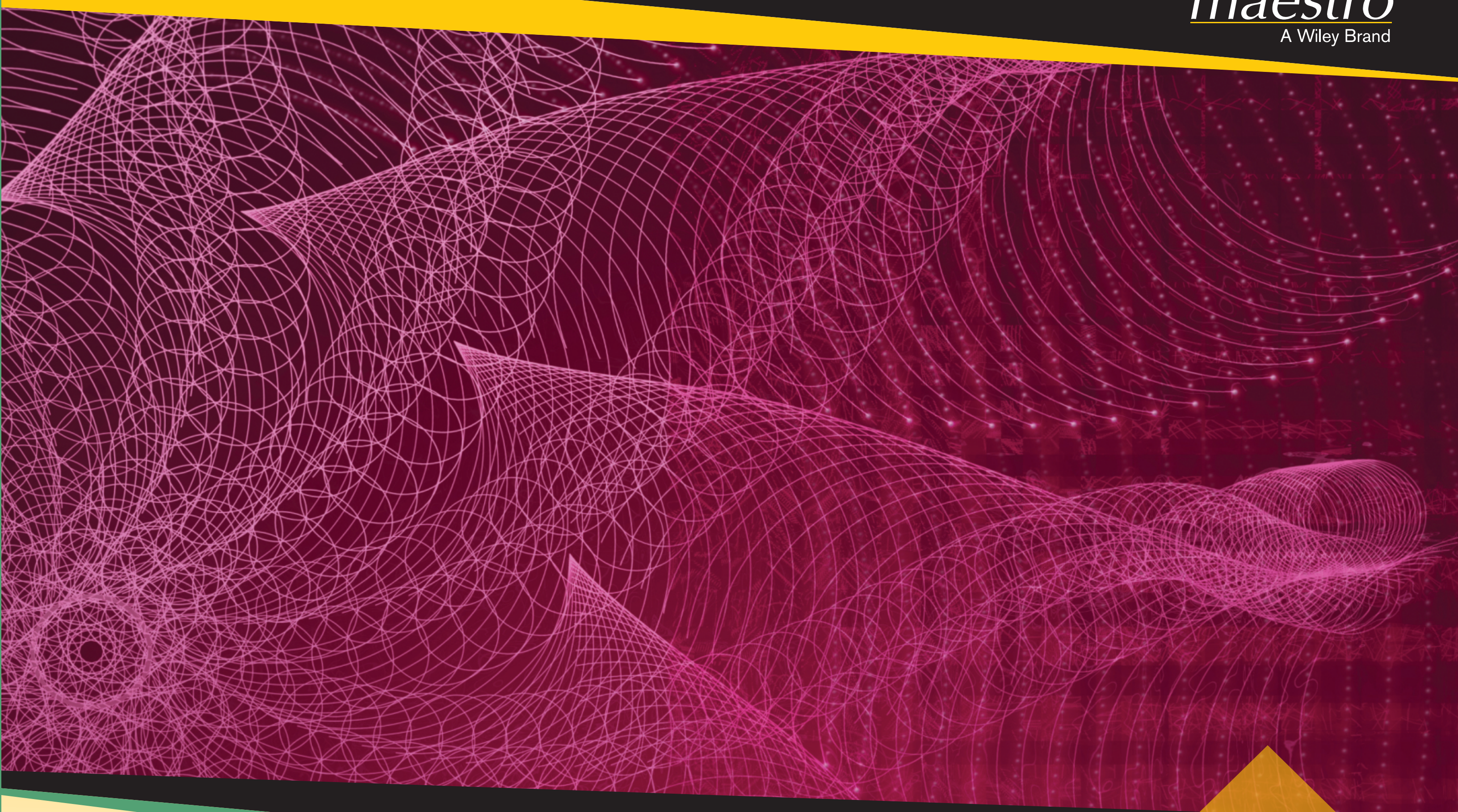
maestro
A Wiley Brand



VOLUME – II

WILEY PROBLEMS IN MATHEMATICS FOR JEE

with Summarized Concepts

- Solved Examples
- Practice Exercises with Complete Solutions
- Covers Solved JEE 2018 (Main and Advanced) Mathematics Questions

WILEY

maestro
A Wiley Brand

WILEY

PROBLEMS IN MATHEMATICS FOR JEE

with Summarized Concepts

VOLUME –II

WILEY

WILEY

PROBLEMS IN MATHEMATICS FOR JEE

with Summarized Concepts

VOLUME – II

Other Wiley Editorial Offices:

John Wiley & Sons, Inc. 111 River Street, Hoboken, NJ 07030, USA

Wiley-VCH Verlag GmbH, Pappellaee 3, D-69469 Weinheim, Germany

John Wiley & Sons Australia Ltd, 42 McDougall Street, Milton, Queensland 4064, Australia

John Wiley & Sons (Asia) Pte Ltd, 1 Fusionpolis Walk #07-01 Solaris, South Tower Singapore 138628

John Wiley & Sons Canada Ltd, 22 Worcester Road, Etobicoke, Ontario, Canada, M9W 1L1

First Edition: 2018

ISBN: 978-81-265-7630-2

ISBN: 978-81-265-8690-5 (ebk)

www.wileyindia.com

Printed at:

Note to the Student

Wiley Mathematics Problem Book is specifically designed to meet the needs of engineering (JEE) aspirants and give an edge to their preparation. The book offers complete coverage of the mathematics curriculum (of Class 12 syllabus) for JEE. It is enriched with unique elements and features that help students recapitulate the concepts, build problem-solving skills and apply them to solve all question-types asked in the engineering entrance examinations. The book is a valuable resource for both JEE (Main) and JEE (Advanced) aspirants. The chapter flow of the book is aligned with JEE Main syllabus and its coverage in the classroom. However, topics specific to JEE (Advanced) and advanced level questions are also covered both as solved examples and practice exercises.

We will now walk you through the target examinations and some key features of the book that enhance the learning experience.

TARGET EXAMINATION

Admission to Undergraduate Engineering Programs at IITs, NITs and other Center and State (participating) funded Technical Institutions use the Joint Entrance Examination Main (JEE Main) score as eligibility/merit criteria. The JEE (Main) is also an eligibility test for the Joint Entrance Examination Advanced [JEE (Advanced)], which is mandatory for the candidate if he/she is aspiring for admission to the undergraduate program offered by the IITs. The JEE (Advanced) scores are used as an eligibility criteria for admission into IITs.

An effective exam strategy for success in these examinations can be based on the detailed analysis of previous years question papers and planning your preparation accordingly. The Mathematics Question Paper of these examinations is a judicious mix of easy, moderate and tough questions. The analysis of question distribution over the units of mathematics syllabus for these examinations is given below.

EXAM ANALYSIS OF PAPERS

Mathematics question paper comes as an amalgamation of easy, moderate and tough questions. This section shows the unit-wise as well as chapter-wise analysis of previous 9 years (2010-2018) JEE Main and JEE Advanced papers.

JEE Main

Unit	Year								
	2010	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**
Algebra	14	13	13	12	12	11	12	13	12
Calculus	8	10	9	8	9	8	7	10	8
Trigonometry	2	1	1	3	2	3	3	2	3
Analytical Geometry	6	6	7	7	7	8	8	5	7

JEE Advanced

Unit	Year								
	2010	**2011**	**2012**	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**
Algebra	16	17	12	14	12	6	12	10	8
Trigonometry	5	1	2	4	3	1	2	1	1
Analytical Geometry	13	8	9	10	7	3	9	7	9
Differential Calculus	2	7	6	2	11	5	7	8	12
Integral Calculus	8	7	10	7	5	4	5	7	4
Vector	3	3	2	3	2	1	1	3	2

MATHEMATICS JEE MAIN PAPERS ANALYSIS (2010-2018)

Unit	Chapter	AIEEE 2010	AIEEE 2011	AIEEE 2012	JEE Main 2013	JEE Main 2014 (Offline)	JEE Main 2015 (Offline)	JEE Main 2016 (Offline)	JEE Main 2017 (Offline)	JEE Main 2018 (Offline)
Algebra	Complex Numbers and Quadratic Equations	2	2	2	3	3	2	2	1	1
	Permutations and Combinations	1	1	1	1		2	1	2	1
	Binomial Theorem		1	1	1	1		1		1
	Sequences and Series	2	1	2	1	2	2	2	3	2
	Statistics	1	1	1	1	1	1	1	1	1
	Mathematical Reasoning	1	1	1	1	1		1		1
	Matrices and Determinants	3	2	2	2	2	2	2	3	3
	Vector Algebra	2	2	2	1	1	1	1	1	1
	Probability	2	2	1	1	1	1	1	2	1
Calculus	Sets, Relations and Functions	1	2	1	1	1	2	1	2	2
	Limits, Continuity and Differentiability	3	3	2	2	3	3	2	2	1
	Application of Derivatives	1	1	3	1	2			2	1
	Integrals	1	2	2	3	2	2	1	2	2
	Application of Integrals	1	1	1	1	1		1	1	1
	Differential Equations	1	1				1	2	1	1
Trigonometry	Trignometric Functions	2	1	1	2	2	2	2	2	3
	Inverse Trignometric Functions				1		1	1		
Analytical Geometry	Conic Sections	2	2	3	3	3	4	5	3	5
	Three-Dimensional Geometry	4	4	4	4	4	4	3	2	2

MATHEMATICS JEE ADVANCED PAPERS ANALYSIS (2010-2018)

Unit	Chapter	IIT-JEE 2010						IIT-JEE 2011						IIT-JEE 2012						JEE Advanced 2013						JEE Advanced 2014						JEE Advanced 2015						JEE Advanced 2016						JEE Advanced 2017						JEE Advanced 2018					
		P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U
Algebra	Complex Numbers		1		1						1		2	1						1	1	2							1								1						1		1						2				
	Quadratic Equations	1						2					1													1	1						1					1								2									1
	Permutations and Combinations	1																																				1											1				1		
	Sequence and Series	1					2						1	1		2					1				1						1						1	2											1						1
	Binomial Theorem	1												1												1											1						1												1
	Logarithms							1											1		1																																		
	Matrices and Determinants	1		3			2	2		3			1	2	1						2						2											1	1				1	1	1				1		1				1
	Probability	2							1	2				1	2					1		2			2	2		2			1			2				1		2				1	1						2				
Trigonometry	Properties and Solution of Triangles (Heights and Distances)		1				1							1							1						1												1																
	Trigonometric Equations						2						1													1																			1										
	Trigonometric Ratios and Identities						1								1						1																	1																	
	Inverse Trigonometric Function (Principal Values Only)																			1			1								1		1																		1				
Analytical Geometry	Rectangular Coordinate System																																																						
	Straight Lines and Pair of Lines	2			1		1	1					1	1						1	2																	1																	
	Conics		1	3			2	2	2				1	1	1				1			2	1		1			2			1		2				1		2	2					2	3					1		1		
	Circle			2				1								2					1					2	1				1								4										1		2	1			
	Three-Dimensional Geometry						1							2	1					1			1																					1							2				2
Differential Calculus	Sets and Relations							2																									1											1							1		1		2
	Limits							1	1					2							1							2			1						1		3				1		1										
	Functions	1					1							1	2												2		1					2			1													1	1		1		1
	Continuity and Differentiability								1				2	1						1						1	2		1		1								1					1	4				1	1	2				1
	Differentiation																																						2																

Unit	Chapter	IIT-JEE 2010						IIT-JEE 2011						IIT-JEE 2012						JEE Advanced 2013						JEE Advanced 2014						JEE Advanced 2015						JEE Advanced 2016						JEE Advanced 2017						JEE Advanced 2018					
		P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U	P	Q	R	S	T	U
Integral Calculus	Application of Derivatives	1							1		1		1						2	1	2	2											1					1							1	3									1
	Indefinite Integration													1																																									
	Definite Integration	1	1					2						1	1	2										1			1		1		1				1	1					1		1										1
	Area Under the Curve			3				1						1	1					1											1							1							1										
	Differential Equations		1				1						1		1					1						1							1						1					1											
Vector	Vectors	2					1	1	1				1	1					1	1			1		1		1				1						1		1					1		2									2

P: Single Correct Choice Type **Q:** One or More Than One Option Correct Type **R:** Paragraph Type

S: Matrix-Match Type **T:** Reasoning Type **U:** Integer Answer Type

FEATURES OF THE BOOK

A. Understand the Concepts

1. All the concepts as per the JEE curriculum are explained in simple steps to develop fundamental understanding of the subject.

21.2 Tangent and Normal

A tangent to a point is a line which touches the curve at that point. A normal to a point is the line which is perpendicular to the tangent at that point.

If the equation of a curve is $y = f(x)$ and a point $A(x_1, y_1)$ lies on it, then the equation of the tangent at point A is

$$y - y_1 = \left(\frac{dy}{dx}\right)\bigg|_A (x - x_1)$$

and the equation of the normal at point A is

$$y - y_1 = -\frac{1}{(dy/dx)|_A}(x - x_1)$$

2. Important points to remember about concepts highlighted as Key Points.

Key Point:
When the curve is given in parametric form, that is, when $x = g(t)$ and $y = h(t)$, the equation of tangent at the point $t = t_1$ is

$$y - h(t_1) = \frac{h'(t_1)}{g'(t_1)}[x - g(t_1)]$$

and the equation of normal is

$$y - h(t_1) = -\frac{g'(t_1)}{h'(t_1)}[x - g(t_1)]$$

B. Every Aspect of the Subject Covered

In form of formulas, figures, graphs and tables to enhance problem-solving skills.

(i) $\sin^{-1}(\sin\theta) = \theta,\ \forall\,\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta,\ \forall\,\theta \in [0, \pi]$

$\tan^{-1}(\tan\theta) = \theta,\ \forall\,\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\cot^{-1}(\cos\theta) = \theta,\ \forall\,\theta \in (0, \pi)$

$\sec^{-1}(\sec\theta) = \theta,\ \forall\,\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

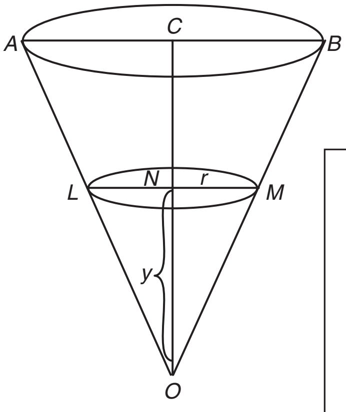


Figure 21.4

Table 17.1 Domain and principal ranges of all the six inverse trigonometric functions

Function	Domain (values of x)	Principal Range (values of y)
x	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
x	$[-1, 1]$	$[0, \pi]$
x	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

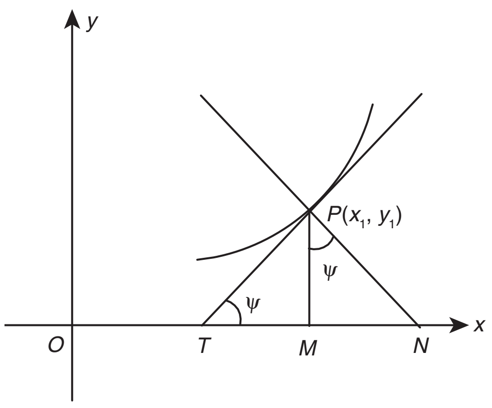


Figure 21.2

C. Reinforce Concepts

1. **Illustrations** pose a specific problem using concepts already presented and then work through the solution.

Illustration 21.1 Find the slope of tangent at the point that has the ordinate –3 on the curve $x^3 = 3y^2$.

Solution: Differentiating the equation of the given curve w.r.t. x, we get

$$3x^2 = 3\times\left(2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{2y}$$

Now, to obtain this value, we require abscissa as well. Substituting $y = -3$ in the equation of curve, we have

2. **Your Turn** within each chapter is present to reinforce and check the understanding of the students.

Your Turn 1

1. Find the slopes of the curve $y = (x + 2)(x - 3)$ at the points where it meets x–axis. **Ans.** $-5, 5$
2. Find the points on the curve $y = x^3 - 2x^2 + x - 2$ when the gradient is zero.

   Ans. $(1, -2)$ and $\left(\frac{1}{3}, -\frac{50}{27}\right)$
3. Find the equation of tangent and normal to the curve $x^3 = y^2$ at the point (1, 1). Also find the length of tangent, normal, subtangent and subnormal.

   Ans. $3x - 2y - 1 = 0,\ 2x + 3y - 5 = 0,\ \frac{\sqrt{13}}{3}, \frac{\sqrt{13}}{2}, \frac{2}{3}, \frac{3}{2}$

3. **Additional Solved Examples** suitable for JEE exams are provided with in-depth solutions for the students to understand the logic behind and formula used.

Additional Solved Examples

1. The number of real solutions of $\cos^{-1} x + \cos^{-1} 2x = -\pi$ is

   (A) 0 **(B)** 1
   (C) 2 **(D)** Infinitely many

Solution:

$$\cos^{-1} x = -(\pi + \cos^{-1} 2x)$$

Range of $\cos^{-1} x \in [0, \pi]$

Since $\cos^{-1} x$ has a range from $[0, \pi]$, thus the sum of two $\cos^{-1}$ cannot be equal to $-\pi$ a negative quantity.

Hence, the correct answer is option (A).

D. Understanding the Exam Pattern

Through Previous Years' Solved JEE Main/AIEEE Questions and Previous Years' Solved JEE Advanced/IIT-JEE Questions.

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Let (x, y) be such that

$$\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$$

Match the statements in Column I with statements in Colum

Column I	Column II
(A) If $a=1$ and $b=0$, then (x, y)	**(P)** lies on the circle x^2+y
(B) If $a=1$ and $b=1$, then (x, y)	**(Q)** lies on $(x^2-1)(y^2-1)=$
(C) If $a=1$ and $b=2$, then (x, y)	**(R)** lies on $y=x$
(D) If $a=2$ and $b=2$, then (x, y)	**(S)** lies on $(4x^2-1)(y^2-1)$

[IIT-JEE 2

Previous Years' Solved JEE Main/AIEEE Questions

1. If $\sin^{-1}\left(\frac{x}{5}\right)+\text{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then a value of x is

(A) 1 **(B)** 3

(C) 4 **(D)** 5 **[AIEEE 2007]**

Solution: We have

$$\sin^{-1}\frac{x}{5}+\sin^{-1}\frac{4}{5}=\frac{\pi}{2}\Rightarrow \sin^{-1}\frac{x}{5}=\cos^{-1}\frac{4}{5}\Rightarrow \sin^{-1}\frac{x}{5}=\sin^{-1}\frac{3}{5}$$

Therefore, $x = 3$.

Hence, the correct answer is option (B).

E. Practice to Complete Your Learning

Through Practice Exercise 1 (JEE Main) and Practice Exercise 2 (JEE Advanced). All questions types as per JEE Main and Advanced covered.

Practice Exercise 1

1. The points on the curve $y = 12x - x^3$ at which the gradient is zero are

(A) (0, 2), (2, 16) **(B)** (0, –2), (2, –16)

(C) (2, –16), (–2, 16) **(D)** (2, 16), (–2, –16)

2. The area of the triangle formed by the coordinate axes and a tangent to the curve $xy=a^2$ at the point (x_1, y_1) on it is

(A) $\frac{a^2x_1}{y_1}$ **(B)** $\frac{a^2y_1}{x_1}$ **(C)** $2a^2$ **(D)** $4a^2$

3. The slope of tangent to the curve $x=t^2+3t-8$, $y=2t^2-2t-5$ at the point (2, –1) is

(A) 22/7 **(B)** 6/7 **(C)** –6 **(D)** None of these

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. For the curve represented parametrically by the equations, $x = 2\ln\cot t + 1$ and $y = \tan t + \cot t$

(A) tangent at $t = \pi/4$ is parallel to x-axis

(B) normal at $t = \pi/4$ is parallel to y-axis

(C) tangent at $t = \pi/4$ is parallel to the line $y = x$

(D) tangent and normal intersect at the point (2, 1)

Matrix Match Type Questions

24. Match the following:

List I	List II
(A) Circular plate is expanded by the heat from the radius 5 cm to 5.06 cm. Approximate increase in the area is	**(p)** 2
(B) If an edge of a cube increases by 1%, then the percentage increase in the volume is	**(q)** 0.6 π
(C) If the rate of decrease of $y = \frac{x^2}{\ }- 2x + 5$ is	**(r)** 3

Comprehension Type Questions

Paragraph for Questions 9–11: Let $a(t)$ is a function of t such that $\frac{da}{dt}=2$ for all the values of t and $a = 0$ when $t = 0$. Further $y = m(t)x + c(t)$ is the tangent to the curve $y = x^2 - 2ax + a^2 + a$ at the point whose abscissa is 0. Then

9. If the ... $+ a^2 +$

(A) 2

10. If the ... then

(A) 1

(C) 1

Integer Type Questions

29. Let α be the angle in radians between $\frac{x^2}{36}+\frac{y^2}{4}=1$ and the circle $x^2 + y^2 = 12$ at their points of the intersection. If $\alpha=\tan^{-1}\frac{k}{2\sqrt{3}}$, then find the value of k^2.

30. Find the minimum value of $(x_1 - x_2)^2 + \left(\sqrt{2-x_1^2}-\frac{9}{x_2}\right)^2$, where $x_1 \in (0, \sqrt{2})$ and $x_2 \in R^+$.

F. Check Your Performance and Problem-Solving Approach

Through Answer Key and Solution to practice exercises provided with explanation.

Answer Key

Practice Exercise 1

1. (D)	**2.** (C)	**3.** (B)	**4.** (C)	**5.** (A)	**6.** (A)
7. (B)	**8.** (C)	**9.** (C)	**10.** (A)	**11.** (B)	**12.** (C)
13. (A)	**14.** (D)	**15.** (C)	**16.** (C)	**17.** (C)	**18.** (A)
19. (A)	**20.** (D)	**21.** (D)	**22.** (B)	**23.** (D)	**24.** (B)
25. (C)					
31. (B)					

Solutions

Practice Exercise 1

1. We have

$$\frac{dy}{dx} = 12 - 3x^2 = 0 \Rightarrow x = \pm 2$$

Hence, the points are (2, 16) and (–2, –16).

2. We have

$$y = \frac{a^2}{x}$$

Therefore,

$$\frac{dy}{dx} = -\frac{a^2}{x^2}$$

Now, at (x_1, y_1).

or $$x = 2x_1$$

Therefore, the point on x-axis is $(2x_1, 0)$. Now, the tangent meets y-axis where $x = 0$. Since

$$x_1^2 y = 2a^2 x_1$$

we have $$y = \frac{2a^2}{x_1}$$

So, the point on the y-axis is

$$\left(0, \frac{2a^2}{x_1}\right)$$

The required area is

$$\frac{1}{2}(2x_1)\left(\frac{2a^2}{x_1}\right) = 2a^2$$

Contents

Chapter 26 Vector Algebra 1163

Chapter 27 Three-Dimensional Geometry 1213

17 Inverse Trigonometry

17.1 Introduction

The inverse of a function $f: A \rightarrow B$ exists if f is one-one onto, that is, a bijection and is given by $f(x) = y \Rightarrow f^{-1}(y) = x$.

Consider the sine function with domain R and range [–1, 1]. Clearly this function is not a bijection and so it is not invertible. If we restrict the domain of it in such a way that it becomes one–one, then it would become invertible. If we consider sine as a function with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and co-domain [–1, 1], then it is a bijection and therefore, invertible. The inverse of sine function is defined as

$$\sin^{-1} x = \theta \Leftrightarrow \sin\theta - x, \text{ where } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } x \in [-1, 1]$$

Hence, $\sin^{-1} x$ is an angle and it denotes the smallest numerical angle, whose sine is x.

17.2 Domain and Range of Inverse Trigonometric Functions

We know that $\tan \frac{\pi}{3} = \sqrt{3}$

This is written in inverse trigonometry as $\frac{\pi}{3} = \tan^{-1}\sqrt{3}$.

But, $\tan \frac{4\pi}{3}$ is also equal to $\sqrt{3}$

Does it mean, $\frac{4\pi}{3} = \tan^{-1}\sqrt{3}$?

The answer is no, $\tan^{-1}\sqrt{3}$ is taken as the numerically least angle whose tangent is $\sqrt{3}$. This is done to associate a single value to $\tan^{-1}\sqrt{3}$ to safeguard the definition of a function.

So, the equations $\tan x = y$ and $x = \tan^{-1} y$ are not identical because the former associates many values of x to a single value of y, while the latter associates a single x to a particular value of y. In the same way, the remaining five inverse trigonometric functions are also defined. To assign a unique angle to a particular value of trigonometric ratio, we introduce a term called 'principal range'. The principal ranges of all the inverse trigonometric functions have been fixed. For example, principal range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, that is, we have to search for an angle in this interval only.

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$ only, although $\sin\frac{5\pi}{6} = \frac{1}{2}$, $\sin\frac{13\pi}{6} = \frac{1}{2}$, etc.

$\left(\text{note that } \sin^{-1}\frac{1}{2} \neq \frac{1}{\sin\frac{1}{2}}\right)$

Similarly, even if $\cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$ but $\cot^{-1}(-\sqrt{3}) \neq -\frac{\pi}{6}$ because principal range of $\cot^{-1}x$ is $(0, \pi)$.

So, $\cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$ only.

Note:

1. See Fig. 17.1. Here, $\sin^{-1} x$, $\operatorname{cosec}^{-1} x$, $\tan^{-1} x$, belong to I and IV quadrants.

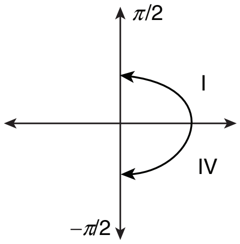


Figure 17.1

2. See Fig. 17.2. Here, $\cos^{-1} x$, $\sec^{-1} x$, $\cot^{-1} x$, belong to I and II quadrant.

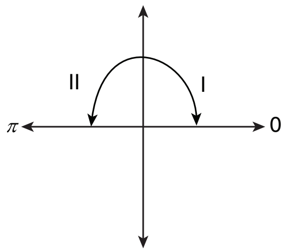


Figure 17.2

3. I quadrant is common to all the inverse functions.
4. III quadrant is not used in inverse functions.
5. IV quadrant is used in the clockwise direction, that is, $-\frac{\pi}{2} \leq y \leq 0$.

The principal range of inverse trigonometric functions is the most important thing in this lesson. All formula and problems are linked in some way or the other to that only.

1. See Fig. 17.3. If $\sin y = x$, then $y = \sin^{-1} x$, under certain condition.

   $-1 \leq \sin y \leq 1$, but $\sin y = x$.

   Hence,

   $$-1 \leq x \leq 1$$

   Again,

   $$\sin y = -1 \Rightarrow y = -\frac{\pi}{2}$$

   and $$\sin y = 1 \Rightarrow y = \frac{\pi}{2}$$

   Keeping in mind numerically smallest angles or real numbers.

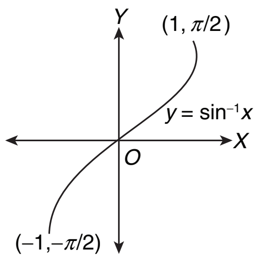


Figure 17.3

Hence,

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

These restrictions on the values of x and y provide us with the domain and range for the function, $y = \sin^{-1} x$.
So,

$$\text{Domain: } x \in [-1,1]$$

$$\text{Range: } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

2. See Fig. 17.4. If $\cos y = x$, then $y = \cos^{-1} x$, under certain conditions.

$$-1 \le \cos y \le 1 \Rightarrow -1 \le x \le 1$$

$$\cos y = -1 \Rightarrow y = \pi$$

$$\cos y = 1 \Rightarrow y = 0$$

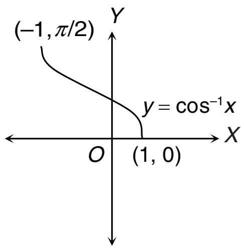


Figure 17.4

Hence, $0 \le y \le \pi$ {as cos x is a decreasing function in [0, π]}.

These restrictions on the values of x and y provide us the domain and range for the function, $y = \cos^{-1} x$.
So,

$$\text{Domain: } x \in [-1,1]$$

$$\text{Range: } y \in [0, \pi]$$

3. See Fig. 17.5. If $\tan y = x$, then $y = \tan^{-1} x$, under certain conditions.
$\tan y \in R \Rightarrow x \in R$,

$$-\infty < \tan y < \infty \Rightarrow -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Thus,

$$\text{Domain: } x \in R;$$

$$\text{Range: } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

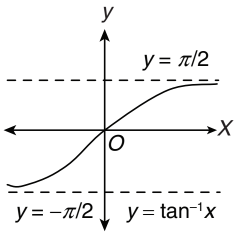


Figure 17.5

4. See Fig. 17.6. If $\cot y = x$, then $y = \cot^{-1} x$, under certain conditions.

$\cot y \in R \Rightarrow x \in R$;

$-\infty < \cot y < \infty \Rightarrow 0 < y < \pi$

These conditions on x and y make the function, $\cot y = x$ one-one and onto, so that the inverse function exists, that is, $y = \cot^{-1} x$ is meaningful.

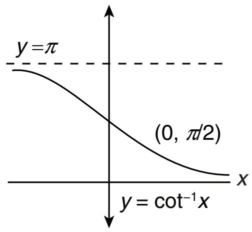


Figure 17.6

So,

$$\text{Domain: } x \in R$$

$$\text{Range: } y \in (0, \pi)$$

5. See Fig. 17.7. If $\sec y = x$, then

$$y = \sec^{-1} x$$

where $|x| \ge 1$ and $0 \le y \le \pi, y \ne \frac{\pi}{2}$

Here,

$$\text{Domain: } x \in R - (-1,1)$$

$$\text{Range: } y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

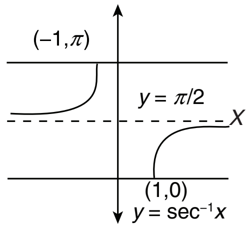


Figure 17.7

6. See Fig. 17.8. If $\text{cosec } y = x$, then

$$y = \text{cosec}^{-1} x$$

where $|x| \ge 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$

Here,

$$\text{Domain: } x \in R - (-1,1)$$

$$\text{Range: } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

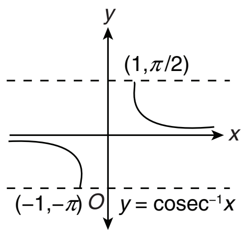


Figure 17.8

We list below (Table 17.1) the domain and principal ranges of all the six inverse trigonometric functions.

Table 17.1 Domain and principal ranges of all the six inverse trigonometric functions

Function	**Domain (values of x)**	**Principal Range (values of y)**
$y = \sin^{-1} x$	[–1, 1]	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	[–1, 1]	$[0, \pi]$
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

Illustration 17.1 Evaluate the following:

(A) $\tan^{-1}(-1)$ **(B)** $\cot^{-1}(-1)$ **(C)** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution:

(A) $\tan\left(\frac{-\pi}{4}\right) = -1$

Hence,

$$\tan^{-1}(-1) = -\frac{\pi}{4} \left\{\text{since } \frac{-\pi}{4} \in \text{range of } \tan^{-1} x\right\}$$

(B) $\cot\left(\frac{3\pi}{4}\right)=-1$

Hence,

$$\cot^{-1}(-1)=\frac{3\pi}{4}\left\{\text{since } \frac{3\pi}{4}\in\text{range of } \cot^{-1}x\right\}$$

(C) $\sin\left(\frac{-\pi}{3}\right)=\frac{-\sqrt{3}}{2}$

Hence,

$$\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)=\frac{-\pi}{3}\left\{\text{since } \frac{-\pi}{3}\in\text{range of } \sin^{-1}x\right\}$$

Illustration 17.2 Simplify

$$\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{-1}{2}\right)-\tan^{-1}(-\sqrt{3})+\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

Solution: The value is

$$\frac{-\pi}{4}+\frac{2\pi}{3}-\left(\frac{-\pi}{3}\right)+\left(\frac{2\pi}{3}\right)$$

$$=-\frac{\pi}{4}+\frac{2\pi}{3}+\frac{\pi}{3}+\frac{2\pi}{3}=\frac{17\pi}{12}$$

17.3 Properties of Inverse Trigonometric Functions

Property 1

(i) $\sin^{-1}(\sin\theta)=\theta,\ \forall\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta)=\theta,\ \forall\ \theta\in[0,\pi]$

(iii) $\tan^{-1}(\tan\theta)=\theta,\ \forall\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(iv) $\cot^{-1}(\cos\theta)=\theta,\ \forall\ \theta\in(0,\pi)$

(v) $\sec^{-1}(\sec\theta)=\theta,\ \forall\theta\in[0,\pi]-\left\{\frac{\pi}{2}\right\}$

(vi) $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta,\ \forall\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Property 2

(i) $\sin(\sin^{-1}x)=x,\ \forall\ x\in[-1,1]$

(ii) $\cos(\cos^{-1}x)=x,\ \forall\ x\in[-1,1]$

(iii) $\tan(\tan^{-1}x)=x,\ \forall\ x\in R$

(iv) $\cot(\cot^{-1}x)=x,\ \forall\ x\in R$

(v) $\sec(\sec^{-1}x)=x,\ \forall\ x\in(-\infty,-1]\cup[1,\infty)$

(vi) $\text{cosec}(\text{cosec}^{-1}x)=x,\ \forall\ x\in(-\infty,-1]\cup[1,\infty)$

Property 3

(i) $\sin^{-1}(-x)=-\sin^{-1}x,\ \forall\ x\in[-1,1]$

(ii) $\cos^{-1}(-x)=\pi-\cos^{-1}x,\ \forall\ x\in[-1,1]$

(iii) $\tan^{-1}(-x)=-\tan^{-1}x,\ \forall\ x\in R$

(iv) $\cot^{-1}(-x)=\pi-\cot^{-1}x,\ \forall\ x\in R$

(v) $\sec^{-1}(-x)=\pi-\sec^{-1}x,\ \forall\ x\in(-\infty,-1]\cup[1,\infty)$

(vi) $\text{cosec}^{-1}(-x)=-\text{cosec}^{-1}x,\ \forall\ x\in(-\infty,-1]\cup[1,\infty)$

Proof (i): $\sin^{-1}(-x)=-\sin^{-1}x$, provided that $-1\le x\le 1$

As $(-x)\in[-1,1]$

$\Rightarrow x\in[-1,1]$

Let $\sin^{-1}(-x)=\theta$. Then $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

$$(-x)=\sin\theta$$
$$\Rightarrow x=-\sin\theta$$
$$\Rightarrow x=\sin(-\theta)$$
$$\Rightarrow(-\theta)=\sin^{-1}x$$
$$\Rightarrow\theta=-\sin^{-1}x$$
$$\Rightarrow\sin^{-1}(-x)=-\sin^{-1}x,\ \forall\ x\in[-1,1]$$

Proof (ii): $\cos^{-1}(-x)=\pi-\cos^{-1}x$, provided that $-1\le x\le 1$

As $(-x)\in[-1,1]$

$\Rightarrow x\in[-1,1]$

Let $\cos^{-1}(-x)=\theta$. Then $\theta\in[0,\pi]$.

$$-x=\cos\theta$$
$$\Rightarrow x=\cos(\pi-\theta)$$
$$\Rightarrow\cos^{-1}x=\pi-\theta$$
$$\Rightarrow\theta=\pi-\cos^{-1}x$$
$$\Rightarrow\cos^{-1}(-x)=\pi-\cos^{-1}x,\ \forall\ x\in[-1,1]$$

Similarly, we can do the remaining ones from (iii) to (vi).

Property 4

(i) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2},\ \forall\ x\in[-1,1]$

(ii) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\ \forall\ x\in R$

(iii) $\sec^{-1}x+\text{cosec}^{-1}x=\frac{\pi}{2},\ \forall\ x\in(-\infty,-1]\cup[1,\infty)$

Proof (i): $\sin^{-1}x+\cos^{-1}x=\pi/2,\ \forall\ x\in[-1,1]$

Let $\sin^{-1}x=\theta$. Then $\forall\ x\in[-1,1]$.

where, $\theta\in[-\pi/2,\pi/2]$

$$-\pi/2\le\theta\le\pi/2$$
$$\Rightarrow-\pi/2\le-\theta\le\pi/2$$
$$\Rightarrow 0\le\pi/2-\theta\le\pi$$
$$\Rightarrow(\pi/2-\theta)\in[0,\pi]$$

Since,

$$\sin^{-1}x=\theta$$
$$\Rightarrow x=\sin\theta$$
$$\Rightarrow x=\cos(\pi/2-\theta)$$
$$\Rightarrow\cos^{-1}x=\pi/2-\theta$$
$$\Rightarrow\theta+\cos^{-1}x=\pi/2$$

Hence, $\sin^{-1}x+\cos^{-1}x=\pi/2$.

Proof (ii): $\tan^{-1}x+\cot^{-1}x=\pi/2,\ \forall\ x\in R$

Let $\tan^{-1}x=\theta$. Then $x\in R$.

where, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$$
$$\Rightarrow-\frac{\pi}{2}<-\theta<\frac{\pi}{2}$$

$\Rightarrow 0 < \frac{\pi}{2} - \theta < \pi$

$\Rightarrow (\pi/2 - \theta) \in (0, \pi)$

Since,

$\tan^{-1}x = \theta$

$\Rightarrow x = \tan\theta$

$\Rightarrow x = \cot(\pi/2 - \theta)$

$\Rightarrow \cot^{-1}x = \pi/2 - \theta$

$\Rightarrow \theta + \cot^{-1}x = \pi/2$

Hence, $\tan^{-1}x + \cot^{-1}x = \pi/2$.

Proof (iii): $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2,\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

Let $\sec^{-1}x = \theta$. Then $x \in (-\infty, -1] \cup [1, \infty)$.

where, $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\Rightarrow \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\Rightarrow \frac{\pi}{2} - \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \frac{\pi}{2} - \theta \neq 0$

Since,

$\sec^{-1}x = \theta$

$\Rightarrow x = \sec\theta$

$\Rightarrow x = \operatorname{cosec}(\pi/2 - \theta)$

$\Rightarrow \operatorname{cosec}^{-1}x = \pi/2 - \theta$

$\Rightarrow \theta + \operatorname{cosec}^{-1}x = \pi/2$

Hence, $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$.

Property 5

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x,\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x,\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, \forall\ x > 0 \\ -\pi + \cot^{-1}x, \forall\ x < 0 \end{cases}$

Proof (i): Let $\operatorname{cosec}^{-1}x = \theta$. Then $x = \operatorname{cosec}\theta$.

where, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\sin\theta = \left(\frac{1}{x}\right)$ for $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow \left(\frac{1}{x}\right) \in [-1, 1] - \{0\}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{x}\right)$

Hence, $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$

Proof (ii): Let $\sec^{-1}x = \theta$. Then $x = \sec\theta$.

where, $\theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\cos\theta = \left(\frac{1}{x}\right)$ for $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow \left(\frac{1}{x}\right) \in [-1, 1] - \{0\}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{x}\right)$

Hence, $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$

Proof (iii): Let $\cot^{-1}x = \theta$

where, $-\infty < x < \infty$ and $0 < \theta < \pi$

Now, consider two cases,

Case I: $x > 0$

$\cot^{-1}x = \theta \Rightarrow \theta \in (0, \pi/2)$

$\Rightarrow x = \cot\theta \Rightarrow \frac{1}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{1}{x}\right)$

Hence, $\tan^{-1}(1/x) = \cot^{-1}x$, for all $x > 0$.

Case II: $x < 0$

$\Rightarrow \theta \in (\pi/2, \pi)$

$\Rightarrow \pi/2 < \theta < \pi$

$\Rightarrow -\pi/2 < (\theta - \pi) < 0$

$\Rightarrow (\theta - \pi) \in (-\pi/2, 0)$

Hence, $\cot^{-1}x = \theta$

$\Rightarrow \cot\theta = x$

$\Rightarrow \frac{1}{x} = \tan\theta$

$\Rightarrow \frac{1}{x} = -\tan(\pi - \theta)$

$\Rightarrow \frac{1}{x} = \tan(\theta - \pi)$

$\Rightarrow (\theta - \pi) = \tan^{-1}\left(\frac{1}{x}\right)$

$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = -\pi + \theta$

So, $\tan^{-1}\left(\frac{1}{x}\right) = -\pi + \cot^{-1}x$, when $x < 0$.

Hence,

$$\tan^{-1}\frac{1}{x} = \begin{Bmatrix} \cot^{-1}x, & \forall & x > 0 \\ -\pi + \cot^{-1}x, & \forall & x < 0 \end{Bmatrix}$$

Note:

Conversion of inverse trigonometric ratio in their domain

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\frac{1}{x}$

$= \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\dfrac{1}{x}\right)$

$$= \sec^{-1}\sqrt{1+x^2} = \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Illustration 17.3 Evaluate the following:

(A) $\sec^{-1}[\sec(-30°)]$ **(B)** $\sin^{-1}\left(\sin\dfrac{5\pi}{3}\right)$ **(C)** $\sin^{-1}\left[\sin\left(\dfrac{2\pi}{3}\right)\right]$

Solution:

(A) $\sec^{-1}[\sec(-30°)] = \sec^{-1}(\sec 30°) = 30°$

(B) $\sin^{-1}\left(\sin\dfrac{5\pi}{3}\right) = \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right) = -\dfrac{\pi}{3}$

(C) $\sin^{-1}\left[\sin\left(\pi - \dfrac{\pi}{3}\right)\right] = \sin^{-1}\left[\sin\left(\dfrac{\pi}{3}\right)\right] = \dfrac{\pi}{3}$

Illustration 17.4 If $\theta = \sin^{-1} x + \cos^{-1} x - \tan^{-1} x$, $x \geq 0$, then the smallest interval in which θ lies is

(A) $\dfrac{\pi}{2} \leq \theta \leq \dfrac{3\pi}{4}$ **(B)** $0 \leq \theta \leq \dfrac{\pi}{4}$

(C) $-\dfrac{\pi}{4} \leq \theta \leq 0$ **(D)** $\dfrac{\pi}{4} \leq \theta \leq \dfrac{\pi}{2}$

Solution:

$$\theta = \sin^{-1} x + \cos^{-1} x - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

We know

$$\tan^{-1} x = A \text{ where } x \in R \text{ and } A \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Hence,

$$\frac{\pi}{4} \leq \frac{\pi}{2} - \tan^{-1} x \leq \frac{\pi}{2} \Rightarrow \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$$

Illustration 17.5 Find the value of x which satisfies the equation $\tan(\cos^{-1} x) = \sin\left[\cot^{-1}\left(\dfrac{1}{2}\right)\right]$.

Solution: Put $\cot^{-1}\left(\dfrac{1}{2}\right) = \theta$. Then $\cot\theta = \dfrac{1}{2}$.

Hence,

$$\sin\theta = \frac{2}{\sqrt{5}}$$

Put $\cos^{-1} x = \phi$, then $x = \cos\phi$.

Also

$$\tan\phi = \sin\theta = \frac{2}{\sqrt{5}}$$

Therefore

$$x = \cos\phi = \frac{\sqrt{5}}{3}$$

Illustration 17.6 If $\sin^{-1}\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{4} - \cdots\right) + \cos^{-1}\left(x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{4} - \cdots\right) = \dfrac{\pi}{2}$ for $0 < |x| < \sqrt{2}$, then find the value of x.

Solution: We know that

$$\sin^{-1} y + \cos^{-1} y = \frac{\pi}{2}, |y| \leq 1$$

Hence, according to question,

$$x - \frac{x^2}{2} + \frac{x^3}{4} - \cdots = x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \cdots$$

$$\Rightarrow \frac{x}{1+\dfrac{x}{2}} = \frac{x^2}{1+\dfrac{x^2}{2}}, (\because 0 < |x| < \sqrt{2}) \Rightarrow \frac{x}{2+x} = \frac{x^2}{2+x^2}$$

$$\Rightarrow 2x + x^3 = 2x^2 + x^3 \Rightarrow x = x^2$$

Hence,

$$x - x^2 = 0 \Rightarrow x(1-x) = 0 \Rightarrow x = 0 \text{ and } x = 1, \text{ but } x \neq 0.$$

So, $x = 1$.

Illustration 17.7 If $\sin^{-1} x + \sin^{-1} y = \dfrac{2\pi}{3}$, then $\cos^{-1} x + \cos^{-1} y =$ ______.

(A) $\dfrac{2\pi}{3}$ **(B)** $\dfrac{\pi}{3}$ **(C)** $\dfrac{\pi}{6}$ **(D)** π

Solution:

$$\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$$

$$\Rightarrow \frac{\pi}{2} - \cos^{-1} x + \frac{\pi}{2} - \cos^{-1} y = \frac{2\pi}{3}$$

$$\Rightarrow \cos^{-1} x + \cos^{-1} y = \frac{\pi}{3}$$

Your Turn 1

1. If $\sin^{-1} x + \cot^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{2}$, then x is

(A) 0 **(B)** $\dfrac{1}{\sqrt{5}}$

(C) $\dfrac{2}{\sqrt{5}}$ **(D)** $\dfrac{\sqrt{3}}{2}$ **Ans.** (B)

2. The value of $\sin(\cos^{-1} x)$ is

(A) $(1+x^2)^{3/2}$ **(B)** $(1+x^2)^{-3/2}$

(C) $(1-x^2)^{1/2}$ **(D)** $(1+x^2)^{-1/2}$

Ans. (C)

3. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\dfrac{\pi}{2}$ is

(A) Zero **(B)** One

(C) Two **(D)** Infinite **Ans.** (C)

4. Evaluate $\cos 2\cos^{-1}x+\sin^{-1}x$ at $x=\dfrac{1}{5}$. **Ans.** $\dfrac{-2\sqrt{6}}{5}$

5. Find the value of $\cos^{-1}\left(\cos\dfrac{5\pi}{3}\right)+\sin^{-1}\left(\sin\dfrac{5\pi}{3}\right)$. **Ans.** 0

6. The equation $2\cos^{-1}x+\sin^{-1}x=\dfrac{11\pi}{6}$ has

(A) No solution **(B)** Only one solution

(C) Two solutions **(D)** Three solutions **Ans.** (A)

17.4 General Values of Inverse Circular Functions

We know that if α is the smallest angle whose sine is x, then all the angles whose sine is x can be written as $n\pi+(-1)^n\alpha$, where $n = 0, 1, 2, \ldots$. Therefore, the general value of $\sin^{-1}x$ can be taken as $n\pi+(-1)^n\alpha$.

Thus, we have

$$\sin^{-1}x=n\pi+(-1)^n\alpha, -1\le x\le 1;$$

if $\sin\alpha = x$, then

$$-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$$

Similarly, general values of other inverse circular functions are given as follows:

$\cos^{-1}x=2n\pi\pm\alpha, -1\le x\le 1;$	If $\cos\alpha=x,\ 0\le\alpha\le\pi$
$\tan^{-1}x=n\pi-\alpha, x\in R;$	If $\tan\alpha=x,\ -\dfrac{\pi}{2}<\alpha<\dfrac{\pi}{2}$
$\cot^{-1}x=n\pi-\alpha, x\in R;$	If $\cot\alpha=x,\ 0\le\alpha\le\pi$
$\sec^{-1}x=2n\pi\pm\alpha, x\ge 1$ or $x\le -1;$	If $\sec\alpha=x, 0\le\alpha\le\pi$ and $\ne\dfrac{\pi}{2}$
$\operatorname{cosec}^{-1}x=n\pi+(-1)^n\alpha, x\ge 1$ or $x\le -1;$	If $\operatorname{cosec}\alpha=x, -\dfrac{\pi}{2}\le\alpha\le\dfrac{\pi}{2}$ and $x\ne 0$

Property 6

(i) $$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{ if } x>0, y>0 \text{ and } xy<1\\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{ if } x>0, y>0 \text{ and } xy>1\\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{ if } x<0, y<0 \text{ and } xy>1\end{cases}$$

(ii) $$\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), \text{ if } xy>-1\\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), \text{ if } x>0, y<0 \text{ and } xy<-1\\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), \text{if } x<0, y>0 \text{ and } xy<-1\end{cases}$$

Note:

(i) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$

(ii) $\tan^{-1}x_1+\tan^{-1}x_2+\cdots+\tan^{-1}x_n=\tan^{-1}\left[\dfrac{S_1-S_3+S_5-\cdots}{1-S_2+S_4-S_6+\cdots}\right]$

where S_k denotes the sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.

Proof (i) Let $\tan^{-1}x=A$ and $\tan^{-1}y=B$ where $x, y\in R$ and $A, B\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$. Then

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}=\frac{x+y}{1-xy}$$

Case (a): When $x, y > 0$ and $xy < 1$, then

$$\tan(A+B)=\frac{x+y}{1-xy}>0$$

Therefore, $\tan(A + B)$ lies in 1st or 3rd quadrant.

$$x>0\Rightarrow A\in\left(0,\frac{\pi}{2}\right), y>0\Rightarrow B\in\left(0,\frac{\pi}{2}\right)$$
$$\Rightarrow A+B\in(0,\pi)$$

As $(A+B)$ lies in 1st or 3rd quadrant. So,

$$A+B\in\left(0,\frac{\pi}{2}\right)$$
$$\Rightarrow\tan^{-1}[\tan(A+B)]=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$
$$A+B=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$
$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Case (b): When $x, y < 0$ and $xy < 1$, then

$$\tan(A+B)=\frac{x+y}{1-xy}<0$$

Therefore, $\tan(A + B)$ lies in 2nd or in 4th quadrant.

$$x<0\Rightarrow A\in\left(-\frac{\pi}{2},0\right), y<0\Rightarrow B\in\left(-\frac{\pi}{2},0\right)$$
$$\Rightarrow A+B\in(-\pi,0)$$

As $(A+B)$ lies in 2nd or 4th quadrant. So,

$$A+B\in\left(-\frac{\pi}{2},0\right)$$

$$\Rightarrow \tan^{-1}[\tan(A+B)]=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$A+B=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Case (c): When $x>0, y<0$

$$x>0\Rightarrow A\in\left(0,\frac{\pi}{2}\right), y<0\Rightarrow B\in\left(-\frac{\pi}{2},0\right)$$

$$\Rightarrow A+B\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$\Rightarrow \tan^{-1}[\tan(A+B)]=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$A+B=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Same for $x<0, y>0$.

Case (d): When $x, y>0$ and $xy>1$, then

$$\tan(A+B)=\frac{x+y}{1-xy}<0$$

Therefore, $\tan(A+B)$ lies in 2nd or in 4th quadrant.

$$x>0\Rightarrow A\in\left(0,\frac{\pi}{2}\right), y>0\Rightarrow B\in\left(0,\frac{\pi}{2}\right)$$

$$\Rightarrow A+B\in(0,\pi)$$

As $(A+B)$ lies in 2nd or 4th quadrant. So,

$$A+B\in\left(\frac{\pi}{2},\pi\right)$$

$$\Rightarrow \tan^{-1}[\tan(A+B-\pi)]=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$A+B-\pi=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Case (e): When $x, y<0$ and $xy>1$, then

$$\tan(A+B)=\frac{x+y}{1-xy}<0$$

Therefore, $\tan(A+B)$ lies in 1st or in 3rd quadrant.

$$x<0\Rightarrow A\in\left(-\frac{\pi}{2},0\right), y<0\Rightarrow B\in\left(-\frac{\pi}{2},0\right)$$

$$\Rightarrow A+B\in(-\pi,0)$$

As $(A+B)$ lies in 1st or 3rd quadrant. So,

$$A+B\in\left(-\pi,-\frac{\pi}{2}\right)$$

$$\Rightarrow \tan^{-1}[\tan(A+B+\pi)]=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$A+B+\pi=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\tan^{-1}x+\tan^{-1}y=-\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Proof (ii): Let $\tan^{-1}x=A$ and $\tan^{-1}y=B$ where

$x, y\in R$ *and* $A, B\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}=\frac{x-y}{1+xy}$$

Case (a): When $xy>-1$, then

$$\tan(A-B)=\frac{x-y}{1+xy}$$

$$\text{if}\quad x>0, A\in\left(0,\frac{\pi}{2}\right), y>0\Rightarrow B\in\left(0,\frac{\pi}{2}\right)$$

$$\Rightarrow A-B\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

So,

$$\tan^{-1}[\tan(A-B)]=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$A-B=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Same for all values of x and y with $xy>-1$.

Case (b): When $x>0,\ y<0$ and $xy<-1$, then

$$\tan(A-B)=\frac{x-y}{1+xy}<0$$

Therefore, $\tan(A-B)$ lies in 2nd or in 4th quadrant.

$$x>0\Rightarrow A\in\left(0,\frac{\pi}{2}\right), y<0\Rightarrow B\in\left(-\frac{\pi}{2},0\right)$$

$$\Rightarrow A-B\in(0,\pi)$$

As $(A-B)$ lies in 2nd or 4th quadrant. So,

$$A-B\in\left(\frac{\pi}{2},\pi\right)$$

$$\Rightarrow \tan^{-1}[\tan(A-B-\pi)]=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$A-B-\pi=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\tan^{-1}x-\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Case (c): When $x<0, y>0$ and $xy<-1$, then

$$\tan(A-B)=\frac{x-y}{1+xy}>0$$

Therefore, $\tan(A-B)$ lies in 1st or in 3rd quadrant.

$$x<0 \Rightarrow A \in \left(-\frac{\pi}{2},0\right), y>0 \Rightarrow B \in \left(0,\frac{\pi}{2}\right)$$

$$\Rightarrow A-B \in (-\pi,0)$$

As $(A-B)$ lies in 1st or 3rd quadrant. So,

$$A-B \in \left(-\pi,-\frac{\pi}{2}\right)$$

$$\Rightarrow \tan^{-1}[\tan(A-B+\pi)] = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$A-B+\pi = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\tan^{-1}x - \tan^{-1}y = -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Illustration 17.8 Prove that

$$\tan^{-1}\left(\frac{1}{2}\tan 2\theta\right)+\tan^{-1}(\cot\theta)+\tan^{-1}(\cot^3\theta) = \begin{cases} 0, \text{if } \frac{\pi}{4}<\theta<\frac{\pi}{2} \\ \pi, \text{if } 0<\theta<\frac{\pi}{4} \end{cases}$$

Solution:

Case (a): If $0<\theta<\frac{\pi}{4}$, then $\cot\theta>1$, $\cot^3\theta>1$.

Hence,

$$\tan^{-1}(\cot\theta)+\tan^{-1}(\cot^3\theta) = \pi+\tan^{-1}\left\{\frac{\cot\theta + \cot^3\theta}{1-\cot^4\theta}\right\}$$

Taking $\cot\theta$ common from numerator and using,

$$(1+\cot^2\theta) = \operatorname{cosec}^2\theta$$

$$= \pi+\tan^{-1}\left\{-\frac{\cot\theta\cdot\operatorname{cosec}^2\theta\cdot\sin^4\theta}{\cos^4\theta-\sin^4\theta}\right\}$$

$$= \pi+\tan^{-1}\left\{\frac{-\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}\right\}$$

$$= \pi+\tan^{-1}\left\{-\frac{1}{2}\tan 2\theta\right\}$$

$$= \pi-\tan^{-1}\left(\frac{1}{2}\tan 2\theta\right)$$

since $2\theta<\frac{\pi}{2}$ and $\tan 2\theta>0$

Hence,

$$\tan^{-1}\left(\frac{1}{2}\tan 2\theta\right)+\tan^{-1}(\cot\theta)+\tan^{-1}(\cot^3\theta)=\pi$$

Case (b): If $\frac{\pi}{4}<\theta<\frac{\pi}{2}$, then $0<\cot\theta<1$, $0<\cot^3\theta<1$

Therefore,

$$\tan^{-1}(\cot\theta)+\tan^{-1}(\cot^3\theta) = \tan^{-1}\left(-\frac{1}{2}\tan 2\theta\right)$$

$$= -\tan^{-1}\left(\frac{1}{2}\tan 2\theta\right) \quad \{\text{since } 2\theta>\pi \text{ and } \tan 2\theta<0\}$$

Hence,

$$\tan^{-1}\left(\frac{\tan 2\theta}{2}\right)+\tan^{-1}(\cot\theta)+\tan^{-1}(\cot^3\theta)=0$$

Illustration 17.9 Find the number of positive integral solutions of the equation $\tan^{-1}x+\cos^{-1}\frac{y}{\sqrt{1+y^2}}=\sin^{-1}\frac{3}{\sqrt{10}}$ or $\tan^{-1}x+\cot^{-1}y=\tan^{-1}3$.

Solution:

$$\tan^{-1}x+\tan^{-1}\frac{1}{y}=\tan^{-1}3 \text{ or } \tan^{-1}\frac{1}{y} = \tan^{-1}3-\tan^{-1}x$$

$$\text{or } \tan^{-1}\frac{1}{y}=\tan^{-1}\frac{3-x}{1+3x} \Rightarrow y=\frac{1+3x}{3-x}$$

As x, y are positive integers, $x=1, 2$ and corresponding $y=2, 7$. Hence, solutions are $(x, y)=(1, 2), (2, 7)$.

Illustration 17.10 Find the value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right]$.

Solution:

$$\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)\right] = \tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$$

$$=\tan\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right) = \tan\left[\tan^{-1}\left(\frac{17}{12}\times\frac{12}{6}\right)\right] = \frac{17}{6}$$

Illustration 17.11 Find the value of $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.

Solution:

$$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\tan^{-1}1=\frac{\pi}{4}$$

Property 7

(1) $$\sin^{-1}x+\sin^{-1}y=\begin{cases} \sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}), \\ \quad \text{if } x,y\in[-1,1] \text{ and } x^2+y^2\le 1 \\ \quad \text{or if } xy<0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}), \\ \quad \text{if } x,y\in(0,1] \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}), \\ \quad \text{if } x,y\in[-1,0) \text{ and } x^2+y^2>1 \end{cases}$$

(2) $$\sin^{-1}x-\sin^{-1}y=\begin{cases} \sin^{-1}(x\sqrt{1-y^2}-y\sqrt{1-x^2}), \\ \quad \text{if } x,y\in[-1,1] \text{ and } x^2+y^2\le 1 \\ \quad \text{or if } xy>0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}(x\sqrt{1-y^2}-y\sqrt{1-x^2}), \\ \quad \text{if } x\in(0,1], y\in[-1,0) \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}(x\sqrt{1-y^2}-y\sqrt{1-x^2}), \\ \quad \text{if } x\in[-1,0), y\in(0,1] \text{ and } x^2+y^2>1 \end{cases}$$

Proof (1): Let $\sin^{-1}x = A$ and $\sin^{-1}y = B$ where

$$x, y \in [-1,1] \text{ and } A, B \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$A + B \in [-\pi, \pi]$$
$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$
$$\sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

Case (a): If $x, y \in [-1, 0)$ and $x^2 + y^2 > 1$, then

$$A + B \in \left[-\pi, -\frac{\pi}{2}\right)$$

and $\sin(A+B) < 0$ and $\cos(A+B) < 0$

$$A + B < -\frac{\pi}{2} \Rightarrow A < -\frac{\pi}{2} - B$$

$$\Rightarrow \cos A < -\sin B \Rightarrow \sqrt{1-x^2} < -y \Rightarrow x^2 + y^2 > 1$$
$$\sin^{-1}[\sin(A+B)] = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$$
$$\sin^{-1}\{\sin[-\pi - (A+B)]\} = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$$
$$-\pi - (A+B) = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$
$$(A+B) = -\pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$
$$\sin^{-1}x + \sin^{-1}y = -\pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

Case (b): If $x, y \in (0,1]$ and $x^2 + y^2 > 1$, then

$$A + B \in \left(\frac{\pi}{2}, \pi\right]$$

and $\sin(A+B) > 0$ and $\cos(A+B) < 0$

$$A + B > \frac{\pi}{2} \Rightarrow A > \frac{\pi}{2} - B$$

$$\Rightarrow \cos A < \sin B \Rightarrow \sqrt{1-x^2} < y \Rightarrow x^2 + y^2 > 1$$
$$\sin^{-1}[\sin(A+B)] = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$$
$$\sin^{-1}\{\sin[\pi - (A+B)]\} = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$$
$$\pi - (A+B) = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$
$$(A+B) = \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$
$$\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

Case (c): If $x, y \in [-1,1]$ and $x^2 + y^2 \le 1$, then

$$A + B \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

and $\sin(A+B) > 0$ and $\cos(A+B) > 0$

$$A + B \le \frac{\pi}{2} \Rightarrow A \le \frac{\pi}{2} - B$$

$$\Rightarrow \cos A \ge \sin B \Rightarrow \sqrt{1-x^2} \ge y \Rightarrow x^2 + y^2 \le 1$$
$$\sin^{-1}(\sin(A+B)) = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$
$$(A+B) = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$
$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

Proof (2): Replace y by $-y$ in Proof (1).

Illustration 17.12 Find the value of x which satisfies the equation $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3} = \sin^{-1}x$.

Solution:

$$\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3} = \sin^{-1}\left[\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right] = \sin^{-1}\left[\frac{\sqrt{5}+4\sqrt{2}}{9}\right]$$

Therefore, $x = \dfrac{\sqrt{5}+4\sqrt{2}}{9}$.

Illustration 17.13 Find the value of C which satisfies the equation $\sin^{-1}\frac{3}{5} + \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}C$.

Solution: Given,

$$\sin^{-1}C = \sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13}$$

Hence,

$$\sin^{-1}C = \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} = \sin^{-1}\left\{\frac{3}{5}\sqrt{1-\frac{25}{169}} + \frac{5}{13}\sqrt{1-\frac{9}{25}}\right\}$$
$$= \sin^{-1}\left(\frac{56}{65}\right) \Rightarrow C = \frac{56}{65}$$

Property 8

(1) $$\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}(xy - \sqrt{1-x^2}\cdot\sqrt{1-y^2}), & \text{if } x, y \in [-1,1] \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}(xy - \sqrt{1-x^2}\cdot\sqrt{1-y^2}), & \text{if } x, y \in [-1,1] \text{ and } x + y \le 0 \end{cases}$$

(2) $$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}(xy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}), & \text{if } x, y \in [-1,1] \text{ and } x \le y \\ -\cos^{-1}(xy + \sqrt{1-x^2}\cdot\sqrt{1-y^2}), & \text{if } x \in (0,1], y \in [-1,0) \text{ and } x \ge y \end{cases}$$

Proof (1): Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$ where

$$x, y \in [-1,1] \text{ and } A, B \in [0, \pi]$$
$$\Rightarrow A + B \in [0, 2\pi]$$
$$\cos(A+B) = \cos A \cos B - \sin B \sin A$$
$$\cos(A+B) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

Case (a): If $x, y \in [-1,1]$ and $x + y \ge 0$, then

$$A + B \in [0, \pi]$$
$$A + B \le \pi \Rightarrow A \le \pi - B$$
$$\Rightarrow \cos A \ge -\cos B \Rightarrow \cos A + \cos B \ge 0 \Rightarrow x + y \ge 0$$
$$\cos^{-1}[\cos(A+B)] = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$
$$A + B = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$
$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$

Case (b): If $x, y \in [-1,1]$ and $x + y \le 0$, then

$$A + B \in [\pi, 2\pi]$$

$$A + B \ge \pi \Rightarrow A \ge \pi - B$$

$$\Rightarrow \cos A \le -\cos B \Rightarrow \cos A + \cos B \le 0 \Rightarrow x + y \le 0$$

$$\cos^{-1}[\cos(A+B)] = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$

$$\cos^{-1}\{\cos[2\pi - (A+B)]\} = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$

$$2\pi - (A+B) = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$

$$\cos^{-1}x + \cos^{-1}y = 2\pi - \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$

Proof (2): Replace y by $-y$ in Proof (1).

Illustration 17.14 If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

Solution: Given,

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$$

$$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z = \cos^{-1}(-z)$$

$$\Rightarrow \cos[\cos^{-1}x + \cos^{-1}y] = \cos[\cos^{-1}(-z)]$$

Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$. Then

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\Rightarrow \cos(A+B) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

$$\Rightarrow (A+B) = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$$

$$\Rightarrow \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = \cos^{-1}(-z)$$

$$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$$

$$\Rightarrow (xy+z)^2 = (1-x^2)(1-y^2) \Rightarrow x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2$$

$$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$$

Hence, proved.

Illustration 17.15 If $\alpha = \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{1}{3}$ and $\beta = \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{1}{3}$, then

(A) $\alpha < \beta$ **(B)** $\alpha = \beta$

(C) $\alpha > \beta$ **(D)** None of these

Solution:

$$\alpha = \sin^{-1}\left[\frac{4}{5}\sqrt{1-\frac{1}{9}} + \frac{1}{3}\sqrt{1-\frac{16}{25}}\right]$$

$$= \sin^{-1}\left[\frac{8\sqrt{2}}{15} + \frac{3}{15}\right] = \sin^{-1}\left(\frac{8\sqrt{2}+3}{15}\right)$$

Since $\frac{8\sqrt{2}+3}{15} < 1$, therefore

$$\alpha < \frac{\pi}{2}$$

$$\beta = \left(\frac{\pi}{2} - \sin^{-1}\frac{4}{5} + \frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) = (\pi - \alpha) > \frac{\pi}{2}$$

$$\Rightarrow \alpha < \beta$$

Illustration 17.16 If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$, then $9x^2 - 12xy\cos\theta + 4y^2$ is equal to

(A) $36\sin^2\theta$ **(B)** $36\cos^2\theta$

(C) $36\tan^2\theta$ **(D)** None of these

Solution:

$$\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$$

$$\Rightarrow \frac{x}{2}\cdot\frac{y}{3} - \sqrt{\left(1-\frac{x^2}{4}\right)}\sqrt{\left(1-\frac{y^2}{9}\right)} = \cos\theta$$

$$\Rightarrow (xy - 6\cos\theta)^2 = (4-x^2)(9-y^2)$$

$$\Rightarrow 9x^2 - 12xy\cos\theta + 4y^2 = 36(1-\cos^2\theta) = 36\sin^2\theta$$

Your Turn 2

1. If $f(x) = \cos^{-1}x + \cos^{-1}\left\{\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right\}$, then

   (A) $f\left(\frac{2}{3}\right) = \frac{\pi}{3}$ **(B)** $f\left(\frac{2}{3}\right) = 2\cos^{-1}\frac{2}{3} - \frac{\pi}{3}$

   (C) $f\left(\frac{1}{3}\right) = \frac{\pi}{3}$ **(D)** $f\left(\frac{1}{3}\right) = 2\cos^{-1}\frac{1}{3} - \frac{\pi}{3}$ **Ans.** (A), (D)

2. If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$, then $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} =$

   (A) $\sin^2\alpha$ **(B)** $\cos^2\alpha$

   (C) $\tan^2\alpha$ **(D)** $\cot^2\alpha$ **Ans.** (A)

3. All possible values of p and q for which $\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$ holds is

   (A) $p = -1, q = \frac{1}{2}$ **(B)** $q > 1, p = \frac{1}{2}$

   (C) $0 \le p \le 1, q = \frac{1}{2}$ **(D)** None of these **Ans.** (C)

4. The number of solutions of $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$ is

   (A) 0 **(B)** 1 **(C)** 2 **(D)** Infinite **Ans.** (B)

5. Obtain the value of $\cos^{-1}\left(-\frac{3}{5}\right) + \sin^{-1}\left(-\frac{5}{13}\right)$ in terms of $\cos^{-1}$ function. **Ans.** $\cos^{-1}\left(-\frac{16}{65}\right)$

6. $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} =$

   (A) 0 **(B)** $\frac{\pi}{2}$ **(C)** π **(D)** $\frac{2\pi}{3}$ **Ans.** (C)

7. If a, b, c be positive real numbers and the value of $\theta = \tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}$, then $\tan\theta$ is

   (A) 0 **(B)** 1 **(C)** $a+b+c$ **(D)** None of these **Ans.** (A)

Property 9

(1) $$2\sin^{-1}x=\begin{cases}-\pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -1\le x<-\dfrac{1}{\sqrt{2}}\\ \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \dfrac{1}{\sqrt{2}}<x\le 1\end{cases}$$

(2) $$3\sin^{-1}x=\begin{cases}-\pi-\sin^{-1}(3x-4x^3), & \text{if } -1\le x<-\dfrac{1}{2}\\ \sin^{-1}(3x-4x^3), & \text{if } -\dfrac{1}{2}\le x\le\dfrac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3), & \text{if } \dfrac{1}{2}<x\le 1\end{cases}$$

Proof (1): Let $\sin^{-1}x=A$ where $x\in[-1,1]$ and $A\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$. Then

$$x=\sin A$$
$$\sin 2A=2\sin A\cos A=2x\sqrt{1-x^2}$$
where $2A\in[-\pi,\pi]$

Case (a): If $2A\in\left[-\pi,-\dfrac{\pi}{2}\right)$, then

$$A\in\left[-\frac{\pi}{2},-\frac{\pi}{4}\right)\Rightarrow x\in\left[-1,-\frac{1}{\sqrt{2}}\right)$$
$$\sin^{-1}(\sin 2A)=\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow \sin^{-1}[(\sin(-\pi-2A)]=\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow -\pi-2A=\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow 2A=-\pi-\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow 2\sin^{-1}x=-\pi-\sin^{-1}(2x\sqrt{1-x^2})$$

Case (b): If $2A\in\left(\dfrac{\pi}{2},\pi\right]$, then

$$A\in\left(\frac{\pi}{4},\frac{\pi}{2}\right]\Rightarrow x\in\left(\frac{1}{\sqrt{2}},1\right]$$
$$\sin^{-1}(\sin 2A)=\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow \sin^{-1}[\sin(\pi-2A)]=\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow \pi-2A=\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow 2A=\pi-\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow 2\sin^{-1}x=\pi-\sin^{-1}(2x\sqrt{1-x^2})$$

Case (c): If $2A\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$, then

$$A\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\Rightarrow x\in\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$$
$$\sin^{-1}(\sin 2A)=\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow 2A=\sin^{-1}(2x\sqrt{1-x^2})$$
$$\Rightarrow 2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$$

Proof (2): Let $\sin^{-1}x=A$ where $x\in[-1,1]$ and $A\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$. Then

$$x=\sin A$$
$$\sin 3A=3\sin A-4\sin^3 A=3x-4x^3$$
where $3A\in\left[-\dfrac{3\pi}{2},\dfrac{3\pi}{2}\right]$

Case (a): If $3A\in\left[-\dfrac{3\pi}{2},-\dfrac{\pi}{2}\right)$, then

$$A\in\left[-\frac{3\pi}{6},-\frac{\pi}{6}\right)\Rightarrow x\in\left[-1,-\frac{1}{2}\right)$$
$$\sin^{-1}(\sin 3A)=\sin^{-1}(3x-4x^3)$$
$$\Rightarrow \sin^{-1}[\sin(-\pi-3A)]=\sin^{-1}(3x-4x^3)$$
$$\Rightarrow -\pi-3A=\sin^{-1}(3x-4x^3)$$
$$\Rightarrow 3A=-\pi-\sin^{-1}(3x-4x^3)$$
$$\Rightarrow 3\sin^{-1}x=-\pi-\sin^{-1}(3x-4x^3)$$

Case (b): If $3A\in\left(\dfrac{\pi}{2},\dfrac{3\pi}{2}\right]$, then

$$A\in\left(\frac{\pi}{6},\frac{\pi}{2}\right]\Rightarrow x\in\left(\frac{1}{2},1\right]$$
$$\sin^{-1}(\sin 3A)=\sin^{-1}(3x-4x^3)$$
$$\Rightarrow \sin^{-1}[\sin(\pi-3A)]=\sin^{-1}(3x-4x^3)$$
$$\Rightarrow \pi-3A=\sin^{-1}(3x-4x^3)$$
$$\Rightarrow 3A=\pi-\sin^{-1}(3x-4x^3)$$
$$\Rightarrow 3\sin^{-1}x=\pi-\sin^{-1}(3x-4x^3)$$

Case (c): If $3A\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$, then

$$A\in\left[-\frac{\pi}{6},\frac{\pi}{6}\right]\Rightarrow x\in\left[-\frac{1}{2},\frac{1}{2}\right]$$
$$\sin^{-1}(\sin 3A)=\sin^{-1}(3x-4x^3)$$
$$\Rightarrow 3A=\sin^{-1}(3x-4x^3)$$
$$\Rightarrow 3\sin^{-1}x=\sin^{-1}(3x-4x^3)$$

Property 10

(1) $$2\cos^{-1}x=\begin{cases}2\pi-\cos^{-1}(2x^2-1), & \text{if } -1\le x<0\\ \cos^{-1}(2x^2-1), & \text{if } 0\le x\le 1\end{cases}$$

(2) $$3\cos^{-1}x=\begin{cases}2\pi+\cos^{-1}(4x^3-3x), & \text{if } -1\le x<-\dfrac{1}{2}\\ 2\pi-\cos^{-1}(4x^3-3x), & \text{if } -\dfrac{1}{2}\le x\le\dfrac{1}{2}\\ \cos^{-1}(4x^3-3x), & \text{if } \dfrac{1}{2}<x\le 1\end{cases}$$

Proof (1): Let $\cos^{-1}x=A$ where $x\in[-1,1]$ and $A\in[0,\pi]$. Then

$$x=\cos A$$
$$\cos 2A=2\cos^2 A-1=2x^2-1$$
where $2A\in[0,2\pi]$

Case (a): If $2A\in[0,\pi]$, then

$$A\in\left[0,\frac{\pi}{2}\right]\Rightarrow x\in[0,1]$$

$$\cos^{-1}(\cos 2A) = \cos^{-1}(2x^2-1)$$
$$\Rightarrow 2A = \cos^{-1}(2x^2-1)$$
$$\Rightarrow 2\cos^{-1}x = \cos^{-1}(2x^2-1)$$

Case (b): If $2A \in (\pi, 2\pi]$, then

$$A \in \left(\frac{\pi}{2}, \pi\right] \Rightarrow x \in [-1, 0)$$

$$\cos^{-1}[\cos(2\pi - 2A)] = \cos^{-1}(2x^2-1)$$
$$\Rightarrow 2\pi - 2A = \cos^{-1}(2x^2-1)$$
$$\Rightarrow 2A = 2\pi - \cos^{-1}(2x^2-1)$$
$$\Rightarrow 2\cos^{-1}x = 2\pi - \cos^{-1}(2x^2-1)$$

Proof (2): Let $\cos^{-1}x = A$ where $x \in [-1, 1]$ and $A \in [0, \pi]$. Then

$$x = \cos A$$
$$\cos 3A = 4\cos^3 A - 3\cos A = 4x^3 - 3x$$

where $3A \in [0, 3\pi]$

Case (a): If $3A \in [0, \pi)$, then

$$A \in \left[0, \frac{\pi}{3}\right) \Rightarrow x \in \left(\frac{1}{2}, 1\right]$$

$$\cos^{-1}(\cos 3A) = \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow 3A = \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow 3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$$

Case (b): If $3A \in [\pi, 2\pi]$, then

$$A \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \Rightarrow x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\cos^{-1}[\cos(2\pi - 3A)] = \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow 2\pi - 3A = \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow 3A = 2\pi - \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow 3\cos^{-1}x = 2\pi - \cos^{-1}(4x^3 - 3x)$$

Case (c): If $3A \in (2\pi, 3\pi)$, then

$$A \in \left(\frac{2\pi}{3}, \pi\right] \Rightarrow x \in \left[-1, -\frac{1}{2}\right)$$

$$\cos^{-1}[\cos(3A - 2\pi)] = \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow 3A - 2\pi = \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow 3A = 2\pi + \cos^{-1}(4x^3 - 3x)$$
$$\Rightarrow 3\cos^{-1}x = 2\pi + \cos^{-1}(4x^3 - 3x)$$

Property 11

(1) $$2\tan^{-1}x = \begin{cases} -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x < -1 \\ \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x > 1 \end{cases}$$

(2) $$3\tan^{-1}x = \begin{cases} -\pi + \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right), & \text{if } x < -\dfrac{1}{\sqrt{3}} \\ \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right), & \text{if } -\dfrac{1}{\sqrt{3}} \le x \le \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right), & \text{if } x > \dfrac{1}{\sqrt{3}} \end{cases}$$

Proof (1): Let $\tan^{-1}x = A$ where $x \in R$ and $A \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then

$$x = \tan A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2x}{1-x^2}$$

where $2A \in (-\pi, \pi)$

Case (a): If $2A \in \left(-\pi, -\frac{\pi}{2}\right)$, then

$$A \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \Rightarrow x \in (-\infty, -1)$$

$$\tan^{-1}(\tan 2A) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow \tan^{-1}[\tan(\pi + 2A)] = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow \pi + 2A = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow 2A = -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow 2\tan^{-1}x = -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Case (b): If $2A \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$$A \in \left[-\frac{\pi}{4}, -\frac{\pi}{4}\right] \Rightarrow x \in [-1, 1]$$

$$\tan^{-1}(\tan 2A) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow 2A = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Case (c): If $2A \in \left(\frac{\pi}{2}, \pi\right)$, then

$$A \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \Rightarrow x \in (1, \infty)$$

$$\tan^{-1}(\tan 2A) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow \tan^{-1}[\tan(-\pi + 2A)] = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow -\pi + 2A = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$
$$\Rightarrow 2A = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\Rightarrow 2\tan^{-1}x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Proof (2): Let $\tan^{-1}x = A$ where $x \in R$ and $A \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $x = \tan A$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A} = \frac{3x-x^3}{1-3x^2}$$

$$3A \in \left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$$

Case (a): If $3A \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$, then

$$A \in \left(-\frac{\pi}{2}, -\frac{\pi}{6}\right) \Rightarrow x \in \left(-\infty, -\frac{1}{\sqrt{3}}\right)$$

$$\tan^{-1}(\tan 3A) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan^{-1}[\tan(\pi+3A)] = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \pi + 3A = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow 3A = -\pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow 3\tan^{-1}x = -\pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Case (b): If $3A \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$$A \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \Rightarrow x \in \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

$$\tan^{-1}(\tan 3A) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow 3A = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow 3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Case (c): If $3A \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, then

$$A \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \Rightarrow x \in \left(\frac{1}{\sqrt{3}}, \infty\right)$$

$$\tan^{-1}(\tan 3A) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \tan^{-1}[\tan(-\pi+3A)] = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow -\pi + 3A = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow 3A = \pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$3\tan^{-1}x = \pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

Illustration 17.17 Let α, β and γ are three angles given by

$$\alpha = 2\tan^{-1}(\sqrt{2}-1), \beta = 3\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\left(-\frac{1}{2}\right) \text{ and } \gamma = \cos^{-1}\left(\frac{1}{3}\right).$$

Then

(A) $\alpha > \beta$ **(B)** $\beta > \gamma$

(C) $\alpha > \gamma$ **(D)** None of these

Solution:

$$\alpha = 2\tan^{-1}(\sqrt{2}-1) = 2\tan^{-1}\tan\frac{\pi}{8}$$

$$= 2\times\frac{\pi}{8} = \frac{\pi}{4} = \cos^{-1}\frac{1}{\sqrt{2}}$$

$$\beta = 3\cdot\frac{\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$$

Therefore, $\beta > \alpha$. Also,

$$\frac{1}{3} < \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos^{-1}\frac{1}{3} > \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

So, $\gamma > \alpha$.

Again $\cos^{-1}\left(\frac{1}{3}\right)$ belongs to the first quadrant and β is in the second quadrant.

Hence, $\beta > \gamma$.

Illustration 17.18 The value of

$$\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right] + \cos\left[\tan^{-1}2\sqrt{2}\right] =$$

(A) $\frac{16}{15}$ **(B)** $\frac{14}{15}$

(C) $\frac{12}{15}$ **(D)** $\frac{11}{15}$

Solution:

$$\sin\left[2\tan^{-1}\left(\frac{1}{3}\right)\right] + \cos[\tan^{-1}(2\sqrt{2})]$$

$$= \sin\left[\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}\right] + \cos[\tan^{-1}(2\sqrt{2})]$$

$$= \sin\left[\tan^{-1}\frac{3}{4}\right] + \cos[\tan^{-1}2\sqrt{2}]$$

$$=\sin\left[\sin^{-1}\frac{3}{5}\right]+\cos\left[\cos^{-1}\frac{1}{3}\right]=\frac{3}{5}+\frac{1}{3}=\frac{14}{15}$$

Property 12

$$2\tan^{-1}x=\begin{cases}-\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{ if } x<-1\\ \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{ if } -1\le x\le 1\\ \pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{ if } x>1\end{cases}$$

Proof: Let $\tan^{-1}x=A$ where $x\in R$ and $A\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then

$$x=\tan A$$

$$\sin 2A=\frac{2\tan A}{1+\tan^2 A}=\frac{2x}{1+x^2}$$

where $2A\in(-\pi,\pi)$

Case (a): If $2A\in\left(-\pi,-\frac{\pi}{2}\right)$, then

$$A\in\left(-\frac{\pi}{2},-\frac{\pi}{4}\right)\Rightarrow x\in(-\infty,-1)$$

$$\sin 2A=\frac{2x}{1+x^2}$$

$$\Rightarrow \sin^{-1}[\sin(-\pi-2A)]=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow -\pi-2A=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow 2A=-\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow 2\tan^{-1}A=-\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Case (b): If $2A\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then

$$A\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\Rightarrow x\in[-1,1]$$

$$\sin 2A=\frac{2x}{1+x^2}$$

$$\Rightarrow \sin^{-1}[\sin(2A)]=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow 2A=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow 2\tan^{-1}A=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Case (c): If $2A\in\left(\frac{\pi}{2},\pi\right)$, then

$$A\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)\Rightarrow x\in(1,\infty)$$

$$\sin 2A=\frac{2x}{1+x^2}$$

$$\Rightarrow \sin^{-1}[\sin(\pi-2A)]=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \pi-2A=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow 2A=\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow 2\tan^{-1}A=\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Illustration 17.19 The solution set of the equation $\sin^{-1}x=2\tan^{-1}x$ is

(A) {1, 2} **(B)** {–1, 2}

(C) {–1,1,0} **(D)** {1, 1/2, 0}

Solution:

$$\sin^{-1}x=2\tan^{-1}x\Rightarrow \sin^{-1}x=\sin^{-1}\frac{2x}{1+x^2}$$

$$\Rightarrow \frac{2x}{1+x^2}=x \Rightarrow x^3-x=0\Rightarrow x(x+1)(x-1)=0\Rightarrow x=\{-1,1,0\}$$

Illustration 17.20 If $2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$ is independent of x, then

(A) $x\in[1,+\infty)$ **(B)** $x\in[-1,1]$

(C) $x\in(-\infty,-1]$ **(D)** None of these

Solution: Let $x=\tan\theta$. Then

$$\sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}(\sin 2\theta)$$

Hence, $2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=2\theta+\sin^{-1}(\sin 2\theta)$.

If $-\frac{\pi}{2}\le 2\theta\le\frac{\pi}{2}$, then

$$2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=2\theta+2\theta=4\tan^{-1}x$$

which is not independent of x

If $-\frac{\pi}{2}\le \pi-2\theta\le\frac{\pi}{2}$, then

$$2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}$$
$$=2\theta+\sin^{-1}[\sin(\pi-2\theta)]=2\theta+\pi-2\theta=\pi$$

which is independent of x

Hence, $\theta\notin\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ but $\theta\in\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$ and from the principal value of $\tan^{-1}x$.

$$\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Hence,

$$\theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$$

$$\Rightarrow \tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=\pi$$

Also at $\theta=\frac{\pi}{4}$,

$$2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=2\cdot\frac{\pi}{4}+\sin^{-1}1=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

Hence, the given function = π = constant if $\theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$, that is, $x\in[1,+\infty)$.

Property 13

$$2\tan^{-1}x=\begin{cases}-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \text{ if } x\le 0\\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \text{ if } x\ge 0\end{cases}$$

Proof: Let $\tan^{-1}x=A$ where $x\in R$ and $A\in\left(\frac{\pi}{2},\frac{\pi}{2}\right)$. Then

$$x=\tan A$$

$$\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}=\frac{1-x^2}{1+x^2}$$

where $2A\in(-\pi,\pi)$

Case (a): If $2A\in(-\pi,0)$, then

$$A\in\left(-\frac{\pi}{2},0\right)\Rightarrow x\in(-\infty,0)$$

$$\cos 2A=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \cos^{-1}[\cos(-2A)]=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow -2A=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow -2A=-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow 2\tan^{-1}x=-\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Case (b): If $2A\in[0,\pi]$, then

$$A\in\left[0,\frac{\pi}{2}\right)\Rightarrow x\in[0,\infty)$$

$$\cos 2A=\frac{1-x^2}{1+x^2}$$

$$\Rightarrow \cos^{-1}[\cos(2A)]=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow 2A=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Illustration 17.21 Write in the simplest form:

$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ where $-\frac{\pi}{2}\le x\le\frac{3\pi}{2}$

Solution:

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right]$$

$$=\tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]=\frac{\pi}{4}-\frac{x}{2}$$

Illustration 17.22 Find the angle

(A) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ **(B)** $\sin^{-1}\sin 5$ (where 5 is in radians).

Solution:

(A) Let $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\theta$

$$\tan^{-1}\tan\left(\pi-\frac{\pi}{4}\right)=\theta$$

$$\tan^{-1}\left(-\tan\frac{\pi}{4}\right)=\theta$$

$$\Rightarrow -\tan^{-1}\tan\frac{\pi}{4}=\theta\quad\left[\text{As} \tan^{-1}\tan\theta=\theta, \text{ if } \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

$$\Rightarrow -\frac{\pi}{4}=\theta$$

(B) We know

$$\sin^{-1}\sin\theta=\theta,\ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\approx[-1.57,1.57]\qquad(1)$$

Hence, $\sin^{-1}\sin 5\ne 5$ as $5\notin[-1.57, 1.57]$.
Therefore,

$$\sin 5=\sin(\pi+\overline{5-\pi})$$
$$=-\sin(5-\pi)$$

Since $(5-\pi)\notin[-1.57, 1.57]$, so we again add and subtract π.

$$\Rightarrow \sin 5=-\sin(\pi+\overline{5-2\pi})$$
$$=+\sin(5-2\pi)\ [\text{Since } (5-2\pi)\in[-1.57, 1.57]]$$

Hence,

$$\sin^{-1}\sin 5=\sin^{-1}\sin(5-2\pi)=5-2\pi$$

Note: To solve this type of problem, the procedure is to add and subtract π till it belongs to the principal value range of respective inverse trigonometric function.

Your Turn 3

1. If $2\tan^{-1}(\cos x) = \tan^{-1}(\operatorname{cosec}^2 x)$, then $x =$

(A) $\frac{\pi}{2}$ (B) π

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{3}$ **Ans.** (D)

2. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} =$

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ (D) None of these **Ans.** (C)

3. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$ is equal to

(A) π (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$ **Ans.** (D)

4. $\frac{a^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{a}{b}\right) + \frac{b^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{b}{a}\right)$ is equal to

(A) $(a-b)(a^2+b^2)$ (B) $(a+b)(a^2-b^2)$

(C) $(a+b)(a^2+b^2)$ (D) None of these **Ans.** (D)

5. If $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}$, then $x =$

(A) a (B) b

(C) $\frac{a+b}{1-ab}$ (D) $\frac{a-b}{1+ab}$ **Ans.** (D)

6. The formula $\cos^{-1}\frac{1-x^2}{1+x^2} = 2\tan^{-1}x$ holds only for

(A) $x \in R$ (B) $|x| \le 1$

(C) $x \in (-1,1]$ (D) $x \in [0,+\infty]$ **Ans.** (D)

Additional Solved Examples

1. The number of real solutions of $\cos^{-1}x + \cos^{-1}2x = -\pi$ is

(A) 0 (B) 1

(C) 2 (D) Infinitely many

Solution:

$$\cos^{-1}x = -(\pi + \cos^{-1}2x)$$

Range of $\cos^{-1}x \in [0, \pi]$

Since $\cos^{-1}x$ has a range from $[0, \pi]$, thus the sum of two $\cos^{-1}$ cannot be equal to $-\pi$ a negative quantity.

Hence, the correct answer is option (A).

2. Number of pairs (x, y) satisfying $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$ and $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$ is

(A) 0 (B) 1

(C) 2 (D) None of these

Solution: We know,

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$

Given,

$$\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$$

$$\Rightarrow \frac{\pi}{2} - \sin^{-1}x - \frac{\pi}{2} + \sin^{-1}y = \frac{\pi}{3}$$

$$\sin^{-1}y - \sin^{-1}x = \frac{\pi}{3}$$

$$\sin^{-1}y + \sin^{-1}x = \frac{2\pi}{3} \quad (1)$$

$$2\sin^{-1}y = \pi$$

$$\sin^{-1}y = \frac{\pi}{2} \quad (2)$$

$$\Rightarrow y = 1$$

Put Eq. (2) in Eq. (1).

$$\frac{\pi}{2} + \sin^{-1}x = \frac{2\pi}{3}$$

$$\sin^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow x = \frac{1}{2}$$

Thus, solution set is $\left(\frac{1}{2}, 1\right)$.

Hence, the correct answer is option (B).

3. Which of the following is the solution set of the equation $\sin^{-1}x = \cos^{-1}x + \sin^{-1}(3x-2)$?

(A) $\left\{\frac{1}{2}, 1\right\}$ (B) $\left[\frac{1}{2}, 1\right]$

(C) $\left[\frac{1}{3}, 1\right]$ (D) $\left\{\frac{1}{3}, 1\right\}$

Solution: We know

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} = 2\cos^{-1}x + \frac{\pi}{2} - \cos^{-1}(3x-2)$$

$$\cos^{-1}(3x-2) = 2\cos^{-1}x$$

$$\cos^{-1}(3x-2) = \cos^{-1}(2x^2-1)$$

$$3x - 2 = 2x^2 - 1$$

$$2x^2 - 3x + 1 = 0$$

$$(2x-1)(x-1) = 0$$

$$\Rightarrow x = \frac{1}{2}$$

$$\Rightarrow x = 1$$

Thus, only at $\left\{\frac{1}{2}, 1\right\}$ above expression has a solution

Hence, the correct answer is option (A).

4. If $\cot^{-1}\left(\frac{1}{a}\right)+\cot^{-1}\left(\frac{1}{b}\right)+\cot^{-1}\left(\frac{1}{c}\right)=\frac{\pi}{2}$, then

(A) $a+b+c=abc$ **(B)** $ab+bc+ca=1$

(C) $ab+bc+ca=abc$ **(D)** None of these

Solution: We know,

$$\tan^{-1}a=\cot^{-1}\frac{1}{a}(a>0)$$

$$\Rightarrow \tan^{-1}a+\tan^{-1}b+\tan^{-1}c=\frac{\pi}{2}$$

$$\tan^{-1}\left(\frac{a+b+c-abc}{1-ab-bc-ca}\right)=\frac{\pi}{2}$$

$$\Rightarrow 1-ab-bc-ca=0$$
$$ab+bc+ca=1$$

Hence, the correct answer is option (B).

5. $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$ holds good for all

(A) $|x|\le 1$ **(B)** $0\le x\le 1$

(C) $|x|\le 1/2$ **(D)** None of these

Solution: We know,

$$\sin^{-1}(z)\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$

So, for

$$3\sin^{-1}(x)=\sin^{-1}(3x-4x^3)$$

$$\frac{-\pi}{2}\le\sin^{-1}\left(3x-4x^3\right)\le\frac{\pi}{2}$$

$$\Rightarrow\frac{-\pi}{2}\le 3\sin^{-1}x\le\frac{\pi}{2}$$

$$\frac{-\pi}{6}\le\sin^{-1}x\le\frac{\pi}{6}$$

$$\Rightarrow x\in\left[-\frac{1}{2},\frac{1}{2}\right]\ \Rightarrow |x|\le\frac{1}{2}$$

Hence, the correct answer is option (C).

6. If $(\tan^{-1}x)^2+(\cos^{-1}x)^2=\frac{5\pi^2}{8}$, then x equals

(A) -1 **(B)** 1

(C) 0 **(D)** None of these

Solution: We know,

$$\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$$

So,

$$(\tan^{-1}x)^2+\left(\frac{\pi}{2}-\tan^{-1}x\right)^2=\frac{5\pi^2}{8}$$

$$2(\tan^{-1}x)^2+\frac{\pi^2}{4}-\pi\tan^{-1}x=\frac{5\pi^2}{8}$$

Let $\tan^{-1}x=t$. Then

$$2t^2-\pi t-\frac{3\pi^2}{8}=0$$

$$\Rightarrow 16t^2-8\pi t-3\pi^2=0$$
$$\Rightarrow (4t+\pi)(4t-3\pi)=0$$
$$\Rightarrow t=\frac{-\pi}{4}, t=\frac{3\pi}{4}$$

So,

$$\tan^{-1}x=\frac{-\pi}{4}, \tan^{-1}x=\frac{3\pi}{4}$$

$$x=-1, \qquad x=-1$$

Thus, $x=-1$.

