-1 + \frac{(b - 1)(b^2 + 1)}{(b + 1)(b + 2)}\right) \geq -1$

$\Rightarrow \quad \left(\frac{(b - 1)(b^2 + 1)}{(b + 1)(b + 2)}\right) \geq 0$



From the sign scheme, the values of b is

$$b \in (-2, -1) \cup [1, \infty)$$

20. Given $f(x) = 1 + 2\sin x + 3\cos^2 x$

$\Rightarrow \quad f'(x) = 2\cos x - 3\sin x \cos x$

$\Rightarrow \quad f''(x) = -2\sin x - 3\cos^2 x + 3\sin^2 x$

Now, $f'(x) = 0$ gives,

$\Rightarrow \quad 2\cos x - 3\sin x \cos x = 0$

$\Rightarrow \quad \cos x(2 - 3\sin x) = 0$

$\Rightarrow \quad \cos x = 0, \sin x = \frac{2}{3}$

$\Rightarrow \quad x = \frac{\pi}{2}, x = \sin^{-1}\left(\frac{2}{3}\right)$

At $x = \frac{\pi}{2}$, $f''\left(\frac{\pi}{2}\right) = -2 + 3 = 1 < 0$

$f(x)$ will provide us the min value and the min value is $f\left(\frac{\pi}{2}\right) = 1 + 2 + 0 = 3$

At $x = \sin^{-1}\left(\frac{2}{3}\right)$,

$f''\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = -\frac{4}{3} - 3 \cdot \frac{\sqrt{5}}{3} + \frac{4}{3} < 0$

$f(x)$ will provide us the maximum value and the maximum value is $f\left(\sin^{-1}\left(\frac{2}{3}\right)\right)$

$$= 1 + 2 \cdot \frac{2}{3} + 3 \cdot \frac{\sqrt{5}}{3} = \frac{7}{3} + \sqrt{5}$$

21. Let $y = \tan A . \tan B$

$\Rightarrow \quad y = \tan A . \tan\left(\frac{\pi}{3} - A\right)$

$\Rightarrow \quad y = \tan A\left(\frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}\right)$

$\Rightarrow \quad y = t\left(\frac{\sqrt{3} - t}{1 + \sqrt{3}t}\right), t = \tan A$

$\Rightarrow \quad y = \left(\frac{t\sqrt{3} - t^2}{1 + \sqrt{3}t}\right)$

$\Rightarrow \quad \frac{dy}{dt} = \frac{(1 + \sqrt{3}t)(\sqrt{3} - 2t) - (t\sqrt{3} - t^2) \cdot \sqrt{3}}{(1 + \sqrt{3}t)^2}$

For extrema, $\frac{dy}{dt} = 0$ gives

$\Rightarrow \quad \frac{(1 + \sqrt{3}t)(\sqrt{3} - 2t) - (t\sqrt{3} - t^2) \cdot \sqrt{3}}{(1 + \sqrt{3}t)^2} = 0$

$\Rightarrow \quad (1 + \sqrt{3}\,t)(\sqrt{3} - 2t) = (t\sqrt{3} - t^2)\cdot\sqrt{3}$

$\Rightarrow \quad \sqrt{3} + 3t - 2t - 2\sqrt{3}\,t^2 = 3t - \sqrt{3}\,t^2$

$\Rightarrow \quad \sqrt{3} - 2t - \sqrt{3}\,t^2 = 0$

$\Rightarrow \quad \sqrt{3}\,t^2 + 2t - \sqrt{3} = 0$

$\Rightarrow \quad \sqrt{3}\,t^2 + (3 - 1)t - \sqrt{3} = 0$

$\Rightarrow \quad \sqrt{3}\,t(t + \sqrt{3}) - 1(t + \sqrt{3}) = 0$

$\Rightarrow \quad (t + \sqrt{3})(\sqrt{3}\,t - 1) = 0$

$\Rightarrow \quad t = -\sqrt{3}, \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \tan A = -\sqrt{3}, \frac{1}{\sqrt{3}}$

$\Rightarrow \quad A = \frac{\pi}{6}\ (\because A > 0)$ and $B = \frac{\pi}{6}$

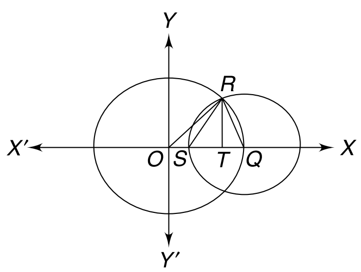
Hence, the min value of $\tan A.\tan B$

$$= \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} = \frac{1}{3}$$

22. Given circle is $x^2 + y^2 = 1$ ...(i)

Centre = (0, 0) and radius = $OP = OQ = 1$

so, $Q = (1, 0)$



Equation of a circle with centre at Q and variable radius is $(x - 1)^2 + y^2 = r^2$. ...(ii)

From (i) and (ii), we get,

$$2x - 1 = 1 - r^2$$

$$x = 1 - \frac{r^2}{2}$$

Now, $RT = \sqrt{OR^2 - OT^2}$

$$= \sqrt{1 - \left(\frac{1 - r^2}{2}\right)^2}$$

$$= \sqrt{r^2 - \frac{r^4}{4}}$$

Now, the area of ΔQSR

$$A = \frac{1}{2}\cdot QS.RT$$

$$A^2 = \frac{1}{4}\cdot(QS)^2(RT)^2$$

$$B = \frac{1}{4}\cdot r^2\left(r^2 - \frac{r^4}{4}\right)$$

$$B = \frac{1}{16}(4r^4 - r^6)$$

$$\frac{dB}{dr} = \frac{1}{16}(16r^3 - 6r^5)$$

$$\frac{d^2B}{dr^2} = \frac{1}{16}(48r^2 - 30r^4)$$

For extremum, $\frac{dB}{dr} = 0$ gives

$$\frac{1}{16}(16r^3 - 6r^5) = 0$$

$$(16r^3 - 6r^5) = 0$$

$$3r^2 = 8$$

$$r = 2\sqrt{\frac{2}{3}}$$

At $r = 2\sqrt{\frac{2}{3}}$, $\frac{d^2B}{dr^2} < 0$ So, B is maximum Thus, area is maximum

Hence, the maximum area = $\frac{4}{3\sqrt{3}}$ sq.u.