Hence, the correct answer is option (A).

7. If $\sin^{-1}6x+\sin^{-1}6\sqrt{3}x=-\frac{\pi}{2}$, then x is equal to

(A) -1 **(B)** $\frac{1}{\sqrt{12}}$

(C) $-\frac{1}{\sqrt{12}}$ **(D)** None of these

Solution: See Fig. 17.9. Given,

$$\sin^{-1}6x+\sin^{-1}6\sqrt{3}x=-\frac{\pi}{2}$$

$$\frac{-\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}$$

Thus sum of two angles α, β can only be negative when both the angles are negative but angles are negative only when x is negative or equal negative.

Putting $x=-1$, we have

$\sin^{-1}(-6)+\sin^{-1}(-6\sqrt{3})$.

Since $\sin^{-1}x$ has $x\in[-1, 0]$ so $x=-1$ cannot be a solution.

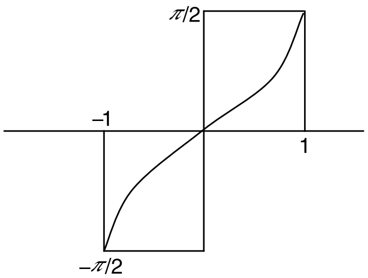


Figure 17.9

Now, putting $x=-\frac{1}{12}$,

$$\sin^{-1}\left[\left(\frac{-1}{12}\right)\right]+\sin^{-1}\left(6\sqrt{3\left(-\frac{1}{12}\right)}\right)$$

$$=\sin^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$=-\frac{\pi}{6}-\frac{\pi}{3}=-\frac{\pi}{2}$$

So,

$$x=-\frac{1}{12}$$

Hence, the correct answer is option (C).

8. $\sin^{-1}x > \cos^{-1}x$ hold for

(A) All values of x **(B)** $x \in \left(0, \frac{1}{\sqrt{2}}\right)$

(C) $x \in \left(\frac{1}{\sqrt{2}}, 1\right]$ **(D)** $x = 0.75$

Solution: See Fig. 17.10.

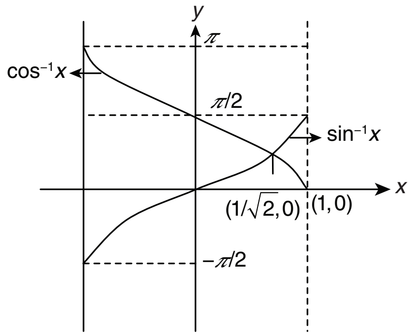


Figure 17.10

$$0 \le \cos^{-1}x \le \pi; \frac{-\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$$

Clearly, $\sin^{-1}x > \cos^{-1}x \, \forall x \in \left(\frac{1}{\sqrt{2}}, 1\right]$.

Hence, the correct answer is option (C).

9. The value of $\tan^{-1} + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is

(A) $\frac{\pi}{4}$ **(B)** $\frac{5\pi}{12}$

(C) $\frac{3\pi}{4}$ **(D)** $\frac{11\pi}{12}$

Solution:

$$\frac{-\pi}{2} < \tan^{-1}x < \frac{\pi}{2}; \frac{-\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}; 0 \le \cos^{-1}x \le \pi$$

$$\tan^{-1}(1) = \frac{\pi}{4} \quad (1)$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \quad (2)$$

$$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{-\pi}{6} \quad (3)$$

Thus, Eqs. (1) + (2) + (3) gives

$$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$$

Alternative Solution: Since

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

and $\tan^{-1}(1) = \frac{\pi}{4}$

So,

$$\frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Main/AIEEE Questions

1. If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then a value of x is

(A) 1 **(B)** 3

(C) 4 **(D)** 5 **[AIEEE 2007]**

Solution: We have

$$\sin^{-1}\frac{x}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2} \Rightarrow \sin^{-1}\frac{x}{5} = \cos^{-1}\frac{4}{5} \Rightarrow \sin^{-1}\frac{x}{5} = \sin^{-1}\frac{3}{5}$$

Therefore, $x = 3$.

Hence, the correct answer is option (B).

2. The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is

(A) $\frac{6}{17}$ **(B)** $\frac{3}{17}$

(C) $\frac{4}{17}$ **(D)** $\frac{5}{17}$ **[AIEEE 2008]**

Solution: Let us consider that, $E = \cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$.

Therefore,

$$E = \cot\left[\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right] = \cot\left[\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}\right)\right]$$

$$= \cot\left(\tan^{-1}\frac{17}{6}\right) = \frac{6}{17}$$

Hence, the correct answer is option (A).

3. Statement I: The equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 - a\pi^3 = 0$ has a solution for all $a \ge \frac{1}{32}$.

Statement II: For any $x \in R$, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ and

$$0 \le \left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \le \frac{9\pi^2}{16}.$$

(A) Both statements I and II are true.
(B) Both statements I and II are false.
(C) Statement I is true and statement II is false.
(D) Statement I is false and statement II is true.

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$(\sin^{-1}x)^3+(\cos^{-1}x)^3-a\pi^3=0$$

$$\Rightarrow(\sin^{-1}x+\cos^{-1}x)[(\sin^{-1}x)^2-\sin^{-1}x\cos^{-1}x+(\cos^{-1}x)^2]=a\pi^3$$

$$\Rightarrow\frac{\cancel{\pi}}{2}[(\sin^{-1}x+\cos^{-1}x)^2-3\sin^{-1}x\cos^{-1}x]=a\cancel{\pi}^3\pi^2$$

$$\Rightarrow\left(\frac{\pi}{2}\right)^2-3\sin^{-1}x\left(\frac{\pi}{2}-\sin^{-1}x\right)=2a\pi^2$$

$$\Rightarrow\frac{\frac{\pi^2}{4}-2a\pi^2}{3}=\frac{\pi}{2}(\sin^{-1}x)-(\sin^{-1}x)^2$$

$$\Rightarrow\frac{\pi^2-8a\pi^2}{12}=\frac{\pi}{2}(\sin^{-1}x)-(\sin^{-1}x)^2$$

$$\Rightarrow(\sin^{-1}x)^2-\frac{\pi}{2}\sin^{-1}x+\left(\frac{\pi}{4}\right)^2=\left(\frac{\pi}{4}\right)^2-\frac{\pi^2-8a\pi^2}{12}$$

$$\Rightarrow\left(\sin^{-1}x-\frac{\pi}{4}\right)^2=\frac{\pi^2}{16}-\frac{\pi^2}{12}+\frac{\cancel{8}^2a\pi^2}{\cancel{12}_3}=\frac{32a\pi^2-\pi^2}{48}$$

Now

$$-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2}\Rightarrow\frac{-\pi}{2}-\frac{\pi}{4}\le\sin^{-1}x-\frac{\pi}{4}\le\frac{\pi}{2}-\frac{\pi}{4}$$

or

$$\frac{-3\pi}{4}\le\sin^{-1}x-\frac{\pi}{4}\le\frac{\pi}{4}$$

$$\Rightarrow 0\le\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\le\frac{9\pi^2}{16}$$

Therefore,

$$0\le\frac{32a\pi^2-\pi^2}{48}\le\frac{9\pi^2}{16}\Rightarrow 0\le\frac{32a-1}{48}\le\frac{9}{16}$$

$$\Rightarrow 0\le 32a-1\le 27\Rightarrow 1\le 32a\le 28$$

$$\Rightarrow\frac{1}{32}\le a\le\frac{28}{32}\Rightarrow a\in\left[\frac{1}{32},\frac{7}{8}\right]$$

Therefore, Statement I is false and II is true.

Hence, the correct answer is option (D).

4. The principal value of $\tan^{-1}\left(\cot\frac{43\pi}{4}\right)$ is

(A) $-\frac{3\pi}{4}$ **(B)** $\frac{3\pi}{4}$

(C) $-\frac{\pi}{4}$ **(D)** $\frac{\pi}{4}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$\cot\frac{43\pi}{4}=\cot\left(\frac{44\pi}{4}-\frac{\pi}{4}\right)=\cot\left(11\pi-\frac{\pi}{4}\right)=\cot\left(\pi-\frac{\pi}{4}\right)$$

$$=\cot\frac{3\pi}{4}=\cot\left(\frac{\pi}{2}+\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=\tan\left(\frac{-\pi}{4}\right)$$

Therefore, $\tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right]=\frac{-\pi}{4}$.

Hence, the correct answer is option (C).

5. Let $\tan^{-1}y=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x|<\frac{1}{\sqrt{3}}$. Then a value of y is

(A) $\frac{3x+x^3}{1-3x^2}$ **(B)** $\frac{3x-x^3}{1+3x^2}$

(C) $\frac{3x+x^3}{1+3x^2}$ **(D)** $\frac{3x-x^3}{1-3x^2}$

[JEE MAIN 2015 (OFFLINE)]

Solution: Since,

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\text{ for }xy<1$$

Now,

$$x\cdot\left(\frac{2x}{1-x^2}\right)=\frac{2x^2}{1-x^2}=-2\left(\frac{1-x^2-1}{1-x^2}\right)=-2+\frac{2}{1-x^2}$$

Further

$$|x|<\frac{1}{\sqrt{3}}\Rightarrow 0\le x^2<\frac{1}{3}\Rightarrow-\frac{1}{3}<-x^2\le 0\Rightarrow\frac{2}{3}<1-x^2\le 1$$

$$\Rightarrow 1\le\frac{1}{1-x^2}<\frac{3}{2}\Rightarrow 2\le\frac{2}{1-x^2}<3\Rightarrow 0\le-2+\frac{2}{1-x^2}<1$$

$$\Rightarrow x\cdot\left(\frac{2x}{1-x^2}\right)\in(0,1)$$

Hence,

$$\tan^{-1}(x)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-\frac{2x^2}{1-x^2}}\right)$$

$$=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=\tan^{-1}(y)\ \text{(given)}$$

$$\Rightarrow y=\left(\frac{3x-x^3}{1-3x^2}\right)$$

Hence, the correct answer is option (D).

6. If $f(x)=2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $x>1$, then $f(5)$ is equal to

(A) $\frac{\pi}{2}$ **(B)** π

(C) $4\tan^{-1}(5)$ **(D)** $\tan^{-1}\left(\frac{65}{156}\right)$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:

$$f(x)=2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right),\ x>1,\ f(5)=?$$

We know that

$$2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & 1\le x\le 1\\ -\pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & x<-1\\ \pi-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & x>1\end{cases}$$

$$\Rightarrow f(x)=2\tan^{-1}x+(\pi-2\tan^{-1}x)\Rightarrow f(x)=\pi\ \forall\ x>1\Rightarrow f(5)=\pi$$

Hence, the correct answer is option (B).

7. Consider $f(x)=\tan^{-1}\left(\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\dfrac{\pi}{2}\right)$. A normal to $y=f(x)$ at $x=\dfrac{\pi}{6}$ also passes through the point

(A) $\left(\dfrac{\pi}{4},0\right)$ **(B)** (0, 0)

(C) $\left(0,\dfrac{2\pi}{3}\right)$ **(D)** $\left(\dfrac{\pi}{6},0\right)$

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\frac{\pi}{2}\right)$$

$$f(x)=\tan^{-1}\left(\sqrt{\frac{\sin^2(x/2)+\cos^2(x/2)+2\sin(x/2)\cos(x/2)}{\sin^2(x/2)+\cos^2(x/2)+2\sin(x/2)\cos(x/2)}}\right)$$

$$f(x)=\tan^{-1}\left(\sqrt{\frac{[\cos(x/2)+\sin(x/2)]^2}{[\cos(x/2)-\sin(x/2)]^2}}\right)$$

$$f(x)=\tan^{-1}\left(\frac{|\cos(x/2)+\sin(x/2)|}{|\cos(x/2)-\sin(x/2)|}\right), x\in\left(0,\frac{\pi}{2}\right)$$

$$f(x)=\tan^{-1}\left[\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\right]$$

$$f(x)=\tan^{-1}\left[\frac{1+\tan(x/2)}{1-\tan(x/2)}\right]$$

$$f(x)=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

$$f(x)=\frac{\pi}{4}+\frac{x}{2}$$

$$f\left(\frac{\pi}{6}\right)=\frac{\pi}{4}+\frac{x}{12}=\frac{3\pi+\pi}{12}=\frac{\pi}{3}$$

The point is $\left(\dfrac{\pi}{6},\dfrac{x}{3}\right)$, Therefore,

$$f'(x)=\frac{1}{2}$$

$$\Rightarrow f'\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

The slope of normal is –2 and the equation of normal is

$$y-\frac{\pi}{3}=-2\left(x-\frac{\pi}{6}\right)$$

$$=-2x+\frac{\pi}{3}$$

Therefore,

$$y=-2x+\frac{2\pi}{3}$$

$$2x+y=\frac{2\pi}{3}$$

$$\Rightarrow\frac{x}{\pi/3}+\frac{y}{2\pi/3}=1$$

So, the normal passes through the point $\left(0,\dfrac{2\pi}{3}\right)$.

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Let (x, y) be such that

$$\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\frac{\pi}{2}$$

Match the statements in Column I with statements in Column II.

Column I	Column II
(A) If $a=1$ and $b=0$, then (x, y)	**(P)** lies on the circle $x^2+y^2=1$
(B) If $a=1$ and $b=1$, then (x, y)	**(Q)** lies on $(x^2-1)(y^2-1)=0$
(C) If $a=1$ and $b=2$, then (x, y)	**(R)** lies on $y=x$
(D) If $a=2$ and $b=2$, then (x, y)	**(S)** lies on $(4x^2-1)(y^2-1)=0$

[IIT-JEE 2007]

Solution: If $a=1$ and $b=0$, then

$$\sin^{-1}x+\cos^{-1}y+\frac{\pi}{2}=\frac{\pi}{2}$$

$$\Rightarrow\sin^{-1}x=-\cos^{-1}y$$

$$\Rightarrow\sin^{-1}x=-\sin^{-1}\sqrt{1-y^2}$$

$$\Rightarrow -x=\sqrt{1-y^2}$$

$$\Rightarrow x^2+y^2=1$$

(A) $\rightarrow$ (P)

If $a=1$ and $b=1$, then

$$\sin^{-1}x+\cos^{-1}y+\cos^{-1}xy=\frac{\pi}{2}$$

$$\cos^{-1}x-\cos^{-1}y=\cos^{-1}xy$$

$$\Rightarrow xy+\sqrt{1-y^2}\sqrt{1-x^2}=xy$$

$$\Rightarrow(x^2-1)(y^2-1)=0$$

(B) $\rightarrow$ (Q)

If $a = 1$ and $b = 2$, then

$$\sin^{-1} x + \cos^{-1} y + \cos^{-1}(2xy) = -\cos^{-1} x - \cos^{-1} y = \cos^{-1}(2xy)$$

$$\Rightarrow xy + \sqrt{1-x^2}\sqrt{1-y^2} = 2xy$$

$$\Rightarrow \sqrt{1-x^2}\sqrt{1-y^2} = xy$$

$$\Rightarrow 1 - x^2 - y^2 + x^2y^2 = x^2y^2$$

$$\Rightarrow x^2 + y^2 = 1$$

(C)→(P)

If $a = 2$ and $b = 2$, we get

$$\sin^{-1} 2x + \cos^{-1} y + \cos^{-1}(2xy) = \frac{\pi}{2}$$

$$\cos^{-1} 2x - \cos^{-1} y = \cos^{-1}(2xy)$$

$$2xy + \sqrt{1-4x^2}\sqrt{1-y^2} = 2xy$$

$$\Rightarrow (4x^2 - 1)(y^2 - 1) = 0$$

(D)→(S)

Hence, the correct matches are (A)→(P); (B)→(Q); (C)→(P); (D)→(S).

2. If $0 < x < 1$, then $\sqrt{1+x^2}\{[x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)]^2 - 1\}^{1/2}$ is equal to

(A) $\dfrac{x}{\sqrt{1+x^2}}$ **(B)** x

(C) $x\sqrt{1+x^2}$ **(D)** $\sqrt{1+x^2}$

Solution: We have

$$\sqrt{1+x^2}\{[x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)]^2 - 1\}^{1/2}$$

$$\sqrt{1+x^2}\left[\left(x\cos\cos^{-1}\frac{1}{\sqrt{1+x^2}} + \sin\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)^2 - 1\right]$$

$$\sqrt{1+x^2}\left[\left(\frac{x^2}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right)^2 - 1\right]^{\frac{1}{2}}$$

$$\sqrt{1+x^2}[(\sqrt{x^2+1})^2 - 1]^{\frac{1}{2}}$$

$$= x\sqrt{1+x^2}$$

Hence, the correct answer is option (C).

3. The value of $\cot\left[\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right]$ is

(A) $\dfrac{23}{25}$ **(B)** $\dfrac{25}{23}$

(C) $\dfrac{23}{24}$ **(D)** $\dfrac{24}{23}$ **[JEE ADVANCED 2013]**

Solution: We have

$$\cot\left[\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right] = \cot\left[\sum_{n=1}^{23}\cot^{-1}(1+n(n+1))\right]$$

$$= \cot\sum_{n=1}^{23}(\tan^{-1}(n+1) - \tan^{-1} n)$$

$$= \cot(\tan^{-1} 24 - \tan^{-1} 1)$$

$$= \cot\left[\tan^{-1}\left(\frac{23}{25}\right)\right]$$

$$= \frac{25}{23}$$

Hence, the correct answer is option (B).

4. Match List I to List II.

List I	List II
P. Let $y(x) = \cos(3\cos^{-1} x)$, $x \in [-1, 1]$, $x \pm \dfrac{\sqrt{3}}{2}$. Then $\dfrac{1}{y(x)}\left\{(x^2-1)\dfrac{d^2y(x)}{dx^2} + x\dfrac{dy(x)}{dx}\right\}$ equals	**1.** 1
Q. Let $A_1, A_T, \ldots, A_n$ ($n > 2$) be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k, $k = 1, 2, \cdots, n$. If $\left\lvert\sum_{k=1}^{n-1}(\overline{a_k} \times \overline{a_{k-1}})\right\rvert = \left\lvert\sum_{k=1}^{n-1}(\overline{a_k} \cdot \overline{a_{k+1}})\right\rvert$, then the minimum value of n is	**2.** 2
R. If the normal from the point $P(h, 1)$ on the ellipse $\dfrac{x^2}{6} + \dfrac{x^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	**3.** 8
S. Number of positive solutions satisfying the equation $\tan^{-1}\left(\dfrac{1}{2x+1}\right) + \tan^{-1}\left(\dfrac{1}{4x+1}\right) = \tan^{-1}\left(\dfrac{2}{x^2}\right)$ is	**4.** 9

	P	Q	R	S
(A)	4	3	2	1
(B)	2	4	3	1
(C)	4	3	1	2
(D)	2	4	1	3

[JEE ADVANCED 2014]

Solution: For (P) in List I:

$$y(x) = \cos(3\cos^{-1} x)$$

Hence,

$$\frac{dy}{dx} = -\{\sin(3\cos^{-1} x)\}3\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{3\sin(\cos^{-1} x)}{\sqrt{1-x^2}}$$

$$= \left\{\sqrt{1-x^2}\frac{dy}{dx}\right\}^2 = \left\{\sin\left(3\cos^{-1} x\right)\right\}^2$$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 9\sin^2(3\cos^{-1} x)$$

$$= 9[1 - \cos^2(3\cos^{-1} x)]$$

$$= 9(1-y^2)$$

Now differentiating

$$(1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = 9\left\{-2y\frac{dy}{dx}\right\}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -9y$$

$$\Rightarrow (x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 9y$$

Therefore,

$$(P) \rightarrow (4)$$

For (Q) in List I:

$$\left|\sum_{k=1}^{n-1}(\vec{a}_k \times \vec{a}_{k+1})\right| = |\vec{a}_1 \times \vec{a}_2 + \vec{a}_2 \times \vec{a}_3 + \cdots + a_{n-1} \times \vec{a}_n|$$

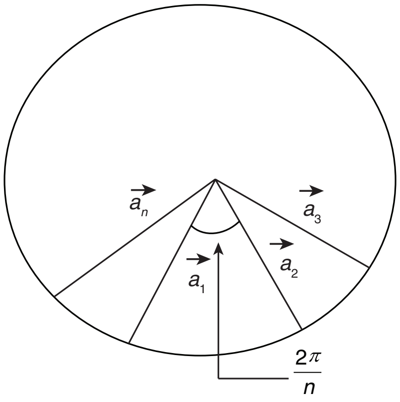


Figure 17.11

$$= (a^2 + a^2 + \cdots + a^2)\sin\frac{2\pi}{n}$$

$$= (n-1)a^2 \sin\frac{2\pi}{n} \qquad (1)$$

(See Fig. 17.11.) Since all $|\vec{a}_1|$ are equal

Also $\left|\sum_{k=1}^{n-1}(\vec{a}_k \cdot \vec{a}_{k+1})\right| = (a^2 + a^2 \cdots + a^2)\cos\frac{2\pi}{n}$

$$= (n-1)a^2 \cos\frac{2\pi}{n} \qquad (2)$$

From Eqs. (1) and (2),

$$\sin\frac{2\pi}{n} = \cos\frac{2\pi}{n}$$

$$\Rightarrow \tan\frac{2\pi}{n} = 1 \Rightarrow \frac{2\cancel{\pi}}{n} = \frac{\cancel{\pi}}{4} \Rightarrow n = 8$$

Therefore,

$$(Q) \rightarrow (3)$$

For (R) in List I:

(See Fig. 17.12.) Equation of normal at $(\sqrt{6}\cos\theta, \sqrt{3}\sin\theta)$ is

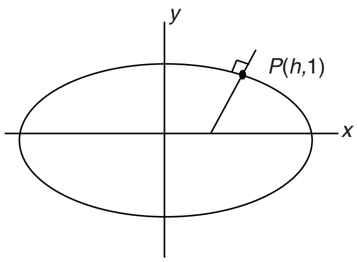


Figure 17.12

$$\frac{\sqrt{6}x}{\cos\theta} - \frac{\sqrt{3}y}{\sin\theta} = 3 \qquad (1)$$

Since,

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

Therefore, slope of this normal is

$$= \left(-\frac{\sqrt{6}}{\cos\theta}\right)\Big/\left(\frac{-\sqrt{3}}{\sin\theta}\right) = \frac{\sqrt{6}}{\sqrt{3}}\frac{\sin\theta}{\cos\theta} = \sqrt{2}\tan\theta$$

Since $(h, 1)$ lies on this normal,

Therefore,

$$\frac{\sqrt{6}h}{\cos\theta} - \frac{\sqrt{3}l}{\sin\theta} = 3 \qquad (2)$$

Now this normal is $\perp$ to $x + y = 8$, Hence, its slope is

$$\frac{-1}{-1} = 1$$

(See Fig. 17.13.) Therefore,

$$\sqrt{2}\tan\theta = 1 \Rightarrow \tan\theta = \frac{1}{\sqrt{2}}$$

So,

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}, \quad \sin\theta = \frac{1}{\sqrt{3}}$$

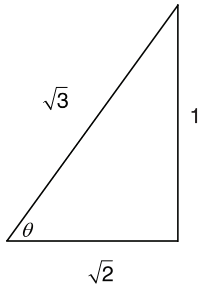


Figure 17.13

So, Eq. (2) becomes

$$\frac{\sqrt{6}h}{\frac{\sqrt{2}}{\sqrt{3}}} - \frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} = 3$$

$$\Rightarrow \left(\sqrt{\frac{\overset{3}{\cancel{6}} \times 3}{\cancel{2}}}\right)h = 6$$

Therefore,

$$h = 2$$

Hence,

$$(R) \rightarrow (2)$$

For (S) in List I:

$$\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \tan^{-1}\left\{\frac{\frac{1}{2x+1} + \frac{1}{4x+1}}{1 - \frac{1}{(2x+1)(4x+1)}}\right\} = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \tan^{-1}\left\{\frac{\frac{4x+1+2x+1}{8x^2+6x+1}}{\frac{8x^2+6x+1-1}{8x^2+6x+1}}\right\} = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{6x+2}{8x^2+6x}\right) = \tan^{-1}\frac{2}{x^2}$$

Therefore,

$$6x^3+2x^2=16x^2+12x$$
$$\Rightarrow 2x(3x^2+x-8x-6)=0$$
$$\Rightarrow x(3x^2-7x-6)=0$$
$$\Rightarrow x(3x^2-9x+2x-6)=0$$
$$\Rightarrow x[3x(x-3)+2(x-3)]=0$$
$$\Rightarrow x(x-3)(3x+2)=0$$

Therefore,

$$x=0,3,-\frac{2}{3}$$

Hence, number of +ve solutions = 1

Therefore,

(S) → (1)

Hence, the correct answer is option (A).

5. If $\alpha=3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta=3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are)

(A) $\cos\beta>0$ **(B)** $\sin\beta<0$
(C) $\cos(\alpha+\beta)>0$ **(D)** $\cos\alpha<0$

[JEE ADVANCED 2015]

Solution:

$$\alpha=3\sin^{-1}\left(\frac{6}{11}\right)>3\sin^{-1}\left(\frac{6}{12}\right)=\frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2}<\alpha<\pi \qquad (1)$$

$$\beta=3\cos^{-1}\left(\frac{4}{9}\right)>3\cos^{-1}\left(\frac{1}{2}\right)=\pi$$

Also

$$\pi<3\cos^{-1}\left(\frac{4}{9}\right)<\frac{3\pi}{2}$$

$$\Rightarrow \pi<\beta<\frac{3\pi}{2} \qquad (2)$$

From Eqs. (1) and (2)

$$\cos\alpha<0, \cos\beta<0, \sin\beta<0,$$

Also,

$$\frac{3\pi}{2}<(\alpha+\beta)<\frac{5\pi}{2}$$

$$\Rightarrow \cos(\alpha+\beta)>0$$

Hence, the correct answer is option (C).

Practice Exercise 1

1. The value of $\tan(\tan^{-1}1+\tan^{-1}2+\tan^{-1}3)=$

(A) 0 **(B)** tan 1
(C) $\tan\frac{1}{3}$ **(D)** $\tan\frac{1}{6}$

2. $\cos^{-1}(\cos x)=\pi+x$, then x belongs to

(A) $(0,\pi)$ **(B)** $(\pi,2\pi)$
(C) $[0,\pi]$ **(D)** None of these

3. $\sin^{-1}\sin(16)$ is equal to

(A) $5\pi-16$ **(B)** $16-5\pi$
(C) $6\pi-16$ **(D)** None of these

4. If $\theta=\cot^{-1}7+\cot^{-1}8+\cot^{-1}18$, then $\cot\theta$ is equal to

(A) 2 **(B)** 3
(C) 4 **(D)** None of these

5. $\cos\left[2\tan^{-1}\left(\frac{1}{7}\right)\right]$ equals

(A) $\sin(4\cot^{-1}3)$ **(B)** $\sin(3\cot^{-1}4)$
(C) $\cos(3\cot^{-1}4)$ **(D)** $\cos(4\cot^{-1}3)$

6. The value of $\tan^{-1}\{2\sin[\sec^{-1}(2)]\}$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$

7. The value of $\sin[\sin^{-1}(\sqrt{5}/4)+\tan^{-1}(\sqrt{5}/11)]$ is

(A) $\frac{\sqrt{5}}{4\sqrt{11}}$ **(B)** $\frac{4}{\sqrt{35}}$
(C) $\frac{\sqrt{55}}{8}$ **(D)** None of these

8. The x satisfying $\sin^{-1}x+\sin^{-1}(1-x)=\cos^{-1}x$ are

(A) 1, 0 **(B)** 1, –1
(C) $0,\frac{1}{2}$ **(D)** None of these

9. The equation $\sin^{-1}x=2\sin^{-1}a$ holds true for

(A) $-1\le a\le 1$ **(B)** $a\ge 0$
(C) $-\frac{1}{\sqrt{2}}\le a\le\frac{1}{\sqrt{2}}$ **(D)** All real values of a

10. The principal value of $\cos^{-1}\left\{\frac{1}{\sqrt{2}}\left[\cos\left(\frac{9\pi}{10}\right)-\sin\left(\frac{9\pi}{10}\right)\right]\right\}$ is

(A) $\frac{3\pi}{20}$ **(B)** $\frac{7\pi}{20}$
(C) $\frac{7\pi}{10}$ **(D** None of these

11. The number of positive integral solutions of the equation

$$\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right) \text{ is}$$

(A) 1 **(B)** 2
(C) 0 **(D)** None of these

12. If $A=\tan^{-1}\frac{1}{7}$, $B=\tan^{-1}\frac{1}{3}$, then

(A) $\cos 2A=\sin 2A$ **(B)** $\cos 2A=\sin 2B$
(C) $\cos 2A=\cos 2B$ **(D)** $\cos 2A=\sin 4B$

13. The equation $2\cos^{-1}x=\sin^{1}(2x\sqrt{1-x^2})$ is valid for all values of x satisfying

(A) $-1\le x\le 1$ **(B)** $0\le x\le 1$
(C) $0\le x\le\frac{1}{\sqrt{2}}$ **(D)** $\frac{1}{\sqrt{2}}\le x\le 1$

14. The value of the expression

$\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{5}}{5}\right)-\cos^{-1}\left(\frac{\sqrt{10}}{10}\right)$ is

(A) $\cot^{-1}\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$ **(B)** $\cot^{-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

(C) $-\pi+\cot^{-1}\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$ **(D)** $\pi-\cot^{-1}\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)$

15. $\cos^{-1}\left[\cos\left(\frac{-17\pi}{5}\right)\right]$ is equal to

(A) $-\frac{17\pi}{5}$ **(B)** $\frac{3\pi}{5}$

(C) $\frac{2\pi}{5}$ **(D)** None of these

16. If $0\le x\le 1$ and $\theta=\sin^{-1}x+\cos^{-1}x-\tan^{-1}x$, then

(A) $\theta\le\frac{\pi}{2}$ **(B)** $\theta\ge\frac{\pi}{4}$

(C) $\theta=\frac{\pi}{4}$ **(D)** $\frac{\pi}{4}\le\theta\le\frac{\pi}{2}$

17. If $\tan^{-1}x=2\cot^{-1}7+\cos^{-1}\left(\frac{3}{5}\right)$, then $x=$

(A) $\frac{117}{44}$ **(B)** $\frac{117}{125}$

(C) $\frac{7}{24}$ **(D)** $\frac{117}{84}$

18. If $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\beta$, then $x=$

(A) $\tan\beta$ **(B)** $\pm\sqrt{\frac{1-\tan\beta}{1+\tan\beta}}$

(C) $\pm\sqrt{\sin 2\beta}$ **(D)** $\pm\sqrt{\cos 2\beta}$

19. If $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$, then $x=$

(A) 3 **(B)** $\frac{1}{3}$

(C) $\pm\frac{1}{\sqrt{2}}$ **(D)** None of these

20. Let $f(x)=\sec^{-1}x+\tan^{-1}x$. Then $f(x)$ is real for

(A) $x\in[-1,1]$ **(B)** $x\in R$

(C) $x\in(-\infty,0)$ **(D)** $(-\infty,-1]\cup[1,\infty)$

21. If $\cos^{-1}x-\sin^{-1}x=0$, then x is equal to

(A) $\pm\frac{1}{\sqrt{2}}$ **(B)** 1

(C) $\frac{1}{\sqrt{2}}$ **(D)** $\sqrt{2}$

22. If $\sum_{i=1}^{10}\cos^{-1}x_i=0$, then $\sum_{i=1}^{10}x_i$ is

(A) 0 **(B)** 10

(C) 5 **(D)** None of these

23. Value of $\cos\left[2\cos^{-1}\left(\frac{4}{5}\right)\right]$ equals

(A) $\frac{6}{25}$ **(B)** $\frac{7}{25}$

(C) $\frac{4}{25}$ **(D)** $\frac{8}{25}$

24. The value of $\cos\left[\tan^{-1}\left(\tan\frac{15\pi}{4}\right)\right]$ is

(A) $\frac{1}{\sqrt{2}}$ **(B)** $-\frac{1}{\sqrt{2}}$

(C) 1 **(D)** None of these

25. Number of solutions to the equation $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$ in the interval $(-2\pi, 2\pi)$ is

(A) 4 **(B)** 3

(C) 7 **(D)** No solution

26. Number of solutions to the equation

$\sin^{-1}x-\cos^{-1}x=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is

(A) One **(B)** Two

(C) Four **(D)** None of these

27. The number of real solutions of the equation

$\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ is

(A) Zero **(B)** One

(C) Two **(D)** Infinite

28. If $\sum_{i=1}^{2n}\sin^{-1}x_i=n\pi$, then $\sum_{i=1}^{2n}x_i$ is

(A) n **(B)** $2n$

(C) $\frac{n(n+1)}{2}$ **(D)** None of these

29. The value of $\sin^{-1}\{\sin[2\cot+^{-1}(\sqrt{2}-1)]\}$ is

(A) $-\frac{\pi}{4}$ **(B)** $\frac{3\pi}{4}$

(C) $\frac{7\pi}{4}$ **(D)** None of these

30. The value of $\left\{\tan\left[\frac{\pi}{4}+\frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right]+\tan\left[\frac{\pi}{4}-\frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)\right]\right\}^{-1}$,

where $0<a<b$ is

(A) $\frac{b}{2a}$ **(B)** $\frac{a}{2a}$

(C) $\frac{\sqrt{b^2-a^2}}{2b}$ **(D)** $\frac{\sqrt{b^2-a^2}}{2a}$

31. If $4\cos^{-1}x+\sin^{-1}x=\pi$, then x equals to

(A) $\frac{1}{2}$ **(B)** $\frac{1}{\sqrt{2}}$

(C) 1 **(D)** $\frac{\sqrt{3}}{2}$

32. $\tan^{-1}\left(\dfrac{\sin 1-1}{\cos 1}\right)$ equals

(A) 0 **(B)** $1-\dfrac{\pi}{2}$

(C) $\dfrac{\pi}{2}-1$ **(D)** $\dfrac{1}{2}-\dfrac{\pi}{4}$

33. If $x \geq 1$, then $2\tan^{-1}x+\sin^{-1}\dfrac{2x}{1+x^2}$ is equal to

(A) $4\tan^{-1}x$ **(B)** π

(C) 0 **(D)** None of these

34. The inequality $\log_2 x < \sin^{-1}(\sin 5)$ hold if

(A) $x \in (0, 2^{5-2\pi})$ **(B)** $x \in (2^{5-2\pi}, \infty)$

(C) $x \in (2^{2\pi-5}, \infty)$ **(D)** None of these

35. If $\alpha \leq \tan^{-1}x+\cot^{-1}x+\sin^{-1}x \leq \beta \ \forall x \in (0,1)$, then

(A) $\alpha=0, \beta=\dfrac{\pi}{2}$ **(B)** $\alpha=0, \beta=\pi$

(C) $\alpha=-1, \beta=\pi$ **(D)** $\alpha=\dfrac{\pi}{2}, \beta=\pi$

36. If $2\sin^{-1}x+3\sin^{-1}y+4\sin^{-1}z=\dfrac{9\pi}{2}$ then $\dfrac{1}{2x}+\dfrac{1}{3y}+\dfrac{1}{4z}$ equals

(A) $\dfrac{11}{12}$ **(B)** $\dfrac{13}{12}$

(C) $\dfrac{15}{12}$ **(D)** $\dfrac{17}{12}$

37. $\tan^{-1}\left[\cos^{-1}\left(\dfrac{4}{5}\right)+\sin^{-1}\left(\dfrac{2}{\sqrt{3}}\right)\right]$ is

(A) $\dfrac{17}{6}$ **(B)** $\dfrac{17}{16}$

(C) $\dfrac{6}{17}$ **(D)** None of these

38. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}$ equals

(A) xyz **(B)** 1

(C) 0 **(D)** $2xyz$

39. The value of $\cot-1\dfrac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}$ if $0° < x < 90°$ is

(A) x **(B)** $\dfrac{x}{2}$

(C) $\pi-\dfrac{x}{2}$ **(D)** $-\dfrac{x}{2}$

40. The value of $\tan^2(\sec^{-1}2)+\cot^2(\text{cosec}^{-1}3)$ is

(A) 13 **(B)** 15

(C) 12 **(D)** 11

41. The principal value of $\sin^{-1}(\sin 10)$ is

(A) 10 **(B)** $10-3\pi$

(C) $3\pi-10$ **(D)** None of these

42. Value of $\sin^{-1}\left\{\sin\left[\cos^{-1}\left(\cos\left\{\sin^{-1}\left[\sin\left(\dfrac{3\pi}{4}\right)\right]\right\}\right)\right]\right\}$ equals

(A) $\dfrac{3\pi}{4}$ **(B)** $\dfrac{5\pi}{4}$

(C) $\dfrac{\pi}{4}$ **(D)** None of these

43. $\tan^{-1}(\tan 4)+\cot^{-1}(\cot 4)$ equals

(A) 8 **(B)** $\pi-8$

(C) $8-2\pi$ **(D)** 0

44. If $\tan^{-1}(x)+\tan^{-1}y+\tan^{-1}z=\dfrac{\pi}{2}$ and $(x-y)^2+(y-z)^2+(z-x)^2=0$, then $x^2+y^2+z^2$ equals

(A) 0 **(B)** 4

(C) 1 **(D)** None of these

45. The greatest of $\tan 1$, $\tan^{-1}1$, $\sin 1$, $\sin^{-1}1$ is

(A) $\tan 1$ **(B)** $\tan^{-1}1$

(C) $\sin 1$ **(D)** $\sin^{-1}1$

46. If $\cot^{-1}\left(\dfrac{n}{\pi}\right)>\dfrac{\pi}{6}$, n being a natural number, then maximum value of n is

(A) 1 **(B)** 5

(C) 9 **(D)** None of these

47. Two angles of a triangle are $\cot^{-1}2$ and $\cot^{-1}3$. Then the third angle is

(A) $\dfrac{\pi}{4}$ **(B)** $\dfrac{3\pi}{4}$

(C) $\dfrac{\pi}{6}$ **(D)** $\dfrac{\pi}{3}$

48. The value of $\cos^{-1}\left(\cos\dfrac{2\pi}{3}\right)-\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$ is equal to

(A) 0 **(B)** $\dfrac{4\pi}{3}$

(C) $-\dfrac{4\pi}{3}$ **(D)** $\dfrac{\pi}{3}$

49. If in ΔABC, $\angle A=\sin^{-1}(x)$, $\angle B=\sin^{-1}(y)$ and $\angle C=\sin^{-1}(z)$, then $x\sqrt{1-y^2}\sqrt{1-z^2}+y\sqrt{1-x^2}\sqrt{1-z^2}+z\sqrt{1-x^2}\sqrt{1-y^2}$ is equal to

(A) xyz **(B)** $x+y+z$

(C) $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}$ **(D)** None of these

50. If $A=\tan^{-1}\left(\dfrac{x\sqrt{3}}{2k-x}\right)$ and $B=\tan^{-1}\left(\dfrac{2x-k}{k\sqrt{3}}\right)$, then the value of $A-B$ (independent of x) is

(A) $\dfrac{\pi}{2}$ **(B)** $\dfrac{\pi}{3}$

(C) $\dfrac{\pi}{6}$ **(D)** $\dfrac{\pi}{8}$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If $y = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cdots + \infty$, then tan y is

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$ **(D)** None of these

2. If $x, y > 0$, then the range of $\sin^{-1}\left(\frac{x}{1+x^2}\right) + \sin^{-1}\left(\frac{y}{1+y^2}\right)$ is

(A) $[0, \pi]$ **(B)** $(-\pi, \pi]$

(C) $\left[0, \frac{2\pi}{3}\right]$ **(D)** $\left(0, \frac{2\pi}{3}\right]$

3. For $0 < \theta < 2\theta$; $\sin^{-1}\sin\theta > \cos^{-1}\sin\theta$ is true when

(A) $\left(\frac{\pi}{4}, \pi\right)$ **(B)** $\left(\pi, \frac{3\pi}{4}\right)$

(C) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ **(D)** $\left(\frac{3\pi}{4}, 2\pi\right)$

4. For which value of x, $\sin[\cot^{-1}(x + 1)] = \cos(\tan^{-1}x)$ is

(A) $\frac{1}{2}$ **(B)** 0

(C) 1 **(D)** $-\frac{1}{2}$

5. The number of integral values of k for which the equation $\sin^{-1}x + \tan^{-1}x = \sin^{-1}\sin x + 2k - 1$ has a real solution is

(A) 1 **(B)** 3

(C) 2 **(D)** 4

6. Let $f(x) = \frac{1}{4 - 3\cos^2 x + 5\sin^2 x}$ and its anti-derivative $F(x) = \frac{1}{3}\tan^{-1}[g(x)] + c$. Then

(A) $g(x)$ is equal to 3 tan$x$ **(B)** $g\left(\frac{\pi}{4}\right)$ is equal to 3

(C) $g'\left(\frac{\pi}{3}\right)$ is equal to 6 **(D)** $g'\left(\frac{\pi}{3}\right)$ is equal to 12

7. $\sin^{-1}(2x\sqrt{1-x^2})$ for $-1 \le x \le 1$ is equal to

(A) $\pi - 2\sin^{-1}(x)$ for $\frac{1}{\sqrt{2}} \le x \le 1$

(B) $2\sin^{-1}(x)$ for $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(C) $-\pi - 2\sin^{-1}(x)$ for $-1 \le x \le -\frac{1}{\sqrt{2}}$

(D) $2\cos^{-1}(x)$ for $\frac{1}{\sqrt{2}} \ge x \le 1$

Matrix Match Type Questions

8. Match the following:

Column I	Column II
(A) If $y = \cos^{-1}\left(\frac{2\cos x - 3\sin x}{\sqrt{13}}\right)$, then $\frac{dy}{dx}$ is	**(i)** $\frac{\pi}{14}$
(B) $\cos^{-1}\left[\sin\left(\frac{46\pi}{7}\right)\right]$ is	**(ii)** $\frac{\pi}{2}$
(C) If $\lvert x\rvert \ge 1$, $a > 0$ and sum of series $\sum_{n=1}^{x}\left(\frac{\sec^{-1}\sqrt{\lvert x\rvert} + \text{cosec}^{-1}\sqrt{\lvert x\rvert}}{a\pi}\right)^n$ is finite, then value of a is	**(iii)** $\left(\frac{1}{2}, \infty\right)$
(D) Let $f(x) = \frac{\text{cosec}^{-1}x + \cos^{-1}\left(\frac{1}{x}\right)}{\text{cosec}\,x}$. Then the greatest value is	**(iv)** 1

9. Match the following:

Column I	Column II
(A) $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \cdots + \infty$	**(i)** $\frac{\pi}{2}$
(B) $\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right)$	**(ii)** $\frac{\pi}{4}$
(C) $\sin^{-1}\left(\frac{4}{5}\right) + 2\tan^{-1}\left(\frac{1}{3}\right)$	**(iii)** π
(D) $\cot^{-1}9 + \text{cosec}^{-1}\left(\frac{\sqrt{41}}{4}\right)$	**(iv)** $\frac{\pi}{3}$

10. Match the following:

Column I	Column II
(A) The maximum value of $\sec^{-1}\left[\frac{7 - 5(x^2+3)}{2(x^2+2)}\right]$ is	**(i)** $\frac{\pi}{6}$

Column I	Column II
(B) The minimum value of $\text{cosec}^{-1}\left[3x^2+\frac{5}{4}\right]+\sec^{-1}\left[3x^2+\frac{1}{4}\right]$ (where [.] denotes the greatest integer function) is	**(ii)** $\frac{1}{3}$
(C) The number of solution of equation $\sin^{-1}(\|x^2-1\|)+\cos^{-1}(\|2x^2-5\|)=\frac{\pi}{2}$ is	**(iii)** $\frac{2\pi}{3}$
(D) If $y=\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}(b)$, $(0<b<1)$ and $0<y\le\frac{\pi}{4}$, then the maximum value of b will be	**(iv)** 2

Integer Type Question

11. The value of $2\cot(\cot^{-1}3+\cot^{-1}7+\cot^{-1}13+\cot^{-1}21)$ is __________.

Answer Key

Practice Exercise 1

1. (A)	**2.** (D)	**3.** (A)	**4.** (B)	**5.** (A)
6. (C)	**7.** (C)	**8.** (C)	**9.** (C)	**10.** (D)
11. (B)	**12.** (D)	**13.** (D)	**14.** (C)	**15.** (B)
16. (D)	**17.** (A)	**18.** (C)	**19.** (C)	**20.** (D)
21. (C)	**22.** (B)	**23.** (B)	**24.** (A)	**25.** (A)
26. (A)	**27.** (C)	**28.** (B)	**29.** (D)	**30.** (C)
31. (D)	**32.** (D)	**33.** (B)	**34.** (A)	**35.** (B)
36. (B)	**37.** (A)	**38.** (D)	**39.** (C)	**40.** (D)
41. (C)	**42.** (C)	**43.** (C)	**44.** (C)	**45.** (D)
46. (B)	**47.** (B)	**48.** (D)	**49.** (A)	**50.** (C)

Practice Exercise 2

1. (C)	**2.** (D)	**3.** (C)	**4.** (D)	**5.** (D)
6. (A), (B), (D)	**7.** (A), (B), (C)	**8.** (A) →(iv); (B) →(i); (C)→(iii); (D) →(ii)	**9.** (A) →(ii); (B) →(iii); (C)→(i); (D) →(ii)	**10.** (A) →(iii); (B) →(i); (C)→(iv); (D) →(ii)
11. 3				

Solutions

Practice Exercise 1

1. Hint: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$\tan\left[\tan^{-1}(1)+\tan^{-1}\left(\frac{2+3}{1-6}\right)\right]$$

$$=\tan\left[\tan^{-1}(1)+\tan^{-1}\left(\frac{5}{-5}\right)\right]$$

$$=\tan[\tan^{-1}(1)+\tan^{-1}(-1)]$$

$$=\tan\left[\frac{\pi}{4}-\frac{\pi}{4}\right]=\tan(0)=0$$

2. Since $0\le\cos^{-1}(x)\le\pi$ and given

$$\cos^{-1}(\cos x)=\pi+x$$

is true only when x belongs to some negative angle but no option is such.

3. Since $5\pi<16<\frac{11\pi}{2}$,

$$\Rightarrow 0 < 16 - 5\pi < \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} < 5\pi - 16 < 0$$

Now,

$$\sin^{-1}(\sin 16) = \sin^{-1}[\sin(5\pi - 16)]$$
$$= 5\pi - 16$$

4. Given

$$\theta = \cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$$
$$= \cot^{-1}\frac{55}{15} + \cot^{-1}18$$
$$= \cot^{-1}\left(\frac{\frac{11}{3}\times 18 - 1}{\frac{11}{3} + 8}\right)$$
$$= \cot^{-1}(3)$$
$$\Rightarrow \cot\theta = 3$$

5. We have

$$\cos\left[2\tan^{-1}\left(\frac{1}{7}\right)\right] = \cos\left(\tan^{-1}\frac{7}{24}\right)$$
$$= \cos\left(\cos^{-1}\frac{24}{25}\right) = \frac{24}{25}$$
$$\sin(4\cot^{-1}3) = \sin\left(4\tan^{-1}\frac{1}{3}\right)$$
$$= \sin\left(2\tan^{-1}\frac{3}{4}\right) = \sin\left(\tan^{-1}\frac{24}{7}\right)$$
$$= \sin\left(\sin^{-1}\frac{24}{25}\right) = \frac{24}{25}$$

Hence, $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin(4\cot^{-1}3)$.

6. See Fig. 17.14.

$$\tan^{-1}\left(2\sin\frac{\pi}{3}\right)$$
$$\tan^{-1}\left(2\times\frac{\sqrt{3}}{2}\right) = \tan^{-1}(\sqrt{3})$$
$$= \frac{\pi}{3}$$

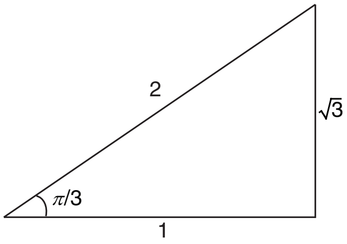


Figure 17.14

7. Hint: $(\sqrt{5})^2 + (\sqrt{11})^2 = 4^2$

See Fig. 17.15.

$\sin[\alpha + \alpha] = \sin 2\alpha$

$= 2\sin\alpha\cos\alpha$

$$= 2\cdot\frac{\sqrt{5}}{4}\cdot\frac{\sqrt{11}}{4} = \frac{\sqrt{55}}{8}$$

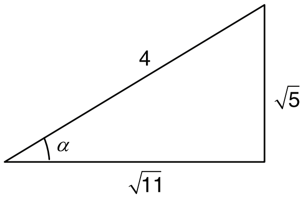


Figure 17.15

8. Hint: $\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$

$$\sin^{-1}x + \sin^{-1}(1-x) = \frac{\pi}{2} - \sin^{-1}x$$
$$2\sin^{-1}x + \sin^{-1}(1-x) = \frac{\pi}{2}$$

Clearly $x = 0$ satisfies equation

$$0 + \frac{\pi}{2} = \frac{\pi}{2}$$

$x = \frac{1}{2}$ also satisfies the equation

$$2\times\frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{2}$$
$$\Rightarrow \left(x = 0, \frac{1}{2}\right)$$

9. Hint: $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

$\sin^{-1}x = \sin^{-1}(2a\sqrt{1-a^2})$ when

$$-1 \le x \le 1$$
$$-1 \le 2a\sqrt{1-a^2} \le 1$$
$$0 \le 4a^2(1-a^2) \le 1$$
$$0 \le a^2(1-a^2) \le \frac{1}{4}$$
$$a^2(1-a^2) \le \frac{1}{4}$$
$$\Rightarrow a^2 \le \frac{1}{2}$$
$$\Rightarrow -\frac{1}{\sqrt{2}} \le a \le +\frac{1}{\sqrt{2}}$$

10. Hint: $\cos^{-1}[\cos(-x)] = \pi - x$

$$\cos^{-1}\left\{\frac{1}{\sqrt{2}}\left[\cos\left(\pi - \frac{\pi}{10}\right) - \sin\left(\pi - \frac{\pi}{10}\right)\right]\right\}$$
$$= \cos^{-1}\left[\frac{1}{\sqrt{2}}\left(-\cos\frac{\pi}{10} - \sin\frac{\pi}{10}\right)\right]$$
$$= \cos^{-1}\left[(-1)\left(\frac{1}{\sqrt{2}}\cos\frac{\pi}{10} + \frac{1}{\sqrt{2}}\sin\frac{\pi}{10}\right)\right]$$
$$= \cos^{-1}\left[(-1)\left(\cos\frac{\pi}{4}\cos\frac{\pi}{10} + \sin\frac{\pi}{4}\cdot\sin\frac{\pi}{10}\right)\right]$$
$$= \cos^{-1}\left[(-1)\cos\left(\frac{\pi}{4} - \frac{\pi}{10}\right)\right]$$
$$= \pi - \cos^{-1}\left(\cos\frac{3\pi}{20}\right) = \pi - \frac{3\pi}{20} = \frac{17\pi}{20}$$

Hence, none of the given alternatives are correct.

11. Hint: $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}$

$$\tan^{-1}x+\cos^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)=\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$$

$$\Rightarrow \tan^{-1}x+\tan^{-1}y=\tan^{-1}3$$

$$\Rightarrow \tan^{-1}\frac{x+y}{1-xy}=\tan^{-1}3$$

$$\Rightarrow \frac{x+y}{1-xy}=3$$

$$\Rightarrow x+y=3-3xy$$
$$\Rightarrow x+y+3xy=3$$

Hence, only integral solution possible is (3, 0) and (0, 3).

12. Hint: $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$, $\sin 2\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}$

$$A=\tan^{-1}\frac{1}{7} \text{ and } B=\tan^{-1}\frac{1}{3}\Rightarrow \tan A=\frac{1}{7} \text{ and } \tan B=\frac{1}{3}$$

$$\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}=\frac{1-\frac{1}{49}}{1+\frac{1}{49}}=\frac{48}{50}=\frac{24}{25}$$

$$\sin 2B=\frac{2\tan B}{1+\tan^2 B}=\frac{2\times\frac{1}{3}}{1+\frac{1}{9}}=\frac{\frac{2}{3}}{\frac{10}{9}}=\frac{6}{10}=\frac{3}{5}$$

$$\sin 4B=2\sin 2B\cdot\cos 2B=2\cdot\frac{3}{5}\cdot\frac{4}{5}=\frac{24}{25}$$

Hence, $\cos 2A=\sin 4B$.

13. Let $x=\cos A, x\in[-1,1]$ and $A\in[0,\pi]$. Then

$$2\cos^{-1}(\cos A)=\sin^{-1}(2\cos A\sqrt{1-\cos^2 A})$$

$$\Rightarrow 2\cos^{-1}(\cos A)=\sin^{-1}(2\cos A\cdot\sin A)=\sin^{-1}(\sin 2A)$$

$$\Rightarrow 2A=\sin^{-1}(\sin 2A)$$

Now, left hand and right hand will be equal for

$$2A\in[0,2\pi]\Rightarrow 2A\in\left[0,\frac{\pi}{2}\right]\Rightarrow A\in\left[0,\frac{\pi}{4}\right]\Rightarrow x\in\left[\frac{1}{\sqrt{2}},1\right]$$

14. We have

$$\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{5}}{5}\right)-\cos^{-1}\left(\frac{\sqrt{10}}{10}\right)$$

$$=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+\tan^{-1}\left(\frac{1}{2}\right)-\tan^{-1}(3)$$

$$=\tan^{-1}\left(\frac{2+\sqrt{2}}{2\sqrt{2}-1}\right)-\tan^{-1}(3)$$

$$=\tan^{-1}\left(\frac{1-\sqrt{2}}{1+\sqrt{2}}\right)$$

$$=-\pi+\cos^{-1}\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right)$$

15. Hint: $\cos(2\pi-\theta)=\cos\theta$

$$\cos^{-1}\left[\cos\left(-\frac{17\pi}{5}\right)\right]$$

$$=\cos^{-1}\left[\cos\left(4\pi-\frac{3\pi}{5}\right)\right]=\cos^{-1}\left(\cos\frac{3\pi}{5}\right)=\frac{3\pi}{5}$$

16. Hint: $\sin^{-1}x+\cos^{-1}x=\dfrac{\pi}{2}$

$$\theta=\sin^{-1}x+\cos^{-1}x-\tan^{-1}x$$

$$=\frac{\pi}{2}-\tan^{-1}x$$

Given,

$$0\le x\le 1 \Rightarrow 0\le \tan^{-1}x\le\frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{2}\ge\frac{\pi}{2}-\tan^{-1}x\ge\frac{\pi}{4}$$

17. Hint: $\tan^{-1}a+\tan^{-1}b=\tan^{-1}\dfrac{a+b}{1-ab}$

$$\tan^{-1}x=2\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{4}{3}$$

$$\tan^{-1}x=\cot^{-1}\left(\frac{49-1}{14}\right)+\tan^{-1}\frac{4}{3}$$

$$=\cot^{-1}\left(\frac{48}{14}\right)+\tan^{-1}\frac{4}{3}$$

$$=\tan^{-1}\frac{7}{24}+\tan^{-1}\frac{4}{3}=\tan^{-1}\frac{\frac{7}{24}+\frac{4}{3}}{1-\frac{7}{24}\cdot\frac{4}{3}}$$

$$=\tan^{-1}\frac{\frac{7}{24}+\frac{4}{3}}{1-\frac{7}{18}}=\tan^{-1}\frac{\frac{39}{24}}{\frac{11}{18}}=\tan^{-1}\frac{117}{44}$$

$$\Rightarrow \tan^{-1}x=\tan^{-1}\frac{117}{44}$$

18. Hint: If $\dfrac{a}{b}=\dfrac{c}{d}$ then $\dfrac{a+b}{a-b}=\dfrac{c+d}{c-d}$

$$\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}=\beta\Rightarrow \tan\beta=\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$$

$$\Rightarrow \frac{1+\tan\beta}{1-\tan\beta}=\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}$$

Squaring both sides, we get

$$\frac{1+\tan^2\beta+2\tan\beta}{1+\tan^2\beta-2\tan\beta}=\frac{1+x^2}{1-x^2}$$

Using componendo and dividendo, we get

$$x^2 = \sin 2\beta$$

$$\Rightarrow x = \pm\sqrt{\sin 2\beta}$$

19. Hint: $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\dfrac{a+b}{1-ab}$

$$\tan^{-1}\frac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\dfrac{x-1}{x-2}\cdot\dfrac{x+1}{x+2}} = \frac{\pi}{4}$$

Hence,

$$\tan\frac{\pi}{4} = \frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}$$

$$1 = \frac{2x^2-4}{-3}$$

$$\frac{2x^2}{2} = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

20. Hint: Value of $\sec x$ and $\operatorname{cosec} x$ does not lie between $+1$ and -1.
If $\phi(x) = \sec^{-1} x$, then we know that $x \in (-\infty, -1] \cup [1, \infty)$.
Also $g(x) = \tan^{-1} x$, then $x \in R$.
Hence, for holding $f(x) = \sec^{-1} x + \tan^{-1} x$, we have

$$x \in (-\infty, -1] \cup [1, \infty)$$

21. Hint: $\sin\dfrac{\pi}{4} = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

$$\cos^{-1} x - \sin^{-1} x = 0$$

$$\Rightarrow \cos^{-1} x - \frac{\pi}{2} + \cos^{-1} x = 0$$

$$\Rightarrow 2\cos^{-1} x = \frac{\pi}{2}$$

Hence,

$$\cos^{-1} x = \frac{\pi}{4} \Rightarrow x = \frac{1}{\sqrt{2}}$$

22. Since, $\cos^{-1} x_2$ has to be zero separately, only the

$$\sum_{i=1}^{10} \cos^{-1} x_i = 0$$

That is,

$$\cos^{-1} x_1 = 0, \cos^{-1} x_2 = 0 \ldots \cos^{-1} x_{10} = 0$$

$$\Rightarrow x_1 = 1 \quad x_2 = 1 \ldots x_{10} = 1$$

Adding all the terms, we get

$$\sum_{i=1}^{10} x_i = 10$$

23. We have

$$\cos\left(2\cos^{-1}\frac{4}{5}\right) = \cos\left[\cos^{-1}\left(2\times\frac{16}{25}-1\right)\right] = \frac{7}{25}$$

24. Hint: $\tan(2\mu\pi - \theta) = -\tan\theta$, $\cot(-\theta) = \cos\theta$

$$\cos\left\{\tan^{-1}\left[\tan\left(4\pi - \frac{\pi}{4}\right)\right]\right\} = \cos\left\{\tan^{-1}\left[-\tan\left(\frac{\pi}{4}\right)\right]\right\}$$

$$= \cos\left[-\tan\left(\tan\frac{\pi}{4}\right)\right] = \cos\left[\tan^{-1}\left(\tan\frac{\pi}{4}\right)\right] = \cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}$$

25. Hint: $2\tan^{-1} x = \tan^{-1}\dfrac{2x}{1-x^2}$

$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$$\Rightarrow \tan^{-1}\frac{2\cos x}{1-\cos^2 x} = \tan^{-1} 2\operatorname{cosec} x$$

$$\Rightarrow \tan^{-1} 2\cot x \cdot \operatorname{cosec} x = \tan^{-1} 2\operatorname{cosec} x$$

$$\cot x \cdot \operatorname{cosec} x - \operatorname{cosec} x = 0 \Rightarrow \operatorname{cosec} x(\cot x - 1) = 0$$

$\Rightarrow \operatorname{cosec} x = 0$ which is not possible, and
$\cot x = 1$ which has 4 solutions.

26. Hint: $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$

$$\sin^{-1} x - \cos^{-1} x = \cos^{-1}\frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin^{-1} x = \cos^{-1} x + \cos^{-1}\frac{\sqrt{3}}{2} = \cos^{-1}\left\{x\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\sqrt{1-x^2}\right\}$$

$$\Rightarrow \cos^{-1}\sqrt{1-x^2} = \cos^{-1}\left\{\frac{\sqrt{3}x}{2} - \frac{1}{2}\sqrt{1-x^2}\right\}$$

$$\Rightarrow \sqrt{1-x^2} = \frac{\sqrt{3}x}{2} - \frac{1}{2}\sqrt{1-x^2}$$

$$\Rightarrow \frac{3}{2}\sqrt{1-x^2} = \frac{\sqrt{3}x}{2}$$

Squaring both sides, we have

$$\frac{9}{4}(1-x^2) = \frac{3x^2}{4} \Rightarrow x = \pm\frac{\sqrt{3}}{2}$$

Only $x = \dfrac{\sqrt{3}}{2}$ satisfies the equation.

27. Hint: $\sin^{-1} x = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2} - \tan^{-1}\sqrt{x(x+1)} = \cot^{-1}\sqrt{x(x+1)}$$

$$\Rightarrow \sin^{-1}\sqrt{x^2+x+1} = \sin^{-1}\frac{1}{\sqrt{x^2+x+1}}$$

$$\Rightarrow \sqrt{x^2+x+1} = \frac{1}{\sqrt{x^2+x+1}} \Rightarrow x^2+x+1 = 1$$

$$\Rightarrow x(x+1) = 0, x = 0, x = -1$$

Both the values satisfies the equation, so there are two solutions.

28. Given

$$\sin^{-1} x_1 + \sin^{-1} x_2 + \cdots + \sin^{-1} x_{2n} = n\pi$$

which is possible only if

$$\sin^{-1}x_1=\sin^{-1}x_2=\cdots=\sin^{-1}x_{2n}=\frac{\pi}{2}$$

$$\Rightarrow x_1=x_2=\cdots=x_{2n}=1$$

Therefore, $x_1+x_2+\ldots+x_{2n}=2n$.

29. We have

$$\sin^{-1}\{\sin[2\cot^{-1}(\sqrt{2}-1)]\}=\sin^{-1}\{\sin[\cot^{-1}(-1)]\}$$

$$=\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$$

$$=\sin^{-1}\left[\sin\left(\pi-\frac{\pi}{4}\right)\right]=\frac{\pi}{4}$$

30. We have $\left[\tan\left(\frac{\pi}{4}+\alpha\right)+\tan\left(\frac{\pi}{4}-\alpha\right)\right]^{-1}$, where $\alpha=\frac{1}{2}\sin^{-1}\left(\frac{a}{b}\right)$

$$=\left(\frac{1+\tan\alpha}{1-\tan\alpha}+\frac{1-\tan\alpha}{1+\tan\alpha}\right)^{-1}$$

$$=\left[\frac{2(1+\tan^2\alpha)}{1-\tan^2\alpha}\right]^{-1}$$

$$=\left(\frac{2}{\cos 2\alpha}\right)^{-1}=\frac{2\cos 2\alpha}{2}$$

Now, $\alpha=\frac{1}{2}\sin^{-1}\frac{a}{b}$

$$\Rightarrow \frac{a}{b}=\sin 2\alpha$$

$$\Rightarrow \cos 2\alpha=\sqrt{1-\frac{a^2}{b^2}}=\frac{\sqrt{b^2-a^2}}{b}$$

Therefore, given expression is $\dfrac{\sqrt{b^2-a^2}}{2b}$.

31. Hint: $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

$$4\cos^{-1}x+\frac{\pi}{2}-\cos^{-1}x=\pi$$

$$\Rightarrow 3\cos^{-1}x=\frac{\pi}{2}\Rightarrow \cos^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$$

32. Hint: $1-\sin A=\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2$

$$\tan^{-1}\left(\frac{\sin 1-1}{\cos 1}\right)=\tan^{-1}\left(-\frac{1-\sin 1}{\cos 1}\right)=-\tan^{-1}\left(\frac{1-\sin 1}{\cos 1}\right)$$

$$=\tan^{-1}\left[\frac{\left(\cos\frac{1}{2}-\sin\frac{1}{2}\right)^2}{\left(\cos^2\frac{1}{2}-\sin^2\frac{1}{2}\right)}\right]=-\tan^{-1}\left[\frac{\cos\frac{1}{2}-\sin\frac{1}{2}}{\cos\frac{1}{2}+\sin\frac{1}{2}}\right]$$

$$=-\tan^{-1}\left[\frac{1-\tan\frac{1}{2}}{1+\tan\frac{1}{2}}\right]$$

$$=-\tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\frac{1}{2}}{1+\tan\frac{\pi}{4}\cdot\tan\frac{1}{2}}\right]=-\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{1}{2}\right)\right]$$

$$=-\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{1}{2}-\frac{\pi}{4}$$

33. Hint: $2\tan^{-1}x=\pi-\sin^{-1}\frac{2x}{1+x^2}$, when $x\geq 1$

With the given condition

$$2\tan^{-1}x=\pi-\sin^{-1}\frac{2x}{1+x^2}$$

Hence,

$$2\tan^{-1}x+\sin^{-1}\frac{2x}{1+x^2}=\pi-\sin^{-1}\frac{2x}{1+x^2}+\sin^{-1}\frac{2x}{1+x^2}=\pi$$

34. Hint: $\sin^{-1}(\sin 5)=\sin^{-1}[\sin(-2\pi+5)]$

$$\log_2 x<\sin^{-1}(\sin 5)$$
$$\Rightarrow \log_2 x<\sin^{-1}[\sin(-2\pi+5)]$$
$$\Rightarrow \log_2 x<-2\pi+5=5-2\pi$$
$$\Rightarrow x<2^{5-2\pi}$$

Also, $x\neq 0$ is positive.

Therefore, required value of x belongs to $x\in(0,2^{5-2\pi})$.

35. Hint:
$$\left.\begin{aligned}\tan^{-1}x&\in\left(0,\frac{\pi}{4}\right)\\ \cot^{-1}x&\in\left(\frac{\pi}{2},\frac{\pi}{4}\right)\\ \sin^{-1}x&\in\left(0,\frac{\pi}{2}\right)\end{aligned}\right\}\forall x\in(0,1)$$

Let $\tan^{-1}x+\cot^{-1}x+\sin^{-1}x=z$.

At $x=0, z=\frac{\pi}{2}$ but $x\neq 0$, thus $\alpha\neq\frac{\pi}{2}$.

At $x=1$,

$$z=\pi\Rightarrow\beta=\pi$$

From option, it is clear that $z\geq 0$

$$\alpha=0,\beta=\pi$$

36. Hint: $\frac{-\pi}{2}\leq\sin^{-1}p\leq\frac{\pi}{2}$

$$2(\sin^{-1}x)+3(\sin^{-1}y)+4(\sin^{-1}z)=9\cdot\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \sin^{-1}x=\sin^{-1}y=\sin^{-1}z=-\frac{\pi}{2}$$

$$\Rightarrow x=y=z=1$$

$$\Rightarrow \frac{1}{2x}+\frac{1}{3y}+\frac{1}{4y}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{13}{12}$$

37. Hint: $\tan^{-1}a+\tan^{-1}b=\tan^{-1}\dfrac{a+b}{1-ab}$

Given equation must be as follows:

$$\tan^{-1}\left\{\cot^{-1}\left(\frac{4}{3}\right)+\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)\right\}$$

$$=\tan\left\{\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right\}$$

$$=\tan\left\{\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right\}=\tan\left\{\tan^{-1}\left(\frac{17}{6}\right)\right\}=\frac{17}{6}$$

38. Given

$$\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$$

Let $\sin^{-1}x=\alpha, \sin^{-1}y=\beta, \sin^{-1}z=y$. Then

$$\alpha+\beta+\gamma=\pi$$

Now,

$$x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}$$

$$=\frac{1}{2}(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)$$

$$=\frac{1}{2}(4\sin\alpha\sin\beta\sin\gamma)$$

$$=2\sin\alpha\sin\beta\sin\gamma$$

$$=2xyz$$

39. We have

$$\cot^{-1}\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}$$

$$\cot\left(\frac{\cos\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}-\cos\frac{x}{2}-\sin\frac{x}{2}}\right)$$

$$=\cot^{-1}\left(-\cot\frac{x}{2}\right)$$

$$=\pi-\frac{x}{2}$$

40. We have

$$\tan^2(sex^{-1}2)=\cot^2(\mathrm{cosec}^{-1}3)$$

$$=\tan^2\left[\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)\right]=\cot^2\left[\cot^{-1}\left(\frac{2\sqrt{2}}{1}\right)\right]$$

$$=(\sqrt{3})^2+(2\sqrt{2})^2=3+8=11$$

41. Since

$$3\pi<10<\frac{7\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2}<3\pi-10<0$$

Now,

$$\sin^{-1}(\sin 10)=\sin^{-1}[\sin(3\pi-10)]$$
$$=3\pi-10$$

42. We have

$$\sin^{-1}\left\{\sin\left[\cos^{-1}\left(\cos\left\{\sin^{-1}\left[\sin\left(\frac{3\pi}{4}\right)\right]\right\}\right)\right]\right\}$$

$$=\sin^{-1}\left\{\sin\left[\cos^{-1}\left(\cos\frac{\pi}{4}\right)\right]\right\}$$

$$=\sin^{-1}\left(\sin\frac{\pi}{4}\right)=\frac{\pi}{4}$$

43. Since

$$\pi<4<\frac{3\pi}{2}$$

$$0<4-\pi<\frac{\pi}{2}$$

Now,

$$\tan^{-1}(\tan 4)+\cot^{-1}(\cot 4)$$

$$=\tan^{-1}[\tan(\pi+4-\pi)]+\cot^{-1}[\cot(\pi+4-\pi)]$$

$$=\tan^{-1}[\tan(4-\pi)]+\cot^{-1}[\cot(4-\pi)]$$

$$=4-\pi+4-\pi$$
$$=8-2\pi$$

44. Given

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$$

Let $\tan^{-1}x=\alpha, \tan^{-1}y=\beta, \tan^{-1}z=y$. Then

$$x=\tan\alpha, y=\tan\beta, z=\tan\gamma$$

Hence,

$$\alpha+\beta+\gamma=\frac{\pi}{2}$$

$$\Rightarrow \tan(\alpha+\beta)=\tan\left(\frac{\pi}{2}-\gamma\right)$$

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{1}{\tan\gamma}\Rightarrow\frac{x+y}{1-xy}=\frac{1}{z}$$

$$\Rightarrow xy+yz=zx=1$$

Now,

$$(x-y)^2+(y-z)^2+(z-x)^2=0$$

$$\Rightarrow x^2+y^2+z^2=xy+yz+zx=1$$

45. $\tan 1, \tan^{-1}1, \sin 1, \sin^{-1}1$

$$\tan 57°, \frac{\pi}{4}, \sin 57°, \frac{\pi}{2}$$

$$1.539, \frac{3.14}{4}, \frac{\sqrt{3}}{2}, \frac{3.14}{2}$$

$$1.539, 0.78, 0.866, 1.57$$

So, $\sin^{-1} 1$ is the greatest value.

46. Given

$$\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$$

$$\Rightarrow \cot\left(\cot^{-1}\frac{n}{\pi}\right) < \cot\frac{\pi}{6}$$

$$\Rightarrow \frac{n}{\pi} < \sqrt{3}$$

$$\Rightarrow n < \sqrt{3}\cdot\pi = 5.5 \text{ (nearly)}$$

So, the maximum value of n is 5.

47. We have

$$\cot^{-1}2 + \cot^{-1}3 = \cot^{-1}\left(\frac{2\times 3 - 1}{2+3}\right) = \frac{\pi}{4}$$

Hence, third angle $= \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$.

48. We have

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right) - \sin^{-1}\left(\frac{2\pi}{3}\right)$$

$$= \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$$

49. Given

$$x = \sin A, y = \sin B, z = \sin C$$

Now,

$\sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B$

$-\sin A \sin B \sin C = \sin(A + B + C) = \sin n\pi = 0$

Therefore, given expression $= \sin A \sin B \sin C = xyz$.

50. We have

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\dfrac{x\sqrt{3}}{2k-x} - \dfrac{2x-k}{k\sqrt{3}}}{1 + \dfrac{x\sqrt{3}}{2k-x}\times\dfrac{2x-k}{k\sqrt{3}}}$$

$$= \frac{3xk - 4kx + 2k^2 + 2x^2 - xk}{2k^2\sqrt{3} - xk\sqrt{3} + 2x^2\sqrt{3} - kx\sqrt{3}}$$

$$= \frac{2x^2 + 2k^2 - 2kx}{\sqrt{3}(2x^2 + 2k^2 - 2kx)} = \frac{1}{\sqrt{3}}$$

Hence, $A - B = \dfrac{\pi}{6}$.

Practice Exercise 2

1. $y = \cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cdots + \infty$

n^{th} term $= \cot^{-1} 2n^2$

$$\Rightarrow t_n = \cot^{-1}(2n-1) - \cot^{-1}(2n+1)$$

$$\left(\begin{aligned} \because \cot^{-1}(2n-1) - \cot^{-1}(2n+1) &= \cot^{-1}\left[\frac{(2n-1)(2n+1)+1}{2n+1-2n+1}\right] \\ &= \cot^{-1} 2n^2 \end{aligned}\right)$$

$$\begin{aligned} t_1 &= \cot^{-1}1 - \cot^{-1}3 \\ t_2 &= \cot^{-1}3 - \cot^{-1}5 \\ t_3 &= \cot^{-1}5 - \cot^{-1}7 \\ &\vdots \qquad \vdots \qquad \vdots \end{aligned}$$

$$\Rightarrow y = \cot^{-1}1 - \cot^{-1}(2n+1)$$

As $n \to \infty$, $\cot^{-1}(2n+1) \to 0$, so

$$y = \frac{\pi}{4}$$

2. $$\frac{1+x^2}{x} \ge 2 \quad (\because x > 0)$$

$$\Rightarrow \frac{x}{1+x^2} \le \frac{1}{2} \Rightarrow \sin^{-1}\left(\frac{x}{1+x^2}\right) \in \left(0, \frac{\pi}{6}\right]$$

Similarly, $\sin^{-1}\left(\dfrac{y}{1+y^2}\right) \in \left(0, \dfrac{\pi}{2}\right]$.

So, range of the given expression is $\left(0, \dfrac{2\pi}{3}\right)$.

3. $$\sin^{-1}\sin\theta > \frac{\pi}{2} - \sin^{-1}\sin\theta$$

$$\Rightarrow \sin^{-1}\sin\theta > \frac{\pi}{4}$$

Therefore,

$$\sin\theta > \frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

4. $$\sin[\cot^{-1}(x+1)] = \sin\left[\sin^{-1}\left(\frac{1}{\sqrt{x^2+2x+2}}\right)\right]$$

$$\Rightarrow \sin[\cot^{-1}(x+1)] = \frac{1}{\sqrt{x^2+2x+2}}$$

$$\cos(\tan^{-1}x) = \cos\left[\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right] = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{1}{x^2+2x+2} = \frac{1}{1+x^2}$$

$$\Rightarrow x = -\frac{1}{2}$$

5. $\sin^{-1}x + \tan^{-1}x - \sin^{-1}\sin x = 2k - 1$

The range of $\sin^{-1}x + \tan^{-1}x - \sin^{-1}\sin x$ is $\left[-\frac{3\pi}{4}-1, \frac{3\pi}{4}+1\right]$.

Then

$$-\frac{3\pi}{4}-1 \le 2k-1 \le \frac{3\pi}{4}+1 \Rightarrow -\frac{3\pi}{8} \le k \le \frac{3\pi}{8}+1$$

Hence, the integral values of k are $-1, 0, 1, 2$.

6. $F(x) = \int \frac{1}{4-3\cos^2 x+5\sin^2 x}dx = \int \frac{1}{9-8\cos^2 x}dx$

$$= \int \frac{\sec^2 x}{9\sec^2 x - 8}dx = \int \frac{\sec^2 x}{1+9\tan^2 x}dx = \frac{1}{3}\tan^{-1}(3\tan x) + c$$

$$\Rightarrow g(x) = 3\tan x,\ g\left(\frac{\pi}{4}\right) = 3,\ g'\left(\frac{\pi}{3}\right) = 12$$

7. Let $y = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta$

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow y = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow y = \begin{cases} 2\sin^{-1}x, & x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \\ \pi - 2\sin^{-1}x & x \in \left[\frac{1}{\sqrt{2}}, 1\right] \\ -\pi - 2\sin^{-1}x & -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

8. (A) $y = \cos^{-1}\left(\frac{2\cos x - 3\sin x}{\sqrt{13}}\right)$

$$y = \cos^{-1}\cos(\alpha + x)\ \forall\ \alpha = \tan^{-1}\frac{3}{2}$$

Therefore, $\frac{dy}{dx} = 1$.

(B) $\cos^{-1}\cos\left(\frac{\pi}{14}\right) = \frac{\pi}{14}$

(C) $\sum_{n=1}^{\infty}\left(\frac{\pi}{2a\pi}\right)^n = \sum_{n=1}^{\infty}\left(\frac{1}{2a}\right)^n = \text{finite} \Rightarrow a > \frac{1}{2}$

(D) $f(x) = \frac{\sin^{-1}\left(\frac{1}{x}\right) + \cos^{-1}\left(\frac{1}{x}\right)}{\operatorname{cosec} x} = \frac{\pi}{2}\sin x$

9. (A) $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \cdots + \infty$

$$T_n = \tan^{-1}\frac{1}{1+n+n^2} = \tan^{-1}\frac{(n+1)-n}{1+(n+1)n}$$

$$= \tan^{-1}(n+1) - \tan^{-1}n$$

Putting $n = 1, 2, 3, \ldots$, and adding, we get

$$S_n = \tan^{-1}(n+1) - \tan^{-1}1$$

$$S_\infty = \tan^{-1}(\infty) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

(B) Since

$$\sin^{-1}\left(\frac{12}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right) \text{ and } \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$$

Therefore,

$$\text{LHS} = \tan^{-1}\frac{12}{5} + \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{63}{16}$$

Since $\frac{12}{5} \times \frac{3}{4} > 1$, we have

$$\tan^{-1}\frac{12}{5} + \tan^{-1}\frac{3}{4} = \pi + \tan^{-1}\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\cdot\frac{3}{4}} = \pi - \tan^{-1}\frac{63}{16}$$

So,

$$\tan^{-1}\frac{12}{5} + \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{63}{16} = \pi$$

(C) $2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2/3}{1-\frac{1}{9}} = \tan^{-1}\frac{3}{4}$

and

$$\sin^{-1}\frac{4}{5} = \tan^{-1}\frac{4}{3}$$

Therefore,

$$\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\frac{4}{3} + \cot^{-1}\frac{4}{3} = \frac{\pi}{2}$$

(D) $\operatorname{cosec}^{-1}x = \cot^{-1}\sqrt{x^2-1}$

Hence,

$$\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \cot^{-1}\sqrt{\frac{41}{16}-1} = \cot^{-1}\left(\frac{5}{4}\right)$$

Therefore,

$$\cot^{-1}9 + \operatorname{cosec}^{-1}\frac{\sqrt{41}}{4} = \cot^{-1}9 + \cot^{-1}\frac{5}{4}$$

$$= \tan^{-1}\frac{1}{9} + \tan^{-1}\frac{4}{5} = \tan^{-1}\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\cdot\frac{4}{5}} = \tan^{-1}1 = \frac{\pi}{4}$$

10. (A) $\sec^{-1}\left(\dfrac{7-5(x^2+3)}{2(x^2+2)}\right) = \sec^{-1}\left[\dfrac{1}{(x^2+2)}-\dfrac{5}{2}\right]$

Since

$$\frac{1}{x^2+2}\le\frac{1}{2}$$

$$\frac{1}{x^2+2}-\frac{5}{2}\le -2$$

Therefore, the maximum value is $\sec^{-1}(-2) = \dfrac{2\pi}{3}$.

(B) Minimum value $= \operatorname{cosec}^{-1}2 + \sec^{-1}1 = \dfrac{\pi}{6}$, when

$$\left[3x^2+\frac{1}{4}\right]=1$$

(C) $$\sin^{-1}|x^2-1|+\cos^{-1}|2x^2-5|=\frac{\pi}{2}$$

$\Rightarrow |x^2-1| = |2x^2-5|$

$\Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$ (Two solutions)

(D) $y=\tan^{-1}\dfrac{1}{2}+\tan^{-1}b$, $(0 < b < 1)$

$\Rightarrow y=\tan^{-1}\left(\dfrac{1/2+b}{1-b/2}\right)$, $\left(\because \dfrac{1}{2}b<1\right)$

$$0<\tan^{-1}\left(\frac{1+2b}{2-b}\right)\le\frac{\pi}{4}$$

$\Rightarrow 0<\left(\dfrac{1+2b}{2-b}\right)\le 1$

$\Rightarrow 0 < (1 + 2b) \le (2 - b), (1 + 2b > 0)$

$\Rightarrow 3b \le 1 \Rightarrow b \le \dfrac{1}{3}$

$\Rightarrow b_{max} = \dfrac{1}{3}$

11. $$\cot\left[\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\tan^{-1}\left(\frac{1}{21}\right)\right]$$

Using, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left[\dfrac{x+y}{1-xy}\right]$, we get

$$\cot\left[\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right]=\frac{3}{2}$$

Solved JEE 2017 Questions

JEE Main 2017

1. The value of $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$, $|x|<\dfrac{1}{2}$, $x\neq 0$, is equal to

(A) $\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}(x^2)$ **(B)** $\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}(x^2)$

(C) $\dfrac{\pi}{4}-\cos^{-1}(x^2)$ **(D)** $\dfrac{\pi}{4}+\cos^{-1}(x^2)$

(ONLINE)

Solution: It is given that

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$$

Substituting $x^2=\cos\theta$, we get

$$\theta=\cos^{-1}(x^2)$$

Using the identities $1+\cos\theta=2\cos^2\dfrac{\theta}{2}$ and $1-\cos\theta=2\sin^2\dfrac{\theta}{2}$, we get

$$\tan^{-1}\left[\frac{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}}\right]=\tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}\right]$$

$$=\tan^{-1}\left[\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right]$$

Dividing the equation by $\cos\dfrac{\theta}{2}$, we get

$$\tan^{-1}\left[\frac{1+\frac{\sin(\theta/2)}{\cos(\theta/2)}}{1-\frac{\sin(\theta/2)}{\cos(\theta/2)}}\right]$$

Now, substituting $\dfrac{\sin(\theta/2)}{\cos(\theta/2)}=\tan\dfrac{\theta}{2}$, we get

$$\tan^{-1}\left[\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right]$$

We know that $\tan\dfrac{\pi}{4}=1$; therefore,

$$\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)$$

because $\tan(a+b)=\dfrac{\tan a+\tan b}{1-\tan a\tan b}$

$$\Rightarrow\frac{\pi}{4}+\frac{\theta}{2}$$

Substituting $\theta=\cos^{-1}x^2$, we get

$$\frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Hence, the correct answer is option (B).

2. A value of x satisfying the equation $\sin[\cot^{-1}(1+x)]=\cos[\tan^{-1}x]$ is

(A) $\dfrac{1}{2}$ **(B)** 0

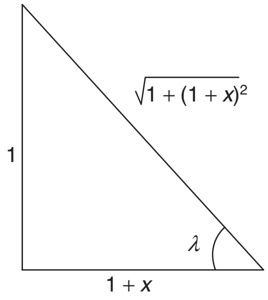
(C) -1 **(D)** $-\dfrac{1}{2}$

(ONLINE)

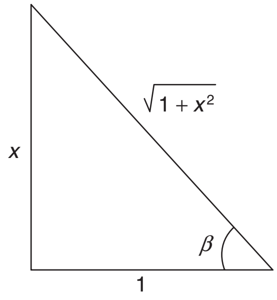
Solution: We have the following two cases:

From Fig. (a): $\sin(\cot^{-1}(1+x))=\dfrac{1}{\sqrt{1+1+x^2+2x}}=\dfrac{1}{\sqrt{x^2+2x+2}}$

From Fig. (b): $\cos(\tan^{-1}x)=\dfrac{1}{\sqrt{1+x^2}}$



(a)



(b)

Therefore, the value of x satisfying the given equation is found as follows:

$$\sin[\cot^{-1}(1+x)]=\cos[\tan^{-1}(x)]$$

$$=x^2+2x+2=1+x^2$$

That is,

$$2x=-1\Rightarrow x=-\frac{1}{2}$$

Hence, the correct answer is option (A).

18 Matrices and Determinants

MATRICES

18.1 Definition of a Matrix

A rectangular array of symbols (which could be real or complex numbers) along rows and columns is called a matrix.

Thus, a system of $m \times n$ symbols arranged in a rectangular formation along m rows and n columns and bounded by the brackets [.] is called an m by n matrix (which is written as $m \times n$ matrix).

Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ -- & ------- & -- \\ -- & ------- & -- \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

is a matrix of order $m \times n$. In a compact form, the above matrix is represented by

$$A = [a_{ij}],\ 1 \le i \le m,\ 1 \le j \le n \text{ or simply } [a_{ij}]_{m\times n}$$

The numbers $a_{11}, a_{12}, \ldots$ of this rectangular array are called the elements of the matrix. The element a_{ij} belongs to the i^{th} row and the j^{th} column and is called the $(i, j)^{\text{th}}$ element of a matrix.

18.2 Order of a Matrix

If any matrix A contains 'm' rows and 'n' columns then $m \times n$ is termed as order of matrix.
Order is generally written as suffix of the array.
Now any matrix of order $m \times n$ will have the notation $[a_{ij}]_{m\times n}$.
That is,

$$A = [a_{ij}]_{m\times n} \text{ or } (a_{ij})_{m\times n} \text{ or } \left\| a_{ij} \right\|_{m\times n}$$

It is obvious that $1 \le i \le m$ and $1 \le j \le n$.

Illustration 18.1 In the inter-sports meet of local colleges, the games to be played are TT, hockey, badminton, tennis and basketball. Three colleges of Meerut sent the following number of players.

Meerut College (MC)—35 players: 5 (TT), 11 (hockey), 5 (badminton), 6 (tennis) and 8 (basketball).

Nanak Chand College (NAS)—22 players: 3 (TT), 13 (hockey), 2 (badminton), 4 (tennis) and none for basketball.

Dev Nagri College (DN)—31 players: 2 (TT), 15 (hockey), 3 (badminton), 5 (tennis) and 6 (basketball).

Put this information in a matrix form.

Solution: The above information can be put in a tabular form as follows.

Colleges	Number of players				
	TT	Hockey	Badminton	Tennis	Basketball
MC (35)	5	11	5	6	8
NAS (22)	3	13	2	4	0
DN (31)	2	15	3	5	6

The number 4 represents the number of players the NAS College has sent for playing tennis. The number 15 represents the number of players the DN College has sent for playing hockey. Similarly, the number 8 represents the number of players that the Meerut College has sent for playing basketball. The above can be put in a rectangular array form as follows:

$$\begin{bmatrix} 5 & 11 & 5 & 6 & 8 \\ 3 & 13 & 2 & 4 & 0 \\ 2 & 15 & 3 & 5 & 6 \end{bmatrix}$$

This is a 3×5 matrix, where 3 represents the number of colleges (number of rows) participating and 5 represents the number of games (number of columns) being played in the meet.

18.3 Types of a Matrix

The elements that appear in the rectangular array are known as entries. Depending upon these entries, matrices are of the following types:

1. **Row matrix:** A single row matrix is called a row matrix or a row vector.
   For example, the matrix $[a_{11}\ a_{12} \ldots a_{1n}]$ is a $1 \times n$ row matrix.
2. **Column matrix:** A single column matrix is called a column matrix or a column vector.

   For example, the matrix $\begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ \ldots \\ a_{m1} \end{bmatrix}$ is an $m \times 1$ column matrix.

3. **Square matrix:** If $m = n$, i.e. if the number of rows and columns of a matrix are equal, say n, then it is called a square matrix of order n.
4. **Null (or zero) matrix:** If all the elements of a matrix are equal to zero, then it is called a null matrix and is denoted by $O_{m\times n}$ or O.
5. **Diagonal matrix:** A square matrix in which all its non-diagonal elements are zero is called a diagonal matrix. Thus, in a diagonal matrix $a_{ij} = 0$ if $i \neq j$.
   The diagonal matrices of orders 2 and 3 are as follows:
   $$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$$
   The elements a_{ij} of a square matrix for which $i = j$ are called the diagonal elements of a matrix and the diagonal along which all these elements lie is called the principal diagonal or the leading diagonal or the diagonal of the matrix.
6. **Scalar matrix:** A square matrix in which all the diagonal elements are equal and all other elements are equal to zero is called a scalar matrix.
   That is, in a scalar matrix $a_{ij} = k$, for $i = j$ and $a_{ij} = 0$ for $i \neq j$. Thus,
   $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is a scalar matrix.
7. **Unit matrix or identity matrix:** A square matrix in which all its diagonal elements are equal to 1 and all other elements are equal to zero is called a unit matrix or an identity matrix, denoted by U or I.
   For example, unit (or identity) matrices of orders 2 and 3 are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, respectively.
8. **Negative of a matrix:** Let $A = [a_{ij}]_{m\times n}$ be a matrix. Then, the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m\times n}$ and is denoted by $-A$.
9. A square matrix in which all elements below leading diagonal or all elements above leading diagonal are zero is called a triangular matrix.
   (i) ***Upper triangular matrix:*** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$, for all $i > j$. Thus, in an upper triangular matrix all elements below diagonals are zero.
   For example, $A = \begin{bmatrix} a & b & c \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$ is an upper triangular matrix.
   (ii) ***Lower triangular matrix:*** A square matrix $A = [a_{ij}]$ is called a lower triangular matrix if $a_{ij} = 0$ for all $i < j$. Thus, in a lower triangular matrix, all elements above diagonal are zero.
   For example, $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ is a lower triangular matrix.

18.4 Equality of Matrices

Two matrices A and B are said to be equal, written as $A = B$, if
1. they both are of the same order, i.e. have the same number of rows and columns and
2. the elements in the corresponding places of the two matrices are the same.

18.5 Addition and Subtraction of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order $m \times n$. Then, their sum (or difference) $A + B$ (or $A - B$) is defined as another matrix of the same order, say $C = [c_{ij}]$ such that any element of C is the sum (or difference) of the corresponding elements of A and B. Therefore,
$$C = A \pm B = [a_{ij} \pm b_{ij}]$$

Illustration 18.2 Find $A + B$ and $A - B$ where $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$

Solution: Here, both A and B are 2×3 matrices. Therefore,
$$A + B = \begin{bmatrix} 1+7 & 2+3 & 4+2 \\ 0+5 & 5+1 & 3+9 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 6 \\ 5 & 6 & 12 \end{bmatrix}$$
and
$$A - B = \begin{bmatrix} 1-7 & 2-3 & 4-2 \\ 0-5 & 5-1 & 3-9 \end{bmatrix} = \begin{bmatrix} -6 & -1 & 2 \\ -5 & 4 & -6 \end{bmatrix}$$

18.5.1 Properties of Matrix Addition

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A + O = O + A = A$; here O {null matrix} will be additive identity.
4. If A is a given matrix, then the matrix $-A$ is the additive inverse of A for $A + (-A) =$ null matrix O.
5. If A, B and C are three matrices of the same order, then
   $$A + B = A + C \Rightarrow B = C \qquad \text{(left cancellation law)}$$
   and
   $$B + A = C + A \Rightarrow B = C \qquad \text{(right cancellation law)}$$

18.6 Multiplication of a Matrix by a Scalar

Let $A = [a_{ij}]_{m\times n}$ be a matrix and k a scalar. Then, the matrix obtained by multiplying each element of matrix A by k is called the scalar multiple of A and is denoted by kA.

18.6.1 Properties of Multiplication of a Matrix by a Scalar

1. If k_1 and k_2 are scalars and A be a matrix, then $(k_1 + k_2)A = k_1A + k_2A$.

2. If k_1 and k_2 are scalars and A be a matrix, then $k_1(k_2 A) = (k_1 k_2)A$.
3. If A and B are two matrices of the same order and k is a scalar, then $k(A + B) = kA + kB$.
   That is, the scalar multiplication of matrices distributes over the addition of matrices.
4. If A is any matrix and k be a scalar, then $(-k)A = -(kA) = k(-A)$.

18.7 Multiplication of Two Matrices

Let $A = [a_{ij}]$ be an $m \times p$ matrix and $B = [b_{ij}]$ be a $p \times n$ matrix. These matrices A and B are such that the number of columns of A are the same as the number of rows of B, each being equal to p. Then, the product AB (in the order it is written) will be a matrix $C = [c_{ij}]$ of the type $m \times n$.

Here c_{ij} will be the element of C occurring in i^{th} row and the j^{th} column, and it will be row by column product of i^{th} row of A having p columns with the j^{th} column of B having p rows, the elements of which are

$$a_{i1}\, a_{i2} \ldots a_{ip} \text{ and } \begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{matrix}$$

Therefore,

$$c_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + \cdots + a_{ip} b_{pj} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

The summation is to be performed with respect to repeated suffix k.

This gives us the particular i–j^{th} element of C which is of order $m \times n$. For obtaining an element of C occurring in the second row and the third column, we shall put $i = 2$ and $j = 3$. Therefore,

$$c_{23} = \sum_{k=1}^{p} a_{2k} b_{k3} = a_{21} b_{13} + a_{22} b_{23} + \cdots + a_{2p} b_{p3}$$

Since there are m rows in A, i can take values from 1 to m. Similarly, there are n columns in B, j can take values from 1 to n. Thus, we shall get all the mn elements of C. Again

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \qquad (18.1)$$

This gives us the i–j^{th} element of AB which is of order $m \times n$ having m rows and n columns.

1. **Elements of the j^{th} column of AB:** For obtaining elements of the j^{th} column, j will remain fixed for the j^{th} column whereas i will change from 1 to m as there are m rows in AB.
   Hence, giving i the values 1, 2, 3, …, m and keeping j fixed in Eq. (18.1) we shall get all the elements of the j^{th} column of AB. Therefore, the j^{th} column of AB is
   $$\sum_{k=1}^{p} a_{1k}\, b_{kj},\ \sum_{k=1}^{p} a_{2k} b_{kj},\ \ldots,\ \sum_{k=1}^{p} a_{mk} b_{kj}$$
2. **An easy way to remember:** If we denote the ordered set of rows of A by R_1, R_2, R_3 each having two elements and ordered set of columns of B by C_1, C_2, each having two elements, then
   $$AB = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}_{3\times1} [C_1\, C_2]_{1\times2} = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \\ R_3C_1 & R_3C_2 \end{bmatrix}_{3\times2}$$
3. **Few important things for the multiplication**
   (a) Condition for product AB to exist or to be defined: If A and B are two matrices then their product is defined or in other words A is *conformable* to B for multiplication if the number of columns of A is the same as the number of rows in B. That is, if A is a matrix of order $m \times p$ and B is a matrix of order $p \times n$, the matrix AB will be of order $m \times n$.
   (b) Pre-multiplication and post-multiplication: When we say multiply A by B then it could mean both AB or BA where A and B are any numbers. But when A and B are matrices then as seen above AB and BA do not necessarily mean the same thing. If AB is defined for matrix multiplication, BA may not be defined. To avoid this, when we say product AB it would mean the matrix A post-multiplied by B and when we say product BA it would mean matrix A pre-multiplied by B. In AB, A is called the *pre-factor* and B the *post-factor*.
   (c) In the case when both A and B are square matrices of the same order then also both AB and BA are defined and the product matrix is also a matrix of the same order but still $AB \neq BA$.
   (d) Again we know that for two scalars a and b when $ab = 0$ it means that either a or b (or both) is zero. But for two matrices A and B, $AB = O$, i.e. a null matrix, does not necessarily imply that either A or $B = O$ as shown above because neither A nor B is null matrix whereas AB is a null matrix.
   $$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
   $$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
   But, $A \neq 0$ and $B \neq 0$

Illustration 18.3 If $A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, then compute AB and BA.

Solution: Here, A is 3×3 and B is 3×3. Hence, both AB and BA are defined and each will be 3×3 matrix. Let

$$AB = C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

where C_{ij} means the product of the element at i^{th} row of A with the element at j^{th} column of B.

For example, C_{23} = product of the second row of A with the third column of B. That is,

$$[-3\ \ 2\ \ -1] \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = -3\cdot3 + 2\cdot6 - 1\cdot3 = 0$$

Similarly, we can find other elements of C.

We can also say that by the product of the first row of A with the three columns of B, we shall get the three elements of the first row of C. That is,

$$R_1C_1, R_1C_2, R_1C_3$$

and similarly take the second row of A and multiply with all the columns of B and we will get the three elements of the second row of C, i.e. R_2C_1, R_2C_2, R_2C_3 and elements of the third row of C will be R_3C_1, R_3C_2, R_3C_3. Therefore,

$$AB = \begin{bmatrix} 1\cdot1-1\cdot2+1\cdot1 & 1\cdot2-1\cdot4+1\cdot2 & 1\cdot3-1\cdot6+1\cdot3 \\ -3\cdot1+2\cdot2-1\cdot1 & -3\cdot2+2\cdot4-1\cdot2 & -3\cdot3+2\cdot6+1\cdot3 \\ 2\cdot1+1\cdot2+0\cdot1 & -2\cdot2+1\cdot4+0\cdot2 & -2\cdot3+1\cdot6+0\cdot3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O \quad \text{(i.e. null matrix)}$$

Similarly, BA can also be computed.

Illustration 18.4 If A and B are matrices such that both AB and $A + B$ are defined, prove that both A and B are square matrices of the same order.

Solution: We know that two matrices A and B are conformable for addition if they are of the same order. Thus, if A be $m \times n$ then B should also be $m \times n$ as $A + B$ is defined.

Again since AB is also defined therefore the number of columns in A (i.e., n) should be equal to the number of rows in B (i.e., m). Hence, $n = m$ and in that case both A and B will be the square matrices of order equal to $m = n$.

Illustration 18.5 If A is any $m \times n$ matrix and both AB and BA are defined prove that B should be an $n \times m$ matrix.

Solution: Since A is $m \times n$ and AB is defined, therefore B should be $n \times p$ because the number of columns of A should be equal to number of rows of B.

Again B is now $n \times p$ and A is $m \times n$.
Since BA is also defined, therefore p would be equal to m by the same argument as above.
Therefore, B is $n \times m$ matrix.

18.7.1 Properties of Matrix Multiplication

1. Multiplication of matrices is distributive with respect to addition of matrices. That is,

$$A(B + C) = AB + AC$$

2. Matrix multiplication is associative if conformability is assured. That is,

$$A(BC) = (AB)C$$

3. The multiplication of matrices is not always commutative. That is, AB is not always equal to BA ($AB \neq BA$).

4. Multiplication of a matrix A by a null matrix conformable with A will give null matrix. Consider

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 1 & 2 \\ 6 & 4 & 2 \\ 7 & 4 & 6 \end{bmatrix}_{4\times3}$$

and

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times2}$$

Then $AO = O$.

5. If A is an $m\times n$ matrix then $I_mA = A = AI_n$ where I_m and I_n are identity matrices of order m and n, respectively. If A is a square matrix of order n and I is the identity matrix of order n, then $AI = A = IA$. Thus, I is the multiplicative identity.

6. For a square matrix A, positive integral powers of A, i.e. A^n, can be obtained by multiplying A by itself n times, i.e.

$$A^2 = A \times A$$
$$A^3 = A \times A \times A = A^2 \times A$$

and so on.

7. **Matrix polynomial:** If $f(x) = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots + a_m$ is a polynomial in x and A is a square matrix of order n, then

$$f(A) = a_0A^m + a_1A^{m-1} + a_2A^{m-2} + \cdots + a_mI_n$$

is called matrix polynomial. For example, if $f(x) = 3x^2 - 2x + 5$ and A is a square matrix of third order then matrix polynomial is $f(A) = 3A^2 - 2A + 5I_3$.

18.8 Operations Regarding Matrices

18.8.1 Transpose of a Matrix

If A is a given matrix of the type $m \times n$ then the matrix obtained by changing the rows of A into columns and columns of A into rows is called **transpose** of matrix A and is denoted by A' or A^T. As there are m rows in A, therefore there will be m columns in A' and similarly as there are n columns in A, there will be n rows in A'.
Thus if $A = [a_{ij}]_{m\times n}$ then

$$A' = A^T = [a_{ji}]_{n\times m}$$

For example,

$$\text{If } A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & 0 & -2 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 7 & -2 \end{bmatrix}.$$

Properties of Transpose

1. $(A')' = A$
2. $(KA)' = KA'$, with K being a scalar
3. $(A \pm B)' = A' \pm B'$
4. $(AB)' = B'A'$
5. $(ABC)' = C'B'A'$

18.8.2 Conjugate of a Matrix

Let $A = [a_{ij}]$ be a given matrix. Then the matrix obtained by replacing all the elements by their conjugate complex is called the conjugate of matrix A. It is represented by $\bar{A}$, i.e. $\bar{A} = [\bar{a}_{ij}]$.

Properties of Conjugates

1. $\overline{(\bar{A})} = A$
2. $\overline{(A+B)} = \bar{A} + \bar{B}$
3. $\overline{(\alpha A)} = \bar{\alpha}\bar{A}$, with α being any number
4. $\overline{(AB)} = \bar{A}\bar{B}$, with A and B being conformable for multiplication.

18.8.3 Transpose of the Conjugate of a Matrix

Transpose of the conjugate of a matrix is equal to the conjugate of the transpose of a matrix A, i.e. $(\bar{A})' = (\bar{A'})$ and is written as A^{θ}.

Properties of Transpose Conjugate

1. $(A^{\theta})^{\theta} = A$
2. $(A+B)^{\theta} = A^{\theta} + B^{\theta}$
3. $(kA)^{\theta} = \bar{k}\ A^{\theta}$, k being any number
4. $(AB)^{\theta} = B^{\theta}A^{\theta}$

18.8.4 Trace of a Matrix

Let A be a square matrix of order n. The sum of the elements of A lying along the principal diagonal is called the trace of A. We shall write the trace of A as tr A. Thus, if $A = [a_{ij}]_{n\times n}$, then

$$\text{tr } A = \sum_{i=1}^{n} a_i = a_{11} + a_{22} + \cdots + a_{nn}.$$

Trace of *a* Matrix

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

18.9 Types of a Matrix on the Basis of Operations

1. **Symmetric matrix:** A square matrix $A = [a_{ij}]$ is said to be symmetric if its $(i, j)^{\text{th}}$ element is the same as its $(j, i)^{\text{th}}$ element, i.e. $a_{ij} = a_{ji}$ for all i, j.
2. **Skew-symmetric matrix:** A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if the $(i, j)^{\text{th}}$ element of A is the negative of the $(j, i)^{\text{th}}$ element of A, i.e. if $a_{ij} = -a_{ji}$ for all i, j.

Properties of Symmetric and Skew-Symmetric Matrices

1. If A is a symmetric matrix, then $A' = A$.
2. If A is a skew-symmetric matrix, then $A' = -A$.
3. Diagonal elements of a skew-symmetric matrix are zero.
4. For any square matrix A,
   (a) $A + A'$ is symmetric.
   (b) $A - A'$ is skew-symmetric.
   (c) AA' and $A'A$ are symmetric matrices.
5. If A is a symmetric matrix, then all positive integral powers of A are symmetric.
6. If A is a skew-symmetric matrix, then all positive even integral powers of A are symmetric and all positive odd integral powers of A are skew-symmetric.

3. **Hermitian matrix:** A square matrix $A = [a_{ij}]$ is said to be Hermitian if the $(i, j)^{\text{th}}$ element of A is equal to the conjugate complex of the $(j, i)^{\text{th}}$ element of A, i.e. $a_{ij} = \bar{a}_{ji}$ for all i and j.
4. **Skew-Hermitian matrix:** A square matrix $A = [a_{ij}]$ is said to be skew-Hermitian if the $(i, j)^{\text{th}}$ element of A is equal to the negative of the conjugate complex of the $(j, i)^{\text{th}}$ element of A, i.e. $a_{ij} = -\bar{a}_{ji}$ for all i and j.

Hermitian and Skew-Hermitian Matrices

A square matrix $A = [a_{ij}]$ is said to be Hermitian matrix if $a_{ij} = \bar{a}_{ji}$ $\forall\ i, j$, i.e. $A = A^{\theta}$.

1. If A is a Hermitian matrix then $a_{ii} = \bar{a}_{ii} \Rightarrow a_{ii}$ is real $\forall\ i$. Thus every diagonal element of a Hermitian matrix must be real.
2. A Hermitian matrix over the set of real numbers is actually a real symmetric matrix. A square matrix $A = [a_{ij}]$ is said to be skew-Hermitian if $a_{ij} = -\bar{a}_{ji}$, $\forall\ i, j$, i.e. $A^{\theta} = -A$.
3. If A is a skew-Hermitian matrix then $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$, i.e. a_{ii} must be purely imaginary or zero.
4. A skew-Hermitian matrix over the set of real numbers is actually a real skew-symmetric matrix.

5. **Orthogonal matrix:** A square matrix A is said to be orthogonal if $A'A = I = AA'$.
6. **Unitary matrix:** A square matrix A is said to be unitary if $A^{\theta}A = I = AA^{\theta}$.
7. **Idempotent matrix:** A square matrix A such that $A^2 = A$ is called an idempotent matrix.
8. **Nilpotent matrix:** A square matrix A will be called a nilpotent matrix if $A^k = O$ (null matrix) where k is a positive integer. If however k is the least positive integer for which $A^k = O$ then k is the **index** of the nilpotent matrix A.
9. **Involutory matrix:** A square matrix A such that $A^2 = I$ is called the involutory matrix.

Your Turn 1

1. If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$ then $AB + BA = O$.

   (True/False) **Ans.** False

2. If $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$ then $x = -3$, $y = -2$ and $z = 4$. (True/False)

Ans. True

3. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$, then $A^3 - 3A^2 - A + 9I$ equals ______.

Ans. Zero

4. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ and $A \times A' = I$, then find $x + y$.

Ans. -3

5. For the three matrices A, B and C,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

verify the following relations:

(a) $A^2 = B^2 = C^2 = I$

(b) $AB = -BA$; $AC = -CA$; $BC = -CB$

6. Use matrix multiplication to divide Rs. 30000 in two parts such that the total annual interest at 9% on the first part and 11% on the second part amounts Rs. 3060.

Ans. First part $\rightarrow$ 12000
Second part $\rightarrow$ 18000

7. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, show that $AB = BA = O_{3\times 3}$.

8. Express the matrix $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Ans. $A = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}$

9. Let A and B be symmetric matrices of the same order. Then, show that

(a) $AB - BA$ is a skew-symmetric matrix.

(b) $AB + BA$ is a symmetric matrix.

10. If $A = \begin{bmatrix} 3 & -4 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$, show that $(AB)^T = B^T A^T$.

DETERMINANTS

18.10 Definition of a Determinant

Every square matrix A can be associated to a number or an expression which is known as the determinant of A and is denoted by $|A|$ or det A.

18.11 Evaluation of Determinants

18.11.1 Determinants of the First Order

If $A = [a_{11}]$, then $|A| = a_{11}$

18.11.2 Determinants of the Second Order

The notation $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ consisting of 2^2 numbers termed as elements, arranged in two rows and two columns, is called a determinant of second order. The elements a_1 and b_2 are said to lie along the principal diagonal; the elements a_2 and b_1 are said to lie along the secondary diagonal.

The value of the determinant is obtained by forming the product of the elements along the principal diagonal and subtracting from it the product of the elements along the secondary diagonal. Thus,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{18.2}$$

18.11.3 Determinants of the Third Order

The notation $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of 3^2 elements, arranged in three rows and three columns, is called a determinant of third order. Its value is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

This may be written as

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

or

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

We can therefore write

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \tag{18.3}$$

Note that each term of a second-order determinant is the product of two quantities and each term of a third-order determinant is the product of three quantities.

18.12 Minors

The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row and the column in which the given element occurs.

The minor of a_1 in Eq. (18.2) is b_2 and b_2 may be considered a determinant of first order. Similarly, the minor of a_2 is b_1.

For example, the minor of a_1 in Eq. (18.3) is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ and the minor of b_2 in Eq. (18.3) is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$.

18.13 Cofactors

In Eq. (18.3), the elements a_1, b_1, c_1 are multiplied by

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

These expressions are called the cofactors of the elements a_1, b_1, c_1.

Generally, the cofactor of an element is its minor with its sign or opposite sign prefixed in accordance with the following rule.

For any determinant if a_{ij} is the element at the intersection of the i^{th} row and j^{th} column, then the cofactor of a_{ij} has positive sign or negative sign before minor of a_{ij} according to $i + j$ is even or odd. The determinant may be expanded along any chosen row or column.

The cofactors of the elements a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 will be denoted by A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3, respectively.

For example, element b_3 in Eq. (18.3) lies at the intersection of the third row and the second column. Since 3 + 2 = 5 is an odd number, we have

$$B_3 = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

The cofactor B_2 of the element b_2 is $+\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ because element b_2 lies at the intersection of the second row and the second column, and $2 + 2 = 4$ is an even number.

Let the determinant in Eq. (18.3) be denoted by Δ. When the cofactors are used, the expansion of the determinant takes the compact form:

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 = a_2A_2 + b_2B_2 + c_2C_2 = a_3A_3 + b_3B_3 + c_3C_3$$

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 = b_1B_1 + b_2B_2 + b_3B_3 = c_1C_1 + c_2C_2 + c_3C_3$$

and

$$a_2A_1 + b_2B_1 + c_2C_1 = 0 = a_2A_3 + b_2B_3 + c_2C_3, \text{ etc.}$$

18.14 Adjoint of a Square Matrix

Let $A = [a_{ij}]_{n\times n}$ be any $n \times n$ matrix. The transpose B' of the matrix $B = [C_{ij}]_{n\times n}$, where C_{ij} denotes the cofactor of the element a_{ij} in the determinant $|A|$, is called the adjoint of the matrix A and is denoted by the symbol adj A.

Illustration 18.6 If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, then find adj A.

Solution: In $|A|$, the cofactor of α is δ and the cofactor of β is $-\gamma$. Also the cofactor of γ is $-\beta$ and the cofactor of δ is α. Therefore, the matrix B formed of the cofactor of the elements of $|A|$ is

$$B = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}$$

Now, adj A = the transpose of the matrix $B = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$.

18.15 Inverse of a Matrix

Let A be any n-rowed square matrix. Then, a matrix B, if it exists, such that $AB = BA = I_n$, is called inverse of A.

The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

If A be an invertible matrix, then the inverse of A is $\frac{1}{|A|}$ adj A. It is usual to denote the inverse of A by A^{-1}.

18.15.1 Theorem (Uniqueness of Inverse)

Theorem: Inverse of a square matrix if it exists is unique.

Proof: Let $A = [a_{ij}]_{n\times n}$ be a square matrix. Let inverse of A exist.
To prove: Inverse of A is unique.
If possible, let B and C be two inverses of A. Then

$$AB = BA = I_n \text{ and } AC = CA = I_n$$

Now

$$B = BI_n = B(AC) \qquad [\text{since } AC = I_n]$$
$$= (BA)C = I_nC = C$$

Hence $B = C$. This implies that the inverse of A is unique.

18.15.2 Properties of Inverse of a Matrix

1. $(AB)^{-1} = B^{-1}A^{-1}$

2. $(A')^{-1} = (A^{-1})'$

3. $(A^{-1})^{\theta} = (A^{\theta})^{-1}$

Illustration 18.7 Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: We find the determinant of A,

$$|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

Expanding along R_1 we get

$$|A| = 0(2-3) - 1(1-9) + 2(1-6) = 8 - 10 = -2$$

Since $|A| \neq 0$, therefore A^{-1} exists.

Now the cofactors of the elements of the first row of $|A|$ are $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$, that is, are –1, 8, –5, respectively.

The cofactors of the elements of the second row of $|A|$ are $-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$, that is, are 1, –6, 3, respectively.

The cofactors of the elements of the third row of $|A|$ are $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$, that is, are –1, 2, –1, respectively.

Therefore, adj A = the transpose of the matrix B where

$$B = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

So,

$$\text{adj } A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Now

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

Here $|A| = -2$. Therefore

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

18.16 Singular and Non-Singular Matrices

A square matrix A is said to be non-singular or singular according to $|A| \neq 0$ or $|A| = 0$.

18.17 Elementary Operations or Elementary Transformations of a Matrix

Any of the following operations is called an *elementary transformation* (*operation*).

1. The interchange of any two rows (or columns).
2. The multiplication of the elements of any row (or column) by a non-zero number.
3. The addition to the elements of any row (or column), the corresponding elements of any other row (or column) multiplied by a non-zero number.

Any elementary transformation is called a row transformation or column transformation considering as it applies to rows or columns.

Clearly, there will be a total of six elementary operations (transformations) on a matrix, three of them are due to rows and are called row operations whereas three of them are due to columns and are called column operations.

1. The elementary operations of interchange of the i^{th} row and the j^{th} row is denoted by $R_i \leftrightarrow R_j$ and the interchange of the i^{th} column and the j^{th} column is denoted by $C_i \leftrightarrow C_j$.

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$$

Applying $R_1 \leftrightarrow R_3$, i.e. interchanging the first row and the third row, matrix A becomes the matrix

$$B = \begin{bmatrix} 2 & 0 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

2. The elementary operation of the multiplication of the elements of the i^{th} row by a non-zero number k is denoted by $R_i \to kR_i$.

Similarly, the multiplication of the elements of the i^{th} column by a non-zero number k is denoted by $C_i \to kC_i$.

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$$

On multiplying the elements of the third column of matrix A by 2, i.e. on applying $C_3 \to 2C_3$, we get the new matrix

$$B = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & 8 \\ 2 & 0 & 10 \end{bmatrix}$$

3. The elementary operation of the addition to the elements of the i^{th} row, the corresponding elements of the j^{th} row multiplied by a non-zero number k is denoted by $R_i \to R_i + kR_j$.

Similarly, the elementary operation of the addition to the elements of the i^{th} column, the corresponding elements of the j^{th} column multiplied by a non-zero number k is denoted by $C_i \to C_i + kC_j$.

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 7 & 4 \\ 2 & 4 & 5 \end{bmatrix}$$

On applying the elementary operation $C_2 \to C_2 + 2C_1$, matrix A becomes matrix B.

18.17.1 Equivalent Matrices

Two matrices A and B are said to be *equivalent* if one can be obtained from other by applying a finite number of elementary operations on the other matrix. If A and B are equivalent matrices, we write $A \sim B$.

Example: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 6 \\ 2 & 0 & 10 \end{bmatrix}$$

Now,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix} \quad [\text{applying } R_1 \leftrightarrow R_2]$$

$$\sim \begin{bmatrix} 2 & 3 & 8 \\ 1 & 2 & 6 \\ 2 & 0 & 10 \end{bmatrix} = B \quad [\text{applying } C_3 \to 2C_3]$$

Here, $A \sim B$ as B has been obtained from A by applying two elementary operations.

18.17.2 Elementary Matrix

A matrix obtained from unit matrix by a single elementary operation is called an elementary matrix.

Example: Let

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad [R_1 \to 2R_1]$$

is an elementary matrix.

18.18 Inverse of a Matrix by Elementary Operations (Elementary Operations on Matrix Equation)

Let *A*, *B* and *X* be three square matrices of the same order such that

$$X = AB \tag{18.4}$$

The matrix Eq. (18.4) will also be valid if we apply a row operation on matrix *X* [occurring on the left-hand side of Eq. (18.4)] and the same row operation on matrix *A* (the first factor of product *AB* on the matrix on the right-hand side).

Thus, on the application of a sequence of row operations on the matrix equation $X = AB$ (these row operations are applied on *X* and on the first matrix *A* of product *AB* simultaneously), the matrix equation is still valid (we assume this fact without proof).

Similarly, a sequence of elementary column operations on the matrix equation $X = AB$ can be applied simultaneously on *X* and on the second matrix *B* of product *AB* and the equation will be still valid.

In view of the above-mentioned fact, it is clear that we can find the inverse of a matrix *A*, if it exists, by using either a sequence of elementary row operations or a sequence of elementary column operations but not both simultaneously.

18.18.1 Using Row Operation

Apply a series of row operations on $A = IA$ till we get $I = BA$.
Now by definition of inverse of a matrix, $B = A^{-1}$.

18.18.2 Using Column Operation

Apply a series of column operations on $A = AI$ till we get $I = BA$.
By definition of inverse, *B* is inverse of *A*.

Illustration 18.8 Obtain the inverse of the matrix using elementary operations, $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: We will use row operation first. We have

$$A = IA$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad (R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad (R_3 \to R_3 - 3R_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad (R_1 \to R_1 - 2R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \qquad (R_3 \to R_3 + 5R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \qquad \left(R_3 \to \frac{1}{2}R_3\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \qquad (R_1 \to R_1 + R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A \qquad (R_2 \to R_2 - 2R_3)$$

Hence,

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

We also show the result using column operation. We have

$$A = AI$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_1 \leftrightarrow C_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_3 \to C_3 - 2C_1)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 2 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (C_3 \to C_3 + C_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 & 1/2 \\ 1 & 0 & -1 \\ 0 & 0 & 1/2 \end{bmatrix} \qquad \left(C_3 \to \frac{1}{2}C_3\right)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} -2 & 1 & 1/2 \\ 1 & 0 & -1 \\ 0 & 0 & 1/2 \end{bmatrix} \qquad (C_1 \to C_1 - 2C_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = A \begin{bmatrix} 1/2 & 1 & 1/2 \\ -4 & 0 & -1 \\ 5/2 & 0 & 1/2 \end{bmatrix} \qquad (C_1 \to C_1 + 5C_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} \qquad (C_2 \to C_2 - 3C_3)$$

Hence,

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

18.19 Rank of a Matrix

A number r is said to be the rank of a matrix A if it possesses the following two properties:

1. There is at least one square submatrix of A of order r whose determinant is not equal to zero.
2. If the matrix A contains any square submatrix of order $r + 1$, then the determinant of every square submatrix of A of order $r+1$ should be zero.

In short, the rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

18.20 Echelon Form of a Matrix

A matrix A is said to be in Echelon form if either A is null matrix or it satisfies the following conditions:

1. Every non-zero row in A precedes every zero row.
2. The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Also rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix.

For example, $A=\begin{bmatrix}0 & 8 & 12 & 4\\ 0 & 0 & -2 & 5\\ 0 & 0 & 0 & 0\end{bmatrix}$ is in its Echelon form with two non-zero rows. Therefore rank of $A = 2$.

To reduce the given matrix $A = [a_{ij}]_{m\times n}$ in Echelon form, use elementary transformations to make

$$a_{21}, a_{31}, \ldots, a_{m1} = 0$$

Then

$$a_{32}, a_{42}, \ldots, a_{m2} = 0$$

and so on.
For example, let

$$A=\begin{bmatrix}1 & 1 & 1 & -3\\ 3 & 1 & -2 & -2\\ 2 & 4 & 7 & 7\end{bmatrix}$$

Operating $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, we get

$$A=\begin{bmatrix}1 & 1 & 1 & -3\\ 0 & -2 & -5 & 7\\ 0 & 2 & 5 & 13\end{bmatrix}$$

Operating $R_3 \to R_3 + R_2$

$$A=\begin{bmatrix}1 & 1 & 1 & -3\\ 0 & -2 & -5 & 7\\ 0 & 0 & 0 & 20\end{bmatrix}$$

which is an Echelon form with 3 non-zero rows. Therefore, rank of $A = 3$.

18.21 Homogeneous Linear Equations

The equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0\\ &\cdots\cdots\\ &\cdots\cdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0\end{aligned}\right\} \tag{18.5}$$

represent a system of m homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$. Let

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ a_{31} & a_{32} & \cdots & a_{3n}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}_{m\times n}, X=\begin{bmatrix}x_1\\ x_2\\ x_3\\ \cdots\\ \cdots\\ x_n\end{bmatrix}_{n\times 1}, O=\begin{bmatrix}0\\ 0\\ 0\\ \cdots\\ \cdots\\ 0\end{bmatrix}_{m\times 1}$$

where A, X, O are $m\times n$, $n\times 1$, $m\times 1$ matrices, respectively. Then, obviously we can write the system of equations (18.3) in the form of a single matrix equation

$$AX = O \tag{18.6}$$

The matrix A is called the coefficient matrix of the system of equations.

1. If $|A| = 0$, the system has infinitely many solutions.
2. If $|A| \neq 0$, the system has zero solution or trivial solutions.

These conclusions can also be written on the basis of the rank method as follows:

Suppose we have m equations in n unknowns. Then the coefficient matrix A will be of the type $m\times n$. Let r be the rank of the matrix A. Obviously, r cannot be greater than n (the number of columns of the matrix A). Therefore, we have either $r = n$ or $r < n$.

1. **Case I:** If $r = n$, the equation $AX = O$ will have $n - n$, i.e. no linearly independent solutions. In this case, the zero solution will be the only solution. We know that zero vector forms a linearly dependent set.
2. **Case II:** If $r < n$, we shall have $n - r$ linearly independent solutions. Any linear combination of these $n - r$ solutions will also be a solution of $AX = O$. Thus, in this case the equation $AX = O$ will have an infinite number of solutions.
3. **Case III:** Suppose $m < n$, i.e. the number of solutions is less than the number of unknowns. Since $r \le m$, therefore r is definitely less than n. Hence, in this case the given system of equations must possess a non-zero solution. The number of solutions of the equation $AX = O$ will be infinite.

Illustration 18.9 Does the following system of equations possess a common non-zero solution?

$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

Solution: Determinant of coefficient matrix is $|A| = -2$ which is non-zero.
Therefore, $x = y = z = 0$ is the only solution.
Alternate method (Using Rank): The given system of equations can be written in the form of the single matrix equation as

$$AX = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

We shall start reducing the coefficient matrix A to triangular form by applying only E-row transformations on it. Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$, the given system of equations is equivalent to

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = O$$

Here, we find that the determinant of the matrix on the left-hand side of this equation is not equal to zero. Therefore, the rank of this matrix is 3. So, there is no need of further applying E-row transformation on the coefficient matrix. The rank of the coefficient matrix A is 3, i.e. equal to the number of unknowns. Therefore, the given system of equations does not possess any linearly independent solution. The zero solution, i.e. $x = y = z = 0$ is the only solution of the given system of equations.

18.21.1 Solution of Homogeneous System of Linear Equations

Let $AX = O$ be a homogeneous system of n linear equations with n unknowns. Now if A is non-singular then the system of equations will have a unique solution, i.e. trivial solution, and if A is a singular, then the system of equations will have infinitely many solutions.

18.22 System of Linear Non-Homogeneous Equations

Let the equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots\cdots \\ &\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\} \quad (18.7)$$

be a system of m non-homogeneous equations in n unknowns x_1, x_2, …, x_n. If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}_{n\times 1}, B = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}_{m\times 1}$$

where A, X, B are $m \times n$, $n \times 1$, $m \times 1$ matrices, respectively, the above equations can be written in the form of a single matrix equation $AX = B$.

Any set of values of $x_1, x_2, \dots, x_n$ which simultaneously satisfy all these equations is called a solution of the system (18.7). When the system of equations has one or more solutions, the equations are said to be consistent, otherwise they are said to be inconsistent.

If $B \neq 0$, the system (18.7) is said to be non-homogenous.

1. If $|A| \neq 0 \to X = A^{-1}B$, where $A^{-1} = \dfrac{\text{adj } A}{|A|}$

   The given system has unique solution.

2. If $|A| = 0$, since $AX = B$, we have

   $$(\text{adj } A)AX = (\text{adj } A)B \Rightarrow |A|X = (\text{adj } A)B$$
   $$\Rightarrow (\text{adj } A)B = 0 \qquad [\text{since } |A| = 0]$$

   which is true for infinite values of X.

   Therefore, for infinitely many solutions to the system we should have

   $$(\text{adj } A)B = 0$$

   Clearly for no solution we should have

   $$(\text{adj } A)B \neq 0$$

   These conclusions can also be written on the basis of the rank method as follows: The matrix

   $$[A\ B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix of the given system of equations.

Suppose the coefficient matrix A is of the type $m \times n$, i.e. we have m equations in n unknowns. Write the augmented matrix $[A\ B]$ and reduce it to an Echelon form by applying only E-row transformations and comparing the ranks of the augmented matrix $[A\ B]$ and the coefficient matrix A. Then, the following different cases arise:

Case I: Rank $A <$ Rank $[A\ B]$
In this case, the equations $AX = B$ are inconsistent, i.e. they have no solution.

Case II: Rank A = Rank $[A\ B] = r$ (say).
In this case, the equations $AX = B$ are consistent, i.e. they possess a solution. If $r < m$, then in the process of reducing the matrix $[A\ B]$ to the Echelon form, $(m - r)$ equations will then be replaced by an equivalent system of r equations. From these r equations, we shall be able to express the values of some r unknowns in terms of the remaining $n - r$ unknowns which can be given any arbitrary chosen values.

If $r = n$, then $n - r = 0$, so that no variable is to be assigned arbitrary values and therefore in this case there will be a unique solution.

If $r < n$, then $n - r$ variables can be assigned arbitrary values. So, in this case there will be an infinite number of solutions. Only $n - r + 1$ solutions will be linearly independent and the rest of the solutions will be linear combinations of them.

If $m < r$, then $r \le m < n$. Thus, in this case $n - r > 0$. Therefore, when the number of equations is less than the number of unknowns, the equations will always have an infinite number of solutions provided they are consistent.

For a non-singular matrix A:

$$AX = B \Rightarrow X = A^{-1}B$$

By comparing entries on both the sides, we have a unique solution for a given system of equations.

Illustration 18.10 Show that the equations $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$ and $6y - 18z + 1 = 0$ are not consistent.

Solution:

$$\Delta = |A| = \begin{vmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{vmatrix} = 0$$

$$\Delta_1 = \begin{vmatrix} -11 & 6 & 0 \\ -3 & 20 & -6 \\ -1 & 6 & -18 \end{vmatrix} \neq 0$$

$$\Delta_2 = \begin{vmatrix} 2 & -11 & 6 \\ 6 & -3 & 20 \\ 0 & -1 & 6 \end{vmatrix} \neq 0$$

$$\Delta_3 = \begin{vmatrix} 2 & 6 & -11 \\ 6 & 20 & -3 \\ 0 & 6 & -1 \end{vmatrix} \neq 0$$

So, the system is inconsistent.

Alternate method: The given system of equations is equivalent to the single matrix equation:

$$AX = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix} = B$$

We shall reduce the coefficient matrix A to triangular form by E-row operations on it and apply the same operations on the right-hand side, i.e. on the matrix B.

Performing $R_2 \to R_2 - 3R_1$, we have

$$\begin{bmatrix} 2 & 6 & 3 \\ 0 & 2 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -1 \end{bmatrix}$$

Performing $R_3 \to R_3 - 3R_2$, we have

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ 30 \\ -91 \end{bmatrix}$$

The last equation of this system is $0x + 0y + 0z = -91$. This shows that the given system is not consistent.

18.22.1 Matrix Method of Solving Non-Homogeneous System of Linear Equations

1. If A is a non-singular matrix, then the system of equations given by $AX = B$ has a unique solution given by $X = A^{-1}B$.
2. If A is a singular matrix and (adj A)$D = 0$, then the system of equations given by $AX = D$ is consistent with infinitely many solutions.
3. If A is a singular matrix and (adj A)$D \neq 0$, then the system of equation given by $AX = D$ is inconsistent.

Your Turn 2

1. Evaluate the determinant $\begin{bmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{bmatrix}$.

   Ans. -37

2. Compute the adjoint of the matrix $A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}$ and verify that A (adj A) = (adj A)$A = |A|I$.

3. Find the inverse of matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$ using elementary transformation.

   Ans. $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$

4. Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 4 & 5 \end{bmatrix}$ in Echelon form and hence find its rank.

   Ans. 2

5. Solve the following system of equations using matrix method:

   $$x + 2y + z = 7$$
   $$x + 3z = 11$$
   $$2x - 3y = 1$$

   Ans. $x = 2, y = 1, z = 3$

6. Solve the following system of homogeneous equations:

   (i) $2x + 3y - z = 0$
   $x - y - 2z = 0$
   $3x + y + 2z = 0$

   (ii) $x + y - 6z = 0$
   $x - y + 2z = 0$
   $-3x + y + 2z = 0$

   Ans. Only trivial solution

18.23 Minor of Any Element of a Matrix

Consider the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

If we leave the row and the column passing through the element a_{ij}, then the second-order determinant thus obtained is called the minor of the element a_{ij} and we shall denote it by M_{ij}. In this way, we can get nine minors corresponding to the nine elements of Δ.

For example,

$$\text{Minor of element } a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$$

$$\text{Minor of element } a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$$

$$\text{Minor of element } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11} \text{ and so on}$$

18.24 Cofactor of Any Element of a Matrix

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called cofactor of the element a_{ij}. We shall denote the cofactor of an element by the C_{ij}. With this notation, cofactor of $a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}$.

18.25 Determinant of Any Matrix

If matrix $A = [a_{ij}]$ is a square matrix of order 'n', then

$$\text{Determinant of } A = \left(\sum_{k=1}^{n} a_{1k}C_{1k}\right) = \left(\sum_{k=1}^{n} a_{2k}C_{2k}\right) = \cdots = \cdots$$
$$= \left(\sum_{k=1}^{n} a_{k1}C_{k1}\right) = \left(\sum_{k=1}^{n} a_{k2}C_{k2}\right) = \cdots = \cdots$$

where C_{ik} represents cofactor of the element of the i^{th} row and the k^{th} column of matrix A.

For 3×3 order matrix A;

$$\begin{aligned} \det A \text{ (or } |A|) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

Thus determinant of a matrix can be obtained by adding the products of elements of any row or column by their cofactors.

Note: If elements of a row (or column) are multiplied by the cofactors of any other row (or column), then the sum of these products is zero. For example,

$$a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23} = 0$$

Illustration 18.11 Evaluate the determinant $\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 5 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Solution: We can do it in two ways.

(a) Expanding along the second row, we have

$$\Delta = -5\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$
$$= -5\,(9-8) - 2\,(6-4) - 1\,(4-3)$$
$$= -5 - 4 - 1 = -10$$

(b) Expanding along the third column, we have

$$\Delta = 4\begin{vmatrix} 5 & -2 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}$$
$$= 4\,(10+2) - 1\,(4-3) + 3\,(-4-15)$$
$$= 48 - 1 - 57 = -10$$

Hence, determinant is -10.

Basic Concepts

1. A determinant of order 3 consisting of three rows and three columns is written as
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$$
2. The numbers a_i, b_i, c_i ($i = 1, 2, 3$) are called the elements of the determinant.
3. The determinant obtained by deleting the i^{th} row and j^{th} column is called the **minor** of element at the i^{th} row and the j^{th} column. The cofactor of this element is $(-1)^{i+j}$ (minor). Note that
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 + b_1B_1 + c_1C_1$$
where A_1, B_1 and C_1 are the cofactors of a_1, b_1 and c_1, respectively.

18.26 Properties of Determinants

1. If two rows (or columns) in a determinant are interchanged, the sign of the determinant changes. For example, by interchanging the two rows of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, we get the determinant $\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}$.

   But we have
$$\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$
2. If the numbers in one row are added m times the numbers in another row, the value of the determinant remains unaltered. For example,
$$\begin{vmatrix} a_1 + ma_2 & b_1 + mb_2 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$
   This rule can be extended to more number of rows for higher order determinants.
3. If rows and columns are interchanged, the value of the determinant remains unaltered. For example,
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
   Another way of saying this is that it makes no difference if we reflect the numbers of the determinant in the line of the principal diagonal. This means that any statement that can truly be made about rows in particular results (1) and (2) can equally well be made about columns.
4. If all the numbers in any row are zeros, the value of the determinant is zero. For example,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$
5. If two rows are identical, the value of the determinant is zero. For example,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$$

6. If the elements of a row are multiplied by any number m, the determinant is multiplied by m. For example,

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

7. **Row–column operations:** The value of determinant remains unchanged when any row (or column) is multiplied by a number or any expression and then added or subtracted from any other row (or column). That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+ma_2-na_3 & a_2 & a_3 \\ b_1+mb_2-nb_3 & b_2 & b_3 \\ c_1+mc_2-nc_3 & c_2 & c_3 \end{vmatrix}$$

The above operation is written using $C_1 \to C_1 + mC_2 - nC_3$ which means C_1 is replaced by $C_1 + mC_2 - nC_3$.

8. Determinant of a triangular matrix is the product of its diagonal elements. For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} = a_1b_2c_3$$

9. If a determinant Δ becomes zero on putting $x = \alpha$, then we say that $(x - \alpha)$ is a factor of Δ. For example, if

$$\Delta = \begin{vmatrix} x & 1 & 3 \\ x^2 & 2 & 9 \\ x^3 & 3 & 27 \end{vmatrix}$$

then

$$\Delta = 0 \text{ if } x = 3$$

Hence, $(x - 3)$ is a factor of Δ.

Illustration 18.12 Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

Solution: Let

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

If b is put equal to a, two rows are exactly alike. Therefore, $\Delta = 0$ when $b = a$. Hence, $(a - b)$ is a factor of Δ [this follows from the factor theorem which states that for $f(x)$, if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$].
Similarly, $(b - c)$ and $(c - a)$ are factors.
Again, Δ is of third degree in a, b and c.
We already know the three linear factors are $(a - b)$, $(b - c)$ and $(c - a)$. If there is another factor, it must be a mere number. Thus

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = N(a-b)(b-c)(c-a), \text{ where } N \text{ is a number}$$

By equating coefficients of bc^2 on both sides, we get $N = 1$. Therefore,

$$\Delta = (a-b)(b-c)(c-a)$$

Alternative method:

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Subtracting the second row from the first and then the third row from the second, we have

$$\Delta = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Now expanding along the first column, we have

$$\Delta = (a-b)(b-c)[(b+c)-(a+b)] = (a-b)(b-c)(c-a)$$

Illustration 18.13 Show that

$$\Delta = \begin{vmatrix} b+c & c & b \\ c & c+a & a \\ b & a & b+a \end{vmatrix} = 4abc$$

Solution:

$$\Delta = \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ c & c+a & a \\ b & a & a+b \end{vmatrix} \text{ by } R_1: R_1 + R_2 + R_3$$

Now take 2 as a common factor and then apply $R_2: R_2 - R_1$ and $R_3: R_3 - R_1$

$$\Delta = 2\begin{vmatrix} b+c & c+a & a+b \\ -b & 0 & -b \\ -c & -c & 0 \end{vmatrix}$$

Now apply $C_2: C_2 - C_1$

$$\Delta = 2\begin{vmatrix} b+c & a-b & a+b \\ -b & b & -b \\ -c & 0 & 0 \end{vmatrix}$$

Now expand through R_3 to get

$$\Delta = 2[(-c)\{-ab + b^2 - ab - b^2\}] = 4abc$$

Illustration 18.14 Show that

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ca & c^2-ab \end{vmatrix} = 0$$

Solution: Applying $C_1: C_1 - C_2$ and $C_2: C_2 - C_3$ we get

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2+c(a-b) & b^2-c^2+a(b-c) & c^2-ab \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b+c & b+c+a & c^2-ab \end{vmatrix}$$

$$= (a-b)(b-c)[(b+c+a)-(a+b+c)] = 0$$

(Expanding along R_1)

Note: If a determinant can be so transformed that two elements in a row or column are made zero, then the determinant can be expanded in terms of that row or column.

Illustration 18.15 Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix}$.

Solution: We have

$$\begin{vmatrix} 1 & bc & b+c \\ 1 & ca & c+a \\ 1 & ab & a+b \end{vmatrix} = \begin{vmatrix} 1 & bc & a+b+c-a \\ 1 & ca & a+b+c-b \\ 1 & ab & a+b+c-c \end{vmatrix}$$

$$= \begin{vmatrix} 1 & bc & a+b+c \\ 1 & ca & a+b+c \\ 1 & ab & a+b+c \end{vmatrix} - \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} + \begin{vmatrix} bc & 1 & a \\ ca & 1 & b \\ ab & 1 & c \end{vmatrix}$$

$$= \begin{vmatrix} bc & 1 & a \\ ca & 1 & b \\ ab & 1 & c \end{vmatrix}, \text{ since the first determinant vanishes}$$

$$= \frac{1}{abc}\begin{vmatrix} abc & a & a^2 \\ abc & b & b^2 \\ abc & c & c^2 \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Illustration 18.16 Without expanding the determinants, prove that $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Solution: The determinant on the left is equal to

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$$= \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & b+c & c+a \\ a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \end{vmatrix} \quad (C_1 \to C_1 - C_2)$$

$$= 2\begin{vmatrix} a & b+c & c+a \\ b & c+a & a+b \\ c & a+b & b+c \end{vmatrix} \quad (C_3 \to C_3 - C_1)$$

$$= 2\begin{vmatrix} a & b+c & c \\ b & c+a & a \\ c & a+b & b \end{vmatrix} \quad (C_2 \to C_2 - C_3)$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Illustration 18.17 Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

Solution: Let Δ stand for the determinant on the left. Then

$$\Delta = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Additional Properties of Determinants:

1. The determinant remains unaltered if its rows are changed into columns and the columns into rows.
2. If all the elements of a row (or column) are zero, then the determinant is zero.
3. If the elements of a row (column) are proportional (or identical) to the elements of any other row (column), then the determinant is zero.
4. The interchange of any two rows (columns) of the determinant changes its sign.
5. If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.
6. A determinant remains unaltered under a column (C_i) operation of the form $C_i + \alpha C_j + \beta C_k$ ($j, k \neq i$) or a row (R_i) operation of the form $R_i + \alpha R_j + \beta R_k$ ($j, k \neq i$).
7. If each element in any row (column) is the sum of r terms, then the determinant can be expressed as the sum of r determinants.
8. If the determinant $\Delta = f(x)$ and $f(a) = 0$, then $(x - a)$ is a factor of the determinant. In other words, if two rows (or two columns) become proportional (identical) for $x = a$, then $(x - a)$ is a factor of determinant. In general, if r rows become identical for $x = a$, then $(x - a)^{r-1}$ is a factor of the determinant.
9. If in a determinant (of order 3 or more) the elements in all the rows (columns) are in AP with same or different common difference, the value of the determinant is zero.
10. The determinant value of an odd-order skew-symmetric determinant is always zero.

18.27 Sum of Determinants

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ be two third-order determinants in which the corresponding second and third columns are identical. Then

$$\Delta_1 + \Delta_2 = \begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix}$$

This fact is evident if we expand all the three determinants in terms of column 1 and compare the results.

Similarly, if $\Delta_3 = \begin{vmatrix} p_1 & q_1 & r_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then

$$\Delta_1 + \Delta_3 = \begin{vmatrix} a_1 + p_1 & b_1 + q_1 & c_1 + r_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Here, we note that the corresponding second and third rows are identical.
Similarly, the determinant

$$\begin{vmatrix} d_1 + e_1 + f_1 & d_2 + e_2 + f_2 & d_3 + e_3 + f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

can be decomposed into the sum of three determinants

$$\begin{vmatrix} d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} f_1 & f_2 & f_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It may be observed that the determinant

$$\begin{vmatrix} a_1 + b_1 & c_1 + d_1 & e_1 + f_1 \\ a_2 + b_2 & c_2 + d_2 & e_2 + f_2 \\ a_3 + b_3 & c_3 + d_3 & e_3 + f_3 \end{vmatrix}$$

can be expressed as sum of $2 \times 2 \times 2 = 8$ determinants.

Illustration 18.18 If $A_k = \begin{vmatrix} 2^{k-1} & x & 2^n - 1 \\ 2(3^{k-1}) & y & 3^n - 1 \\ 3(4^{k-1}) & z & 4^n - 1 \end{vmatrix}$, prove that $\sum_{k=1}^{n} A_k = 0$.

Solution: Observe that all the determinants $A_1, A_2, \ldots, A_n$ have identical second and third columns. Hence,

$$\sum_{k=1}^{n} A_k = \begin{vmatrix} \sum_{k=1}^{n} 2^{k-1} & x & 2^n - 1 \\ \sum_{k=1}^{n} 2(3^{k-1}) & y & 3^n - 1 \\ \sum_{k=1}^{n} 3(4^{k-1}) & z & 4^n - 1 \end{vmatrix}$$

Now,

$$\sum_{k=1}^{n} 2^{k-1} = 1 + 2 + 2^2 + \cdots + 2^{n-1} = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

(sum of n terms of a GP)

$$\sum_{k=1}^{n} 2(3^{k-1}) = 2\{1 + 3 + 3^2 + \cdots + 3^{n-1}\} = \frac{2(3^n - 1)}{3 - 1} = 3^n - 1$$

and

$$\sum_{k=1}^{n} 3(4^{k-1}) = 4^n - 1$$

Hence,

$$\sum_{k=1}^{n} A_k = \begin{vmatrix} 2^n - 1 & x & 2^n - 1 \\ 3^n - 1 & y & 3^n - 1 \\ 4^n - 1 & z & 4^n - 1 \end{vmatrix} = 0 \text{ (since } C_1 = C_3)$$

Illustration 18.19 If $f(r) = \begin{vmatrix} 2r + 1 & {}^nC_r & 1 \\ n^2 + 2n + 1 & 2^n & n + 1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$, $0 \le r \le n$, then prove that $\sum_{r=0}^{n} f(r) = 0$.

Solution: Since R_2 and R_3 are constants (independent of the variable r), we have

$$\sum_{r=0}^{n} f(r) = \begin{vmatrix} \sum_{r=0}^{n} (2r+1) & \sum_{r=0}^{n} {}^nC_r & \sum_{r=0}^{n} 1 \\ n^2 + 2n + 1 & 2^n & n + 1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix}$$

Now,

$$\sum_{r=0}^{n} (2r + 1) = 1 + 3 + \cdots + (2n + 1) = (n + 1)^2$$

$$\sum_{r=0}^{n} {}^nC_r = {}^nC_0 + {}^nC_1 + \cdots + {}^nC_n = 2^n$$

$$\sum_{r=0}^{n} 1 = 1 + 1 + \cdots (n+1) \text{ times} = n + 1$$

Hence

$$\sum_{r=0}^{n} f(r) = \begin{vmatrix} (n+1)^2 & 2^n & n+1 \\ (n+1)^2 & 2^n & n+1 \\ \cos^2(n^2) & \cos^2 n & \cos^2(n+1) \end{vmatrix} = 0 \text{ (since } R_1 = R_2)$$

18.28 Multiplication of Determinants

Two determinants of the same order, i.e. each consisting of the same number of rows and equal number of columns, can be multiplied to give a determinant of the same order. Thus, if A is a 2×2 determinant and B is another 2×2 determinant, $A \times B = C$ is also 2×2 determinant. The multiplication is performed by a method of working the row of A on the columns of B.

The method is as follows: If

$$A = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}; B = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}$$

then $$AB = \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 & a_1\alpha_2 + a_2\beta_2 \\ b_1\alpha_1 + b_2\beta_1 & b_1\alpha_2 + b_2\beta_2 \end{vmatrix}$$

To cite a numerical example for a 3×3 determinant, we have

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 6 \\ 3 & 0 & 2 \end{vmatrix} \times \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1\times2+3\times0+4\times1 & 1\times1+3\times1+4\times2 & 1\times2+3\times3+4\times4 \\ 2\times2+(-1)\times0+6\times1 & 2\times1+(-1)\times1+6\times2 & 2\times2+(-1)\times3+6\times4 \\ 3\times2+0\times0+2\times1 & 3\times1+0\times1+2\times2 & 3\times2+0\times3+2\times4 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 12 & 27 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix}$$

(The first row is obtained by working the first row elements 1, 3, 4, respectively, on 2, 0, 1 the first column; then on 1, 1, 2 the second column; then on 2, 3, 4 the third column. Likewise for the second and the third rows.)

Verification:

$$A = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 6 \\ 3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 \\ 0 & -7 & -2 \\ 0 & -9 & -10 \end{vmatrix} \begin{matrix} \to (R_2 - 2R_1) \\ \to (R_5 - 3R_1) \end{matrix}$$

$$= 70 - 18 = 52$$

$$B = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -3 & -6 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} \to (R_1 - 2R_3)$$

$$= -9 + 6 = -3$$

$$C = \begin{vmatrix} 6 & 12 & 27 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 9 \\ 10 & 13 & 25 \\ 8 & 7 & 14 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 9 \\ 0 & 2 & 2 \\ 8 & 7 & 14 \end{vmatrix} \to R_2 - (R_1 + R_3)$$

$$= 3\begin{vmatrix} 2 & 4 & 5 \\ 0 & 2 & 0 \\ 8 & 7 & 7 \end{vmatrix} = 6(14 - 40) = -156$$

Therefore, $AB = -156 = C$

Multiplication can also be performed row by row, column by row or column by column.

Illustration 18.20 Show that

$$\begin{vmatrix} yz - x^2 & zx - y^2 & xy - z^2 \\ zx - y^2 & xy - z^2 & yz - x^2 \\ xy - z^2 & yz - x^2 & zx - y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix}.$$

where $r^2 = x^2 + y^2 + z^2$ and $u^2 = yz + zx + xy$.

Solution: Consider the determinant

$$\Delta = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

Minor of x is $yz - x^2$ of y is $y^2 - zx$, of z is $xy - z^2$.
The left-hand side determinant in the problem is therefore

$$\begin{vmatrix} X & -Y & Z \\ -Y & Z & -X \\ Z & -X & Y \end{vmatrix}$$

where the capital letters denote the minor of the corresponding small letters. Therefore,

$$\text{LHS} = \begin{vmatrix} X & -Y & Z \\ -Y & Z & -X \\ Z & -X & Y \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} x^2+y^2+z^2 & xy+yz+zx & xz+xy+yz \\ xy+yz+zx & x^2+y^2+z^2 & xy+yz+zx \\ xy+yz+zx & xy+yz+zx & x^2+y^2+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix} \text{ in the notation of the problem}$$

Illustration 18.21 For all values of A, B, C and P, Q, R, show that

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A+R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

Solution: The given determinant is the product of

$$\Delta_1 = \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} \cos P & \cos Q & \cos R \\ \sin P & \sin Q & \sin R \\ 0 & 0 & 0 \end{vmatrix}$$

and $\Delta_1 = \Delta_2 = 0$ and hence $\Delta_1 \cdot \Delta_2 = 0$.

Alternately

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \cos P \begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$+ \sin P \begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= (\cos P)\, A_1 + (\sin P)\, B_1$$

where

$$A_1 = \begin{vmatrix} \cos A & \sin A \sin Q & \sin A \sin R \\ \cos B & \sin B \sin Q & \sin B \sin R \\ \cos C & \sin C \sin Q & \sin C \sin R \end{vmatrix}$$

(using $C_2 \to C_2 - (\cos Q)\, C_1$, $C_3 \to C_3 - (\cos R)\, C_1$)

So

$$A_1 = \sin Q \sin R \begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin C & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix} = 0$$

(second and third columns are identical)

Similarly, it may be proved that $B_1 = 0$.

Product of Two Determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Here, we have multiplied rows by rows. We can also multiply rows by columns or columns by rows, or columns by columns.
Note: If $\Delta = |a_{ij}|$ is a determinant of order n, then the value of the determinant $|A_{ij}|$, where A_{ij} is the cofactor of a_{ij}, is Δ^{n-1}. This is known as *power cofactor formula*.

18.29 Differentiation of Determinants

Following is the differentiation of a determinant whose elements are functions of a variable x. Let

$$F(x) = \begin{vmatrix} f(x) & g(x) \\ h(x) & u(x) \end{vmatrix}$$

Then

$$F(x) = f(x) \times u(x) - g(x) \times h(x)$$

and

$$F'(x) = \frac{d}{dx} F(x)$$

$$= \{f(x) \times u'(x) + u(x) \times f'(x)\} - \{g(x) \times h'(x) + h(x)\, g'(x)\}$$

$$= \begin{vmatrix} f'(x) & g'(x) \\ h(x) & u(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ h'(x) & u'(x) \end{vmatrix}$$

Thus, $F'(x)$ is the sum of two determinants of which the first one is obtained by differentiating the elements of the first row alone and retaining the second row without any change and the second one is obtained by differentiating the elements of the second row.
Similarly, if

$$F(x) = \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$$

then

$$F'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) & h_1'(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2'(x) & g_2'(x) & h_2'(x) \\ f_3(x) & g_3(x) & h_3(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3'(x) & g_3'(x) & h_3'(x) \end{vmatrix}$$

Illustration 18.22 If α is a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$ and $C(x)$ are polynomials of degrees 3, 4 and 5, respectively, then show that

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

is divisible by $f(x)$, where the prime symbol denotes the derivatives.

Solution: Let

$$g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Then

$$g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Now

$$g(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Since two rows are identical, we have $g(\alpha) = 0$

$$g'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

Since two rows are identical, we have $g'(\alpha) = 0$.
Since $g(\alpha) = 0$ and also $g'(\alpha) = 0$, a is a repeated root of $g(x) = 0$
Therefore,

$$g(x) = (x - \alpha)^2 h(x) \tag{18.8}$$

Since α is a repeated root of $f(x) = 0$, we have

$$f(x) = N(x - \alpha)^2 \tag{18.9}$$

where N is some number. From Eqs. (18.8) and (18.9), we find that $g(x)$, i.e. the given determinant is divisible by $f(x)$.

Differentiation of a Determinant

Let $\Delta(x) = \begin{vmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{vmatrix}$. Then

$$\Delta'(x) = \begin{vmatrix} a_1'(x) & b_1'(x) \\ a_2(x) & b_2(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & b_1(x) \\ a_2'(x) & b_2'(x) \end{vmatrix}$$

where the prime symbol denotes the derivative with respect to x.

18.30 Special Determinants

18.30.1 Symmetric Determinant

If the elements of a determinant are such that $a_{ij} = a_{ji}$ (where a_{ij} is the element of i^{th} row and j^{th} column), then the determinant is said to be a symmetric determinant. The elements situated at equal distances from the diagonal are equal both in magnitude and sign. For example,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

18.30.2 Skew-Symmetric Determinant

If $a_{ij} = -a_{ji}$ (where a_{ij} is the element of i^{th} row and j^{th} column), then the determinant is said to be a skew-symmetric determinant, which means that all the diagonal elements are zero and the elements situated at equal distances from the diagonal are equal in magnitude but opposite in sign. The value of a skew-symmetric determinant of odd order is zero. For example,

$$A = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 4 \\ -5 & -4 & 0 \end{bmatrix}$$

$$|A| = 0$$

18.30.3 Circulant Determinants

In these determinants, the elements of the rows (or columns) are in cyclic arrangement. For example,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

$$= -\frac{1}{2}(a+b+c) \times \{(a-b)^2 + (b-c)^2 + (c-a)^2\}$$

Illustration 18.23 Evaluate the determinant $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ and show that it is negative for all positive values of a, b and c.

Solution: Expanding along the first row, we have

$$\Delta = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$$

$$\Delta = a(bc - a^2) - b(b^2 - ca) + c(ab - c^2) = 3abc - a^3 - b^3 - c^3$$

$$= -(a^3 + b^3 + c^3 - 3abc) = -(a + b + c)\{a^2 + b^2 + c^2 - ab - bc - ca\}$$

$$= -\frac{(a + b + c)}{2}\{(a - b)^2 + (b - c)^2 + (c - a)^2\}$$

is negative if a, b and c are positive.

18.31 Solution of System of Linear Equations

18.31.1 Solution of System of Two Linear Equations in Two Unknowns

Consider the system of two linear equations in two unknowns:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Solving the system we get

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}};$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Note: The given equations are consistent and independent if and only if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$.

Illustration 18.24 Solve the system $4x + y = 13$, $3x - 2y = 7$ using determinants.

Solution: The solution requires the values of three determinants. The denominator Δ is formed by writing the coefficients of x and y in order

$$\Delta = \begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -8 - 3 = -11$$

Δ_1, the numerator of x, is formed by replacing the coefficients of x by the constant terms

$$\Delta_1 = \begin{vmatrix} 13 & 1 \\ 7 & -2 \end{vmatrix} = -26 - 7 = -33$$

Δ_2, the numerator of y, is formed by replacing the coefficients of y by the constant terms

$$\Delta_2 = \begin{vmatrix} 4 & 13 \\ 3 & 7 \end{vmatrix} = 28 - 39 = -11$$

Then

$$x = \frac{\Delta_1}{\Delta} = \frac{-33}{-11} = 3$$

and

$$y = \frac{\Delta_2}{\Delta} = \frac{-11}{-11} = 1$$

18.31.2 Solution of System of Three Linear Equations in Three Unknowns

Consider the system of three linear equations in three unknowns:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Consider

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

1. If $\Delta \neq 0$, system has unique solution given by

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

That is, system is consistent with independent solution.

2. If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$ then system has infinite many solutions. That is, system is consistent with dependent solution.
3. If $\Delta = 0$ and any of $\Delta_1, \Delta_2, \Delta_3$ are non-zero then the system has no solution. That is, system is inconsistent.

18.31.3 Solution of System of Three Equations in Two Unknowns

The following system of equations

$$a_1x + b_1y + c_1 = 0;\quad a_2x + b_2y + c_2 = 0;\quad a_3x + b_3y + c_3 = 0$$

is consistent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Illustration 18.25 Find those values of c for which the equations $2x + 3y = 3; (c+2)x + (c+4)y = (c+6)$ and $(c+2)^2 x + (c+4)^2 y = (c+6)^2$ are consistent. Also solve the equations for those values of c.

Solution: The condition for consistency is

$$\Rightarrow \begin{vmatrix} 2 & 3 & 3 \\ c+2 & c+4 & c+6 \\ (c+2)^2 & (c+4)^2 & (c+6)^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 3 & 0 \\ -2 & c+4 & 2 \\ -2(2c+6) & (c+4)^2 & 2(2c+10) \end{vmatrix} = 0 \ (C_1 \to C_1 - C_2)$$

$$\Rightarrow (-1)\{(c+4)(2c+10) - (c+4)^2\} - 3\{-2(2c+10) + 2(2c+6)\} = 0$$

$$\Rightarrow c^2 + 8c + 16 - 2c^2 - 18c - 40 + 12c + 60 - 12c - 36 = 0$$

$$\Rightarrow -c^2 - 10c = 0 \Rightarrow c = 0 \text{ or } c = -10$$

For $c = 0$, the three equations are

$$2x + 2y = 3;\ 2x + 4y = 6;\ 4x + 16y = 36$$

and the solution is $x = -3; y = 3$. For $c = -10$, the equations are

$$2x + 3y = 3$$
$$-8x - 6y = -4 \Rightarrow 4x + 3y = 2$$
$$64x - 36y = 16 \Rightarrow 16x + 9y = 4$$

and the corresponding solution is $x = -\dfrac{1}{2}; y = \dfrac{4}{3}$.

18.31.4 Cramer's Rule

Consider the system of n linear equations in n unknowns given by

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Let

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Let D_j be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{vmatrix}$.

Then, if $D \neq 0$, we have

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D}, \dots, x_n = \frac{D_n}{D}$$

When $D = 0$, we have the following cases:

Case 1: If $D = 0$ and the other determinants $D_1 = D_2 = \cdots = D_n = 0$, then system of equation has infinitely many solutions if all cofactors of $D_1, D_2, \dots, D_n$ and D are zero. If any one cofactor of $D_1, D_2, D_3, \dots, D_n$ is non-zero then system has no solution.

Example: $x + 3y + 2z = 1;\ 2x + 6y + 4z = 5;\ 3x + 9y + 6z = 9$

Here, $D_x = D_y = D_z = D = 0$ yet system has no solution whereas

$x + 3y + 2z = 1;\ 2x + 6y + 4z = 2;\ 3x + 9y + 6z = 3$

has infinitely many solutions.

Case 2: If $D = 0$ but any one of the $D_1, D_2, \dots, D_n$ is not equal to zero then the system has no solution, hence is inconsistent.

Cramer's Rule

If

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

then solution of linear equations $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$ and $a_3x + b_3y + c_3z = d_3$ is given by {where $(d_1, d_2, d_3) \neq (0,0,0)$}

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta},\ z = \frac{\Delta_z}{\Delta}$$

where

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\ \Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},$$

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

1. If any of $\Delta_x, \Delta_y, \Delta_z \in R$ and $\Delta \neq 0$, the system of equation will have unique solution and is said to be consistent independent.
2. If $\Delta_x = \Delta_y = \Delta_z = 0$ and Δ is also zero, then the system of equation will have infinitely many solutions and is said to be consistent dependent.
3. If $\Delta_x, \Delta_y, \Delta_z$ are non-zero and Δ is zero, then the system of equations will have no solution and is said to be inconsistent.

Illustration 18.26 Solve the following system using determinants:

$$x + 4y + 4z = 7$$
$$3x + 2y + 2z = 6$$
$$9x + 6y + 2z = 14$$

Solution: The solution requires the values of four determinants: The denominator

$$\Delta = \begin{vmatrix} 1 & 4 & 4 \\ 3 & 2 & 2 \\ 9 & 6 & 2 \end{vmatrix} = 40$$

Δ_1, the numerator of x is

$$\Delta_1 = \begin{vmatrix} 7 & 4 & 4 \\ 6 & 2 & 2 \\ 14 & 6 & 2 \end{vmatrix} = 40$$

Δ_2, the numerator of y is

$$\Delta_2 = \begin{vmatrix} 1 & 7 & 4 \\ 3 & 6 & 2 \\ 9 & 14 & 2 \end{vmatrix} = 20$$

Δ_3, the numerator of z is

$$\Delta_3 = \begin{vmatrix} 1 & 4 & 7 \\ 3 & 2 & 6 \\ 9 & 6 & 14 \end{vmatrix} = 40$$

Then

$$x = \frac{\Delta_1}{\Delta} = \frac{40}{40} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{20}{40} = \frac{1}{2}$$

$$z = \frac{\Delta_3}{\Delta} = \frac{40}{40} = 1$$

18.31.5 System of Homogeneous Linear Equations

Existence of non-trivial solution: If the three equations (homogeneous)

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
and
$$a_3x + b_3y + c_3z = 0$$

are considered then there always exists a solution, i.e. $x = y = z = 0$. This is called *trivial solution*.

If the three equations are to have a solution other than $x = 0 = y = z$, such a solution is known as *non-trivial* solution. The condition required for the existence of such a solution is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Illustration 18.27 Let λ and α be real. Find the set of all values of λ for which the system of linear equations

$$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$$
$$x + (\cos\alpha)y + (\sin\alpha)z = 0$$
$$-x + (\sin\alpha)y - (\cos\alpha)z = 0$$

has a non-trivial solution. For $\lambda = 1$, find all the values of α.

Solution: The condition for the existence of non-trivial solution (trivial solution is $x = y = z = 0$) is

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & \sin\alpha\,(\lambda+1) & \cos\alpha\,(1-\lambda) \\ 0 & \cos\alpha + \sin\alpha & \sin\alpha - \cos\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)\sin\alpha\,(\sin\alpha - \cos\alpha) - (1-\lambda)\cos\alpha\,(\cos\alpha + \sin\alpha) = 0$$

$$\Rightarrow \lambda(\sin^2\alpha + \cos^2\alpha) + \sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha = 0$$

$$\Rightarrow \lambda = \sin 2\alpha + \cos 2\alpha = \sqrt{2}\sin\left(2\alpha + \frac{\pi}{4}\right)$$

$$\Rightarrow -1 \le \frac{\lambda}{\sqrt{2}} \le 1 \Rightarrow -\sqrt{2} \le \lambda \le \sqrt{2}$$

For $\lambda = 1$,

$$\sin\left(2\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

since, $2\alpha + \frac{\pi}{4} = \frac{\pi}{4}$

General solution:

$$2\alpha + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{4}$$

$$2\alpha = n\pi + (-1)^n\frac{\pi}{4} - \frac{\pi}{4}$$

If n is even, $2\alpha = n\pi$

If n is odd, $2\alpha = n\pi - \frac{\pi}{2}$.

Your Turn 3

1. Evaluate the determinant

$$\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

Ans. Zero

2. If $\begin{vmatrix} x & 3 & 6 \\ 3 & 6 & x \\ 6 & x & 3 \end{vmatrix} = \begin{vmatrix} 2 & x & 7 \\ x & 7 & 2 \\ 7 & 2 & x \end{vmatrix} = \begin{vmatrix} 4 & 5 & x \\ 5 & x & 3 \\ x & 4 & 5 \end{vmatrix} = 0$ then what is the value of x?

Ans. $x = -9$

3. If $\Delta = \begin{vmatrix} my+nz & mq+nr & mb+nc \\ kz-mx & kr-mp & kc-ma \\ nx+ky & np+kq & na+kb \end{vmatrix}$ and Δ is the product of two determinants one of which is $\begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$, then find the other one. Also show that $\Delta = 0$.

Ans. $\begin{vmatrix} 0 & m & n \\ -m & 0 & k \\ n & k & 0 \end{vmatrix}$

4. If the equations $x = ay + z$, $y = az + x$ and $z = ax + y$ are consistent having non-trivial solution, then prove that $a^3 + 3a = 0$.

5. If $f(x)=\begin{vmatrix} x^2-x & x^3 & x^4-1 \\ 2x-1 & 3x^2 & 4x^3 \\ 2 & 6x & 12x^2 \end{vmatrix}$, then find the coefficient of x in $f(x)$. **Ans.** 6

Additional Solved Examples

1. Determine the values of α, β, γ when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.

Solution: Let

$$A=\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \Rightarrow A'=\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

Given A is orthogonal. Then $AA'=I$. Hence

$$\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}\begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2 \\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 \\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we have

$$4\beta^2+\gamma^2=1 \quad (1)$$
$$2\beta^2-\gamma^2=0 \quad (2)$$
$$\alpha^2+\beta^2+\gamma^2=1 \quad (3)$$

From Eqs. (1) and (2), we get

$$6\beta^2=1 \Rightarrow \beta^2=\frac{1}{6}$$

So,

$$\gamma^2=\frac{1}{3}$$

From Eq. (3),

$$\alpha^2=1-\beta^2-\gamma^2=1-\frac{1}{6}-\frac{1}{3}=\frac{1}{2}$$

Hence,

$$\alpha=\pm\frac{1}{\sqrt{2}},\ \beta=\pm\frac{1}{\sqrt{6}} \text{ and } \gamma=\pm\frac{1}{\sqrt{3}}$$

2. Show that the product of two triangular matrices is itself triangular.

Solution: Let $A=[a_{ij}]_{n\times n}$ and $B=[b_{jk}]_{n\times n}$ be two triangular matrices. Then $a_{ij}=0$ when $i>j$. Also

$$b_{jk}=0 \text{ when } j>k$$

Let $AB=[c_{ik}]_{n\times n}$. Then, $c_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}$.

Suppose that $i>k$:

(1) If $j<i$, then $a_{ij}=0$ and therefore $c_{ik}=0$.

(2) If $i<j$, then $j>k$ because $i>k$. In this case, $b_{jk}=0$.

and therefore $c_{ik}=0$.

Thus, $c_{ik}=0$ whenever $i>k$.

Hence, the matrix AB is also a triangular matrix.

3. If $A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2-4A-5I=0$, where I and 0 are the unit matrix and the null matrix of order 3, respectively. Use this result to find A^{-1}.

Solution: Given

$$A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Therefore,

$$A^2=A\cdot A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

So,

$$A^2-4A-5I=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}-4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}-5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}-4\begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}-5\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$=\begin{bmatrix} 9-4-5 & 8-8-0 & 8-8-0 \\ 8-8-0 & 9-4-5 & 8-8-0 \\ 8-8-0 & 8-8-0 & 9-4-5 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore,

$$A^2-4A-5I=0 \Rightarrow 5I=A^2-4A$$

By multiplying by A^{-1}, we get

$$5A^{-1}=A-4I$$

$$=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}-\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 1-4 & 2-0 & 2-0 \\ 2-0 & 1-4 & 2-0 \\ 2-0 & 2-0 & 1-4 \end{bmatrix}=\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Therefore,

$$A^{-1}=\frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}=\begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$$

4. Find a square matrix A of order 2×2 such that $A^2=I_2$.

Solution: Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the required matrix. Then, $A^2=I$. So

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing respective entries we get

$$a^2+bc=1 \quad (1)$$
$$b+bd=0 \quad (2)$$
$$ac+cd=0 \quad (3)$$
$$cb+d^2=1 \quad (4)$$

These must hold simultaneously.

If $\alpha + d = 0$, the above four equations hold simultaneously if $d=-a$ and $a^2+bc=1$.

Hence, one possible square root of I is $A=\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ where α, β, γ are the three numbers related by the condition $\alpha^2+\beta\gamma=1$.

If $a+d \neq 0$, then the above four equations hold simultaneously if $b=0, c=0, a=1, d=1$ or if $b=0, c=0, a=-1, d=-1$.

Hence, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, i.e. $\pm I$ are the values of A.

5. Show that every square matrix A can be uniquely expressed as $P+iQ$, where P and Q are Hermitian matrices.

Solution: Let

$$P=\frac{1}{2}(A+A^\theta) \text{ and } Q=\frac{1}{2i}(A-A^\theta)$$

Then

$$A=P+iQ \quad (1)$$

Now

$$P^\theta = \left\{\frac{1}{2}(A+A^\theta)\right\}^\theta = \frac{1}{2}(A+A^\theta)^\theta$$
$$= \frac{1}{2}\{A^\theta+(A^\theta)^\theta\} = \frac{1}{2}(A^\theta+A) = \frac{1}{2}(A+A^\theta) = P$$

Therefore, P is a Hermitian matrix.

Also

$$Q^\theta = \left\{\frac{1}{2i}(A-A^\theta)\right\}^\theta = \left(\overline{\frac{1}{2i}}\right)(A-A^\theta)^\theta = -\frac{1}{2i}\{A^\theta-(A^\theta)^\theta\}$$
$$= -\frac{1}{2i}(A^\theta-A) = \frac{1}{2i}(A-A^\theta) = Q$$

Therefore, Q is also a Hermitian matrix.

Thus, A can be expressed in the form (1).

For A to be unique, let $A=R+iS$, where R and S are both Hermitian matrices. We have

$$A^\theta = (R+iS)^\theta = R^\theta + (iS)^\theta = R^\theta + \bar{i}\,S^\theta = R^\theta - iS^\theta$$
$$= R - iS \text{ (since } R \text{ and } S \text{ are both Hermitian matrices)}$$

Therefore,

$$A+A^\theta = (R+iS)+(R-iS) = 2R$$
$$\Rightarrow R = \frac{1}{2}(A+A^\theta) = P$$

Also,

$$A-A^\theta = (R+iS)-(R-iS) = 2iS$$
$$\Rightarrow S = \frac{1}{2i}(A-A^\theta) = Q$$

Hence, expression (1) for A is unique.

6. Discuss for all values of λ, the system of equations $x+y+4z=6$, $x+2y-2z=6$, $x\lambda+y+z=6$ with regards to existence and nature of solutions.

Solution: The matrix form of the given system is

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ \lambda & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

The given system of equations will have a unique solution iff the coefficient matrix is non-singular. Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - \lambda R_1$, we get

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 1-\lambda & 1-4\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6-6\lambda \end{bmatrix} \quad (1)$$

Therefore, the coefficient matrix will be non-singular iff

$$1-4\lambda+6-6\lambda \neq 0 \Rightarrow \lambda \neq \frac{7}{10}$$

Thus, the given system will have a unique solution if $\lambda \neq \frac{7}{10}$.

In case $\lambda = \frac{7}{10}$, Eq. (1) becomes

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 3/10 & -18/10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 18/10 \end{bmatrix}$$

Using $R_3 \to R_3 - \frac{3}{10}R_2$ gives

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 18/10 \end{bmatrix}$$

This shows that the equations are not consistent in this case.

7. Let a, b, c be positive real numbers with $abc=1$.

Let $A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$. If $A^TA=I$ where A^T is the transpose of A and I is the identity matrix, then determine the value of $a^3+b^3+c^3$.

Solution: We have

$$\det(A^TA) = \det(I) = 1$$

This implies that

$$[\det(A)]^2 = 1 \text{ or } \det(A) = \pm 1$$

Now,

$$\det(A) = 3abc - (a^3+b^3+c^3)$$
$$= -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] < 0$$

as a, b, c are positive.

Hence,

$$\det(A) = -1$$
$$\Rightarrow 3abc - (a^3+b^3+c^3) = -1$$
$$\Rightarrow a^3+b^3+c^3 = 4$$

8. If M is a 3×3 matrix, where $M^TM = I$ and $\det(M) = 1$, then prove that $\det(M - I) = 0$.

Solution:

$$(M-I)^T = M^T - I = M^T - M^TM = M^T(I-M)$$
$$\Rightarrow |(M-I)^T| = |M-I| = |M^T|\,|I-M| = |I-M|$$
$$\Rightarrow |M-I| = 0$$

Alternate method

$$\det(M-I) = \det(M-I)\det(M^T) = \det(MM^T - M^T)$$
$$= \det(I - M^T) = -\det(M^T - I) = -\det(M-I)^T = -\det(M-I)$$
$$\Rightarrow \det(M-I) = 0$$

9. $A = \begin{bmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{bmatrix}$, $B = \begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$. If there is a vector matrix X, such that $AX = U$ has infinitely many solutions, then prove that $BX = V$ cannot have a unique solution. If $afd \neq 0$ then prove that $BX = V$ has no solution.

Solution: $AX = U$ has infinite solutions. This implies $|A| = 0$ which gives

$$\begin{vmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{vmatrix} = 0 \Rightarrow ab = 1 \text{ or } c = d$$

and

$$|A_1| = \begin{vmatrix} a & 0 & f \\ 1 & c & g \\ 1 & d & h \end{vmatrix} = 0 \Rightarrow g = h$$

$$|A_2| = \begin{vmatrix} a & f & 1 \\ 1 & g & b \\ 1 & h & b \end{vmatrix} = 0 \Rightarrow g = h$$

$$|A_3| = \begin{vmatrix} f & 0 & 1 \\ g & c & b \\ h & d & b \end{vmatrix} = 0$$

$$\Rightarrow g = h, c = d \Rightarrow c = d \text{ and } g = h$$

Now $BX = V$.

$$|B| = \begin{vmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{vmatrix} = 0 \text{ (since } C_2 \text{ and } C_3 \text{ are equal)}$$

This means $BX = V$ has no unique solution.

$$|B_1| = \begin{vmatrix} a^2 & 1 & 1 \\ 0 & d & c \\ 0 & g & h \end{vmatrix} = 0 \qquad \text{(since } c = d, g = h\text{)}$$

$$|B_2| = \begin{vmatrix} a & a^2 & 1 \\ 0 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf = a^2df \qquad \text{(since } c = d\text{)}$$

$$|B_3| = \begin{vmatrix} a & 1 & a^2 \\ 0 & d & 0 \\ f & g & 0 \end{vmatrix} = a^2df$$

If $adf \neq 0$, then $|B_2| = |B_3| \neq 0$. Hence, no solution exists.

10. Show that if A and B are symmetric and commute, then

(a) $A^{-1}B$ **(b)** AB^{-1}

(c) $A^{-1}B^{-1}$ are symmetric.

Solution:

(a) Since A and B commute: $AB = BA$

Pre- and post-multiplying both sides by A^{-1}, we get

$$A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1}$$
$$\Rightarrow (A^{-1}A)(BA^{-1}) = A^{-1}B(AA^{-1}) \qquad \text{(by associativity)}$$
$$\Rightarrow I(BA^{-1}) = (A^{-1}B)I$$
$$\Rightarrow BA^{-1} = A^{-1}B$$

Now,

$$(A^{-1}B)' = (BA^{-1})' = (A^{-1})'B' \qquad \text{(by reversal law)}$$
$$= A^{-1}B \text{ [as } B' = B \text{ (symmetric) and } (A^{-1})' = (A')^{-1} = A^{-1}]$$

Hence, $A^{-1}B$ is symmetric.

(b) Pre-and post-multiplying by B^{-1}, we get

$$B^{-1}(AB)B^{-1} = B^{-1}(BA)B^{-1}$$
$$\Rightarrow (B^{-1}A)BB^{-1} = B^{-1}B(AB^{-1})$$
$$\Rightarrow B^{-1}A = AB^{-1}$$

Now,

$$(AB^{-1})' = (B^{-1}A)' = A'B^{-1})'$$
$$= AB^{-1} \text{ [as } A = A' \text{ (symmetric) and } (B^{-1})' = (B')^{-1} = B^{-1}]$$

Hence, AB^{-1} is symmetric.

(c) Since A and B are symmetric, we have

$$AB = BA$$
$$\Rightarrow (BA)^{-1} = (AB)^{-1}$$
$$\Rightarrow A^{-1}B^{-1} = B^{-1}A^{-1}$$
$$\Rightarrow (A^{-1}B^{-1})' = (B^{-1}A^{-1})' = (A^{-1})' \cdot (B^{-1})' = A^{-1}B^{-1}$$
$$\text{[as } (A^{-1})' = A^{-1} \text{ and } (B^{-1})' = B^{-1}]$$

Hence, $A^{-1}B^{-1}$ is symmetric.

11. Let $a > 0$, $d > 0$. Find the value of the determinant

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

Solution:

$$\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

Take common $\dfrac{1}{a(a+d)(a+2d)}$ from R_1

$$\frac{1}{(a+d)(a+2d)(a+3d)} \text{ from } R_2$$

$$\frac{1}{(a+2d)(a+3d)(a+4d)} \text{ from } R_3$$

$$\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & a+3d & a+d \\ (a+3d)(a+4d) & a+4d & a+2d \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_2$ in Δ' where

$$\Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta',$$

we get

$$\Delta' = 4d^4$$

$$\Rightarrow \Delta = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

12. Given that $a = \cos\theta + i\sin\theta$, $b = \cos 2\theta - i\sin 2\theta$, $c = \cos 3\theta + i\sin 3\theta$ and if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, show that $\theta = 2n\pi$, $n \in Z$.

Solution:

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -\frac{1}{2}(a+b+c)\,((a-b)^2+(b-c)^2+(c-a)^2) = 0$$

$$\Rightarrow a+b+c = 0 \text{ or } a = b = c$$

If $a+b+c=0$, we have

$$\cos\theta + \cos 2\theta + \cos 3\theta = 0,$$
$$\sin\theta - \sin 2\theta + \sin 3\theta = 0$$

This gives

$$\cos 2\theta(2\cos\theta + 1) = 0$$

and $\quad \sin 2\theta(1 - 2\cos\theta) = 0 \quad (1)$

which is not possible as $\cos 2\theta = 0$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq 1/2$ and $\cos\theta = -1/2$ gives $\sin 2\theta \neq 0$, $\cos\theta \neq 1/2$.

Therefore, Eq. (1) does not hold simultaneously, and so

$$a+b+c \neq 0$$

Therefore,

$$a = b = c \text{ or } e^{i\theta} = e^{-2i\theta} = e^{3i\theta}$$

which is satisfied only by $e^{i\theta} = 1$, i.e. $\cos\theta = 1$, $\sin\theta = 0$. So $\theta = 2n\pi$, $n \in Z$.

13. If x, y, z are not all zero and if $ax + by + cz = 0$, $bx + cy + az = 0$, $cx + ay + bz = 0$, prove that $x{:}y{:}z = 1{:}1{:}1$ or $1{:}\omega{:}\omega^2$ or $1{:}\omega^2{:}\omega$, where ω is the complex cube roots of unity.

Solution: For non-trivial solution,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c)\,(a^2+b^2+c^2-ab-bc-ca) = 0 \quad (1)$$
$$\Rightarrow (a+b+c)\,(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c) = 0 \quad (2)$$

Now, by using Cramer's rule

$$a+b+c = 0 \Rightarrow x:y:z = 1:1:1$$

or $\quad a+b\omega+c\omega^2 = 0 \Rightarrow x:y:z = 1:\omega:\omega^2$

or $\quad a+b\omega^2+c\omega = 0 \Rightarrow x:y:z = 1:\omega^2:\omega$

14. If $\begin{vmatrix} x^3+1 & x^2 & x \\ y^3+1 & y^2 & y \\ z^3+1 & z^2 & z \end{vmatrix} = 0$ and x, y, z are all different, then prove that $xyz = -1$.

Solution:

$$\text{LHS} = \begin{vmatrix} x^3 & x^2 & x \\ y^3 & y^2 & y \\ z^3 & z^2 & z \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} = xyz\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$$

$$= (xyz+1)\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$$

Operate $R_3 - R_1$ and $R_2 - R_1$ on the determinants. We get

$$\text{LHS} = (xyz+1)\begin{vmatrix} x^2 & x & 1 \\ y^2-x^2 & y-x & 0 \\ z^2-x^2 & z-x & 0 \end{vmatrix}$$

$$= (xyz+1)(x+y)(x-z)(z-x) = 0, \text{ given } x \neq y \neq z$$
$$\Rightarrow xyz+1 = 0 \Rightarrow xyz = -1$$

15. If $a^2+b^2+c^2 = 1$, prove that

$$\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

is independent of a, b and c.

Solution: Let

$$\Delta = \begin{vmatrix} a^2+(b^2+c^2)\cos\phi & ab(1-\cos\phi) & ac(1-\cos\phi) \\ ba(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & bc(1-\cos\phi) \\ ca(1-\cos\phi) & cb(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Multiplying C_1, C_2, C_3 by a, b, c, respectively, and taking a, b, c common from R_1, R_2, R_3, respectively, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2+(b^2+c^2)\cos\phi & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2(1-\cos\phi) & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ we have

$$\Delta = \begin{vmatrix} a^2+b^2+c^2 & b^2(1-\cos\phi) & c^2(1-\cos\phi) \\ a^2+b^2+c^2 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi) \\ a^2+b^2+c^2 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi \end{vmatrix}$$

Taking $a^2+b^2+c^2$ common from C_1, we get

$$\Delta=(a^2+b^2+c^2)\begin{vmatrix}1 & b^2(1-\cos\phi) & c^2(1-\cos\phi)\\ 1 & b^2+(c^2+a^2)\cos\phi & c^2(1-\cos\phi)\\ 1 & b^2(1-\cos\phi) & c^2+(a^2+b^2)\cos\phi\end{vmatrix}$$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, we get

$$\Delta=1\begin{vmatrix}1 & b^2(1-\cos\phi) & c^2(1-\cos\phi)\\ 0 & (b^2+c^2+a^2)\cos\phi & 0\\ 0 & 0 & (c^2+a^2+b^2)\cos\phi\end{vmatrix}$$

(since $a^2+b^2+c^2=1$)

$=(a^2+b^2+c^2)^2\cos^2\phi$ {by property since all elements are zero below leading diagonal}

$=1^2\cos^2\phi=\cos^2\phi$, which is independent of a, b and c.

16. If the system of equations $x=cy+bz$, $y=az+cx$ and $z=bx+ay$ has a non-zero solution and at least one of a, b, c is a proper fraction, prove that $a^2+b^2+c^2<3$ and $abc>-1$.

Solution: We are given that system of equations has non-trivial solution. So

$$\begin{vmatrix}1 & -c & -b\\ -c & 1 & -a\\ -b & -a & 1\end{vmatrix}=0$$

$$\Rightarrow 1(1-a^2)+c(-c-ab)-b(ac+b)=0$$

$$\Rightarrow 1-2abc-a^2-b^2-c^2=0 \text{ or } a^2+b^2+c^2+2abc=1$$

$$\Rightarrow a^2+b^2c^2+2abc=1-b^2-c^2+b^2c^2=(1-b^2)(1-c^2)$$

$$\Rightarrow (a+bc)^2=(1-b^2)(1-c^2)$$

Similarly,

$$(b+ac)^2=(1-a^2)(1-b^2) \text{ and } (c+ab)^2=(1-a^2)(1-b^2)$$

Hence, $(1-a^2)$, $(1-b^2)$ and $(1-c^2)$ all have same sign. Since at least one of them is proper fraction, it implies all of them are positive. So

$$1-a^2>0,\ 1-b^2>0,\ 1-c^2>0$$

$$\Rightarrow a^2+b^2+c^2<3 \Rightarrow 1-2abc<3 \Rightarrow abc>-1$$

17. If numbers n, r are two different positive integers such that $n\ge r+2$ and it is given that

$$\Delta(n,r)=\begin{vmatrix}{}^nC_r & {}^nC_{r+1} & {}^nC_{r+2}\\ {}^{n+1}C_r & {}^{n+1}C_{r+1} & {}^{n+1}C_{r+2}\\ {}^{n+2}C_r & {}^{n+2}C_{r+1} & {}^{n+2}C_{r+2}\end{vmatrix}$$

then show that

$$\Delta(n,r)=\frac{{}^{n+2}C_3}{{}^{r+2}C_3}\Delta(n-1,r-1)$$

Solution: We know that

$${}^mC_k=\frac{m}{k}\,{}^{m-1}C_{k-1}$$

Now

$$\Delta(n,r)=\begin{vmatrix}\frac{n}{r}\,{}^{n-1}C_{r-1} & \frac{n}{r+1}\,{}^{n-1}C_r & \frac{n}{r+2}\,{}^{n-1}C_{r+1}\\ \frac{n+1}{r}\,{}^nC_{r-1} & \frac{n+1}{r+1}\,{}^nC_r & \frac{n+2}{r+2}\,{}^nC_{r+1}\\ \frac{n+2}{r}\,{}^{n+1}C_{r-1} & \frac{n+2}{r+1}\,{}^{n+1}C_r & \frac{n+2}{r+2}\,{}^{n+1}C_{r+1}\end{vmatrix}$$

$$=\frac{n(n+1)(n+2)}{r(r+1)(r+2)}\begin{vmatrix}{}^{n-1}C_{r-1} & {}^{n-1}C_r & {}^{n-1}C_{r+1}\\ {}^nC_{r-1} & {}^nC_r & {}^nC_{r+1}\\ {}^{n+1}C_{r-1} & {}^{n+1}C_r & {}^{n+1}C_{r+1}\end{vmatrix}$$

$$=\frac{{}^{n+2}C_3}{{}^{r+2}C_3}\Delta(n-1,\ r-1)$$

18. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then show that

$$\begin{vmatrix}\vec{a} & \vec{b} & \vec{c}\\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\end{vmatrix}=\vec{0}$$

Solution: Since $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar there must exist k_1, k_2, k_3 not all zero (say $k_1\ne 0$) such that $k_1\vec{a}+k_2\vec{b}+k_3\vec{c}=\vec{0}$. Now by operating $C_1\to k_1C_1+k_2C_2+k_3C_3$, we have

$$\text{LHS}=\frac{1}{k_1}\begin{vmatrix}k_1\vec{a}+k_2\vec{b}+k_3\vec{c} & \vec{b} & \vec{c}\\ k_1\vec{a}\cdot\vec{a}+k_2\vec{a}\cdot\vec{b}+k_3\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ k_1\vec{b}\cdot\vec{a}+k_2\vec{b}\cdot\vec{b}+k_3\vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\end{vmatrix}\ (k_1\ne 0)$$

$$=\frac{1}{k_1}\begin{vmatrix}k_1\vec{a}+k_2\vec{b}+k_3\vec{c} & \vec{b} & \vec{c}\\ \vec{a}\cdot[k_1\vec{a}+k_2\vec{b}+k_3\vec{c}] & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{b}\cdot[k_1\vec{a}+k_2\vec{b}+k_3\vec{c}] & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\end{vmatrix}$$

$$=\frac{1}{k_1}\begin{vmatrix}\vec{0} & \vec{b} & \vec{c}\\ \vec{0} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c}\\ \vec{0} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c}\end{vmatrix}=\vec{0}$$

19. Let a, b, c be real numbers with $a^2+b^2+c^2=1$. Show that the equation

$$\begin{vmatrix}ax-by-c & bx+ay & cx+a\\ bx+ay & -ax+by-c & cy+b\\ cx+a & cy+b & -ax-by+c\end{vmatrix}=0$$

represents a straight line.

Solution: Given

$$\begin{vmatrix}ax-by-c & bx+ay & cx+a\\ bx+ay & -ax+by-c & cy+b\\ cx+a & cy+b & -ax-by+c\end{vmatrix}=0$$

Using $C_1\to aC_1+bC_2+cC_3$ gives

$$\begin{vmatrix}(a^2+b^2+c^2)x & bx+ay & cx+a\\ (a^2+b^2+c^2)y & -ax+by-c & cy+b\\ (a^2+b^2+c^2) & cy+b & -ax-by+c\end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix}x & bx+ay & cx+a\\ y & -ax+by-c & cy+b\\ 1 & cy+b & -ax-by+c\end{vmatrix}=0$$

Applying $C_2\to C_2-bC_1$ and $C_3\to C_3-cC_1$, we get

$$\Delta=\begin{vmatrix}x & ay & a\\ y & -ax-c & b\\ 1 & cy & -ax-by\end{vmatrix}=0$$

Applying $R_3 \to R_3 + xR_1 + yR_2$, we get

$$\Delta = \begin{vmatrix} x & ay & a \\ y & -ax-c & b \\ x^2+y^2+1 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (x^2+y^2+1)(aby + a^2x + ac) = 0$$
$$\Rightarrow ax + by + c = 0$$

as $(x^2+y^2+1) \neq 0$ being sum of three positive numbers.