23. Given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let $P(x, y)$ be any point and the foci are F_1 (–ae, 0) and F_2 (ae, 0) respectively. Then the area of the triangle PF_1F_2

$$= A = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -ae & 0 & 1 \\ ae & 0 & 1 \end{vmatrix}$$

$$A = \frac{1}{2}(2aey) = aye = aeb\sqrt{1 - \frac{x^2}{a^2}}$$

$$A = eb\sqrt{a^2 - x^2}$$

So A is maximum, when $x = 0$ and the maximum area = $aeb = b\sqrt{a^2 - b^2}$

24. Equation of any line passing through (h, k) is

$$(y - h) = m(x - k)$$

Put $y = 0$, then $x = \left(k - \frac{h}{m}\right)$

So, the point P is $\left(\left(k - \frac{h}{m}\right), 0\right)$

Put $x = 0$, then $y = h - mk$

so, the point Q is $(0, (h - mk))$

Hence, the area of the triangle OPQ

$$= A = \frac{1}{2}\cdot OP.OQ$$

$\Rightarrow \quad A = \frac{1}{2}\cdot\left(k - \frac{h}{m}\right)\cdot(h - mk)$

$\Rightarrow \quad \frac{dA}{dm} = \frac{1}{2}\left((h - mk)\left(\frac{h}{m^2}\right) + \left(k - \frac{h}{m}\right)\cdot(-k)\right).$

For extrema, $\frac{dA}{dm} = 0$ gives

$$\Rightarrow \quad \frac{1}{2}\left[(h - mk)\left(\frac{h}{m^2}\right) + \left(k - \frac{h}{m}\right)\cdot(-k)\right]. = 0$$

$$\Rightarrow \quad \left[(h - mk)\left(\frac{h}{m^2}\right) - \left(k - \frac{h}{m}\right)\cdot k)\right] = 0$$

$$\Rightarrow \quad (h - mk)\left(\frac{h}{m^2}\right) = \left(k - \frac{h}{m}\right)\cdot k$$

$$\Rightarrow \quad (h - mk)\left(\frac{h}{m^2}\right) = -\left(\frac{h - km}{m}\right)\cdot k$$

$$\Rightarrow \quad (h - mk)\left(\frac{h}{m^2} + \frac{k}{m}\right) = 0$$

$$\Rightarrow \quad (h - mk)(h + km) = 0$$

$$\Rightarrow \quad m = \frac{h}{k}, -\frac{h}{k}$$

Since $h > 0$, $k > 0$ and $m < 0$, so the value of m is $m = -\frac{h}{k}$

Hence, the min area of the triangle OPQ

$$= \frac{1}{2}\left(k + \frac{h}{\frac{h}{k}}\right)\left(h + \frac{h}{k}\cdot k\right)$$

$$= \frac{1}{2}.2k.2h = 2hk$$

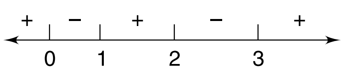
25. Ans. (b)

Given function is $f(x) = x^{25}(1 - x)^{75}$

$$\Rightarrow \quad f'(x) = 25x^{24}(1 - x)^{75} - 75x^{25}(1 - x)^{74}$$

$$\Rightarrow \quad f'(x) = 25x^{24}(1 - x)^{74}(1 - x - 3x)$$

$$\Rightarrow \quad f'(x) = -25x^{24}(x - 1)^{74}(4x - 1)$$



Clearly $x = 1/4$ is the point of maxima.

26. Given $f(x) = \frac{1}{8}\log x - bx + x^2$, $x > 0$

$$\Rightarrow \quad f'(x) = \frac{1}{8x} - b + 2x$$

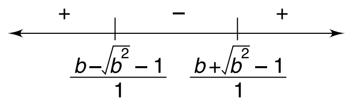
For max or min, $f'(x)$ gives

$$\frac{1}{8x} - b + 2x = 0$$

$$\Rightarrow \quad 16x^2 - 8bx + 1 = 0$$

$$\Rightarrow \quad x = \frac{8b \pm \sqrt{64b^2 - 64}}{32} = \frac{b \pm \sqrt{b^2 - 1}}{4}$$

Clearly, root is real, if $b \geq 1$



Thus, $x = \frac{b - \sqrt{b^2 - 1}}{b}$ is the point of maxima and $x = \frac{b + \sqrt{b^2 - 1}}{b}$ is the point of minima.

27. No questions asked in 1997.

28. Ans. (d)

Given $f(x) = \frac{x^2 - 1}{x^2 + 1}$

$$\Rightarrow \quad f'(x) = \frac{(x^2 + 1)\cdot 2x - (x^2 - 1)2x}{(x^2 + 1)^2}$$

$$\Rightarrow \quad f'(x) = \frac{4x}{(x^2 + 1)^2}$$

Clearly, $x = 0$ is the point of minima

Thus, the minimum value = –1.

29. Ans. (a)

Given $f(x) = \cos x + \cos(x\sqrt{2})$

The maximum value of $f(x)$ occurs when

$$\Rightarrow \quad \cos x = 1 \ \& \ \cos(x\sqrt{2}) = 1$$

$$\Rightarrow \quad x = 2n\pi \ \& \ x\sqrt{2} = 2m\pi, \ m, n, \in I$$

$$\Rightarrow \quad x = 2n\pi \ \& \ x = \sqrt{2}m\pi, \ m, n \in I$$

$$\Rightarrow \quad 2n\pi = \sqrt{2}m\pi$$

$$\Rightarrow \quad m = \sqrt{2}n$$

It is possible only when $x = 0$.

Thus, the function will provide us the maximum value at $x = 0$.

and the maximu value is 2.

30. Given

$$\Rightarrow f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 2(ax + b) & 2ax + 2b + 1 & 2ax + b \end{vmatrix}$$

$$\Rightarrow f'(x) = \begin{vmatrix} 2ax & 2ax - 1 & 2ax + b + 1 \\ b & b + 1 & -1 \\ 2b & 2b + 2 & -1 \end{vmatrix}$$

$(R_3 \to R_3 - R_1)$

$$\Rightarrow f'(x) = \begin{vmatrix} 2ax & -1 & b + 1 \\ b & 1 & -1 - b \\ 2b & 2 & -1 - 2b \end{vmatrix}$$

$(C_3 \to C_3 - C_1, C_2 \to C_2 - C_1)$

$$\Rightarrow f'(x) = \begin{vmatrix} 2ax & -1 & b+1 \\ 2ax+b & 0 & 0 \\ 2b & 2 & -1-2b \end{vmatrix}$$

$(R_2 \to R_2 + R_1)$

$$\Rightarrow f'(x) = -(2ax + b)\begin{vmatrix} -1 & b+1 \\ 2 & -1-2b \end{vmatrix}$$

$\Rightarrow f'(x) = -(2ax + b)(1 + 2b - 2b - 2)$

$\Rightarrow f'(x) = (2ax + b)$

$\Rightarrow f(x) = ax^2 + bx + c$

Given $f(0) = 2 \Rightarrow c = 2$

and $f(1) = 1 \Rightarrow a + b + c = 1$

$\Rightarrow \quad a + b + 2 = 1$

$\Rightarrow \quad a + b = -1$ ...(i)

Also, $\quad f'\left(\frac{5}{2}\right) = 0$

$\Rightarrow \quad 5a + b = 0$

$\Rightarrow \quad b = -5a$ ...(ii)

On solving, we get, $a = 1/4$ and $b = -5/4$

Thus, the given function $f(x)$ is

$$f(x) = \frac{x^2}{4} - \frac{5x}{4} + 2$$

31. Let the point P be $(r\cos\theta, r\sin\theta)$

Given curve is $ax^2 + 2bxy + ay^2 = c$ ...(i)

(i) reduces to $ar^2\cos^2\theta + r^2 b\sin 2\theta + ar^2\sin^2\theta = c$

$$\Rightarrow \quad r^2 = \frac{c}{(a\cos^2\theta + b\sin 2\theta + a\sin^2\theta)}$$

$$\Rightarrow \quad r^2 = \frac{c}{a + b\sin 2\theta}$$

$\Rightarrow r^2$ will be minimum when $(a + b\sin 2\theta)$ is maximum

$$\sin 2\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

Therefore, $r^2 = \dfrac{c}{a + b}$

Thus, the points are

$$= (r\cos\theta, r\sin\theta)$$

$$= \left(r\cos\left(\frac{\pi}{4}\right), r\sin\left(\frac{\pi}{4}\right)\right), \left(r\cos\left(\frac{5\pi}{4}\right), r\sin\left(\frac{5\pi}{4}\right)\right)$$

$$= \left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right), \left(-\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}\right)$$

32. Given curve is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Equation of any point $P(a\cos\theta, b\sin\theta)$ is

$$\frac{a\cos\theta x}{a^2} + \frac{b\sin\theta y}{b^2} = 1$$

$$\Rightarrow \quad \frac{x}{a\sec\theta} + \frac{y}{b\operatorname{cosec}\theta} = 1 \quad \text{...(ii)}$$

ON is perpendicular to the tangent

Thus, equation of ON is

$$\frac{x}{b\operatorname{cosec}\theta} - \frac{y}{a\sec\theta} = 0$$

$$\Rightarrow \quad \frac{x}{b\operatorname{cosec}\theta} = \frac{y}{a\sec\theta}$$

$$\Rightarrow \quad y = \frac{ax}{b}\tan\theta \quad \text{...(iii)}$$

On solving (ii) and (iii), we get,

$$x = \frac{ab^2\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

and $$y = \frac{a^2\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

Thus, the co-ordinates of N

$$= \left(\frac{ab^2\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}, \frac{a^2 b\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta}\right)$$

Therefore, the area of the triangle OPN = A

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ a\cos\theta & b\sin\theta & 1 \\ \dfrac{ab^2\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta} & \dfrac{a^2 b\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times \frac{a\cos\theta . b\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta}\begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ b^2 & a^2 & 1 \end{vmatrix}$$

$$= \frac{1}{4} \times \frac{ab\sin 2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}(a^2 - b^2)$$

$$= \frac{(a^2 - b^2)ab}{4} \times \frac{2}{a^2\tan\theta + b^2\cot\theta}$$

$$\le \frac{(a^2 - b^2)ab}{2} \times \frac{1}{2ab} = \left(\frac{a^2 - b^2}{4}\right)$$

Hence, the co-ordinates of the point P

$$= \left(\frac{ab}{\sqrt{a^2 + b^2}}, \frac{ab}{\sqrt{a^2 + b^2}}\right)$$

33. Any normal to the ellipse at $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - \text{by}\operatorname{cosec}\theta = a^2 - b^2$

Here, $a = 3$ and $b = 2$, so the normal reduces to $3x\sec\theta - 2y\operatorname{cosec}\theta = 3^2 - 2^2 = 5$

If p be the length of the perpendicular from the origin, so $p = \dfrac{5}{\sqrt{9\sec^2\theta + 4\operatorname{cosec}^2\theta}}$

$\Rightarrow \quad p = \dfrac{5}{\sqrt{9(1+\tan^2\theta)+4(1+\cos^2\theta)}}$

$\Rightarrow \quad p = \dfrac{5}{\sqrt{13+(9\tan^2\theta+4\cot^2\theta)}}$

$\Rightarrow \quad f(\theta) = (9\tan^2\theta + 4\cot^2\theta)$

$\Rightarrow \quad f(\theta) = (3\tan\theta)^2 + (2\cos\theta)^2$

$\Rightarrow \quad f(\theta) = ((3\tan\theta) - (2\cos\theta))^2 + 12\tan\theta.\cot\theta$

$\Rightarrow \quad f(\theta)\ ((3\tan\theta) - (2\cos\theta)^2 + 12$

$f(\theta)$ is minimum, when

$\Rightarrow \quad ((3\tan\theta) - (2\cot\theta))^2 = 0$

$\Rightarrow \quad (3\tan\theta) = (2\cot\theta)$

$\Rightarrow \quad \tan^2\theta = \dfrac{2}{3}$

$\Rightarrow \quad \tan\theta = \sqrt{\dfrac{2}{3}}$

Thus, p is maximum, when $f(\theta)$ is minimum

where $\tan\theta = \sqrt{\dfrac{2}{3}}$

So, $\tan\theta = \sqrt{\dfrac{5}{3}}$, $\operatorname{cosec}\theta = \sqrt{\dfrac{5}{2}}$

Hence, the equation of the normal is

$$3\cdot\sqrt{\frac{5}{3}}\cdot x - 2\cdot\sqrt{\frac{5}{2}}\cdot y = 5$$

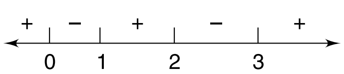
$\Rightarrow \quad x\cdot\sqrt{3} - y\cdot\sqrt{2} = \sqrt{5}.$

34. Ans. (b, d)

Given function is

$$f(x) = \int_{-1}^{x} t(e^t-1)(t-1)(t-2)(t-3)^5\,dt$$

$\Rightarrow \quad f'(x) = x(e^x-1)(x-1)(x-2)^3(x-3)^5$



From the sign scheme, it is clear that, $f(x)$ has a local max at $x = 0$ and $x = 2$ and local minima at $x = 1$ and $x = 3$.