20. If $f(x)$ is a polynomial of degree < 3, then prove that

$$\begin{vmatrix} 1 & a & \dfrac{f(a)}{(x-a)} \\ 1 & b & \dfrac{f(b)}{(x-b)} \\ 1 & c & \dfrac{f(c)}{x-c} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \frac{f(x)}{(x-a)(x-b)(x-c)}$$

Solution:

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)} \qquad (1)$$

$$A = -\frac{f(a)}{(a-b)(c-a)},\ B = -\frac{f(b)}{(a-b)(b-c)} \text{ and } C = -\frac{f(c)}{(b-c)(c-a)}$$

Therefore,

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{\dfrac{(c-b)f(a)}{(x-a)} - \dfrac{(c-a)f(b)}{(x-b)} + \dfrac{(b-a)f(c)}{(x-c)}}{(a-b)(b-c)(c-a)}$$

$$= \begin{vmatrix} 1 & a & \dfrac{f(a)}{(x-a)} \\ 1 & b & \dfrac{f(b)}{(x-b)} \\ 1 & c & \dfrac{f(c)}{(x-c)} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Previous Years' Solved JEE Main/AIEEE Questions

1. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $x \neq 0, y \neq 0$ then D is

(A) divisible by neither x nor y **(B)** divisible by both x and y
(C) divisible by x but not y **(D)** divisible by y but not x

[AIEEE 2007]

Solution: We have

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$$

Therefore, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ imply that

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix} = xy$$

Hence, D is divisible by both x and y.

Hence, the correct answer is option (B).

2. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $|A^2| = 25$, then $|\alpha|$ equals

(A) 5^2 **(B)** 1 **(C)** 1/5 **(D)** 5

[AIEEE 2007]

Solution: We have

$$A^2 = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 25\alpha+5\alpha^2 & 5\alpha+25\alpha^2+5\alpha \\ 0 & \alpha^2 & 5\alpha^2+25\alpha \\ 0 & 0 & 25 \end{bmatrix}$$

$$\Rightarrow |A|^2 = 25(25\alpha^2) = 62525\alpha^{22}$$
$$\Rightarrow 25 = 62525\alpha^{22}$$
$$\Rightarrow |\alpha| = \frac{1}{5}$$

Hence, the correct answer is option (C).

3. Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by tr (A), the sum of diagonal entries of A. Assume that $A^2 = I$.

Statement-1: If $A \neq I$ and $A \neq -I$, then det $A = -1$.
Statement-2: If $A \neq I$ and $A \neq -I$, then tr $(A) \neq 0$.

(A) Statement-1 is false, Statement-2 is true.

(B) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(C) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(D) Statement-1 is true, Statement-2 is false.

[AIEEE 2008]

Solution: Let us consider that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Therefore,

$$A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2 + bc = 1 = bc + d^2;$$

and $\quad (a+d)c = 0 = (a+d)b$

As $A \neq I$, $A \neq -I$, $a = -d$, we have

$$\det A = \begin{vmatrix} \sqrt{1-bc} & b \\ c & -\sqrt{1-bc} \end{vmatrix} = -1 + bc - bc = -1$$

Therefore, Statement-1 is true. However, tr(A) = 0 and therefore Statement-2 is false.

Hence, the correct answer is option (D).

4. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$. Then $a^2 + b^2 + c^2 + 2abc$ is equal to

(A) 2 **(B)** -1 **(C)** 0 **(D)** 1

[AIEEE 2008]

Solution: The system of equations $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ have non-trivial solution if

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow 1(1-a^2) + c(-c-ab) - b(ca+b) = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$$

Hence, the correct answer is option (D).

5. Let A be a square matrix all of whose entries are integers. Then which one of the following is true?

(A) If det(A) = ± 1, then A^{-1} exists but all its entries are not necessarily integers.

(B) If det(A) ≠ ± 1, then A^{-1} exists and all its entries are non-integers.

(C) If det(A) = ± 1, then A^{-1} exists and all its entries are integers.

(D) If det(A) = ± 1, then A^{-1} need not exist.

[AIEEE 2008]

Solution: It is given that each entry of A is integer. Therefore, the cofactor of every entry is an integer and so each entry in the adjoint of matrix A is an integer. So

$$\det A = \pm 1 \text{ and } A^{-1} = \frac{1}{\det(A)}(\text{adj}\, A)$$

This implies that all entries in A^{-1} are integers.

Hence, the correct answer is option (C).

6. Let a, b, c be such that $b(a + c) \neq 0$. If

$$\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0, \text{ then the}$$

value of 'n' is

(A) zero **(B)** any even integer

(C) any odd integer **(D)** any integer

[AIEEE 2009]

Solution: We have

$$\text{LHS} = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix}$$

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$$

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix}$$

$$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+2} \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} \quad (C_1 \leftrightarrow C_2)$$

Now,

$$\text{LHS} = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} \left[1 + (-1)^{n+2}\right] = 0$$

$$\Rightarrow 4b(a+c)(1+(-1)^{n+2}) = 0$$

$$\Rightarrow 4b(a+c) \neq 0 \Rightarrow 1 + (-1)^{n+2} = 0$$

which is true only if $n + 2$ is odd, that is, n is odd integer.

Hence, the correct answer is option (C).

7. Let A be a 2 × 2 matrix

Statement-1: adj(adj A)= A

Statement-2: $|\text{adj}\, A| = |A|$

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(C) Statement-1 is true, Statement-2 is false.

(D) Statement-1 is false, Statement-2 is true.

[AIEEE 2009]

Solution: We have

$$|\text{adj}\, A| = |A|^{n-1} = |A|^{2-1} = |A|$$

Now, $$\text{adj}\,(\text{adj}\, A) = |A|^{n-2} A = |A|^0 A = A$$

Hence, the correct answer is option (B).

8. The number of 3 × 3 non-singular matrices, with four entries as 1 and all other entries as 0, is

(A) 5 **(B)** 6

(C) at least 7 **(D)** less than 4

[AIEEE 2010]

Solution: Let us consider the following matrix:

$$\begin{bmatrix} 1 & X & X \\ X & 1 & X \\ X & X & 1 \end{bmatrix}$$

which are six non-singular matrices because six blanks (i.e. X) can be filled by five zeros and one 1. In the same manner, we have the matrix

$$\begin{bmatrix} X & X & 1 \\ X & 1 & X \\ 1 & X & X \end{bmatrix}$$

which are six non-singular matrices. Therefore, in the required case, there are more than 7.

Hence, the correct answer is option (C).

9. Let A be a 2 × 2 matrix with non-zero entries and let $A^2 = I$, where I is 2 × 2 identity matrix. Define tr (A) = sum of diagonal elements of A and $|A|$ = determinant of matrix A.

Statement-1: $\text{tr}(A) = 0$

Statement-2: $|A| = 1$

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is false.
(C) Statement-1 is false, Statement-2 is true.
(D) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

[AIEEE 2010]

Solution: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \neq 0$$

Therefore,

$$A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

$$\Rightarrow a^2 + bc = 1,\ bc + d^2 = 1 \text{ and } ab + bd = ac + cd = 0$$

Therefore, $c \neq 0$ and $b \neq 0$ $\Rightarrow a + d = 0$. Trace $A = a + d = 0$. Thus, $|A| = ad - bc = -a^2 - bc = -1$.

Hence, the correct answer is option (B).

10. The number of values of k for which the linear equations

$$4x + ky + 2z = 0$$
$$kx + 4y + z = 0$$
$$2x + 2y + z = 0$$

possess a non-zero solution is

(A) 2 **(B)** 1
(C) zero **(D)** 3

[AIEEE 2011]

Solution:

$$\begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0 \Rightarrow k^2 - 6k + 8 = 0 \Rightarrow k = 4 \text{ and } 2$$

Hence, the correct answer is option (A).

11. Let A and B be two symmetric matrices of order 3.

Statement-1: $A(BA)$ and $(AB)A$ are symmetric matrices.

Statement-2: AB is symmetric matrix if matrix multiplication of A and B is commutative.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is false.
(C) Statement-1 is false, Statement-2 is true.
(D) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

[AIEEE 2011]

Solution: We have, $A^T = A$ and $B^T = B$. Therefore,

$$(A(BA))^T = (BA)^T A^T = (A^T B^T)A = (AB)A = A(BA)$$

and $$((AB)A)^T = A^T (AB)^T = A(B^T A^T) = A(BA) = (AB)A$$

Therefore, Statement-1 is correct.

Also $(AB)^T = B^T A^T = BA = AB$ (since AB is commutative)

Therefore, Statement-2 is also correct, but it is not a correct explanation of Statement-1.

Hence, the correct answer is option (A).

12. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then $u_1 + u_2$ is equal to

(A) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ **(B)** $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

(C) $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ **(D)** $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

[AIEEE 2012]

Solution: We have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Let us consider that $u_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$; $u_2 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$. Therefore

$$Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Therefore,

$$u_1 + u_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Hence, the correct answer is option (D).

13. Let P and Q be 3×3 matrices with $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then determinant of $(P^2 + Q^2)$ is equal to

(A) -2 **(B)** 1 **(C)** 0 **(D)** -1

[AIEEE 2012]

Solution: We have $P^3 = Q^3$. Therefore,

$$P^3 - P^2Q = Q^3 - Q^2P \Rightarrow P^2(P - Q) = Q^2(Q - P)$$
$$\Rightarrow P^2(P - Q) + Q^2(P - Q) = O$$
$$\Rightarrow (P^2 + Q^2)(P - Q) = O$$
$$\Rightarrow |P^2 + Q^2| = 0$$

Hence, the correct answer is option (C).

14. If $\alpha, \beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1 + f(1) & 1 + f(2) \\ 1 + f(1) & 1 + f(2) & 1 + f(3) \\ 1 + f(2) & 1 + f(3) & 1 + f(4) \end{vmatrix} = K(1 - \alpha)^2 (1 - \beta)^2 (\alpha - \beta)^2,$$

then K is equal to

(A) 1 **(B)** –1 **(C)** $\alpha\beta$ **(D)** $\frac{1}{\alpha\beta}$

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$$

$$= \{(1-\alpha)(1-\beta)(\alpha-\beta)^2\}$$

On comparison with the given equation, we get $K = 1$.

Hence, the correct answer is option (A).

15. If A is a 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' equals

(A) B^{-1} **(B)** (B^{-1}) **(C)** $I+B$ **(D)** I

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$BB' = A^{-1}A'(A^{-1}A')' = A^{-1}A'A(A^{-1})'$$

$$= A^{-1}AA'(A^{-1})' = IA'(A^{-1})' = I(A^{-1}A)' = I \cdot I' = I^2 = I$$

Hence, the correct answer is option (D).

16. If a, b, c are non-zero real numbers and if the system of equations

$$(a-1)x = y+z$$
$$(b-1)y = z+x$$
$$(c-1)z = x+y$$

has a non-trivial solution, then $ab+bc+ca$ equals

(A) $a+b+c$ **(B)** abc **(C)** 1 **(D)** –1

[JEE MAIN 2014 (ONLINE SET 1)]

Solution: For the non-trivial solution

$$\begin{vmatrix} 1-a & 1 & 1 \\ 1 & 1-b & 1 \\ 1 & 1 & 1-c \end{vmatrix} = 0$$

$$\Rightarrow (1-a)\{(1-b)(1-c)-1\} - 1(1-c-1) + 1(1-1+b)$$

$$\Rightarrow (1-a)\{1-c-b+bc-1\} + c + b = 0$$

$$\Rightarrow -\cancel{c} - \cancel{b} + bc + ac + ab - abc + \cancel{c} + \cancel{b} = 0$$

$$\Rightarrow ab+bc+ca = abc$$

Hence, the correct answer is option (B).

17. If B is a 3×3 matrix such that $B^2 = 0$, then $\det[(I+B)^{50} - 50B]$ is equal to

(A) 1 **(B)** 2 **(C)** 3 **(D)** 50

[JEE MAIN 2014 (ONLINE SET 1)]

Solution:

$$\det[(I+B)^{50} - 50B] \Rightarrow [(I+B)^{50} = I + 50B]$$

[using induction process $(I+B)^n = I+nB$) (assuming $B^2 = 0$)]

Therefore,

$$\det[(I+B)^{50} - 50B] = \det[I + 50B - 50B] = 1$$

Hence, the correct answer is option (A).

18. Let A be a 3×3 matrix such that

$$A\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then A^{-1} is

(A) $\begin{bmatrix} 3 & 1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **(B)** $\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ **(D)** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$

[JEE MAIN 2014 (ONLINE SET 2)]

Solution:

$$A\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Applying $C_1 \leftrightarrow C_3$ on both matrices we get

$$A\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Applying $C_2 \leftrightarrow C_3$ on both matrices we get

$$A\begin{bmatrix} 3 & 1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Hence, the correct answer is option (A).

18. Let for $i = 1, 2, 3$, $p_i(x)$ be a polynomial of degree 2 in x, $p_i'(x)$ and $p_i''(x)$ be the first-and second-order derivatives of $p_i(x)$, respectively. Let $A(x) = \begin{bmatrix} p_1(x) & p_1'(x) & p_1''(x) \\ p_2(x) & p_2'(x) & p_2''(x) \\ p_3(x) & p_3'(x) & p_3''(x) \end{bmatrix}$

and $B(x) = [A(x)]^T A(x)$. Then, the determinant of $B(x)$

(A) is a polynomial of degree 6 in x.

(B) is a polynomial of degree 3 in x.

(C) is a polynomial of degree 2 in x.

(D) does not depend on x.

[JEE MAIN 2014 (ONLINE SET 2)]

Solution:

$$|B(x)| = |A(x)||A(x)| = |A(x)|^2$$

Now highest power in the determinant of $A(x)$ can be 3, as $p_i'(x)$ is of degree 1 and $p_i''(x)$ is constant. Hence, $|B(x)|$ must have maximum degree 6.

Hence, the correct answer is option (A).

19. If $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+\lambda)^2 & (b+\lambda)^2 & (c+\lambda)^2 \\ (a-\lambda)^2 & (b-\lambda)^2 & (c-\lambda)^2 \end{vmatrix} = k\lambda\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$, $\lambda \neq 0$, then

k is equal to

(A) $4\lambda abc$ **(B)** $-4\lambda abc$ **(C)** $4\lambda^2$ **(D)** $-4\lambda^2$

[JEE MAIN 2014 (ONLINE SET 3)]

Solution:

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+\lambda)^2 & (b+\lambda)^2 & (c+\lambda)^2 \\ (a-\lambda)^2 & (b-\lambda)^2 & (c-\lambda)^2 \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ 4a\lambda & 4b\lambda & 4c\lambda \\ \lambda^2-2a\lambda & \lambda^2-2b\lambda & \lambda^2-2c\lambda \end{vmatrix}$$

$(R_2 \to R_2 - R_3 \text{ and } R_3 \to R_3 - R_1)$

$$= 4\lambda \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ \lambda^2-2a\lambda & \lambda^2-2b\lambda & \lambda^2-2c\lambda \end{vmatrix}$$

$$= 4\lambda \left\{ \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ \lambda^2 & \lambda^2 & \lambda^2 \end{vmatrix} + \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ -2a\lambda & -2b\lambda & -2c\lambda \end{vmatrix} \right\}$$

$$= 4\lambda^3 \left\{ \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} + 0 \right\}$$

(since, two rows are proportional)

$$= 4\lambda^3 \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \qquad (\text{since, } k\lambda = 4\lambda^3 \Rightarrow k = 4\lambda^2)$$

Hence, the correct answer is option (C).

20. If $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$ be such that $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, then

(A) $y = 2x$ **(B)** $y = -2x$ **(C)** $y = x$ **(D)** $y = -x$

[JEE MAIN 2014 (ONLINE SET 3)]

Solution: We have

$$AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}_{2\times 3} \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} y+2x+x \\ 3y-x+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Thus,

$$y + 3x = 6 \text{ and } 3y - x = 6$$
$$\Rightarrow y + 3x = 3y - x$$
$$\Rightarrow 2y = 4x$$
$$\Rightarrow y = 2x$$

Hence, the correct answer is option (A).

21. Let A and B be any two 3×3 matrices. If A is symmetric and B is skew-symmetric, then the matrix $AB - BA$ is

(A) skew-symmetric
(B) symmetric
(C) neither symmetric nor skew-symmetric
(D) I or $-I$, where I is an identity matrix

[JEE MAIN 2014 (ONLINE SET 4)]

Solution: We have

$$(AB - BA)' = (AB)' - (BA)' = B'A' - A'B' = (-B)(A) - A(-B) = AB - BA$$

Therefore, $AB - BA$ is symmetric.

Hence, the correct answer is option (B).

22. If $\begin{vmatrix} r & 2r-1 & 3r-2 \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{1}{2}n(n-1) & (n-1)^2 & \dfrac{1}{2}(n-1)(3n+4) \end{vmatrix}$, then the value of $\sum_{r=1}^{n-1} \Delta_r$

(A) depends only on a
(B) depends only on n
(C) depends both on a and n
(D) is independent of both a and n

[JEE MAIN 2014 (ONLINE SET 4)]

Solution:

$$\sum_{r=1}^{n-1} \Delta_r = \Delta_1 + \Delta_2 + \cdots + \Delta_{n-1}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{n}{2}(n-1) & (n-1)^2 & \left(\dfrac{n-1}{2}\right)(3n+4) \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & 3 & 4 \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{n}{2}(n-1) & (n-1)^2 & \left(\dfrac{n-1}{2}\right)(3n+4) \end{vmatrix}$$

$$+ \cdots + \begin{vmatrix} n-1 & 2(n-1)-1 & 3(n-1)-2 \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{n}{2}(n-1) & (n-1)^2 & \left(\dfrac{n-1}{2}\right)(3n+4) \end{vmatrix}$$

$$= \begin{vmatrix} 1+2+\cdots n-1 & 1+3+\cdots 2(n-1)-1 & 1+4+\cdots 3(n-1)-2 \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{n}{2}(n-1) & (n-1)^2 & \left(\dfrac{n-1}{2}\right)(3n+4) \end{vmatrix}$$

$$= \begin{vmatrix} (n-1)\dfrac{n}{2} & (n-1)^2 & \left(\dfrac{n-1}{2}\right)(3n-4) \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{n(n-1)}{2} & (n-1)^2 & \dfrac{1}{2}(n-1)(3n+4) \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & \left(\dfrac{n-1}{2}\right)(\cancel{3n}-4-\cancel{3n}-4) \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{n(n-1)}{2} & (n-1)^2 & \dfrac{1}{2}(n-1)(3n+4) \end{vmatrix} \quad (R_1 \to R_1 - R_2)$$

$$= \frac{-8(n-1)}{2}\left\{\frac{n}{2}(n-1)^2 - (n-1)\frac{n(n-1)}{2}\right\} = 0$$

Hence, the correct answer is option (D).

23. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is a 3×3 identity matrix, then the ordered pair (a, b) is equal to

(A) $(-1, 1)$ **(B)** $(2, 1)$
(C) $(-2, -1)$ **(D)** $(2, -1)$

[JEE MAIN 2015 (OFFLINE)]

Solution: We have

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$$

$$A \cdot A^T = 9I$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & (a+4+2b) \\ 0 & 9 & (2a+2-2b) \\ (a+4+2b) & (2a+2-2b) & (a^2+4+b^2) \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow a + 2b = -4; a^2 + b^2 + 4 = 9; 2a - 2b = -2;$
$\Rightarrow a = -2, b = -1 \Rightarrow (a, b) \equiv (-2, -1)$

Hence, the correct answer is option (C).

24. The set of all values of λ for which the system of linear equations

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$
$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$
$$-x_1 + 2x_2 = \lambda x_3$$

has a non-trivial solution

(A) is a singleton.
(B) contains two elements.
(C) contains more than two elements.
(D) is an empty set.

[JEE MAIN 2015 (OFFLINE)]

Solution: The system of linear equation

$$2x_1 - 2x_2 + x_3 = \lambda x_1$$
$$2x_1 - 3x_2 + 2x_3 = \lambda x_2$$
$$-x_1 + 2x_2 = \lambda x_3$$

has a non-trivial solution if

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\,[(3\lambda + \lambda^2) - 4] + 2(-2\lambda + 2) + 1(4 - 3 - \lambda) = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 5\lambda + 3 = 0 \Rightarrow \lambda = 1, 1, 3$$

Therefore, $\lambda \in \{1, 3\}$.

Hence, the correct answer is option (B).

25. The least value of the product xyz for which the determinant $\begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix}$ is non–negative is

(A) $-2\sqrt{2}$ **(B)** $-16\sqrt{2}$
(C) -8 **(D)** -1

[JEE MAIN 2015 (ONLINE SET 1)]

Solution: To find the least value of xyz where

$$\begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix} \geq 0$$

$$\Rightarrow x(yz - 1) - 1\,(z - 1) + (1 - y) = 0$$
$$\Rightarrow xyz - x - y - z + 2 \geq 0$$
$$\Rightarrow xyz \geq x + y + z - 2 \quad (1)$$

For x, y, z,

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} \Rightarrow x + y + z \geq 3\sqrt[3]{xyz} \quad (2)$$

From Eqs. (1) and (2)

$$xyz \geq 3\sqrt[3]{xyz} - 2$$
$$\Rightarrow t^3 \geq 3t - 2 \text{ where } t = \sqrt[3]{xyz}$$
$$\Rightarrow t^3 - 3t + 2 \geq 0 \Rightarrow (t-1)\,(t^2 + t - 2) \geq 0$$
$$\Rightarrow (t-1)\,(t+2)\,(t-1) \geq 0$$
$$\Rightarrow (t-1)^2\,(t+2) \geq 0 \Rightarrow t + 2 \geq 0 \Rightarrow t \geq -2$$
$$\Rightarrow \sqrt[3]{xyz} \geq -2 \Rightarrow xyz \geq -8$$

Hence, the correct answer is option (C).

26. If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then which one of the following statements is not correct?

(A) $A^4 - I = A^2 + I$ **(B)** $A^3 - I = A(A - I)$
(C) $A^2 + I = A(A^2 - I)$ **(D)** $A^3 + I = A(A^3 - I)$

[JEE MAIN 2015 (ONLINE SET 1)]

Solution:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Statement (A): $A^4 - I = (-I)^2 - I = I - I = 0$ and $A^2 + I = 0$. So, option (A) is correct.

Statement (B): $A^3 - I = A \cdot A^2 - I = A(-I) - I = -A - I$ and $A(A - I) = A^2 - AI = -I - A$. So, option (B) is correct.

Statement (C): $A^2 + I = 0$ and $A(A^2 - I) = A(-I - I) = -2AI = -2A$. So, option (C) is incorrect.

Statement (D): $A^3 + I = A \cdot A^2 + I = A(-I) + I = -A + I$ and $A(A^3 - I) = A(-A - I) = -A^2 - A = -(-I) - A = I - A$. So, option (D) is correct.

Hence, the correct answer is option (C).

27. If A is a 3×3 matrix such that $|5 \text{ adj } A| = 5$, then $|A|$ is equal to

(A) $\pm\frac{1}{5}$ **(B)** ± 5 **(C)** ± 1 **(D)** $\pm\frac{1}{25}$

[JEE MAIN 2015 (ONLINE SET 2)]

Solution: Order of A is 3; $|5 \text{ adj } A| = 5$

$$|5 (\text{adj } A)| = 5 \Rightarrow (5)^3 |\text{adj } A| = 5 \Rightarrow (5)^2 |A|^{3-1} = 1$$

$$\Rightarrow |A|^2 = \frac{1}{25} \Rightarrow |A| = \pm\frac{1}{5}$$

Hence, the correct answer is option (A).

28. If $\begin{vmatrix} x^2+1 & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = ax-12$, then '$a$' is equal to

(A) 12 **(B)** 24 **(C)** −12 **(D)** −24

[JEE MAIN 2015 (ONLINE SET 2)]

Solution:

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = ax-12 \quad (1)$$

Operating $R_2 \to R_2 - (R_1 + R_3)$ gives

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

$$\Rightarrow \Delta = -(-4)\begin{vmatrix} x+1 & x-2 \\ 2x-1 & 2x-1 \end{vmatrix} = 4(2x-1)\begin{vmatrix} x+1 & x-2 \\ 1 & 1 \end{vmatrix}$$

$$= 4(2x-1)(x+1-x+2) = 4(2x-1)(3) = 24x - 12 \quad (2)$$

Now from Eqs. (1) and (2)

$$\Delta = ax - 12 = 24x - 12$$

$$\Rightarrow a = 24$$

Hence, the correct answer is option (B).

29. The system of linear equations

$$x\,\lambda y - z = 0$$
$$\lambda x - y - z = 0$$
$$x + y - \lambda z = 0$$

has a non-trivial solution for

(A) exactly three values of λ.
(B) infinitely many values of λ.
(C) exactly one value of λ.
(D) exactly two values of λ.

[JEE MAIN 2016 (OFFLINE)]

Solution: For non-trivial solution, we have

$$\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow 1(\lambda+1) - \lambda(-\lambda^2+1) - 1(\lambda+1) = 0$$
$$\Rightarrow \lambda(\lambda^2 - 1) = 0$$
$$\Rightarrow \lambda = -1, 0, 1$$

Hence, the correct answer is option (A).

30. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \text{ adj } A = AA^T$, then $5a + b$ is equal to

(A) 13 **(B)** −1 **(C)** 5 **(D)** 4

[JEE MAIN 2016 (OFFLINE)]

Solution: We have the matrix

$$A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$$

$$A \text{ adj } A = AA^T \quad (1)$$

Now

$$\text{adj } A = \begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$$

Substituting these values into Eq. (1), we get

$$\begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & b \\ -3 & 5a \end{bmatrix} = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 10a+3b & 5ab-5ab \\ 0 & 3b+10a \end{bmatrix} = \begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 10a+3b & 0 \\ 0 & 10a+3b \end{bmatrix} = \begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix}$$

$$10a + 3b = 25a^2 + b^2 \quad (2)$$
$$10a + 3b = 13 \quad (3)$$

and $$15a - 2b = 0 \Rightarrow b = \frac{15a}{2} \quad (4)$$

Substituting the value of b into Eq. (3), we get

$$10a + 3\left(\frac{15a}{2}\right) = 13$$

$$\Rightarrow \frac{20a + 45a}{2} = 13$$

$$\Rightarrow 65a = 13 \times 2$$

$$\Rightarrow a = \frac{2}{5}$$

Substituting the value of a in Eq. (4), we get

$$b = \frac{15}{2} \times \frac{2}{5} = 3$$

Therefore

$$5a + b = 5\left(\frac{2}{5}\right) + 3 = 5$$

Hence, the correct answer is option (C).

31. The number of distinct real roots of the equation $\begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$ in the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is

(A) 1 **(B)** 4 **(C)** 2 **(D)** 3

[JEE MAIN 2016 (ONLINE SET 1)]

Solution:

$$\begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \cos x - \sin x & 0 & \sin x \\ 0 & \cos x - \sin x & \sin x \\ \sin x - \cos x & \sin x - \cos x & \cos x \end{vmatrix} = 0$$

$$(C_1 \to C_1 - C_3, C_2 \to C_2 - C_3)$$

$$\Rightarrow (\cos x - \sin x)^2 \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 1 & \sin x \\ -1 & -1 & \cos x \end{vmatrix} = 0$$

$$\Rightarrow (\cos x - \sin x)^2 \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 1 & \sin x \\ -1 & 0 & \cos x + \sin x \end{vmatrix} = 0$$

$$(R_3 \to R_3 + R_2)$$

$$\Rightarrow (\cos x - \sin x)^2 (\cos x + 2\sin x) = 0$$

Therefore,

$$2\sin x + \cos x = 0 \text{ or } \sin x = \cos x$$

For $2\sin x + \cos x = 0$, $\tan x = -\frac{1}{2}$; therefore, one solution in $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

For $\sin x = \cos x$, one solution in $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Therefore, the total number of solutions is 2.

Hence, the correct answer is option (C).

32. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T Q^{2015} P$ is

(A) $\begin{bmatrix} 0 & 2015 \\ 0 & 0 \end{bmatrix}$ **(B)** $\begin{bmatrix} 2015 & 0 \\ 1 & 2015 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$ **(D)** $\begin{bmatrix} 2015 & 1 \\ 0 & 2015 \end{bmatrix}$

[JEE MAIN 2016 (ONLINE SET 1)]

Solution: We have

$$Q = PAP^T$$

$$Q^2 = PAP^T PAP^T$$

$$PP^T = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \text{ and so on}$$

Now

$$Q^2 = PA^2P^T$$

$$Q^{2015} = PA^{2015}P^T$$

$$P^T Q^{2015} P = P^T P A^{2015} P^T P = A^{2015} = \begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$$

Hence, the correct answer is option (C).

33. If $A = \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix}$, then the determinant of the matrix $(A^{2016} - 2A^{2015} - A^{2014})$ is

(A) -175 **(B)** 2014 **(C)** 2016 **(D)** -25

[JEE MAIN 2016 (ONLINE SET 2)]

Solution: We have

$$A = \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix}$$

Therefore,

$$A^2 = \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 3 \\ -9 & -2 \end{bmatrix}$$

$$|A^{2016} - 2A^{2015} - A^{2014}| = |A|^{2014} \times |A^2 - 2A - I| = \begin{bmatrix} 20 & 5 \\ -15 & -5 \end{bmatrix}$$

Therefore,

$$|A^2 - 2A - I| = \begin{vmatrix} 20 & 5 \\ -15 & -5 \end{vmatrix} = -100 + 75 = -25$$

$$|A| = \begin{vmatrix} 4 & -1 \\ 3 & -1 \end{vmatrix} = -4 + 3 = -1 \Rightarrow |A|^{2014} = 1$$

$$|A^{2016} - 2A^{2015} - A^{2014}| = (-25) \times 1 = -25$$

Hence, the correct answer is option (D).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Consider the system of equations

$$-2y + 3z = -1$$
$$-x + y - 2z = k$$
$$x - 3y + 4z = 1$$

Statement-1: The system of equations has no solution for $k \neq 3$

Statement-2: The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$ for $k \neq 3$

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

(C) Statement-1 is true, Statement-2 is false.

(D) Statement-1 is false, Statement-2 is true.

[IIT-JEE 2008]

Solution: We have

$$x - 2y + 3z = -1$$
$$-x + y - 2z = k$$
$$x - 3y + 4z = 1$$

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$$

$$\Delta_x = \begin{vmatrix} -1 & -2 & 3 \\ k & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (3 - k)$$

$$\Delta_y = \begin{vmatrix} 1 & -1 & 3 \\ -1 & k & -2 \\ 1 & 1 & 4 \end{vmatrix} = (k - 3)$$

$$\Delta_z = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 1 & k \\ 1 & -3 & 1 \end{vmatrix} = (k - 3)$$

Statement-1: If $k \neq 3$, then $\Delta = 0$ and $\Delta_x, \Delta_y, \Delta_z \neq 0$.
Hence, the system of equation has no solution.
Therefore Statement-1 is true.
Statement-2 is true and is a correct explanation of Statement-1.
Hence, the correct answer is option (A).

Paragraph for Questions 2 to 4: Let $\mathbb{A}$ be the set of all 3 × 3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and 4 of them are 0.

[IIT-JEE 2009]

2. The number of matrices in $\mathbb{A}$ is

(A) 12 **(B)** 6 **(C)** 9 **(D)** 3

Solution: If two zeros are the entries in the diagonal, then ${}^3C_2 \times {}^3C_1$.
If all the entries in the principle diagonal is 1, then ${}^3C_1$.
So total matrices = 12.
Hence, the correct answer is option (A).

3. The number of matrices A in $\mathbb{A}$ for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution is

(A) less than 4 **(B)** at least 4 but less than 7
(C) at least 7 but less than 10 **(D)** at least 10

Solution: Let

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 1 \end{bmatrix}$$

For unique solution, $|A| \neq 0$.

Either $b = 0$ or $c = 0 \Rightarrow |A| \neq 0 \Rightarrow$ 2 matrices

$\begin{bmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 0 \end{bmatrix}$ either $a = 0$ or $c = 0 \Rightarrow |A| \neq 0 \Rightarrow$ 2 matrices

$\begin{bmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ either $a = 0$ or $b = 0 \Rightarrow |A| \neq 0 \Rightarrow$ 2 matrices

$\begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$

If $a = b = 0 \Rightarrow |A| = 0$

If $a = c = 0 \Rightarrow |A| = 0$

If $b = c = 0 \Rightarrow |A| = 0$

So, there will be only 6 matrices.
Hence, the correct answer is option (B).

4. The number of matrices A in $\mathbb{A}$ for which the system of linear equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent is

(A) 0 **(B)** more than 2 **(C)** 2 **(D)** 1

Solution: The six matrices A for which $|A| = 0$ are

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow$ inconsistent

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow$ inconsistent

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow$ infinite solutions

$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ inconsistent

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow$ inconsistent

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow$ infinite solutions

Hence, the correct answer is option (B).

5. The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions is

(A) 0 **(B)** $2^9 - 1$ **(C)** 168 **(D)** 2

[IIT-JEE 2010]

Solution: Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

where a_i, b_i, c_i for i = 1, 2, 3 have values 0 or 1. Then the given system is equivalent to

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

which represents three distinct planes.
However, three planes cannot intersect at two distinct points. Therefore, the number of such 3 × 3 matrices will be zero.
Hence, the correct answer is option (A).

Paragraph for Questions 6 to 8: Let p be an odd prime number and T_p be the following set of 2 × 2 matrices:

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c \in \{0, 1, \ldots, p-1\} \right\}$$

[IIT-JEE 2010]

6. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det(A) divisible by p is

(A) $(p-1)^2$ **(B)** $2(p-1)$
(C) $(p-1)^2 + 1$ **(D)** $2p - 1$

Solution: If A is symmetric then $b = c$. So

$$|A| = a^2 - b^2 = (a+b)(a-b)$$

which is divisible by p if $(a+b)$ is divisible by p or $(a-b)$ is divisible by p.
Now $(a+b)$ is divisible by p if (a, b) can take values $(1, p-1)$, $(2, p-2)$, $(3, p-3)$, $\ldots$, $(p-1, 1)$.
Therefore, $(p-1)$ ways.
Also $(a-b)$ is divisible by p only when $a - b = 0$, that is, $a = b$. Then (a, b) can take values $(0, 0)$, $(1, 1)$, $(2, 2)$, $\ldots$, $(p-1, p-1)$.
Therefore, p ways.
If A is skew-symmetric then $a = 0$ and $b = -c$ or $b + c = 0$, which gives $|A| = 0$ when $b^2 = 0 \Rightarrow b = 0$, $c = 0$. But this possibility is already included when A is symmetric and $(a, b) = (0,0)$.
Again if A is both symmetric and skew-symmetric, then clearly A is null matrix which case is already included. Hence total number of ways $= p + p - 1 = 2p - 1$.
Hence, the correct answer is option (D).

7. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p is
[**Note:** The trace of a matrix is the sum of its diagonal entries.]
(A) $(p-1)(p^2 - p + 1)$ **(B)** $p^3 - (p-1)^2$
(C) $(p-1)^2$ **(D)** $(p-1)(p^2 - 2)$

Solution: As tr(A) is not divisible by $p \Rightarrow a \neq 0$.
det(A) is divisible by $p \Rightarrow a^2 - bc$ is divisible by p.
The number of ways of selection of a, b and c is

$$(p-1)[(p-1)\times 1] = (p-1)^2$$

Hence, the correct answer is option (C).

8. The number of A in T_p such that det (A) is not divisible by p is
(A) $2p^2$ **(B)** $p^3 - 5p$ **(C)** $p^3 - 3p$ **(D)** $p^3 - p^2$

Solution: The total number of $A = p \times p \times p = p^3$.
The number of A such that det(A) is divisible by p equals

$$\begin{aligned}&(p-1)^2 + \text{ number of } A \text{ in which } a = 0\\ &= (p-1)^2 + p + p - 1\\ &= p^2\end{aligned}$$

The required number is $p^3 - p^2$.
Hence, the correct answer is option (D).

9. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det(adj A) + det(adj B) $= 10^6$, then $[k]$ is equal to ______.
[**Note:** adj M denotes the adjoint of a square matrix M and $[k]$ denotes the largest integer less than or equal to k].
[IIT-JEE 2010]

Solution:
$|A| = [2k+1]^3$, $|B| = 0$ (Since B is a skew-symmetric matrix of order 3)

$$\begin{aligned}&\Rightarrow \det(\text{adj } A) + \det(\text{adj } B) = |A|^{n-1}\\ &= [(2k+1)^3]^2 = 10^6 \Rightarrow 2k+1 = 10 \Rightarrow 2k = 9\\ &[k] = 4\end{aligned}$$

Hence, the correct answer is (4).

10. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to
(A) M^2 **(B)** $-N^2$ **(C)** $-M^2$ **(D)** MN
[IIT-JEE 2011]

Solution: We have $MN = NM$. Then

$$\begin{aligned}&M^2N^2(M^TN)^{-1}(MN^{-1})^T = M^2N^2N^{-1}(M^T)^{-1}(N^{-1})^T \cdot M^T\\ &= M^2N \cdot (M^T)^{-1}(N^{-1})^T M^T = -M^2 \cdot N(M)^{-1}(N^T)^{-1}M^T\\ &= +M^2NM^{-1}N^{-1}M^T = -M \cdot NMM^{-1}N^{-1}M = -MNN^{-1}M = -M^2\end{aligned}$$

Note: A skew-symmetric matrix of order 3 cannot be non-singular, hence the question is incorrect.
Hence, the correct answer is option (C).

11. Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$, where each of a, b, and c is either ω or ω^2. Then, the number of distinct matrices in the set S is
(A) 2 **(B)** 6 **(C)** 4 **(D)** 8
[IIT-JEE 2011]

Solution: For being non-singular

$$\begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$$

$$\begin{aligned}&\Rightarrow ac\omega^2 - (a+c)\omega + 1 \neq 0 \Rightarrow (a\omega - 1)(c\omega - 1) \neq 0\\ &\Rightarrow a \neq \omega^2 \text{ and } c \neq \omega^2 \Rightarrow a = \omega \text{ and } c = \omega \text{ and } b = \omega \text{ or } \omega^2\end{aligned}$$

Hence, the number of possible triplets of (a, b, c) is 2, that is, $(\omega, \omega^2, \omega)$ and (ω, ω, ω).
Hence, the correct answer is option (A).

12. Let M be a 3×3 matrix satisfying

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

Then, the sum of the diagonal entries of M is ______.
[IIT-JEE 2011]

Solution: Let

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}\Rightarrow b=-1, e=2, h=3$$

$$M\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}\Rightarrow a=0, d=3, g=2$$

$$M\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}\Rightarrow g+h+i=12\Rightarrow i=7$$

Therefore, the sum of diagonal elements = 9.

Hence, the correct answer is (9).

13. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is

(A) 2^{10} **(B)** 2^{11} **(C)** 2^{12} **(D)** 2^{13}

[IIT-JEE 2012]

Solution:

$$|Q|=\begin{vmatrix}2^2a_{11} & 2^3a_{12} & 2^4a_{13}\\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23}\\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33}\end{vmatrix}$$

$$\Rightarrow |Q|=2^2\cdot 2^3\cdot 2^4\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ 2a_{21} & 2a_{22} & 2a_{23}\\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33}\end{vmatrix}$$

$$\Rightarrow |Q|=2^9\cdot 2\cdot 2^2\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$\Rightarrow |Q|=2^{12}|P|$$

$$\Rightarrow |Q|=2^{13}$$

Hence, the correct answer is option (D).

14. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X=\begin{bmatrix}x\\y\\z\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\end{bmatrix}$ such that

(A) $PX=\begin{bmatrix}0\\0\\0\end{bmatrix}$ **(B)** $PX = X$ **(C)** $PX = 2X$ **(D)** $PX = -X$

[IIT-JEE 2012]

Solution: Given

$$P^T = 2P + I$$
$$\Rightarrow P = 2P^T + I = 2(2P + I) + I$$
$$\Rightarrow P + I = 0$$
$$\Rightarrow PX + X = 0$$
$$\Rightarrow PX = -X$$

Hence, the correct answer is option (D).

15. If the adjoint of a 3×3 matrix P is $\begin{bmatrix}1 & 4 & 4\\ 2 & 1 & 7\\ 1 & 1 & 3\end{bmatrix}$, then the possible value(s) of the determinant of P is (are)

(A) -2 **(B)** -1 **(C)** 1 **(D)** 2

[IIT-JEE 2012]

Solution:

$$|\operatorname{adj} P| = |P|^2 \quad \text{as } (|\operatorname{adj}(P)| = |P|^{n-1})$$

$$|\operatorname{adj} P| = 1(3-7) - 4(6-7) + 4(2-1) = 4$$

Hence, $|P| = 2$ or -2

Hence, the correct answers are options (A) and (D).

16. For 3×3 matrices M and N, which of the following statement(s) is (are) NOT correct?

(A) N^TMN is symmetric or skew-symmetric, according to M is symmetric or skew-symmetric

(B) $MN - NM$ is skew-symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) (adj M)·(adj N) = adj(MN) for all invertible matrices M and N

[JEE ADVANCED 2013]

Solution: We have

$$(N^TMN)^T = -N^TM^T(N^T)^T = N^TM^TN$$

(A) If M is skew symmetric, then $(N^TMN)^T = -N^TMN$. Therefore, it is concluded that it is skew-symmetric.
If M is symmetric, then $(M^TMN)^T = N^TMN$. Therefore, it is concluded that it is symmetric. Hence, option (A) is correct.

(B) We have

$$\begin{aligned}(MN-NM)^T &= (MN)^T - (NM)^T\\ &= N^TM^T - M^TN^T\\ &= -(M^TM^T - N^TM^T)\\ &= -(MN - NM)\end{aligned}$$

Therefore, it is concluded that it is skew-symmetric and hence option (B) is correct.

(C) $(MN)^T = N^TM^T$. Symmetricity and skew-symmetricity depend on the nature of M and N; therefore, option (C) is incorrect.

(D) adj(MN) = adj(N) adj M; therefore, option (D) is incorrect.

Hence, the correct answers are options (C) and (D).

17. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be an $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then, $P^2 \neq 0$, when n = ?

(A) 57 **(B)** 55 **(C)** 58 **(D)** 56

[JEE ADVANCED 2013]

Solution: We have $P = [P_{ij}]_{n\times n}, P^2 \neq 0.$ Now

$$P_{ij} = \omega^{\hat{i}+\hat{j}}$$

$$P=\begin{bmatrix}\omega^2 & 1 & \omega & \omega^2 & 1 & \dots\\ 1 & \omega & \omega^2 & 1 & \omega & \dots\\ \omega & \omega^2 & 1 & \omega & \omega^2 & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots\end{bmatrix}_{n\times n}$$

$$P^2 = \begin{bmatrix} \omega^2 & 1 & \omega & \omega^2 & 1 & \cdots \\ 1 & \omega & \omega^2 & 1 & \omega & \cdots \\ \omega & \omega^2 & 1 & \omega & \omega^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \omega^2 & 1 & \omega & \omega^2 & 1 & \cdots \\ 1 & \omega & \omega^2 & 1 & \omega & \cdots \\ \omega & \omega^2 & 1 & \omega & \omega^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Therefore,

$$(\omega^4+1+\omega^2)+(\omega^4+1+\omega^2)+\cdots=0$$

This is possible only when n is a multiple of 3. Therefore, n can be 55, 58, 56 ($P^2 \neq 0$).

Hence, the correct answers are options (B), (C) and (D).

18. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if

(A) The first column of M is the transpose of the second row of M

(B) The second row of M is the transpose of the first column of M

(C) M is a diagonal matrix with non-zero entries in the main diagonal

(D) The product of entries in the main diagonal of M is not the square of an integer

[JEE ADVANCED 2014]

Solution: Let

$$M = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \quad a,b,d \in I$$

(A) $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} \Rightarrow a=b=d \Rightarrow |M| = \begin{vmatrix} a & a \\ a & a \end{vmatrix} = 0$

$\Rightarrow$ Not invertible. Therefore, (A) is false.

(B) $\begin{bmatrix} b & d \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix} \Rightarrow a=b=d$

$\Rightarrow |M|=0 \Rightarrow$Not invertible. Therefore, (B) is false.

(C) If M is a diagonal matrix, then $M = \begin{bmatrix} a & o \\ o & d \end{bmatrix} \Rightarrow |M| = ad \neq 0$

$\Rightarrow M$ invertible. Therefore, (C) is correct .

(D) Given $ad \neq b^2$. Now $|M| = ad - b^2 \neq 0$ for M to be invertible. Therefore, (D) is true.

Hence, the correct answers are options (C) and (D).

19. Let M and N be two 3×3 matrices such $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(A) Determinant of $(M^2 + MN^2)$ is 0.

(B) There is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix.

(C) Determinant of $(M^2 + MN^2) \neq 1$.

(D) For a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix.

[JEE ADVANCED 2014]

Solution: Given $MN = NM$, therefore $a^2 - b^2 = (a+b)(a-b)$ of algebra of numbers is applicable

Therefore,

$$M^2 = N^4$$
$$\Rightarrow M^2 - N^4 = 0 \text{ (null matrix)}$$
$$\Rightarrow (M+N^2)(M-N^2) = 0$$

Now since $M \neq N^2$ (given), therefore, the possibilities are

$$(M+N^2) = 0 \text{ and } M - N^2 \neq 0 \qquad (1)$$

or

$$(M+N^2) \neq 0 \text{ and } M - N^2 \neq 0 \qquad (2)$$

Now we know, if A and B are non-null square matrix and $AB = 0$ then A and B both are singular, i.e. $|A| = 0$ and $|B| = 0$ and $AB = 0$

Note: For example, let A be non-singular. Then

$$B = \mathrm{I}(B) = A^{-1}AB = 0$$

So $AB = 0$ assumed. Therefore, B is singular, which is a contradiction. So, A has to be singular. Similarly, then B also has to be singular. Therefore, from Eqs. (1) and (2), we conclude the only possibility is $|M+N^2| = 0$

Now checking options:

(A) $|M^2 + MN^2| = |M||M+N^2| = 0$

Therefore, (A) is correct.

(B) $(M^2 + MN^2)U = 0$

Since $M^2 + MN^2$ is singular, therefore, U has infinitely many possible values (non-trivial solutions). So (B) is true.

(C) False, since $|M^2 + MN^2| = 0$

(D) False. Since $|M^2 + MN^2| = 0$, therefore, U is not a necessarily a zero matrix.

Hence, the correct answers are options (A) and (B).

20. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew-symmetric?

(A) $Y^3Z^4 - Z^4Y^3$ **(B)** $X^{44} + Y^{44}$

(C) $X^4Z^3 - Z^3X^4$ **(D)** $X^{23} + Y^{23}$

[JEE ADVANCED 2015]

Solution:

$X, Y \to$ skew-symmetric matrices of order 3×3

$Z \to$ symmetric matrix of order 3×3

and $X, Y, Z \neq 0$

Checking all the options:

Option (A) $(Y^3Z^4 - Z^4Y^3)^T = (Y^3Z^4)^T - (Z^4Y^3)^T$

$= (Z^4)^T(Y^3)^T - (Y^3)^T(Z^4)^T$

$= (Z^T)^4(Y^T)^3 - (Y^T)^3(Z^T)^4 = -Z^4Y^3 + Y^3Z^4$

Option (B) $(X^{44} + Y^{44})^T = (X^T)^{44} + (Y^T)^{44}$

$= X^{44} + Y^{44} \Rightarrow$ (symmetric)

Option (C) $(X^4Z^3 - Z^3X^4)^T = (Z^T)^3(X^T)^4 - (X^T)^4(Z^T)^3$

$= Z^3X^4 - X^4Z^3 \Rightarrow$ (skew-symmetric)

Option (D) $(X^{23} + Y^{23})^T = (X^T)^{23} + (Y^T)^{23}$

$= (-X)^{23} + (-Y)^{23} = -(X^{23} + Y^{23}) \Rightarrow$ (skew-symmetric)

Hence, the correct answers are options (C) and (D).

21. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$$

(A) -4 **(B)** 9 **(C)** -9 **(D)** 4

[JEE ADVANCED 2015]

Solution:

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha^2 \end{vmatrix} = -648\alpha$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$ give

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (3+2\alpha) & (3+4\alpha) & (3+6\alpha) \\ (5+2\alpha) & (5+4\alpha) & (5+6\alpha) \end{vmatrix} = -648\alpha$$

$R_3 \to R_3 - R_2$ gives

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$ give

$$\begin{vmatrix} (1+\alpha)^2 & \alpha(2+3\alpha) & \alpha(2+5\alpha) \\ (3+2\alpha) & 2\alpha & 2\alpha \\ (2) & 0 & 0 \end{vmatrix} = -648\alpha$$

$$\Rightarrow 2(2\alpha^2)[(2+3\alpha)-(2+5\alpha)] = -648\alpha$$
$$\Rightarrow -8\alpha^3 = -648\alpha$$
$$\Rightarrow \alpha^2 = 81 \text{ or } \alpha = 0$$
$$\Rightarrow \alpha = 0 \text{ or } \pm 9$$

Hence, the correct answers are options (B) and (C).

22. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in R$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\dfrac{k}{8}$ and $\det(Q) = \dfrac{k^2}{2}$, then

(A) $\alpha = 0, k = 8$ **(B)** $4\alpha - k + 8 = 0$

(C) $\det(P \operatorname{adj}(Q)) = 2^9$ **(D)** $\det(Q \operatorname{adj}(P)) = 2^{13}$

[JEE ADVANCED 2016]

Solution: It is given that

$$P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix} \text{ and } Q = [q_{ij}]$$

Now,

$$C_{32} = -\begin{vmatrix} 3 & -2 \\ 2 & \alpha \end{vmatrix} = -(3\alpha + 4)$$

Here, $|P| = 12\alpha + 20$,

$$PQ = kI \Rightarrow Q = kIP^{-1}$$

Now, $|P|\,|Q| = k^3$. Therefore,

$$(12\alpha + 20)\frac{k^2}{2} = k^3$$

$$\Rightarrow 6\alpha + 10 = k \qquad (1)$$
$$\Rightarrow |P| = 2k$$

So,

$$Q = kI\frac{(\operatorname{adj} P)}{|P|} = \frac{(\operatorname{adj} P)}{2}$$

Now,

$$q_{23} = \frac{1}{2}C_{32} = \frac{-(3\alpha+4)}{2} = \frac{-k}{8} \Rightarrow 12\alpha + 16 = k \quad (2)$$

Solving Eqs. (1) and (2), we get

$$-4 = -k \Rightarrow k = 4$$

Using the value of k in Eq. (1), we get

$$6\alpha + 10 = 4 \Rightarrow \alpha = -1$$

That is, $\alpha = -1$ and $k = 4$.

Hence, option (A) is incorrect.

The values of $\alpha = -1$ and $k = 4$ satisfy the equation given in option (B).

Hence, option (B) is correct.

Now,

$$\det Q = \frac{k^2}{2} = 8$$

Therefore,

$$\det(P \operatorname{adj} Q) = (\det P)\det(\operatorname{adj} Q) = (2 \times 4)(\det Q)^2$$
$$= 8 \times 8^2 = 2^3 \times 2^6 = 2^9$$

Hence, the correct answers are options (B) and (C).

23. The total number of distinct $x \in R$ for which $\begin{vmatrix} x & x^2 & 1+x^2 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$ is __________.

[JEE ADVANCED 2016]

Solution: It is given that

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & (2x)^2 & 1+(2x)^3 \\ 3x & (3x)^2 & 1+(3x)^3 \end{vmatrix} = 10$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ 2x & 4x^2 & 1 \\ 3x & 9x^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 2x & 4x^2 & 8x^3 \\ 3x & 9x^2 & 27x^3 \end{vmatrix} = 10$$

$$\Rightarrow x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 1 \\ 3 & 3^2 & 1 \end{vmatrix} + x^6\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix} = 10$$

$$\Rightarrow x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 1 \\ 3 & 3^2 & 1 \end{vmatrix} + 6x^6\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2^2 & 2^3 \\ 3 & 3^2 & 3^3 \end{vmatrix} = 10$$

$$\Rightarrow x^3(1+6x^3)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 10$$

Using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ we get

$$x^3(1+6x^3)\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 1 & 2 & 8 \end{vmatrix} = 10$$

$$\Rightarrow x^3(1+6x^3)\ 1(8-6) = 10$$
$$\Rightarrow x^3(1+6x^3) = 5$$
$$\Rightarrow 6(x^3)^2 + x^3 - 5 = 6$$
$$\Rightarrow 6(x^3)^2 + 6x^3 - 5x^3 - 5 = 0$$
$$\Rightarrow 6x^3(x^3+1) - 5(x^3+1) = 0$$
$$\Rightarrow (6x^3 - 5)\ (x^2 - x + 1)\ (x+1) = 0$$

Therefore, $x = -1$ and $(5/6)^{1/3}$.
Hence, there are two district values for x. Thus, 2 is the answer.
Hence, the correct answer is (2).

24. Let $z = \dfrac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then, the total number of ordered pairs (r, s) for which $P^2 = -I$ is ______.

[JEE ADVANCED 2016]

Solution: It is given that

$$z = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{i2\pi/3} = \omega$$

(ω is cube root of unity), where $r, s \in \{1, 2, 3\}$.

It is also given that $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since $P^2 = -I$, we have

$$P^2 = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & zr \end{bmatrix}\begin{bmatrix} (z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-z)^{2r} + z^{4s} & (-z)^r \cdot z^{2s} + z^r z^{2s} \\ (-z)^r z^{2s} + z^r \cdot z^{2s} & z^{4s} + (z)^{2r} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

That is, we have

$$(-z)^{2r} + z^{4s} = -1 \text{ and } z^{4s} + z^{2r} = -1$$

and $\quad ((-z)^r + z^r)z^{2s} = 0$ and $z^{2r} + z^{4s} = -1$

$$\Rightarrow ((-\omega)^r + (\omega)^r)\cdot \omega^{2s} = 0$$

Now, $\omega^{2s} \neq 0$; therefore,

$$(-\omega)^r + (\omega)^r = 0$$

where r is the odd number and hence $r = 1, 3$.
When $r = 1$ $\quad (-\omega)^2 + \omega^{4s} = -1 \Rightarrow \omega^{4s} = -1 - \omega^2 = +\omega$

Now, s can be 1 (since $s \neq 3$).
That is, $(r, s) = (1, 1)$, that is, the total number of ordered pair (r, s) is one (single) for which $P^2 = -I$.
Hence, the correct answer is (1).

25. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\dfrac{q_{31}+q_{32}}{q_{21}}$ equals

(A) 52 **(B)** 103 **(C)** 201 **(D)** 205

[JEE ADVANCED 2016]

Solution: It is given that

$$P^{50} - Q = I$$
$$\Rightarrow Q = P^{50} - I = [q_{ij}]_{3\times 3}$$

The given matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{bmatrix}$$

Therefore,

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 3a^2 & 2a & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 3a^2 & 2a & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3a & 1 & 0 \\ 6a^2 & 3a & 1 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1 & 0 & 0 \\ 4a & 1 & 0 \\ 10a^2 & 4a & 1 \end{bmatrix}$$

$$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 50a & 1 & 0 \\ T_{50} & 50a & 1 \end{bmatrix}$$

Therefore, using difference method, we get

$$S_{50} = a^2 + 3a^2 + 6a^2 + 10a^2 + \cdots + T_{50}$$
$$S_{50} = a^2 + 3a^2 + 6a^2 + \cdots + T_{49} + T_{50}$$
$$\overline{0 = a^2 + 2a^2 + 3a^2 + \cdots - T_{50}}$$

$$\Rightarrow T_{50} = a^2 + 2a^2 + 3a^2 + \cdots + 50a^2$$

$$T_{50} = a^2(1 + 2 + 3 + \cdots + 50) = \frac{a^2(50)(51)}{2} = a^2(25)(51)$$

Therefore,

$$I^{50} - I = \begin{bmatrix} 0 & 0 & 0 \\ 50a & 0 & 0 \\ T_{50} & 50a & 0 \end{bmatrix} = [q_{ij}]_{3\times 3}$$

Hence, we get the following values:

$$q_{31} = T_{50} = 5(51)a^2$$
$$q_{32} = 50a$$
$$q_{21} = 50a$$

Therefore,

$$\frac{q_{31}+q_{32}}{q_{21}} = \frac{(25)(51)(a^2)}{50a} + 1 = 1 + \frac{(25)(51)(16)}{(50)(4)} = 103$$

Hence, the correct answer option is (B).

Practice Exercise 1

1. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, then A^2 is equal to

(A) Unit matrix **(B)** Null matrix **(C)** A **(D)** $-A$

2. If $A=\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then $|A|\,|\text{adj } A|$ is equal to

(A) a^{25} **(B)** a^{27} **(C)** a^{81} **(D)** None of these

3. If A is a 3×3 skew-symmetric matrix, then $|A|$ is given by

(A) 0 **(B)** -1 **(C)** 1 **(D)** None of these

4. If A and B are two square matrices of the same order and m is a positive integer, then

$(A+B)^m = {}^mC_0A^m + {}^mC_1A^{m-1}B + {}^mC_2A^{m-2}B^2 + \cdots + {}^mC^{m-1}AB^{m-1} + {}^mC_mB^m$ if

(A) $AB=BA$ **(B)** $AB+BA=0$

(C) $A^m=0, B^m=0$ **(D)** None of these

5. If $A=(a_{ij})_{3\times 3}$ is a skew-symmetric matrix, then

(A) $a_{ii}=0, \forall\, i$ **(B)** $A-A'$ is null matrix

(C) $|A|\neq 0$ **(D)** None of these

6. $\begin{bmatrix} 7 & 1 & 2 \\ 9 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ is equal to

(A) $\begin{bmatrix} 43 \\ 44 \end{bmatrix}$ **(B)** $\begin{bmatrix} 43 \\ 45 \end{bmatrix}$

(C) $\begin{bmatrix} 45 \\ 44 \end{bmatrix}$ **(D)** None of these

7. For any matrix A of order 2×2, if $A(\text{adj } A)=\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A|$ is equal to

(A) 20 **(B)** 100 **(C)** 10 **(D)** 0

8. If a matrix A is symmetric as well as skew-symmetric, then A is a

(A) Diagonal matrix **(B)** Null matrix

(C) Unit matrix **(D)** None of these

9. If A and B be two square matrices such that $AB=O$, then

(A) $\det A=0$ or $\det B=0$ **(B)** $\det B=0$

(C) $B=A^{-1}$ **(D)** $\det A=0$

10. If $A=\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then

(A) $A^2=A$ **(B)** $A^2=0$

(C) $A^2=I$ **(D)** $A^3=O$

11. If $A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, then A is

(A) An invertible matrix **(B)** An idempotent matrix

(C) A nilpotent matrix **(D)** None of these

12. If A and B are symmetric matrices of the same order, then

(A) AB is a symmetric matrix

(B) $A-B$ is skew-symmetric matrix

(C) $AB+BA$ is a symmetric matrix

(D) $AB-BA$ is a symmetric matrix

13. If A is any square matrix, then

(A) $A+A'$ is skew-symmetric **(B)** $A-A'$ is symmetric

(C) AA' is symmetric **(D)** None of these

14. If A is a square matrix such that $A^3=I$ then A^{-1} is equal to

(A) I **(B)** A **(C)** A^2 **(D)** None of these

15. If A is any square matrix then which of the following is not symmetric?

(A) $A+A'$ **(B)** $A-A'$ **(C)** AA' **(D)** $A'A$

16. Let A be a skew-symmetric matrix of order n then

(A) $|A|=0$ if n is even **(B)** $|A|=0$ if n is odd

(C) $|A|=0$ for all $n\in N$ **(D)** None of these

17. Each diagonal element of skew-symmetric matrix is

(A) Zero **(B)** Positive

(C) Non-real **(D)** Negative

18. If $A=\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, then the value of $(A-2I)(A-3I)$ is

(A) Unit matrix **(B)** Non-singular matrix

(C) Null matrix **(D)** None of these

19. Matrix A has m rows and $n+5$ columns, matrix B has m rows and $11-n$ columns. If both AB and BA exist, then

(A) AB and BA are square matrices

(B) AB and BA are of order 8×8 and 3×13, respectively

(C) $AB=BA$

(D) None of these

20. If $A=\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then A^{-1} is equal to

(A) $\begin{bmatrix} 7 & -3 & -3 \\ 0 & 1 & 0 \\ -1 & 0 & 5 \end{bmatrix}$ **(B)** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 7 & -3 & -3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **(D)** None of these

21. If $A=\begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then A^8 equals

(A) $128B$ **(B)** $132B$ **(C)** $116B$ **(D)** $8B$

22. If α, β, γ are three real numbers and

$A=\begin{bmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\beta-\alpha) & 1 & \cos(\beta-\gamma) \\ \cos(\gamma-\alpha) & \cos(\gamma-\beta) & 1 \end{bmatrix}$ then

(A) A is skew-symmetric **(B)** A is invertible

(C) A is non-singular **(D)** $|A|=0$

23. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then the number of solutions of the system of equations $a_1x+b_1y+c_1z=0$, $a_2x+b_2y+c_2z=0$ and $a_3x+b_3y+c_3z=0$ is

(A) Infinite number of solutions
(B) Only one unique solution
(C) More than one solution
(D) None of these

24. Let A and B be two square matrices of the same dimension and let $[A, B] = AB - BA$. Then for three 2×2 matrices A, B, C, $[[A, B], C] + [[B, C], A] + [[C, A], B]$ is equal to

(A) 1 **(B)** 0
(C) $ABC - CBA$ **(D)** None of these

25. If the matrices $A, B, (A+B)$ are non-singular, then $[A(A+B)^{-1}B]^{-1}$ is equal to

(A) $A+B$ **(B)** $A^{-1}+B^{-1}$
(C) $(A+B)^{-1}$ **(D)** None of these

26. If A and B matrices commute then

(A) A^{-1} and B also commute
(B) B^{-1} and A also commute
(C) A^{-1} and B^{-1} also commute
(D) All the above

27. If A, B and C are three matrices conformable for multiplication, then $(ACB)^{-1}$ is equal to

(A) $A^{-1}B^{-1}C^{-1}$ **(B)** $B^{-1}C^{-1}A^{-1}$
(C) $C^{-1}B^{-1}A^{-1}$ **(D)** Cannot be determined

28. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, then A^2 is equal to

(A) Unit matrix **(B)** Null matrix **(C)** A **(D)** $-A$

29. Trace of a skew-symmetric matrix is always equal to

(A) $\sum a_{ij}$ **(B)** $\sum a_{ii}$ **(C)** Zero **(D)** None of these

30. The matrix of the transformation reflection in the line $x+y=0$ is

(A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **(B)** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(C)** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **(D)** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

31. If $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = (y-z)(z-x)(x-y)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$, then

(A) $n=2$ **(B)** $n=-2$ **(C)** $n=-1$ **(D)** $n=1$

32. If $\Delta_r = \begin{vmatrix} (2r) & x & N(N+1) \\ (6r^2-1) & y & N^2(2N+3) \\ (4r^3-2Nr) & z & N^3(N+1) \end{vmatrix}$, where $N \in$ natural numbers, then $\sum_{r=1}^{N} \Delta_r$ is equal to

(A) N **(B)** N^2 **(C)** Zero **(D)** None of these

33. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then, the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is

(A) 3ω **(B)** $3\omega(\omega-1)$ **(C)** $3\omega^2$ **(D)** $3\omega(1-\omega)$

34. If $\Delta(x) = \begin{vmatrix} \sin 2x & e^x \sin x + x\cos x & \sin x + x^2 \cos x \\ \cos x + \sin x & e^x + x & 1+x^2 \\ e^x \cos x & e^{2x} & e^x \end{vmatrix}$, then

(A) $\Delta'(0)=0$ **(B)** $\Delta'\left(\frac{\pi}{2}\right)=0$
(C) $\Delta'\left(\frac{\pi}{4}\right)=0$ **(D)** All the above

35. Let $g(x) = \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$, where α is a constant. Then $\lim_{x\to 0} \frac{g(x)}{x}$ is equal to

(A) 0 **(B)** 1 **(C)** -1 **(D)** None of these

36. If $\Delta(x) = \begin{vmatrix} 4x-4 & (x-2)^2 & x^3 \\ 8x-4\sqrt{2} & (x-2\sqrt{2})^2 & (x+1)^3 \\ 12x-4\sqrt{3} & (x-2\sqrt{3})^2 & (x-1)^3 \end{vmatrix}$, then the coefficient of x in $\Delta(x)$ is

(A) $64(5-\sqrt{2}-\sqrt{3})$ **(B)** $64(5+\sqrt{2}-\sqrt{3})$
(C) $64(5+\sqrt{2}+\sqrt{3})$ **(D)** None of these

37. If $\Delta_1 = \begin{vmatrix} a^2+x^2 & ab-cx & ac+bx \\ ab+cx & b^2+x^2 & bc+ax \\ ac-bx & bc-ax & c^2+x^2 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$, then

(A) $\Delta_1 = \Delta_2$ **(B)** $\Delta_1 = \Delta_2^2$
(C) $\Delta_1 = 2\Delta_2$ **(D)** None of these

38. If a, b and c are even natural numbers, then

$$\Delta = \begin{vmatrix} a-1 & a & a+1 \\ b-1 & b & b+1 \\ c-1 & c & c+1 \end{vmatrix}$$ is equal to

(A) $a+b+c$ **(B)** $a^2+b^2+c^2$
(C) abc **(D)** None of these

39. If $f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$, then $\int_0^{\pi/2} f(x)\,dx$ equals

(A) 1/4 **(B)** 1/3 **(C)** 1/2 **(D)** 1

40. If $f(x) = \begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 2 \end{vmatrix}$, then the value of $\int_{-\pi/2}^{\pi/2} f(x)dx$ is equal to

(A) 0 **(B)** 1 **(C)** 2 **(D)** None of these

41. If the expression $\begin{vmatrix} x^2+x+3 & 1 & 4 \\ 2x^4+x^3+2x+1 & 2 & 3 \\ x^2+x & 1 & 1 \end{vmatrix}$ is equal to $ax^4+bx^3+cx^2+dx+e$, then the value of e is equal to

(A) zero **(B)** 1 **(C)** 2 **(D)** None of these

42. The determinant $\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero if

(A) a, b, c are in AP **(B)** a, b, c are in GP
(C) a, b, c are in HP **(D)** None of these

43. If α, β, γ are the roots of the equation $x^3+px+q=0$ (where $p \neq 0, q \neq 0$), then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(A) p **(B)** q **(C)** p^2-2q **(D)** None of these

44. The number of values of k for which the system of equations $(k+1)x+8y=4k$, $kx+(k+3)y=3k-1$ has infinitely many solutions is

(A) 0 **(B)** 1 **(C)** 2 **(D)** infinite

45. The determinant $\Delta = \begin{vmatrix} a^2(1+x) & ab & ac \\ ab & b^2(1+x) & bc \\ ac & bc & c^2(1+x) \end{vmatrix}$ is divisible by

(A) $1+x$ **(B)** $(1+x)^2$ **(C)** x^2 **(D)** None of these

46. If the system of equations $ax+y+z=0$, $x+by+z=0$ and $x+y+cz=0$ $(a, b, c \neq 1)$ has a non-trivial solution, then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ is

(A) -1 **(B)** 0 **(C)** 1 **(D)** None of these

47. If the system of equations $x+ay=0$, $az+y=0$ and $ax+z=0$ has infinite solutions, then the value of a is

(A) -1 **(B)** 1 **(C)** 0 **(D)** No real values

48. If $A+B+C=\pi$, then the value of determinant $\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$ is equal to

(A) 0 **(B)** 1 **(C)** -1 **(D)** None of these

49. For $A=a^2+b^2+c^2$, $B=ab+bc+ca$, $(a^3+b^3+c^3-3abc)^2$ is equal to

(A) $\begin{vmatrix} B & A & B \\ B & B & A \\ A & B & B \end{vmatrix}$ **(B)** $\begin{vmatrix} A & B & B \\ B & B & A \\ B & A & B \end{vmatrix}$

(C) $\begin{vmatrix} B & B & A \\ B & A & B \\ A & B & B \end{vmatrix}$ **(D)** None of these

50. If A, B, C are angles of a triangle ABC, then the value of the determinant $\begin{vmatrix} \sin\frac{A}{2} & \sin\frac{B}{2} & \sin\frac{C}{2} \\ \sin(A+B+C) & \sin\frac{B}{2} & \cos\frac{A}{2} \\ \cos\frac{(A+B+C)}{2} & \tan(A+B+C) & \sin\frac{C}{2} \end{vmatrix}$ is less than or equal to

(A) 1/2 **(B)** 1/4 **(C)** 1/8 **(D)** None of these

51. The sum of two non-integral roots of $\begin{vmatrix} x & 2 & 5 \\ 3 & x & 3 \\ 5 & 4 & x \end{vmatrix} = 0$ is

(A) 5 **(B)** -5 **(C)** -18 **(D)** None of these

52. The value of $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ (2^x+2^{-x})^2 & (3^x+3^{-x})^2 & (5^x+5^{-x})^2 \\ (2^x-2^{-x})^2 & (3^x-3^{-x})^2 & (5^x-5^{-x})^2 \end{vmatrix}$ is

(A) 0 **(B)** 30^x **(C)** 30^{-x} **(D)** None of these

53. There are three points (a, x), (b, y) and (c, z) such that the straight lines joining any two of them are not equally inclined to the coordinate axes where $a, b, c, x, y, z \in R$. If $\begin{vmatrix} x+a & y+b & z+c \\ y+b & z+c & x+a \\ z+c & x+a & y+b \end{vmatrix} = 0$ and $a+c=-b$, then $x, -\frac{y}{2}, z$ are in

(A) AP **(B)** GP **(C)** HP **(D)** None of these

54. If $\begin{vmatrix} \sum_{k=0}^{n-2} 1 & n(n-1) & n^2 \\ \sum_{k=1}^{n} 1 & (n+1)(n-1) & n(n+1) \\ \sum_{k=1}^{n-1} 1 & n^2+1 & n^2 \end{vmatrix} = 72$, then n is equal to

(A) 6 **(B)** 9 **(C)** 8 **(D)** None of these

55. $\Delta = \begin{vmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix}$ equals

(A) $(a+b+c)^6$ **(B)** $(a-b)^2(b-c)^2(c-a)^2$
(C) $4(a-b)(b-c)(c-a)$ **(D)** None of these

56. If in a triangle ABC,

$$\begin{vmatrix} 1 & 1 & 1 \\ \cot(A/2) & \cot(B/2) & \cot(C/2) \\ \tan(B/2)+\tan(C/2) & \tan(C/2)+\tan(A/2) & \tan(A/2)+\tan(B/2) \end{vmatrix} = 0$$

then the triangle must be

(A) Equilateral **(B)** Obtuse angle
(C) Isosceles **(D)** None of these

57. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(A) 0 **(B)** 2 **(C)** 1 **(D)** 3

58. If $f(x) = \frac{\ln x}{x}$, then $\begin{vmatrix} \ln x & x & 0 \\ 1/x & 1 & x \\ -1/x^2 & 0 & 2 \end{vmatrix}$ is

(A) $x^3 f'(x)$ **(B)** $x^2 f'''(x)$
(C) $x^3 f'''(x)$ **(D)** None of these

59. If the system of equations $\lambda x + (b-a)y + (c-a)z = 0$, $(a-b)x + \lambda y + (c-b)z = 0$ and $(a-c)x + (b-c)y + \lambda z = 0$ has a non-trivial solution, then the value of λ is

(A) $\lambda = 0$ **(B)** $\lambda = 1$ **(C)** $\lambda = -1$ **(D)** None of these

60. If A' is the transpose of a square matrix A, then

(A) $|A| \ne |A'|$ **(B)** $|A| = |A'|$
(C) $|A| + |A'| = 0$ **(D)** $|A| = |A'|$ only when A is symmetric

61. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then B equals

(A) $I\cos\theta + J\sin\theta$ **(B)** $I\sin\theta + J\cos\theta$
(C) $I\cos\theta - J\sin\theta$ **(D)** $-I\cos\theta + J\sin\theta$

62. If I_n is the identity matrix of order n, then $(I_n)^{-1}$

(A) does not exist **(B)** is equal to I_n
(C) equals O **(D)** nI_n

63. If for a matrix A, $A^2 + I = O$ where I is the identity matrix, then A equals

(A) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **(B)** $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ **(C)** $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ **(D)** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

64. The number of non-zero diagonal matrices of order 4 satisfying $A^2 = A$ is

(A) 2 **(B)** 4 **(C)** 16 **(D)** 15

65. If A and B are symmetric matrices of order n $(A \ne B)$, then

(A) $A + B$ is skew-symmetric **(B)** $A + B$ is symmetric
(C) $A + B$ is a diagonal matrix **(D)** $A + B$ is a zero matrix

66. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, then AB is equal to

(A) A^3 **(B)** B^2 **(C)** O **(D)** I

67. If $\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \left\{\begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-1}\right\}^2$, then the value of x is

(A) $a/125$ **(B)** $2a/25$ **(C)** $2a/125$ **(D)** None of these

68. The values of x for which the matrix $\begin{bmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{bmatrix}$ is non-singular are

(A) $R - \{0\}$ **(B)** $R - \{-(a+b+c)\}$
(C) $R - \{0, -(a+b+c)\}$ **(D)** None of these

69. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, then A^{-1} is equal to

(A) A' **(B)** $2A$ **(C)** $\frac{1}{2}A$ **(D)** $\frac{1}{19}A$

70. The inverse of a skew-symmetric matrix is

(A) A symmetric matrix if it exists
(B) A skew-symmetric matrix if it exists
(C) Transpose of the original matrix
(D) May not exist

71. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$, then adj (adj A) is

(A) $\begin{bmatrix} 3 & 3 & 3 \\ 6 & 9 & 15 \\ 9 & 15 & 36 \end{bmatrix}$ **(B)** $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 6 & 9 \\ 3 & 9 & 15 \\ 3 & 15 & 36 \end{bmatrix}$ **(D)** None of these

72. Let A be a square matrix of order 3 such that transpose of inverse of A is A itself. Then |adj (adj A)| is equal to

(A) 9 **(B)** 27 **(C)** 4 **(D)** 1

73. If $A = \begin{bmatrix} \alpha & a \\ \beta & b \\ \gamma & c \end{bmatrix}$, then AA^T is

(A) A non-singular matrix **(B)** A singular matrix
(C) An identity matrix **(D)** None of these

74. If A and B are two non-singular square matrices of the same order, the adjoint of AB is equal to

(A) (adj A) (adj B) **(B)** (adj B) (adj A)
(C) adj(BA) **(D)** adj A + adj B

75. If $A^k = 0$ (null matrix) for some positive integral value of k, then $I + A + A^2 + \cdots + A^{k-1}$ is equal to

(A) Null matrix **(B)** $(I + A)^k$
(C) $(I - A)^{-1}$ **(D)** None of these

76. The matrix X for which $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix} X = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$ is

(A) $\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$ **(B)** $\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix}$

(C) $\begin{bmatrix} 6 & 2 \\ \frac{11}{2} & 2 \end{bmatrix}$ **(D)** $\begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$

77. If $A=\begin{bmatrix}\alpha & 0\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}$, then the value of a for which $A^2=B$ is

(A) 1 **(B)** −1 **(C)** 4 **(D)** No real values

78. If $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & b & -1\end{bmatrix}$, then the trace of the matrix A is equal to

(A) 1 **(B)** −1 **(C)** 0 **(D)** $a+b$

79. Let A and B are the non-singular square matrices, then which of the following is always correct?

(A) $(AB)^{\theta}=A^{\theta}B^{\theta}$ **(B)** $(AB)'=B'A'$

(C) $A(\text{adj } B)=B(\text{adj } A)$ **(D)** $|\text{adj } A|=|A|^{n-2}$

80. If $l_r^2+m_r^2+n_r^2=1$, where $r=1, 2, 3$ and $l_1l_2+m_1m_2+n_1n_2=0$ …etc, then, $\Delta=\begin{vmatrix}l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{vmatrix}^2$ is equal to

(A) −1 **(B)** 1 **(C)** ±1 **(D)** 3

81. If α is a root of $x^4=1$ with negative principal argument, then the principal argument of $\Delta(\alpha)$ where

$$\Delta(\alpha)=\begin{vmatrix}1 & 1 & 1\\ \alpha^n & \alpha^{n+1} & \alpha^{n+3}\\ \dfrac{1}{\alpha^{n+1}} & \dfrac{1}{\alpha^n} & 0\end{vmatrix} \text{ is}$$

(A) $\frac{5\pi}{4}$ **(B)** $-\frac{3\pi}{4}$ **(C)** $\frac{\pi}{4}$ **(D)** $-\frac{\pi}{4}$

82. If $a, b>0$ and $\Delta(x)=\begin{vmatrix}x & a & a\\ b & x & a\\ b & b & x\end{vmatrix}$, then

(A) $\Delta(x)$ is increasing in $(-\sqrt{ab}, \sqrt{ab})$

(B) $\Delta(x)$ is decreasing in $(\sqrt{ab}, \infty)$

(C) $\Delta(x)$ has a local minimum at $x=\sqrt{ab}$

(D) None of these

83. If $f(x)=\begin{vmatrix}\cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma)\\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma)\\ \sin(\beta-\alpha) & \sin(\gamma-\alpha) & \sin(\alpha-\beta)\end{vmatrix}$ and $f(2)=5$, then $\sum_{r=1}^{20} f(r)$ is equal to

(A) 5 **(B)** 10 **(C)** 100 **(D)** None of these

84. The value of $f\left(\frac{\pi}{6}\right)$ where

$$f(q)=\begin{vmatrix}\cos^2\theta & \cos\theta\sin\theta & -\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta & \cos\theta\\ \sin\theta & -\cos\theta & 0\end{vmatrix} \text{ is}$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

85. The system of equations $102x+95y+88z=81$, $3x+10y+17z=24$, $57x+50y+43z=36$ has

(A) Many solutions **(B)** No solution

(C) A unique solution **(D)** None of these.

86. If $ax^3+bx^2+cx+d=\begin{vmatrix}x^2 & (x-1)^2 & (x-2)^2\\ (x-1)^2 & (x-2)^2 & (x-3)^2\\ (x-2)^2 & (x-3)^2 & (x-4)^2\end{vmatrix}$, then

(A) $a=1, b=2, c=3, d=-8$

(B) $a=-1, b=2, c=3, d=-8$

(C) $a=0, b=0, c=0, d=8$

(D) $a=0, b=0, c=0, d=-8$

87. If in a ΔABC, $\begin{vmatrix}1 & \sin A & \sin^2 A\\ 1 & \sin B & \sin^2 B\\ 1 & \sin C & \sin^2 C\end{vmatrix}=0$, then the triangle is

(A) Equilateral or isosceles

(B) Equilateral or right angled

(C) Right angled or isosceles

(D) None of these

88. If a, b and c are sides of a ΔABC and

$$\begin{vmatrix}a^2 & b^2 & c^2\\ (a+1)^2 & (b+1)^2 & (c+1)^2\\ (a-1)^2 & (b-1)^2 & (c-1)^2\end{vmatrix}=0, \text{ then}$$

(A) ABC is an equilateral triangle

(B) ABC is a right angled triangle

(C) ABC is an Isosceles triangle

(D) None of these

89. The value of $\begin{vmatrix}^{10}C_4 & ^{10}C_5 & ^{11}C_m\\ ^{11}C_6 & ^{11}C_7 & ^{12}C_{m+2}\\ ^{12}C_8 & ^{12}C_9 & ^{13}C_{m+4}\end{vmatrix}$ is zero, when m is

(A) 6 **(B)** 4 **(C)** 5 **(D)** None of these

90. If $D=\begin{vmatrix}2 & 1 & [\sin^2\theta]\\ [\sin^2\theta] & \cos\theta & i\\ i & 1 & \sin\theta\end{vmatrix}$ (here $[\cdot]$ is greatest integer function and $i=\sqrt{-1}$), then $\arg(D)\in$

(A) $\{-\tan^{-1}2, -\pi+\tan^{-1}2\}$

(B) $\left(\frac{-3\pi}{4}, \frac{-\pi}{2}\right)\cup\{-\tan^{-1}2, -\pi+\tan^{-1}2\}$

(C) $\left[\frac{-3\pi}{4}, \frac{-\pi}{2}\right]$

(D) None of these

91. If $\Delta(x)=\begin{vmatrix} e^x & \sin 2x & \tan x^2 \\ \ln(1+x) & \cos x & \sin x \\ \cos x^2 & e^x-1 & \sin x^2 \end{vmatrix}=A+Bx+Cx^2+\cdots$, then B is equal to

(A) 0 **(B)** 1 **(C)** 2 **(D)** None of these

92. Let $x_1y_1z_1$, $x_2y_2z_2$ and $x_3y_3z_3$ be three 3-digit even numbers and $\Delta=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$. Then, Δ is

(A) Divisible by 2 but not necessarily by 4
(B) Divisible by 4 but not necessarily by 8
(C) Divisible by 8
(D) None of these

93. If α, β, γ are the roots of $x^3+ax^2+b=0$, then the determinant $\Delta=\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ equals

(A) $-a^3$ **(B)** a^3-3b **(C)** a^2+3b **(D)** a^3

94. Let a, b, c be cube roots of unity and $\Delta=\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$, then

(A) $\text{Re}(\Delta)=0$ **(B)** $\text{Im}(\Delta)=0$
(C) $\text{Re}(\Delta)+\text{Im}(\Delta)=0$ **(D)** $\text{Re}(\Delta)\text{Im}(D)=4$

95. Given $q^2-pr<0$, $p>0$, then the value of $\Delta=\begin{vmatrix} p & q & px+qy \\ q & r & qx+ry \\ px+qy & qx+ry & 0 \end{vmatrix}$ is

(A) 0 **(B)** Positive **(C)** Negative **(D)** q^2+pr

96. If p, q, r are in AP, then the determinant

$$\begin{vmatrix} a^2+2^{n+1}+2p & b^2+2^{n+2}+3q & c^2+p \\ 2^n+p & 2^{n+1}+q & 2q \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix}$$ is equal to

(A) 1 **(B)** 0
(C) $a^2b^2c^2-2^n$ **(D)** $(a^2+b^2+c^2)-2^nq$

97. For the determinant $\Delta=\begin{vmatrix} -1 & a & a^2 \\ \sin x & \cos y & \sin x \\ \cos y & \sin x & \cos y \end{vmatrix}$ $(a\in R)$

(A) If Δ is non-zero, it is independent of a
(B) Δ is always independent of a
(C) If Δ is independent of a, then $\sin x+\sin y$ can be equal to $\frac{3}{2}$
(D) None of these

98. If a, b and c are positive integers and $x=cy+bz$, $y=az+cx$, $z=bx+ay$, where x, y and z are not all zero, then number of ordered triplet (a, b, c) satisfying above is

(A) 0 **(B)** 1
(C) Finitely many **(D)** Infinitely many

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If A is matrix of size $n\times n$ such that $A^2+A+2I=0$, then

(A) A is non-singular **(B)** A is symmetric
(C) $|A|\neq 0$ **(D)** $A^{-1}=-\frac{1}{2}(A+I)$

2. If $\Delta_r=\begin{vmatrix} 2^{r-1} & \frac{1}{r(r+1)} & \sin r\theta \\ x & y & z \\ 2^n-1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\sin\frac{n}{2}\theta}{\sin\frac{\theta}{2}} \end{vmatrix}$ then $\sum_{r=1}^{n}\Delta_r$ is equal to

(A) 0 **(B)** Independent of n
(C) Independent of θ **(D)** Independent of x, y and z

3. The value of x satisfying $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}=0$ is

(A) 0 **(B)** $a+b+c$
(C) $-(a+b+c)$ **(D)** None of these

4. Let $A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$. Then

(A) $A^2-4A-5I_3=0$ **(B)** $A^{-1}=\frac{1}{5}(A-4I_3)$
(C) A^3 is not invertible **(D)** A^2 is invertible

5. If A and B are invertible square matrices of the same order, then which of the following is correct?

(A) $\text{adj}(AB)=(\text{adj}B)(\text{adj}A)$
(B) $(\text{adj}A)'=(\text{adj}A')$
(C) $|\text{adj}A|=|A|^{n-1}$, where n is the order of matrix A
(D) $\text{adj}(\text{adj}B)=|B|^{n-2}B$, where n is the order of matrix B

6. If $A=\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$. Then

(A) $\text{adj}(\text{adj}A)=A$ **(B)** $|\text{adj}(\text{adj}A)|=1$
(C) $|\text{adj}A|=1$ **(D)** None of these

7. System of equation

$$x+3y+2z=6$$
$$x+\lambda y+2z=7$$
$$x+3y+2z=m$$

has

(A) Unique solution if $\lambda = 2, \mu \neq 6$
(B) Infinitely many solution if $\lambda = 4, \mu = 6$
(C) No solution if $\lambda = 5, \mu = 7$
(D) No solution if $\lambda = 3, \mu = 5$

8. Which of the following statement is always true?
(A) Adjoint of a symmetric matrix is a symmetric matrix
(B) Adjoint of a unit matrix is unit matrix
(C) $A\,(\text{adj } A) = (\text{adj } A)\, A$
(D) Adjoint of a diagonal matrix is diagonal matrix

9. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then
(A) $A^3 = 9A$ **(B)** $A^3 = 27A$
(C) $A + A = A^2$ **(D)** A^{-1} does not exist

Comprehension Type Questions

Paragraph for Questions 10–12: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$. If U_1, U_2 and U_3 are columns matrices satisfying

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix},\ AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

and U is 3×3 matrix whose columns are U_1, U_2 and U_3 then

10. The value of $|U|$ is
(A) 3 **(B)** –3 **(C)** 3/2 **(D)** 2

11. The sum of the elements of U^{-1} is
(A) –1 **(B)** 0 **(C)** 1 **(D)** 3

12. The value of $[3\ \ 2\ \ 0]\, U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is
(A) 5 **(B)** 5/2 **(C)** 4 **(D)** 3/2

Paragraph for Questions 13–15: Consider the determinant

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

M_{ij} = Minor of the element of i^{th} row and j^{th} column
C_{ij} = Cofactor of the element of i^{th} row and j^{th} column

13. Value of $b_1 \times C_{31} + b_2 \times C_{32} + b_3 \times C_{33}$ is
(A) 0 **(B)** Δ **(C)** 2Δ **(D)** Δ^2

14. If all the elements of the determinant are multiplied by 2, then the value of new determinant is
(A) 0 **(B)** 8Δ **(C)** 2Δ **(D)** $2^9 \times \Delta$

15. $a_3 M_{13} - b_3 \times M_{23} + d_3 \times M_{33}$ is equal to
(A) 0 **(B)** 4Δ **(C)** 2Δ **(D)** Δ

Paragraph for Questions 16–18: Let A be an $m \times n$ matrix. If there exists a matrix L of type $n \times m$ such that $LA = I_n$, then L is called left inverse of A. Similarly, if there exists a matrix R of type $n \times m$ such that $AR = I_m$, then R is called right inverse of A.

For example, to find right inverse of matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$$

we take

$$R = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}$$

and solve $AR = I_3$ that is,

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{array}{lll} x - u = 1 & y - v = 0 & z - w = 0 \\ x + u = 0 & y + v = 1 & z + w = 0 \\ 2x + 3u = 0 & 2y + 3v = 0 & 2z + 3w = 1 \end{array}$$

As this system of equations is inconsistent, we say there is no right inverse for matrix A.

16. Which of the following matrices is not left inverse of matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$?

(A) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ **(B)** $\begin{bmatrix} 2 & -7 & 3 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

(C) $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ **(D)** $\begin{bmatrix} 0 & 3 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

17. The number of right inverses for the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ is
(A) 0 **(B)** 1 **(C)** 2 **(D)** Infinite

18. For which of the following matrices number of left inverses is greater than the number of right inverses?

(A) $\begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix}$ **(B)** $\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix}$ **(D)** $\begin{bmatrix} 3 & 3 \\ 1 & 1 \\ 4 & 4 \end{bmatrix}$

Paragraph for Questions 19–21: Let $A = [a_{ij}]_3$ be a square matrix of order 3 whose elements are distinct integers from 1, 2, …, 9. The matrix is formed so that the sum of numbers in every row, column and diagonal is a multiple of 9.

19. The number of possible combinations of three distinct numbers from 1 to 9 that have a sum of 9 or 18 is
(A) 10 **(B)** 7 **(C)** 8 **(D)** 9

20. The element a_{22} must be a multiple of
(A) 2 **(B)** 3 **(C)** 4 **(D)** 9

21. The maximum value of trace of the matrix A is
(A) 18 **(B)** 19 **(C)** 12 **(D)** none

Matrix Match Type Questions

22. Match the following:

Column I	Column II
(A) A is a real skew-symmetric matrix such that $A^2 + I = 0$. Then	**(p)** $BA - AB$
(B) A is a matrix such that $A_2 = A$. If $(I + A)^n = I + \lambda A$, then λ equals ($n \in N$)	**(q)** A is of even order
(C) If for a matrix A, $A^2 = A$, and $B = I - A$, then $AB + BA + I - (I - A)^2$ equals	**(r)** A
(D) A is a matrix with complex entries and A^* stands for transpose of complex conjugate of A. If $A^* = A$ and $B^* = B$, then $(AB - BA)^*$ equals	**(s)** $2^n - 1$
	(t) ${}^nC_1 + {}^nC_2 \cdots + {}^nC_n$

23. Match the following:

Column I	Column II
(A) Let $\lvert A\rvert = \lvert a_{ij}\rvert_{3\times 3} \neq 0$. Each element a_{ij} is multiplied by k^{i-j}. Let $\lvert B\rvert$ the resulting determinant, where $k_1\lvert A\rvert + k_2\lvert B\rvert = 0$. Then $k_1 + k_2 =$	**(p)** 0
(B) The maximum value of a third-order determinant each of its entries are ± 1 equals	**(q)** 4
(C) $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$	**(r)** 1
(D) $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B$, where A and B are determinants of order 3. Then $A + 2B =$	**(s)** 2
	(t) $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

Integer Type Questions

24. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$ and $B = (\text{adj } A)$ and $C = 5A$, then find the value of $\dfrac{|\text{adj} B|}{|C|}$.

25. If $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $\phi(x) = (1 + x)(1 - x)^{-1}$ and $\phi(A) = -\lambda A$, then find the value of λ.

26. If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $abc = 1$, $A'A = I$, then find maximum value of $a^3 + b^3 + c^3$.

27. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $A^2 = 8A + KI_2$, then find the value of $|k|$.

28. Find the coefficient of x in the determinant

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix}, \text{ where } a_i, b_j \in N$$

29. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$ and $f(2) = 6$, then find $\dfrac{1}{5}\sum_{r=1}^{25} f(r)$.

30. Let $f(x) = \begin{vmatrix} x & 1 & 1 \\ \sin 2\pi x & 2x^2 & 1 \\ x^3 & 3x^4 & 1 \end{vmatrix}$. If $f(x)$ be an odd function and its odd value is equal to $g(x)$, then find the value of λ. Also $f(1)\, g(1) = -4\lambda$

31. If $f(x)$ satisfies the equation $\begin{vmatrix} f(x+1) & f(x+8) & f(x+1) \\ 1 & 2 & -5 \\ 2 & 3 & \lambda \end{vmatrix} = 0$ for all real x and if f is periodic with period 7, then find the value of $|\lambda|$.