35. Given $-1 \le p \le 1$

and $4x^3 - 3x - p = 0$

$\Rightarrow \quad p = 3x - 4x^3$

$\Rightarrow \quad p = \cos 3\theta$, where $x = \cos\theta$

Since $\quad x = \cos\theta \in \left[\dfrac{1}{2}, 1\right]$

$\Rightarrow \quad 0 \le \theta \le \dfrac{\pi}{3}$

$\Rightarrow \quad 0 \le 3\theta \le \pi$

$\Rightarrow \quad -1 \le \cos(3\theta) \le 1$

$\Rightarrow \quad -1 \le p \le 1$

Hence, the result.

Also, $\quad \cos 3\theta = p$

$\Rightarrow \quad 3\theta = \cos^{-1}(p)$

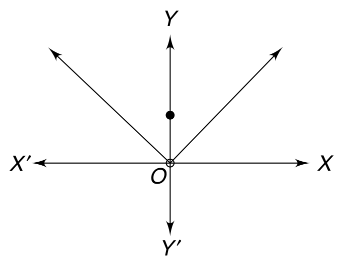
$\Rightarrow \quad \theta = \dfrac{1}{3}\cos^{-1}(p)$

Thus, $x = \cos\theta = \cos\left(\dfrac{1}{3}\cos^{-1}(p)\right)$

36. Ans. (d)

Given $f(x) = \begin{cases} |x| & : 0 < |x| \le 2 \\ 1 & : x = 0 \end{cases}$

$\Rightarrow \quad f(x) = \begin{cases} |x| & : -2 \le x \le 2 - \{0\} \\ 1 & : x = 0 \end{cases}$



From the graph, it is clear that, $f(x)$ has neither a local maximum nor a local minimum at $x = 0$.

37. Ans. (d)

Given $\quad f(x) = (1+b^2)x^2 + 2bx + 1$

$\Rightarrow \quad f'(x) = 2(1+b^2)x + 2b$

$\Rightarrow \quad f''(x) = 2(1+b^2) > 0$

So $f(x)$ will provide us the minimum value.

Now, $f'(x) = 0$ gives $2(1+b^2)x + 2b = 0$

$\Rightarrow \quad x = -\dfrac{2b}{2(1+b^2)} = -\dfrac{b}{(1+b^2)}$

Thus the min value of $f(x) = m(b)$

$$= (1+b^2)\left(-\frac{b}{(1+b^2)}\right)^2 + 2b\left(-\frac{b}{(1+b^2)}\right) + 1$$

$$= \left(\frac{b^2}{(1+b^2)}\right) - \left(\frac{2b^2}{(1+b^2)}\right) + 1$$

$$= \left(1 - \frac{b^2}{1+b^2}\right)$$

$$= \frac{1}{1+b^2}$$

Thus, the range of $m(b) = (0, 1]$

38. Equation of any line passing through (8, 2)

is $\quad y - 2 = m(x - 8)$

Put $\quad y = 0$, then $x = 8 - \dfrac{2}{m}$

So, $\quad OP = \left(8 - \frac{2}{m}\right)$

Put $x = 0$, then $y = 2 - 8m$

So, OQ = $(2 - 8m)$.

Let $\quad S = OP + OQ$

$\Rightarrow \quad S = \left(8 - \frac{2}{m}\right) + (2 - 8m)$

$\Rightarrow \quad \frac{dS}{dm} = \frac{2}{m^2} - 8$

$\Rightarrow \quad \frac{d^2S}{dm^2} = -\frac{4}{m^3}$

Now, $\frac{dS}{dm} = 0$ gives $\frac{2}{m^2} - 8 = 0$

$\Rightarrow \quad 8m^2 = 2$

$\Rightarrow \quad 4m^2 = 1$

$\Rightarrow \quad m = \pm\frac{1}{2}$

Since slope is negative, so $m = -1/2$

when $m = -1/2$, $\frac{d^2S}{dm^2} > 0$.

So, the function will provide us the minimum value
Thus, the minimum value

$= (8 + 4) + (2 + 4) = 18$

39. Ans. (b)

Equation of tangent to the ellipse $\frac{x^2}{27} + \frac{y^2}{1} = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ is

$$\frac{3\sqrt{3}\cos\theta}{27}x + \frac{\sin\theta}{1}y = 1$$

$\Rightarrow \quad \frac{x}{3\sqrt{3}\sec\theta} + \frac{y}{\operatorname{cosec}\theta} = 1$

Let $\quad S$ = sum of the intersepts

$\Rightarrow \quad S = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$

$\Rightarrow \quad \frac{dS}{d\theta} = 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta$

Now $\frac{dS}{d\theta} = 0$ gives

$$3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta = 0$$

$\Rightarrow \quad 3\sqrt{3}\sec\theta\tan\theta = \operatorname{cosec}\theta\cot\theta$

$\Rightarrow \quad \tan^3\theta = \frac{1}{3\sqrt{3}}$

$\Rightarrow \quad \tan^3\theta = \left(\frac{1}{\sqrt{3}}\right)^3$

$\Rightarrow \quad \tan\theta = \left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \quad \tan\theta = \left(\frac{1}{\sqrt{3}}\right) = \tan\left(\frac{\pi}{6}\right)$

$\Rightarrow \quad \theta = \left(\frac{\pi}{6}\right)$

40. Given $f(x) = x^2 + 2bx + 2c^2$

$= (x + b)^2 + (2c^2 - b^2)$

Thus, the min value of $f(x)$ is $(2c^2 - b^2)$

Also, $g(x) = -x^2 - 2cx + b^2$

$= b^2 + c^2 - (x + c)^2$

Thus, the max value of $g(x)$ is $2b^2$

Given condition is,

Min value of $f(x)$ > max value of $g(x)$

$\Rightarrow \quad 2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \quad c^2 > 2b^2$

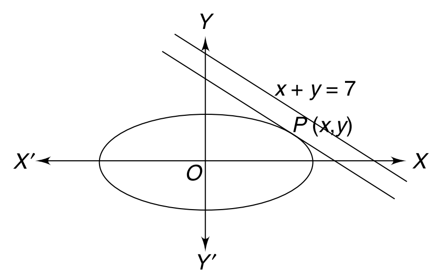
$\Rightarrow \quad \sqrt{c^2} > \sqrt{2b^2}$

$\Rightarrow \quad |c| > \sqrt{2}\,|b|$

41. Given curve is $x^2 + 2y^2 = 6$

$\Rightarrow \quad \frac{x^2}{6} + \frac{y^3}{3} = 1$

and the given line is $x + y = 7$.



So the tangent is parallel to the line, so slope of the tangent to the curve is equal to the slope of the line.

$\Rightarrow \quad -\frac{x}{2y} = -1$

$\Rightarrow \quad x = 2y$

Put $x = 2y$ in the curve $\frac{x^2}{6} + \frac{y^2}{3} = 1$, we get,

$\Rightarrow \quad \frac{4y^2}{6} + \frac{y^2}{3} = 1$

$\Rightarrow \quad \frac{2y^2}{3} + \frac{y^2}{3} = 1$

$\Rightarrow \quad 3y^2 = 3$

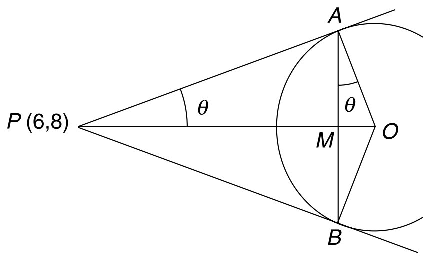
$\Rightarrow \quad y^2 = 1$

$\Rightarrow \quad y = \pm 1$

when $y = \pm 1$, then $x = \pm 2$

So, the point can be $(\pm 2, \pm 1)$

42. Given circle is $x^2 + y^2 = r^2$



We have $OP = \sqrt{6^2 + 8^2} = 10$

$BM = r\cos\theta,\ OM = r\sin\theta$

where $0 < \theta < \frac{\pi}{2}$

Also, $\sin\theta = \frac{r}{10}$

If A denote the area of the triangle PAB, then

$\Rightarrow \quad A = 2ar(\Delta PBM)$

$\Rightarrow \quad A = 2 \times \frac{1}{2} \times PM \times BM$

$\Rightarrow \quad A = PM \times BM$

$= (OP - OM).BM$

$= (10 - r\sin\theta).r\cos\theta$

$= (10 - 10\sin\theta.\sin\theta)(10\sin\theta.\cos\theta)$

$= 100\sin\theta\cos^3\theta$

$$\frac{dA}{d\theta} = 100\cos\theta\cos^3\theta - 300\sin^2\theta\cos^2\theta$$

$$= 300\cos^4\theta\left(\frac{1}{\sqrt{3}} - \tan\theta\right)\left(\frac{1}{\sqrt{3}} + \tan\theta\right)$$

For maxima and minima, $\frac{dA}{d\theta} = 0$ gives

$\Rightarrow \quad \frac{1}{\sqrt{3}} - \tan\theta = 0$

$\Rightarrow \quad \theta = \frac{\pi}{6}$

Thus, $\frac{dA}{d\theta} = \begin{cases} > 0 & : 0 < \theta < \frac{\pi}{6} \\ < 0 & : \frac{\pi}{6} < \theta < \frac{\pi}{6} \end{cases}$

Therefore A is maximum, when $\theta = \frac{\pi}{6}$

and $r = 10.\sin\left(\frac{\pi}{6}\right) = 5$

43. Ans. (a)

Given $f(x) = x^3 + bx^2 + cx + d$

$\Rightarrow \quad f'(x) = 3x^2 + 2bx + c$

Now, $D = 4b^2 - 12c$

$= 4(b^2 - c) - 8c$

$= 4(b^2 - c) + 8(-c)$

< 0

Thus, $a > 0$, $D < 0 \Rightarrow f'(x) > 0,\ \forall\ x \in R$

Therefore, f is strictly increasing function.

44. Ans. ab

Given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Any point on the curve is $P(a\cos\theta, b\sin\theta)$

Equation of any tangent to the given curve is

$$\frac{a\cos\theta}{a^2}x + \frac{b\sin\theta}{b^2}y = 1$$

$$\Rightarrow \quad \frac{x}{a\sec\theta} + \frac{y}{b\,\text{cosec}\,\theta} = 1$$

x-intercept $= a\sec\theta$ and y-intercept $= b\,\text{cosec}\,\theta$

Area of the triangle $= \frac{1}{2} \times a\sec\theta \times b\,\text{cosec}\,\theta$

$= \frac{ab}{2\sin\theta\cos\theta}$

$= \frac{ab}{\sin 2\theta}$

Minimum area of the triangle $= ab$,

when $\sin 2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$.

45. Ans. $4\sqrt{65}$ units.

Let the polynomial be

$f(x) = ax^3 + bx^2 + cx + d$

According to the given conditions,

$f(-1) = -a + b + c + d = 10$

$f(1) = a + b + c + d = 6$

Also, $\quad f'(-1) = 3a - 2b + c = 0$

$\Rightarrow \quad f''(1) = 6a + 2b = 0$

$\Rightarrow \quad a + 3b = 0$

On solving, we get,

$a = 1, b = -3, c = -9, d = 5$

Thus, $\quad f(x) = x^3 - 3x^2 - 9x + 5$

$\Rightarrow \quad f'(x) = 3x^2 - 6x + 9$

$\Rightarrow \quad f'(x) = 3(x^2 - 2x - 3) = 3(x + 1)(x - 3)$

Therefore, $x = -1$ is point of maxima and $x = 3$ is the point of minima

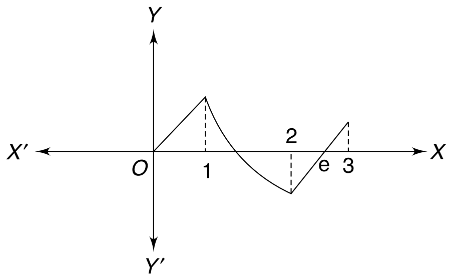
Now, $f(-1) = -1 - 3 + 9 + 5 = 10$, pt is $(-1, 10)$

$f(3) = 27 - 27 - 27 + 5 = -22$, pt is $(3, -22)$

Distance $= \sqrt{(-22 - 10)^2 + (3 + 1)^2}$

$= \sqrt{1024 + 16} = \sqrt{1040} = 4\sqrt{65}$

46. Ans. (a, d)



Given $g(x) = \int_0^x f(t)\,dt,\ x \in [1, 3]$

$\Rightarrow \quad g'(x) = f(x)$

$\Rightarrow \quad g'(x) = 0$ gives $f(x) = 0$

$\Rightarrow \quad 2 = e^{x-1}$ or $x = e$

$\Rightarrow \quad x - 1 = \log 2$ or $x = e$

$\Rightarrow \quad x = 1 + \log 2$ or $x = e$

As $g'(x) = \begin{cases} > 0 : & 0 < x < 1 + \log 2 \\ < 0 : & 1 + \log 2 < x < e \\ > 0 : & e < x < 3 \end{cases}$

Thus $g(x)$ has a local maximum at $x = 1 + \log 2$ and a local minima at $x = e$.