Answer Key

Practice Exercise 1

1. (A)	**2.** (D)	**3.** (A)	**4.** (A)	**5.** (A)	**6.** (A)
7. (C)	**8.** (B)	**9.** (A)	**10.** (D)	**11.** (C)	**12.** (C)
13. (C)	**14.** (C)	**15.** (B)	**16.** (B)	**17.** (A)	**18.** (C)

19. (A)	**20.** (B)	**21.** (A)	**22.** (D)	**23.** (B)	**24.** (B)
25. (D)	**26.** (C)	**27.** (B)	**28.** (A)	**29.** (C)	**30.** (D)
31. (C)	**32.** (C)	**33.** (B)	**34.** (D)	**35.** (A)	**36.** (D)
37. (B)	**38.** (D)	**39.** (B)	**40.** (A)	**41.** (A)	**42.** (B)
43. (D)	**44.** (B)	**45.** (C)	**46.** (C)	**47.** (A)	**48.** (A)
49. (A)	**50.** (C)	**51.** (B)	**52.** (A)	**53.** (A)	**54.** (C)
55. (B)	**56.** (C)	**57.** (C)	**58.** (A)	**59.** (A)	**60.** (B)
61. (A)	**62.** (B)	**63.** (B)	**64.** (D)	**65.** (B)	**66.** (C)
67. (C)	**68.** (C)	**69.** (D)	**70.** (B)	**71.** (A)	**72.** (D)
73. (B)	**74.** (B)	**75.** (C)	**76.** (C)	**77.** (D)	**78.** (A)
79. (B)	**80.** (B)	**81.** (B)	**82.** (C)	**83.** (C)	**84.** (B)
85. (A)	**86.** (D)	**87.** (A)	**88.** (C)	**89.** (C)	**90.** (D)
91. (A)	**92.** (A)	**93.** (D)	**94.** (B)	**95.** (C)	**96.** (B)
97. (D)	**98.** (A)				

Practice Exercise 2

1. (A), (C), (D)	**2.** (A), (B), (C), (D)	**3.** (A), (D)	**4.** (A), (B), (D)	**5.** (A), (B), (C), (D)	**6.** (A), (B), (C)
7. (B), (C), (D)	**8.** (A), (B), (C), (D)	**9.** (A), (D)	**10.** (A)	**11.** (B)	**12.** (A)
13. (A)	**14.** (B)	**15.** (D)	**16.** (C)	**17.** (D)	**18.** (C)
19. (A)	**20.** (B)	**21.** (A)	**22.** (A) → (q), (B) → (s, t), (C) → (r), (D) → (p)		
23. (A) → (p, t), (B) → (q), (C) → (r), (D) → (p, t)			**24.** 1	**25.** 1	**26.** 2, 4
27. 7	**28.** 0	**29.** 150	**30.** −1	**31.** 4	

Solutions

Practice Exercise 1

1. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ (Unit matrix)

2. $|A||\text{adj}A| = |A||A|^2 = |A|^3 = (a^3)^3 = a^9$

3. $|A| = 0$ as the determinant of any skew-symmetric matrix of odd order is zero.

4. If we consider $m = 2$

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$$
$$= A^2 + 2AB + B^2 \text{ (if } AB = BA)$$

Similarly

$$(A+B)^m = {}^mC_0A^m + {}^mC_1A^{m-1}B + {}^mC_2A^{m-2}B^2 + \cdots + {}^mC_mB^m$$

holds if $AB = BA$.

5. We know that $[a_{ij}]' = -[a_{ij}]$ for skew-symmetric matrix. Then

$$[a_{ji}] + [a_{ij}] = 0$$
$$\Rightarrow a_{ji} + a_{ij} = 0$$

For $i = j$, we get

$$a_{ii} + a_{ii} = 0 \Rightarrow a_{ii} = 0$$

This means diagonal elements of every skew-symmetric matrix are zero.

6. $\begin{bmatrix} 7 & 1 & 2 \\ 9 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 21+4+10 \\ 27+8+5 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 35 \\ 40 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 43 \\ 44 \end{bmatrix}$

7. $A\,(\text{adj } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 10I$. Also $A(\text{adj } A) = |A|I$.

Hence, $|A| = 10$.

8. Let A be symmetric as well as skew-symmetric matrix. Then
$A' = A$ and $A' = -A$
$\Rightarrow A = -A$ or $2A = 0$ or $A = 0$

9. $AB = 0 \Rightarrow |AB| = |0| \Rightarrow |A||B| = 0 \Rightarrow$ either $|A| = 0$ or $|B| = 0$

10. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $A^3 = 0$.

11. $A=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}$

Since $A^2=0$, hence, A is a nilpotent matrix with index 2.

12. A and B are symmetric matrices of same order, i.e. $A=A'$, $B=B'$. Then

$$(AB+BA)^T=(AB)^T+(BA)^T$$
$$=B^T\times A^T+A^T\times B^T=BA+AB=AB+BA$$

So, $AB+BA$ is a symmetric matrix.

13. A is a square matrix

$$(AA')'=(A')'\cdot A'=AA'$$

Hence, AA' is a symmetric matrix.

14. A is a square matrix and $A^3=I\Rightarrow A^3=I\Rightarrow A^2\times A=I$. So A^2 is the inverse of A

Hence, $A^{-1}=A^2$.

15. A is a square matrix

$$(A-A')'=A'-(A')'=A'-A$$

So, $A-A'$ is not a symmetric matrix.

16. Because determinant of every skew-symmetric matrix of odd order is zero, therefore, $|A|=0$ if n is odd.

17. Each diagonal element of skew-symmetric matrix is zero.

18. $A=\begin{bmatrix}4&2\\-1&1\end{bmatrix}$

$$A-2I=\begin{bmatrix}4&2\\-1&1\end{bmatrix}+\begin{bmatrix}-2&0\\0&-2\end{bmatrix}=\begin{bmatrix}2&2\\-1&-1\end{bmatrix}$$

$$A-3I=\begin{bmatrix}4&2\\-1&1\end{bmatrix}+\begin{bmatrix}-3&0\\0&-3\end{bmatrix}=\begin{bmatrix}1&2\\-1&-2\end{bmatrix}$$

Hence,

$$(A-2I)\,(A-3I)=\begin{bmatrix}2&2\\-1&-1\end{bmatrix}=\begin{bmatrix}1&2\\-1&-2\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

19. $O(A)=m\times(n+5)$ and $O(B)=m\times(11-n)$

AB exists $\Rightarrow n+5=m\Rightarrow m-n=5$ (1)

BA exists $\Rightarrow 11-n=m\Rightarrow m+n=11$ (2)

Solving Eqs. (1) and (2), we have

$$m=8 \text{ and } n=3$$

Therefore, $O(A)=8\times 8$ and $O(B)=8\times 8$

Therefore, AB and BA both are square matrices of order 8×8.

20. $A=\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}$

$$|A|=\begin{vmatrix}1&3&3\\1&4&3\\1&3&4\end{vmatrix}$$
$$=1(16-9)-3(4-3)+3(3-4)$$
$$=7-3-3=1$$

$$C_A=\begin{bmatrix}7&-1&-1\\-3&1&0\\-3&0&1\end{bmatrix}\Rightarrow \text{adj } A=\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}$$

Therefore,

$$A^{-1}=\frac{1}{|A|}\text{adj } A=\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}$$

21. $A=\begin{bmatrix}i&-i\\-i&i\end{bmatrix}$ and $B=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$

$$A^2=\begin{bmatrix}i&-i\\-i&i\end{bmatrix}\begin{bmatrix}i&-i\\-i&i\end{bmatrix}=\begin{bmatrix}-2&2\\2&-2\end{bmatrix}=-2\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=-2B$$

$$A^4=4B^2=4\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=4\begin{bmatrix}2&-2\\-2&2\end{bmatrix}=8\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=8B$$

$$A^8=64B^2=64\begin{bmatrix}2&-2\\-2&2\end{bmatrix}=128\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=128B$$

22. Because $\cos(-\theta)=\cos\theta\Rightarrow A$ is a symmetric matrix.

Alternative solution:

Given determinant can also be written as the product of two determinants as follows:

$$\begin{vmatrix}1&\cos(\alpha-\beta)&\cos(\alpha-\gamma)\\\cos(\beta-\alpha)&1&\cos(\beta-\gamma)\\\cos(\gamma-\alpha)&\cos(\gamma-\beta)&1\end{vmatrix}$$

$$=\begin{vmatrix}\cos\alpha&\sin\alpha&0\\\cos\beta&\sin\beta&0\\\cos\gamma&\sin\gamma&0\end{vmatrix}\times\begin{vmatrix}\cos\alpha&\sin\alpha&0\\\cos\beta&\sin\beta&0\\\cos\gamma&\sin\gamma&0\end{vmatrix}\quad\text{(row by row)}$$

$=0\times 0=0$

Also $|A|=\begin{vmatrix}1&a&c\\a&1&b\\b&c&1\end{vmatrix}$ where $a=\cos(\alpha-\beta)$, $b=\cos(\beta-\gamma)$,

$c=\cos(\gamma-\alpha)$

$=1-a^2-b^2-c^2+2abc$

$=1-[\cos^2(\alpha-\beta)+\cos^2(\beta-\gamma)+\cos^2(\gamma-\alpha)-2\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\gamma-\alpha)]$

$=1-[1+\cos^2(\alpha-\beta)-\sin^2(\beta-\gamma)+\cos(\gamma-\alpha)\{\cos(\gamma-\alpha)-2\cos(\alpha-\beta)\cos(\beta-\gamma)\}]$

$=1-[1+\cos(\alpha-\gamma)\cos(\alpha-2\beta+\gamma)+\cos(\gamma-\alpha)\{\cos(\gamma-\alpha)-\cos(\alpha-\gamma)\cos(\alpha-2\beta+\gamma)\}]$

$=1-[1+\cos(\alpha-\gamma)\cos(\alpha-2\beta+\gamma)-\cos(\gamma-\alpha)\cos(\alpha-2\beta+\gamma)]$

$=0$

23. If $|A|\neq 0$, then homogeneous system of linear equations $AX=0$ has only trivial solution, i.e. $X=0$.

24. $[[A,B],C]+[[B,C],A]+[[C,A],B]$

$=[AB-BA,C]+[BC-CB,A]+[CA-AC,B]$

$=(AB-BA)C-C(AB-BA)+(BC-CB)A-A(BC-CB)+(CA-AC)B$

$- B(CA - AC)$

$= \cancel{ABC} - \cancel{BAC} - \cancel{CAB} + \cancel{CBA} + \cancel{BCA} - \cancel{CBA} - \cancel{ABC}$
$\quad + \cancel{ACB} + \cancel{CAB} - \cancel{ACB} - \cancel{BCA} + \cancel{BAC} = 0$

25. $[A(A+B)^{-1}B]^{-1} = B^{-1}(A+B)A^{-1}$ which cannot be simplified further, in general.

26. A and B matrices commute so $AB = BA$.

$$A^{-1} \times B^{-1} = (BA)^{-1} = (AB)^{-1} = B^{-1} \times A^{-1}$$

So, A^{-1} and B^{-1} also commute.

27. A, B, C are three conformable matrices for multiplication $(ACB)^{-1} = B^{-1} \times C^{-1} \times A^{-1}$.

28. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A^2 is a unit matrix of third order.

29. Trace of a skew-symmetric matrix is always equal to zero. Now

$$A = -A' \Rightarrow \sum a_{ii} = 0$$

30. After reflection in line $x + y = 0$, y becomes x. Therefore, we need a matrix which when multiplied by

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \text{ changes it to } \begin{bmatrix} -y_1 & -x_1 \\ -y_2 & -x_2 \end{bmatrix}$$

We observe that

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -y_1 & -x_1 \\ -y_2 & -x_2 \end{bmatrix}$$

Hence, matrix of transformation is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

31. The degree of any term of the determinant is

$$n + n + 2 + n + 3 = 2 \Rightarrow n = -1$$

Hence, (C) is the correct answer.

32. $\sum_{r=1}^{N} 2r = 2\left(\dfrac{N(N+1)}{2}\right) = N(N+1)$

$\sum_{r=1}^{N} (6r^2 - 1) = 6N\left(\dfrac{(N+1)(2N+1)}{6}\right) - N$

$= N(2N^2 + 2N + N + 1) - N = 2N^3 + 3N^2 = N^2(2N+3)$

$\sum_{r=1}^{N} (4r^3 - 2Nr) = N^3(N+1)$

Hence, (C) is the correct answer.

33. Since $1 + \omega + \omega^2 = 0$, the given determinant is

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} \qquad (R_1 \to R_1 + R_2 + R_3)$$

$= 3(\omega^2 - \omega^4) = 3(\omega^2 - \omega) = 3\omega(\omega - 1)$

Hence, (B) is the correct answer.

34. We can write $\Delta(x)$ as product of two determinants as follows:

$$\Delta(x) = \begin{vmatrix} \sin x & \cos x & 1 \\ 1 & 1 & 1 \\ e^x & 0 & 0 \end{vmatrix} \times \begin{vmatrix} \cos x & \sin x & 0 \\ e^x & x & 0 \\ 1 & x^2 & 0 \end{vmatrix} = 0$$

which is independent of $(x) \Rightarrow \Delta'(x) = 0 \ \forall\, x$.

35. $\lim_{x \to 0} \dfrac{g(x)}{x} \qquad \left[\dfrac{0}{0} \text{ form as } g(0) = 0\right]$

$\lim_{x \to 0} \dfrac{g'(x)}{1} = g'(0) = 0$

Hence, (A) is the correct answer.

36. Note that $\Delta(x)$ is a polynomial of degree at most 6 in x.
If $\Delta(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_6 x^6$, then
$\Delta'(x) = a_1 + 2a_2 x + \cdots + 6a_6 x^5 \Rightarrow a_1 = \Delta'(0)$.

Now

$$\Delta'(x) = \begin{vmatrix} 4 & (x-2)^2 & x^3 \\ 8 & (x-2\sqrt{2})^2 & (x+1)^3 \\ 12 & (x-2\sqrt{3})^2 & (x-1)^3 \end{vmatrix}$$
$$+ \begin{vmatrix} 4x-4 & 2(x-2) & x^3 \\ 8x-4\sqrt{2} & 2(x-2\sqrt{2}) & (x+1)^3 \\ 12x-4\sqrt{3} & 2(x-2\sqrt{3}) & (x-1)^3 \end{vmatrix}$$
$$+ \begin{vmatrix} 4x-4 & (x-2)^2 & 3x^2 \\ 8x-4\sqrt{2} & (x-2\sqrt{2})^2 & 3(x+1)^2 \\ 12x-4\sqrt{3} & (x-2\sqrt{3})^2 & 3(x-1)^2 \end{vmatrix}$$

$$\Rightarrow \Delta'(0) = 0 + 0 + \begin{vmatrix} -4 & 4 & 0 \\ -4\sqrt{2} & 8 & 3 \\ -4\sqrt{3} & 12 & 3 \end{vmatrix}$$

$\Rightarrow \Delta'(0) = 48(1 + \sqrt{2} - \sqrt{3})$

37. We have $\Delta_2 = \begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix}$

Cofactors of 1st row are: $x^2 + a^2, ab + cx, ac - bx$
Cofactors of 2nd row are: $ab - cx, x^2 + b^2, ac + bx$
Cofactors of 3rd row are: $ac + bx, bc - ax, x^2 + c^2$
Therefore, determinant of cofactors of Δ_2 is

$$\begin{vmatrix} x^2+a^2 & ab+cx & ac-bx \\ ab-cx & x^2+b^2 & ax+bc \\ ac+bx & bc-ax & x^2+c^2 \end{vmatrix}$$

$$= \begin{vmatrix} x^2+a^2 & ab-cx & ac+bx \\ ab+cx & x^2+b^2 & bc-ax \\ ac-bx & ax+bc & x^2+c^2 \end{vmatrix}$$

$= \Delta^2_2 \qquad (\because |\text{adj } A| = |A|^2 \text{ where } A \text{ is of order})$

Thus, $\Delta_1 = \Delta^2_2$.

38. By applying $C_1 \to C_1 + C_2 + C_3$, we get $\Delta = 0$.
Hence, (D) is the correct answer.

39. Direct expansion gives $f(x) = \cos 3x$. Hence, $\int_0^{\pi/2} \cos 3x\,dx = -\frac{1}{3}$.

Hence, (B) is the correct answer.

40. $f(-x) = -f(x)$. It is an odd function. Hence, $\int_{-\pi/2}^{\pi/2} f(x) = 0$.

Hence, (A) is the correct answer.

41. Putting $x = 0$, we get $e = \begin{vmatrix} 3 & 1 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$.

$$C_3 \to C_3 - C_1 \text{ gives } \begin{vmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

42. By applying $R_3 \to R_3 - \alpha R_1 - R_2$, we get

$$(b^2 - ac)(aa^2 + 2ba + c) = 0$$

Hence, (B) is the correct answer.

43. Use concept of polynomial roots. Sum of the roots = 0.
By applying $R_1 \to R_1 + R_2 + R_3$ and using $\alpha + \beta + \gamma$, we get $\Delta = 0$.

44. Here $\Delta = 0$ for $k = 3, 1$; $\Delta_x = 0$ for $k = 2, 1$, $\Delta_y = 0$ for $k = 1$.
Hence, $k = 1$.

Alternate method:
For infinitely many solutions the two equations become identical, so

$$\frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1} \quad \Rightarrow \quad k = 1$$

45. $\Delta(0) = 0$ and $\Delta'(0) = 0 \Rightarrow x$ is a repeated root of $\Delta \Rightarrow \Delta$ is divisible by x^2.

46. If given homogeneous system has non-trivial solution then $C_1 - C_2, C_2 - C_3$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a-1 & 0 & 1 \\ 1-b & b-1 & 1 \\ 0 & 1-c & c \end{vmatrix} = 0$$

Expanding along R_1 we get

$$(a-1)[(b-1)c - (1-c)] + (1-b)(1-c) = 0$$

$$\Rightarrow (1-a)(1-b)c + (1-a)(1-c) + (1-b)(1-c) = 0$$

Dividing by $(1-a)(1-b)(1-c)$, we get

$$\frac{c}{1-c} + \frac{1}{1-b} + \frac{1}{1-a} = 0$$

$$\Rightarrow -\left(\frac{1-c-1}{1-c}\right) + \frac{1}{1-b} + \frac{1}{1-a} = 0$$

$$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

47. $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1 + a(a^2) = 0 \Rightarrow a^3 = -1 \Rightarrow a = -1$

Hence, (A) is the correct answer.

48. $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} \sin(A+B) & \sin(A-B) & \cot A - \cot B & 0 \\ \sin(B+C) & \sin(B-C) & \cot B - \cot C & 0 \\ & \sin^2 C & \cot C & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \sin C & \sin(A-B) & \dfrac{\sin(B-A)}{\sin A \cdot \sin B} & 0 \\ \sin A & \sin(B-C) & \dfrac{\sin(C-B)}{\sin B \cdot \sin C} & 0 \\ & \sin^2 C & \cot C & 1 \end{vmatrix}$$

Expanding along the third column, we have

$$\Delta = \frac{\sin(A-B)\cdot\sin(C-B)}{\sin B} - \frac{\sin(B-C)\cdot\sin(B-A)}{\sin B}$$

$$\frac{\sin(A-B)}{\sin B}[-\sin(B-C) + \sin(B-C)] = 0$$

49. We know that

$$(a^3 + b^3 + c^3 - 3abc)^2 = \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} A & B & B \\ B & A & B \\ B & B & A \end{vmatrix} = \begin{vmatrix} B & A & B \\ B & B & A \\ A & B & B \end{vmatrix}$$

50. $\sin(A+B+C) = \cos\left(\frac{A+B+C}{2}\right) = 0$. Hence

$$\Delta = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$$

51. $\Delta = \begin{vmatrix} x-5 & 2 & 5 \\ 0 & x & 3 \\ 5-x & 4 & x \end{vmatrix}$

$$\Rightarrow (x-5)\begin{vmatrix} 1 & 2 & 5 \\ 0 & x & 3 \\ -1 & 4 & x \end{vmatrix} = 0 \qquad (C_1 \to C_1 - C_3)$$

$$\Rightarrow (x-5)\begin{vmatrix} 0 & 6 & 5+x \\ 0 & x & 3 \\ -1 & 4 & x \end{vmatrix} = 0 \quad (R_1 \to R_1 + R_3)$$

$$\Rightarrow (x-5)[18 - x(5+x)] = 0$$

Therefore, the non-integral roots are the roots of

$$x^2 + 5x - 18 = 0$$

52. Using $R_2 \to R_2 - R_3$ we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2\cdot 2^x \cdot 2 \cdot 2^{-x} & 2\cdot 3^x \cdot 2\cdot 3^{-x} & 2\cdot 5^x \cdot 2 \cdot 5^{-x} \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix} = 4$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ (2^x - 2^{-x})^2 & (3^x - 3^{-x})^2 & (5^x - 5^{-x})^2 \end{vmatrix} = 0$$

Hence, (A) is the correct answer.

53. From the given conditions,

$$\frac{y-x}{b-a}\neq\pm 1,\quad \frac{z-y}{c-b}\neq\pm 1,\ \frac{z-x}{c-a}\neq\pm 1$$

$$\Rightarrow x+a\neq y+b\neq z+c$$

The determinant is a symmetric one. The determinant will be equal to zero if $x+a+y+b+z+c=0$

But $a+b+c=0$ (given). So

$$x+y+z=0$$

$$\Rightarrow x+z=2\left(-\frac{y}{2}\right)\Rightarrow x,\ -\frac{y}{2},\ z \text{ are in AP}$$

Hence, (A) is the correct answer.

54. The determinant $=\begin{vmatrix} n-1 & n^2-n & n^2\\ n & n^2-1 & n^2+n\\ n-1 & n^2+1 & n^2\end{vmatrix}$

$$=\begin{vmatrix} n-1 & n^2-n & n^2\\ n & n^2-1 & n^2+n\\ 0 & n+1 & 0\end{vmatrix}\qquad (R_3\to R_3-R_1)$$

$=-(n+1)[n(n^2-1)-n^3]=n(n+1)=72=8\times 9$

55. $\Delta=\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}\begin{vmatrix} 1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$

$$=\begin{vmatrix} 1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}^2=[(a-b)(b-c)(c-a)]^2$$

Hence, (B) is the correct answer.

56. $\begin{vmatrix} 1 & 1 & 1\\ \cot A/2 & \cot B/2 & \cot C/2\\ \tan B/2+\tan C/2 & \tan C/2+\tan A/2 & \tan A/2+\tan B/2\end{vmatrix}=0$

Operating C_1-C_2 and C_2-C_3

$$\begin{vmatrix} 0 & 0 & 1\\ \dfrac{\tan B/2-\tan A/2}{\tan A/2\tan B/2} & \dfrac{\tan C/2-\tan B/2}{\tan B/2\tan C/2} & \cot C/2\\ \tan B/2-\tan A/2 & \tan C/2-\tan B/2 & \tan A/2+\tan B/2\end{vmatrix}=0$$

$\Rightarrow (\tan B/2-\tan A/2)(\tan C/2-\tan B/2)\times$

$$\begin{vmatrix} 0 & 0 & 1\\ \dfrac{1}{\tan A/2\tan B/2} & \dfrac{1}{\tan B/2\tan C/2} & \cot C/2\\ 1 & 1 & \tan A/2+\tan B/2\end{vmatrix}=0$$

Expanding along R_1 we get

$(\tan B/2-\tan A/2)(\tan C/2-\tan B/2)(\tan C/2-\tan A/2)=0$

$\Rightarrow A=B$ or $B=C$ or $C=A$

$\Rightarrow\Delta$ must be an isosceles triangle

57. Let $\Delta=\begin{vmatrix} \sin x & \cos x & \cos x\\ \cos x & \sin x & \cos x\\ \cos x & \cos x & \sin x\end{vmatrix}$

Applying $C_1\to C_1+C_2+C_3$, we get

$$\Delta=\begin{vmatrix} \sin x+2\cos x & \cos x & \cos x\\ \sin x+2\cos x & \sin x & \cos x\\ \sin x+2\cos x & \cos x & \sin x\end{vmatrix}$$

$$=(\sin x+2\cos x)\begin{vmatrix} 1 & \cos x & \cos x\\ 1 & \sin x & \cos x\\ 1 & \cos x & \sin x\end{vmatrix}$$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$, we get

$$\Delta=(\sin x+2\cos x)\begin{vmatrix} 1 & \cos x & \cos x\\ 0 & \sin x-\cos x & 0\\ 0 & 0 & \sin x-\cos x\end{vmatrix}$$

$$\Delta=(\sin x+2\cos x)(\sin x-\cos x)^2$$

Thus, $\Delta=0$. This gives

$(\sin x+2\cos x)(\sin x-\cos x)^2=0$

$\Rightarrow\tan x=-2$ and $\tan x=1$

As $-\dfrac{\pi}{4}\le x\le\dfrac{\pi}{4}$, we get $-1\le\tan x\le 1$. Since, $\tan x=1$ we have

$$x=\frac{\pi}{4}$$

Hence, (C) is the correct answer.

58. $f'(x)=\dfrac{1-\ln x}{x^2}$, $f''(x)=\dfrac{x^2\left(-\dfrac{1}{x}\right)-(1-\ln x)2x}{x^4}=\dfrac{-1-2+2\ln x}{x^3}$

$\Rightarrow x^3f''(x)=2\ln x-3$

$\Delta=\ln x(2)-x\left(\dfrac{2}{x}+\dfrac{1}{x}\right)=2\ln x-3\Rightarrow\Delta=x^3f''(x)$

Hence, (A) is the correct answer.

59. As the given system of equations has a non-trivial solution,

$$\begin{vmatrix} \lambda & b-a & c-a\\ a-b & \lambda & c-b\\ a-c & b-c & \lambda\end{vmatrix}=0$$

When $\lambda=0$ then determinants become skew-symmetric determinants of odd order, which is equal to zero. Thus, $\lambda=0$.

60. $A=A'$. Then $|A|=|A'|$ because the expansion of a determinant row-wise is same as the expansion of a determinant column-wise.

61. $I=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}$, $J=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}$

$B=\begin{bmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\cos\theta+\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}\sin\theta$

$=I\cos\theta+J\sin\theta$

62. I_n is an identity matrix.

63. $A^2+I=0\Rightarrow A^2=\begin{bmatrix} -1 & 0\\ 0 & -1\end{bmatrix}$ of (I is of second order) $\Rightarrow$ $\begin{pmatrix} i & 0\\ 0 & i\end{pmatrix}$.

64. Each diagonal element is either 0 or 1. So number of matrices $=2^4-1$.

65. A and B are symmetric. $A = A'$, $B = B'$. So

$$(A+B)' = A' + B' = A + B$$

66. $$AB = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

67. Using $A^{-1} = \dfrac{1}{|A|}\text{adj } A$, we get

$$\begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-1} = \frac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-1}\right\}^2 = \frac{1}{625}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$$

$$= \frac{1}{625}\begin{bmatrix} 25 & 0 \\ 10a & 25 \end{bmatrix}$$

$$= \begin{bmatrix} 1/25 & 0 \\ 2a/125 & 1/25 \end{bmatrix}$$

Now

$$\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} 1/25 & 0 \\ 2a/125 & 1/25 \end{bmatrix} \Rightarrow x = \frac{2a}{125}$$

68. $$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$$

$$\Rightarrow (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$

$$\Rightarrow (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = 0$$

$\Rightarrow x^2(x+a+b+c) = 0$

Hence, $x = 0$ or $x = -(a+b+c)$.

69. $$A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} \Rightarrow |A| = -4 - 15 = -19$$

$$C_A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix} \Rightarrow \text{adj } A = C'_A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19}A$$

70. If A is a skew-symmetric matrix of odd order, then $|A| = 0$. So, inverse does not exist.
Let A be of even order. Then

$$AA^{-1} = A^{-1}A = I_n \Rightarrow (AA^{-1})^T = (A^{-1}A)^T = I_n$$

$$\Rightarrow (A^{-1})^T A^T = A^T(A^{-1})^T = I_n \Rightarrow (A^{-1})^T(-A) = (-A)(A^{-1})^T = I_n$$

So, $(A^{-1})^T = -A^{-1}$ (inverse of a matrix is unique).

71. $\text{adj}(\text{adj } A) = |A|^{n-2}A$, where n is the order of matrix. Since $n = 3$ here

$$\text{adj}(\text{adj } A) = |A|A = \begin{bmatrix} 3 & 6 & 9 \\ 3 & 9 & 15 \\ 3 & 15 & 36 \end{bmatrix} \qquad (\text{as } |A| = 3)$$

72. Given that $(A^{-1})^T = A \Rightarrow (A^T)^{-1} = A \Rightarrow AA^T = I \Rightarrow |A| = \pm 1$

Now, $|\text{adj}(\text{adj } A)| = |A|^{2^2} = |A|^4 = 1$

73. $$\begin{bmatrix} \alpha & a \\ \beta & b \\ \gamma & c \end{bmatrix}\begin{bmatrix} \alpha & \beta & \rho \\ a & b & c \end{bmatrix} = \begin{bmatrix} \alpha^2 + a^2 & \alpha\beta + ab & \alpha\gamma + ac \\ \alpha\beta + ab & \beta^2 + b^2 & \beta\gamma + bc \\ \alpha\gamma + ac & \beta\gamma + bc & \gamma^2 + c^2 \end{bmatrix}$$

$$= |AA^T| = \begin{vmatrix} \alpha & a & 0 \\ \beta & b & 0 \\ \gamma & c & 0 \end{vmatrix}\begin{vmatrix} \alpha & \beta & \gamma \\ a & b & c \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Therefore, AA^T is a singular matrix.

74. A and B are non-singular, so AB is non-singular. Hence

$$AB\,\text{adj}(AB) = |AB|\,I \qquad (1)$$

$$\begin{aligned} AB(\text{adj } B\text{ adj } A) &= A(B\text{ adj } B)\text{ adj } A \\ &= A(|B|\,I)\text{ adj } A \\ &= |B|\,(A(\text{adj } A)) \\ &= |B|\,|A|\,I \end{aligned} \qquad (2)$$

$$\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$$

75. Let $B = I + A + A^2 + \cdots + A^{k-1}$ so that

$$\begin{aligned} B(I-A) &= (I + A + A^2 + \cdots + A^{k-1})(I-A) \\ &= I - A + A - A^2 + \cdots - A^{k-1} + A^{k-1} + A^{k-1} - A^k \\ &= I - A^k = I - 0 = I \Rightarrow B = (1-A)^{-1} \end{aligned}$$

76. Let $A = \begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$. Then the matrix equation is $AX = B$.

Since $|A| = \begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix} = -2 + 12 \neq 0$; A is an invertible matrix.

Let C_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$. Then

$$\begin{aligned} C_{11} &= (-1)^{1+1}(-2) = -2 \\ C_{12} &= (-1)^{1+2}\,3 = -3 \\ C_{21} &= (-1)^{2+1}(-4) = 4 \\ C_{22} &= (-1)^{2+2}\,1 = 1 \end{aligned}$$

Therefore, $\text{adj } A = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$, so

$$A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$$

Now, $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow X = A^{-1}B$

$$\Rightarrow X = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ \dfrac{11}{2} & 2 \end{bmatrix}$$

77. $$A^2 = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix}$$

Clearly, no real value of α.

78. Trace (A) = sum of diagonal elements $= 1$

79. If A and B are the non-singular matrices, then $(AB)' = B'A'$ is always correct.

80. Multiply determinant row to row and solve.
Hence, (B) is the correct answer.

81. Clearly, $\alpha = -i$, where $i^2 = -1$. So

$$\Delta(a) = a^n \times \frac{1}{\alpha^n}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^3 \\ \frac{1}{\alpha} & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -i & i \\ i & 1 & 0 \end{vmatrix}$$

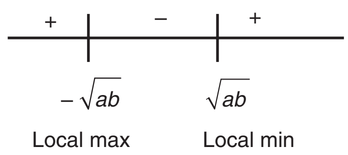
$$= 1(-i) + 1\,(i^2) + (1 + i^2) = -1 - i, \text{ arg is } -\frac{3\pi}{4}$$

82. Given that $\Delta(x) = \begin{vmatrix} x & a & a \\ b & x & a \\ b & b & x \end{vmatrix}$. We have

$$\Delta'(x) = \begin{vmatrix} 1 & a & a \\ 0 & x & a \\ 0 & b & x \end{vmatrix} + \begin{vmatrix} x & 0 & a \\ b & 1 & a \\ b & 0 & x \end{vmatrix} + \begin{vmatrix} x & a & 0 \\ b & x & 0 \\ b & b & 1 \end{vmatrix}$$

$\Rightarrow \Delta'(x) = 3(x^2 - ab)$

Now sign scheme for $\Delta'(x)$



Since $\Delta(x)$ is increasing in $(-\infty, -\sqrt{ab}) \cup (\sqrt{ab}, \infty)$, $\Delta(x)$ is decreasing in $(-\sqrt{ab}, \sqrt{ab})$

$\Delta(x)$ has a local minimum at $x = \sqrt{ab}$

$\Delta(x)$ has a local maximum at $x = -\sqrt{ab}$

83. Clearly, $f'(x) = 0 \Rightarrow f(x) =$ constant. But $f(2) = 5$. Therefore, $f(x) = 5$. Now

$$\sum_{r=1}^{20} f(r) = \sum_{r=1}^{20} 5 = 5 \times 20 = 100$$

Hence, (C) is the correct answer.

84. $C_1 \to C_1 - \sin\theta\, C_3$ and $C_2 \to C_2 + \sin\theta\, C_3$

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ \sin\theta & -\cos\theta & 0 \end{vmatrix}$$

Again $R_3 - \sin\theta\, R_1 + \cos\theta\, R_2$, we get

$$f(\theta) = \begin{vmatrix} 1 & 0 & -\sin\theta \\ 0 & 1 & \cos\theta \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow f\left(\frac{\pi}{6}\right) = 1$$

85. Here determinant of coefficient matrix

$$\Delta = \begin{vmatrix} 102 & 95 & 88 \\ 3 & 10 & 17 \\ 57 & 50 & 43 \end{vmatrix}$$

$= 0$ (using $C_1 \to C_1 + C_3 - 2C_2$)

Similarly,

$\Delta_1 = \Delta_2 = \Delta_3 = 0$

Hence, system has infinite many solutions.

86. Apply $C_1 \to C_1 - C_2$; $C_2 \to C_2 - C_3$, we get

$$\begin{vmatrix} (2x-1) & (2x-3) & (x-2)^2 \\ (2x-3) & (2x-5) & (x-3)^2 \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix}$$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ gives

$$\begin{vmatrix} 2 & 2 & (2x-5) \\ 2 & 2 & (2x-7) \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix}$$

$R_1 \to R_1 - R_2$ gives

$$\begin{vmatrix} 0 & 0 & 2 \\ 2 & 2 & (2x-7) \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix} = -8$$

Therefore, the value of determinant is independent of x.
$a = b = c = 0$ and $d = -8$.

87. Since $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$

$$\begin{vmatrix} 1 & \sin A & \sin^2 A \\ 1 & \sin B & \sin^2 B \\ 1 & \sin C & \sin^2 C \end{vmatrix} = \frac{1}{8R^3}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$\Rightarrow (a-b)(b-c)(c-a) = 0$

$\Rightarrow a = b$ or $b = c$ or $c = a$

Therefore, at least two of a, b, c are equal. So, the triangle is isosceles or equilateral.
Hence, (A) is the correct answer.

88. When $a = b$ or $b = c$ or $c = a$, the determinant reduces to zero. It is not necessary that $a = b = c$ for determinant to be zero. Therefore, triangle is isosceles.
Hence, (C) is the correct answer.

89. $C_2 \to C_2 + C_1$

$$\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{11}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{12}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{13}C_9 & {}^{13}C_{m+4} \end{vmatrix}$$

For $m = 5$, $C_2 \equiv C_3$.
Hence, (C) is the correct answer.

90. $[\sin^2\theta] = \begin{cases} 0, \sin^2\theta \neq 1 \\ 1, \sin^2\theta = 1 \end{cases}$

If $\sin^2\theta \neq 1 \Rightarrow D = 2\sin\theta\cos\theta - 2i - 1$

$\text{Re}(D) = 2\sin\theta\cos\theta - 1$

$-2 \le \text{Re}(D) \le 0$

$$-\frac{3\pi}{4} \le \arg D \le -\frac{\pi}{2}$$

If $\sin^2\theta = 1$, $\sin\theta = \pm 1$, $\cos\theta = 0$

$\arg(D) = \arg(1 - 2i)$ or $\arg(-1 - 2i)$

91. $$\Delta'(x) = \begin{vmatrix} e^x & 2\cos 2x & 2x\sec^2 x^2 \\ \ln(1+x) & \cos x & \sin x \\ \cos x^2 & e^x - 1 & \sin x^2 \end{vmatrix}$$

$$+\begin{vmatrix} e^x & \sin 2x & \tan x^2 \\ \frac{1}{(1+x)} & -\sin x & \cos x \\ \cos x^2 & e^x-1 & \sin x^2 \end{vmatrix} + \begin{vmatrix} e^x & \sin 2x & \tan x^2 \\ \ln(1+x) & \cos x & \sin x \\ -2x\sin x^2 & e^x & 2x\cos x^2 \end{vmatrix}$$

$= B + 2Cx + \cdots$

Put $x = 0$. So

$$B = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

92. $$\Delta = \begin{vmatrix} 100x_1+10y_1+z_1 & y_1 & z_1 \\ 100x_2+10y_2+z_2 & y_2 & z_2 \\ 100x_3+10y_3+z_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} 2A & y_1 & z_1 \\ 2B & y_2 & z_2 \\ 2C & y_3 & z_3 \end{vmatrix},$$

where $A, B, C \in I$

$$= 2\begin{vmatrix} A & y_1 & z_1 \\ B & y_2 & z_2 \\ C & y_3 & z_3 \end{vmatrix} \in I$$

which is divisible by 2 but not necessarily by 4 or 8.

93. Because α, β, γ are roots of $x^3 + ax^2 + b = 0$, therefore,

$\alpha+\beta+\gamma = -a,\ \alpha\beta+\beta\gamma+\gamma\alpha = 0,\ \alpha\beta\gamma = -\beta$

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = -(\alpha+\beta+\gamma)[\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha]$$

$= -(-a)[(-a)^2 - 3\times 0]$

$= a \times a^2 = a^3$

94. $$\Delta = \begin{vmatrix} 0 & -2b^2 & -2a^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix} \quad (R_1 \to R_1 - R_2 - R_3)$$

$$= -2\begin{vmatrix} 0 & b^2 & a^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

$$= -2\begin{vmatrix} 0 & b^2 & a^2 \\ a^2 & c^2 & 0 \\ b^2 & 0 & c^2 \end{vmatrix} \quad (R_1 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$= 4a^2b^2c^2 = 4$ [as a, b, c are the cube roots of unity]

Therefore, $\mathrm{Im}(\Delta) = 0$

95. Applying $R_3 \to R_3 - xR_1 - yR_2$ we get

$$\Delta = \begin{vmatrix} p & q & px+qy \\ q & r & qx+ry \\ 0 & 0 & -(px^2+2qxy+ry^2) \end{vmatrix}$$

$\Rightarrow (q^2 - pr)(px^2 + 2qxy + ry^2) < 0$

As $q^2 - pr < 0 \Rightarrow$ Discriminant of quad < 0

$\Rightarrow$ (–ve) (+ve) < 0

96 Applying $R_1 \to R_1 - R_3$ and $2q = p + r$ we get

$$\begin{vmatrix} 2^{n+1}-2^n+p & 2^{n+2}-2^{n+1}+q & p+r \\ 2^n+p & 2^{n+1}+q & p+r \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix}$$

$$= \begin{vmatrix} 2^n+p & 2^{n+1}+q & p+r \\ 2^n+p & 2^{n+1}+q & p+r \\ a^2+2^n+p & b^2+2^{n+1}+2q & c^2-r \end{vmatrix} = 0$$

97. $\Delta = 1(\sin^2 x - \cos^2 y) + a^2(\sin^2 x - \cos^2 y)$

$= (1 + a^2)(\sin^2 x - \cos^2 y)$

98. The given equations are $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$.

It has non-trivial solutions so

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow 1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$

$\Rightarrow 1 - 2abc = a^2 + b^2 + c^2 > 0$

$\Rightarrow abc < \frac{1}{2}$

So, $0 < abc < \frac{1}{2}$. Clearly, no triplet (a, b, c) of positive integers can satisfy it.

Practice Exercise 2

1. $A(A + I) = -2I$. Now

$|A(A + I)| = |-2I|$

$\Rightarrow |A|\,|A + I| = (-2)^n \neq 0$

$\Rightarrow |A| \neq 0,\ A\left\{-\frac{1}{2}(A+I)\right\} = I$

$\Rightarrow A^{-1} = -\frac{1}{2}(A + I)$

2. $$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum 2^{r-1} & \sum \frac{1}{r(r+1)} & \sum \sin r\theta \\ x & y & z \\ 2^n - 1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\ \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \end{vmatrix}$$

$$= \begin{vmatrix} 2^n-1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\ \sin\frac{n\theta}{2}}{\sin\ \theta/2} \\ x & y & z \\ 2^n-1 & \frac{n}{n+1} & \frac{\sin\left(\frac{n+1}{2}\right)\theta\ \sin\frac{n\theta}{2}}{\sin\theta/2} \end{vmatrix} = 0$$

3. Operating $C_1 \to C_1 + C_2 + C_3$ we get

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & c \\ x+a+b+c & b & x+c \end{vmatrix} = 0$$

$$\Rightarrow (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix} = 0$$

Now $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get
$x^2(x+a+b+c) = 0$
$\Rightarrow$ Either $x = 0$ or $x = -(a+b+c)$

4. $$A^2 - 4A - 5I_3 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} + \begin{bmatrix} -4 & -8 & -8 \\ -8 & -4 & -8 \\ -8 & -8 & -4 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Therefore,

$$A^2 - 4A - 5I_3 = 0$$
$$\Rightarrow A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I_3 = 0$$
$$\Rightarrow (A^{-1}A)A - 4I_3 - 5A^{-1} = 0$$
$$\Rightarrow IA - 4I_3 - 5A^{-1} = 0$$
$$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I_3)$$

Also,

$$|A^2| = \begin{vmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{vmatrix} = 9(81-64) - 8(72-64) + 8(64-72)$$

$$= 9 \times 17 - 8 \times 8 + 8 \times (-8) = 133 - 128 = 5 \neq 0$$

Therefore, A^2 is invertible.

$$A^3 = A \times A^2 = A \times (4A - 5I_3) = 4A^2 - 5A$$

$$= \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix} + \begin{bmatrix} -5 & -10 & -10 \\ -10 & -5 & -10 \\ -10 & -10 & -5 \end{bmatrix} = \begin{bmatrix} 31 & 22 & 22 \\ 22 & 31 & 22 \\ 22 & 22 & 31 \end{bmatrix}$$

Therefore, $|A^3| \neq 0$ and so $|A^3|$ is invertible.

5. Here, (A), (B), (C), (D) are the properties of adjoint. Hence, (A), (B), (C) and (D) are the correct answers.

6. We have

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} = 3(-3+4) - 2(-3+4) + 0 = 1$$

adj(adj A) $= |A|^{3-2}A = A$ and $|$adj (adj A)$| = |A| = 1$
Also, $|$adj $A| = |A|^{3-1} = |A|^2 = 1^2 = 1$

7.
$$x + 3y + 2z = 6 \quad (1)$$
$$x + \lambda y + 2z = 7 \quad (2)$$
$$x + 3y + 2z = m \quad (3)$$

(A) If $\lambda = 2$, then $D = 0$. Therefore, unique solution is not possible

(B) If $\lambda = 4, \mu = 6$

$$x + 3y = 6 - 2z$$
$$x + 4y = 7 - 2z$$

Therefore, $y = 1$ and $x = 3 - 2z$.
Substituting in Eq. (3), we get $3 - 2z + 3 + 2z = 6$ is satisfied.
Therefore, infinite solutions.

(C) $\lambda = 5, \ \mu = 7$
Consider Eqs. (2) and (3).

$$x + 5y = 7 - 2z$$
$$x + 3y = 7 - 2z$$

Therefore, $y = 0$, $x = 7 - 2z$ are the solutions.
Substituting in Eq. (1) we have $7 - 2z + 2z = 6$ is not satisfied.
Therefore, no solution.

(D) If $\lambda = 3, \mu = 5$, then Eqs. (1) and (2) have no solution. Therefore, no solution.

8. By the properties of adjoint of a matrix, adjoint of a symmetric matrix is again a symmetric matrix.

Similarly,

$$A\,(\text{adj } A) = (\text{adj } A)\,A = |A|I$$

Also adjoints of unit and diagonal matrices are also unit and diagonal matrices.

Hence, all options are correct.

9. $$A^2 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 3A$$

$A^3 = A^2A = 3A \times A = 3A^2 = 3 \times (3A) = 9A$ and $|A| = 0$

Therefore, A^{-1} does not exist.

10. Let U_1 be $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Similarly, $U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$, $U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$.

Hence, $U = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix}$ and $|U| = 3$

11. $$\text{adj } U = \begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix}$$

Hence, $U^{-1} = \dfrac{\text{adj}U}{3}$ and the sum of the elements of $U^{-1} = 0$

12. The value of

$$[3 \ \ 2 \ \ 0]U\begin{bmatrix}3\\2\\0\end{bmatrix} = [3 \ \ 2 \ \ 0]\begin{bmatrix}1 & 2 & 2\\-2 & -1 & -1\\1 & -4 & -3\end{bmatrix}\begin{bmatrix}3\\2\\0\end{bmatrix}$$

$$= [-1 \ \ 4 \ \ 4]\begin{bmatrix}3\\2\\0\end{bmatrix} = -3+8=5$$

13. $b_1 \times C_{31} + b_2 \times C_{32} + b_3 \times C_{33}$

$$= b_1\begin{vmatrix}a_2 & a_3\\b_2 & b_3\end{vmatrix} - b_2\begin{vmatrix}a_1 & a_3\\b_1 & b_3\end{vmatrix} + b_3\begin{vmatrix}a_1 & a_2\\b_1 & b_2\end{vmatrix} = 0$$

14. The value of new determinant $= 2^3\Delta = 8\Delta$

15. $a_3 M_{13} - b_3 \cdot M_{23} + d_3 \cdot M_{33} = a_3 C_{13} + b_3 \cdot C_{23} + d_3 \cdot C_{33} = \Delta$ by definition.

16. As second row of all the options is same, we look at the elements of the first row.

Let left inverse be $\begin{bmatrix}a & b & c\\d & e & f\end{bmatrix}$. Then

$$\begin{bmatrix}a & b & c\\d & e & f\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\\2 & 3\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Therefore,

$$a+b+2c=1$$
$$-a+b+3c=0$$
$$\Rightarrow b = \frac{1-5c}{2} \text{ and } a = \frac{1+c}{2}$$

Thus, matrices in options (A), (B) and (D) are the inverses and matrix in option (C) is not the left inverse.

17. Let right inverse be

$$\begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}$$

Now,

$$a-c+2e=1$$
$$b-d+2f=0$$
$$2a-c+e=0$$
$$2b-d+f=1$$

Therefore, infinite solutions.

18. By observation, there cannot be any left inverse for (B) and (D), so we will check for (A) and (C) only.

For (A) let left inverse be $\begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}$. Then

$$\begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}\begin{bmatrix}1 & 2 & 4\\-3 & 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Now, $a-3b=1$, $2a+2b=0$ and $4a+b=0$, which is not possible.

For (C)

$$\begin{bmatrix}a & b & c\\d & e & f\end{bmatrix} = \begin{bmatrix}1 & 4\\2 & -3\\5 & 4\end{bmatrix}$$

$$\Rightarrow a+2b+5c=1$$
$$4a-3b+4c=0$$
$$d+2e+5f=0$$
$$4d-3e+4f=1$$

Therefore, there are infinite number of left inverses

Right inverse:

$$\begin{bmatrix}1 & 4\\2 & -3\\5 & 4\end{bmatrix}\begin{bmatrix}a & b & c\\d & e & f\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$\Rightarrow \quad a+4d=1$, $2a-3d=0$ and $5a+4d=0$

which is not possible.

Therefore, there is no right inverse.

19. In a matrix $\begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$, a_{22} must be a multiple of 3 (3, 6 or 9) because from the above possible combinations only 3, 6 and 9 are repeated four times in a row or column or diagonal.

20. Clearly the maximum value of the sum of the diagonal elements is 18 which are called the trace or the matrix A.

21. Possible combinations are (1, 2, 6), (1, 3, 5), (1, 8, 9), (2, 3, 4), (2, 7, 9), (3, 6, 9), (3, 7, 8), (4, 5, 9), (4, 6, 8), (5, 6, 7)

Hence, total 10 combinations are possible.

22. **(A) → (q), (B) → (s, t), (C) → (r), (D) → (p)**

(A) $A^2 = -I$, therefore A is of even order

(B) $(I+A)^n = C_0 I^n + C_1 I A + C_2 I A^2 + \cdots + C_n I A^n$

$= C_0 I + C_1 A + C_2 A + \cdots + C_n A = I + (2^n - 1)A$

Therefore, $\lambda = 2^n - 1$

(C) $A^2 = A$ and $B = I - A$

$AB + BA + I - (I + A^2 - 2A) = AB + BA - A + 2A$

$= AB + BA + A$

$= A(I-A) + (I-A)A + A = A - A + A - A + A = A$

(D) $A^* = A$, $B^* = B$

$(AB - BA)^* = B^* A^* - A^* B^* = BA - AB$

23. **(A) → (p, t), (B) → (q), (C) → (r), (D) → (p, t)**

(A) $|A| = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{vmatrix}$

$$|B| = \begin{vmatrix}a_{11} & k^{-1}a_{12} & k^{-2}a_{13}\\k\,a_{21} & a_{22} & k^{-1}a_{23}\\k^2 a_{31} & k\,a_{32} & a_{33}\end{vmatrix} = \frac{1}{k^3}\begin{vmatrix}k^2 a_{11} & k\,a_{12} & a_{13}\\k^2 a_{21} & k a_{22} & a_{23}\\k^2 a_{31} & k a_{32} & a_{33}\end{vmatrix} = |A|$$

$k_1|A| + k_2|B| = 0$

$k_1 + k_2 = 0$

(B) $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$

(C) $\begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$

$\Rightarrow \sin^2\gamma - \cos\alpha(\cos\alpha - \cos\beta\cos\gamma) + \cos\beta(\cos\alpha\cos\gamma - \cos\beta)$
$= -\cos\alpha(-\cos\beta\cos\gamma) + \cos\beta(\cos\alpha\cos\gamma)$

$\Rightarrow \sin^2\gamma - \cos^2\alpha + 2\cos\alpha\cos\beta\cos\gamma - \cos^2\beta$
$= 2\cos\alpha\cos\beta\cos\gamma$

$\Rightarrow \sin^2\gamma = \cos^2\alpha + \cos^2\beta \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(D) $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

$R_2 \to R_2 - (R_1 + R_3)$ gives

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = 4\begin{vmatrix} x+1 & x-2 \\ 2x-1 & 2x-1 \end{vmatrix}$$

$= 4\begin{vmatrix} x+1 & -3 \\ 2x-1 & 0 \end{vmatrix} = (24x - 12)$

Therefore, $A = 24$, $B = -12$ and $A + 2B = 0$.

24. $\dfrac{|\text{adj}B|}{|C|} = \dfrac{|\text{adj}(\text{adj}A)|}{|5A|} = \dfrac{|A|^{(3-1)^2}}{5^3|A|} = \dfrac{|A|^3}{125}$

Now, $|A| = 5$. Therefore,

$$\frac{|\text{adj}B|}{|C|} = 1$$

25. $I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$

and $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$

Now, $|I - A| = 0 - 2 = -2$. So

$\text{adj}(I - A) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$(I - A)^{-1} = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & 0 \end{bmatrix}$

As

$$\phi(x) = (1 + x)(1 - x)^{-1}$$

Therefore,

$\phi(A) = (I + A)(I - A)^{-1} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} = -A$

Now, comparing the above equation with $\phi(A) = -\lambda A$, we have $\lambda = 1$

26. $A'A = I$. Therefore

$$|A'A| = |I| \Rightarrow |A| = \pm 1$$

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 1$$

$\Rightarrow 3abc - a^3 - b^3 - c^3 = \pm 1$
$\Rightarrow a^3 + b^3 + c^3 = 2$ and 4

27. Here, $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 \\ -1 & 7-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda)(7 - \lambda) = 0$
$\Rightarrow \lambda^2 - 8\lambda + 7 = 0$
$\Rightarrow A^2 - 8A + 7I_2 = 0$
$\Rightarrow A^2 = 8A - 7I_2$
$\Rightarrow k = -7 \Rightarrow |k| = 7$

28. Let

$$\begin{vmatrix} (1+x)^{a_1b_1} & (1+x)^{a_1b_2} & (1+x)^{a_1b_3} \\ (1+x)^{a_2b_1} & (1+x)^{a_2b_2} & (1+x)^{a_2b_3} \\ (1+x)^{a_3b_1} & (1+x)^{a_3b_2} & (1+x)^{a_3b_3} \end{vmatrix} = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \cdots$$

For λ_1, differentiate w.r.t. x and put $x = 0$. So $\lambda_1 = 0$.

29. Clearly $f'(x) = 0$. Therefore

$$f(x) = c = 6$$

Therefore,

$$\sum_{r=1}^{25} f(r) = \sum_{r=1}^{25} 6 = 150$$

30. $f(-x) = -f(x) = g(x)$. Therefore,

$f(x) \times g(x) = -(f(x))^2$ or $f(1)\,g(1) = -(f(1))^2$

$$= -\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix}^2 = -4$$

$\Rightarrow \lambda f(1)\,g(1) = 4 \Rightarrow \lambda(-4) = 4 \Rightarrow \lambda = -1$

31. On solving, we get

$(2\lambda + 15)\,f(x + 1) - (\lambda + 10)\,f(x + 8) - f(x + 1) = 0$
$\Rightarrow (2\lambda + 14)\,f(x + 1) = (\lambda + 10)\,f(x + 8)$

Since, f is periodic with period 7, therefore

$f(x + 1) = f(x + 8)$
$\Rightarrow 2\lambda + 14 = \lambda + 10$
$\Rightarrow |\lambda| = 4$

Solved JEE 2017 Questions

JEE Main 2017

1. If $A=\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj$(3A^2+12A)$ is equal to

(A) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ **(B)** $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

(C) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ **(D)** $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$

(OFFLINE)

Solution: The given matrix is

$$A=\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$

Therefore,

$$A^2=\begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$$

That is,

$$3A^2=\begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix}$$

Also,

$$12A=\begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$$

Hence,

$$3A^2+12A=\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$

Therefore,

$$\text{adj}(3A^2+12A)=\begin{bmatrix} 51 & 68 \\ 84 & 72 \end{bmatrix}$$

Hence, the correct answer is option (A).

2. If S is the set of distinct values of b for which the following system of linear equations

$$x+y+z=1$$
$$x+ay+z=1$$
$$ax+by+z=0$$

has no solution, then S is

(A) an infinite set.
(B) a finite set containing two or more elements.
(C) a singleton.
(D) an empty set.

(OFFLINE)

Solution: For $\Delta = 0$ (and at the one of the solutions of Δ_1, Δ_2, $\Delta_3 \neq 0$):

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix}=0$$

$$1(a-b)-1(1-a)+1(b-a^2)=0$$
$$2a-b-1+b-a^2=0$$
$$a^2-2a+1=0 \Rightarrow a=1$$

Using $a=1$ in the given system of equations, we get

$$x+y+z=1$$
$$x+y+z=1$$
$$x+by+z=0$$

We see that there is only one value of b; therefore, S is singleton set.

Hence, the correct answer is option (C).

3. Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point

(A) $\left(1, \frac{3}{4}\right)$ **(B)** $\left(1, -\frac{3}{4}\right)$

(C) $\left(2, \frac{1}{2}\right)$ **(D)** $\left(2, -\frac{1}{2}\right)$

(OFFLINE)

Solution: We can write the given vertices of the triangle in the following form:

$$\frac{1}{2}\begin{vmatrix} k & -3k & 1 \\ 5 & k & 1 \\ -k & 2 & 1 \end{vmatrix}=28$$

That is,

$$5k^2+13k-46=0$$

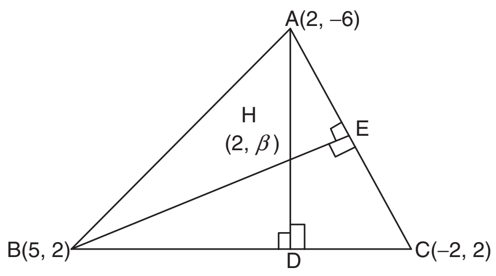
or

$$5k^2+13k+66=0$$

From the above, we confirm that no real solution exists. Therefore,

$$k=\frac{-23}{5} \text{ or } k=2$$

Since it is given that k is an integer, we consider only $k=2$.
Therefore, the vertices are obtained as (2, –6), (5, 2) and (–2, 2) as depicted in the following figure.



Thus, solving the equations of two altitudes, the orthocentre of the triangle is obtained as $\left(2, \frac{1}{2}\right)$.

Hence, the correct answer is option (C).

4. If $S=\left\{x\in[0,2\pi]: \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}=0\right\}$, then

$\sum_{x\in S} \tan\left(\frac{\pi}{3}+x\right)$ is equal to

(A) $-2+\sqrt{3}$ **(B)** $4+2\sqrt{3}$
(C) $-4-2\sqrt{3}$ **(D)** $-2-\sqrt{3}$

(ONLINE)

Solution: Solving the determinant

$$\begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix} = 0$$

we get

$$0[0-\sin x\cos x]-\cos x[0-\cos^2 x]-\sin x[\sin^2 x-0]=0$$
$$\Rightarrow \cos^3 x-\sin^3 x=0$$

Using $a^3-b^3=(a-b)(a^2+b^2+ab)$. We get

$$\cos^3 x-\sin^3 x=(\cos x-\sin x)(\cos^2 x+\sin^2 x+\sin x\cos x)=0$$

Using $\cos^2 x+\sin^2 x=1$. Now,

$$(\cos x-\sin x)(1+\sin x\cos x)=0$$
$$\Rightarrow \cos x-\sin x=0 \Rightarrow \cos x=\sin x$$
$$\Rightarrow \tan x=1$$
$$\Rightarrow x=\frac{\pi}{4} \qquad (1)$$

Now, evaluating $\sum_{x\in S}\tan\left(\frac{\pi}{3}+x\right)$ using, $\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}$, we get

$$\sum_{x=\pi/4}\tan\left(\frac{\pi}{3}+x\right)=\tan\left(\frac{\pi}{3}+\frac{\pi}{4}\right) \qquad \text{[from Eq. (1)]}$$

$$\Rightarrow \tan\left(\frac{\pi}{3}+\frac{\pi}{4}\right)=\frac{\tan(\pi/3)+\tan(\pi/4)}{1-\tan(\pi/3)\tan(\pi/4)}=\frac{\sqrt{3}+1}{1-\sqrt{3}\times 1}=\frac{1+\sqrt{3}}{1-\sqrt{3}}$$

Multiplying and dividing by $1+\sqrt{3}$, we get

$$\sum\tan\left(\frac{\pi}{3}+x\right)=\frac{1+\sqrt{3}}{1-\sqrt{3}}\times\frac{1+\sqrt{3}}{1+\sqrt{3}}$$

Using $a^2-b^2=(a+b)(a-b)$

$$\sum\tan\left(\frac{\pi}{3}+x\right)=\frac{(1+\sqrt{3})^2}{(1-3)}=\frac{1+3+2\sqrt{3}}{-2}=\frac{4+2\sqrt{3}}{-2}$$
$$=\frac{4}{(-2)}+\frac{2\sqrt{3}}{(-2)}=-2-\sqrt{3}$$

Hence, the correct answer is option (D).

5. The number of real values of λ, for which the system of linear equations

$$2x+4y-\lambda z=0$$
$$4x+\lambda y+2z=0$$
$$\lambda x+2y+2z=0$$

has infinitely many solutions, is

(A) 0 **(B)** 1
(C) 2 **(D)** 3

(ONLINE)

Solution: The system of equations can be written in the matrix form as

$$\begin{bmatrix} 2 & 4 & -\lambda \\ 4 & \lambda & 2 \\ \lambda & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

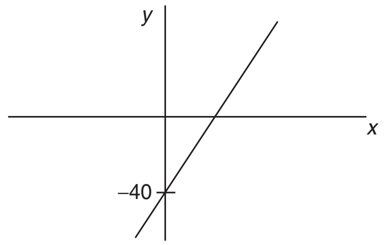
The system has infinite solutions; thus, we get

$$\begin{vmatrix} 2 & 4 & -\lambda \\ 4 & \lambda & 2 \\ \lambda & 2 & 2 \end{vmatrix}=0$$

$$\Rightarrow 0=2(2\lambda-4)-4(8-2\lambda)-\lambda(8-\lambda^2)$$
$$\Rightarrow 4\lambda-8-32+8\lambda-8\lambda+\lambda^3=0$$
$$\Rightarrow \lambda^3+4\lambda-40=0$$

We can solve this by graphical method

$$y=x^3+4x-40$$



For $x=0$, $y=-40$: If we take $y=-40$, then we have

$$-40=x^3+4x-40$$
$$\Rightarrow x^3+4x=0$$
$$\Rightarrow x(x^2+4)=0$$
$$\Rightarrow x=0,\ x^2+4=0$$
$$\Rightarrow x=\pm 2i$$

The given equation of line intersects x only at one point; therefore, the real value of λ is only one.
Hence, the correct answer is option (B).

6. Let A be any 3×3 invertible matrix. Then, which one of the following is not always true?

(A) $\text{adj}(\text{adj}(A))=|A|\cdot(\text{adj}(A))^{-1}$

(B) $\text{adj}(\text{adj}(A))=|A|^2\cdot(\text{adj}(A))^{-1}$

(C) $\text{adj}(A)=|A|\cdot A^{-1}$

(D) $\text{adj}(\text{adj}(A))=|A|\cdot A$

(ONLINE)

Solution: From the properties of invertible matrices, option (1) is not true.

Hence, the correct answer is option (A).

7. For two 3×3 matrices A and B, let $A+B=2B'$ and $3A+2B=I_3$, where B' is the transpose of B and I_3 is 3×3 identity matrix. Then

(A) $10A+5B=3I_3$ **(B)** $5A+10B=2I_3$

(C) $3A + 6B = 2I_3$ **(D)** $B + 2A = I_3$

(ONLINE)

Solution: It is given that

$$A + B = 2B' \quad (1)$$

Taking transpose on both sides, we get

$$A' + B' = 2B \quad (2)$$
$$3A + 2B = I_3 \quad (3)$$

That is,

$$3A' + 2B' = I_3 \quad (4)$$

Substituting Eq. (2) in Eq. (4), we get

$$3(2B - B') + 2B' = I_3$$

That is,

$$6B - B' = I_3$$

Writing as $B' = \dfrac{A+B}{2}$, we get

$$6B - \frac{A+B}{2} = I_3$$

Therefore,

$$12B - A - B = 2I_3$$
$$11B - A = 2I_3$$
$$11B - A = 6A + 4B \quad \text{[from Eq. (3)]}$$

Therefore,

$$7B = 7A \Rightarrow A = B$$
$$I_3 = 3A + 2A = 5A = 5B$$
$$10A + 5B = 10A + 5A = 15A = 3I_3$$

Hence, the correct answer is option (A).

JEE Advanced 2017

1. Which of the following is(are) NOT the square of a 3×3 matrix with real entries?

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **(B)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **(D)** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Solution: For a matrix to be a square of matrix with real entries, its determinant should be positive.

- **Option (A):** The determinant $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is possible: $1(1) - 0(0) + 0(0) = +1$.
- **Option (B):** The determinant $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is not possible: $1(-1) - 0(0) + 0(0) = -1$.
- **Option (C):** The determinant $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is possible: $1(1) - 0(0) + 0(0) = +1$.
- **Option (D):** The determinant $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is not possible: $-1(1) - 0(0) + 0(0) = -1$.

Hence, the correct answers are options (B) and (D).

2. For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$ _______.

Solution: It is given that

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} = 0$$

$$\Rightarrow 1(1-\alpha^2) - \alpha(\alpha - \alpha^3) + \alpha^2(\alpha^2 - \alpha^2) = 0$$

$$\Rightarrow (1-\alpha^2) - \alpha^2(1-\alpha^2) = 0 \Rightarrow (1-\alpha^2)(1-\alpha^2) = 0$$

$$\Rightarrow (1-\alpha^2)^2 = 0$$

$$\Rightarrow \alpha^2 = 1 \Rightarrow \alpha = \pm 1$$

For $\alpha = 1$, the given system of linear equations has no solution. That is,

$$\begin{bmatrix} +1 & +1 & +1 \\ +1 & +1 & +1 \\ +1 & +1 & +1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$x + y + z = 1$$
$$x + y + z = -1$$
$$x + y + z = 1$$

Since two planes are parallel, $\alpha = 1$ is rejected and for $\alpha = -1$, the given system of linear equations has coincident planes.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} x - y + z = 1 \\ -x + y - z = -1 \Rightarrow x - y + z = 1 \\ x - y + 1 = 1 \end{cases}$$

Therefore, $\alpha = -1$ holds well. Therefore,

$$1 + \alpha + \alpha^2 = 1 + (-1) + (-1)^2$$
$$= 1 - 1 + 1 = 1$$

Hence, the correct answer is (1).

3. How many 3×3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of M^TM is 5?

(A) 126 **(B)** 198

(C) 162 **(D)** 135

Solution: Let us consider a 3×3 matrix

$$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Therefore,

$$M^T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

It is given that sum of diagonal of M^TM is 5. Therefore,

$$M^TM = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1^2 + b_1^2 + c_1^2 & a_1a_2 + b_1b_2 + c_2 & a_1a_3 + b_1b_3 + c_1c_3 \\ a_2a_1 + b_2b_1 + c_2c_1 & a_2^2 + b_2^2 + c_2^2 & a_2a_3 + b_2b_3 + c_2c_3 \\ a_3a_1 + b_3b_1 + c_3c_1 & a_3a_2 + b_3b_2 + c_3c_2 & a_3^2 + b_3^2 + c_3^2 \end{bmatrix}$$

$$\Rightarrow (a_1^2 + b_1^2 + c_1^2) + (a_2^2 + b_2^2 + c_2^2) + (a_3^2 + b_3^2 + c_3^2) = 5$$

There are two possible cases:

(i) $0^2 + 0^2 + 0^2 + 0^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 = 5$

This has ${}^9C_5$ combinations possible.

(ii) $1^2 + 2^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 = 5$

This has ${}^9C_7 \times {}^2C_1$ possible combinations.
Therefore,

$${}^9C_5 + {}^9C_7 \times {}^2C_1 = \frac{9!}{5! \times 4!} + \frac{9!}{7!2!}\frac{2!}{1!1!}$$

$$\Rightarrow \frac{9 \times 8 \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2 \times 1} + \frac{9 \times 8 \times 7!}{7! \times 2 \times 1} \times 2 \Rightarrow 126 + 72 = 198$$

Thus, the total number of 3×3 matrices is 198.
Hence, the correct answer is option (B).

19 Limit, Continuity and Differentiability

19.1 Limit of a Function

The concept of limit is used to discuss the behaviour of a function close to a certain point. For example,

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Clearly the function is not defined at $x = 1$, but for values close to $x = 1$ the function can be written as

$$f(x) = x + 1$$

As x approaches 1 (written as $x \to 1$), $f(x)$ approaches the value 2 [$f(x) \to 2$]. We write this as

$$\lim_{x \to 1} f(x) = 2$$

It must be noted that it is not necessary for the function to be undefined at the point where limit is calculated. In the above example, $\lim_{x \to 2} f(x)$ is the same as the value of function at $x = 2$, that is, 3.

Sometimes, functions approach different values as x-approaches x_0 from left and right. By left we mean $x < x_0$ and right means $x > x_0$. This is written as $x \to x_0^-$ and $x \to x_0^+$, respectively. For example,

$$f(x) = [x] \text{ (greatest integer function)}$$

For any integer n,

$$\lim_{x \to n^-} f(x) = n - 1 \qquad (19.1)$$

and

$$\lim_{x \to n^+} f(x) = n \qquad (19.2)$$

In such cases we say that $\lim_{x \to n} f(x)$ does not exist. The limit in Eq. (19.1) is said to be the left hand limit (L.H.L.) at $x = n$ and that in Eq. (19.2) is called the right hand limit (R.H.L.) at $x = n$.

19.2 Definition

19.2.1 Informal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that function approaches the limit L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L$$

This definition is "informal" because phrases like arbitrarily close and sufficiently close are imprecise and their meaning depends on the context.

The definition is clear enough and enables us to recognize and evaluate limits of specific functions.

19.2.2 Formal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possible at x_0 itself, we say that $f(x)$ approaches the limit L as x approaches x_0 and write $\lim_{x \to x_0} f(x) = L$, if for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \in$$

Illustration 19.1 Show that $\lim_{x \to 1} (5x - 3) = 2$.

Solution: See Fig. 19.1. Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\in > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, if

$$0 < |x - 1| < \delta,$$

then $f(x)$ is within the distance $\in$ of $L = 2$, that is

$$|f(x) - 2| < \in$$

We find δ by working backwards from the $\in$ inequality

$$|(5x - 3) - 2| = |5x - 5| < \in$$
$$5|x - 1| < \in$$
$$|x - 1| < (\in/5)$$

Thus, we can take $\delta = \in/5$.

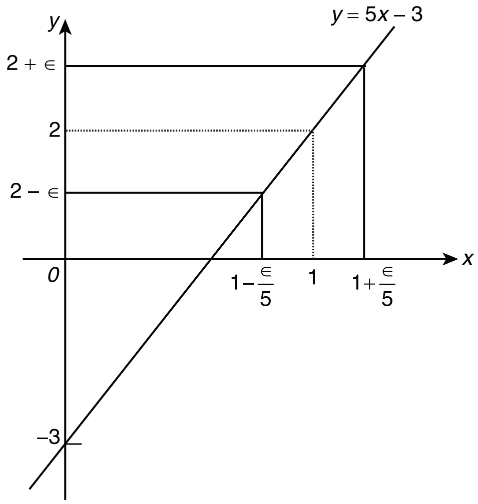


Figure 19.1

If $0 < |x - 1| < \delta = \in/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\in/5) = \in$$

This proves that $\lim_{x \to 1}(5x - 3) = 2$.

The value of $\delta = \in/5$ is not the only value that will make $0 < |x-1| < \delta$ imply $|5x-5| < \in$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one, that will work.

For limit L to exist as x approaches x_0, a function f must be defined on both sides of x_0, and its values $f(x)$ must approach L as x approaches x_0 from either side. Because of this, ordinary limits are sometimes called two-side limits.

19.2.3 Right Hand Limit

If $\lim\limits_{x \to x_0^+} f(x) = L$ for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all x.

$$x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \in$$

Then we call it right hand limit. For example:

$$\lim_{x \to 0^+} \frac{x}{|x|} = +1$$

19.2.4 Left Hand Limit

If $\lim\limits_{x \to x_0^-} f(x) = L$ for every number $\in > 0$, there exists a corresponding number $\delta > 0$ such that for all x.

$$x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \in$$

Then we call it left hand limit. For example:

$$\lim_{x \to 0^-} \frac{x}{|x|} = -1$$

Through graph of $\frac{x}{|x|}$, we can easily visualize the things written above.

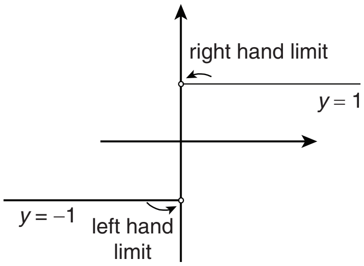


Figure 19.2

Now, from the discussions we have just gone through we can easily say that limit of a function will exist iff LHL and RHL both are finite, unique and equal (Fig. 19.2).

For example: $\lim\limits_{x \to 1} [x]$ will not exist as LHL = $\lim\limits_{x \to 1^-} [x] = 0$; RHL = $\lim\limits_{x \to 1^+} [x] = 1$

This can be seen graphically, in Fig. 19.3.

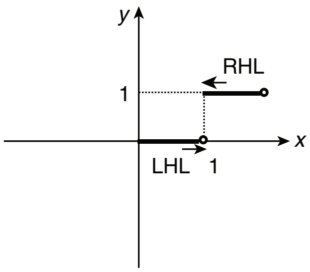


Figure 19.3

Left and Right Limits: Let $y = f(x)$ be a given function, and $x = a$ is the point under consideration.
Left tendency of $f(x)$ at $x = a$ is called its left limit and right tendency is called its right limit.
$f(a+0) = \lim\limits_{h \to 0} f(a+h)$ and $f(a-0) = \lim\limits_{h \to 0} f(a-h)$ where 'h' is a small positive number.
Thus, for the existence of the limit of $f(x)$ at $x = a$, $f(a-0) = f(a+0)$

19.3 Algebra of Limits

Let $\lim\limits_{x \to a^-} f(x) = l_1$ and $\lim\limits_{x \to a^-} g(x) = l_2$. Then

1. $\lim\limits_{x \to a} [c_1 f(x) \pm c_2 g(x)] = \lim\limits_{x \to a} [c_1 f(x)] \pm \lim\limits_{x \to a} [c_2 g(x)] = c_1 l_1 \pm c_2 l_2$, where c_1 and c_2 are given constants.
2. $\lim\limits_{x \to a} f(x) \cdot g(x) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = l_1 \cdot l_2$
3. $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{l_1}{l_2}, \quad l_2 \neq 0$
4. $\lim\limits_{x \to a} f[g(x)] = f[\lim\limits_{x \to a} g(x)] = f(l_2)$, if and only if $f(x)$ is continuous at $x = l_2$.
   In particular, $\lim\limits_{x \to a} \ln [g(x)] = \ln l_2$ if $l_2 > 0$.
   All these theorems must be used with utmost care. For example, we have assumed that l_1 and l_2 are finite. If these are not finite, the given theorems will not be applicable.
   For example: $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$, and if we try to apply the theorems, we get $\lim\limits_{x \to 0} \frac{\sin x}{x} = \lim\limits_{x \to 0} \sin x \cdot \lim\limits_{x \to 0} \frac{1}{x}$, which does not exist.
   Which of course is an absurd result, we are getting this absurd result because in this case the given limit cannot be written as the product of two limits as $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist. Similarly, $\lim\limits_{x \to 0} \left[\frac{\sin x}{x}\right]$, where [·] denotes the greatest integer function ≠ $\left[\lim\limits_{x \to 0} \frac{\sin x}{x}\right]$. Here $[x]$ is not continuous at $x = 1$.
5. $\lim\limits_{x \to a} [1 + f(x)]^{\frac{1}{g(x)}} = e^{\lim\limits_{x \to a} \frac{\log[1+f(x)]}{g(x)}}$

19.4 Evaluation of Limits

Following are indeterminate forms:

1. $\frac{0}{0}$ **2.** $\frac{\infty}{\infty}$ **3.** $0 \times \infty$ **4.** $\infty - \infty$

5. 0° **6.** ∞° **7.** 1^∞

We shall divide the ways of evaluation of limits in five categories:

19.4.1 Simplification

In this method, we can use:

1. Direct substitution
2. Rationalisation
3. Factorization
4. Use of formulas like binomial expansion, trigonometric formulas, etc.

1. **Direct substitution:** We can directly substitute the number at which limit is to be find. For example

   - $\lim_{x\to1}(x^2+3x-2)$ can be find out by this method.

     $\lim_{x\to1}(x^2+3x-2)=2$

   - $\lim_{x\to-2}|x|=2$

   But before using this method, we have to see that LHL should remain equal to RHL.

   For example: In $\lim_{x\to1}\sec^{-1}x$, if we directly substitute, we will get $\lim_{x\to1}\sec^{-1}x$ as 0. But LHL of $\lim_{x\to1}\sec^{-1}x$ will not exist. So, answer should be, limit does not exist.

2. **Rationalisation method:** Rationalisation is followed when we have powers in fractions on expressions in numerator and denominator or in both. After rationalization, the terms are factorised, which on cancellation give the result.

Illustration 19.2 Find $\lim_{x\to\infty}(x-\sqrt{x^2+x})$.

Solution: This is apparently of the form ∞ minus ∞ and can be converted to $\frac{\infty}{\infty}$ form by multiplying numerator and the denominator by the conjugate. Therefore,

$$\lim_{x\to\infty}(x-\sqrt{x^2+x})=\lim_{x\to\infty}\frac{(x-\sqrt{x^2+x})(x+\sqrt{x^2+x})}{x+\sqrt{x^2+x}}$$

$$=\lim_{x\to\infty}\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}=\lim_{x\to\infty}\left(\frac{-x}{x+\sqrt{x^2+x}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{-1}{1+\sqrt{1+\frac{1}{x}}}\right)=\frac{-1}{1+1}=\frac{-1}{2}$$

Illustration 19.3 Find $\lim_{x\to1}\left(\frac{\sqrt{x^2+8}-\sqrt{10-x^2}}{\sqrt{x^2+3}-\sqrt{5-x^2}}\right)$.

Solution: This is of the form $\frac{3-3}{2-2}=\frac{0}{0}$ if we put $x=1$.

To eliminate the $\frac{0}{0}$ factor, multiply and divide by the conjugate of numerator and the conjugate of the denominator. Therefore,

$$\text{Limit}=\lim_{x\to1}(\sqrt{x^2+8}-\sqrt{10-x^2})\frac{(\sqrt{x^2+8}+\sqrt{10-x^2})}{(\sqrt{x^2+8}+\sqrt{10-x^2})}$$

$$\times\frac{(\sqrt{x^2+3}+\sqrt{5-x^2})}{(\sqrt{x^2+3}+\sqrt{5-x^2})(\sqrt{x^2+3}-\sqrt{5-x^2})}$$

$$=\lim_{x\to1}\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\times\frac{(x^2+8)-(10-x^2)}{(x^2+3)-(5-x^2)}$$

$$=\lim_{x\to1}\left(\frac{\sqrt{x^2+3}+\sqrt{5-x^2}}{\sqrt{x^2+8}+\sqrt{10-x^2}}\right)\times1=\frac{2+2}{3+3}=\frac{2}{3}$$

3. **Factorization method:** We factorize numerator and denominator of the rational function, so that common factors in numerator and denominator cancel out. By doing this, we in turn are eliminating the factors which are making the function in one of the indeterminate form. For example:

$$\lim_{x\to16}\frac{\sqrt{x}-4}{x^{1/4}-2}=\lim_{x\to16}\frac{(x^{1/4}-2)(x^{1/4}+2)}{(x^{1/4}-2)}$$

$$=\lim_{x\to16}(x^{1/4}+2)=4$$

4. **Use of formulas:** We can use formulas which we have studied in other different topics to make functions simplified. For example:

$$\lim_{x\to0}\frac{\sin x}{\sin\frac{x}{2}}=\lim_{x\to0}\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$=\lim_{x\to0}2\cos\frac{x}{2}=2$$

19.5 Use of Standard Limits

These standard forms are used in case $f(x)\to0$ when $x\to a$.

1. $\lim_{x\to a}\frac{\sin f(x)}{f(x)}=1$
2. $\lim_{x\to a}\cos f(x)=1$
3. $\lim_{x\to a}\frac{\tan f(x)}{f(x)}=1$
4. $\lim_{x\to a}\frac{\sin^{-1}[f(x)]}{f(x)}=1$
5. $\lim_{x\to a}\frac{\tan^{-1}[f(x)]}{f(x)}=1$

Illustration 19.4 Evaluate $\lim_{n\to\infty}\frac{\sin\frac{a}{n}}{\tan\frac{b}{n+1}}$.

Solution: As $n\to\infty, \frac{1}{n}\to0$ and $\frac{a}{n}$ also tends to zero.

$\sin\frac{a}{n}$ should be written as $\frac{\sin\frac{a}{n}}{\frac{a}{n}}$ so that it looks like $\lim_{\theta\to0}\frac{\sin\theta}{\theta}$.

$$\text{The given limit}=\lim_{n\to\infty}\left(\frac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\frac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right)\cdot\frac{a(n+1)}{n\cdot b}$$

$$=\lim_{n\to\infty}\left(\frac{\sin\frac{a}{n}}{\frac{a}{n}}\right)\left(\frac{\frac{b}{n+1}}{\tan\frac{b}{n+1}}\right)\cdot\frac{a}{b}\left(1+\frac{1}{n}\right)$$

$$=1\times1\times\frac{a}{b}x1=\frac{a}{b}$$

Illustration 19.5 Show that $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3}=\frac{1}{2}$.

Solution:

$$\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3}=\lim_{x\to 0}\frac{\sin x\cdot 2\sin^2\frac{x}{2}}{x^3}$$

$$=\lim_{x\to 0}4\cos\frac{x}{2}\cdot\frac{\sin^3\frac{x}{2}}{x^3}=\lim_{x\to 0}4(1)\left(\frac{1}{2}\right)^3=\frac{1}{2}$$

Illustration 19.6 If $f(x)=\frac{3x-\sin^{-1}x}{4x-\tan^{-1}x}$, then $\lim_{x\to 0}f(x)$ is equal to ______.

Solution:

$$\lim_{x\to 0}\frac{3x-\sin^{-1}x}{4x-\tan^{-1}x}=\lim_{x\to 0}\frac{3-\frac{\sin^{-1}x}{x}}{4-\frac{\tan^{-1}x}{x}}=\frac{3-1}{4-1}=\frac{2}{3}$$

19.6 Some More Standard Forms

These standard forms are used in case $f(x)\to 0$ when $x\to a$.

6. $\lim_{x\to a}[1+f(x)]^{\frac{1}{f(x)}}=e$

7. $\lim_{x\to a}\frac{b^{f(x)}-1}{f(x)}=\log_e b\ \ (b>0)$

8. $\lim_{x\to a}\frac{\log[1+f(x)]}{f(x)}=1$

9. $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

Illustration 19.7 Evaluate $\lim_{x\to 0}\frac{x\cdot 2^x-x}{1-\cos x}$.

Solution: This is $\frac{0}{0}$ form, so the given limit becomes

$$\lim_{x\to 0}\frac{x^2(2^x-1)}{\left(2\sin^2\frac{x}{2}\right)x}=\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\frac{1}{2}\left(\frac{x}{\sin\frac{x}{2}}\right)^2$$

$$=\log_e 2\times\frac{1}{2}\left(\frac{1}{1/2}\right)^2=2\log_e 2=\log_e 4$$

Illustration 19.8 Evaluate $\lim_{x\to\infty}\left(\frac{x+6}{x+1}\right)^{x+4}$.

Solution: The problem depends upon reducing the given expression to the form $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ which is equal to e.

The given limit $=\lim_{x\to\infty}\left(\frac{1+\frac{6}{x}}{1+\frac{1}{x}}\right)^{x+4}$

$$=\frac{\lim_{x\to\infty}\left[\left(1+\frac{6}{x}\right)^{\frac{x}{6}}\right]^{\frac{(x+4)6}{x}}}{\lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{\frac{x+4}{x}}}=\frac{e^6}{e^1}$$

(since $\frac{6(x+4)}{x}=6+\frac{24}{x}$, which tends to 6 and $\frac{x+4}{x}=1+\frac{4}{x}$ which tends to 1.)

$$=e^5$$

Illustration 19.9 Evaluate $\lim_{x\to 0}\frac{e^{\tan x}-e^x}{\tan x-x}$.

Solution:

$$\lim_{x\to 0}\frac{e^{\tan x}-e^x}{\tan x-x}=\lim_{x\to 0}\frac{e^x\times e^{(\tan x-x)}-e^x}{\tan x-x}$$

$$=\lim_{x\to 0}\frac{e^x(e^{\tan x-x}-1)}{\tan x-x}$$

$$=e^0\times 1\qquad[\text{as } x\to 0,\ \tan x-x\to 0]$$

$$=1\times 1=1$$

Reasons for the non-existence of the limit: $\lim_{x\to a}f(x)$ will not exist due to any of these three reasons:

1. $f(x)$ is not defined in the neighbourhood of $x=a$.
2. $f(x)$ does not have a unique tendency.
3. Left and right tendencies of $f(x)$ are not the same.

Some standard limits of indeterminate forms

1. $\lim_{x\to 0}\frac{\sin ax}{x}=a\ \forall a\in R$, where x in radian
2. $\lim_{x\to 0}\frac{\tan ax}{x}=a\ \forall a\in R$, where x in radian
3. $\lim_{x\to 0}\frac{\sin^{-1}ax}{x}=a\ \forall\ a\in R$
4. $\lim_{x\to 0}\frac{\tan^{-1}ax}{x}=a\ \forall a\in R$
5. $\lim_{x\to 0}(1+ax)^{1/x}=e^a=\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x\ \forall a\in R$
6. $\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\log_a e\ (a>0, a\neq 1)$
7. $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a\ , a>0$
8. $\lim_{x\to\infty}\frac{\log x}{x^m}=0\ \ (m>0)$
9. $\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\frac{m}{n}a^{m-n}$, where m, n are rational numers
10. $\lim_{x\to 0}\frac{(1+x)^m-1}{m}=m$

19.7 Use of Expansion

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2. $a^x = 1 + \frac{x\log a}{1!} + \frac{x^2(\log a)^2}{2!} + \cdots \quad (a > 0)$

3. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (-1 < x < 1)$

4. $(1+x)^{1/x} = e\left(1 - \frac{x}{2} + \frac{11x^2}{24} + \cdots\right)$

5. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

6. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Illustration 19.10 Evaluate $\lim\limits_{x\to 0} \frac{\log(5+x) - \log(5-x)}{x}$.

Solution:

$$\lim_{x\to 0} \frac{\log\left[5\left(1+\frac{x}{5}\right)\right] - \log\left[5\left(1-\frac{x}{5}\right)\right]}{x}$$

$$= \lim_{x\to 0} \frac{\log 5 + \log\left(1+\frac{x}{5}\right) - \log 5 - \log\left(1-\frac{x}{5}\right)}{x}$$

$$= \lim_{x\to 0} \frac{\log\left(1+\frac{x}{5}\right)}{5\left(\frac{x}{5}\right)} - \frac{\log\left(1-\frac{x}{5}\right)}{-5\left(-\frac{x}{5}\right)} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

Illustration 19.11 Evaluate $\lim\limits_{x\to 0} \frac{ae^x - b}{x} = 2$. Find a and b.

Solution:

$$\lim_{x\to 0} \frac{a\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \infty\right) - b}{x} = 2$$

$$\lim_{x\to 0} \frac{(a-b) + xa + \frac{ax^2}{2!} + \cdots + \infty}{x} = 2$$

Since, limit is finite, $(a-b) = 0 \Rightarrow b = a$

Therefore,

$$\lim_{x\to 0} \frac{xa + \frac{ax^2}{2!} + \cdots + \infty}{x} = 2$$

$$\lim_{x\to 0} a + \frac{ax}{2!} + \cdots + \infty = 2$$

$$\Rightarrow a = 2$$

Hence, $b = 2$.

Some more expansions

7. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

8. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

9. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

10. $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2\,3^2}{5!}x^5 + \cdots$

11. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

12. $\cos^{-1} x = \frac{\pi}{2} - x - \frac{1^2}{3!}x^3 - \frac{1^2 3^2}{5!}x^5 - \cdots$

Illustration 19.12 Find the value of a, b and c such that

$$\lim_{x\to 0} \frac{axe^x - b\log(1+x) + cxe^{-x}}{x^2 \sin x} = 2$$

Solution: Using the expansion, we have

$$\frac{\lim\limits_{x\to 0} ax\left(1 + x + \frac{x^2}{2!} + \cdots\right) - b\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) + cx\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)}{x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} = 2$$

$$\Rightarrow \lim_{x\to 0} \frac{x(a-b+c) + x^2\left(a + \frac{b}{2} - c\right) + x^3\left(\frac{a}{2} - \frac{b}{3} + \frac{c}{2}\right) + \cdots}{x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} = 2$$

Now, above limit would exist if least power in numerator is greater than or equal to least power in denominator.
That is, coefficient of x and x^2 must be zero and coefficient of x^3 should be 2. That is,

$$a - b + c = 0,\ a + \frac{b}{2} - c = 0,\ \frac{a}{2} - \frac{b}{3} + \frac{c}{2} = 2$$

On solving, we get $a = 3$, $b = 12$, $c = 9$.

Illustration 19.13 Evaluate $\lim\limits_{x\to 0} \frac{\tan^{-1} x - \sin^{-1} x}{\sin^3 x}$.

Solution:

$$\lim_{x\to 0} \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) - \left(x + \frac{1^2\,x^3}{3!} + \frac{1^2\,3^2}{5!}x^5 + \cdots\right)}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)^3}$$

$$= \lim_{x\to 0} \frac{\left(-\left(\frac{1}{3} + \frac{1^2}{3!}\right)x^3 + \left(\frac{1}{5} - \frac{1^2\,3^2}{5!}\right)x^5 + \cdots\right)}{x^3\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)^3}$$

$$= \lim_{x\to 0} \frac{-\left(\frac{1}{3}+\frac{1^2}{3!}\right)+\left(\frac{1}{5}-\frac{1^2\,3^2}{5!}\right)x^2+\cdots}{\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}\cdots\right)^3} = \frac{-1}{2}$$

19.8 L'Hospital's Rule

L'Hospital's rule is applicable only in $\frac{0}{0}$ and $\frac{\infty}{\infty}$ indeterminate forms. For other forms, first we have to convert them into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, then we can use it. It states that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \cdots$$

till we are getting $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Proof: For $\frac{0}{0}$ form:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} \quad [\text{as } f(a)=0 \text{ and } g(a)=0]$$

$$= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ and so on}$$

For $\frac{\infty}{\infty}$ form:

First put $f(x)=\frac{1}{F(x)}$ and $g(x)=\frac{1}{G(x)}$.

Then proceed in same way as $\frac{0}{0}$ form's proof.

L'Hospital's Rule for calculating limits: Let f and g be differentiable functions in the neighbourhood of a which satisfy

1. $g'(x) \neq 0$ for any x in the neighbourhood
2. $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$. Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

This is L'Hospital's Rule for $\frac{0}{0}$ form. However, if $\lim_{x\to a} f(x) \to \infty$ and $\lim_{x\to a} g(x) \to \infty$, then again we can apply the L'Hospital's Rule.

Illustration 19.14 Evaluate $A = \lim_{x\to 0^+} (\sin x)^x$.

Solution:

$$A = e^{\lim_{x\to 0^+} x\log(\sin x)} = e^{\lim_{x\to 0^+}\frac{\log(\sin x)}{1/x}} = e^{\lim_{x\to 0^+}\frac{\cos x/\sin x}{-1/x^2}}$$

$$= e^{\lim_{x\to 0^+} -x^2\frac{\cos x}{\sin x}} = e^{\lim_{x\to 0^+} -2x\cos^2 x} = e^0 = 1$$

Illustration 19.15 Evaluate $\lim_{x\to 0}\frac{p^x-q^x}{r^x-s^x}$.

Solution:

$$\lim_{x\to 0}\frac{p^x-q^x}{r^x-s^x} \quad \left(\frac{0}{0}\text{ form}\right)$$

Applying L'Hospital's rule, we have

$$\lim_{x\to 0}\frac{p^x\log p - q^x\log q}{r^x\log r - s^x\log s} = \frac{\log p-\log q}{\log r - \log s} = \frac{\log(p/q)}{\log(r/s)}$$

19.9 Sandwich Theorem (Squeeze Play Theorem)

Sandwich theorem helps in calculating the limits, when limits cannot be calculated using the above discussed methods.

Sandwich theorem: See Fig. 19.4. If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that,

$$f(x) \le g(x) \le h(x)\ \forall x \in \text{ neighbourhood of } x=a$$

and $\lim_{x\to a} f(x) = \lim_{x\to a}[h(x)] = l$ (say)

Then $\lim_{x\to a} g(x) = l$

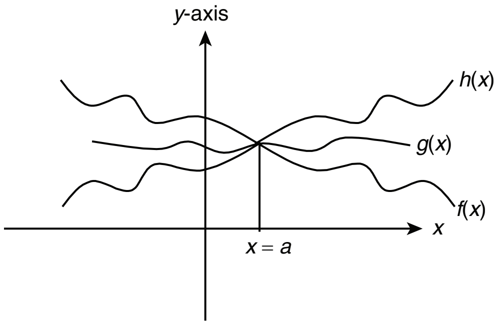


Figure 19.4

Sandwich theorem: Let f, g and h be three functions such that in the neighbourhood of a, $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$. Then $\lim_{x\to a} g(x) = l$.

Illustration 19.16 Evaluate $\lim_{x\to\infty}\frac{\log x}{[x]}$.