47. Ans. (a, b, c)

Let $\quad f'(x) = a(x + 1)(x - \alpha)$

$\Rightarrow \quad f''(x) = a(x + 1) + a(x - \alpha)$

$\Rightarrow \quad f'''(x) = a(2x + 1 - \alpha)$

Since $f'(x)$ has a local minimum at $x = 0$,

$f'(x)$ has a local minima at $x = 0$,

so $\quad f''(0) = 0$ and $f''(0) > 0$

$\Rightarrow \quad a(1 - \alpha) = 0,\ a > 0$

$\Rightarrow \quad \alpha = 1$

Thus, $f'(x) = a(x + 1)(x - 1) = a(x^2 - 1)$

$\Rightarrow \quad f(x) = a\left(\frac{x^3}{3} - x\right) + k$

As $f(-1) = 2$ and $f(3) = 18$, we get,

$$a\left(-\frac{1}{3} + 1\right) + k = 2 \text{ and } 6a + k = 18$$

$$a = 3,\ k = 0$$

Therefore, $f(x) = x^3 - 3x$

$\Rightarrow \quad f'(x) = 3x^2 - 3$

$\Rightarrow \quad f''(x) = 6x$

Now, $f'(x) = 0$ gives $x = 1, -1$.

As $f''(x) = 6 > 0$, so $f(x)$ has a local minimum at $x = 1$.

Thus, $a = 1$ and $f(0) = -2$.

Distance between $(-1, 2)$ and $(1, -2)$

$$= \sqrt{(1 + 1)^2 + (2 + 2)^2}$$

$$= \sqrt{20}$$

$$= 2\sqrt{5}$$

Again, $f'(x) = \begin{cases} > 0 : & x < -1 \\ < 0 : & -1 < x < 1 \\ > 0 : & x > 1 \end{cases}$

Thus, $f(x)$ is increasing for $x \in \lfloor 1, 2\sqrt{5} \rfloor$.

48. We have $g(x) = (f'(x))^2 + g'(x)f''(x)$

$$g(x) = \frac{d}{dx}(f(x).f'(x))$$

By the Rolle's theorem, between any two roots of a polynomial, there is a root of its derivative.

Now, $f(x).f'(x) = 0$

Either $f(x) = 0$ or $f'(x) = 0$

Thus, $f(x)$ has four zeroes at $x = a$, between b and c, between c and d and at e

So, $f'(x)$ has atleast 3 zeroes.

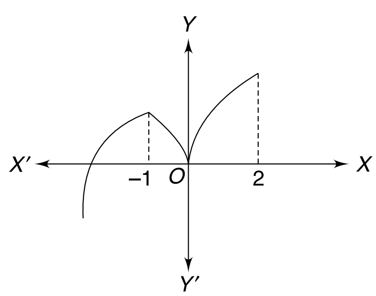
Thus, $f(x)f'(x)$ has atleast 7 zeroes.

Therefore, $g(x)$ has atleast 6 zeroes.

49. No questions asked in 2007.

50. Ans. (c)

Given $f(x) = \begin{cases} (2 + x)^3 & : -3 < x \le -1 \\ x^{2/3} & : -1 < x < 2 \end{cases}$



From the graph, it is clear that, the function $f(x)$ will have a local max value at $x = -1$ and local min value at $x = 0$.

Thus, the total numbers of extrema = 2.

51. Ans. (i) (a), (ii) (a), (iii) (b)

$\Rightarrow f'(x) =$

$$\frac{(x^2 + ax + 1)(2x - a) - (x^2 - ax + 1).(2x + a)}{(x^2 + ax + 1)^2}$$

$\Rightarrow f'(x) = \dfrac{2ax^2 - 2a}{(x^2 + ax + 1)^2}$

$\Rightarrow f''(x) =$

$$\frac{(x^2 + ax + 1)^2 .4ax - 4ax(x^2 - 1)(2x + a)(x^2 + ax + 1)}{(x^2 + ax + 1)^4}$$

$\Rightarrow f''(1) = \dfrac{4a}{(a + 2)^2}$ and $f''(-1) = \dfrac{-4a}{(a - 2)^2}$

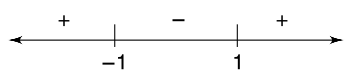
$\Rightarrow (2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$

(ii) Ans. (a)

Given $\quad f(x) = \dfrac{x^2 - ax + 1}{x^2 + ax + 1}$

$\Rightarrow \quad f'(x) = \dfrac{2ax^2 - 2a}{(x^2 + ax + 1)^2}$

$\Rightarrow \quad f'(x) = \dfrac{2a(x + 1)(x - 1)}{(x^2 + ax + 1)^2}$



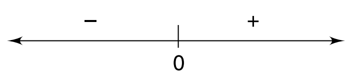
From the sign scheme of $f'(x)$, we get, $f(x)$ decreasing in $(-1, 1)$ and a local minima at $x = 1$.

Ans. (b)

Given $g(x) = \displaystyle\int_0^{e^x} \left(\frac{f'(t)}{1 + t^2}\right) dt$

$\Rightarrow \quad g'(x) = \dfrac{f'(e^x) \cdot e^x}{1 + e^{2x}}$

$\Rightarrow \quad g'(x) = \dfrac{2a(e^{2x} - 1) \cdot e^x}{(e^{2x} + ae^x + 1)^2 (1 + e^{2x})}$



From the sign scheme, it is clear that, $g'(x)$ is positive in $(0, \infty,)$ and negative in $(-\infty, 0)$

52. Ans. 7

Given $\quad A = \{x : x^2 + 20 \le 9x\}$

$= \{x : x^2 - 9x + 20 \le 0\}$

$= \{x : (x - 4)(x - 5) \le 0\}$

$= \{x : 4 \le x \le 5\}$

$= [4, 5]$

Also, $\quad f(x) = 2x^3 - 15x^2 + 36x - 48$

$f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

$= 6(x - 2)(x - 3)$

Clearly, $f(x)$ is increasing in $[4, 5]$

Thus, max value of $f(x)$ is

$= f(5)$

$= 2.125 - 15.\ 25 + 36.\ 5 - 48$

$= 250 - 375 + 180 - 48$

$= 430 - 423$

$= 7.$

53. Given $\displaystyle\lim_{x \to 0} \left(1 + \frac{p(x)}{x^2}\right) = 2$

$\Rightarrow \quad \displaystyle\lim_{x \to 0} \left(\frac{p(x)}{x^2}\right) = 2 - 1 = 1$

Let $p(x) = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4$

$$\lim_{x \to 0} \left(\frac{a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4}{x^2}\right) = 1$$

It is possible onle when $a_4 = 0,\ a_3 = 0,\ a_2 = 1$

Thus, $\quad p(x) = a_0 x^4 + a_1 x^3 + x^2$

$\Rightarrow \quad p'(x) = 4a_0 x^3 + 3a_1 x^2 + 2x$

As $p(x)$ has extremum at $x = 1$ and $x = 2$.

So, $\quad p'(1) = 0\ \&\ p'(2) = 0$

$4a_0 + 3a_1 = -2\ \&\ 32a_0 + 12a_1 = -4$

On solving, we get, $a_0 = \dfrac{1}{4}\ \&\ a_1 = -1$

Therefore, $p(x) = \dfrac{1}{4} x^4 - x^3 + x^2$

Now, $p(2) = \dfrac{1}{4}(2)^4 - (2)^3 + (2)^2$

$= 4 - 8 + 4 = 0$

54. Ans. at $x = 2009$.

Given $f(x) = \ln(g(x))$

$\Rightarrow \quad g(x) = e^{f(x)}$

$\Rightarrow \quad g'(x) = e^{f(x)}.f'(x)$

For local max or min, $g'(x) = 0$ gives

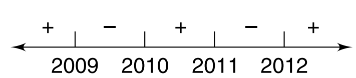
$\Rightarrow \quad e^{f(x)}.f'(x) = 0$

$\Rightarrow \quad f'(x) = 0$

$\Rightarrow \quad f'(x) = 2010(x - 2009)(x - 2010)^2.$

$(x - 2011)^3 (x - 2012)^4 = 0$

$\Rightarrow \quad x = 2009, 2010, 2011, 2012$



From, the sign scheme, it is clear that, $f(x)$ has a local maximum at $x = 2009$. and local minimum at $x = 2011$.

55. Ans. (d).

Given $\quad f(x) = e^{x^2} + e^{-x^2}$

$\Rightarrow \quad f'(x) = 2x(e^{x^2} - e^{-x^2}) \leq 0, \ \forall \ x \in [0, 1]$

Also, $\quad g'(x) = e^{x^2} + 2x^2 e^{x^2} - 2xe^{-x^2} \geq 0$

for all x in $[0, 1]$

and $h'(x) = 2xe^{x^2} + 2x^3 ex^2 - 2xe^{-x^2} \geq 0$

for all x in $[0, 1]$

Clearly, for $0 \leq x \leq 1$, $f(x) \geq g(x) \geq h(x)$

Thus, $\quad f(1) = g(1) = h(1) = e + \dfrac{1}{e}$

$\Rightarrow \quad a = b = c$

56. We have $x^4 - 4x^3 + 12x^2 + x - 1 = 0$

Now, $\quad f(-1) = 1 + 4 + 12 - 1 - 1 = 15 > 0$

$f(0) = -1 < 0$

and $\quad f(1) = 1 - 4 + 12 + 1 - 1 = 9 > 0$

So $f(x) = 0$ has a root in $(-1, 0)$ and a root in $(0, 1)$.

Thus, $f(x) = 0$ has atleast two distinct real roots.

Also, $\quad f'(x) = 4x^3 - 12x^2 + 24x + 1$

and $\quad f'(x) = 12x^2 - 24x + 24x$

$= 12(x^2 - 2x + 1) + 1$

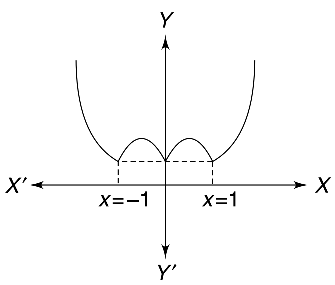
$= 12(x - 1)^2 + 1 > 0, \ \forall \ x \in R$

$\Rightarrow f'(x)$ increases on R.

$\Rightarrow y = f'(x)$ intersects the x-axis exactly once.

$\Rightarrow y = f(x)$ has exactly three real roots.

57. Given $f(x) = |x| + |x^2 - 1|$



Number of points at which f attains either a local max or local min value = 3 at $x = -1, -1/2, 0, 1/2, 1$.

58. Let $p'(x) = k(x - 1)(x - 3)$

$\Rightarrow \quad p'(x) = k(x^2 - 4x + 3)$

$\Rightarrow \quad p(x) = k\left(\dfrac{x^3}{3} - 2x^2 + 3x\right) + c$

Now, $\quad p(1) = 6$

$\Rightarrow \quad c + \dfrac{4}{3}k = 6 \qquad$ ...(i)

Also, $\quad p(3) = 2$

$\Rightarrow \quad c + k(9 - 18 + 9) = 2$

$\Rightarrow \quad c = 2$

Thus $k = 3$

So, $\quad p'(0) = 3k = 9$.

59. Ans. (a, b, c, d)

We have $f(x) = \int_0^x e^{t^2}(t - 2)(t - 3)dt$

$f'(x) = e^{x^2}(x - 2)(x - 3)$

Clearly, has maximum at $x = 2$ and minimum at $x = 3$ and $f(x)$ decreasing in $(2, 3)$

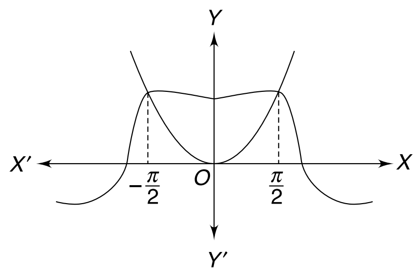
So, by Rolle's theorem $f'(x) = 0$ for $x = 2$ and $x = 3$.

Thus, there exist a point $c \in (2, 3)$ such that $f'(c) = 0$.

60. Ans. (c)

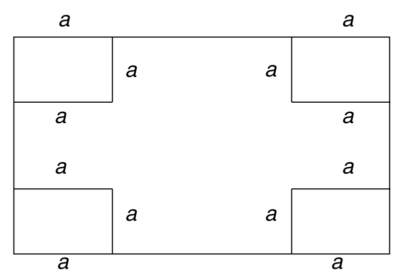
Given $x^2 - x\sin x - \cos x = 0$

$x^2 = x\sin x + \cos x$



Thus the number of points = 2.