Solution: We can have

$$f(x) = \frac{\log x}{x} \text{ and } h(x) = \frac{\log x}{x-1} \text{ as } x-1 < [x] \le x.$$

$$\lim_{x\to\infty}\frac{\log x}{x} = \lim_{x\to\infty}\frac{1/x}{1} = 0 \quad \text{(using L'Hospital's rule)}$$

$$\lim_{x\to\infty}\frac{\log x}{x-1} = \lim_{x\to\infty}\frac{1/x}{1} = 0 \quad \text{(using L'Hospital's rule)}$$

Therefore,

$$\lim_{x\to\infty}\frac{\log x}{[x]} = 0$$

Your Turn 1

1. Does $\lim_{x\to 0}\{x\}$ exists. ({ } represents fractional part function). **Ans.** No
2. Find the value of $\lim_{x\to 0}\frac{|x-1|}{x+1}$. **Ans.** 1

3. Evaluate $\lim\limits_{x\to 1}\dfrac{x-1}{x^3-1}$. **Ans.** 1/3

4. Find the value of $\lim\limits_{x\to 1^+}\dfrac{\sec^{-1}(2-x)}{x^2}$. **Ans.** Limit does not exist

5. Find the value of $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\sin x\right)$. **Ans.** 0

6. Find the value of $\lim\limits_{x\to 5}(x^3-3x^2+2x-1)$. **Ans.** 59

7. Find the value of $\lim\limits_{x\to 0}\dfrac{|x|^{\alpha}}{e^x}$. $(\alpha>0)$ **Ans.** 0

8. Write any five indeterminate forms.

Ans. $\dfrac{0}{0}, \dfrac{\infty}{\infty}, \infty-\infty, 1^{\infty}, 0\times\infty$

9. $\lim\limits_{x\to a}\dfrac{[f(x)]}{[g(x)]}$ is always equal to $\dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$. (True/False)

Ans. False

10. Find the value of $\lim\limits_{x\to 1}\dfrac{x}{[x]}$. **Ans.** Limit does not exist

Some theorems on limits

1. Let $f(x)$ and $g(x)$ are two functions such that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist finitely. Then $\lim\limits_{x\to a}(f\pm g)(x)$, $\lim\limits_{x\to a} f(x)g(x)$ exist and if $\lim\limits_{x\to a} g(x)\neq 0$, $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ exists. However, the converse of any of the above is not necessarily true.
2. $\lim\limits_{x\to a} f[g(x)] = f[\lim\limits_{x\to a} g(x)]$ provided $\lim\limits_{x\to a} g(x)$ exists and lies in the domain of $f(x)$.

Methods for Calculating the Limits of the Form $\lim\limits_{x\to a}[f(x)]^{g(x)}$ when $\lim\limits_{x\to a} g(x)=\infty$

Here, we discuss two different cases:

1. When $\lim\limits_{x\to a} f(x)=1$. In this case, $\lim\limits_{x\to a}[f(x)]^{g(x)} = e^{\lim\limits_{x\to a} g(x)(f(x)-1)}$.
2. When $\lim\limits_{x\to a} f(x)\neq 1$ but $f(x)$ is positive in the neighbourhood of $x=a$. In this case we write,

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)} \Rightarrow \lim_{x\to a}[f(x)]^{g(x)} = e^{\lim\limits_{x\to a} g(x)\cdot\ln[f(x)]}$$

19.10 Continuity

19.10.1 Continuity of a Function

A function $f(x)$ is said to be continuous at $x=a$ if

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

That is, L.H.L = R.H.L. = Value of the function at 'a', that is, $\lim\limits_{x\to a} f(x)=f(a)$.

If $f(x)$ is not continuous at $x=a$, we say that $f(x)$ is discontinuous at $x=a$.

19.10.2 Geometrical Meaning of Continuity

Function $f(x)$ will be continuous at $x=c$ if there is no break in the graph of function $f(x)$ at the point $[c, f(c)]$.

In an interval, function is said to be continuous if there is no break in graph of function in the entire interval.

For example:

- $f(x)=\sin x$ is continuous in its entire domain (Fig. 19.5).

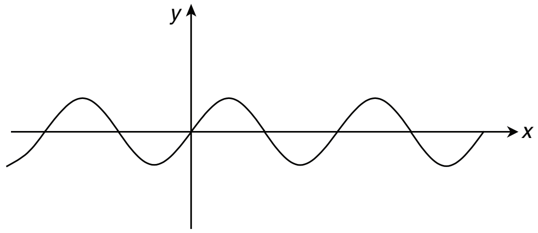


Figure 19.5

- $f(x)=\tan x$ is discontinuous at $x=(2n+1)\dfrac{\pi}{2}$ where $n\in I$. (See Fig. 19.6.)

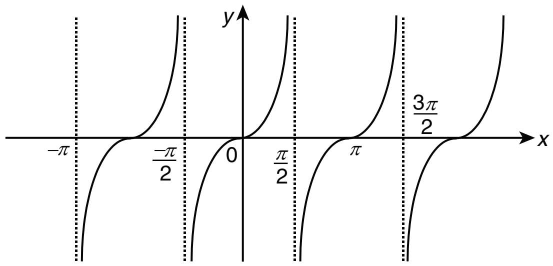


Figure 19.6

$f(x)$ will be discontinuous at $x=a$, in any of the following cases:

(i) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist but are not equal.

For example, $y=[x]$ at $x\in I$. (See Fig. 19.7.)

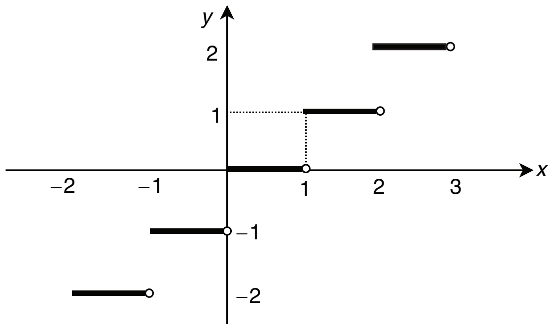


Figure 19.7

(ii) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist but not equal to $f(a)$.

For example, $f(x)=\begin{cases}\dfrac{x^2-4}{x-2} & x\neq 2\\ 5 & x=2\end{cases}$ at $x=2$. (See Fig. 19.8.)

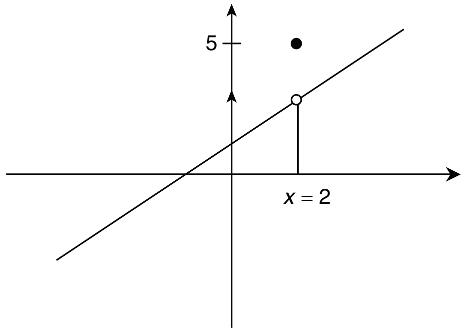


Figure 19.8

(iii) $f(a)$ is not defined.

For example, $y = \dfrac{1}{x}$ at $x = 0$. (See Fig. 19.9.)

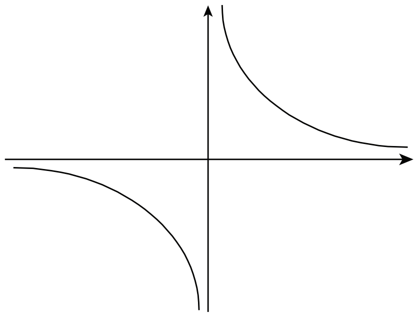


Figure 19.9

(iv) At least one of the limits does not exist.

For example, $y = \sin\left(\dfrac{1}{x}\right)$ at $x = 0$. (See Fig. 19.10.)

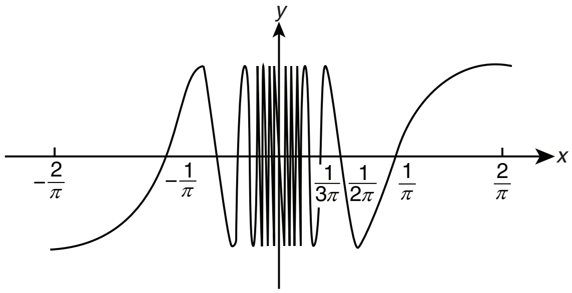


Figure 19.10

Illustration 19.17 $f(x) = \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}$ for all x in $\left(0, \dfrac{\pi}{2}\right)$ except at $x = \dfrac{\pi}{4}$. Define $f\left(\dfrac{\pi}{4}\right)$ so that $f(x)$ may be continuous at $x = \dfrac{\pi}{4}$.

Solution: $f(x)$ will be continuous at $x = \dfrac{\pi}{4}$ if

$$\lim_{x\to\pi/4} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \lim_{x\to\pi/4} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

$$= \lim_{x\to\pi/4} \frac{(\sqrt{2}\cos x - 1)\sin x}{\cos x - \sin x}$$

$$= \lim_{x\to\pi/4} \frac{(\sqrt{2}\cos x - 1)(\sqrt{2}\cos x + 1)}{(\sqrt{2}\cos x + 1)(\cos x - \sin x)} \cdot \frac{(\cos x + \sin x)}{(\cos x + \sin x)} \cdot \sin x$$

$$= \lim_{x\to\pi/4} \left(\frac{2\cos^2 x - 1}{\cos^2 x - \sin^2 x}\right) \frac{(\cos x + \sin x)\sin x}{(\sqrt{2}\cos x + 1)}$$

$$= \lim_{x\to\pi/4} \frac{\sin x(\cos x + \sin x)}{\sqrt{2}\cos x + 1}$$

$$= \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2}\cdot\frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$$

19.10.3 Continuity in an Open Interval

A function $f(x)$ is said to be continuous in an open interval (a, b) if it is continuous at each point of (a, b).

19.10.4 Continuity in a Closed Interval

See Fig. 19.11. A function $f(x)$ is said to be continuous in a closed interval $[a, b]$ if it is

- continuous at each point in (a, b),

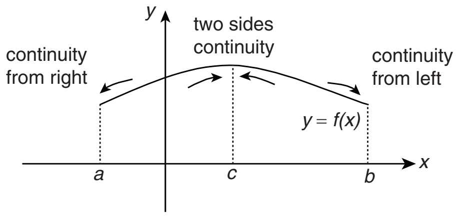


Figure 19.11 Continuity at points a, b and c

- $f(x)$ is continuous from right at $x = a$, that is,

$$\lim_{x\to a^+} f(x) = f(a)$$

- $f(x)$ is continuous from left at $x = b$, that is,

$$\lim_{x\to b^-} f(x) = f(b)$$

Continuity of a Function: A function $f(x)$ is said to be continuous at x = a if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$, that is, L.H.L. = R.H.L. = Value of the function at a, that is, $\lim_{x\to a} f(x) = f(a)$.

$f(x)$ will be discontinuous at $x = a$ in any of the following cases:

1. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but are not equal.
2. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal but not equal to $f(a)$.
3. $f(a)$ is not defined.
4. At least one of the limits does not exist.

If f and g are continuous at $x = a$, then $f + g$, kf and fg are continuous at $x = a$. Moreover, if $g(a) \neq 0$, then f/g is also continuous at $x = a$. Further if g is continuous at a and f is continuous at $g(a)$, then the composition of $f(g(a))$ is continuous at $x = a$.

Illustration 19.18 If $f(x)=\lim\limits_{n\to\infty}\dfrac{\log(x+2)-x^{2n}\sin x}{x^{2n}+1}$, then examine continuity of $f(x)$.

Solution:

If $|x|<1$, then

$$f(x)=\lim_{n\to\infty}\frac{\log(x+2)-x^{2n}\sin x}{x^{2n}+1}=\frac{\log(x+2)}{1}=\log(x+2)$$

If $|x|>1$, then

$$f(x)=\lim_{n\to\infty}\frac{\log(x+2)-x^{2n}\sin x}{x^{2n}+1}$$

$$=\lim_{n\to\infty}\frac{\frac{\log(x+2)}{x^{2n}}-\sin x}{1+\frac{1}{x^{2n}}}=-\sin x$$

If $|x|=1$, then

$$f(x)=\lim_{n\to\infty}\frac{\log(x+2)-x^{2n}\sin x}{x^{2n}+1}=\frac{\log(x+2)-\sin x}{2}$$

$$\text{Thus, } f(x)=\begin{cases}-\sin x, & x<-1\\ \log(x+2), & -1<x<1\\ -\sin x, & x>1\\ \dfrac{\log(x+2)-\sin x}{2}, & x=\pm1\end{cases}$$

Obviously $f(x)$ is discontinuous at $x=\pm 1$.

Illustration 19.19 Let $f(x)=[x]\sin\left(\dfrac{\pi}{[x+1]}\right)$ where $[\cdot]$ denotes greatest integer function. Then find domain of f and the points of discontinuity of f in the domain.

Solution: Let $f(x)=[x]\sin\left(\dfrac{\pi}{[x+1]}\right)$. Then domain of $f(x)$ is $x\in R$ excluding the points where $[x+1]=0$ (As denominator cannot be zero).

$$0\le x+1<1$$
$$\Rightarrow -1\le x<0$$

That is, for all $x\in[-1,0)$, denominator is zero. So, domain is

$$x\in R-[-1,0)$$

Points of Discontinuity

Check the continuity at $x=a$ (where $a\in I$)

$$\text{LHL}=\lim_{h\to0}[a-h]\sin\left(\frac{\pi}{[a+1-h]}\right)$$

$$\Rightarrow \text{LHL}=(a-1)\sin\left(\frac{\pi}{a}\right)\tag{19.3}$$

$$\text{RHL}=\lim_{h\to0}[a+h]\sin\left(\frac{\pi}{[a+1+h]}\right)$$

$$\Rightarrow \text{RHL}=a\sin\left(\frac{\pi}{a+1}\right)\tag{19.4}$$

From Eqs. (19.3) and (19.4), we get LHL ≠ RHL. So, $f(x)$ is discontinuous at $x=a$ (That is, at integral values of x).

So, points of discontinuity are

$x\in I\cap D$ (That is, integer lying in the set of domain)

$$\Rightarrow x\in I-\{-1\}$$

19.10.5 Properties of Continuous Functions

Let $f(x)$ and $g(x)$ are continuous functions at $x=a$. Then

1. $c\,f(x)$ is continuous at $x=a$ where c is any constant
2. $f(x)\pm g(x)$ is continuous at $x=a$
3. $f(x)\cdot g(x)$ is continuous at $x=a$
4. $f(x)/g(x)$ is continuous at $x=a$, provided $g(a)\neq 0$

19.10.6 Intermediate Value Theorem

If c is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation $f(x)=c$ in the open interval (a, b), if $y=f(x)$ is continuous in the interval. (See Fig. 19.12.)

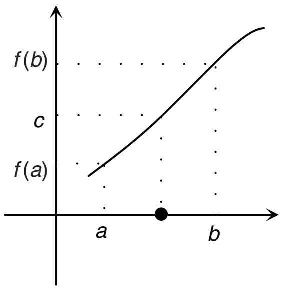


Figure 19.12

19.10.7 Types of Discontinuities

Basically there are two types of discontinuity:

1. **Removable discontinuity:** If $\lim\limits_{x\to a} f(x)$ exists but is not equal to $f(a)$, then $f(x)$ has a removable discontinuity at $x=a$ and it can be removed by redefining $f(x)$ for $x=a$.

Properties of a continuous function: A function $f(x)$, continuous on the interval $[a, b]$, possesses the following properties:

(i) $f(x)$ is bounded on $[a, b]$, that is, there exist m and M such that $m\le f(x)\le M$ for all $x\in[a, b]$, m and M are respectively minimum and maximum values of $f(x)$.

(ii) Let A be any number such that $\min f(x)\le A\le \max f(x)$. Then there exists a point $x_0\in[a, b]$ such that $f(x_0)=A$. It is called **intermediate value theorem**.

Illustration 19.20 Redefine the function $f(x)=[x]+[-x]$ in such a way that it could become continuous for $x\in(0, 2)$.

Solution: Here $\lim\limits_{x\to1} f(x)=-1$ but $f(1)=0$.

Hence, $f(x)$ has a removable discontinuity at $x=1$.

To remove this we define $f(x)$ as

$$f(x)=[x]+[-x],\ x\in(0,1)\cup(1,2)$$
$$f(x)=-1,\ x=1$$

Now, $f(x)$ is continuous for $x\in(0, 2)$.

2. **Non-removable discontinuity:** If $\lim\limits_{x\to a} f(x)$ does not exist, then we cannot remove this discontinuity. So this becomes a non-removable discontinuity or essential discontinuity.

Illustration 19.21 Prove that $f(x) = [x]$ has essential discontinuity at any $x \in I$.

Solution: Proof is obvious as $\lim_{x \to a} f(x)$ does not exist for any $a \in I$.
Hence, $f(x) = [x]$ has essential discontinuity at any $x \in I$.

Your Turn 2

1. Is $f(x) = \frac{1}{x}$ continuous in (0, 2)? **Ans.** Yes
2. For continuity $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$ should be equal to $f(a)$. (True/False) **Ans.** False
3. $f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous in its domain. (True/False) **Ans.** True
4. Is $f(x) = \frac{1}{x} + \frac{1}{|x|}$ continuous in $(-\infty, 0]$? **Ans.** No
5. A continuous function can have some points where limit does not exists. (True/False) **Ans.** False
6. If $f(x) = \begin{cases} ax+1 & x \geq 1 \\ x+2 & x < 1 \end{cases}$ is continuous, then 'a' should be ______. **Ans.** 2
7. $y = |x|$ is a continuous function. (True/False) **Ans.** True
8. Number of points of discontinuities of $f(x) = \frac{[x]}{1+|x|}$ in [1, 3] are ______. **Ans.** 2 points
9. $f(x) = \{x\} + [x]$ is a continuous function. (True/False) **Ans.** True
10. If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function.
    (A) $f(x) + g(x)$, **(B)** $f(x) - g(x)$,
    (C) $f(x)\, g(x)$ **(D)** $\frac{g(x)}{f(x)}$ **Ans.** (D)

19.11 Differentiability

Let $y = f(x)$ be a given function. If at some point, abscissa is x_1 and at other point abscissa is x_2, then it is quite natural that ordinate can be represented by y_1 and y_2, respectively at those points.

$\Delta y = y_2 - y_1$, Δy represents change in 'y'.
$\Delta x = x_2 - x_1$, Δx represents change in 'x'.

then $\Delta y = f(x_1 + \Delta x) - f(x_1)$

clearly increment can be positive, negative or may even be zero. Differential coefficient of $y = f(x)$, with respect to x is defined as the limiting value of $\frac{\Delta y}{\Delta x}$ as Δx tends to zero.

It is usually denoted by $\frac{dy}{dx}$ or $f'(x)$ symbolically.

The derivative of the function with respect to x is the function $f'(x)$ whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. That is,

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^-} \frac{f(x-h) - f(x)}{-h}$$

The function is said to be differentiable at $x = a$ if Right hand derivative (RHD) at $x = a$ denoted by

$$f'(a+0) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{-h}$$

exists and Left hand derivative (LHD) at $x = a$ denoted by

$$f'(a-0) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

also exists.

In both these cases, we have assumed $h > 0$.

A function whose graph is otherwise smooth will fail to have a derivative where the graph has

1. A corner, where the one-sided derivatives differ (Fig. 19.13).

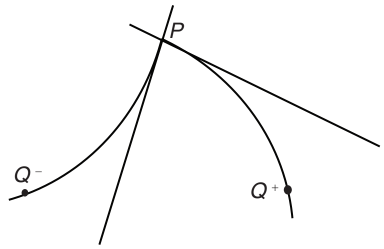


Figure 19.13

2. A cusp, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other (Fig. 19.14).

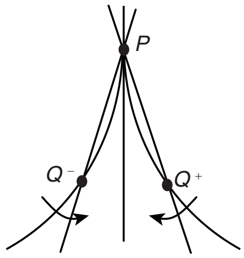


Figure 19.14

3. A vertical tangent, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$) (see Fig. 19.15).

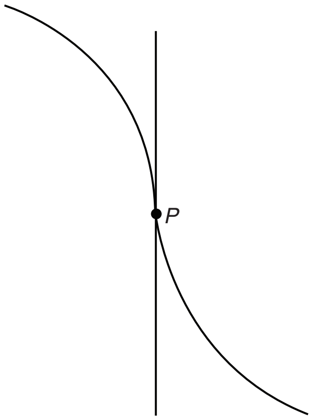


Figure 19.15

4. A discontinuity (Fig. 19.16).

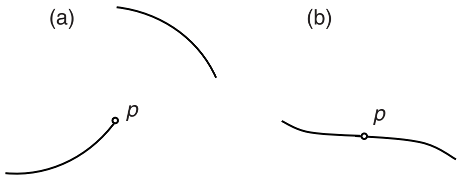


Figure 19.16

A function is continuous at every point where it has a derivative.

Proof: Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or, equivalently, that $\lim_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$f(c+h) = f(c) + [(f(c+h) - f(c)]$$
$$= f(c) + \frac{f(c+h)-f(c)}{h}\cdot h$$

Now take limits as $h \to 0$.

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\cdot \lim_{h\to 0} h$$
$$= f(c) + f'(c)\cdot 0 = f(c) + 0 = f(c)$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

19.11.1 Differentiability in an Interval

1. **Differentiability in an open interval (a, b):** The function of $y = f(x)$ is said to be differentiable in (a, b) if it is differentiable at each point $x \in (a, b)$.
2. **In a closed interval [a, b]:** The function $y = f(x)$ is said to be differentiable in $[a, b]$ if $f'(a+0)$, $f'(b-0)$ exist and $f'(x)$ exist for all $x \in (a, b)$.

Illustration 19.22 If $f(x) = \begin{cases} |x-1|([x]-x), & x \neq 1 \\ 0, & x = 1 \end{cases}$

Test the differentiability at $x = 1$, where [.] denotes the greatest integer function.

Solution: Check the differentiability at $x = 1$

$$R\left[f'(1)\right] = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \qquad (x > 1)$$
$$= \lim_{h\to 0}\frac{|1+h-1|\left\{[1+h]-(1+h)\right\}-0}{h}$$
$$= \lim_{h\to 0}\frac{h(1-1-h)}{h} = \lim_{h\to 0}\frac{h(-h)}{h} = 0$$
$$L[f'(1)] = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} \qquad (x < 1)$$
$$= \lim_{h\to 0}\frac{f(1-h)-0}{-h}$$
$$= \lim_{h\to 0}\frac{|1-h-1|\left\{[1-h]-(1-h)\right\}-0}{-h}$$
$$= \lim_{h\to 0}\frac{h(0-1+h)}{-h} = 1$$
$$Lf'(1) \neq Rf'(1)$$

Hence, $f(x)$ is not differentiable at $x = 1$.

19.11.2 Properties of Differentiability

Let $f(x)$ and $g(x)$ are differentiable functions at $x = a$. Then

1. $cf(x)$ is differentiable at $x = a$ where c is any constant
2. $f(x) \pm g(x)$ is differentiable at $x = a$
3. $f(x)\cdot g(x)$ is differentiable at $x = a$
4. $f(x)/g(x)$ is differentiable at $x = a$, provided $g(a) \neq 0$

Differentiability

Let $y = f(x)$ be a continuous function at a point $x = a$. It is said to be differentiable at $x = a$ if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists and is finite.

If this limit exists we call it the derivative of $y = f(x)$ at a

Right Hand Derivative:

RHD at $f(x)$ at $x = a$ is, $Rf'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, $h > 0$.

Left Hand Derivative:

LHD at $f(x)$ at $x = a$ is, $\lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$, $h > 0$.

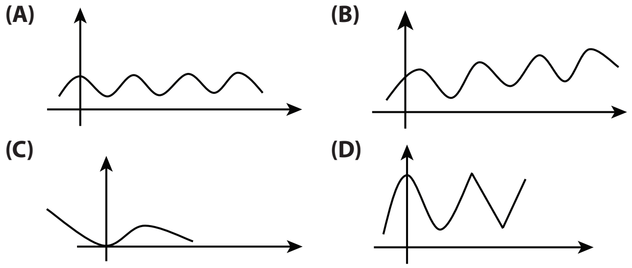
Clearly, $f(x)$ is differentiable at $x = a$ if and only if $Rf'(a) = Lf'(a)$.

Notes: If a function $f(x)$ is differentiable at $x = a$, then it is also continuous at $x = a$. But if a function is continuous at a point, it is not necessarily differentiable at that point. Let us consider the function $f(x) = |x|$.

Geometrically, we interpret $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ as the slope of the graph at the point $[a, f(a)]$. The line through $[a, f(a)]$ which has this slope is called the tangent line at $[a, f(a)]$. Thus, if there is no tangent line at a certain point, the function is not differentiable at that point. In other words, a function is not differentiable at a point where the curve suddenly changes direction (corner point) or the tangent is vertical at some point.

Your Turn 3

1. Which of the following functions, whose graphs are given, will have derivatives at all points:



Ans. (A)

2. Is $|x+1|$ is differentiable at $x = -1$? **Ans.** No
3. Derivative of $y = \frac{1}{x}\sin\left(\frac{1}{x}\right)$ at $x = 0$ is $\frac{1}{2}$. (True/False) **Ans.** False
4. If $f(x) = \begin{cases} x^2 + 3x + b & x \geq 1 \\ 2ax + 3 & x < 1 \end{cases}$ is continuous and differentiable, then find out value of a and b? **Ans.** $a = 5/2$ and $b = 4$
5. If $f(x) = 3x + x\sin x$, then $f'(1)$ is ________. **Ans.** $3 + \cos 1 + \sin 1$

6. If $f(x) = x^2$ and $g(x) = x$, then derivative of $\dfrac{f(x)}{g(x)}$ at $x = 0$ is 1. (True/False)

Ans. False

7. If $\lim\limits_{x\to 1}\dfrac{f(x)}{x-1}=2$ and $f(1)=0$, then $f'(1)=$ ________.

Ans. 2

8. Find values of 'x' for which derivative of $f(x) = \{x\}$ exists. (where $\{.\}$ denotes fractional part of x) **Ans.** $R - I$

9. $f''(x) = \lim\limits_{h\to 0}\dfrac{f'(x+h)-f'(x)}{h}$. (True/False) **Ans.** True

10. Which of the following functions is/are differentiable

(A) $y=\log|x|$ **(B)** $y=|x|$

(C) $y=|x|^2+2x+3$ **(D)** $\dfrac{1}{1-|x|^2}$ **Ans.** (C)

Additional Solved Examples

1. If $\lim\limits_{n\to\infty} n\cos\left(\dfrac{\pi}{4n}\right)\sin\left(\dfrac{\pi}{4n}\right)=k$, then k is

(A) $\dfrac{\pi}{4}$ **(B)** $\dfrac{\pi}{3}$ **(C)** π **(D)** None of these

Solution:

$$k=\lim_{n\to\infty}\frac{n}{2}\sin\frac{\pi}{2n}=\lim_{n\to\infty}\frac{n}{2}\frac{\sin\frac{\pi}{2n}}{\pi/2n}\cdot\frac{\pi}{2n}=\frac{\pi}{4}$$

Hence, the correct answer is option (A).

2. If $\lim\limits_{x\to 0}(1+a\sin x)^{\operatorname{cosec} x}=3$, then a is

(A) ln 2 **(B)** ln 3 **(C)** ln 4 **(D)** log 2

Solution:

$$3=\lim_{x\to 0}(1+a\sin x)^{\operatorname{cosec} x}\ [1^\infty \text{ form}]$$

$$=\lim_{x\to 0}e^{\operatorname{cosec} x\,.\,a\sin x}=e^a$$

Hence,

$$e^a=3\Rightarrow a=\log_e 3=\ln 3$$

Hence, the correct answer is option (B).

3. $\lim\limits_{x\to 1}\dfrac{x\sin\{x\}}{x-1}$, where $\{x\}$ denotes the fractional part of x, is equal to

(A) -1 **(B)** 0 **(C)** 1 **(D)** Does not exist

Solution:

$$\lim_{x\to 1-0}\{x\}=\lim_{x\to 1-0}(x-[x])=1-0=1$$

$$\lim_{x\to 1+0}\{x\}=\lim_{x\to 1+0}(x-[x])=1-1=0$$

Therefore,

$$\lim_{x\to 1-0}\frac{x\sin\{x\}}{x-1}=\lim_{x\to 1-0}\frac{x}{x-1}\sin\{x\}=-\infty\cdot\sin(1)=-\infty$$

$$\lim_{x\to 1+0}\frac{x\sin\{x\}}{x-1}=\lim_{x\to 1+0}\frac{x\sin\{x\}}{\{x\}}\cdot\frac{\{x\}}{x-1}=\lim_{x\to 1+0}\frac{x\sin\{x\}}{\{x\}}\,\frac{x-1}{x-1}$$

$$=1\times 1\times 1=1$$

Since

$$\text{L.H. limit} \neq \text{R.H. limit}$$

Therefore, limit does not exist.

Hence, the correct answer is option (D).

4. If $\lim\limits_{n\to\infty}\dfrac{n^k\sin^2 n!}{n+1}=0$ for

(A) All k **(B)** $0\le k<1$

(C) $k=1$ **(D)** For $k>1$

Solution:

$$\lim_{n\to\infty}\frac{n^k\sin^2 n!}{n+1}=\lim_{n\to\infty}\frac{n^k}{n+1}\cdot(\sin^2 n!)$$

$\lim\limits_{n\to\infty}\sin^2 n!$ does not exist but if

$\lim\limits_{n\to\infty}\dfrac{n^k}{n+1}=0$, then $\lim\limits_{n\to\infty}\dfrac{n^k\sin^2 n!}{n+1}=0$

But

$$\lim_{n\to\infty}\frac{n^k}{n+1}=0\Leftrightarrow 0\le k<1$$

Hence, the correct answer is option (B).

5. If $f(x)=\begin{cases}(\cos x)^{1/\sin x} & \text{for } x\neq 0\\ k & \text{for } x=0\end{cases}$

The value of k, so that f is continuous at $x = 0$ is

(A) 0 **(B)** 1 **(C)** 1/2 **(D)** None of these

Solution: Given, $f(0) = k$. Now

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}(\cos x)^{1/\sin x}\ [1^\infty \text{ form}]$$

$$=\lim_{x\to 0}e^{\frac{1}{\sin x}(\cos x-1)}$$

$$=\lim_{x\to 0}(e)^{(-2\sin^2 x/2)/((2\sin x/2)\cos x/2)}$$

$$=\lim_{x\to 0}e^{-\tan\frac{x}{2}}=e^0=1$$

For $f(x)$ to be continuous at $x = 0$, k should be equal to 1.

Hence, the correct answer is option (B).

6. If $f(x) = [\sqrt{2}\sin x]$, where $[x]$ denotes the greatest integer function, then

(A) $f(x)$ is continuous at $x = 0$

(B) Maximum value of $f(x)$ is 1 in interval $[-2\pi, 2\pi]$

(C) $f(x)$ is discontinuous at $x=\dfrac{n\pi}{2}+\dfrac{\pi}{4}, n\in I$

(D) $f(x)$ is differentiable at $x = n\pi, n \in I$

Solution:

$$f(x)=[\sqrt{2}\sin x]$$

Hence,

$$f(x)=-2, -\frac{\pi}{2}\le x<-\frac{\pi}{4}$$

$$f(x)=-1, -\frac{\pi}{4}\le x<0$$

$$f(x)=0, 0\le x<\frac{\pi}{4}$$

$$f(x) = 1, \frac{\pi}{4} \le x \le \frac{\pi}{2}$$

$$f(x) = 1, \frac{\pi}{2} \le x \le \frac{3\pi}{4}$$

$$f(x) = 0, \frac{3\pi}{4} < x \le \pi$$

Clearly $f(x)$ is discontinuous at

$$x = -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{3\pi}{4}, \text{etc.}$$

General value corresponding to

$$x = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4} \text{ is } \frac{n\pi}{2} + \frac{\pi}{4}$$

Maximum value of $f(x)$ in $[-2\pi, 2\pi]$ is 1 at $x = \frac{\pi}{2}$.

$f(x)$ is discontinuous and non-differentiable at $x = 0$, therefore choice (D) is not correct.

Hence, the correct answer is option (B).

7. The value of $f(0)$, so that the function $f(x) = \dfrac{(27-2x)^{1/3}-3}{9-3(243+5x)^{1/5}}$ $(x \ne 0)$ is continuous is

(A) 2/3 **(B)** 6 **(C)** 2 **(D)** 4

Solution:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(27-2x)^{1/3}-(27)^{1/3}}{3[3-(243+5x)^{1/5}]}$$

$$= \lim_{x\to 0} \frac{\dfrac{(27-2x)^{1/3}-(27)^{1/3}}{(27-2x)-27}(-2x)}{-3\left[\dfrac{(243+5x)^{1/5}-(243)^{1/5}}{243+5x-243}\cdot 5x\right]}$$

$$= \frac{2}{15}\cdot\frac{\frac{1}{3}(27)^{-2/3}}{\frac{1}{5}(243)^{-4/5}} = \frac{2}{15}\cdot\frac{5}{3}\cdot\frac{1}{9}\cdot 81 = 2$$

Hence, the correct answer is option (C).

8. Let $f(x) = \begin{cases} x^p \sin\dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Then $f(x)$ is continuous but not differentiable at $x = 0$ if

(A) $p < 0$ **(B)** $p = 0$ **(C)** $0 < p \le 1$ **(D)** $p \ge 1$

Solution:

$$f(0) = 0$$

For $f(x)$ to be continuous at $x = 0$

$$\lim_{x\to 0} f(x) = 0 \Rightarrow \lim_{x\to 0} x^p \sin\frac{1}{x} = 0$$

This is possible only when $p > 0$. (1)

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^p \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0} h^{p-1}\sin\frac{1}{h}$$

$f'(0)$ will exist only when $p > 1$.

Hence, $f(x)$ will not be differentiable if $p \le 1$. (2)

From Eqs. (1) and (2), for $f(x)$ to be not differentiable but continuous at $x = 0$, possible values of p are given by $0 < p \le 1$.

Hence, the correct answer is option (C).

9. $\displaystyle\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{\tan x} =$

(A) e **(B)** $e/2$ **(C)** e^2 **(D)** None of these

Solution:

$$\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{(\tan x)\cdot\frac{x}{x}}$$

$$= \lim_{x\to 0}\frac{e-e\left(1-\frac{x}{2}+\frac{11}{24}x^2-\cdots\right)}{x} \quad \text{[using expansion]}$$

$$= \lim_{x\to 0}\frac{\frac{e^x}{2}\left(1-\frac{11}{12}x+\cdots\right)}{x} = \frac{e}{2}$$

Hence, the correct answer is option (B).

10. $\displaystyle\lim_{n\to\infty}\left(\tan\theta + \frac{1}{2}\tan\frac{\theta}{2} + \frac{1}{2^2}\tan\frac{\theta}{2^2} + \cdots + \frac{1}{2^n}\tan\frac{\theta}{2^n}\right) =$

(A) $\dfrac{1}{\theta}$ **(B)** $\dfrac{1}{\theta} - 2\cot 2\theta$

(C) $2\cot 2\theta$ **(D)** None of these

Solution:

$$\tan\theta = \cot\theta - 2\cot 2\theta$$

Therefore,

$$\frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2} - \cot\theta$$

$$\frac{1}{2^n}\tan\frac{\theta}{2^n} = \frac{1}{2^n}\cot\frac{\theta}{2^n} - \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}}$$

Hence,

$$S_n = \frac{1}{2^n}\cot\frac{\theta}{2^n} - 2\cot 2\theta$$

So, required limit is

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{\dfrac{2^n\tan\theta/2^n}{\theta/2^n}\cdot\dfrac{\theta}{2^n}} - 2\cot 2\theta\right) = \frac{1}{\theta} - 2\cot 2\theta$$

Hence, the correct answer is option (B).

11. Let $f(x) = \dfrac{|x^3-6x^2+11x-6|}{x^3-6x^2+11x-6}$. Find the set of points 'a', where $\displaystyle\lim_{x\to a} f(x)$ does not exists.

Solution: We write,

$$f(x) = \frac{|x-1|}{x-1}\cdot\frac{|x-2|}{x-2}\cdot\frac{|x-3|}{x-3} = \begin{cases} -1, & x < 1 \\ 1, & 1 < x < 2 \\ -1, & 2 < x < 3 \\ 1, & x > 3 \end{cases}$$

Therefore, the limits exists at all points except at $x = 1, 2, 3$.

As $\lim_{x\to 1^-} f(x) = -1$,

$$\lim_{x\to 1^+} f(x) = 1$$

So, $\lim_{x\to 1} f(x)$ does not exist.

Similarly for $x = 2$ and $x = 3$.
Therefore, required set is {1, 2, 3}.

Hence, the correct answer is {1, 2, 3}.

12. Evaluate $\lim_{x\to 0} \dfrac{(1+x)^{1/x} + e(x-1)}{x}$.

Solution:

$$\lim_{x\to 0} \frac{(1+x)^{1/x} + e(x-1)}{x} = \lim_{x\to 0} \frac{e^{\frac{1}{x}\ln(1+x)} - e}{x} + e$$

$$= \lim_{x\to 0} \frac{e^{1-\frac{x}{2}+\frac{x^2}{3}-\cdots} - e}{x} + e = e\lim_{x\to 0} \frac{e^{-\frac{x}{2}+\frac{x^2}{3}-\cdots} - 1}{x} + e$$

$$= e\lim_{x\to 0} \frac{\left(e^{-\frac{x}{2}+\frac{x^2}{3}-\cdots}\right)\left(-\frac{1}{2}+\frac{2x}{3}-\cdots\right)}{1} + e \text{ (using L'H rule)}$$

$$= -\frac{e}{2} + e = \frac{e}{2}$$

Hence, the correct answer is (e/2).

13. Check the function $f(x) = \lim_{x\to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$ for continuity and differentiability at $x = 0$.

Solution: Let $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$. Then,

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1} = \lim_{h\to 0} \frac{(1/e^{1/h}-1)}{(1/e^{1/h}+1)} = \frac{0-1}{0+1} = -1$$

$$\left[\text{as } h\to 0 \Rightarrow \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty \Rightarrow 1/e^{1/h} \to 0\right] \quad (1)$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h\to 0} \frac{(1-1/e^{1/h})}{(1+1/e^{1/h})}$$

[Dividing numerator and denominator by $e^{1/h}$]

$$= \frac{1-0}{1+0} = 1 \quad \text{[using Eq. (1)]}$$

Clearly, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Hence, $\lim_{x\to 0} f(x)$ does not exist.

As $\lim_{x\to 0} f(x)$ does not exist, function is neither continuous nor differentiable at $x = 0$.

Hence, the correct answer is the function which is neither differentiable nor continuous at $x = 0$.

14. Find a polynomial of least degree, such that

$$\lim_{x\to 0}\left(1+\frac{x^2+f(x)}{x^2}\right)^{1/x} = e^2$$

Solution: Now,

$$\lim_{x\to 0}\left(1+\frac{x^2+f(x)}{x^2}\right)^{1/x}$$

$$\Rightarrow \lim_{x\to 0}\left(1+\frac{x^2+f(x)}{x^2}\right)^{1/x} = L \text{ (say)}$$

exists only when $\lim_{x\to 0} \dfrac{x^2+f(x)}{x^2} = 0$ (i.e., it converts to 1^∞ form).

So, the least degree in $f(x)$ is of degree 2. That is,

$$f(x) = a_2x^2 + a_3x^3 + \cdots$$

Now,

$$L = \lim_{x\to 0}\left(1+\frac{x^2+f(x)}{x^2}\right)^{1/x} = e^2$$

$$= e^{\lim_{x\to 0}\left(\frac{x^2+f(x)}{x^2}\right)\frac{1}{x}} = e^2 = e^{\lim_{x\to 0}\frac{x^2+f(x)}{x^3}} = e^2$$

Hence,

$$\lim_{x\to 0} \frac{x^2+f(x)}{x^3} = 2 \Rightarrow \lim_{x\to 0} \frac{x^2+a_2x^2+a_3x^3+\cdots}{x^3} = 2$$

So, $a_2 = -1$, $a_3 = 2$ and a_4, a_5 are any arbitrary constants. Since, we want polynomial of least degree. Hence,

$$f(x) = -x^2 + 2x^3$$

Hence, the correct answer is $f(x) = -x^2 + 2x^3$.

15. Evaluate $\lim_{x\to 0} \dfrac{(2^x-1)^3}{\sin 2x \ln\left(1+\dfrac{x^2}{2}\right)}$.

Solution:

$$\frac{(2^x-1)^3}{\sin 2x} \cdot \frac{1}{\ln\left(1+\frac{x^2}{2}\right)} = \frac{\left(\frac{2^x-1}{x}\right)^3}{\frac{\sin 2x}{x} \cdot \frac{1}{x^2}\ln\left(1+\frac{x^2}{2}\right)}$$

$$\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)^3 = \left[\lim_{x\to 0}\left(\frac{2^x-1}{x}\right)\right]^3 = (\ln 2)^3$$

$$\lim_{x\to 0} \frac{\sin 2x}{x} = 2 \text{ and}$$

$$\lim_{x\to 0} \frac{1}{x^2}\ln\left(1+\frac{x^2}{2}\right)$$

$$= \lim_{x\to 0} \ln\left[\left(1+\frac{x^2}{2}\right)^{\frac{1}{x^2}}\right]$$

$$= \ln \lim_{x\to 0}\left(1+\frac{x^2}{2}\right)^{\frac{1}{x^2}} = \ln_e e^{1/2} = \frac{1}{2}$$

Therefore, required limit $= \dfrac{(\ln 2)^3}{2\times\frac{1}{2}} = (\ln 2)^3$

Hence, the correct answer is (ln 2)³.

16. Let $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then test whether

(A) $f(x)$ is continuous at $x = 0$
(B) $f(x)$ is differentiable at $x = 0$

Solution:

(A) $f(0+) = \lim_{x\to 0+} xe^{-2/x}$ (since $|x| = x$)

$$= \lim_{x\to 0}\frac{x}{e^{2/x}} = 0$$

$$f(0-) = \lim_{x\to 0-} x e^{0} = \lim_{x\to 0} x = 0$$

$$f(0) = 0$$

Hence, $f(x)$ is continuous at $x = 0$.

(B) Regarding differentiability

$$f'(0+) = \lim_{x\to 0}\frac{f(x)-f(0)}{x}$$

$$= \lim_{x\to 0}\frac{xe^{-2/x}}{x} = \lim_{x\to 0}\frac{1}{e^{2/x}} = 0$$

$$f'(0-) = \lim_{x\to 0}\frac{f(x)-f(0)}{x}$$

$$= \lim_{x\to 0}\frac{x-0}{x} = 1$$

The two values are different.
Hence, $f(x)$ is not differentiable at $x = 0$.

17. Discuss the continuity of $f(x)$ in [0, 2] where

$$f(x) = \lim_{n\to\infty}\left(\sin\frac{\pi x}{2}\right)^{2n}$$

Solution: Since

$$\lim_{n\to\infty} x^{2n} = \begin{cases} 0; & |x| < 1 \\ 1; & |x| = 1 \end{cases}$$

Therefore,

$$f(x) = \lim_{n\to\infty}\left(\sin\frac{\pi x}{2}\right)^{2n} = \begin{cases} 0; & \left|\sin\frac{\pi x}{2}\right| < 1 \\ 1; & \left|\sin\frac{\pi x}{2}\right| = 1 \end{cases}$$

Thus, $f(x)$ is continuous for all x, except for those values of x for which $\left|\sin\frac{\pi x}{2}\right| = 1$, that is, x is an odd integer.

So, $x = (2n + 1)$ where $n \in I$.

Check continuity at $x = (2n + 1)$

$$\text{LHL} = \lim_{x\to 2n+1} f(x) = 0 \quad (1)$$

and $\quad f(2n + 1) = 1 \quad (2)$

from Eqs. (1) and (2), we get

$$\text{LHL} \neq f(2n + 1)$$

Therefore, $f(x)$ is discontinuous at $x = 2n + 1$ (That is, at odd integers).

Hence, $f(x)$ is discontinuous at $x = (2n + 1)$.

Hence, in the given interval, $f(x)$ is discontinuous at $x = 1$.

18. Show that the function $f(x)$ is continuous at $x = 0$ but its derivative does not exists at $x = 0$ if

$$f(x) = \begin{cases} x\sin(\log x^2); & x \neq 0 \\ 0; & x = 0 \end{cases}$$

Solution: Test for continuity

$$\text{LHL} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}(-h)\sin\log(-h)^2$$

$$= -\lim_{h\to 0} h\sin\log h^2$$

As $h \to 0$, $\log h^2 \to -\infty$

Hence, sin log h^2 oscillates between −1 and +1. So,

$$\text{LHL} = -\lim_{h\to 0}(h)\times\lim_{h\to 0}(\sin\log h^2)$$

$$= -0\times(\text{number between } -1 \text{ and } +1) = 0$$

$$\text{RHL} = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} h\sin\log h^2 = \lim_{h\to 0} h\cdot\lim_{h\to 0}\sin\log h^2$$

$$= 0\times(\text{oscillating between } -1 \text{ and } +1) = 0$$

$$f(0) = 0 \text{ (given)}$$

$$\Rightarrow \text{LHL} = \text{RHL} = f(0)$$

Hence, $f(x)$ is continuous at $x = 0$.

Test for differentiability

$$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{-h\sin\log(-h)^2 - 0}{-h} = \lim_{h\to 0}\sin(\log h^2)$$

As the expression oscillates between −1 and +1, the limit does not exists. Therefore, Left hand derivative is not defined.
Hence, the function is not differentiable at $x = 0$.

19. If $f(x) = \begin{cases} \dfrac{5e^{1/x}+2}{3-e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Examine **(i)** $f(x)$ **(ii)** $x^2 f(x)$ for continuity and differentiability at $x = 0$.

Solution:

(i) For $f(x)$

$$\text{RHL} = \lim_{x\to 0+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{5e^{1/h}+2}{3-e^{1/h}}$$

$$= \lim_{h\to 0} \frac{5+2e^{-1/h}}{3e^{-1/h}-1} = \frac{5+0}{0-1} = -5$$

Value of function $= f(0) = 0$

RHL $\neq$ Value of function

Hence, $f(x)$ is not continuous and hence not differentiable also.

(ii) Let $F(x) = x^2 f(x)$. Then

$$L[F'(0)] = \lim_{h\to 0} \frac{F(0-h)-F(0)}{-h} = \lim_{h\to 0} \frac{h^2 f(-h)-0}{-h}$$

$$= \lim_{h\to 0} hf(-h) = 0$$

$$RF'(0) = \lim_{h\to 0} \frac{F(0+h)-F(0)}{h} = \lim_{h\to 0} \frac{h^2 f(h)-0}{h} = \lim_{h\to 0} hf(h) = 0$$

Hence, $LF'(0) = RF'(0)$.

Hence, $F(x)$ is differentiable at $x = 0$, so it is always continuous at $x = 0$.

Previous Years' Solved JEE Main/AIEEE Questions

1. The function $f: R-\{0\} \to R$ given by

$$f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$$

can be made continuous at $x = 0$ by defining $f(0)$ as

(A) 2 **(B)** -1 **(C)** 0 **(D)** 1

[AIEEE 2007]

Solution: We have

$$\lim_{x\to 0} \frac{1}{x} - \frac{2}{e^{2x}-1} \Rightarrow \lim_{x\to 0} \frac{e^{2x}-1-2x}{x(e^{2x}-1)}$$

$$\Rightarrow \lim_{x\to 0} \frac{2e^{2x}-2}{(e^{2x}-1)+2xe^{2x}} \Rightarrow \lim_{x\to 0} \frac{4e^{2x}}{4e^{2x}+4xe^{2x}} = 1$$

Hence, the correct answer is option (D).

2. Let $f(x) = \begin{cases} (x-1)\sin\left(\dfrac{1}{x-1}\right), & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$. Then which one of the following is true?

(A) f is neither differentiable at $x = 0$ nor at $x = 1$

(B) f is differentiable at $x = 0$ and at $x = 1$

(C) f is differentiable at $x = 0$ but not at $x = 1$

(D) f is differentiable at $x = 1$ but not at $x = 0$ **[AIEEE 2008]**

Solution: We have

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \Rightarrow f'(1) = \lim_{h\to 0} \frac{(1+h-1)\sin\left(\dfrac{1}{1+h-1}\right)}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}\sin\left(\frac{1}{h}\right) \Rightarrow f'(1) = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

As the limit does not exist, therefore, f is not differentiable at $x = 1$. Similarly,

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} \Rightarrow f'(0) = \lim_{h\to 0} \frac{(h-1)\sin\left(\dfrac{1}{h-1}\right) - \sin 1}{h}$$

This limit also does not exist, which implies that f is also not differentiable at $x = 0$.

Hence, the correct answer is option (A).

3. Let $f(x) = x|x|$ and $g(x) = \sin x$.

Statement-1: gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement-2: gof is twice differentiable at $x = 0$.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(C) Statement-1 is true, Statement-2 is false

(D) Statement-1 is false, Statement-2 is true **[AIEEE 2009]**

Solution: We have,

$$f(x) = x|x| \text{ and } g(x) = \sin x$$

$$gof(x) = \sin(x|x|) = \begin{cases} -\sin x^2, & x < 0 \\ \sin x^2, & x \geq 0 \end{cases}$$

Therefore,

$$(gof)'(x) = \begin{cases} -2x\cos x^2, & x < 0 \\ 2x\cos x^2, & x \geq 0 \end{cases}$$

$$(gof)''(x) = \begin{cases} -2\cos x^2 + 4x^2\sin x^2, & x < 0 \\ 2\cos x^2 - 4x^2\sin x^2, & x \geq 0 \end{cases}$$

Now LHD at 0,

$$\lim_{h\to 0} \frac{gof(0-h)-gof(0)}{-h} = \frac{-\sin h^2 - 0}{-h} = \lim_{h\to 0}\left(\frac{\sin h^2}{h^2}\right)\times h = 0$$

And RHD at 0,

$$\lim_{h\to 0} \frac{gof(0+h)-gof(0)}{h} = \frac{\sin h^2 - 0}{h} = \lim_{h\to 0}\left(\frac{\sin h^2}{h^2}\right)\times h = 0$$

Clearly, $L(gof)'(0) = 0 = R(gof)'(0)$

Therefore, gof is differentiable at $x = 0$ and also its derivative is continuous at $x = 0$.

Now,

$$(gof)''(x) = -2[x(-\sin x^2)2x + \cos x^2], x < 0$$

$$= 2[x(-\sin x^2)2x + \cos x^2], x > 0$$

Now, $gof''(0)$

$$\text{LHD} = \lim_{h\to 0} \frac{-\cos(h^2)2(-h)-0}{-h} = -2(1) = -2$$

$$\text{RHD} = \lim_{h\to 0} \frac{\cos(h^2)2(h)-0}{h} = 2$$

$L(gof)''(0) = -2$ and $R(gof)''(0) = 2$. Hence $L(gof)''(0) \neq R(gof)''(0)$

That is, $gof(x)$ is not twice differentiable at $x = 0$.

Hence, the correct answer is option (C).

4. Let $f: R \to R$ be a continuous function defined by $f(x) = \dfrac{1}{e^x + 2e^{-x}}$.

Statement-1: $f(c) = \dfrac{1}{3}$, for some $c \in R$.

Statement-2: $0 < f(x) \leq \dfrac{1}{2\sqrt{2}}$, for all $x \in R$.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is false
(C) Statement-1 is false, Statement-2 is true
(D) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation for Statement-1

[AIEEE 2010]

Solution:

$$f(x)=\frac{1}{e^x+2e^{-x}}=\frac{e^x}{e^{2x}+2}\Rightarrow f'(x)=\frac{(e^{2x}+2)e^x-2e^{2x}\cdot e^x}{(e^{2x}+2)^2}$$

$$f'(x)=0\Rightarrow e^{2x}+2=2e^{2x}\Rightarrow e^{2x}=2\Rightarrow e^x=\sqrt{2}$$

Maximum $f(x)=\frac{\sqrt{2}}{2+2}=\frac{\sqrt{2}}{4}=\frac{1}{2\sqrt{2}}$. That is, $0<f(x)\le\frac{1}{2\sqrt{2}}\,\forall x\in R$

Since $0<\frac{1}{3}<\frac{1}{2\sqrt{2}}$, therefore for some $c\in R$, we have, $f(c)=\frac{1}{3}$.

Therefore, we conclude that Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation for Statement-1
Hence, the correct answer is option (D).

5. $\lim\limits_{x\to 2}\left(\frac{\sqrt{1-\cos[2(x-2)]}}{x-2}\right)$

(A) Equals $\sqrt{2}$ **(B)** Equals $-\sqrt{2}$
(C) Equals $\frac{1}{\sqrt{2}}$ **(D)** Does not exist

[AIEEE 2011]

Solution: We have

$$\lim_{x\to 2}\frac{\sqrt{2\sin^2(x-2)}}{x-2}=\lim_{x\to 2}\frac{\sqrt{2}\,|\sin(x-2)|}{x-2}$$

If $x\to 2-$, then $|\sin(x-2)|=-\sin(x-2)$;
If $x\to 2^+$, then $\Rightarrow |\sin(x-2)|=\sin(x-2)$
Therefore, RHL $\ne$ LHL
Thus, it is concluded that the limit does not exist.
Hence, the correct answer is option (D).

6. The value of p and q for which the function

$$f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x}, & x<0\\ q, & x=0\\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}, & x>0\end{cases}$$ is continuous for all x in R is

(A) $p=\frac{5}{2}, q=\frac{1}{2}$ **(B)** $p=-\frac{3}{2}, q=\frac{1}{2}$
(C) $p=\frac{1}{2}, q=\frac{3}{2}$ **(D)** $p=\frac{1}{2}, q=-\frac{3}{2}$

[AIEEE 2011]

Solution:

$$\text{LHL}=\lim_{x\to 0}\frac{\sin(p+1)x+\sin x}{x}=\lim_{x\to 0}(p+1)\cos(p+1)x+\cos x$$
$$=p+1+1=p+2$$

$$\text{RHL}=\lim_{x\to 0}\frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}=\lim_{x\to 0}\frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}\times\frac{\sqrt{x+x^2}+\sqrt{x}}{\sqrt{x+x^2}+\sqrt{x}}$$
$$=\lim_{x\to 0}\frac{x+x^2-x}{x^{3/2}(\sqrt{x+x^2}+\sqrt{x})}=\frac{\sqrt{x}}{(\sqrt{x+x^2}+\sqrt{x})}=\frac{1}{2}$$

Therefore,

$$q=\frac{1}{2}\Rightarrow p+2=\frac{1}{2}\Rightarrow p=-\frac{3}{2}$$

Hence, the correct answer is option (B).

7. If $f:R\to R$ is a function defined by $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is
(A) Continuous for every real x
(B) Discontinuous only at $x=0$
(C) Discontinuous only at non-zero integral values of x
(D) Continuous only at $x=0$

[AIEEE 2012]

Solution:

$$f(x)=[x]\cos\left(\frac{2x-1}{2}\right)\pi=[x]\cos\left(x-\frac{1}{2}\right)\pi=[x]\sin\pi x$$

$f(x)=[x]\sin\pi x$, that is, f is continuous for every real x.

Hence, the correct answer is option (A).

8. $\lim\limits_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

(A) $\frac{1}{2}$ **(B)** 1 **(C)** 2 **(D)** $-\frac{1}{4}$

[JEE MAIN 2013, 2015(OFFLINE)]

Solution: We have

$$\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$$
$$=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}\,\frac{(3+\cos x)x}{\frac{\tan 4x}{4x}\times 4x}=\frac{2(3+1)}{1\times 4}=2$$

Hence, the correct answer is option (C).

9. $\lim\limits_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to

(A) $-\pi$ **(B)** π **(C)** $\frac{\pi}{2}$ **(D)** 1

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}=\lim_{x\to 0}\frac{\sin[\pi(1-\sin^2 x)]}{x^2}=\lim_{x\to 0}\frac{\sin(\pi-\pi\sin^2 x)}{x^2}$$
$$=\lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{x^2}=\lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\frac{\pi\sin^2 x}{x^2}=1\times\pi=\pi$$

Hence, the correct answer is option (B).

10. If $f(x)$ is continuous and $f\left(\frac{9}{2}\right)=\frac{2}{9}$, then $\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right)$ is equal to

(A) 9/2 **(B)** 2/9 **(C)** 0 **(D)** 8/9

[JEE MAIN 2014 (ONLINE SET-1)]

Solution:

$$\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right)=\lim_{x\to 0} f\left\{\frac{2\sin^2\frac{3x}{2}}{x^2}\right\}$$

$$\Rightarrow \lim_{x\to 0} f\left[\frac{2\sin^2(3x/2)}{(9x^2/4)\times(4/9)}\right]=\lim_{x\to 0} f\left\{\left[\frac{\sin(3x/2)}{(3x/2)}\right]^2\times\frac{9}{2}\right\}$$

$$\Rightarrow f\left(\frac{9}{2}\right)=\frac{2}{9}$$

Hence, the correct answer is option (B).

11. If $\lim_{x\to 2}\frac{\tan(x-2)[x^2+(k-2)x-2k]}{x^2-4x+4}=5$, then k is equal to

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$\lim_{x\to 2}\frac{\tan(x-2)}{(x-2)}\cdot\frac{x^2+kx-2x-2k}{(x-2)}=5$$

$$\Rightarrow \lim_{x\to 2}\frac{\tan(x-2)}{x-2}\lim_{x\to 2}\frac{x(x-2)+k(x-2)}{(x-2)}=5$$

$$\Rightarrow \lim_{x\to 2}\frac{\tan(x-2)}{x-2}\lim_{x\to 2}\frac{\cancel{(x-2)}(x+k)}{\cancel{(x-2)}}=5 \Rightarrow 2+k=5\Rightarrow k=3$$

Hence, the correct answer is option (D).

12. Let $f(x)=x|x|$, $g(x)=\sin x$ and $h(x)=(gof)(x)$. Then

(A) $h(x)$ is not differentiable at $x=0$.

(B) $h(x)$ is differentiable at $x=0$, but $h'(x)$ is not continuous at $x=0$.

(C) $h'(x)$ is continuous at $x=0$ but it is not differentiable at $x=0$.

(D) $h'(x)$ is differentiable at $x=0$.

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$f(x)=\begin{cases}x^2, & x>0\\ 0, & x=0\\ -x^2, & x<0\end{cases}$$

$$g(x)=\sin x$$

$$h(x)=g(f(x))=\begin{cases}\sin x^2, & x>0\\ 0, & x=0\\ -\sin x^2, & x<0\end{cases}$$

Now

$$h'(x)=\begin{cases}2x\cos x^2, & x>0\\ 0, & x=0 \text{ (since, LHD = RHD at 0)}\\ -2x\cos x^2, & x<0\end{cases}$$

$$h''(x)=\begin{cases}-4x^2\sin x^2+2\cos x^2, & x>0\\ 0, & x=0\\ 4x^2\sin x^2-2\cos x^2, & x<0\end{cases}$$

At $x=0$,

LHD $=-2$

RHD $=2$

Therefore, $h'(x)$ is continuous at $x=0$ but is not differentiable at $x=0$.

Hence, the correct answer is option (C).

13. Let $f, g: R\to R$ be two functions defined by

$$f(x)=\begin{cases}x\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases} \text{ and } g(x)=xf(x)$$

Statement I: f is a continuous function at $x=0$.

Statement II: g is a differentiable function at $x=0$.

(A) Both statements I and II are false.

(B) Both statements I and II are true.

(C) Statement I is true, statement II is false.

(D) Statement I is false, statement II is true.

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$f(x)=x\sin\frac{1}{x},\ x\neq 0;\quad g(x)=x^2\sin\frac{1}{x}, x\neq 0$$

$$f(x)=0,\quad x=0;\quad g(x)=0,\quad x=0$$

$$\lim_{x\to 0} f(x)=f(0)$$

$$g'(0)=\lim_{h\to 0}\frac{(0+h)^2\sin\frac{1}{(0+h)}-g(0)}{0+h}=\lim_{h\to 0}\frac{h^{\cancel{2}}\sin\frac{1}{h}}{\cancel{h}}=0$$

Hence, the correct answer is option (B).

14. If the function

$$f(x)=\begin{cases}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}, & x\neq\pi\\ k, & x=\pi\end{cases}$$

is continuous at $x=\pi$, then k equals

(A) 0 **(B)** $\frac{1}{2}$ **(C)** 2 **(D)** $\frac{1}{4}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}=\lim_{h\to 0}\frac{\sqrt{2+\cos(\pi+h)}-1}{(\pi-\pi-h)^2}=\lim_{h\to 0}\frac{\sqrt{2-\cos h}-1}{h^2}$$

$$=\lim_{h\to 0}\frac{(2-\cos h-1)}{h^2(\sqrt{2-\cos h}+1)}=\lim_{h\to 0}\frac{1-\cos h}{h^2(\sqrt{2-\cos h}+1)}=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{h^2(\sqrt{2-\cos h}+1)}$$

$$=\lim_{h\to 0}\frac{2\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4(\sqrt{2-\cos h}+1)}=\lim_{h\to 0}2\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\times\lim_{h\to 0}\frac{1}{4(\sqrt{2-\cos h}+1)}$$

$$=\cancel{2}\times\frac{1}{4(\cancel{2})}=\frac{1}{4}$$

Therefore, $k=\frac{1}{4}$

Hence, the correct answer is option (D).

15. Let $f: R \to R$ be a function such that $|f(x)| \le x^2$, for all $x \in R$. Then at $x = 0$ is

(A) Continuous but not differentiable
(B) Continuous as well as differentiable
(C) Neither continuous nor differentiable
(D) Differentiable but not continuous

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$|f(x)| \le x^2$$

When $x = 0$

$$|f(0)| \le 0 \Rightarrow |f(0)| \text{ cannot be negative.}$$

Now,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

Now,

$$|f(h)| \le h^2 = |h|^2 \Rightarrow \left|\frac{f(h)}{h}\right| \le |h|$$

$$\Rightarrow \lim_{h\to 0} -|h| \le \lim_{h\to 0} \frac{f(h)}{h} \le \lim_{h\to 0} |h|$$

Therefore, by Squeeze principle $\lim_{h\to 0} \frac{f(h)}{h} = 0$, that is, $f'(0) = 0$.

Thus, $f(x)$ is differentiable at 0 and hence continuous.

Hence, the correct answer is option (B).

16. Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x\to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to

(A) -4 **(B)** 0 **(C)** 4 **(D)** -8

[JEE MAIN 2015 (OFFLINE)]

Solution:

$$f'(x) = a(x-1)(x-2)(x-k)$$

$$\Rightarrow f'(x) = a[x^3 - (3+k)x^2 + (2+3k)x - 2k]$$

$$\Rightarrow f(x) = a\left[\frac{x^4}{4} - \frac{(3+k)x^3}{3} + \frac{(2+3k)x^2}{2} - 2kx\right] + C$$

$$\Rightarrow \frac{f(x)}{x^2} = a\left[\frac{x^2}{4} - \frac{(3+k)x}{3} + \frac{(2+3k)}{2} - \frac{2k}{x}\right] + \frac{C}{x^2}$$

$$\lim_{x\to 0}\left(1 + \frac{f(x)}{x^2}\right) = \lim_{x\to 0}\left\{1 + a\left[\frac{(2+3k)x^2 - 4kx + 2C}{2x^2}\right]\right\} = 3 \text{ (given)}$$

$$\Rightarrow \lim_{x\to 0}\left\{\frac{[a(2+3k)+2]x^2 - 4kax + 2Ca}{2x^2}\right\} = 3$$

$$\Rightarrow Ca = 0 \qquad (1)$$

and $$\lim_{x\to 0}\left\{\frac{[a(2+3k)+2](2x) - 4ka}{4x}\right\} = 3$$

$$\Rightarrow ka = 0$$

and $$\lim_{x\to 0}\left\{\frac{[(2+3k)+2](2)}{4}\right\} = 3 \Rightarrow a(2+3k) + 2 = 6$$

$$\Rightarrow 2a + 3ak = 4 \Rightarrow a = 2 \text{ (As } ka = 0)$$
$$\Rightarrow C = 0 \text{ and } k = 0$$

$$\Rightarrow f(x) = 2\left[\frac{x^4}{4} - x^3 + x^2\right] = \frac{x^4}{2} - 2x^3 + 2x^2$$

$$\Rightarrow f(2) = 8 - 16 + 8 = 0$$

Hence, the correct answer is option (B).

17. $\lim_{x\to 0} \frac{e^{x^2} - \cos x}{\sin^2 x}$ is equal to

(A) 3 **(B)** $\frac{3}{2}$ **(C)** $\frac{5}{4}$ **(D)** 2

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{\sin^2 x}\left(\frac{0}{0}\text{form}\right) = \lim_{x\to 0} \frac{2xe^{x^2} + \sin x}{\sin 2x}\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{2x(2xe^{x^2}) + 2e^{x^2} + \cos x}{2\cos 2x} = \frac{3}{2} \text{ (Using L'Hospital's rule)}$$

Hence, the correct answer is option (B).

18. For $x \in R$ $f(x) = |\log 2 - \sin x|$ and $g(x) = f[f(x)]$, then

(A) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$.
(B) g is not differentiable at $x = 0$.
(C) $g'(0) = \cos(\log 2)$.
(D) $g'(0) = -\cos(\log 2)$.

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$f(x) = |\log 2 - \sin x|$$
$$g(x) = f(f(x))$$

Therefore,

$$g(x) = |\log 2 - \sin f(x)|$$
$$f(x) = \log 2 - \sin x \; (\log 2 - \sin x) > 0 \text{ in neighbourhood of } x = 0$$

That is,

$$g(x) = |\log 2 - \sin(\log 2 - \sin x)|$$
$$= \log 2 - \sin(\log 2 - \sin x) \; [g(x) \text{ is constant function at } x = 0]$$

Therefore,

$$g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$$
$$= \cos x \cdot \cos(\log 2 - \sin x)$$

Substituting $x = 0$, we get

$$g'(0) = \cos(\log 2)$$

Hence, the correct answer is option (C).

19. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)\cdots 3n}{n^{2n}}\right)^{1/n}$ is equal to

(A) $3\log 3 - 2$ **(B)** $\frac{18}{e^4}$ **(C)** $\frac{27}{e^2}$ **(D)** $\frac{9}{e^2}$

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$y = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)\cdots(n+2n)}{n^{2n}}\right)^{1/n}$$

$$\Rightarrow \ln y = \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(\frac{n+1}{n}\right) + \ln\left(\frac{n+2}{n}\right) + \cdots + \ln\left(\frac{n+2n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{2}{n}\right) + \cdots + \ln\left(1+\frac{2n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} \ln\left(1+\frac{r}{n}\right)$$

$$= \int_0^2 \ln(1+x)dx = (x+1)\ln(x+1) - (x+1)\Big|_0^2$$

$$= 3(\ln 3) - 3 + 1 = 3(\ln 3) - 2 = \ln 27 - \ln e^2$$

$$\ln y = \ln\left(\frac{27}{e^2}\right) \Rightarrow y = \frac{27}{e^2}$$

Hence, the correct answer is option (C).

20. If $f(x)$ is a differentiable function in the interval $(0, \infty)$ such that $f(1) = 1$ and $\lim_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$, for each $x > 0$, then $f(3/2)$ is equal to

(A) $\frac{23}{18}$ **(B)** $\frac{13}{6}$ **(C)** $\frac{25}{9}$ **(D)** $\frac{31}{18}$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We have

$$\lim_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$$

Applying L'Hospital's rule, we get

$$\lim_{t\to x} \frac{2t\, f(x) - x^2 f'(t)}{1} = 1$$

$$2x\, f(x) - x^2 f'(x) = 1$$

$$x^2 f'(x) - 2x\, f(x) + 1 = 0$$

Therefore,

$$f'(x) - \frac{2}{x} f(x) = -\frac{1}{x^2}$$

Integrating factor is

$$e^{-\int \frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$\left(f(x)\cdot\frac{1}{x^2}\right) = -\int \frac{1}{x^4} + C = -\frac{x^{-4+1}}{-4+1} + C = \left(\frac{1}{3}\times\frac{1}{x^3}\right) + C$$

Therefore,

$$f(x) = \frac{1}{3x} + Cx^2$$

$$f(1) = 1 \Rightarrow 1 = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}$$

$$f(x) = \frac{1}{3}\left(\frac{1}{x} + 2x^2\right)$$

$$\Rightarrow f\left(\frac{3}{2}\right) = \frac{1}{3}\left[\frac{2}{3} + \left(2\times\frac{9}{4}\right)\right] = \frac{31}{18}$$

Hence, the correct answer is option (D).

21. If the function

$$f(x) = \begin{cases} -x, & x < 1 \\ a + \cos^{-1}(x+b), & 1 \le x \le 2 \end{cases}$$

is differentiable at $x = 1$, then a/b is equal to

(A) $\frac{\pi+2}{2}$ **(B)** $\frac{\pi-2}{2}$ **(C)** $\frac{-\pi-2}{2}$ **(D)** $-1-\cos^{-1}(2)$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We have

$f(1^+) = f(1) = f(1^-)$ and $f'(1^-) = f'(1^+)$

$$-1 = -\frac{1}{\sqrt{1-(1+b)^2}} \Rightarrow 1-(1+b)^2 = 1 \Rightarrow (1+b) = 0 \Rightarrow b = -1$$

$$-1 = a + \cos^{-1}(0) = a + \frac{\pi}{2}$$

$$a = -1 - \frac{\pi}{2} = -\left(\frac{\pi+2}{2}\right)$$

Therefore,

$$\frac{a}{b} = \left(\frac{\pi+2}{2}\right) = 1 + \frac{\pi}{2}$$

Hence, the correct answer is option (A).

22. If $\lim_{x\to\infty}\left(1 + \frac{a}{x} - \frac{4}{x^2}\right)^{2x} = e^3$, then a is equal to

(A) 2 **(B)** $\frac{3}{2}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We have

$$e^{\lim_{x\to\infty}\left(\frac{a}{x} - \frac{4}{x^2}\right)^{2x}} = e^3$$

$$\Rightarrow \lim_{x\to\infty}\left(2a - \frac{8}{x}\right) = 3$$

$$\Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$$

Hence, the correct answer is option (B).

23. Let $a, b \in \mathbb{R}$, $(a \neq 0)$. If the function f defined as

$$f(x) = \begin{cases} \frac{2x^2}{a}, & 0 \le x < 1 \\ a, & 1 \le x < \sqrt{2} \\ \frac{2b^2 - 4b}{x^3}, & \sqrt{2} \le x < \infty \end{cases}$$

is continuous in the interval $[0, \infty)$, then an ordered pair (a, b) is

(A) $(-\sqrt{2}, 1-\sqrt{3})$ **(B)** $(\sqrt{2}, -1+\sqrt{3})$

(C) $(\sqrt{2}, 1-\sqrt{3})$ **(D)** $(-\sqrt{2}, 1+\sqrt{3})$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have

$$f(x) = \begin{cases} \frac{2x^2}{a}, & 0 \le x < 1 \\ a, & 1 \le x < \sqrt{2} \\ \frac{2b^2 - 4b}{x^3}, & \sqrt{2} \le x < \infty \end{cases}$$

Now,

$$f(1^-) = f(1) = f(1^+)$$

$$f(1)=\lim_{h\to 0}\frac{2(1-h)^2}{a}=\frac{2}{a}$$

$$f(1)=f(1^+)=a$$

That is,

$$\frac{2}{a}=a\Rightarrow a^2=2\Rightarrow a=-\sqrt{2},\sqrt{2}$$

$$f(\sqrt{2}^-)=f(\sqrt{2})=f(\sqrt{2}^+)$$

$$f(\sqrt{2})=f(\sqrt{2}^+)=\lim_{h\to 0}\frac{2b^2-4b}{(\sqrt{2}+h)^3}=\frac{2b(b-2)}{2\sqrt{2}}=\frac{b(b-2)}{\sqrt{2}}$$

$$f(\sqrt{2}^-)=a$$

Therefore,

$$\frac{b(b-2)}{\sqrt{2}}=a\Rightarrow b^2-2b=\sqrt{2}a$$

- When $a=\sqrt{2}$, we get

$$b^2-2b=2$$
$$b^2-2b-2=0$$
$$b=\frac{2\pm\sqrt{4+8}}{2}$$
$$b=1\pm\sqrt{3}$$

- When $a=-\sqrt{2}$, we get

$$b^2-2b=-2$$
$$b^2-2b+2=0$$
$$(b-1)^2+1=0 \quad \text{(Not possible)}$$

Therefore, the possible ordered pair is $(\sqrt{2},1+\sqrt{3})$, $(\sqrt{2},1-\sqrt{3})$.

Hence, the correct answer is option (C).

24. $\lim_{x\to 0}\dfrac{(1-\cos 2x)^2}{2x\tan x - x\tan 2x}$ is

(A) 2 **(B)** $-\dfrac{1}{2}$ **(C)** -2 **(D)** $\dfrac{1}{2}$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have

$$\lim_{x\to 0}\frac{(2x)^4\left(\dfrac{1-\cos 2x}{4x^2}\right)^2}{2x\left(\tan x-\dfrac{\tan 2x}{2}\right)}$$

$$\Rightarrow \lim_{x\to 0}\frac{\left(16\times x^4\times\dfrac{1}{4}\right)}{2x\left(\tan x-\dfrac{\tan x}{1-\tan^2 x}\right)}$$

$$\Rightarrow \lim_{x\to 0}\frac{2x^3(1-\tan^2 x)}{(\tan x-\tan^3 x-\tan x)}=-2$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Let $f(x)$ be differentiable on the interval $(0,\infty)$ such that $f(1)=1$, and

$\lim_{t\to x}\dfrac{t^2f(x)-x^2f(t)}{t-x}=1$ for each $x>0$. Then $f(x)$ is

(A) $\dfrac{1}{3x}+\dfrac{2x^2}{3}$ **(B)** $\dfrac{-1}{3x}+\dfrac{4x^2}{3}$ **(C)** $\dfrac{-1}{x}+\dfrac{2}{x^2}$ **(D)** $\dfrac{1}{x}$

[IIT-JEE 2007]

Solution: We have

$$\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$$

$$\Rightarrow \lim_{t\to x}\frac{2t\,f(x)-x^2f'(t)}{1-0}=1$$

$$\Rightarrow 2x\,f(x)-x^2f'(x)=1$$

$$\Rightarrow \frac{x^2f'(x)-2x\,f(x)}{x^4}=\frac{-1}{x^4}$$

$$\Rightarrow d\left(\frac{f(x)}{x^2}\right)=\frac{-1}{x^4} \quad (1)$$

On integrating both sides of this equation, we get

$$\frac{f(x)}{x^2}=\frac{1}{3x^3}+c$$

Now, since

$$f(1)=1\Rightarrow c=\frac{2}{3}$$

we get

$$f(x)=\frac{1}{3x}+\frac{2}{3}x^2$$

Hence, the correct answer is option (A).

2. In the following $[x]$ denotes the greatest integer less than or equal to x.

Column I	Column II
(A) $x\lvert x\rvert$	**(P)** continuous in $(-1, 1)$
(B) $\sqrt{\lvert x\rvert}$	**(Q)** differentiable in $(-1, 1)$
(C) $x+[x]$	**(R)** strictly increasing in $(-1, 1)$
(D) $\lvert x-1\rvert+\lvert x+1\rvert$	**(S)** not differentiable at least at one point in $(-1, 1)$

[IIT-JEE 2007]

Solution:

(A) $\to$(P), (Q), (R)

$f(x)=x\lvert x\rvert$

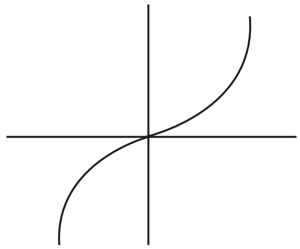


Figure 19.17

(B) $\to$(P), (S)

$f(x)=\sqrt{\lvert x\rvert}$

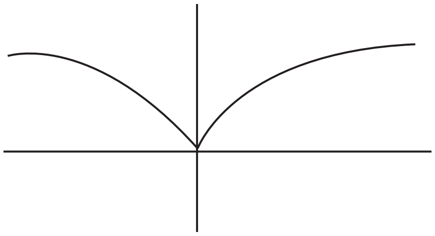


Figure 19.18

(C)→(R), (S)

$$f(x) = x + [x]$$
$$= \begin{cases} x-1, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$$

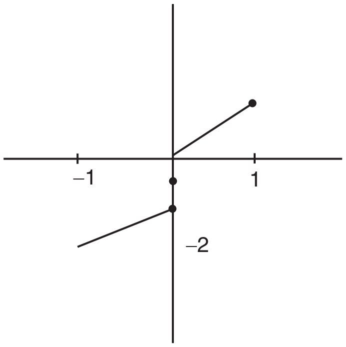


Figure 19.19

(D)→(P), (Q)

$$f(x) = |x-1| + |x+1|$$

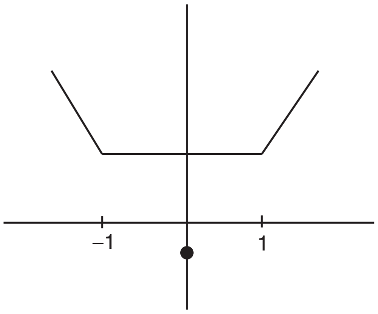


Figure 19.20

Hence, the correct matches are (A)→(P, Q, R); (B)→(P, S); (C)→(R, S); (D)→(P, Q).

3. Let $f(x) = 2 + \cos x$ for all real x.

Statement-1: For each real t, there exists a point c in $[t, t + \pi]$ such that $f'(c) = 0$. Because

Statement-2: $f(t) = f(t + 2\pi)$ for each real t.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True, Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

[IIT-JEE 2007]

Solution: We have $f(x) = 2 + \cos x$ for all values of x.

$$f'(x) = -\sin x$$

Statement-1:

$$f'(t) = -\sin t$$
$$f'(\pi + t) = \sin t$$

Also, since

$$f'(t) \cdot f'(\pi + t) = -\sin^2 t$$

which is negative, the equation $f'(t) = 0$ has at least one solution in $[t, t + \pi]$.

Statement-2:

Since $f(x) = 2 + \cos x$ is a periodic function with period 2π, we get

$$f(2\pi + t) = f(t)$$

Hence, the correct answer is option (B).

4. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then

(A) $f''(x)$ vanishes at least twice on [0, 1]

(B) $f'\left(\frac{1}{2}\right) = 0$

(C) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$

(D) $\int_0^{1/2} f(t) e^{\sin \pi t} dt = \int_{1/2}^{1} f(1-t) e^{\sin \pi t} dt$

[IIT-JEE 2008]

Solution: We have

$$f(x) = f(1-x) \tag{1}$$

Put $x = \frac{1}{2} + x$. Then

$$f\left(\frac{1}{2} + x\right) = f\left(\frac{1}{2} - x\right)$$

So, $f\left(\frac{1}{2} + x\right)$ is an even function and $\sin x \cdot f\left(\frac{1}{2} + x\right)$ is an odd function. Therefore,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$$

Differentiate equation (1), we get

$$f'(x) = -f'(1-x) \tag{2}$$

Put $x = \frac{1}{2}$, we get

$$f'\left(\frac{1}{2}\right) = -f'\left(\frac{1}{2}\right)$$
$$\Rightarrow f'\left(\frac{1}{2}\right) = 0$$

Now, put $x = \frac{1}{4}$ in Eq. (2)

$$f'\left(\frac{1}{4}\right) = -f'\left(\frac{3}{4}\right) = 0$$
$$\Rightarrow f'\left(\frac{1}{4}\right) = f'\left(\frac{1}{2}\right) = f'\left(\frac{3}{4}\right)$$

Using Rolle's theorem, $f'''(x) = 0$ has at least one solution in $\left(\frac{1}{4},\frac{1}{2}\right)$ and also in $\left(\frac{1}{2},\frac{3}{4}\right)$.

So, $f''(x) = 0$ vanishes at least twice on [0, 1]. Now,

$$\int_0^{\frac{1}{2}} f(t)e^{\sin \pi t}dt$$

Put $t = 1 - x$. Then

$$\int_1^{\frac{1}{2}} f(1-x)\cdot e^{\sin \pi(1-x)}\cdot(-dx)$$

$$= \int_{\frac{1}{2}}^{1} f(1-x)e^{\sin\pi x}\cdot dx$$

Hence, the correct answers are options (A), (B), (C) and (D).

5. Let f and g be real valued functions defined on interval (–1, 1) such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x) \sin x$.

Statement-1: $\lim_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$

Statement-2: $f'(0) = g(0)$

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True; Statement-2 is not a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

[IIT-JEE 2008]

Solution: We have

$$f(x) = g(x)\sin x$$
$$f'(x) = g(x)\cos x + g'(x)\cdot\sin x$$
$$\Rightarrow f'(0) = g(0)$$
$$f''(x) = 2g'(x)\cos x - g(x)\sin x + \sin x\cdot g''(x)$$
$$\Rightarrow f''(0) = 2g'(0) = 0$$

Now,

$$\lim_{x\to 0}[g(x)\cot x - g(0)\operatorname{cosec} x]$$
$$= \lim_{x\to 0}\frac{g(x)\cdot\cos x - g(0)}{\sin x}\quad\left(\frac{0}{0}\text{form}\right)$$
$$= \lim_{x\to 0}\frac{g'(x)\cos x - g(x)\sin x}{\cos x}$$
$$= g'(0) = f''(0)$$

Hence, the correct answer is option (B).

6. Let $L = \lim_{x\to 0}\dfrac{a - \sqrt{a^2 - x^2} - \dfrac{x^2}{4}}{x^4}, a > 0$. If L is finite, then

(A) $a = 2$ **(B)** $a = 1$ **(C)** $L = \dfrac{1}{64}$ **(D)** $L = \dfrac{1}{32}$

[IIT-JEE 2009]

Solution:

$$L = \lim_{x\to 0}\frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4} = \lim_{x\to 0}\frac{1}{x^2(a + \sqrt{a^2 - x^2})} - \frac{1}{4x^2}$$
$$= \lim_{x\to 0}\frac{(4-a) - \sqrt{a^2 - x^2}}{4x^2(a + \sqrt{a^2 - x^2})}$$

Numerator $\to 0$ if $a = 2$, then $L = \dfrac{1}{64}$.

Hence, the correct answers are options (A) and (C).

7. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t}\, dt$. Then which of the following statement(s) is (are) true?

(A) $f''(x)$ exists for all $x \in (0, \infty)$
(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$
(C) There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$
(D) There exists $\beta > 0$ such that $|f(x)| + |f'(x)| \leq \beta$ for all $x \in (0, \infty)$

[IIT-JEE 2010]

Solution:

$$f'(x) = \frac{1}{x} + \sqrt{1 + \sin x}$$

$f'(x)$ is not differentiable at $\sin x = -1$ or $x = 2n\pi - \dfrac{\pi}{2}, n \in N$.

In $x \in (1, \infty)$ $\quad f(x) > 0, f'(x) > 0$

Consider,

$$f(x)\ f = \ln x + \int_0^x \sqrt{1 + \sin t}\ dt - \frac{1}{x} - \sqrt{1 + \sin x}$$
$$= \left(\int_0^x \sqrt{1 + \sin t}\ dt - \sqrt{1 + \sin x}\right) + \ln x - \frac{1}{x}$$

Consider, $g(x) = \int_0^x \sqrt{1 + \sin t}\ dt - \sqrt{1 + \sin x}$

It can be proved that $g(x) \geq 2\sqrt{2} - \sqrt{10} \quad \forall x \in (0, \infty)$

Now there exists some $\alpha > 1$ such that $\dfrac{1}{x} - \ln x \leq 2\sqrt{2} - \sqrt{10}$ for all $x \in (\alpha, \infty)$ as $\dfrac{1}{x} - \ln x$ is strictly decreasing function.

So, $g(x) \geq \dfrac{1}{x} - \ln x$.

Hence, the correct answers are options (B) and (C).

8. Let $f: R \to R$ be a function such that $f(x + y) = f(x) + f(y), \forall x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then

(A) $f(x)$ is differentiable only in a finite interval containing zero
(B) $f(x)$ is continuous $\forall x \in R$
(C) $f'(x)$ is constant $\forall x \in R$
(D) $f(x)$ is differentiable except at finitely many points

[IIT-JEE 2011]

Solution: Since $f(0) = 0$ and

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h} = f'(0) = k \text{ (say)}$$

$$\Rightarrow f(x) = kx + c \Rightarrow f(x) = kx \qquad (\text{As } f(0) = 0).$$

Therefore, $f(x)$ is continuous for all $x \in R$ and $f'(x) = k$, that is, constant for all $x \in R$.

Hence, the correct answers are options (B) and (C).

9. If $\lim_{x\to 0} [1 + x\ln(1+b^2)]^{1/x} = 2b\sin^2\theta, b > 0$ and $\theta \in (-\pi, \pi]$, then the value of θ is

(A) $\pm\frac{\pi}{4}$ **(B)** $\pm\frac{\pi}{3}$ **(C)** $\pm\frac{\pi}{6}$ **(D)** $\pm\frac{\pi}{2}$

([**IIT-JEE 2011]**

Solution:

$$e^{\ln(1+b^2)} = 2b\sin^2\theta$$

$$\Rightarrow \sin^2\theta = \frac{1+b^2}{2b}$$

$$\Rightarrow \sin^2\theta = 1 \text{ as } \frac{1+b^2}{2b} \ge 1$$

$$\theta = \pm\pi/2$$

Hence, the correct answer is option (D).

10. If $f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \le -\frac{\pi}{2} \\ -\cos x & -\frac{\pi}{2} < x \le 0 \\ x - 1 & 0 < x \le 1 \\ \ln x & x > 1 \end{cases}$, then

(A) $f(x)$ is continuous at $x = -\pi/2$
(B) $f(x)$ is not differentiable at $x = 0$
(C) $f(x)$ is differentiable at $x = 1$
(D) $f(x)$ is differentiable at $x = -3/2$

[IIT-JEE 2011]

Solution:

$$\lim_{x\to -\frac{\pi^-}{2}} f(x) = 0 = f(-\pi/2)$$

$$\lim_{x\to -\frac{\pi^+}{2}} f(x) = \cos\left(-\frac{\pi}{2}\right) = 0$$

$$f'(x) = \begin{cases} -1, & x \le -\pi/2 \\ \sin x, & -\pi/2 < x \le 0 \\ 1, & 0 < x \le 1 \\ 1/x, & x > 1 \end{cases}$$

Clearly, $f(x)$ is not differentiable at $x = 0$ as $f'(0^-) = 0$ and $f'(0^+) = 1$. $f(x)$ is differentiable at $x = 1$ as $f'(1^-) = f'(1^+) = 1$.

Hence, the correct answers are options (A), (B), (C) and (D).

11. Let $f: (0,1) \to \mathbb{R}$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on (0, 1)
(B) $f \ne f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$
(C) $f = f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$
(D) f^{-1} is differentiable on (0, 1)

[IIT-JEE 2011]

Solution:

$$f(x) = \frac{b-x}{1-bx}$$

Let $y = \frac{b-x}{1-bx}$. Then

$$x = \frac{b-y}{1-by}$$

$$0 < x < 1 \Rightarrow 0 < \frac{b-y}{1-by} < 1$$

$$\frac{b-y}{1-by} > 0 \Rightarrow y < b \text{ or } y > \frac{1}{b}$$

$$\frac{b-y}{1-by} - 1 > 0 \Rightarrow -1 < y < \frac{1}{b}$$

$$\Rightarrow -1 < y < b$$

Hence, the correct answer is option (A).

12. If $\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$, then

(A) $a = 1, b = 4$ **(B)** $a = 1, b = -4$
(C) $a = 2, b = -3$ **(D)** $a = 2, b = 3$

[IIT-JEE 2012]

Solution: Given

$$\lim_{x\to\infty}\left(\frac{x^2+x+1}{x+1} - ax - b\right) = 4$$

$$\Rightarrow \lim_{x\to\infty} \frac{x^2+x+1-ax^2-ax-bx-b}{(x+1)} = 4$$

$$\Rightarrow \lim_{x\to\infty} \frac{(1-a)x^2+(1-a-b)x+(1-b)}{(x+1)} = 4$$

$$\Rightarrow 1-a = 0 \text{ and } 1-a-b = 4 \Rightarrow b = -4, a = 1$$

Hence, the correct answer is option (B).

13. Let $f(x) = \begin{cases} x^2\left|\cos\frac{\pi}{x}\right|, & x \ne 0 \\ 0, & x = 0 \end{cases}$ $x \in \mathbb{R}$. Then f is

(A) Differentiable both at $x = 0$ and at $x = 2$
(B) Differentiable at $x = 0$ but not differentiable at $x = 2$

(C) Not differentiable at $x = 0$ but differentiable at $x = 2$

(D) Differentiable neither at $x = 0$ nor at $x = 2$

[IIT-JEE 2012]

Solution:

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^2\left|\cos\frac{\pi}{h}\right| - 0}{h}$$

$$= \lim_{h\to 0} h\cos\left(\frac{\pi}{h}\right) = 0$$

so, $f(x)$ is differentiable at $x = 0$

$$f'(2^+) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{(2+h)^2\left|\cos\frac{\pi}{2+h}\right| - 0}{h}$$

$$= \lim_{h\to 0}\frac{(2+h)^2\cos\left(\frac{\pi}{2+h}\right)}{h}$$

$$f'(2^+) = \lim_{h\to 0}\frac{(2+h)^2}{h}\sin\left(\frac{\pi}{2}-\frac{\pi}{2+h}\right)$$

$$= \lim_{h\to 0}\frac{(2+h)^2}{h}\sin\left[\frac{\pi\cdot h}{2(2+h)}\right]$$

$$= \lim_{h\to 0}\frac{(2+h)^2}{h}\frac{\sin(\pi h/2(2+h))}{(\pi h/2(2+h))}\times\frac{\pi h}{2(2+h)} = \pi$$

Again, $f'(2^-) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$

$$= \lim_{h\to 0}\frac{(2-h)^2\left|\cos\left(\frac{\pi}{2-h}\right)\right|}{-h}$$

$$= \lim_{h\to 0}\frac{-(2-h)^2\cos\left(\frac{\pi}{2-h}\right)}{-h}$$

$$= \lim_{h\to 0}\frac{(2-h)^2\sin\left[\frac{\pi}{2}-\frac{\pi}{2-h}\right]}{h}$$

$$= \lim_{h\to 0}\frac{(2-h)^2}{h}\cdot\sin\left[\frac{-\pi h}{2(2-h)}\right]$$

$$= -\lim_{h\to 0}\frac{(2-h)^2}{\frac{\pi h}{2(2-h)}}\cdot\sin\frac{\pi h}{2(2-h)}\times\frac{\pi}{2(2-h)} = -\pi$$

Hence, the correct answer is option (B).

14. For every integer n, let a_n and b_n be real numbers. Let function $f:\mathbb{R}\to\mathbb{R}$ be given by

$$f(x) = \begin{cases} a_n + \sin\pi x, & \text{for } x\in[2n, 2n+1] \\ b_n + \cos\pi x, & \text{for } x\in(2n-1, 2n) \end{cases},$$ for all integers n. If f is continuous, then which of the following hold(s) for all n?

(A) $a_{n-1} - b_{n-1} = 0$ **(B)** $a_n - b_n = 1$

(C) $a_n - b_{n+1} = 1$ **(D)** $a_{n-1} - b_n = -1$

[IIT-JEE 2012]

Solution: At $x = 2n$

$$\text{L.H.L} = \lim_{h\to 0}[b_n + \cos\pi(2n-h)] = b_n + 1$$

$$\text{R.H.L} = \lim_{h\to 0}[a_n + \sin\pi(2n+h)] = a_n$$

$$f(2n) = a_n$$

For continuity $b_n + 1 = a_n$

At $x = 2n+1$

$$\text{L.H.L} = \lim_{h\to 0}[a_n + \sin\pi(2n+1-h)] = a_n$$

$$\text{R.H.L} = \lim_{h\to 0}\{b_{n+1} + \cos[\pi(2n+1-h)]\} = b_{n+1} - 1$$

$$f(2n+1) = a_n$$

For continuity

$$a_n = b_{n+1} - 1$$

$$a_{n-1} - b_n = -1$$

Hence, the correct answers are options (B) and (D).

15. $a\in\mathbb{R}$ (the set of all real numbers), $a\neq -1$,

$$\lim_{n\to\infty}\frac{(1^a + 2^a + \cdots + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+\cdots+(na+n)]} = \frac{1}{60}$$

then $a = ?$

(A) 5 **(B)** 7 **(C)** $\frac{-15}{2}$ **(D)** $\frac{-17}{2}$

[JEE ADVANCED 2013]

Solution: We have

$$\lim_{n\to\infty}\frac{\sum_{r=1}^{n}(r)^a}{(n+1)^{a-1}\left[\sum_{r=1}^{n}(na+r)\right]} = \frac{1}{60}$$

That is,

$$\lim_{n\to\infty}\frac{n^a\sum\left(\frac{r}{n}\right)^a}{n(n+1)^{a-1}\sum\left(a+\frac{r}{n}\right)} = \frac{1}{60} \Rightarrow \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{a-1}}\frac{\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{\sum_{r=1}^{n}\left(a+\frac{r}{n}\right)} = \frac{1}{60}$$

$$= \frac{\int_0^1 x^a\,dx}{\int_0^1 (a+x)\,dx} = \frac{1}{60}$$

$$\frac{x^{a+1}\Big|_0^1}{(a+1)\left[\left(ax+\frac{x^2}{2}\right)\Big|_0^1\right]} = \frac{1}{60}$$

$$\frac{2}{(2a+1)(a+1)} = \frac{1}{60}$$

$$2a^2 + 3a + 1 = 120$$

$$2a^2 + 3a - 119 = 0$$

Therefore,

$$a = \frac{-3 \pm \sqrt{9+8(119)}}{4} = \frac{-3 \pm \sqrt{961}}{4} = \frac{-3 \pm 31}{4}$$

Thus, $a = 7, -\frac{17}{2}$.