61. Ans (a, c)



Let $l = 15\,x$ and $b = 8x$

Then Volume $= V = (8x - 2a)(15x - 2a).a$

$= 4a^3 - 46a^2x + 120ax^2$

$\dfrac{dV}{da} = 6a^2 - 46ax + 60x^2$

$\dfrac{d^2V}{da^2} = 12a - 46x$

Now, $\dfrac{dV}{da} = 0$ gives

$\Rightarrow \quad 6a^2 - 46ax + 60x^2 = 0$

$\Rightarrow \quad 30x^2 - 23ax + 3a^2 = 0$

$\Rightarrow \quad 30x^2 - 18ax - 5ax + 3a^2 = 0$

$\Rightarrow \quad 6x(5x - 3a) - a(5x - 3a) = 0$

$\Rightarrow \quad (6x - a)(5x - 3a) = 0$

$\Rightarrow \quad x = \dfrac{a}{6}, \dfrac{3a}{5}$

$\Rightarrow \quad x = \dfrac{5}{6}, 3$ when $a = 5$

When $\quad x = 3,\ a = 5,\ \dfrac{d^2V}{da^2} < 0$

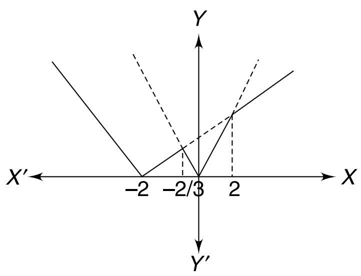
So the Volume is maximum.

Hence, the lengths are $l = 15.3 = 45$ and $b = 8.3 = 24$.

62. Ans. (a, b)

Given $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$

$= \min\{2|x|, |x + | 2\}$



From the graph, it is clear that, $f(x)$ has minimum value at $x = -2$, 0 and maximum value at $x = -2/3$.

62. Ans.(d)

(i) $\quad f''(x) - 2f'(x) + f(x) \ge e^x$

$\Rightarrow \quad f''(x)e^{-x} - 2f'(x)e^{-x} + f(x).e^{-x} \ge 1$

$\Rightarrow \quad \dfrac{d}{dx}(f'(x)e^{-x}) - \dfrac{d}{dx}(f(x).e^{-x}) \ge 1$

$\Rightarrow \quad \dfrac{d}{dx}(f'(x)e^{-x}) - f(x).e^{-x}) \ge 1$

$\Rightarrow \quad \dfrac{d^2}{dx^2}(f'(x)e^{-x}) - \ge 1,\ \forall\ x \in [0, 1]$

Let $\varphi(x) = e^{-x}f(x)$

Thus, $\varphi(x)$ is concave upward.

Now, $\quad f(0) = 0 = f(1)$

$\Rightarrow \quad \varphi(0) = 0 = \varphi(1)$

$\Rightarrow \quad \varphi(x) < 0$

$\Rightarrow \quad f(x) < 0$

(ii) Ans. (c)

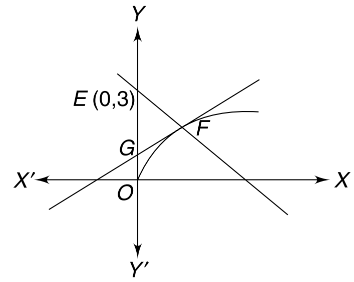
Here, $\varphi'(x) < 0,\ x \in \left(0, \dfrac{1}{4}\right)$

and $\varphi'(x) < 0,\ x \in \left(\dfrac{1}{4}, 1\right)$

$\Rightarrow \quad e^{-x}f'(x) - e^{-x}f(x) < 0,\ x \in \left(0, \dfrac{1}{4}\right)$

$\Rightarrow \quad f'(x) < f(x),\ x \in \left(0, \dfrac{1}{4}\right)$

63. Ans (a).



Tangent at F, $yt = x + 4t^2$

Area of the triangle $EFG = A$

$$= \frac{1}{2}\begin{vmatrix} 0 & 3 & 1 \\ 0 & 4t & 1 \\ 4t^2 & 8t & 1 \end{vmatrix}$$

$$= \frac{1}{2}\,4t^2(3 - 4t)$$

$$= 2t^2(3 - 4t)$$

$\Rightarrow \quad A = (6t^2 - 8t^3)$

$\Rightarrow \quad \dfrac{dA}{dt} = 12t - 23t^2$

$\Rightarrow \quad \dfrac{d^2A}{dt^2} = 12 - 48t$

Now, $\dfrac{dA}{dt} = 0$ gives, $t = 0, \dfrac{1}{2}$

when $t = 1/2$, $\dfrac{d^2A}{dt^2} < 0$

So area is maximum.

Maximum Area $= \left(\dfrac{6}{4} - \dfrac{8}{8}\right) = \dfrac{1}{2}$ sq unit.

$G = (0, 4t) = (0, 2)$

Thus, $\quad y_1 = 2$

$F = (x_0, y_0) = (4t^2, 8t) = (1, 4)$

So, $\quad y_0 = 4$

Also, the point $(4t^2, 8t) = (1, 4)$ satisfies the line $y = mx + 3$.

So, $\quad 4 = m + 3 \Rightarrow m = 1$

64. Ans. (b, d)

Given $\quad f(x) = x^5 - 5x + a$

$\Rightarrow \quad f'(x) = 5x^4 - 5 = 5(x^4 - 1)$

$\Rightarrow \quad f'(x) = 5(x^2 + 1)(x + 1)(x - 1)$

$f'(x) = 0$ has the roots $-1, 1$

Now, by Rolle's theorem the position and sign of the roots of $f(x) = 0$ can be found as follows:

$x =$	$-\infty$	-1	1	∞
$f(x) =$	∞	$a + 4$	$a - 4$	∞

If $f(x) = 0$ has three real roots, we must have $a + 4 > 0$ and $a - 4 < 0$

Thus, $-4 < a < 4$.

Also, if it has only one real root, then $a - 4 > 0 \Rightarrow a > 4$.

65. Given $f(x) = \sin(x^2) + \cos(x^2)$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos(x^2) + \frac{1}{\sqrt{2}}\sin(x^2)\right)$$

$$= \sqrt{2}\left(\cos\left(x^2 - \frac{\pi}{4}\right)\right)$$

It will provide us the max value, when

$$\Rightarrow \quad \cos\left(x^2 - \frac{\pi}{4}\right) = 1$$

$$\Rightarrow \quad \left(x^2 - \frac{\pi}{4}\right) = 2n\pi$$

$$\Rightarrow \quad x^2 = \frac{\pi}{4} + 2n\pi$$

$$\Rightarrow \quad x^2 = \frac{\pi}{4}, \frac{9\pi}{4}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{\pi}{4}}, \pm\sqrt{\frac{9\pi}{4}} \text{ as } x \in \lfloor -\sqrt{13}, \sqrt{13} \rfloor$$

Thus, the number of points = 4.