Hence, the correct answers are options (B) and (D).

16. For every pair of continuous function f, $g : [0,1] \rightarrow \mathbb{R}$ such that max $\{f(x) : x \in [0,1]\}$ = max $\{g(x) : x \in [0,1]\}$, the correct statement(s) is (are):

(A) $[f(c)]^2 + 3f(c) = [g(c)]^2 + 3g(c)$ for some $c \in [0, 1]$

(B) $[f(c)]^2 + f(c) = [g(c)]^2 + 3g(c)$ for some $c \in [0, 1]$

(C) $[f(c)]^2 + 3f(c) = [g(c)]^2 + g(c)$ for some $c \in [0, 1]$

(D) $[f(c)]^2 = [g(c)]^2$ for some $c \in [0, 1]$

[JEE ADVANCED 2014]

Solution: Suppose $f(x)$ is maximum at c_1 and $g(x)$ is maximum at c_2. When $f(x)$ is maximum $g(x)$ may or may not be maximum (Fig. 19.21).

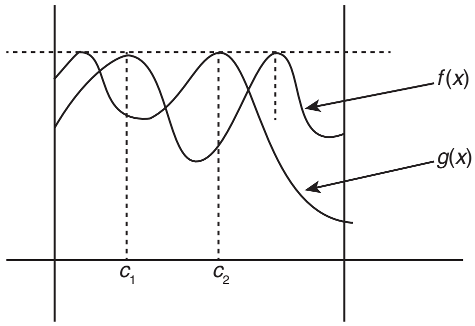


Figure 19.21

Therefore, in the function $h(x) = f(x) - g(x)$, we get

$$h(c_1) = f(c_1) - g(c_1) \geq 0 \text{ and } h(c_2) = f(c_2) - g(c_2) \leq 0$$

Hence, $h(x) = 0$ for some $c \in [0, 1]$. (See Figs. 19.22 and 19.23.)

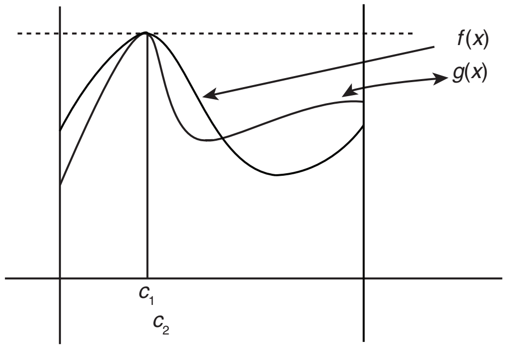


Figure 19.22

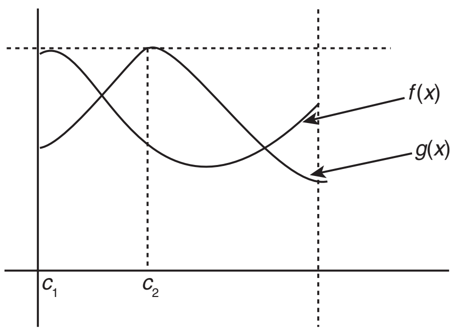


Figure 19.23

Therefore,

$$h(c) = 0$$
$$\Rightarrow f(c) - g(c) = 0$$
$$\Rightarrow f(c) = g(c)$$

Hence, the correct answers are options (A) and (D).

17. Let $f : [a, b] \rightarrow [1, \infty)$ be a continuous function and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a, \\ \int_a^x f(t)dt & \text{if } a \leq x \leq b, \text{ then} \\ \int_a^b f(t)dt & \text{if } x > b \end{cases}$$

(A) $g(x)$ is continuous but not differentiable at a

(B) $g(x)$ is differentiable on $\mathbb{R}$

(C) $g(x)$ is continuous but not differentiable at b

(D) $g(x)$ is continuous and differentiable at either a or b but not both

[JEE ADVANCED 2014]

Solution: Checking continuity of g (Fig. 19.24).

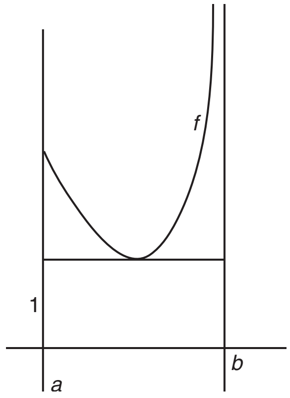


Figure 19.24

$$\lim_{h \to 0} g(a-h) = 0$$

$$\lim_{h \to 0} g(a+h) = \int_a^{a+h} \lim_{h \to 0} f(t)dt = \int_a^a f(t)dt$$

$$g(a) = \int_a^a f(t)dt = 0$$

Hence, g is continuous at a.

Similarly, g is continuous at b. As

$$\underset{h \to 0}{g(b-h)} = g(b) = \underset{h \to 0}{g(b+h)} = \int_a^b f(t)dt$$

Now

$$g'(x) = \begin{cases} 0, & x < a \\ f(x), & a < x < b \\ 0, & x > b \end{cases}$$

Since $f(x) \geq 1$ in $[a, b]$ given, so as we cross a and b according to $g(x)$ function, there are sharp edges encountered due to abrupt change in the slopes from 0 to k and then from k to 0, where $k \geq 1$ (Fig. 19.25).

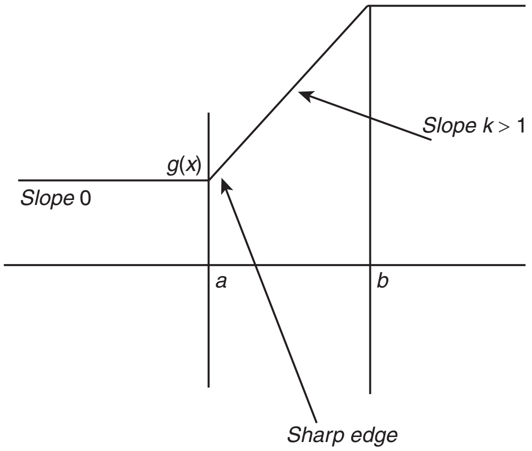


Figure 19.25

$$g'(a+) = f(a+h) \geq 1, \text{ etc.}$$

Hence, the correct answers are options (A) and (C).

18. The largest value of the non-negative integer a for which

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4} \text{ is \_\_\_\_\_.}$$

[JEE ADVANCED 2014]

Solution:

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$

$$\Rightarrow \lim_{x\to 1}\left\{\frac{\sin(x-1)-a(x-1)}{\sin(x-1)+(x-1)}\right\}^{(1+\sqrt{x})} = \frac{1}{4}$$

$$\Rightarrow \lim_{x\to 1}\left\{\frac{\frac{\sin(x-1)}{x-1}-a}{\frac{\sin(x-1)}{x-1}+1}\right\}^{1+\sqrt{x}} = \frac{1}{4}$$

$$\Rightarrow \left\{\frac{1-a}{1+1}\right\}^{1+1} = \frac{1}{4}$$

$$\Rightarrow \left(\frac{1-a}{2}\right)^2 = \frac{1}{4}$$

Hence,

$$\frac{1-a}{2} = \pm\frac{1}{2}$$

$$\Rightarrow 1-a = \pm 1 \Rightarrow a = 0, 2$$

Therefore, largest value = 2.

Hence, the correct answer is (2).

19. Let $f: [0, 2] \to \mathbb{R}$ be a function which is continuous on [0, 2] and is differentiable on (0, 2) with $f(0) = 1$.

Let $F(x) = \int_0^{x^2} f(\sqrt{t})\,dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals

(A) $e^2 - 1$ **(B)** $e^4 - 1$ **(C)** $e - 1$ **(D)** e^4

[JEE ADVANCED 2014]

Solution: Since,

$$F(x) = \int_0^{x^2} f(\sqrt{t})\,dt$$

Hence,

$$f'(x) = f(\sqrt{x^2})\frac{d}{dx}(x^2) - f(\sqrt{0})\frac{d}{dx}0$$

(By Newton–Leibnitz rule)

$$= 2x\,f(\sqrt{x^2}) \qquad (1)$$

Now according to question

$$F'(x) = f'(x) \Rightarrow 2x\,f(\sqrt{x^2}) = f'(x)$$

$$\Rightarrow 2xf(x) = f'(x) \qquad (\text{As } x \in (0,2))$$

Hence,

$$\frac{f'(x)}{f(x)} = 2x \Rightarrow \int\frac{f'(x)}{f(x)}dx = 2\int x\,dx$$

$$\Rightarrow \log_e f(x) = \frac{\cancel{2}x^2}{\cancel{2}} + c$$

$$\Rightarrow f(x) = e^{x^2+c} = e^{x^2}\cdot e^c$$

By initial condition $f(0) = 1$

Hence,

$$1 = e^{o^2}\cdot e^c \Rightarrow e^c = 1$$

$$\Rightarrow f(x) = e^{x^2}$$

Therefore,

$$f(\sqrt{t}) = e^{(\sqrt{t})^2} = e^t$$

Now

$$F(x) = \int_0^{x^2} e^t dt = [e^t]_0^{x^2} = e^{x^2} - 1$$

$$\Rightarrow F(2) = e^{2^2} - 1 = e^4 - 1$$

Hence, the correct answer is option (B).

20. Let $f'(x) = \dfrac{192x^3}{2+\sin^4 \pi x}$ for all $x \in \mathbb{R}$ with $f\left(\dfrac{1}{2}\right) = 0.$ If $m \leq \displaystyle\int_{1/2}^{1} f(x)dx \leq M,$ then the possible values of m and M are

(A) $m = 13, M = 24$ **(B)** $m = \frac{1}{4}, M = \frac{1}{2}$

(C) $m = -11, M = 0$ **(D)** $m = 1, M = 12$

[JEE ADVANCED 2015]

Solution:

$$f'(x) = \frac{192x^3}{2+\sin^4\pi x}\ \forall x \in \mathbb{R},$$

As

$$\frac{1}{2} \leq x \leq 1$$

Therefore,

$$8 \leq f'(x) \leq 96$$

$$\Rightarrow \int_{1/2}^{x} 8 \le \int_{1/2}^{x} f'(x) \le \int_{1/2}^{x} 96$$

$$\Rightarrow 8x-4 \le f(x)-f\left(\frac{1}{2}\right) \le 96x-48$$

$$\Rightarrow \int_{1/2}^{1} 8x-4 \le \int_{1/2}^{1} f(x) \le \int_{1/2}^{1} 96x-48$$

$$\Rightarrow 1 \le \int_{1/2}^{1} f(x) \le 12$$

Hence, the correct answer is option (D).

21. Match the Column I to Column II

Column I	Column II
(A) In $\mathbb{R}^3$, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha = 2+\sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	**(P)** 1
(B) Let a and b be real numbers such that the function $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\ge 1\end{cases}$ is differentiable for all $x\in\mathbb{R}$. Then possible value(s) of a is (are)	**(Q)** 2
(C) Let $\omega \ne 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4x+3}+(2+3\omega-3\omega^2)^{4x+3}+(-3+2\omega+3\omega^2)^{4x+3}=0$. Then possible value(s) of x is (are)	**(R)** 3
(D) Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is an arithmetic progression, then the value(s) of $\lvert q-a\rvert$ is (are)	**(S)** 4
	(T) 5

[JEE ADVANCED 2015]

Solution:

(A) Let $\vec{a}=\alpha\hat{i}+\beta\hat{j}$ and $\vec{b}=\sqrt{3}\hat{i}+\hat{j}$.

Therefore, magnitude of projection of $\vec{a}$ on $\vec{b}=\dfrac{|\vec{a}\cdot\vec{b}|}{|\vec{b}|}$

$$=\frac{|\sqrt{3}\alpha+\beta|}{\sqrt{3+1}}=\sqrt{3} \Rightarrow \sqrt{3}\alpha+\beta=\pm 2\sqrt{3}$$

$$\Rightarrow \sqrt{3}(2+\sqrt{3}\beta)+\beta=\pm 2\sqrt{3}$$

$$\Rightarrow \beta=0 \text{ or } \beta=-\sqrt{3} \Rightarrow \alpha=2 \text{ or } \alpha=-1$$

$$\Rightarrow |\alpha|=2 \text{ or } 1$$

$$\Rightarrow (A) \to (P), (Q)$$

(B) $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x>1\end{cases}$

Therefore, $f(x)$ is differentiable $\forall x\in\mathbb{R}$. So,

$$f(1^-)=f(1^+)$$
$$\Rightarrow -3a-2=b+a^2$$
$$\Rightarrow a^2+3a+2=-b$$
$$\Rightarrow (a+2)(a+1)=-b \quad (1)$$

Also,

$$f'(x)=\begin{cases}-6ax; & x<1\\ b; & x>1\end{cases}$$

$$\Rightarrow f'(1^-)=f'(1^+)$$
$$\Rightarrow -6a=b \quad (2)$$

Hence, from Eqs. (1) and (2),

$$a^2+3a+2=6a$$
$$\Rightarrow a=1 \text{ or } a=2$$
$$\Rightarrow (B) \to (P), (Q)$$

(C) $(3-3\omega+2\omega^2)^{4x+3}+(2+3\omega-3\omega^2)^{4x+3}+(-3+2\omega+3\omega^2)^{4x+3}=0$

$\Rightarrow [1-3\omega+2(1+\omega^2)]^{4x+3}+[2(1+\omega)+\omega-3\omega^2]^{4x+3}+[-3+\omega^2+2(\omega+\omega^2)]^{4x+3}=0$

$\Rightarrow [1-3\omega-2\omega]^{4x+3}+[-5\omega^2+\omega]^{4x+3}+(\omega^2)^{4x+3}(1-5\omega)^{4x+3}=0$

$\Rightarrow (1-5\omega)^{4x+3}(1+\omega^x+\omega^{2x})=0$

$\omega(1-5\omega)^{4x+3}\ne 0 \Rightarrow 1+\omega^x+\omega^{2x}=0$

$\Rightarrow x=3k+1$ or $x=3k+2$; $k\in z$

$\Rightarrow x\in\{1, 2, 4, 5\}$

$\Rightarrow (C) \to (P), (Q), (S), (T)$

(D) HM of 'a' and 'b' $=\dfrac{2ab}{a+b}=4$, where $a, b>0$.

Now, $a, 5, q, b$ are in AP, where $q>0$. Therefore

$$a+b=5+q \quad (3)$$

Also $a+q=10$ and $q=\dfrac{5+b}{2}$ (4)

$$\Rightarrow b=2q-5 \quad (5)$$

Therefore, from Eqs. (3), (4) and (5),

$$a=\frac{5}{2} \text{ or } a=6$$

$$\Rightarrow q=\frac{15}{2} \text{ or } 4 \Rightarrow |q-a|=5 \text{ or } 2$$

$$\Rightarrow (D) \to (Q), (T)$$

Hence, the correct matches are (A) → (P, Q); (B) → (P, Q); (C) → (P, Q, S, T); (D) → (Q, T)

22. Let m and n be two positive integers greater than 1.

If $\lim\limits_{\alpha\to 0}\left(\dfrac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\dfrac{e}{2}\right)$, then the value of $\dfrac{m}{n}$ is ______.

[JEE ADVANCED 2015]

Solution: $m, n\in\mathbb{N}$ and $m, n>1$

$$L=\lim_{\alpha\to 0}\left[\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right]=\lim_{\alpha\to 0}\left[\frac{e(e^{\cos\alpha^n-1}-1)}{\alpha^m}\right]$$

$$= \lim_{\alpha \to 0}\left[\frac{e(e^{\cos\alpha^n - 1} - 1)}{(\cos\alpha^n - 1)} \cdot \frac{(\cos\alpha^n - 1)}{\alpha^m}\right]$$

$$= e.\lim_{\alpha \to 0}\left(\frac{\cos\alpha^n - 1}{\alpha^m}\right)$$

$$= e \cdot \lim_{\alpha \to 0}\left[\frac{-\sin\alpha^n \cdot n\alpha^{n-1}}{m \cdot \alpha^{m-1}}\right]$$

$$= e \cdot \left(\frac{-n}{m}\right) \cdot \lim_{\alpha \to 0}\left[\left(\frac{\sin\alpha^n}{\alpha^n}\right)\frac{\alpha^{2n-1}}{\alpha^{m-1}}\right]$$

$$= \frac{-n}{m} \cdot e \lim_{\alpha \to 0}(\alpha^{2n-m}) = \frac{-e}{2} \text{ (given)}$$

$$\Rightarrow \lim_{\alpha \to 0}\alpha^{2n-m} = \frac{m}{2n}$$

$$\Rightarrow m = 2n$$

$$\Rightarrow \frac{m}{n} = 2$$

Hence, the correct answer is (2).

23. Let f, g: $[-1, 2] \to \mathbb{R}$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table:

e	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is (are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$

(C) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$

(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

[JEE ADVANCED 2015]

Solution: f, g: $[-1, 2] \to \mathbb{R}$,

$f(x)$ is twice differentiable on $(-1, 2)$

$f(-1) = 3, g(-1) = 0,$

$f(0) = 6, g(0) = 1$

$f(2) = 0, g(2) = -1$

$(f - 3g)'' \neq 0$ on $(-1, 0)$ and $(0, 2)$

Number of solutions of $f'(x) - 3g'(x) = 0$ in $(-1, 0) \cup (0, 2) = ?$

Let $h(x) = f(x) - 3g(x)$. Then

$h(-1) = f(-1) - 3g(-1) = 3,$

$h(0) = f(0) - 3g(0) = 6 - 3(1) = 3$

Therefore, by Rolle's theorem, $h'(x)$ that is, $f'(x) - 3g'(x) = 0$ has at least one root in $(-1, 0)$.

Also $h(2) = f(2) - 3g(2) = 0 - 3(-1) = 3$

Hence, again by Rolle's theorem, $f'(x) - 3g'(x) = 0$ has at least one root in $(0, 2)$.

That is, $f'(x) - 3g'(x) = 0$ has at least 2 roots in $(-1, 2)$.

Since $(f - 3g)'' \neq 0$ for $(-1, 0)$ and $(0, 2)$

So, $f(x)$ has no point of inflexion in $(-1, 0)$ and $(0, 2)$. Therefore, $(f' - 3g')(x) \neq 0$ in $(-1, 0)$ and $(0, 2)$, that is, $(f' - 3g')(x) \neq 0$ exactly once in $(-1, 0)$ and exactly once in $(0, 2)$.

Hence, the correct answers are options (B) and (C).

Paragraph for Questions 24 and 25: Let $F: \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in R$.

[JEE ADVANCED 2015]

24. The correct statement(s) is (are)

(A) $f'(1) < 0$

(B) $f(2) < 0$

(C) $f'(x) \neq 0$ for any $x \in (1, 3)$

(D) $f'(x) = 0$ for some $x \in (1, 3)$

Solution: $F: \mathbb{R} \to \mathbb{R}$, thrice differentiable,

$F(1) = 0, F(3) = 4,$

$F'(x) < 0\ \forall \in (1/2, 3),$

$f(x) = xF(x)\ \ \forall x \in \mathbb{R}$

Since

$$F'(x) < 0\ \ \forall x \in \left(\frac{1}{2}, 3\right)$$

So, $F(x)$ is a decreasing function on $\left(\frac{1}{2}, 3\right)$. Therefore,

$$F(1) > F(x) > F(3)\ \forall\ 1 < x < 3$$
$$\Rightarrow 0 > F(x) > -4\ \forall\ x \in (1, 3)$$

Hence, $f'(x) = \underbrace{xF'(x)}_{-ve} + \underbrace{F(x)}_{-ve} < 0\ \ \forall\ x \in (1, 3).$

Also

$$f'(1) = F'(1) + F(1) = F'(1) + 0$$

$$\Rightarrow f'(1) = F'(1) < 0 \text{ as } F'(x) < 0\ \ \forall x \in \left(\frac{1}{2}, 3\right)$$

Further $f(2) = \underbrace{2F(2)}_{-ve} < 0.$

Hence, the correct answers are options (A), (B) and (C).

25. If $\int_1^3 x^2F'(x)dx = -12$ and $\int_1^3 x^3F''(x)dx = 40$, then the correct expression(s) is (are)

(A) $9f'(3) + f'(1) - 32 = 0$

(B) $\int_1^3 f(x)dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$

(D) $\int_1^3 f(x)dx = -12$

Solution:

$$\int_1^3 x^2F'(x)dx = -12 \text{ and } \int_1^3 x^3F''(x)dx = 40;$$

$$9f'(3) + f'(1) \pm 32 = ?,\ \int_1^3 f(x)dx = P$$

Here,

$$9f'(3) \pm f'(1) = 9(3F'(3) + F(3) \pm (F'(1) + F(1))$$
$$= 9[3F'(3) - 4] \pm F'(1) = 27F'(3) \pm F'(1) - 36 \quad (1)$$

Now

$$\int_1^3 x^2F'(x)dx = \left|x^2F(x)\right|_1^3 - \int_1^3 2xF(x)dx$$

$$= [9F(3) - F(1)] - 2\int_1^3 f(x)dx$$

$$\Rightarrow -12 = -36 - 2\int_1^3 f(x)dx$$

$$\Rightarrow \int_1^3 f(x)dx = -12 \quad (2)$$

Also,

$$\int_1^3 x^3F''(x)dx = \left|x^3F'(x)\right|_1^3 - \int_1^3 3x^2F'(x)dx$$

$$\Rightarrow 40 = 27F'(3) - F'(1) - 3(-12)$$

$$\Rightarrow 27F'(3) - F'(1) = 4 \quad (3)$$

Hence, from Eqs. (1) and (3), we get

$$27F'(3) - F'(1) - 36 = 4 - 36 = -32$$

$$\Rightarrow 9f'(3) - f'(1) + 32 = 0$$

Hence, the correct answers are options (C) and (D).

26. Let $f:(0,\infty) \to \mathbb{R}$ be a differentiable function such that

$f'(x) = 2 - \dfrac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$. Then

(A) $\lim_{x\to 0^+} f'\left(\dfrac{1}{x}\right) = 1$ **(B)** $\lim_{x\to 0^+} xf\left(\dfrac{1}{x}\right) = 2$

(C) $\lim_{x\to 0^+} x^2 f'(x) = 0$ **(D)** $|f(x)| \le 2$ for all $x \in (0,2)$

[JEE ADVANCED 2016]

Solution: It is given that

$$f:(0,\infty) \to \mathbb{R}, f'(x) = 2 - \frac{f(x)}{x}$$

Now, the linear differential equation is

$$f'(x) + \frac{f(x)}{x} = 2$$

The integrating factor is

$$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

Therefore,

$$\int d(xf(x)) = \int 2x\,dx + c$$

$$xf(x) = x^2 + c \Rightarrow f(x) = \left(x + \frac{c}{x}\right) \quad \forall x \in (0,\infty)$$

Now,

$$f(1) \neq 1 \Rightarrow 1 \neq 1 + c \Rightarrow c \neq 0$$

and

$$f'(x) = 1 - \frac{c}{x^2} \Rightarrow \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}\left(1 - \frac{c}{x^2}\right) = 1$$

$$\Rightarrow \lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = \lim_{x\to 0^+}(1 - cx^2) = 1$$

Hence, option (A) is correct.

Now,

$$\lim_{x\to 0^+} x f\left(\frac{1}{x}\right) = \lim_{x\to 0^+} x\left(\frac{1}{x} + cx\right) = \lim_{x\to 0^+}(1 + (cx^2)) = 1$$

Hence, option (B) is incorrect.

Now,

$$\lim_{x\to 0^+} x^2 f'(x) = \lim_{x\to 0^+} x^2\left(1 - \frac{c}{x^2}\right) = \lim_{x\to 0^+}(x^2 - c) = -c \neq 0$$

Hence, option (C) is incorrect.

We cannot say anything about $|f(x)| \le 2 \ \forall x \in (0,2)$ because we do not know the value of c.

Hence, option (D) is incorrect.

Hence, the correct answer is option (A).

27. Let $\alpha, \beta \in \mathbb{R}$ be such that $\lim_{x\to 0}\dfrac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$. Then $6(\alpha + \beta)$ equals ________.

[JEE ADVANCED 2016]

Solution: It is given that $\alpha, \beta \in \mathbb{R}$ such that $\lim_{x\to 0}\dfrac{x^2 \sin\beta x}{\alpha x - \sin x} = 1$.

Therefore,

$$\lim_{x\to 0}\frac{x^2\sin\beta x}{\alpha x - \sin x} = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{x^2\left(\dfrac{\sin\beta x}{\beta x}\right)\beta x}{\alpha x - \sin x} = 1$$

$$\Rightarrow \beta \lim_{x\to 0}\frac{x^3}{\alpha x - \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots\right)} = 1$$

$$\Rightarrow \beta \lim_{x\to 0}\frac{x^3}{x(\alpha - 1) + \dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \cdots} = 1$$

For finite limit $\alpha = 1$,

$$3! \times \beta = 1 \Rightarrow \beta = \frac{1}{6}$$

Then,

$$6(\alpha + \beta) = 6\left(1 + \frac{1}{6}\right) = 6 + 1 = 7$$

Hence, the correct answer is (7).

28. Let $a, b \in \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$. Then f is

(A) Differentiable at $x = 0$ if $a = 0$ and $b = 1$.

(B) Differentiable at $x = 1$ if $a = 1$ and $b = 0$.

(C) NOT differentiable at $x = 0$ if $a = 1$ and $b = 0$.

(D) NOT differentiable at $x = 1$ if $a = 1$ and $b = 1$.

[JEE ADVANCED 2016]

Solution: The given function is

$$f(x) = a\cos(|x^3 - x|) + b|x|\sin(|x^3 + x|)$$

which is an even function.

$$f(x) = a\cos(x^3 - x) + bx\sin(x^3 + x)$$

For a function to be differentiable at $x = 0$, the function must be continuous.

$$f(0) = a\cos(0-0) + b(0)\sin(0) = a$$

Therefore,

$$f(0^+) = \lim_{h\to 0}[a\cos(h^3-h) + bh\sin(h^3+h)] = a$$

$$f(0^-) = \lim_{h\to 0}[a(\cos(-h^3+h) + b(-h)\sin(-h^3-h)]$$

$$= \lim_{h\to 0}[a\cos(h^3-h) + bh\sin(h^3+h)] = a$$

which is continuous at $x = 0$; hence, $f(x)$ is differentiable for all values of a and b. Therefore,

$$f(1) = a\cos(1-1) + b\,1\sin(1+1) = a + b\sin 2$$

$$f(1^+) = \lim_{h\to 0} a\cos\left[(1+h)^3-(1+h)\right] + b(1+h)\sin\left[(1+h)^3+(1+h)\right] = f(1)$$

$$f(1^-) = \lim_{h\to 0} a\cos(1-h)^3-(1-h) + b(1-h)\sin(1-h)^3+(1-h) = f(1)$$

Thus, $f(x)$ is continuous and we can also see that f is differentiable at $x = 0$ and $x = 1$.

Hence, the correct answers are options (A) and (B).

29. Let $f(x) = \lim_{n\to\infty}\left(\dfrac{n^n(x+n)\left(x+\frac{n}{2}\right)\cdots\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right)\cdots\left(x^2+\frac{n^2}{n^2}\right)}\right)^{x/n}$, for all $x > 0$. Then

(A) $f\left(\frac{1}{2}\right) \ge f(1)$ **(B)** $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$

(C) $f'(2) \le 0$ **(D)** $\dfrac{f'(3)}{f(3)} \ge \dfrac{f'(2)}{f(2)}$

[JEE ADVANCED 2016]

Solution: The given function is

$$f(x) = \lim_{n\to\infty}\left[\frac{n^n(x+n)(x+n/2)\cdots(x+n/n)}{n!(x^2+n^2)(x^2+n^2/4)\cdots(x^2+n^2/n^2)}\right]^{x/n} \quad \forall x > 0$$

$$\Rightarrow f(x) = \lim_{n\to\infty}\left[\frac{n^n n^n\left(1+\frac{x}{n}\right)\left(1+\frac{2x}{n}\right)\cdots\left(1+\frac{nx}{n}\right)}{n!n!\left(1+\frac{x^2}{n^2}\right)\left(1+\frac{4x^2}{n^2}\right)\cdots\left(1+\frac{n^2x^2}{n^2}\right)}\times\frac{(n!)^2}{(n^2)^n}\right]^{x/n}$$

$$f(x) = \lim_{n\to\infty}\left[\frac{\left(1+\frac{x}{n}\right)\left(1+\frac{2x}{n}\right)\cdots\left(1+\frac{nx}{n}\right)}{\left(1+\frac{x^2}{n^2}\right)\left(1+\frac{4x^2}{n^2}\right)\cdots\left(1+\frac{n^2x^2}{n^2}\right)}\right]^{x/n}$$

$$= \lim_{n\to\infty}\left[\prod_{r=1}^{n}\left(\frac{1+\frac{rx}{n}}{1+\frac{r^2x^2}{n^2}}\right)\right]^{x/n}$$

Therefore,

$$\ln f(x) = x\lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{1}{n}\ln\left(\frac{1+\frac{rx}{n}}{1+\frac{r^2x^2}{n^2}}\right)\right]$$

$$\ln f(x) = x\int_0^1 \ln\left(\frac{1+xy}{1+x^2y^2}\right)dy$$

Substituting $xy = t$, we get $\Rightarrow dy = \dfrac{dt}{x}$. Therefore,

$$\ln f(x) = \frac{x}{x}\int_0^x \ln\left(\frac{1+t}{1+t^2}\right)dt = \int_0^x \ln\left(\frac{1+t}{1+t^2}\right)dt$$

Applying Newton–Leibniz rule, we get

$$\frac{f'(x)}{f(x)} = \ln\left(\frac{1+x}{1+x^2}\right) \Rightarrow f'(x) = f(x)\ln\left(\frac{1+x}{1+x^2}\right)$$

It is obvious that $f(x) > 0\ \forall x > 0$.

$$\text{For } x = 2\text{:}\quad \frac{f'(2)}{f(2)} = \ln\left(\frac{3}{5}\right) < 0 \Rightarrow f'(2) \le 0$$

Hence, option (C) is correct.

That is, $f'(x) \ge 0\ \forall x \in [0,1]$, so $f(x)$ is an increasing function.

That is,

$$f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$$

Hence, option (B) is correct.

Hence, the correct answers are options (B) and (C).

30. Let $f:\left[-\frac{1}{2}, 2\right] \to \mathbb{R}$ and $g:\left[-\frac{1}{2}, 2\right] \to \mathbb{R}$ be function defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|f(x) + |4x-7|f(x)$, $[\cdot]$ denotes the greatest integer less than or equal to y for $y \in \mathbb{R}$. Then

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$.

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$.

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$.

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$.

[JEE ADVANCED 2016]

Solution: It is given that

$$f:\left[\frac{1}{2}, 2\right] \to \mathbb{R}$$

$$g:\left[-\frac{1}{2}, 2\right] \to \mathbb{R}$$

Therefore,

$f(x) = [x^2 - 3]$, $g(x) = [|x| + |4x-7|]f(x)$

$f(x) = [x^2] - 3$

$$f(x)=[x^2]-3=\begin{cases}-3 & x\in\left[-\frac{1}{2},0\right)\\ -3 & x\in[0,1)\\ -2 & x\in[1,\sqrt{2})\\ -1 & x\in[\sqrt{2},\sqrt{3})\\ 0 & x\in[\sqrt{3},2)\\ 1 & x=2\end{cases}=\begin{cases}-3, & x\in\left[-\frac{1}{2},1\right)\\ -2, & x\in[1,\sqrt{2})\\ -1, & x\in[\sqrt{2},\sqrt{3})\\ 0, & x\in[\sqrt{3},2)\\ 1, & x=2\end{cases}$$

$$g(x)=(|x|+|4x-7|)\cdot f(x)$$

Let $h(x)=|x|+|4x-7|$. Then

$$g(x)=h(x)\cdot f(x)$$

where $h(x)$ is continuous for all x and it has sharp edge at 0 and 7/4; $f(x)$ is discontinuous at

$$x=1,\sqrt{2},\sqrt{3},2$$

Hence, option (B) is correct.

Thus, $g(x)$ is non-differentiable at

$$x=0,1,\sqrt{2},\sqrt{3},2$$

Hence, option (D) is correct.

Hence, the correct answers are options (B) and (D).

Practice Exercise 1

1. If $f(x)=\begin{cases}x\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$, then $\lim\limits_{x\to 0}f(x)=$

(A) 1 **(B)** 0 **(C)** –1 **(D)** None of these

2. $\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}=$

(A) –1/10 **(B)** 1/10 **(C)** –1/8 **(D)** None of these

3. If $f(9)=9$, $f'(9)=4$, then $\lim\limits_{x\to 9}\dfrac{\sqrt{f(x)}-3}{\sqrt{x}-3}=$

(A) 2 **(B)** 4 **(C)** –2 **(D)** –4

4. $\lim\limits_{x\to 0}\dfrac{|x|}{x}=$

(A) 1 **(B)** –1 **(C)** 0 **(D)** Does not exist

5. $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}=$

(A) $\dfrac{1}{2\sqrt{x}}$ **(B)** $\dfrac{1}{\sqrt{x}}$ **(C)** $2\sqrt{x}$ **(D)** $\sqrt{x}$

6. $\lim\limits_{x\to 0}\dfrac{2^x-1}{(1+x)^{1/2}-1}=$

(A) log 2 **(B)** log 4 **(C)** $\log\sqrt{2}$ **(D)** None of these

7. $\lim\limits_{x\to 1}\dfrac{x-1}{2x^2-7x+5}=$

(A) 1/3 **(B)** 1/11 **(C)** –1/3 **(D)** None of these

8. $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=$

(A) 1 **(B)** 0 **(C)** Does not exist **(D)** None of these

9. $\lim\limits_{x\to 0}\dfrac{2\sin^2 3x}{x^2}=$

(A) 6 **(B)** 9 **(C)** 18 **(D)** 3

10. $\lim\limits_{\alpha\to\pi/4}\dfrac{\sin\alpha-\cos\alpha}{\alpha-\frac{\pi}{4}}=$

(A) $\sqrt{2}$ **(B)** $1/\sqrt{2}$ **(C)** 1 **(D)** None of these

11. $\lim\limits_{x\to\pi/2}(\sec\theta-\tan\theta)=$

(A) 0 **(B)** 1/2 **(C)** 2 **(D)** ∞

12. $\lim\limits_{x\to 0}\dfrac{\tan 2x-x}{3x-\sin x}=$

(A) 0 **(B)** 1 **(C)** 1/2 **(D)** 1/3

13. $\lim\limits_{x\to\pi/2}\dfrac{2x-\pi}{\cos x}=$

(A) 2 **(B)** 1 **(C)** –2 **(D)** None of these

14. $\lim\limits_{h\to 0}\dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}=$

(A) $a\cos a+a^2\sin a$ **(B)** $a\sin a+a^2\cos a$
(C) $2a\sin a+a^2\cos a$ **(D)** $2a\cos a+a^2\sin a$

15. $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/2}-(1-x)^{1/2}}{x}=$

(A) 0 **(B)** 1/2 **(C)** 1 **(D)** –1

16. $\lim\limits_{x\to a}\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}=$

(A) $\dfrac{1}{\sqrt{3}}$ **(B)** $\dfrac{2}{3\sqrt{3}}$ **(C)** $\dfrac{2}{\sqrt{3}}$ **(D)** $\dfrac{2}{3}$

17. $\lim\limits_{x\to 0}\dfrac{x.2^x-x}{1-\cos x}=$

(A) 0 **(B)** log 4 **(C)** log 2 **(D)** None of these

18. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=$

(A) $\dfrac{1}{2}$ **(B)** $-\dfrac{1}{2}$ **(C)** $\dfrac{2}{3}$ **(D)** None of these

19. $\lim\limits_{x\to 0}x^x=$

(A) 0 **(B)** 1 **(C)** e **(D)** None of these

20. $\lim\limits_{x\to\infty}\dfrac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}}=$

(A) 16 **(B)** 24 **(C)** 32 **(D)** 8

21. $\lim\limits_{x\to 0}\left[\dfrac{x}{\tan^{-1}2x}\right]=$

(A) 0 **(B)** $\dfrac{1}{2}$ **(C)** 1 **(D)** ∞

22. $\lim_{x\to 0}\frac{e^{x^2}-\cos x}{x^2}=$

(A) $\frac{3}{2}$ **(B)** $-\frac{1}{2}$ **(C)** 1 **(D)** None of these

23. $\lim_{x\to 0}\frac{\log(a+x)-\log a}{x}+k\lim_{x\to e}\frac{\log x-1}{x-e}=1,$ then

(A) $k=e\left(1-\frac{1}{a}\right)$ **(B)** $k=e(1+a)$

(C) $k=e(2-a)$ **(D)** The equality is not possible

24. $\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x}=$

(A) 1 **(B)** -1 **(C)** 0 **(D)** None of these

25. $\lim_{x\to 0}\frac{\sin x+\log(1-x)}{x^2}$ is equal to

(A) 0 **(B)** $\frac{1}{2}$ **(C)** $-\frac{1}{2}$ **(D)** None of these

26. $\lim_{x\to\infty}\sqrt{\frac{x+\sin x}{x-\cos x}}=$

(A) 0 **(B)** 1 **(C)** -1 **(D)** None of these

27. $\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{1/x}=$

(A) 1 **(B)** -1 **(C)** e^2 **(D)** e

28. $\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2}=$

(A) e^2 **(B)** e **(C)** e^{-2} **(D)** e^{-1}

29. The value of $\lim_{x\to-\infty}\frac{\sqrt{4x^2+5x+8}}{4x+5}$ is

(A) $-1/2$ **(B)** 0 **(C)** 1/2 **(D)** 1

30. $\lim_{x\to 0}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$ is

(A) 2 **(B)** -2 **(C)** $\frac{1}{2}$ **(D)** $-\frac{1}{2}$

31. For $x\in R$, $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x$ is equal to

(A) e **(B)** e^{-1} **(C)** e^{-5} **(D)** e^5

32. If $\lim_{x\to 0}\frac{[(a-n)nx-\tan x]\sin nx}{x^2}=0,$ where n is a non-zero real number, then a is equal to

(A) 0 **(B)** $\frac{n+1}{n}$ **(C)** n **(D)** $n+\frac{1}{n}$

33. Given that $f'(2)=6$ and $f'(1)=4$, then

$\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

(A) Does not exist **(B)** Is equal to $-3/2$
(C) Is equal to 3/2 **(D)** Is equal to 3

34. $\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$ is

(A) 0 **(B)** 1 **(C)** 2 **(D)** Non-existent

35. $\lim_{x\to\pi/6}\left[\frac{3\sin x-\sqrt{3}\cos x}{6x-\pi}\right]=$

(A) $\sqrt{3}$ **(B)** $1/\sqrt{3}$ **(C)** $-\sqrt{3}$ **(D)** $-1/\sqrt{3}$

36. $\lim_{x\to 0}\frac{\cos(\sin x)-1}{x^2}=$

(A) 1 **(B)** -1 **(C)** 1/2 **(D)** $-1/2$

37. $\lim_{n\to\infty}(3^n+4^n)^{\frac{1}{n}}=$

(A) 3 **(B)** 4 **(C)** ∞ **(D)** e

38. If $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2,$ then the values of a and b are

(A) $a=1, b=2$ **(B)** $a=1, b\in R$
(C) $a\in R, b=2$ **(D)** $a\in R, b\in R$

39. $\lim_{\theta\to\frac{\pi}{2}}\frac{\frac{\pi}{2}-\theta}{\cot\theta}=$

(A) 0 **(B)** -1 **(C)** 1 **(D)** ∞

40. $\lim_{x\to\infty}\left(1-\frac{4}{x-1}\right)^{3x-1}=$

(A) e^{12} **(B)** e^{-12} **(C)** e^4 **(D)** e^3

41. $\lim_{x\to 0}\left[\frac{e^x-e^{\sin x}}{x-\sin x}\right]$ is equal to

(A) -1 **(B)** 0 **(C)** 1 **(D)** None of these

42. The value of $\lim_{x\to-1}\frac{x^2+3x+2}{x^2+4x+3}$ is equal to

(A) 0 **(B)** 1 **(C)** 2 **(D)** 1/2

43. The value of $\lim_{x\to 0}\frac{2}{x}\log(1+x)$ is equal to

(A) e **(B)** e^2 **(C)** $\frac{1}{2}$ **(D)** 2

44. The value of $\lim_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}$ is equal to

(A) $e^{-1/3}$ **(B)** $e^{-2/3}$ **(C)** e^{-1} **(D)** e^{-2}

45. The value of $\lim_{x\to\infty}\frac{(x+1)(3x+4)}{x^2(x-8)}$ is equal to

(A) 2 **(B)** 3 **(C)** 1 **(D)** 0

46. If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, \text{ when } [x]\neq 0\\ 0, \text{ when } [x]=0\end{cases}$ where $[x]$ is the greatest integer function, then $\lim\limits_{x\to 0} f(x)=$

(A) -1 **(B)** 1 **(C)** 0 **(D)** None of these

47. If $\lim\limits_{n\to\infty}\dfrac{1-(10)^n}{1+(10)^{n+1}}=\dfrac{-\alpha}{10}$, then the value of α is

(A) 0 **(B)** -1 **(C)** 1 **(D)** 2

48. The value of $\lim\limits_{x\to 0}\dfrac{\log[1+x^3]}{\sin^3 x}=$

(A) 0 **(B)** 1 **(C)** 3 **(D)** None of these

49. $\lim\limits_{\theta\to 0}\dfrac{4\theta(\tan\theta-\sin\theta)}{(1-\cos 2\theta)^2}$ is

(A) $1/\sqrt{2}$ **(B)** $1/2$ **(C)** 1 **(D)** 2

50. The value of $\lim\limits_{x\to 0}\dfrac{27^x-9^x-3^x+1}{\sqrt{5}-\sqrt{4+\cos x}}$ is

(A) $\sqrt{5}(\log 3)^2$ **(B)** $8\sqrt{5}\log 3$

(C) $16\sqrt{5}\log 3$ **(D)** $8\sqrt{5}(\log 3)^2$

51. The value of $\lim\limits_{n\to\infty}\dfrac{x^n}{x^n+1}$ where $x<-1$ is

(A) 1/2 **(B)** $-1/2$ **(C)** 1 **(D)** None of these

52. The value of $\lim\limits_{n\to\infty}\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dfrac{1}{5.7}+\dfrac{1}{7.9}+\cdots+\dfrac{1}{(2n-1)(2n+1)}$ is equal to

(A) 1/2 **(B)** 1/3 **(C)** 1/4 **(D)** None of these

53. The value of the constant α and β such that $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-\alpha x-\beta\right)=0$ are respectively

(A) (1, 1) **(B)** (–1, 1) **(C)** (1, –1) **(D)** (0, 1)

54. Let $f:R\to R$ be a differentiable function having $f(2)=6, f'(2)=\left(\dfrac{1}{48}\right)$. Then $\lim\limits_{x\to 2}\dfrac{\int_6^{f(x)}4t^3dt}{x-2}$ equals

(A) 12 **(B)** 18 **(C)** 24 **(D)** 36

55. $\lim\limits_{n\to\infty}\left[\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+\cdots+\dfrac{n}{1-n^2}\right]$ is equal to

(A) 0 **(B)** $-\dfrac{1}{2}$ **(C)** $\dfrac{1}{2}$ **(D)** None of these

56. $\lim\limits_{n\to\infty}\left[\dfrac{1}{n^3+1}+\dfrac{4}{n^3+1}+\dfrac{9}{n^3+1}+\cdots+\dfrac{n^2}{n^3+1}\right]=$

(A) 1 **(B)** 2/3 **(C)** 1/3 **(D)** 0

57. If $S_n=\sum\limits_{k=1}^{n}a_k$ and $\lim\limits_{n\to\infty}a_n=a$, then $\lim\limits_{n\to\infty}\dfrac{S_{n+1}-S_n}{\sqrt{\sum\limits_{k=1}^{n}k}}$ is equal to

(A) 0 **(B)** a **(C)** $\sqrt{2}a$ **(D)** $2a$

58. If $a_1=1$ and $a_{n+1}=\dfrac{4+3a_n}{3+2a_n}, n\geq 1$ and if $\lim\limits_{n\to\infty}a_n=a$, then the value of a is

(A) $\sqrt{2}$ **(B)** $-\sqrt{2}$ **(C)** 2 **(D)** None of these

59. The value of $\lim\limits_{n\to\infty}\cos\left(\dfrac{x}{2}\right)\cos\left(\dfrac{x}{4}\right)\cos\left(\dfrac{x}{8}\right)\cdots\cos\left(\dfrac{x}{2^n}\right)$ is

(A) 1 **(B)** $\dfrac{\sin x}{x}$ **(C)** $\dfrac{x}{\sin x}$ **(D)** None of these

60. $\lim\limits_{n\to\infty}\dfrac{1}{2}+\dfrac{1}{2^2}+\dfrac{1}{2^3}+\cdots+\dfrac{1}{2^n}$ equals

(A) 2 **(B)** -1 **(C)** 1 **(D)** 3

61. $\lim\limits_{n\to\infty}\left\{\dfrac{1}{n^2}+\dfrac{2}{n^2}+\dfrac{3}{n^2}+\cdots+\dfrac{n}{n^2}\right\}$ is

(A) 1/2 **(B)** 0 **(C)** 1 **(D)** ∞

62. The value of $\lim\limits_{n\to\infty}\dfrac{1-n^2}{\sum n}$ will be

(A) -2 **(B)** -1 **(C)** 2 **(D)** 1

63. If $x_n=\dfrac{1-2+3-4+5-6+\cdots-2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$, then $\lim\limits_{n\to\infty}x_n$ is equal to

(A) $\dfrac{1}{3}$ **(B)** $-\dfrac{2}{3}$ **(C)** $\dfrac{2}{3}$ **(D)** 1

64. $\lim\limits_{x\to\infty}\dfrac{(x+1)^{10}+(x+2)^{10}+\cdots+(x+100)^{10}}{x^{10}+10^{10}}$ is equal to

(A) 0 **(B)** 1 **(C)** 10 **(D)** 100

65. The value of $\lim\limits_{n\to\infty}\dfrac{1+2+3+\cdots+n}{n^2+100}$ is equal to

(A) ∞ **(B)** $\dfrac{1}{2}$ **(C)** 2 **(D)** 0

66. The value of $\lim\limits_{x\to 0}\dfrac{\int_0^x\cos t^2}{x}dt$ is

(A) 0 **(B)** 1 **(C)** -1 **(D)** None of these

67. If $f(x)=|x-2|$, then

(A) $\lim\limits_{x\to 2+}f(x)\neq 0$ **(B)** $\lim\limits_{x\to 2-}f(x)\neq 0$

(C) $\lim\limits_{x\to 2+}f(x)\neq\lim\limits_{x\to 2-}f(x)$ **(D)** $f(x)$ is continuous at $x=2$

68. If the function $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{when } x\neq\dfrac{\pi}{2}\\ 3, & \text{when } x=\dfrac{\pi}{2}\end{cases}$ be continuous at $x=\dfrac{\pi}{2}$, then $k=$

(A) 3 **(B)** 6 **(C)** 12 **(D)** None of these

69. The function $f(x)=\dfrac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at $x=0$. The value which should be assigned to f at $x=0$ so that it is continuous at $x=0$ is

(A) $a-b$ **(B)** $a+b$ **(C)** $\log a+\log b$ **(D)** $\log a-\log b$

70. Let $f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2}, & \text{if } x\neq 2\\ k, & \text{if } x=2\end{cases}$. If $f(x)$ be continuous for all x, then $k=$

(A) 7 **(B)** -7 **(C)** ± 7 **(D)** None of these

71. Let $f(x)=\begin{cases}x^2+k, & \text{when } x\geq 0\\ -x^2-k, & \text{when } x<0\end{cases}$. If the function $f(x)$ be continuous at $x=0$, then $k=$

(A) 0 **(B)** 1 **(C)** 2 **(D)** -2

72. In order that the function $f(x)=(x+1)^{1/x}$ is continuous at $x=0$, $f(0)$ must be defined as

(A) $f(0)=0$ **(B)** $f(0)=e$
(C) $f(0)=1/e$ **(D)** $f(0)=1$

73. If $f(x)=\begin{cases}x, & \text{when } 0<x<1/2\\ 1, & \text{when } x=1/2\\ 1-x, & \text{when } 1/2<x<1\end{cases}$, then

(A) $\lim\limits_{x\to 1/2+} f(x)=2$

(B) $\lim\limits_{x\to 1/2-} f(x)=2$

(C) $f(x)$ is continuous at $x=\dfrac{1}{2}$

(D) $f(x)$ is discontinuous at $x=\dfrac{1}{2}$

74. If $f(x)=\begin{cases}(x^2/a)-a, & \text{when } x<a\\ 0, & \text{when } x=a\\ a-(x^2/a), & \text{when } x>a\end{cases}$, then

(A) $\lim\limits_{x\to a} f(x)=a$
(B) $f(x)$ is continuous at $x=a$
(C) $f(x)$ is discontinuous at $x=a$
(D) None of these

75. If $f(x)=\begin{cases}e^{1/x}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$, then

(A) $\lim\limits_{x\to 0+} f(x)=e$ **(B)** $\lim\limits_{x\to 0+} f(x)=0$
(C) $f(x)$ is discontinuous at $x=0$ **(D)** None of these

76. If $f(x)=\begin{cases}\dfrac{x^2-4x+3}{x^2-1}, & \text{for } x\neq 1\\ 2, & \text{for } x=1\end{cases}$, then

(A) $\lim\limits_{x\to 1+} f(x)=2$ **(B)** $\lim\limits_{x\to 1-} f(x)=3$
(C) $f(x)$ is discontinuous at $x=1$ **(D)** None of these

77. The points at which the function $f(x)=\dfrac{x+1}{x^2+x-12}$ is discontinuous are

(A) $-3, 4$ **(B)** $3, -4$ **(C)** $-1, -3, 4$ **(D)** $-1, 3, 4$

78. If $f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & \text{when } x\neq 0\\ 2, & \text{when } x=0\end{cases}$, then

(A) $\lim\limits_{x\to 0+} f(x)\neq 2$ **(B)** $\lim\limits_{x\to 0-} f(x)=0$
(C) $f(x)$ is continuous at $x=0$ **(D)** None of these

79. If $f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$, then

(A) $f(0+0)=1$ **(B)** $f(0-0)=1$
(C) f is continuous at $x=0$ **(D)** None of these

80. The value of k so that the function

$f(x)=\begin{cases}k(2x-x^2), & \text{when } x<0\\ \cos x, & \text{when } x\geq 0\end{cases}$ is continuous at $x=0$ is

(A) 1 **(B)** 2 **(C)** 4 **(D)** None of these

81. If $f(x)=\begin{cases}\dfrac{x}{e^{1/x}+1}, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$, then

(A) $\lim\limits_{x\to 0+} f(x)=1$ **(B)** $\lim\limits_{x\to 0-} f(x)=1$
(C) $f(x)$ is continuous at $x=0$ **(D)** None of these

82. If $f(x)=\begin{cases}(1+2x)^{1/x}, & \text{for } x\neq 0\\ e^2, & \text{for } x=0\end{cases}$, then

(A) $\lim\limits_{x\to 0+} f(x)=e$ **(B)** $\lim\limits_{x\to 0-} f(x)=e^2$
(C) $f(x)$ is discontinuous at $x=0$ **(D)** None of these

83. If $f(x)=\begin{cases}2^{1/x}, & \text{for } x\neq 0\\ 3, & \text{for } x=0\end{cases}$, then

(A) $\lim\limits_{x\to 0+} f(x)=0$ **(B)** $\lim\limits_{x\to 0-} f(x)=\infty$
(C) $f(x)$ is continuous at $x=0$ **(D)** None of these

84. If $f(x)=\begin{cases}\frac{1}{x}\sin x^2, x\neq 0\\ 0, x=0\end{cases}$, then

(A) $\lim_{x\to 0+} f(x)\neq 0$ **(B)** $\lim_{x\to 0-} f(x)\neq 0$

(C) $f(x)$ is continuous at $x=0$ **(D)** None of these

85. If $f(x)=\begin{cases}x-1, & x<0\\ \frac{1}{4}, & x=0\\ x^2, & x>0\end{cases}$, then

(A) $\lim_{x\to 0+} f(x)=1$ **(B)** $\lim_{x\to 0-} f(x)=1$

(C) $f(x)$ is discontinuous at $x=0$ **(D)** None of these

86. Which of the following statements is true for graph $f(x)=\log x$?

(A) Graph shows that function is continuous

(B) Graph shows that function is discontinuous

(C) Graph finds for negative and positive values of x

(D) Graph is symmetric along x-axis

87. If function $f(x)=\begin{cases}\frac{x^2-1}{x-1}, & \text{when } x\neq 1\\ k, & \text{when } x=1\end{cases}$ is continuous at $x=1$, then the value of k will be

(A) -1 **(B)** 2 **(C)** -3 **(D)** -2

88. At which points the function $f(x)=\frac{x}{[x]}$, where [·] is greatest integer function, is discontinuous

(A) Only positive integers

(B) All positive and negative integers and (0, 1)

(C) All rational numbers

(D) None of these

89. For the function $f(x)=\begin{cases}\frac{\sin^2 ax}{x^2}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$ which one is a true statement

(A) $f(x)$ is continuous at $x=0$

(B) $f(x)$ is discontinuous at $x=0$, when $a\neq \pm 1$

(C) $f(x)$ is continuous at $x=a$

(D) None of these

90. If $f(x)=\begin{cases}-x^2, & \text{when } x\leq 0\\ 5x-4, & \text{when } 0<x\leq 1\\ 4x^2-3x, & \text{when } 1<x<2\\ 3x+4, & \text{when } x\geq 2\end{cases}$, then

(A) $f(x)$ is continuous at $x=0$

(B) $f(x)$ is continuous at $x=2$

(C) $f(x)$ is discontinuous at $x=1$

(D) None of these

91. If $f(x)=\begin{cases}\sin^{-1}|x|, & \text{when } x\neq 0\\ 0, & \text{when } x=0\end{cases}$, then

(A) $\lim_{x\to 0+} f(x)\neq 0$ **(B)** $\lim_{x\to 0-} f(x)\neq 0$

(C) $f(x)$ is continuous at $x=0$ **(D)** None of these

92. If $f(x)=\begin{cases}\frac{\sin 2x}{5x}, & \text{when } x\neq 0\\ k, & \text{when } x=0\end{cases}$ is continuous at $x=0$, then the value of k will be

(A) 1 **(B)** $\frac{2}{5}$ **(C)** $-\frac{2}{5}$ **(D)** None of these

93. If $f(x)=\begin{cases}1+x^2, & \text{when } 0\leq x\leq 1\\ 1-x, & \text{when } x>1\end{cases}$, then

(A) $\lim_{x\to 1^+} f(x)\neq 0$ **(B)** $\lim_{x\to 1^-} f(x)\neq 2$

(C) $f(x)$ is discontinuous at $x=1$ **(D)** None of these

94. If $f(x)=\begin{cases}\frac{x^2-1}{x+1}, & \text{when } x\neq -1\\ -2, & \text{when } x=-1\end{cases}$, then

(A) $\lim_{x\to(-1)^-} f(x)=-2$ **(B)** $\lim_{x\to(-1)^+} f(x)=-2$

(C) $f(x)$ is continuous at $x=-1$ **(D)** All the above are correct

95. If $f(x)=\begin{cases}\frac{5}{2}-x, & \text{when } x<2\\ 1, & \text{when } x=2\\ x-\frac{3}{2}, & \text{when } x>2\end{cases}$, then

(A) $f(x)$ is continuous at $x=2$

(B) $f(x)$ is discontinuous at $x=2$

(C) $\lim_{x\to 2} f(x)=1$

(D) None of these

96. If $f(x)=|x-b|$, then function

(A) Is continuous at $x=1$

(B) Is continuous at $x=b$

(C) Is discontinuous at $x=b$

(D) None of these

97. If $f(x)=\begin{cases}\frac{|x-a|}{x-a}, & \text{when } x\neq a\\ 1, & \text{when } x=a\end{cases}$, then

(A) $f(x)$ is continuous at $x=a$

(B) $f(x)$ is discontinuous at $x=a$

(C) $\lim_{x\to a} f(x)=1$

(D) None of these

98. If $f(x)=\begin{cases}x^2, & \text{when } x\neq 1\\ 2, & \text{when } x=1\end{cases}$, then

(A) $\lim_{x\to 1} f(x)=2$

(B) $f(x)$ is continuous at $x=1$

(C) $f(x)$ is discontinuous at $x = 1$
(D) None of these

99. If $f(x) = \begin{cases} 1+x, \text{ when } x \le 2 \\ 5-x, \text{when } x > 2 \end{cases}$, then

(A) $f(x)$ is continuous at $x = 2$
(B) $f(x)$ is discontinuous at $x = 2$
(C) $f(x)$ is continuous at $x = 3$
(D) None of these

100. If $f(x) = \begin{cases} 1, \text{ when } 0 < x \le \dfrac{3\pi}{4} \\ 2\sin\dfrac{2}{9}x, \text{when} \dfrac{3\pi}{4} < x < \pi \end{cases}$, then

(A) $f(x)$ is continuous at $x = 0$
(B) $f(x)$ is continuous at $x = \pi$
(C) $f(x)$ is continuous at $x = \dfrac{3\pi}{4}$
(D) $f(x)$ is discontinuous at $x = \dfrac{3\pi}{4}$

101. If $f(x) = \begin{cases} x\sin x, \text{when } 0 < x \le \dfrac{\pi}{2} \\ \dfrac{\pi}{2}\sin(\pi + x), \text{when} \dfrac{\pi}{2} < x < \pi \end{cases}$, then

(A) $f(x)$ is discontinuous at $x = \pi/2$
(B) $f(x)$ is continuous at $x = \pi/2$
(C) $f(x)$ is continuous at $x = 0$
(D) None of these

102. If $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{when } x < 0 \\ a, & \text{when } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{(16+\sqrt{x})}-4}, & \text{when } x > 0 \end{cases}$,

is continuous at $x = 0$, then the value of 'a' will be

(A) 8 **(B)** -8 **(C)** 4 **(D)** None of these

103. If $f(x) = \begin{cases} ax^2 - b, \text{ when } 0 \le x < 1 \\ 2, \text{ when } x = 1 \\ x+1, \text{ when} 1 < x \le 2 \end{cases}$ is continuous at $x = 1$, then

the most suitable value of a, b are

(A) $a = 2, b = 0$ **(B)** $a = 1, b = -1$
(C) $a = 4, b = 2$ **(D)** All the above

104. If $f(x) = \begin{cases} \dfrac{x - |x|}{x}, \text{when } x \ne 0 \\ 2, \text{when } x = 0 \end{cases}$, then

(A) $f(x)$ is continuous at $x = 0$
(B) $f(x)$ is discontinuous at $x = 0$
(C) $\lim_{x \to 0} f(x) = 2$
(D) None of these

105. If $f(x) = \begin{cases} \dfrac{x^4 - 16}{x-2}, \text{ when } x \ne 2 \\ 16, \text{when } x = 2 \end{cases}$, then

(A) $f(x)$ is continuous at $x = 2$
(B) $f(x)$ is discountinous at $x = 2$
(C) $\lim_{x \to 2} f(x) = 16$
(D) None of these

106. If $f(x) = \begin{cases} x^2, \text{when } x \le 1 \\ x+5, \text{when } x > 1 \end{cases}$, then

(A) $f(x)$ is continuous at $x = 1$
(B) $f(x)$ is discontinuous at $x = 1$
(C) $\lim_{x \to 1} f(x) = 1$
(D) None of these

107. If $f(x) = \begin{cases} \dfrac{x^2 + 3x - 10}{x^2 + 2x - 15}, & \text{when } x \ne -5 \\ a, & \text{when } x = -5 \end{cases}$ is continuous at $x = -5$,

then the value of 'a' will be

(A) $\dfrac{3}{2}$ **(B)** $\dfrac{7}{8}$ **(C)** $\dfrac{8}{7}$ **(D)** $\dfrac{2}{3}$

108. If $f(x) = \begin{cases} x + \lambda, x < 3 \\ 4, x = 3 \\ 3x - 5, x > 3 \end{cases}$ is continuous at $x = 3$, then $\lambda =$

(A) 4 **(B)** 3 **(C)** 2 **(D)** 1

109. The value of k which makes $f(x) = \begin{cases} \sin\dfrac{1}{x}, x \ne 0 \\ k, x = 0 \end{cases}$ continuous at $x = 0$ is

(A) 8 **(B)** 1 **(C)** -1 **(D)** None of these

110. If $f(x) = \begin{cases} \sin x, x \ne n\pi, n \in Z \\ 2, \text{otherwise} \end{cases}$ and $g(x) = \begin{cases} x^2 + 1, x \ne 0, 2 \\ 4, x = 0 \\ 5, x = 2 \end{cases}$,

then $\lim_{x \to 0} g[f(x)]$ is

(A) 5 **(B)** 6 **(C)** 7 **(D)** 1

111. Let $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, x < 4 \\ a + b, x = 4 \\ \dfrac{x-4}{|x-4|} + b, x > 4 \end{cases}$. Then $f(x)$ is continuous at $x = 4$ when

(A) $a = 0, b = 0$ **(B)** $a = 1, b = 1$
(C) $a = -1, b = 1$ **(D)** $a = 1, b = -1$

112. Let $f(x)=\begin{cases}\dfrac{x^4-5x^2+4}{|(x-1)(x-2)|}, & x\neq 1,2\\ 6, & x=1\\ 12, & x=2\end{cases}$

Then $f(x)$ is continuous on the set

(A) R **(B)** $R-\{1\}$ **(C)** $R-\{2\}$ **(D)** $R-\{1, 2\}$

113. Function $f(x)=\begin{cases}x-1, x<2\\ 2x-3, x\geq 2\end{cases}$ is a continuous function

(A) For all real values of x
(B) For $x=2$ only
(C) For all real values of x such that $x\neq 2$
(D) For all integral values of x only

114. If the function $f(x)=\begin{cases}1+\sin\dfrac{\pi x}{2}, & \text{for} -\infty<x\leq 1\\ ax+b, & \text{for } 1<x<3\\ 6\tan\dfrac{x\pi}{12}, & \text{for } 3\leq x<6\end{cases}$ is continuous in the interval $(-\infty, 6)$, then the values of a and b are respectively

(A) 0, 2 **(B)** 1, 1 **(C)** 2, 0 **(D)** 2, 1

115. If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]+1}, & \text{for } x>0\\ \dfrac{\cos\frac{\pi}{2}[x]}{[x]}, & \text{for } x<0\\ k, & \text{at } x=0\end{cases}$; where $[x]$ denotes the greatest integer less than or equal to x, then in order that f be continuous at $x=0$, the value of k is

(A) Equal to 0 **(B)** Equal to 1
(C) Equal to –1 **(D)** Indeterminate

116. The function $f(x)=\begin{cases}x+2, & 1\leq x\leq 2\\ 4, & x=2\\ 3x-2, & x>2\end{cases}$ is continuous at

(A) $x=2$ only **(B)** $x\leq 2$
(C) $x\geq 2$ **(D)** None of these

117. If the function $f(x)=\begin{cases}5x-4, & \text{if } 0<x\leq 1\\ 4x^2+3bx, & \text{if } 1<x<2\end{cases}$ is continuous at every point of its domain, then the value of b is

(A) –1 **(B)** 0 **(C)** 1 **(D)** None of these

118. The values of A and B such that the function

$$f(x)=\begin{cases}-2\sin x, & x\leq -\dfrac{\pi}{2}\\ A\sin x+B, & -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}\\ \cos x, & x\geq \dfrac{\pi}{2}\end{cases}$$, is continuous every where

are

(A) $A=0, B=1$ **(B)** $A=1, B=1$
(C) $A=-1, B=1$ **(D)** $A=-1, B=0$

119. If $f(x)=\dfrac{x^2-10x+25}{x^2-7x+10}$ for $x\neq 5$ and f is continuous at $x=5$, then $f(5)=$

(A) 0 **(B)** 5 **(C)** 10 **(D)** 25

120. In order that the function $f(x)=(x+1)^{\cot x}$ is continuous at $x=0$, $f(0)$ must be defined as

(A) $f(0)=\dfrac{1}{e}$ **(B)** $f(0)=0$
(C) $f(0)=e$ **(D)** None of these

121. The function $f(x)=\sin|x|$ is

(A) Continuous for all x
(B) Continuous only at certain points
(C) Differentiable at all points
(D) None of these

122. If $f(x)=|x|$, then $f(x)$ is

(A) Continuous for all x
(B) Differentiable at $x=0$
(C) Neither continuous nor differentiable at $x=0$
(D) None of these

123. If $f(x)=\begin{cases}\dfrac{1-\sin x}{\pi-2x}, & x\neq\dfrac{\pi}{2}\\ \lambda, & x=\dfrac{\pi}{2}\end{cases}$, be continuous at $x=\pi/2$, then value of λ is

(A) –1 **(B)** 1 **(C)** 0 **(D)** 2

124. Let $f(x)=\begin{cases}\dfrac{\sin\pi x}{5x}, & x\neq 0\\ k, & x=0\end{cases}$. If $f(x)$ is continuous at $x=0$, then $k=$

(A) $\dfrac{\pi}{5}$ **(B)** $\dfrac{5}{\pi}$ **(C)** 1 **(D)** 0

125. If $f(x)=\dfrac{2-\sqrt{x+4}}{\sin 2x}, (x\neq 0)$, is continuous function at $x=0$, then $f(0)$ equals

(A) $\dfrac{1}{4}$ **(B)** $-\dfrac{1}{4}$ **(C)** $\dfrac{1}{8}$ **(D)** $-\dfrac{1}{8}$

126. If function $f(x)=\begin{cases}x, & \text{if } x \text{ is rational}\\ 1-x, & \text{if } x \text{ is irrational}\end{cases}$, then $f(x)$ is continuous at ______ number of points.

(A) ∞ **(B)** 1 **(C)** 0 **(D)** None of these

127. If $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & \text{if } x\neq 3\\ 2x+k, & \text{otherwise}\end{cases}$, is continuous at $x=3$, then k equals to

(A) 3 **(B)** 0 **(C)** –6 **(D)** 1/6

128. The function defined by $f(x)=\begin{cases}\left(x^2+e^{\frac{1}{2-x}}\right)^{-1}, & x\neq 2\\ k, & x=2\end{cases}$, is continuous from right at the point $x=2$, then k is equal to

(A) 0 **(B)** 1/4 **(C)** –1/4 **(D)** None of these

129. For the function $f(x)=\dfrac{\log_e(1+x)-\log_e(1-x)}{x}$ to be continuous at $x=0$ the value of $f(0)$ should be

(A) –1 **(B)** 0 **(C)** –2 **(D)** 2

130. If $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{for } -1\le x<0\\ 2x^2+3x-2, & \text{for } 0\le x\le 1\end{cases}$, is continuous at $x=0$, then $k=$

(A) -4 **(B)** -3 **(C)** -2 **(D)** -1

131. The function $f(x)=\dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$ is not defined at $x=\pi$. The value of $f(\pi)$, so that $f(x)$ is continuous at $x=\pi$, is

(A) $-\dfrac{1}{2}$ **(B)** $\dfrac{1}{2}$ **(C)** -1 **(D)** 1

132. If $f(x)=\begin{cases}\dfrac{1-\cos x}{x}, x\neq 0\\ k,\ x=0\end{cases}$ is continuous at $x=0$, then $k=$

(A) 0 **(B)** $\dfrac{1}{2}$ **(C)** $\dfrac{1}{4}$ **(D)** $-\dfrac{1}{2}$

133. A function f on R into itself is continuous at a point a in R, iff for each $\in>0$, there exists, $\delta>0$ such that

(A) $|f(x)-f(a)|<\in\ \Rightarrow |x-a|<\delta$
(B) $|f(x)-f(a)|>\in\ \Rightarrow |x-a|>\delta$
(C) $|x-a|>\delta\Rightarrow |f(x)-f(a)|>\in$
(D) $|x-a|<\delta\Rightarrow |f(x)-f(a)|<\in$

134. For the function $f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1}, & x\neq 0\\ 0, & x=0\end{cases}$, which of the following is correct

(A) $\lim\limits_{x\to 0} f(x)$ does not exist
(B) $f(x)$ is continuous at $x=0$
(C) $\lim\limits_{x\to 0} f(x)=1$
(D) $\lim\limits_{x\to 0} f(x)$ exists but $f(x)$ is not continuous at $x=0$

135. The function 'f' is defined by $f(x)=2x-1$, if $x>2$, $f(x)=k$ if $x=2$ and x^2-1, if $x<2$ is continuous, then the value of k is equal to

(A) 2 **(B)** 3 **(C)** 4 **(D)** –3

136. If the function, $f(x)=\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x},(x\neq 0)$ is continuous at each point of its domain, then the value of $f(0)$ is

(A) 2 **(B)** 1/3 **(C)** 2/3 **(D)** –1/3

137. The function $f(x)=|x|+\dfrac{|x|}{x}$ is

(A) Continuous at the origin
(B) Discontinuous at the origin because $|x|$ is discontinuous there
(C) Discontinuous at the origin because $\dfrac{|x|}{x}$ is discontinuous there
(D) Discontinuous at the origin because both $|x|$ and $\dfrac{|x|}{x}$ are discontinuous there

138. The value of f at $x=0$ so that the function

$f(x)=\dfrac{2^x-2^{-x}}{x}, x\neq 0$, is continuous at $x=0$ is

(A) 0 **(B)** log 2 **(C)** 4 **(D)** log 4

139. The function $f(x)=\dfrac{2x^2+7}{x^3+3x^2-x-3}$ is discontinuous for

(A) $x=1$ only
(B) $x=1$ and $x=-1$ only
(C) $x=1, x=-1, x=-3$ only
(D) $x=1, x=-1, x=-3$ and other values of x

140. If $f(x)=\begin{cases}\dfrac{1-|x|}{1+x}, & x\neq -1\\ 1, & x=-1\end{cases}$, then the value of $f([2x])$ will be (where [ ] shows the greatest integer function)

(A) Continuous at $x=-1$ **(B)** Continuous at $x=0$
(C) Discontinuous at $x=\dfrac{1}{2}$ **(D)** All of these

141. If the function $f(x)=\dfrac{1-\cos 4x}{8x^2}$, where $x\neq 0$ and $f(x)=k$ where $x=0$ is a continuous function at $x=0$, then the value of k will be

(A) $k=0$ **(B)** $k=1$ **(C)** $k=-1$ **(D)** None of these

142. If $f(x)=\begin{cases}e^x; & x\le 0\\ |1-x|; & x>0\end{cases}$, then

(A) $f(x)$ is differentiable at $x=0$
(B) $f(x)$ is continuous at $x=0$
(C) $f(x)$ is differentiable at $x=1$
(D) $f(x)$ is continuous at $x=1$

143. Which of the following statements is true:

(A) A continuous function is an increasing function.
(B) An increasing function is continuous.
(C) A continuous function is differentiable.
(D) A differentiable function is continuous.

144. If $f(x)=\begin{cases} x+1, \text{when } x<2 \\ 2x-1, \text{when } x\ge 2\end{cases}$, then $f'(2)$ equals

(A) 0 **(B)** 1 **(C)** 2 **(D)** Does not exist

145. If $f(x)=\begin{cases} x\dfrac{e^{(1/x)}-e^{(-1/x)}}{e^{(1/x)}+e^{(-1/x)}}, x\ne 0 \\ 0, \quad x=0\end{cases}$ then which of the following is true

(A) f is continuous and differentiable at every point
(B) f is continuous at every point but is not differentiable
(C) f is differentiable at every point
(D) f is differentiable only at the origin

146. If $f(x)=|x-3|$, then f is

(A) Discontinuous at $x=2$
(B) Not differentiable $x=2$
(C) Differentiable at $x=3$
(D) Continuous but not differentiable at $x=3$

147. Let $h(x)=\min\{x, x^2\}$, for every real number of x. Then

(A) h is continuous for all x
(B) h is differentiable for all x
(C) $h'(x)=1$, for all $x>1$
(D) h is not differentiable at two values of x

148. There exists a function $f(x)$ satisfying $f(0)=1$, $f'(0)=-1$, $f(x)>0$ for all x and

(A) $f(x)<0, \forall x$ **(B)** $-1<f''(x)<0, \forall x$
(C) $-2<f''(x)\le -1, \forall x$ **(D)** $f''(x)<-2, \forall x$

149. The function $f(x)=\begin{cases} x, \text{if } 0\le x\le 1 \\ 1, \text{if } 1<x\le 2\end{cases}$ is

(A) Continuous at all x, $0\le x\le 2$ and differentiable at all x, except $x=1$ in the interval [0, 2]
(B) Continuous and differentiable at all x in [0, 2]
(C) Not continuous at any point in [0, 2]
(D) Not differentiable at any point [0, 2]

150. The function $f(x)=|x|$ at $x=0$ is

(A) Continuous but non-differentiable
(B) Discontinuous and differentiable
(C) Discontinuous and non-differentiable
(D) Continuous and differentiable

151. Consider $f(x)=\begin{cases} \dfrac{x^2}{|x|}, x\ne 0 \\ 0, x=0\end{cases}$

(A) $f(x)$ is discontinuous everywhere
(B) $f(x)$ is continuous everywhere
(C) $f'(x)$ exists in $(-1, 1)$
(D) $f'(x)$ exists in $(-2, 2)$

152. At the point $x=1$, the given function

$$f(x)=\begin{cases} x^3-1; 1<x<\infty \\ x-1; -\infty<x\le 1\end{cases} \text{ is}$$

(A) Continuous and differentiable
(B) Continuous and not differentiable
(C) Discontinuous and differentiable
(D) Discontinuous and not differentiable

153. Let $[x]$ denotes the greatest integer less than or equal to x. If $f(x)=[x \sin \pi x]$, then $f(x)$ is

(A) Continuous at $x=0$ **(B)** Continuous in $(-1, 0)$
(C) Differentiable in $(-1,1)$ **(D)** All the above

154. The function defined by $f(x)=\begin{cases} |x-3|; & x\ge 1 \\ \dfrac{1}{4}x^2-\dfrac{3}{2}x+\dfrac{13}{4}; & x<1\end{cases}$ is

(A) Continuous at $x=1$ **(B)** Continuous at $x=3$
(C) Differentiable at $x=1$ **(D)** All the above

155. If $f(x)=\begin{cases} e^x+ax, \; x<0 \\ b(x-1)^2, \; x\ge 0\end{cases}$ is differentiable at $x=0$, then (a, b) is

(A) $(-3, -1)$ **(B)** $(-3, 1)$ **(C)** $(3, 1)$ **(D)** $(3, -1)$

156. The function $y=|\sin x|$ is continuous for any x but it is not differentiable at

(A) $x=0$ only
(B) $x=\pi$ only
(C) $x=k\pi$ (k is an integer) only
(D) $x=0$ and $x=k\pi$ (k is an integer)

157. The function $y=e^{-|x|}$ is

(A) Continuous and differentiable at $x=0$
(B) Neither continuous nor differentiable at $x=0$
(C) Continuous but not differentiable at $x=0$
(D) Not continuous but differentiable at $x=0$

158. A function $f(x)=\begin{cases} 1+x, & x\le 2 \\ 5-x, & x>2\end{cases}$ is

(A) Not continuous at $x=2$
(B) Differentiable at $x=2$
(C) Continuous but not differentiable at $x=2$
(D) None of these

159. The left-hand derivative of $f(x)=[x]\sin(\pi x)$ at $x=k$, k is an integer and $[x]=$ greatest integer $\le$ x, is

(A) $(-1)^k (k-1)\pi$ **(B)** $(-1)^{k-1}(k-1)\pi$
(C) $(-1)^k k\pi$ **(D)** $(-1)^{k-1} k\pi$

160. Let $f(x)=\begin{cases} 0, & x<0 \\ x^2, & x\ge 0\end{cases}$, then for all values of x

(A) f is continuous but not differentiable
(B) f is differentiable but not continuous
(C) f' is continuous but not differentiable
(D) f' is continuous and differentiable

161. Which of the following is not true:
(A) A polynomial function is always continuous
(B) A continuous function is always differentiable
(C) A differentiable function is always continuous
(D) e^x is continuous for all x

162. The function $f(x)=x^2\sin\dfrac{1}{x}, x\ne 0, f(0)=0$ at $x=0$

(A) Is continuous but not differentiable
(B) Is discontinuous
(C) Is having continuous derivative
(D) Is continuous and differentiable

163. If $f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5} & \text{for } x\neq 1\\ -\dfrac{1}{3} & \text{for } x=1\end{cases}$, then $f'(1)=$

(A) $-1/9$ **(B)** $-2/9$ **(C)** $-1/3$ **(D)** $1/3$

164. If $f(x)=\dfrac{x}{1+|x|}$ for $x\in R$, then $f'(0)=$.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

165. The value of m for which the function $f(x)=\begin{cases}mx^2, & x\le 1\\ 2x, & x>1\end{cases}$ is differentiable at $x=1$ is

(A) 0 **(B)** 1 **(C)** 2 **(D)** Does not exist

166. Let $f(x)=\begin{cases}\sin x, & \text{for } x>0\\ 1-\cos x, & \text{for } x\le 0\end{cases}$ and $g(x)=e^x$. Then $(gof)'(0)$ is

(A) 1 **(B)** -1 **(C)** 0 **(D)** None of these

167. Suppose $f(x)$ is differentiable at $x=1$ and $\lim\limits_{h\to 0}\dfrac{1}{h}f(1+h)=5$, then $f'(1)$ equals

(A) 5 **(B)** 6 **(C)** 3 **(D)** 4

168. If f is a real-valued differentiable function satisfying $|f(x)-f(y)|\le(x-y)^2$, $x, y\in R$ and $f(0)=0$, then $f(1)$ is equal to

(A) 2 **(B)** 1 **(C)** -1 **(D)** 0

169. Let f be differentiable for all x. If $f(1)=-2$ and $f'(x)\ge 2$ for $x\in[1, 6]$, then

(A) $f(6)<5$ **(B)** $f(6)=5$ **(C)** $f(6)\ge 8$ **(D)** $f(6)<8$

170. $f(x)=||x|-1|$ is not differentiable at

(A) 0 **(B)** $\pm 1, 0$ **(C)** 1 **(D)** ± 1

171. If $f(x)$ is twice differentiable polynomial function such that $f(1)=1$, $f(2)=-4$, $f(3)=9$, then

(A) $f''(x)=2$, $\forall x\in R$
(B) There exist at least one $x\in(1, 3)$ such that $f''(x)=2$
(C) There exist at least one $x\in(2, 3)$ such that $f'(x)=5=f''(x)$
(D) There exist at least one $x\in(1, 2)$ such that $f(x)=3$

172. If $f(x)$ is a differentiable function such that $f: R\to R$ and $f\left(\dfrac{1}{n}\right)=0\ \forall\, n\ge 1, n\in I$, then

(A) $f(x)=0\ \forall\, x\in(0, 1)$
(B) $f(0)=0=f'(0)$
(C) $f(0)=0$ but $f'(0)$ may or may not be 0
(D) $|f(x)|\le 1\ \forall\, x\in(0, 1)$

173. Let f be continuous on $[1, 5]$ and differentiable in $(1, 5)$. If $f(1)=-3$ and $f'(x)\ge 9$ for all $x\in(1, 5)$, then

(A) $f(5)\ge 33$ **(B)** $f(5)\ge 36$
(C) $f(5)\le 36$ **(D)** $f(5)\ge 9$

174. Let $f(x+y)=f(x)\,f(y)$ and $f(x)=1+\sin(3x)g(x)$ where $g(x)$ is continuous. Then $f'(x)$ is

(A) $f(x)g(0)$ **(B)** $3g(0)$
(C) $f(x)\cos 3x$ **(D)** $3f(x)g(0)$

175. Let $f(x)=\begin{cases}1 & \forall x<0\\ 1+\sin x & \forall 0\le x\le \pi/2\end{cases}$. Then what is the value of $f'(x)$ at $x=0$?

(A) 1 **(B)** -1 **(C)** ∞ **(D)** Does not exist

176. If $f(x)=x^2-2x+4$ and $\dfrac{f(5)-f(1)}{5-1}=f'(c)$. Then value of c will be

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

177. Let $f(x+y)=f(x)+f(y)$ and $f(x)=x^2g(x)$ for all $x, y\in R$, where $g(x)$ is a continuous function. Then $f'(x)$ is equal to

(A) $g'(x)$ **(B)** $g(0)$ **(C)** $g(0)+g'(x)$ **(D)** 0

178. The function $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$ is not differentiable at

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2

179. The function which is continuous for all real values of x and differentiable at $x=0$ is

(A) $|x|$ **(B)** $\log x$ **(C)** $\sin x$ **(D)** $x^{\frac{1}{2}}$

180. Which of the following is not true:

(A) Every differentiable function is continuous.
(B) If derivative of a function is zero at all points, then the function is constant.
(C) If a function has maximum or minima at a point, then the function is differentiable at that point and its derivative is zero.
(D) If a function is constant, then its derivative is zero at all points.

181. If $f(x)=\begin{cases}x+2, & -1<x<3\\ 5, & x=3\\ 8-x, & x>3\end{cases}$, then at $x=3$, $f'(x)=$

(A) 1 **(B)** -1 **(C)** 0 **(D)** Does not exist

182. If $f(x)=\begin{cases}x, & 0\le x\le 1\\ 2x-1, & 1<x\end{cases}$, then

(A) f is discontinuous at $x=1$
(B) f is differentiable at $x=1$
(C) f is continuous but not differentiable at $x=1$
(D) None of these

183. If $f(x)=\begin{cases}ax^2+b; & x\le 0\\ x^2; & x>0\end{cases}$ possesses derivative at $x=0$, then

(A) $a=0, b=0$ **(B)** $a>0, =0$
(C) $a\in R, =0$ **(D)** None of these

184. The set of all those points, where the function $f(x)=\dfrac{x}{1+|x|}$ is differentiable, is

(A) $(-\infty, \infty)$ **(B)** $[0, \infty]$
(C) $(-\infty, 0)\cup(0, \infty)$ **(D)** $(0, \infty)$

185. Function $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is not differentiable for

(A) $|x|<1$ **(B)** $x=1,-1$
(C) $|x|>1$ **(D)** None of these

186. If $f(x)=x(\sqrt{x}-\sqrt{x+1})$, then

(A) $f(x)$ is continuous but non-differentiable at $x=0$
(B) $f(x)$ is differentiable at $x=0$
(C) $f(x)$ is not differentiable at $x=0$
(D) None of these

187. The number of points at which the function $f(x)=|x-0.5|+|x-1|+\tan x$ does not have a derivative in the interval (0, 2), is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

188. If $\lim\limits_{x\to5}\frac{x^\lambda-5^\lambda}{x-5}=500$, then positive values of λ is

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6

189. If $f(x)=\begin{cases}x, & x<0\\ 1, & x=0\\ x^2, & x>0\end{cases}$, then $\lim\limits_{x\to0}f(x)$ is equal to

(A) 0 **(B)** 1 **(C)** 2 **(D)** Does not exist

190. Let $f(x)=[\alpha+\beta\sin x]$, $x\in(0,\pi)$, $\alpha\in I$, β is a prime number and [.] denotes G.I.F. The number of points at which $f(x)$ is not differentiable is

(A) β **(B)** $2\beta+1$ **(C)** $2\beta-1$ **(D)** $\beta+1$

191. $\lim\limits_{x\to n}\left[x+[x]\right]$, $n\in I$, is equal to ([·] denotes greatest integer function)

(A) 0 **(B)** 1 **(C)** -1 **(D)** Does not exist

192. $\lim\limits_{x\to0}\left(\sin\frac{x}{m}+\cos\frac{3x}{m}\right)^{2m/x}$ is

(A) 1 **(B)** e^2 **(C)** e^{6m} **(D)** $\log 6m$

193. The function max $\{1-x, 1+x, 2\}$ is

(A) Continuous at all points
(B) Differentiable at all points
(C) Continuous at all points except at -1, 1
(D) None of these

194. The value of $\lim\limits_{x\to0}\frac{1}{x}\log\left(\frac{1+ax}{1-bx}\right)$

(A) 0 **(B)** a/b **(C)** $a+b$ **(D)** $e^{a/b}$

195. $\lim\limits_{n\to\infty}(3^n+5^n+7^n)^{\frac{1}{n}}$ is equal to

(A) e^3 **(B)** e^5 **(C)** 5 **(D)** 7

196. Which of the following functions have finite number of points of discontinuity?

(A) $\tan x$ **(B)** $x[x]$ **(C)** $\frac{|x|}{x}$ **(D)** $\sin[n\pi x]$

197. $\lim\limits_{x\to2}(-1)^{[x]}$, where $[x]$ is the greatest integer function, is equal to

(A) 1 **(B)** -1 **(C)** ±1 **(D)** None of these

198. $\lim\limits_{x\to0}\frac{1+\sin x-\cos x+\log(1-x)}{x^3}$ equals

(A) 1/2 **(B)** $-1/2$ **(C)** 0 **(D)** none of these

199. If $f(x)=\frac{\sin(e^{x-2}-1)}{\log(x-1)}$, then $\lim\limits_{x\to2}f(x)$ is given by

(A) -2 **(B)** 0 **(C)** 1 **(D)** -1

200. $\lim\limits_{x\to0}\frac{x\cos x-\log(1+x)}{x^2}$ is equal to

(A) 1/2 **(B)** 1 **(C)** 0 **(D)** 3

201. $\lim\limits_{x\to0}\frac{|x+1|+|x-1|-2}{x}$ equals

(A) 1 **(B)** -1 **(C)** 2 **(D)** 0

202. $\lim\limits_{x\to0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$ is equal to $(a>0)$

(A) a **(B)** $\ln a$ **(C)** $\ln(1/a)$ **(D)** $1/a$

203. $\lim\limits_{n\to\infty}\left(\frac{1+e^{1/n}+e^{2/n}+\cdots+e^{(n-1)/n}}{n}\right)$ is equal to

(A) 0 **(B)** 1 **(C)** $e-1$ **(D)** e

204. $\lim\limits_{x\to-1+}\frac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}}$ is given by

(A) $\frac{1}{\sqrt{\pi}}$ **(B)** $\frac{1}{\sqrt{2\pi}}$ **(C)** 1 **(D)** 0

205. Let $f:R\to R$ be such that $f(1)=3$ and $f'(1)=6$. Then $\lim\limits_{x\to0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals

(A) 1 **(B)** $e^{1/2}$ **(C)** e^2 **(D)** e^3

206. If $f(x)=\begin{cases}\sin x, & x\ne n\pi,\ n\in I\\ 2, & \text{otherwise}\end{cases}$ and $g(x)=\begin{cases}x^2+1, & x\ne0,\ 2\\ 2, & x=0\\ 4, & x=2\end{cases}$,

then $\lim\limits_{x\to0}g[f(x)]$ is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 6

207. The function $f(x)=\begin{cases}\sin\left(\frac{\pi x}{2}\right), & x<1\\ |2x-3|\ [x], & x\ge1\end{cases}$

(A) Is continuous at $x=1$
(B) Is differentiable at $x=1$
(C) Is continuous but not differentiable at $x=1$
(D) None of these

208. The integer n for which $\lim_{x\to 0}\dfrac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

209. $\lim_{n\to\infty}\{\log_{n-1}(n)\cdot\log_n(n+1)\cdots\log_{n^k-1}(n^k)\}$, where $k\in N$.

(A) $k/2$ **(B)** $2k$ **(C)** k **(D)** None of these

210. Let $f(x)=\int_{-1}^{x}|t|\,dt, x\geq -1$. Then

(A) f and f' are continuous for $x+1>0$
(B) f is continuous but f' is not continuous for $x+1>0$
(C) f and f' are derivable at $x=0$
(D) f is continuous at $x=0$ but f' is not

211. If $f(x)=(x-x_0)\,\phi(x)$ and $\phi(x)$ is continuous at $x=0$, then $f'(x_0)$ is equal to

(A) $\phi'(x_0)$ **(B)** $\phi(x_0)$ **(C)** $x_0\phi(x_0)$ **(D)** None of these

212. If $f(x)=\{|x|-|x-1|\}^2$, then $f'(x)$ equals

(A) 0 for all x
(B) $2\{|x|-|x-1|\}$
(C) $\begin{cases}0 & \text{for } x<0 \text{ and for } x>1\\ 4\ (2x-1) & \text{for } 0<x<1\end{cases}$
(D) $\begin{cases}0 & \text{for } x<0\\ 4\ (2x-1) & \text{for } x>0\end{cases}$

213. If α and β be the roots of the equation $(ax^2+bx+1)=0$, then $\lim_{x\to\alpha}(1+ax^2+bx+c)^{1/(x-\alpha)}$ is

(A) $a\,(\alpha-\beta)$ **(B)** $\ln|a\,(\alpha-\beta)|$
(C) $e^{a(\alpha-\beta)}$ **(D)** None of these

214. If $f(x)$ is differentiable and strictly increasing function and $f'(0)\neq 0$, then the value of $\lim_{x\to 0}\dfrac{f(x^2)-f(x)}{f(x)-f(0)}$ is

(A) 1 **(B)** 0 **(C)** -1 **(D)** 2

215. The function $f(x)=\begin{cases}1, & |x|\geq 1\\ \dfrac{1}{n^2}, & \dfrac{1}{n}<|x|<\dfrac{1}{n-1},\quad n=2,3,\cdots.\\ 0, & x=0\end{cases}$

(A) Is discontinuous at infinitely many points
(B) Is continuous every where
(C) Is discontinuous only at $x=\dfrac{1}{n}$
(D) None of these

216. If $f(x)=1+x-[x]$, $g(x)=\begin{cases}\dfrac{|x|}{x}, & x\neq 0\\ 0, & x=0\end{cases}$. If $h(x)=g(f(x))$, then $h'(1)$ and $h'(-1)$ are

(A) Equal to zero each
(B) Non-existent
(C) Equal to 1 and -1, respectively
(D) None of these

217. If $f(x)=\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$, then $\lim_{x\to\infty}f(x)$ is

(A) e **(B)** e^2 **(C)** e^3 **(D)** e^4

218. If $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$ and $g(x)=\sin x+\cos x$, then points of discontinuity of $f\{g(x)\}$ in $(0,2\pi)$ is

(A) $\left\{\dfrac{\pi}{2},\dfrac{3\pi}{4}\right\}$ **(B)** $\left\{\dfrac{2\pi}{3},\dfrac{5\pi}{3}\right\}$
(C) $\left\{\dfrac{3\pi}{4},\dfrac{7\pi}{4}\right\}$ **(D)** $\left\{\dfrac{5\pi}{4},\dfrac{7\pi}{3}\right\}$

219. The value of the derivative of $|x-1|+|x-3|$ at $x=2$ is

(A) -2 **(B)** 0 **(C)** 2 **(D)** None of these

220. $g: R\to R$, $g(x)=\cos^{-1}[\sin f(x)]$ has exactly two elements in range set. Then

(A) $f(x)$ must be discontinuous function
(B) $f(x)$ may be continuous function
(C) It's not possible to have such a function
(D) $f(x)$ is discontinuous at finite points only

221. $\lim_{x\to a}\left(\dfrac{\cos x-\cos a}{\cot x-\cot a}\right)$ is equal to

(A) $\sin^3 a$ **(B)** $\operatorname{cosec}^3 a$ **(C)** $\dfrac{\sin^3 a}{2}$ **(D)** $\dfrac{\operatorname{cosec}^3 a}{2}$

222. $f(x+y)=f(x)\cdot f(y)\ \forall x$ and y. If $f(3)=3$ and $f'(0)=11$, then $f'(3)$ is given by

(A) 33 **(B)** 28 **(C)** 44 **(D)** 68

223. $\lim_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$ is equal to

(A) $\dfrac{1}{e}$ **(B)** 1 **(C)** 2 **(D)** e

224. Let $f: R\to R$ be a differentiable function and $f(1)=4$. Then the value of $\lim_{x\to 1}\int_4^{f(x)}\dfrac{2t}{x-1}dt$ is

(A) $8f'(1)$ **(B)** $4f'(1)$ **(C)** $2f'(1)$ **(D)** $f'(1)$

225. The value of $f(0)$, so that the function

$$f(x)=\frac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$$ becomes continuous for all x is given by

(A) $\sqrt{a}$ **(B)** $-\sqrt{a}$ **(C)** $a^{3/2}$ **(D)** $-a^{3/2}$

226. The $\lim_{x\to 0}\left(2\sin^2\frac{x}{2}\right)^{\ln\cos x}$ is equal to

(A) Does not exist **(B)** 1 **(C)** 1/2 **(D)** 2

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. Let $S_n = \lim_{n\to\infty}\left(\frac{1}{n^6}+\frac{32}{n^6}+\cdots+\frac{1}{n^6}\right)$ and

$T_n = \lim_{n\to\infty}\left(\frac{1}{n^6}+\frac{32}{n^6}+\cdots+\frac{(n-1)^5}{n^6}\right)$. Then which of the following is/are true?

(A) $S_n \to \frac{1}{6}^+$ **(B)** $(S_n+T_n)<\frac{1}{3}$

(C) $(S_n+T_n)>\frac{1}{3}$ **(D)** $T_n \to \frac{1}{6}^-$

2. Let $f(x)=|x|$, $g(n,x)=\sin[\pi([n]+[n]^2)^{1/x}]$ and $h(x)=\dfrac{\ln\left(\dfrac{e^x+e^{2x}+e^{3x}}{3}\right)}{x}$. Then (where [.] represent greatest integer function)

(A) $\lim_{n\to\infty}\lim_{x\to 0} f\{g[n,h(x)]\}=1$

(B) $\lim_{n\to\infty}\lim_{x\to 0} f\{g[n,h(x)]\}$ does not exist

(C) $\lim_{n\to\infty}\lim_{x\to 0} g[n,h(x)]$ does not exist

(D) $\lim_{n\to\infty}\lim_{x\to 0} g[n,h(x)]=1$

3. If $f(x)=\lim_{n\to\infty} e^{x\tan\left(\frac{1}{n}\right)\ln\left(\frac{1}{n}\right)}$ and $\int \dfrac{f(x)dx}{\sqrt[3]{\sin^{11}x\cos x}}=g(x)+C$, then

(A) $g\left(\frac{\pi}{4}\right)=\frac{3}{2}$

(B) $g(x)$ is continuous for all x

(C) $g\left(\frac{\pi}{4}\right)=\frac{-15}{8}$

(D) $g(x)$ is non-differentiable at infinitely many points

4. $\lim_{x\to 0} x^5\left[\frac{1}{x^3}\right]$ is (where [.] represents greatest integer function)

(A) A non-zero real number **(B)** A rational number

(C) An integer **(D)** Zero

5. Let $f(x)=\begin{cases}\dfrac{a(1-x\sin x)+b\cos x+5}{x^2} & x<0\\ 3 & x=0\\ \left[1+\left(\dfrac{cx+dx^3}{x^2}\right)\right]^{\frac{1}{x}} & x>0\end{cases}$. If f is continuous at $x=0$, then

(A) $a=-1$ **(B)** $b=-4$ **(C)** $c=0$ **(D)** $d=\log_e 5$

6. If $x\cos\alpha + y\sin\alpha = x\cos\beta + y\sin\beta = 2\lim_{n\to\infty}(a^n+b^n)^{1/n}$ (where $x^2+y^2\neq 0$ and $0<b<a$), then

(A) $\cos\alpha+\cos\beta=\cos\alpha\cos\beta$ is a parabola

(B) $\cos\alpha+\cos\beta=\sin\alpha+\sin\beta$ is a straight line

(C) $\cos\alpha+\cos\beta=\sin\alpha\sin\beta$ are pair of line

(D) $\cos\alpha\cos\beta+\sin\alpha\sin\beta=0$ is a circle

7. $f(x)=\left(\dfrac{x}{2+x}\right)^{2x}$, then

(A) $\lim_{x\to\infty} f(x)=-4$ **(B)** $\lim_{x\to\infty} f(x)=2$

(C) $\lim_{x\to\infty} f(x)=e^{-4}$ **(D)** $\lim_{x\to 1} f(x)=\frac{1}{9}$

8. Which of the following is/are true?

(A) If $\lim_{x\to a}\{f(x)+g(x)\}$ exists, then both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

(B) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a}\{f(x)+g(x)\}$ exists.

(C) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x)\,g(x)$ exists.

(D) If $\lim_{x\to a}\{f(x)\,g(x)\}$ exists, then both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.

9. If x is a real number in [0, 1], then the value of $\lim_{m\to\infty}\lim_{n\to\infty}[1+\cos^{2m}(n!\,\pi x)]$ is

(A) 1 if $x\notin Q$ **(B)** 2 if $x\notin Q$

(C) 1 if $x\in Q$ **(D)** 2 if $x\in Q$

10. Which of the following limits tend to unity?

(A) $\lim_{t\to 0}\dfrac{\sin(\tan t)}{\sin t}$ **(B)** $\lim_{x\to\pi/2}\dfrac{\sin(\cos x)}{\cos x}$

(C) $\lim_{x\to 0}\dfrac{\sqrt{x^2}}{x}$ **(D)** $\lim_{x\to\pi/2}\left(\dfrac{1-\cos x}{x^2}\right)$

11. $f(x)=\begin{cases}x+\dfrac{1}{2}, & x<0\\ 2x+\dfrac{1}{3}, & x\geq 0\end{cases}$, identify the correct statement(s) ([ ] denotes greatest integer function)

(A) $\lim_{x\to 0}[f(x)]=0$ **(B)** $\lim_{x\to 0} f(x)$ does not exist

(C) $\left[\lim_{x\to 0} f(x)\right]$ exists **(D)** $\lim_{x\to 0}\dfrac{[f(x)]}{x}$ does not exist

12. Which of the following function(s) has/have removable discontinuity at $x = 1$?

(A) $f(x) = \dfrac{1}{\ln|x|}$ **(B)** $f(x) = \dfrac{x^2-1}{x^3-1}$

(C) $f(x) = 2^{-2^{\frac{1}{1-x}}}$ **(D)** $f(x) = \dfrac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

13. A function $f(x)$ satisfies the relation $f(x+y) = f(x) + f(y) + xy(x+y)\ \forall\ x, y \in R$. If $f'(0) = -1$, then

(A) $f(x)$ is a polynomial function.
(B) $f(x)$ is an exponential function.
(C) $f(x)$ is twice differentiable for all $x \in R$.
(D) $f'(3) = 8$.

14. Let $f(x) = \int_{-2}^{x} |t+1|\,dt$. Then

(A) $f(x)$ is continuous in $[-1, 1]$.
(B) $f(x)$ is differentiable in $[-1, 1]$.
(C) $f'(x)$ is continuous in $[-1, 1]$.
(D) $f'(x)$ is differentiable in $[-1, 1]$.

15. $f(x) = \dfrac{[x]+1}{\{x\}+1}$ for $f: \left[0, \dfrac{5}{2}\right) \to \left(\dfrac{1}{2}, 3\right]$, where [.] represents greatest integer function and { . } represents fractional part of x, then which of the following is true.

(A) f(x) is injective discontinuous function.
(B) $f(x)$ is surjective non-differentiable function.
(C) $\min\left[\lim_{x\to 1-} f(x), \lim_{x\to 1+} f(x)\right] = f(1)$.
(D) max (x values of point of discontinuity) $= f(1)$.

16. If $f(x) = 0$ for $x < 0$ and $f(x)$ is differentiable at $x = 0$, then for $x > 0$, $f(x)$ may be

(A) x^2 **(B)** x **(C)** $\sin x$ **(D)** $-x^{3/2}$

Comprehension Type Questions

Paragraph for Questions 18–20: A tangent line is drawn to a circle of radius unity at the point A and a segment AB is laid off whose length is equal to that of the arc AC, a straight line BC is drawn to intersect the extension of the diameter AO at the point P.

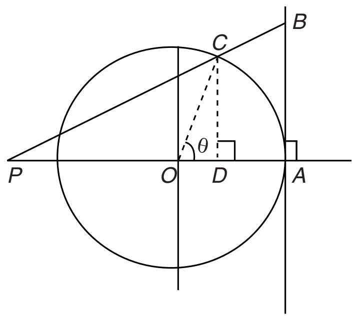


Figure 19.26

17. The value of $\lim_{\theta\to 0^+} PA$ is

(A) $\dfrac{1}{3}$ **(B)** 3 **(C)** 0 **(D)** None of these

18. If tangent at C intersect extended PA at Q, then the area of ΔCPQ is

(A) $\dfrac{1}{2}\left\{\tan\theta - \dfrac{\sin^2\theta(1+\theta\cot\theta)}{\theta-\sin\theta}\right\}$

(B) $\dfrac{1}{2}\left\{\tan\theta + \dfrac{\sin^2\theta(1+\theta\cot\theta)}{\theta-\sin\theta}\right\}$

(C) $\dfrac{1}{2}\left\{\tan\theta + \dfrac{\sin^2\theta(1-\theta\cot\theta)}{\theta-\sin\theta}\right\}$

(D) $\dfrac{1}{2}\left\{\tan\theta - \dfrac{\sin^2\theta(1+\theta\cot\theta)}{\theta-\sin\theta}\right\}$

19. The value of $\lim_{\theta\to 0^+} \dfrac{\text{Area}(\Delta CPQ)}{\sin^2\theta}$ is

(A) $\dfrac{1}{3}$ **(B)** 3 **(C)** 0 **(D)** Not defined

Paragraph for Questions 21–23: Let $f(x) = \lim_{n\to\infty}\left(\cos\sqrt{\dfrac{x}{n}}\right)^n$, $g(x) = \lim_{n\to\infty}(1-x+x\sqrt[n]{e})^n$. Now, consider the function $y = h(x)$, where $h(x) = \tan^{-1}[g^{-1} f^{-1}(x)]$.

20. $\lim_{x\to 0^+} \dfrac{\ln[f(x)]}{\ln[g(x)]}$ is equal to

(A) $\dfrac{1}{2}$ **(B)** $-\dfrac{1}{2}$ **(C)** 0 **(D)** 1

21. Domain of the function $y = h(x)$ is

(A) $(0, \infty)$ **(B)** R **(C)** $(0, 1)$ **(D)** $[0, 1]$

22. Range of the function $y = h(x)$ is

(A) $\left(0, \dfrac{\pi}{2}\right)$ **(B)** $\left(-\dfrac{\pi}{2}, 0\right)$ **(C)** R **(D)** $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Paragraph for Questions 24–26: Let $f(x) = \max\{a, b, c\}$, where

$$a = \lim_{n\to\infty}\lim_{\alpha\to 1^+} \frac{\alpha^n|\sin x| + \alpha^{-n}|\cos x|}{\alpha^n+\alpha^{-n}}$$

$$b = \lim_{n\to\infty}\lim_{\alpha\to 1^-} \frac{\alpha^{-n}|\sin x| + \alpha^{n}|\cos x|}{\alpha^n+\alpha^{-n}}$$

$$c = \lim_{n\to\infty} \frac{\pi}{4n}\left[1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\cdots+\cos\frac{(n-1)\pi}{2n}\right].$$ Then

23. The value of a is

(A) $2|\sin x|$ **(B)** $|\cos x|$ **(C)** $|\sin x|$ **(D)** $\dfrac{1}{2}$

24. The value of $b + c - \dfrac{1}{2}$ is

(A) $|\cos x|$ **(B)** $2|\cos x| - 1$
(C) $|\sin x| + 1$ **(D)** $|\sin x| + |\cos x|$

25. Range of $f(x)$ is

(A) $[0, 1]$ **(B)** $\left[\frac{1}{2}, 1\right]$ **(C)** $\left[\frac{1}{\sqrt{2}}, 1\right]$ **(D)** $\left[\frac{1}{2}, 2\right]$

Paragraph for Questions 27–29: Let a function is defined as

$$f(x)=\begin{cases}[x], & -2 \le x \le -\frac{1}{2}\\ 2x^2-1, & -\frac{1}{2} < x \le 2\end{cases}$$, where [.] denotes greatest integer function.

26. The number of points of discontinuity of $f(x)$ is

(A) 1 **(B)** 2 **(C)** 3 **(D)** None of these

27. The function $f(x-1)$ is discontinuous at the points

(A) $-1, -\frac{1}{2}$ **(B)** $-\frac{1}{2}, 1$ **(C)** $0, \frac{1}{2}$ **(D)** $0, 1$

28. Number of points where $|f(x)|$ is not differentiable is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

Matrix Match Type Questions

29. Match the following:

List I	List II
(A) If $\lim\limits_{n\to\infty} \frac{\left(\sum\limits_{x=1}^{n} x^4\right)\left(\sum\limits_{x=1}^{n} x^5\right)}{\left(\sum\limits_{x=1}^{n} x^t\right)\left(\sum\limits_{x=1}^{n} x^{9-t}\right)} = \frac{4}{5}$, then t can be	**(p)** 6
(B) If m be the slope of tangent to the curve $x^y = y^x$ at (e, e), then $(3-m)$ can be	**(q)** 2
(C) If $f: R-\{0\} \to R$, $f(x)f(y) = f(xy) + 3\left(\frac{1}{x}+\frac{1}{y}\right)$, then $2\left\|f\left(-\frac{1}{2}\right)\right\|$ can be	**(r)** 3
(D) If $\lim\limits_{n\to\infty} \left\{(\sqrt{2}+1)^{n!+K}\right\} = 1$, then K can be (where {.} is a fractional part function)	**(s)** 4
	(t) 7

30. If $\lim\limits_{x\to\infty} \left(\sqrt{x^4+ax^3+3x^2+bx+2} - \sqrt{x^4+2x^3-cx^2+3x-d}\right) = 4$, then match the values of a, b, c and d.

List I	List II
(A) a	**(p)** 5
(B) b	**(q)** 0
(C) c	**(r)** 1
(D) d	**(s)** 2
	(t) $\frac{1}{2}$

31. Match the following:

List I	List II
(A) Let $f: R \to R$ be a differentiable function and $f(1) = 1$, $f'(1) = 3$. Then the value of $\lim\limits_{x\to 1} \int_1^{x^2} \frac{(f(t)-t)}{(x-1)^2}\, dt$ is	**(p)** 0
(B) $\lim\limits_{n\to\infty} \left(\frac{1+\sqrt[n]{4}}{2}\right)^n$ is equal to	**(q)** -1
(C) If $f(x) = \lim\limits_{n\to\infty} \frac{2x}{\pi} \cdot \tan^{-1}(nx), x > 0$, then $\lim\limits_{x\to 0^+} [f(x)-1]$ is {where [.] represents greatest integer function}	**(r)** 2
(D) $\lim\limits_{n\to\infty} \left[\sum\limits_{r=1}^{n} \frac{1}{2^r}\right] =$ (where [.] denotes the greatest integer function)	**(s)** 1
	(t) 4

32. Match the following:

List I	List II
(A) Let $f: R \to R$ be such that $f(a) = 1, f'(a) = 2$ and $\lim\limits_{x\to 0} \left(\frac{f^2(a+x)}{f(a)}\right)^{1/x} = e^k$. Then $k =$	**(p)** 0
(B) $\lim\limits_{x\to \frac{\pi}{2}^+} \frac{\cos[\tan^{-1}(\tan x)]}{x-\frac{\pi}{2}} =$	**(q)** 1
(C) $\lim\limits_{x\to\pi} \frac{\sin(\cos x+1)}{\cos\left(\frac{x}{2}\right)} =$	**(r)** 4
(D) $\lim\limits_{x\to 0} \frac{xe^{\sin x} - e^x \sin^{-1}(\sin x)}{\sin^2 x - x\sin x} =$	**(s)** 3
	(t) Does not exist

Integer Type Questions

33. $f: R \to R$ be a twice differentiable function satisfying $f''(x) - 5f'(x) + 6f(x) \ge 0\ \forall\, x \ge 0$, if $f(0) = 1, f'(0) = 0$, If $f(x)$ satisfies $f(x) \ge ah(bx) - bh(ax), \forall\, x \ge 0$, then find $(a+b)\, h(0)$.

34. If a and b are positive numbers and $\lim\limits_{x\to 0} \frac{(1+a^3)+8e^{\frac{1}{x}}}{1+(2+b+b^2)e^{\frac{1}{x}}} = 2$, then find the value of a^2+b^2.

35. Let $f(x)$be a differentiable function such that $f'(x) + f(x) = 4xe^{-x} \cdot \sin 2x$ and $f(0) = 0$. If $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(k\pi) = \frac{-p\pi e^{\pi}}{(e^{\pi}-1)^2}$, then find value of p.

36. Let $f(x) = \frac{\tan x}{x}$ and $\lim\limits_{x\to 0} ([f(x)] + x^2)^{\frac{1}{\{f(x)\}}} = e^{\lambda}$. Then find λ.

(where [.] and {.} denotes greatest integer and fractional part function respectively)

Answer Key

Practice Exercise 1

1. (B)	**2.** (A)	**3.** (B)	**4.** (D)	**5.** (A)	**6.** (B)	**7.** (C)	**8.** (B)	**9.** (C)	**10.** (A)
11. (A)	**12.** (C)	**13.** (C)	**14.** (C)	**15.** (C)	**16.** (B)	**17.** (B)	**18.** (A)	**19.** (B)	**20.** (C)
21. (B)	**22.** (A)	**23.** (A)	**24.** (D)	**25.** (C)	**26.** (B)	**27.** (C)	**28.** (A)	**29.** (A)	**30.** (C)
31. (C)	**32.** (D)	**33.** (D)	**34.** (C)	**35.** (B)	**36.** (D)	**37.** (B)	**38.** (B)	**39.** (C)	**40.** (B)
41. (C)	**42.** (D)	**43.** (D)	**44.** (B)	**45.** (D)	**46.** (D)	**47.** (C)	**48.** (B)	**49.** (B)	**50.** (D)
51. (C)	**52.** (A)	**53.** (C)	**54.** (B)	**55.** (B)	**56.** (C)	**57.** (A)	**58.** (A)	**59.** (B)	**60.** (C)
61. (A)	**62.** (A)	**63.** (B)	**64.** (D)	**65.** (B)	**66.** (B)	**67.** (D)	**68.** (B)	**69.** (B)	**70.** (A)
71. (A)	**72.** (B)	**73.** (D)	**74.** (B)	**75.** (C)	**76.** (C)	**77.** (B)	**78.** (C)	**79.** (C)	**80.** (D)
81. (C)	**82.** (B)	**83.** (D)	**84.** (C)	**85.** (C)	**86.** (A)	**87.** (B)	**88.** (B)	**89.** (B)	**90.** (B)
91. (C)	**92.** (B)	**93.** (C)	**94.** (D)	**95.** (B)	**96.** (B)	**97.** (B)	**98.** (C)	**99.** (A)	**100.** (C)
101. (A)	**102.** (A)	**103.** (D)	**104.** (B)	**105.** (B)	**106.** (B)	**107.** (B)	**108.** (D)	**109.** (D)	**110.** (D)
111. (D)	**112.** (D)	**113.** (A)	**114.** (C)	**115.** (A)	**116.** (C)	**117.** (A)	**118.** (C)	**119.** (A)	**120.** (C)
121. (A)	**122.** (A)	**123.** (C)	**124.** (A)	**125.** (D)	**126.** (C)	**127.** (B)	**128.** (B)	**129.** (D)	**130.** (C)
131. (C)	**132.** (A)	**133.** (A)	**134.** (D)	**135.** (B)	**136.** (B)	**137.** (C)	**138.** (D)	**139.** (C)	**140.** (D)
141. (B)	**142.** (B), (D)	**143.** (D)	**144.** (D)	**145.** (B)	**146.** (D)	**147.** (A), (C), (D)	**148.** (D)	**149.** (A)	**150.** (A)
151. (B)	**152.** (B)	**153.** (D)	**154.** (D)	**155.** (B)	**156.** (D)	**157.** (C)	**158.** (C)	**159.** (A)	**160.** (C)
161. (B)	**162.** (D)	**163.** (B)	**164.** (B)	**165.** (D)	**166.** (C)	**167.** (A)	**168.** (D)	**169.** (C)	**170.** (B)
171. (B)	**172.** (B)	**173.** (B)	**174.** (D)	**175.** (D)	**176.** (D)	**177.** (D)	**178.** (D)	**179.** (C)	**180.** (C)
181. (D)	**182.** (C)	**183.** (C)	**184.** (A)	**185.** (B)	**186.** (C)	**187.** (C)	**188.** (B)	**189.** (D)	**190.** (C)
191. (D)	**192.** (B)	**193.** (A)	**194.** (C)	**195.** (D)	**196.** (C)	**197.** (D)	**198.** (D)	**199.** (C)	**200.** (A)
201. (D)	**202.** (B)	**203.** (C)	**204.** (B)	**205.** (C)	**206.** (B)	**207.** (C)	**208.** (B)	**209.** (C)	**210.** (A)
211. (B)	**212.** (C)	**213.** (C)	**214.** (C)	**215.** (A)	**216.** (A)	**217.** (D)	**218.** (C)	**219.** (B)	**220.** (A)
221. (A)	**222.** (A)	**223.** (B)	**224.** (A)	**225.** (B)	**226.** (B)				

Practice Exercise 2

1. (A), (C), (D)	**2.** (A), (C)	**3.** (C), (D)	**4.** (B), (C), (D)	**5.** (A), (B), (C)	**6.** (A), (B), (C), (D)
7. (C)	**8.** (B), (C)	**9.** (A), (D)	**10.** (A), (B)	**11.** (A), (B)	**12.** (B), (D)
13. (A), (C), (D)	**14.** (A), (B), (C), (D)	**15.** (A), (B), (D)	**16.** (A), (D)	**17.** (B)	**18.** (C)
19. (D)	**20.** (B)	**21.** (C)	**22.** (D)	**23.** (C)	**24.** (A)
25. (C)	**26.** (B)	**27.** (C)	**28.** (C)	**29.** (A) → (q), (t); (B)→ (q), (s); (C)→ (q), (r); (D)→ (p), (q), (s)	
30. (A)→ (s); (B)→ (p), (q), (r), (s), (t); (C)– (p); (D)→ (p), (q), (r), (s), (t)				**31.** (A)→ (t), (B)→ (r), (C)→ (q), (D)→ (p)	
32. (A)→ (r), (B)→ (q), (C)→ (p), (D)→ (q)		**33.** 5	**34.** 2	**35.** 2	**36.** 3

Solutions

Practice Exercise 1

1. Here $f(0) = 0$

Since

$$-1 \le \sin\frac{1}{x} \le 1 \Rightarrow -|x| \le x\sin\frac{1}{x} \le |x|$$

We know that $\lim\limits_{x\to 0}|x| = 0$ and $\lim\limits_{x\to 0} -|x| = 0.$

In this way $\lim\limits_{x\to 0} f(x) = 0.$

2. $$\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)\times(\sqrt{x}+1)}{(x-1)(2x+3)\times(\sqrt{x}+1)} = \frac{-1}{5.2} = \frac{-1}{10}$$

3. Applying L'Hospital's rule,

$$\lim_{x\to 9}\frac{\frac{1}{2\sqrt{f(x)}}\cdot f'(x)}{\frac{1}{2\sqrt{x}}} = \frac{\frac{f'(9)}{\sqrt{f(9)}}}{\frac{1}{\sqrt{9}}} = \frac{\frac{4}{3}}{\frac{1}{3}} = 4$$

4. Since $\lim\limits_{x\to 0^-}\frac{|x|}{x}=-1$ and $\lim\limits_{x\to 0^+}\frac{|x|}{x}=1$, hence limit does not exist.

5. $\lim\limits_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim\limits_{h\to 0}\frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}=\frac{1}{2\sqrt{x}}$

Alternate solution: Apply L'Hospital's rule,

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{1}{2\sqrt{x+h}}=\frac{1}{2\sqrt{x}}$$

6. $\lim\limits_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}=\lim\limits_{x\to 0}\frac{2^x\log 2}{\frac{1}{2}(1+x)^{-1/2}}=2\log 2=\log 4$

$$\left\{\text{As}\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\right\}$$

7. $$\lim_{x\to 1}\frac{x-1}{(x-1)(2x-5)}=-\frac{1}{3}$$

Alternate solution: Apply L'Hospital's rule,

$$\lim_{x\to 1}\frac{x-1}{2x^2-7x+5}=\frac{1}{4x-7}=-\frac{1}{3}$$

8. $$\lim_{x\to\infty}\frac{\sin x}{x}$$

Let $x=\frac{1}{y}$ or $y=\frac{1}{x}$. Then

$$x\to\infty\Rightarrow y\to 0$$

Hence,

$$\lim_{x\to\infty}\left(\frac{\sin x}{x}\right)=\lim_{y\to 0}\left(y\cdot\sin\frac{1}{y}\right)=\lim_{y\to 0}y\times\lim_{y\to 0}\sin\frac{1}{y}=0\times\cdots=0$$

9. $\lim\limits_{x\to 0}\frac{2\times 9\sin^2 3x}{(3x)^2}=18$

10. $\lim\limits_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4}$

$$=\lim_{\alpha\to\pi/4}\left\{\frac{\sqrt{2}\left(\sin\alpha\cdot\frac{1}{\sqrt{2}}-\cos\alpha\cdot\frac{1}{\sqrt{2}}\right)}{\left(\alpha-\frac{\pi}{4}\right)}\right\}$$

$$=\sqrt{2}\lim_{\alpha\to\pi/4}\frac{\sin\left(\alpha-\frac{\pi}{4}\right)}{\left(\alpha-\frac{\pi}{4}\right)}=\sqrt{2}\times 1=\sqrt{2}$$

Alternate solution: Apply L'Hospital's rule,

$$\lim_{\alpha\to\pi/4}\frac{\sin\alpha-\cos\alpha}{\alpha-(\pi/4)}=\lim_{\alpha\to\pi/4}\frac{\cos\alpha+\sin\alpha}{1}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$$

11. $\lim\limits_{\theta\to\pi/2}\frac{1-\sin\theta}{\cos\theta}=\lim\limits_{\theta\to\pi/2}\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}=0$

12. $\lim\limits_{x\to 0}\frac{\tan 2x-x}{3x-\sin x}=\lim\limits_{x\to 0}\left\{\frac{\frac{2\tan 2x}{2x}-1}{3-\frac{\sin x}{x}}\right\}=\frac{1}{2}$

Alternate solution: Apply L'Hospital's rule,

$$\lim_{x\to 0}\frac{\tan 2x-x}{3x-\sin x}=\lim_{x\to 0}\frac{2\sec^2 2x-1}{3-\cos x}=\frac{2-1}{3-1}=\frac{1}{2}$$

13. $\lim\limits_{x\to\pi/2}2\left[\frac{x-\frac{\pi}{2}}{\sin\left(\frac{\pi}{2}-x\right)}\right]=-2$

Alternate solution: Apply L'Hospital's rule.

$$\lim_{x\to\pi/2}\frac{2x-\pi}{\cos x}=\lim_{x\to\pi/2}\frac{2}{-\sin x}=-2$$

14. Apply L'Hospital's rule,

$$\lim_{h\to 0}\frac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$$

$$=\lim_{h\to 0}\frac{2(a+h)\sin(a+h)+(a+h)^2\cos(a+h)}{1}$$

$$=2a\sin a+a^2\cos a$$

15. Apply L'Hospital's rule,

$$\lim_{x\to 0}\frac{(1+x)^{1/2}-(1-x)^{1/2}}{x}=\lim_{x\to 0}\frac{1}{2\sqrt{1+x}}+\frac{1}{2\sqrt{1+x}}=1$$

16. $\lim\limits_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$

$$=\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\times\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}\times\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}$$

$$=\lim_{x\to a}\frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})}=\frac{2}{3\sqrt{3}}$$

17. $\lim\limits_{x\to 0}\frac{x\cdot(2^x-1)}{1-\cos x}=\lim\limits_{x\to 0}\frac{2^x-1}{x}\cdot\frac{x^2}{1-\cos x}$

$$=\log 2\cdot\lim_{x\to 0}\frac{x^2}{2\sin^2\frac{x}{2}}=(\log 2)\cdot 2=2\log 2=\log 4$$

18. $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\frac{\sin x-\sin x\cos x}{x^3\cos x}$

$$= \lim_{x\to 0}\frac{\sin x\left(2\sin^2\frac{x}{2}\right)}{x^3\cos x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{2}{\cos x}\cdot\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{1}{4}\right] = \frac{1}{2}$$

19. Let $y = x^x$. Then $\log y = x\log x$.

Hence,

$$\lim_{y\to 0}\log y = \lim_{x\to 0} x\log x = 0 = \log 1 \Rightarrow \lim_{x\to 0} x^x = 1$$

20. $$\lim_{x\to\infty}\frac{(2x+1)^{40}(4x-1)^5}{(2x+3)^{45}} = \lim_{x\to\infty}\frac{\left(2+\frac{1}{x}\right)^{40}\left(4-\frac{1}{x}\right)^5}{\left(2+\frac{3}{x}\right)^{45}}$$

$$= \frac{2^{40}\cdot 4^5}{2^{45}} = 32$$

21. Let $\tan^{-1}2x = \theta$. Then $x = \frac{1}{2}\tan\theta$ and as $x \to 0$, $\theta \to 0$. So,

$$\lim_{x\to 0}\frac{x}{\tan^{-1}2x} = \lim_{\theta\to 0}\frac{\frac{1}{2}\tan\theta}{\theta} = \frac{1}{2}$$

22. Apply L'Hospital's rule,

$$\lim_{x\to 0}\frac{2xe^{x^2}+\sin x}{2x} = \lim_{x\to 0} e^{x^2} + \lim_{x\to 0}\frac{\sin x}{2x} = 1+\frac{1}{2} = \frac{3}{2}$$

23. Let $f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}$

Therefore, given function $= f'(a) + kf'(e) = 1$

$$\Rightarrow \frac{1}{a}+\frac{k}{e} = 1 \Rightarrow k = e\left(\frac{a-1}{a}\right)$$

Alternate solution: Apply L'Hospital's rule to find both the limits.

$$\lim_{x\to 0}\frac{1}{a+x} + k\lim_{x\to e}\frac{1}{x} = 1$$

$$\frac{1}{a}+\frac{k}{e} = 1 \Rightarrow k = e\left(\frac{a-1}{a}\right)$$

24. $$\lim_{x\to 0}\frac{\sqrt{\frac{1}{2}(1-\cos 2x)}}{x} = \lim_{x\to 0}\frac{|\sin x|}{x}$$

So, $\lim_{x\to 0^+}\frac{|\sin x|}{x} = 1$ and $\lim_{x\to 0^-}\frac{|\sin x|}{x} = -1$.

Hence, limit does not exist.

25. Apply L'Hospital's rule, we get

$$\lim_{x\to 0}\frac{\cos x - \frac{1}{1-x}}{2x} = \lim_{x\to 0}\frac{-\sin x - \frac{1}{(1-x)^2}}{2} = -\frac{1}{2}$$

Alternate solution:

$$\lim_{x\to 0}\frac{\sin x + \log(1-x)}{x^2}$$

$$= \lim_{x\to 0}\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}{x^2} + \lim_{x\to 0}\frac{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right)}{x^2}$$

$$\left(\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ and } \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right)$$

$$= \lim_{x\to 0}\frac{\frac{-x^2}{2} - x^3\left(\frac{1}{3!}+\frac{1}{3}\right) - \frac{x^4}{4}\cdots}{x^2} = -\frac{1}{2}$$

26. $$\lim_{x\to\infty}\sqrt{\left(\frac{x+\sin x}{x-\cos x}\right)} = \lim_{x\to\infty}\sqrt{\left(\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}\right)} = \lim_{x\to\infty}\sqrt{1} = 1$$

$$\left[\text{As } \lim_{x\to\infty}\frac{\sin x}{x} \text{ and } \lim_{x\to\infty}\frac{\cos x}{x} \text{ both are equal to } 0\right]$$

27. Given limit $= \lim_{x\to 0}\left(\frac{1+\tan x}{1-\tan x}\right)^{1/x}$

$$= \lim_{x\to 0}\frac{\{(1+\tan x)^{1/\tan x}\}^{(\tan x)/x}}{\{(1-\tan x)^{1/\tan x}\}^{(\tan x)/x}} = \frac{e}{e^{-1}} = e^2$$

28. $$\lim_{x\to 0}\left(\frac{1+5x^2}{1+3x^2}\right)^{1/x^2} = \frac{\lim_{x\to 0}[(1+5x^2)^{1/5x^2}]^5}{\lim_{x\to 0}[(1+3x^2)^{1/3x^2}]^3} = \frac{e^5}{e^3} = e^2$$

[As $\lim_{x\to 0}(1+x)^{1/x} = e$]

29. $$\lim_{x\to -\infty}\frac{\sqrt{4x^2+5x+8}}{4x+5} = \lim_{h\to 0}\frac{\sqrt{4(-1/h)^2+5(-1/h)+8}}{4(-1/h)+5}$$

$$= \lim_{h\to 0}\frac{(1/h)\sqrt{4-5h+8h^2}}{(1/h)(-4+5h)} = \frac{\sqrt{4}}{-4} = -\frac{1}{2}$$

30. $$\lim_{x\to 0}\frac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$$

$$= \lim_{x\to 0}\frac{x(\tan 2x - 2\tan x)}{(2\sin^2 x)^2} = \lim_{x\to 0}\frac{1}{4}\frac{x(\tan 2x - 2\tan x)}{\sin^4 x}$$

$$= \lim_{x\to 0}\frac{1}{4}\frac{x\left\{\left(2x+\frac{1}{3}(2x)^3+\frac{2}{15}(2x)^5+\cdots\right)-2\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\cdots\right)\right\}}{x^4\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)^4}$$

$$= \frac{1}{4}\cdot\left(\frac{8}{3}-\frac{2}{3}\right) = \frac{2}{4} = \frac{1}{2}$$

31. $$\lim_{x\to\infty}\left(\frac{x+2-5}{x+2}\right)^x = \lim_{x\to\infty}\left[\left(1-\frac{5}{x+2}\right)^{\frac{x+2}{-5}}\right]^{-\frac{5x}{x+2}} = e^{-5}$$

$$\left(\text{As } \lim_{x\to\infty}\frac{-5x}{x+2} = \lim_{x\to\infty}\frac{-5}{1+\frac{2}{x}} = -5\right)$$

32. $\lim\limits_{x\to 0} n\dfrac{\sin nx}{nx}\cdot\lim\limits_{x\to 0}\left((a-n)n-\dfrac{\tan x}{x}\right)=0$

$$\Rightarrow n((a-n)n-1)=0\Rightarrow (a-n)n=1\Rightarrow a=n+\frac{1}{n}$$

33. $\lim\limits_{h\to 0}\dfrac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}=\lim\limits_{h\to 0}\dfrac{f'(2h+2+h^2)(2+2h)}{f'(h-h^2+1)(1-2h)}$

$$=\frac{6\times 2}{4\times 1}=3$$

34. $$y=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$$

$$\Rightarrow y=\lim_{x\to 0}\frac{\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right]-\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\cdots\right]}{\sin x}$$

$$\Rightarrow y=\lim_{x\to 0}\frac{2\left[\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right]}{\sin x}$$

$$\Rightarrow y=\lim_{x\to 0}\frac{2\left[1+\frac{x^2}{3!}+\frac{x^4}{4!}+\cdots\right]}{\frac{\sin x}{x}}$$

$$\Rightarrow y=\frac{\lim\limits_{x\to 0}2\left[1+\frac{x^2}{2!}+\cdots\right]}{\lim\limits_{x\to 0}\frac{\sin x}{x}}\Rightarrow y=\frac{2}{1}=2$$

Alternate solution: Applying L'Hospital's rule,

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=\frac{e^0+\frac{1}{e^0}}{\cos 0}=\frac{1+1}{1}=2$$

35. Using L'Hospital's rule,

$$\lim_{x\to\pi/6}\frac{3\cos x+\sqrt{3}\sin x}{6}=\frac{3\cdot\frac{\sqrt{3}}{2}+\sqrt{3}\cdot\frac{1}{2}}{6}=\frac{1}{\sqrt{3}}$$

36. $\lim\limits_{x\to 0}\dfrac{\cos(\sin x)-1}{x^2}=\lim\limits_{x\to 0}\dfrac{-2\sin^2\left(\frac{\sin x}{2}\right)}{x^2}=-2\cdot\dfrac{1}{4}=\dfrac{-1}{2}$

37. $\lim\limits_{n\to\infty}(3^n+4^n)^{\frac{1}{n}}=\lim\limits_{n\to\infty}(4^n)^{\frac{1}{n}}\left[\dfrac{3^n}{4^n}+1\right]^{\frac{1}{n}}$

$$=\lim_{n\to\infty}4\left[1+\frac{1}{\left(\frac{4}{3}\right)^n}\right]^{1/n}=4\lim_{n\to\infty}\left[1+\frac{1}{\left(\frac{4}{3}\right)^n}\right]^{1/n}$$

$$=4\left[1+\frac{1}{\infty}\right]^0=4\times(1)^0=4\times 1=4$$

38. Since, $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}+\dfrac{b}{x^2}\right)^{2x}=e^2$

Hence,

$$\lim_{x\to\infty}\left[\left(1+\frac{ax+b}{x^2}\right)^{\frac{x^2}{ax+b}}\right]^{\frac{2(ax+b)}{x}}=e^2$$

$$\Rightarrow\lim_{x\to\infty}e^{\frac{2(ax+b)}{x}}=e^2\Rightarrow\lim_{x\to\infty}\frac{2(ax+b)}{x}=2\Rightarrow 2a=2\Rightarrow a=1$$

Thus, $a=1$ and $b\in R$.

39. Using L'Hospital's rule, $\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{-1}{-\operatorname{cosec}^2\theta}=1$

40. $\lim\limits_{x\to\infty}\left(1-\dfrac{4}{x-1}\right)^{3x-1}=\lim\limits_{x\to\infty}\left[\left(1+\dfrac{(-4)}{x-1}\right)^{\left(\frac{x-1}{-4}\right)}\right]^{\left(\frac{-4}{x-1}\right)(3x-1)}$

$$=e^{\lim\limits_{x\to\infty}\left[\frac{-4\left(3-\frac{1}{x}\right)}{\left(1-\frac{1}{x}\right)}\right]}=e^{-12}$$

41. $\lim\limits_{x\to 0}\left[\dfrac{e^x-e^{\sin x}}{x-\sin x}\right],\left(\dfrac{0}{0}\text{form}\right)$

Using L'Hospital's rule three times, then

$$\lim_{x\to 0}\frac{e^x-e^{\sin x}\cdot\cos x}{1-\cos x}=\lim_{x\to 0}\frac{e^x-e^{\sin x}\cos^2 x+\sin x\cdot e^{\sin x}}{\sin x}$$

$$=\lim_{x\to 0}\frac{e^x-e^{\sin x}\cdot\cos^3 x+e^{\sin x}2\cos x\sin x+e^{\sin x}\cdot\cos x\sin x+e^{\sin x}\cdot\cos x}{\cos x}$$

$=1$

42. $\lim\limits_{x\to -1}\dfrac{x^2+3x+2}{x^2+4x+3}=\lim\limits_{x\to -1}\dfrac{x^2+2x+x+2}{x^2+3x+x+3}$

$$=\lim_{x\to -1}\frac{(x+1)(x+2)}{(x+1)(x+3)}=\lim_{x\to -1}\frac{x+2}{x+3}=\frac{1}{2}$$

43. $\lim\limits_{x\to 0}\dfrac{2}{x}\log(1+x)=\lim\limits_{x\to 0}2\log(1+x)^{\frac{1}{x}}$

$$=\lim_{x\to 0}2\log_e e=2\left\{\text{As}\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\log_e e=1\right\}$$

44. $\lim\limits_{x\to\infty}\left(\dfrac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}=\lim\limits_{x\to\infty}\left(\dfrac{3x+2-6}{3x+2}\right)^{\frac{x+1}{3}}$

$$=\lim_{x\to\infty}\left(1-\frac{6}{3x+2}\right)^{\frac{x+1}{3}}=\lim_{x\to\infty}\left[\left(1-\frac{6}{3x+2}\right)^{\frac{3x+2}{-6}}\right]^{\frac{-6}{3x+2}\cdot\frac{x+1}{3}}$$

$$=\lim_{x\to\infty}e^{\frac{-2(x+1)}{3x+2}}=e^{-2/3}\qquad\left\{\text{As}\lim_{x\to\infty}\frac{-2(x+1)}{3x+2}=\frac{-2}{3}\right\}$$

45. $\lim\limits_{x\to\infty}\dfrac{(x+1)(3x+4)}{x^2(x-8)}=\lim\limits_{x\to\infty}\left[\dfrac{x\left(1+\frac{1}{x}\right)x\left(3+\frac{4}{x}\right)}{x^3\left(1-\frac{8}{x}\right)}\right]$

$$=\lim_{x\to\infty}\left(\frac{1}{x}\frac{\left(1+\frac{1}{x}\right)\left(3+\frac{4}{x}\right)}{\left(1-\frac{8}{x}\right)}\right)=0$$

46. In closed interval of $x=0$ at right hand side $[x]=0$ and at left hand side $[x]=-1$. Also $[0]=0$. Therefore, function is defined as

$$f(x)=\begin{cases}\dfrac{\sin[x]}{[x]} & (-1\le x<0)\\ 0 & (0\le x<1)\end{cases}$$

Hence,

left hand limit $=\lim\limits_{x\to0-}f(x)=\lim\limits_{x\to0-}\dfrac{\sin[x]}{[x]}$

$$=\frac{\sin(-1)}{-1}=\sin 1$$

Right hand limit $=0$. Hence, limit does not exist.

47. $\lim\limits_{n\to\infty}\dfrac{1-(10)^n}{1+(10)^{n+1}}=\lim\limits_{x\to\infty}\dfrac{(10)^n\left[\frac{1}{(10)^n}-1\right]}{(10)^{n+1}\left(1+\frac{1}{10^{n+1}}\right)}=-\dfrac{1}{10}$

Hence, $\alpha=1$.

48. $\lim\limits_{x\to0}\dfrac{\log(1+x^3)}{\sin^3 x}=\lim\limits_{x\to0}\dfrac{3x^2/(1+x^3)}{3\sin^2 x\cos x}$

[By using L'Hospital's rule]

$$=\lim_{x\to0}\left[\frac{1}{1+x^3}\left(\frac{x}{\sin x}\right)^2\cdot\frac{1}{\cos x}\right]=\frac{1}{1+0}\cdot(1)^2\cdot\frac{1}{1}=1$$

49. $\lim\limits_{\theta\to0}\dfrac{4\theta(\tan\theta-\sin\theta)}{(1-\cos2\theta)^2}=\lim\limits_{\theta\to0}\dfrac{4\theta\sin\theta(1-\cos\theta)}{4\sin^4\theta\cos\theta}$

$$=\lim_{\theta\to0}\left(\frac{\theta}{\sin\theta}\right)\frac{2\sin^2\theta/2}{\sin^2\theta\cos\theta}$$

$$=\lim_{\theta\to0}\frac{2\sin^2\theta/2}{[2\sin(\theta/2)\cos(\theta/2)]^2}\frac{1}{\cos\theta}$$

$$=\lim_{\theta\to0}\frac{1}{2}\frac{1}{\cos^2(\theta/2)\cdot\cos\theta}=\frac{1}{2}$$

50. Applying L'Hospital's Rule, we have

$$\lim_{x\to0}\frac{27^x\log27-9^x\log9-3^x\log3}{-\frac{1}{2\sqrt{4+\cos x}}(-\sin x)}$$

$$\Rightarrow\lim_{x\to0}\frac{2(27^x\log27-9^x\log9-3^x\log3)\sqrt{4+\cos x}}{\sin x}$$

$$\Rightarrow\lim_{x\to0}\frac{2[27^x(\log27)^2-9^x(\log9)^2-3^x(\log3)^2]\sqrt{4+\cos x}+2(27^x\log27-9^x\log9-3^x\log3)\frac{1}{2\sqrt{4+\cos x}}(-\sin x)}{\cos x}$$

Applying limit, we have

$$=\frac{2[(\log27)^2-(\log9)^2-(\log3)^2]}{1}\sqrt{4+1}$$

$$=2\left[\frac{9(\log3)^2-4(\log3)^2-(\log3)^2}{1}\right]\sqrt{5}$$

$$=\sqrt{5}.8(\log3)^2$$

$$=8\sqrt{5}(\log3)^2$$

51. $\lim\limits_{n\to\infty}\dfrac{x^n}{x^n\left(1+\frac{1}{x^n}\right)}=\lim\limits_{n\to\infty}\dfrac{1}{\left(1+\frac{1}{x^n}\right)}=1$

52. $\lim\limits_{n\to\infty}\dfrac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{(2n-1)}-\frac{1}{(2n+1)}\right)\right]$

$$=\lim_{n\to\infty}\frac{1}{2}\left[1-\frac{1}{2n+1}\right]=\frac{1}{2}$$

53. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-\alpha x-\beta\right)=0$

$$\Rightarrow\lim_{x\to\infty}\frac{x^2(1-\alpha)-x(\alpha+\beta)+1-\beta}{x+1}=0$$

Since the limit of the given expression is zero, therefore degree of the polynomial in numerator must be less than denominator. Hence,

$1-\alpha=0$ and $\alpha+\beta=0$

$\Rightarrow\alpha=1$ and $\beta=-1$

54. $\lim\limits_{x\to2}\dfrac{\int_6^{f(x)}4t^3dt}{x-2}\ (0/0\text{ form})=\lim\limits_{x\to2}\dfrac{4[f(x)]^3\times f'(x)}{1}$

$=4(f(2))^3\times f'(2)=18$

55. $\lim\limits_{n\to\infty}\left[\dfrac{1}{1-n^2}+\dfrac{2}{1-n^2}+\cdots+\dfrac{n}{1-n^2}\right]$

$$\lim_{n\to\infty}\frac{\Sigma n}{1-n^2}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2+n}{1-n^2}=-\frac{1}{2}$$

56. Given limit $=\lim\limits_{n\to\infty}\dfrac{1^2+2^2+3^2+\cdots+n^2}{1+n^3}=\lim\limits_{n\to\infty}\dfrac{\Sigma n^2}{1+n^3}$

$$=\lim_{n\to\infty}\frac{1}{6}\frac{n(n+1)(2n+1)}{1+n^3}=\lim_{n\to\infty}\frac{1}{6}\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{\left(\frac{1}{n^3}+1\right)}$$

$$=\frac{1}{6}\cdot1\cdot\frac{2}{(1)}=\left(\frac{1}{3}\right)$$

57. We have

$$\lim_{n\to\infty}\frac{s_{n+1}-s_n}{\sqrt{\sum_{k=1}^{n}k}}=\lim_{n\to\infty}\frac{a_{n+1}}{\sqrt{\frac{n(n+1)}{2}}}=0$$

(Since $n\to\infty$, numerator $\to a$ while denominator $\to\infty$)

58. We have

$$a_{n+1}=\frac{4+3a_n}{3+2a_n}$$

$$\Rightarrow \lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}\frac{4+3a_n}{3+2a_n}$$

$$\Rightarrow a=\frac{4+3a}{3+2a}\Rightarrow 2a^2=4\Rightarrow a=\sqrt{2}$$

$$a\neq-\sqrt{2}$$

because each $a_n>0$, therefore $\lim_{n\to\infty} a_n = a > 0$.

59. We know that

$$\cos A\cos 2A\cos 4A\cdots\cos 2^{n-1}A=\frac{\sin 2^n A}{2^n\sin A}$$

Taking $A=\frac{x}{2^n}$, we get

$$\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\cdots\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right)=\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

Hence, $\lim_{n\to\infty}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cdots\cos\left(\frac{x}{2^{n-1}}\right)\cos\left(\frac{x}{2^n}\right)$

$$=\lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}=\lim_{n\to\infty}\frac{\sin x}{x}\cdot\frac{(x/2^n)}{\sin(x/2^n)}=\frac{\sin x}{x}$$

60. $y=\lim_{n\to\infty}\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}=\lim_{n\to\infty}\frac{1}{2}\frac{\left[1-\left(\frac{1}{2}\right)^n\right]}{\left(1-\frac{1}{2}\right)}$

$$\Rightarrow y=\lim_{n\to\infty}\left[1-\frac{1}{2^n}\right]=1-0=1$$

61.

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n}{n^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{1+2+3+\cdots+n}{n^2}\right)=\lim_{n\to\infty}\frac{\frac{n}{2}(n+1)}{n^2}$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{n+1}{n}=\frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=\frac{1}{2}$$

62. $\lim_{n\to\infty}\frac{1-n^2}{\Sigma n}=\lim_{n\to\infty}\frac{(1-n)(1+n)}{\frac{1}{2}n(n+1)}=\lim_{n\to\infty}\frac{2(1-n)}{n}$

$$=\lim_{n\to\infty}2\left(\frac{1}{n}-1\right)=2(0-1)=-2$$

63. $\lim_{n\to\infty}\frac{1-2+3-4+5-6+\cdots-2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$

$$f(x)=y=\frac{-2}{1+2}=\frac{-2}{3}$$

64. $\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+\cdots+(x+100)^{10}}{x^{10}+10^{10}}$

$$=\lim_{x\to\infty}\frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\cdots+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\frac{10^{10}}{x^{10}}\right]}=100$$

65. We have,

$$\lim_{n\to\infty}\frac{1+2+3+\cdots+n}{n^2+100}$$

$$=\lim_{n\to\infty}\frac{n(n+1)}{2(n^2+100)}=\lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)}{2n^2\left(1+\frac{100}{n^2}\right)}=\frac{1}{2}$$

66.

$$\lim_{x\to 0}\frac{\int_0^x\cos t^2dt}{x}$$

Applying L'Hospital's rule, we get

$$\lim_{x\to 0}\frac{\int_0^x\cos t^2 d}{x}=\lim_{x\to 0}\frac{\cos x^2}{1}=1$$

67. Here $f(2)=0$

$$\lim_{x\to 2-}f(x)=\lim_{h\to 0}f(2-h)=\lim_{h\to 0}|2-h-2|=0$$

$$\lim_{x\to 2+}f(x)=\lim_{h\to 0}f(2+h)=\lim_{h\to 0}|2+h-2|=0$$

Hence, it is continuous at $x=2$.

68. $f(\pi/2)=3$. Since $f(x)$ is continuous at $x=\pi/2$, so

$$\lim_{x\to\pi/2}\left(\frac{k\cos x}{\pi-2x}\right)=f\left(\frac{\pi}{2}\right)\Rightarrow\frac{k}{2}=3\Rightarrow k=6$$

69. Since limit of the function is $a+b$ as $x\to 0$, therefore to be continuous at a function, its value must be

$$a+b \text{ at } x=0\Rightarrow f(0)=a+b$$

70. For, f(x) to be continuous

$$\lim_{x\to 2}f(x)=f(2)=k$$

$$\Rightarrow k=\lim_{x\to 2}\frac{x^3+x^2-16x+20}{(x-2)^2}$$

$$=\lim_{x\to 2}\frac{(x^2-4x+4)(x+5)}{(x-2)^2}=7$$

71. Here, $\lim_{x\to 0+}f(x)=k$, $\lim_{x\to 0-}f(x)=-k$ and $f(0)=k$.

But $f(x)$ is continuous at $x=0$, therefore k must be zero.

72. $\lim_{x\to 0}f(x)=f(0)=\lim_{x\to 0}(1+x)^{1/x}=e$

73. Since $\lim\limits_{x\to 1/2} f(x) \neq f\left(\frac{1}{2}\right)$

Hence, the correct answer is option (D).

74. $f(a) = 0$

$$\lim_{x\to a-} f(x) = \lim_{x\to a-}\left(\frac{x^2}{a} - a\right) = \lim_{h\to 0}\left\{\frac{(a-h)^2}{a} - a\right\} = 0$$

and, $$\lim_{x\to a+} f(x) = \lim_{h\to 0}\left\{a - \frac{(a+h)^2}{a}\right\} = 0$$

Hence, it is continuous at $x = a$.

75. $f(0) = 0$

$\lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} e^{-1/h} = 0$ and $\lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0} e^{1/h} = \infty$

Hence, function is discontinuous at $x = 0$.

76. $f(x) = \left\{\frac{x^2 - 4x + 3}{x^2 - 1}\right\}$, for $x \neq 1$

$f(x) = 2$, for $x = 1$

$$f(1) = 2, f(1+) = \lim_{x\to 1+}\frac{x^2 - 4x + 3}{x^2 - 1} = \lim_{x\to 1+}\frac{(x-3)}{(x+1)} = -1$$

$$f(1-) = \lim_{x\to 1-}\frac{x^2 - 4x + 3}{x^2 - 1} = -1 \Rightarrow f(1) \neq f(1-)$$

Hence, the function is discontinuous at $x = 1$.

77. $f(x) = \dfrac{x+1}{(x-3)(x+4)}$. Hence, the points are 3, –4.

78. $f(0+) = f(0-) = 2$ and $f(0) = 2$

Hence, $f(x)$ is continuous at $x = 0$.

79. $\lim\limits_{x\to 0+} f(x) = x^2 \sin\frac{1}{x}$, but $-1 \leq \sin\frac{1}{x} \leq 1$ and $x \to 0$

Therefore,

$$\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x) = f(0)$$

Hence, $f(x)$ is continuous at $x = 0$.

80. $f(0-) = \lim\limits_{x\to 0-} k(2x - x^2) = 0; f(0+) = \lim\limits_{x\to 0+} \cos x = 1$

Hence, $f(0) = \cos x = 1$

Hence, no value of k can make $f(0-) = 1$.

81. $f(0) = 0;\ f(0-) = \lim\limits_{h\to 0}\dfrac{-h}{e^{-1/h} + 1} = \lim\limits_{h\to 0}\dfrac{-h}{1 + \dfrac{1}{e^{1/h}}} = 0$

$$f(0+) = \lim_{h\to 0}\frac{h}{e^{1/h} + 1} = 0$$

82. $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0}[(1+2x)^{1/2x}]^2 = e^2$

83. $\lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0} 2^{1/h} = \infty$

$$\lim_{x\to 0-} f(x) = \lim_{h\to 0} 2^{-1/h} = \lim_{h\to 0}\frac{1}{2^{1/h}} = 0$$

84. $f(0) = \lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0} x\left[\dfrac{\sin x^2}{x^2}\right] = 0$

85. Clearly from curve (Fig. 19.26) drawn of the given function $f(x)$ is discontinuous at $x = 0$.

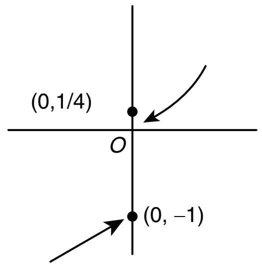


Figure 19.26

86. It is obvious that the correct answer is option (A).

87. $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} x + 1 = 2 = k$

88. (i) When $0 \leq x < 1$, $f(x)$ doesn't exist as $[x] = 0$ here.

(ii) Also $\lim\limits_{x\to 1+} f(x)$ and $\lim\limits_{x\to 1-} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at all integers and also in (0, 1).

89. $\lim\limits_{x\to 0} f(x) = \dfrac{\sin^2 ax}{(ax)^2} a^2 = a^2$ and $f(0) = 1$

Hence, $f(x)$ is discontinuous at $x = 0$, when $a \neq \pm 1$.

90. $\lim\limits_{x\to 0-} f(x) = 0,$

$$f(0) = 0, \lim_{x\to 0+} f(x) = -4$$

$f(x)$ discontinuous at $x = 0$.

and $\lim\limits_{x\to 1-} f(x) = 1$ and $\lim\limits_{x\to 1+} f(x) = 1, f(1) = 1$

Hence, $f(x)$ is continuous at $x = 1$.

Also, $\lim\limits_{x\to 2-} f(x) = 4(2)^2 - 3.2 = 10$

$f(2) = 10$ and $\lim\limits_{x\to 2+} f(x) = 3(2) + 4 = 10$

Hence, $f(x)$ is continuous at $x = 2$.

91. $\lim\limits_{x\to 0} f(x) = \sin^{-1}(0) = 0$ and $f(0) = 0$

Hence, $f(x)$ is continuous at $x = 0$.

92. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{2\sin 2x}{2x.5} = \dfrac{2}{5} = k$

93. $\lim\limits_{x\to 1+} f(x) = 0$ and $\lim\limits_{x\to 1-} f(x) = 1 + 1 = 2.$

Hence, $f(x)$ is discontinuous at $x = 1$.

94. $\lim\limits_{x\to(-1)} f(x) = -2$ and $f(-1) = -2$

95. $\lim\limits_{x\to 2-} f(x) = \dfrac{1}{2}$ and $\lim\limits_{x\to 2+} f(x) = \dfrac{1}{2}$ and $f(2) = 1$

96. Obviously $\lim\limits_{x\to b} f(x) = f(b) = 0$

97. $\lim_{x\to a-} f(x) = -1$, $\lim_{x\to a+} f(x) = 1$, $f(a) = 1$

98. $\lim_{x\to 1} f(x) = 1$, $f(1) = 2$

99. $\lim_{x\to 2-} f(x) = 3$, $\lim_{x\to 2+} f(x) = 3$ and $f(2) = 3$

100. Here, $f\left(\frac{3\pi}{4}\right) = 1$ and $\lim_{x\to 3\pi/4-} f(x) = 1$

$$\lim_{x\to 3\pi/4+} f(x) = \lim_{h\to 0} 2\sin\frac{2}{9}\left(\frac{3\pi}{4}+h\right) = 2\sin\frac{\pi}{6} = 1$$

Hence, $f(x)$ is continuous at $x = \frac{3\pi}{4}$.

101. $\lim_{x\to \pi/2-} f(x) = \frac{\pi}{2}$, $\lim_{x\to \pi/2+} f(x) = \frac{-\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

102. $\lim_{x\to 0-} f(x) = \lim_{x\to 0-}\left(\frac{2\sin^2 2x}{(2x)^2}\right)4 = 8$

$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sqrt{16+\sqrt{x}} + 4 = 8$. Hence, $a = 8$.

103. $\lim_{x\to 1-} f(x) = a - b$, $\lim_{x\to 1+} f(x) = 2 \Rightarrow a - b = 2$

All the given sets of a, b make $f(x)$ continuous at $x = 1$.

104. $\lim_{x\to 0-} f(x) = 1 + 1 = 2$ $\lim_{x\to 0+} f(x) = 0, f(0) = 2$

105. $\lim_{x\to 2} f(x) = \lim_{x\to 2} (x+2)(x^2+4) = 32$, $f(2) = 16$

106. $\lim_{x\to 1-} f(x) = 1$, $\lim_{x\to 1+} f(x) = 6$

107. $\lim_{x\to -5} f(x) = \frac{(x-2)(x+5)}{(x+5)(x-3)} = \frac{-7}{-8} = \frac{7}{8} = a$

108. By definition of continuity, we know that

$$\lim_{x\to 3+} f(x) = f(3) = \lim_{x\to 3-} f(x)$$

$$\Rightarrow \lim_{x\to 3-} f(x) = 4 \text{ or } \lim_{h\to 0} 3 - h + \lambda = 4$$

$$\Rightarrow 3 + \lambda = 4 \Rightarrow \lambda = 1$$

109. If $x \to 0$, then the value of $\sin\frac{1}{x}$ passes through $[-1, 1]$ infinitely many ways, therefore limit of the function does not exist at $x = 0$. Hence, there is no value of k for which the function is continuous at $x = 0$.

110. As we are given $f(x) = \sin x$, if $x \neq n\pi$, that is, $x \neq 0, \pi, 2\pi, \ldots$ and $f(x) = 2$ otherwise. Hence, $\lim_{x\to 0^+} g\{f(x)\} = \lim_{x\to 0^+} g\{\sin x\}$ $= \lim_{x\to 0^+} (\sin^2 x + 1) = 1$.

Similarly, $\lim_{x\to 0-} g\{f(x)\} = 1$

111. $\lim_{x\to 4^-} f(x) = \lim_{h\to 0} f(4-h) = \lim_{h\to 0} \frac{4-h-4}{|4-h-4|} + a$

$$= \lim_{h\to 0} -\frac{h}{h} + a = a - 1$$

$$\lim_{x\to 4+} f(x) = \lim_{h\to 0} f(4+h) = \lim_{h\to 0} \frac{4+h-4}{|4+h-4|} + b = b + 1$$

and $f(4) = a + b$

Since $f(x)$ is continuous at $x = 4$.

Therefore,

$$\lim_{x\to 4-} f(x) = f(4) = \lim_{x\to 4+} f(x)$$

$$\Rightarrow a - 1 = a + b = b + 1 \Rightarrow b = -1 \text{ and } a = 1$$

112. For any $x \neq 1, 2$, we find that $f(x)$ is the quotient of two polynomials and a polynomial is everywhere continuous. Therefore, $f(x)$ is continuous for all $x \neq 1, 2$. Check continuity at $x = 1, 2$.

113. Since $\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x) = f(2) = 1$.

Also it is continuous for all values of x, less than 2 and greater than 2.

114. Given function is continuous at all point in $(-\infty, 6)$ and at $x = 1$, $x = 3$ function is continuous.

If function $f(x)$ is continuous at $x = 1$, then

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) \Rightarrow 1 + \sin\frac{\pi}{2} = a + b$$

$$\Rightarrow a + b = 2 \quad (1)$$

If at $x = 3$, function is continuous, then

$$\lim_{x\to 3^-} f(3) = \lim_{x\to 3^+} f(x) \Rightarrow 3a + b = 6\tan\frac{3\pi}{12}$$

$$\Rightarrow 3a + b = 6 \quad (2)$$

From Eqs. (1) and (2), $a = 2$, $b = 0$

115. If f is continuous at $x = 0$, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \Rightarrow f(0) = \lim_{x\to 0^-} f(x)$$

$$k = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[0-h]}{[0-h]}$$

$$k = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h]}{[-h]} = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}\{-h-1\}}{\{-h-1\}}$$

$$k = \lim_{h\to 0} \frac{\cos\left(-\frac{\pi}{2}\right)}{-1} \Rightarrow k = 0$$

116. Clearly the function is defined only in the interval $[1, \infty)$ hence option (B) cannot even apply. For $x > 2$, $y = 3x - 2$ which is a straight line, hence continuous. Further $y = 4$ at $x = 2$. Hence, the function is continuous at $x = 2$ also (but not at $x = 2$ only).

117. $f(x)$ is continuous at every point of its domain, so

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow 5 \times 1 - 4 = 4 \times 1 + 3 \times b \times 1$$

$$\Rightarrow 1 = 4 + 3b \Rightarrow 3b = -3 \Rightarrow b = -1$$

118. For continuity at all $x \in R$, we must have

$$f\left(-\frac{\pi}{2}\right) = \lim_{x\to(-\pi/2)^-} (-2\sin x) = \lim_{x\to(-\pi/2)^+} (A\sin x + B)$$

$$\Rightarrow 2 = -A + B \quad (1)$$

and $$f\left(\frac{\pi}{2}\right) = \lim_{x\to(\pi/2)^-} (A\sin x + B) = \lim_{x\to(\pi/2)^+} (\cos x)$$

$$\Rightarrow 0 = A + B \quad (2)$$

From Eqs. (1) and (2), we get $A = -1$ and $B = 1$.

119. $f(5) = \lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{x^2 - 10x + 25}{x^2 - 7x + 10}$

$$= \lim_{x\to 5} \frac{(x-5)^2}{(x-2)(x-5)} = \frac{5-5}{5-2} = 0$$

120. For continuity at $x = 0$, we must have

$$f(0) = \lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0} (x+1)^{\cot x} = \lim_{x\to 0} \left[(1+x)^{\frac{1}{x}} \right]^{x\cot x}$$

$$= \lim_{x\to 0} \left[(1+x)^{\frac{1}{x}} \right]^{\lim_{x\to 0}\left(\frac{x}{\tan x}\right)} = e^1 = e$$

121. It is obvious that the correct answer is option (A).

122. It is obvious that $|x|$ is continuous for all x. Now,

$$Rf'(x) = \lim_{h\to 0} \frac{|0+h| - 0}{h} = 1$$

$$Lf'(x) = \lim_{h\to 0} \frac{|0-h| - 0}{-h} = -1$$

Hence, $f(x) = |x|$ is not differentiable at $x = 0$.

123. $f(x)$ is continuous at $x = \frac{\pi}{2}$, then

$$\lim_{x\to \pi/2} f(x) = f\left(\frac{\pi}{2}\right) \text{ or } \lambda = \lim_{x\to \pi/2} \frac{1-\sin x}{\pi - 2x}, \left(\frac{0}{0}\text{form}\right)$$

Applying L'Hospital's rule,

$$\lambda = \lim_{x\to \pi/2} \frac{-\cos x}{-2} \Rightarrow \lambda = \lim_{x\to \pi/2} \frac{\cos x}{2} = 0$$

124. Since $f(x)$ is continuous at $x = 0$, therefore

$$\lim_{x\to 0} f(x) = f(0) \Rightarrow \lim_{x\to 0} \frac{\sin \pi x}{5x} = k$$

$$\Rightarrow \lim_{x\to 0} \left(\frac{\sin \pi x}{\pi x}\right) \cdot \frac{\pi}{5} = k \Rightarrow (1) \cdot \frac{\pi}{5} = k \Rightarrow k = \frac{\pi}{5}$$

125. If $f(x)$ is continuous at $x = 0$, then

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{2 - \sqrt{x+4}}{\sin 2x}, \left(\frac{0}{0}\text{form}\right)$$

Using L'Hospital's rule,

$$f(0) = \lim_{x\to 0} \frac{\left(-\frac{1}{2\sqrt{x+4}}\right)}{2\cos 2x} = -\frac{1}{8}$$

126. At no point, function is continuous.

127. $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3} (x+3) = 6$ and $f(3) = 2(3) + k = 6 + k$

As f is continuous at $x = 3$; Therefore

$$6 + k = 6 \Rightarrow k = 0$$

128. $f(x) = \left[x^2 + e^{\frac{1}{2-x}} \right]^{-1}$ and $f(2) = k$

If $f(x)$ is continuous from right at $x = 2$, then

$$\lim_{x\to 2^+} f(x) = f(2) = k$$

$$\Rightarrow \lim_{x\to 2^+} \left[x^2 + e^{\frac{1}{2-x}} \right]^{-1} = k \Rightarrow k = \lim_{h\to 0} f(2+h)$$

$$\Rightarrow k = \lim_{h\to 0} \left[(2+h)^2 + e^{\frac{1}{2-(2+h)}} \right]^{-1}$$

$$\Rightarrow k = \lim_{h\to 0} [4 + h^2 + 4h + e^{-1/h}]^{-1}$$

$$\Rightarrow k = [4 + 0 + 0 + e^{-\infty}]^{-1} \Rightarrow k = \frac{1}{4}$$

129. By L'Hospital's rule, $\lim_{x\to 0} f(x)$ is 2. Therefore, for $f(x)$ to be continuous, the value of function should be 2.

130. $$\text{L.H.L.} = \lim_{x\to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} = k$$

$$\text{R.H.L.} = \lim_{x\to 0^+} (2x^2 + 3x - 2) = -2$$

Since it is continuous,

$$\text{L.H.L.} = \text{R.H.L.} \Rightarrow k = -2$$

131. $f(x) = \dfrac{2\cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} = \dfrac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}$

$$= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \text{ at } x = \pi,\ f(\pi) = -\tan\frac{\pi}{4} = -1$$

132. $f(x) = \begin{cases} \dfrac{1-\cos x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ continuous at $x = 0$

$$\lim_{x\to 0^+} f(x) = f(0) \Rightarrow \lim_{x\to 0} \frac{2\cdot \sin^2(x/2)}{x} = k$$

$$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 x/2}{(x/2)^2} \cdot \frac{x}{4} = k \Rightarrow k = 0$$

133. It is obvious that the correct answer is option (A).

134. Given $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\Rightarrow \lim_{x\to 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \lim_{x\to 0^+} \frac{1 - \frac{1}{e^{1/x}}}{1 + \frac{1}{e^{1/x}}} = 1$$

$$\lim_{x\to 0^-}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}=\lim_{x\to 0^+}\frac{1-\frac{1}{e^{1/x}}}{1+\frac{1}{e^{1/x}}}=\frac{1-e^{\infty}}{1+e^{\infty}}=1$$

So, $\lim_{x\to 0} f(x)$ exists at $x=0$, but at $x=0$ it is not continuous.

135. We have

$f(x) = 2x - 1$, if $x > 2$,

$f(x) = k$, If $x = 2$ and

$x^2 - 1$, if $x < 2$,

Therefore,

$$\lim_{x\to 2} f(x)=f(2)\Rightarrow \lim_{x\to 2}(2x-1)=k\Rightarrow k=3$$

136. $f(x)=\lim_{x\to 0}\left(\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\right)=f(0)$, $\quad\left(\frac{0}{0}\text{ form}\right)$

Applying L'Hospital's rule,

$$f(0)=\lim_{x\to 0}\frac{\left(2-\frac{1}{\sqrt{1-x^2}}\right)}{\left(2+\frac{1}{1+x^2}\right)}=\frac{2-1}{2+1}=\frac{1}{3}$$

Alternate solution: $f(0)=\lim_{x\to 0}\frac{2-\frac{\sin^{-1}x}{x}}{2+\frac{\tan^{-1}x}{x}}=\frac{2-1}{2+1}=\frac{1}{3}$

137. $|x|$ is continuous at $x=0$ and $\frac{|x|}{x}$ is discontinuous at $x=0$.

Hence, $f(x)=|x|+\frac{|x|}{x}$ is discontinuous at $x=0$.

138. $f(0)=\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(\frac{2^x-2^{-x}}{x}\right)=\lim_{x\to 0}\left[\frac{(2^x+2^{-x})\log_e 2}{1}\right]$

$=(2^0+2^0)\log_e 2$

$=(1+1)\log_e 2$

$=2\log_e 2=\log_e 4$

139. $f(x)=\frac{2x^2+7}{x^2(x+3)-1(x+3)}=\frac{2x^2+7}{(x^2-1)(x+3)}$

$=\frac{2x^2+7}{(x-1)(x+1)(x+3)}$

Hence, points of discontinuity are $x=1, x=-1$ and $x=-3$ only.

140. $f(x)=\begin{cases}\frac{1-|x|}{1+x}, & x\neq -1\\ 1, & x=-1\end{cases}$ and $f(x)=\begin{cases}1 & x<0\\ \frac{1-x}{1+x}, & x\geq 0\end{cases}$

$$f([2x])=\begin{cases}1, & x<0\\ \frac{1-[2x]}{1+[2x]}, & x>0\end{cases}\Rightarrow f([2x])=\begin{cases}1, & x<0\\ 1, & 0\le x<\frac{1}{2}\\ 0, & \frac{1}{2}\le x\le 1\\ -\frac{1}{3}, & 1\le x<\frac{3}{2}\end{cases}$$

Therefore, $f([2x])$, for all values of x where $x<\frac{1}{2}$ a continuous function and for $x=\frac{1}{2}$ and $x=1$, $f(x)$ be a discontinuous function.

141. $f(x)=\begin{cases}\frac{1-\cos 4x}{8x^2}, & x\neq 0\\ k, & x=0\end{cases}$

If $f(x)$ is a continuous function at point $x=0$, then

$\lim_{x\to 0+} f(x)=\lim_{x\to 0-}[f(x)]=\lim_{x\to 0}[f(x)]=\lim_{h\to 0-}[f(0+h)]$

$=\lim_{h\to 0}[f(h)]=\lim_{h\to 0}\frac{1-\cos 4h}{8h^2}=\lim_{h\to 0}\frac{\sin^2 2h}{4h^2}$

$=\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2=(1)^2=1$

$=\lim_{x\to 0^-} f(x)=\lim_{h\to 0}[f(0-h)]=\lim_{h\to 0}[f(-h)]=\lim_{h\to 0}\frac{1-\cos 4(-h)}{8(-h)^2}$

$=\lim_{h\to 0}\frac{1-\cos 4h}{8h^2}=1$

$\Rightarrow f(0)=1\Rightarrow k=1$

142. $f(x)=\begin{cases}e^x; & x\le 0\\ 1-x; & 0<x\le 1\\ x-1; & x>1\end{cases}$

$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{1-h-1}{h}=-1$

$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{e^{-h}-1}{-h}=1$

So, it is not differentiable at $x=0$.
Similarly, it is not differentiable at $x=1$.
But it is continuous at $x=0, 1$.

143. Statement (D) is true, because differentiable function is always continuous.

144. As $Lf'(2)\neq Rf'(2)$, hence the correct answer is option (D).

145. $f(0+0)=\lim_{h\to 0} f(x)=\lim_{h\to 0} f(0+h)$

$=\lim_{h\to 0}(0+h)\frac{e^{1/0+h}-e^{-1/0+h}}{e^{1/0+h}+e^{-1/0+h}}=\lim_{h\to 0} h\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}=0$

and $f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}-h\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}=0$

and $f(0)=0$; hence, $f(0+0)=f(0-0)=f(0)$
Hence, f is continuous at $x=0$.
At remaining points $f(x)$ is obviously continuous.
Thus, it is everywhere continuous. Again,

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{h\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}}{-h}=-1$$

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\dfrac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}}{h} = 1$$

Since $Lf'(0) \neq Rf'(0)$.
Hence, f is not differentiable at $x = 0$.

146. $$\lim_{x\to 3} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0}|3-h-3| = 0$$
$$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0}|3+h-3| = 0$$

Since
$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

Hence, f is continuous at $x = 3$. (See Fig. 19.27.)

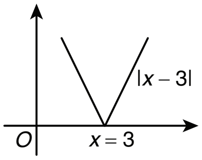


Figure 19.27

Now

$$Lf'(3) = \lim_{h\to 0}\frac{f(3-h)-f(3)}{-h} = \lim_{h\to 0}\frac{|3-h-3|-0}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1$$

$$Rf'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{|3+h-3|-0}{h} = 1$$

Since $Lf'(3) \neq Rf'(3)$. Hence f is not differentiable at $x = 3$.

Trick: Can be seen by graph it is continuous but tangent is not defined at $x = 3$.

147. $x \le x^2 \Rightarrow x(1-x) \le 0 \Rightarrow x(x-1) \ge 0$

$$\Rightarrow x \le 0 \text{ or } x \ge 1; \text{ therefore } h(x) = \begin{cases} x: & x \le 0 \\ x^2: & 0 < x < 1 \\ x: & x \ge 1 \end{cases}$$

$h(x)$ is continuous for every x but not differentiable at $x = 0$ and 1. Also

$$h'(x) = \begin{cases} 1 & x < 0 \\ \text{not exists} & x = 0 \\ 2x & 0 < x < 1 \\ \text{not exists} & x = 1 \\ 1 & x > 1 \end{cases}$$

Therefore, $h'(x) = 1$ for all $x > 1$.

148. It is obvious that the correct answer is option (D).

149. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & 1 \le x \le 2 \end{cases}$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h) = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = 1$$

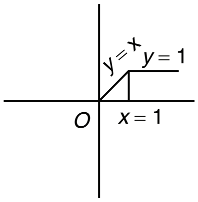


Figure 19.28

Hence, function is continuous in (0, 2).
Now

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}(0+h) = 0 = f(0)$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0}(2-h) = 1 = f(2)$$

Hence, function is continuous in [0, 2]
Clearly, from graph (Fig. 19.28) it is not differentiable at $x = 1$.

150. Since this function is continuous at $x = 0$.
Now for differentiability

$$f(x) = |x| = |0| \text{ and } f(0+h) = f(h) = |h|$$

Hence,
$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{|h|}{h} = -1$$

and
$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{|h|}{h} = 1$$

Therefore, it is continuous and non-differentiable.

151. We have $f(x) = \begin{cases} \dfrac{x^2}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} \dfrac{x^2}{x} = x, & x > 0 \\ 0, & x = 0 \\ \dfrac{x^2}{-x} = -x, & x < 0 \end{cases}$

We have
$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} x = 0$ and $f(0) = 0$

So, $f(x)$ is continuous at $x = 0$.
Also $f(x)$ is continuous for all other values of x.
Hence, $f(x)$ is continuous everywhere. Clearly, $Lf'(0) = -1$ and $Rf'(0) = 1$. Therefore $f(x)$ is not differentiable at $x = 0$.

152. We have

$$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{[(1+h)^3-1]-0}{h} = 3$$

$$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{[(1-h)-1]-0}{-h} = 1$$

Hence,
$$Rf'(1) \neq Lf'(1) \Rightarrow f(x) \text{ is not differentiable at } x = 1$$

Now,
$$f(1+0) = \lim_{h\to 0} f(1+h) = 0$$

and
$$f(1-0) = \lim_{h\to 0} f(1-h) = 0$$

Hence,
$$f(1+0) = f(1-0) = f(0) \Rightarrow f(x) \text{ is continuous at } x = 1$$

Hence, at $x = 1$, $f(x)$ is continuous and not differentiable.

153. Here, when $-1 \le x \le 1$, $0 \le x \sin \pi x < 1$. So,

$$f(x) = [x \sin \pi x] = 0 \text{ for } -1 \le x \le 1,$$

That is, $f(x)$ is constant function (equal to zero) in $[-1, 1]$.
Therefore, $f(x)$ is differentiable in $(-1, 1)$.

154. Since $|x-3| = x-3$, if $x \ge 3$;

$$|x-3| = -x+3, \text{ if } x < 3$$

Hence, the given function can be defined as

$$f(x) = \begin{cases} \dfrac{1}{4}x^2 - \dfrac{3}{2}x + \dfrac{13}{4}, & x < 1 \\ 3-x, & 1 \le x < 3 \\ x-3, & x \ge 3 \end{cases}$$

Now proceed to check the continuity and differentiability at $x = 1$.

$$\lim_{x\to 1} f(x) = f(1) = 2$$

and

$$\lim_{x\to 1} f(x) = -1$$

So, $f(x)$ is continuous and differentiable at $x = 1$.
Also,

$$\lim_{x\to 3} f(x) = f(3) = 0$$

So, $f(x)$ is also continuous at $x = 3$.

155. Given $f(x)$ is differentiable at $x = 0$. Hence, $f(x)$ will be continuous at $x = 0$. Therefore,

$$\lim_{x\to 0-} (e^x + ax) = \lim_{x\to 0+} b(x-1)^2$$

$$\Rightarrow e^0 + a\times 0 = b(0-1)^2 \Rightarrow b = 1 \quad (1)$$

But $f(x)$ is differentiable at $x = 0$, then

$$Lf'(x) = Rf'(x) \Rightarrow \frac{d}{dx}(e^x + ax) = \frac{d}{dx} b(x-1)^2$$

$$\Rightarrow e^x + a = 2b(x-1)$$

At $x = 0$,

$$e^0 + a = -2b \Rightarrow a + 1 = -2b \Rightarrow a = -3$$

$$\Rightarrow (a, b) = (-3, 1)$$

156. It can be easily seen from the graph (Fig. 19.29) of $f(x) = |\sin x|$ that it is everywhere continuous but not differentiable at integer multiples of π and at $x = 0$.

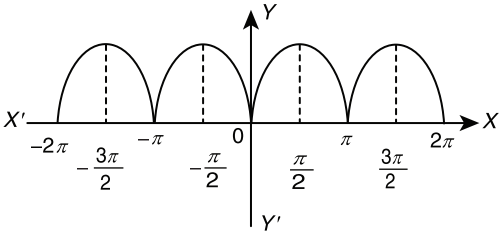


Figure 19.29

157. We have, $f(x) = \begin{cases} e^{-x}, & x \ge 0 \\ e^x, & x < 0 \end{cases}$

Now

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = 1,\ \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f(x)e^{-x} = 1$$

Also, $f(0) = e^0 = 1$. So, $f(x)$ is continuous for all x.

$$(\text{LHD at } x = 0) = \left[\frac{d}{dx}(e^x)\right]_{x=0} = 1$$

$$(\text{RHD at } x = 0) = \left[\frac{d}{dx}(e^{-x})\right]_{x=0} = -1$$

So, $f(x)$ is not differentiable at $x = 0$.

Hence, $f(x) = e^{-|x|}$ is everywhere continuous but not differentiable at $x = 0$.

158. $\lim_{h\to 0^-} 1 + (2-h) = 3,\ \lim_{h\to 0^+} 5 - (2+h) = 3, f(2) = 3$

Hence, f is continuous at $x = 2$.
Now,

$$Rf'(x) = \lim_{h\to 0} \frac{5-(2+h)-3}{h} = -1$$

$$Lf'(x) = \lim_{h\to 0} \frac{1+(2-h)-3}{-h} = 1$$

Since, $Rf'(x) \ne Lf'(x)$; therefore, f is not differentiable at $x = 2$.

159. $f'(k-0) = \lim_{h\to 0} \frac{[k-h]\sin\pi(k-h) - [k]\sin\pi k}{-h}$

$$= \lim_{h\to 0} \frac{(-1)^{k-1}(k-1)\sin\pi h - k\times 0}{-h}$$

$$= \lim_{h\to 0} \frac{(-1)^{k-1}(k-1)\sin\pi h}{-h} = (-1)^k\cdot(k-1)\pi$$

160. $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \ge 0 \end{cases};$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = 0$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}(0+h)^2 = 0$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(0)$$

Hence, $f(x)$ is continuous function at $x = 0$.

$$Lf'(x) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-0}{-h} = \lim_{h\to 0} \frac{0-0}{-h} = 0$$

$$Rf'(x) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\Rightarrow Lf'(x) = Rf'(x)$

Hence, $f(x)$ is differentiable at $x = 0$.

Now

$$f'(x) = \begin{cases} 0, & x < 0 \\ 2x, & x \ge 0 \end{cases};$$

$$\lim_{x\to 0^-} f'(x) = \lim_{h\to 0} f'(0-h) = 0$$

and

$$\lim_{x\to 0^+} f'(x) = \lim_{h\to 0} f'(0+h) = \lim_{h\to 0} 2(0+h) = 0$$

$$\Rightarrow \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 0$$

Hence, $f'(x)$ is continuous function at $x = 0$.

$$Lf''(x) = \lim_{h\to 0^-} \frac{f'(x)-f'(0)}{x-0} = \lim_{h\to 0} \frac{f'(0-h)-f'(0)}{-h}$$

$$= \lim_{h\to 0} \frac{0-0}{-h} = 0$$

$$Rf''(x) = \lim_{h\to 0^+} \frac{f'(x)-f'(0)}{x-0} = \lim_{h\to 0} \frac{f'(0+h)-f'(0)}{h}$$

$$= \lim_{h\to 0} \frac{2(0+h)-0}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$

$$\Rightarrow Lf''(x) \ne Rf''(x)$$

Hence, $f'(x)$ is not differentiable at $x = 0$.

161. A continuous function may or may not be differentiable. So, (B) is not true.

162. $\lim\limits_{x\to 0} f(x) = x^2 \sin\left(\dfrac{1}{x}\right)$, but $-1 \le \sin\left(\dfrac{1}{x}\right) \le 1$ and $x \to 0$

Hence,

$$\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x) = f(0)$$

Therefore, $f(x)$ is continuous at $x = 0$. Also, the function $f(x) = x^2 \sin\dfrac{1}{x}$ is differentiable because

$$Rf'(x) = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h} = 0,\ Lf'(x) = \lim_{h\to 0} \frac{h^2 \sin(1/-h)}{-h} = 0$$

163. By definition,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{2(1+h)-5} - \left(\dfrac{-1}{3}\right)}{h} = \lim_{h\to 0} \frac{\left(\dfrac{1}{2h-3} + \dfrac{1}{3}\right)}{h}$$

$$= \lim_{h\to 0}\left[\frac{3+2h-3}{3h(2h-3)}\right] = \lim_{h\to 0}\left[\frac{2h}{3h(2h-3)}\right]$$

$$= \lim_{h\to 0} \frac{2}{3(2h-3)} = \frac{2}{3(-3)} = \frac{-2}{9}$$

164. Let $x < 0$. Then

$$|x| = -x \Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}$$

$$\Rightarrow [f'(x)]_{x=0} = 1$$

Again

$$x > 0 \Rightarrow |x| = x$$

$$f'(x) = \frac{d}{dx}\left(\frac{x}{1+x}\right) = \frac{1}{(1+x)^2} \Rightarrow [f'(x)]_{x=0} = 1$$

$$\Rightarrow f'(0) = 1$$

165. $Lf'(1) = \lim\limits_{h\to 0} \dfrac{f(1-h) - f(1)}{-h}$

$$= \lim_{h\to 0} \frac{m(1-h)^2 - m}{-h} = \lim_{h\to 0} \frac{m[1+h^2-2h-1]}{-h}$$

$$= \lim_{h\to 0} m(2-h) = 2m$$

and $Rf'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0} \dfrac{2(1+h)-m}{h}$

For differentiability, $Lf'(1) = Rf'(1)$.

But for any value of m, $Rf'(1) = Lf'(1)$ not possible.

166. $(gof)(x) = g[f(x)] = g[1-\cos x] = e^{1-\cos x}$, for $x \le 0$
$(gof)'(x) = e^{1-\cos x} \cdot \sin x$, for $x \le 0$
$(gof)'(0) = 0$

167. $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$; as function is differentiable so it is continuous as it is given that $\lim\limits_{h\to 0} \dfrac{f(1+h)}{h} = 5$ and hence $f(1) = 0$. Therefore,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)}{h} = 5$$

168. $\lim\limits_{x\to y} \left|\dfrac{f(x)-f(y)}{x-y}\right| \le \lim\limits_{x\to y} |x-y| \text{ or } |f'(x)| \le 0$

$\Rightarrow f'(x) = 0 \Rightarrow f(x)$ is constant

As $f(0) = 0$, hence $f(1) = 0$.

169. As $f(1) = -2$ and $f'(x) \ge 2\ \forall x \in [1, 6]$

Applying Lagrange's mean value theorem,

$$\frac{f(6)-f(1)}{5} = f'(c) \ge 2$$

$$\Rightarrow f(6) \ge 10 + f(1) \Rightarrow f(6) \ge 10 - 2 \Rightarrow f(6) \ge 8$$

170. $f(x) = \begin{cases} |x|-1, & |x|-1 \ge 0 \\ -|x|+1, & |x|-1 < 0 \end{cases}$

$$f(x) = \begin{cases} |x|-1, & x \le -1 \text{ or } x \ge 1 \\ -|x|+1, & -1 < x < 1 \end{cases}$$

$$f(x) = \begin{cases} -x-1, & x \le -1 \\ x+1, & -1 < x < 0 \\ -x+1, & 0 \le x < 1 \\ x-1, & x \ge 1 \end{cases}$$

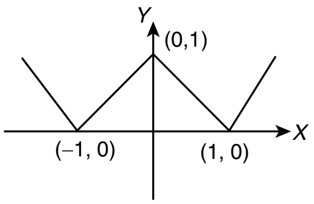


Figure 19.30

From the graph (Fig. 19.30), it is clear that $f(x)$ is not differentiable at $x = -1$, 0 and 1.

171. Let a function be $g(x) = f(x) - x^2$. Then $g(x)$ has at least 3 real roots which are $x = 1, 2, 3$, so $g'(x)$ has at least 2 real roots in $x \in (1, 3)$ and $g''(x)$ has at least 1 real roots in $x \in (1, 3)$
Therefore, $f''(x) = 2$ for at least one $x \in (1, 3)$.

172. $f(1) = f\left(\dfrac{1}{2}\right) = f\left(\dfrac{1}{3}\right) = \cdots = \lim\limits_{n\to\infty} f\left(\dfrac{1}{n}\right) = 0$

Since there are infinitely many points in $x \in (0,1)$ where

$$f(x) = 0 \text{ and } \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0$$

$$\Rightarrow f(0) = 0$$

And since there are infinitely many points in the neighbourhood of $x = 0$. Such that $f(x)$ remains constant in the neighbourhood of $x = 0$. Therefore,

$$f'(0) = 0$$

173. $f(1)=-3; f'(x)\geq 9$ for all $x\in(1,5)$; hence, $f(5)\geq 36$.

174. We know,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Given $f(x+y)=f(x)\,f(y)$, so

$$f'(x)=\lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{f(x)(f(h)-1)}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{f(x)(1+\sin(3h)g(h)-1)}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}3f(x)g(h)=3f(x)g(0)$$

175. $f(x)=\begin{cases}1 & \forall x<0\\ 1+\sin x, & \forall 0\leq x\leq\dfrac{\pi}{2}\end{cases}$

$$\Rightarrow f'(x)=\begin{cases}0, & \forall\ x<0\ \text{(LHD)}\\ \cos x, & 0\leq x\leq\pi/2,\ \text{(RHD)}\end{cases}$$

$Lf'(0)=0$, $Rf'(0)=1$
So, derivative does not exist at $x=0$.

176. $f(x)=x^2-2x+4; f'(x)=2x-2$
At $x=c$, $f'(c)=2c-2$
$f(5)=5^2-2(5)+4=19; f(1)=1^2-2(1)+4=3$

$$\frac{f(5)-f(1)}{5-1}=f(c)\Rightarrow\frac{19-3}{5-1}=2c-2\Rightarrow\frac{16}{4}=2c-2$$

$\Rightarrow 4=2c-2\Rightarrow 2c=6$ or $c=3$

177. We have $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{f(x)+f(h)-f(x)}{h}$

[As $f(x+y)=f(x)+f(x)$]

$$=\lim_{h\to 0}\frac{f(h)}{h}=\lim_{h\to 0}\frac{h^2g(h)}{h}=0.\ g(0)=0$$

[As g is continuous therefore $\lim\limits_{h\to 0}g(h)=g(0)$]

178. Since function $|x|$ is not differentiable at $x=0$. So,

$$|x^2-3x+2|=|(x-1)(x-2)|$$

Hence, the function is not differentiable at $x=1$ and 2.
Now $f(x)=(x^2-1)|x^2-3x+2|+\cos(|x|)$ is not differentiable at $x=2$.
For $1<x<2$, $f(x)=-(x^2-1)(x^2-3x+2)+\cos x$
For $2<x<3$, $f(x)=+(x^2-1)(x^2-3x+2)+\cos x$
$Lf'(x)=-(x^2-1)(2x-3)-2x(x^2-3x+2)-\sin x$
$Lf'(2)=-3-\sin 2$
$Rf'(x)=(x^2-1)(2x-3)+2x(x^2-3x+2)-\sin x$
$Rf'(2)=(4-1)(4-3)+0-\sin 2=3-\sin 2$
Hence, $Lf'(2)\neq Rf'(2)$.

179. Since $\dfrac{dy}{dx}=\cos x$ which is defined at $x=0$ and no other differential coefficient is defined at $x=0$.

180. It is a fundamental concept. Hence, the correct answer is option (C).

181. If $f(x)=\begin{cases}x+2, & -1<x<3\\ 5, & x=3\\ 8-x, & x>3\end{cases}$ and $f(3)=5$

$$\text{L.H.D}=\lim_{x\to 3^-}\frac{f(x)-f(3)}{x-3}=\lim_{h\to 0}\frac{f(3-h)-f(3)}{-h}$$

$$=\lim_{h\to 0}\frac{(3-h+2)-5}{-h}=\lim_{h\to 0}\frac{-h}{-h}=1$$

$$\text{R.H.D}=\lim_{x\to 3^+}\frac{f(x)-f(3)}{x-3}=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

$$=\lim_{h\to 0}\frac{8-(3+h)-5}{h}=\lim_{h\to 0}\frac{-h}{h}=-1$$

L.H.D $\neq$ R.H.D $f(x)$ is not differentiable.

182. $f(x)=\begin{cases}x, & 0\leq x\leq 1\\ 2x-1, & x>1\end{cases}$

$$\lim_{x\to 1^-}f(x)=\lim_{h\to 0}f(1-h)=\lim_{h\to 0}(1-h)=1$$

$$\lim_{x\to 1^+}f(x)=\lim_{h\to 0}f(1+h)=\lim_{h\to 0}2(1+h)-1=1$$

Since,

$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}f(x)=1$$

Hence, function is continuous at $x=1$

$$Lf'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{(1-h)-1}{-h}=1$$

$$Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{-h}=\lim_{h\to 0}\frac{2+2h-1-1}{h}=2$$

Therefore, $Lf'(1)\neq Rf'(1)$
Hence, function is not differentiable at $x=1$.

183. $f(x)$ possesses derivative at $x=0$, so it is both continuous and differentiable at $x=0$. Now $f(0+0)=0$, $f(0-0)=b$, $f(0)=b$.
Therefore,

$$b=0$$

Also $Rf'(0)=0$, $Lf'(0)=0, \forall a\in R$.
Therefore, $f'(0)=0$ if $b=0$.

184. Let $h(x)=x, x\in(-\infty,\infty)$; $g(x)=1+|x|, x\in(-\infty,\infty)$
Here h is differentiable in $(-\infty,\infty)$ but $|x|$ is not differentiable at $x=0$
Therefore, g is differentiable in $(-\infty,0)\cup(0,\infty)$ and $g(x)\neq 0$, $\forall x\in(-\infty,\infty)$, therefore $f(x)=\dfrac{h(x)}{g(x)}=\dfrac{x}{1+|x|}$

It is differentiable in $(-\infty,0)\cup(0,\infty)$ for $x=0$

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h-0}=\lim_{h\to 0}\frac{\dfrac{h}{1+|h|}-0}{h}=\lim_{h\to 0}\frac{1}{1+|h|}=1$$

Therefore f is differentiable at $x=0$, so f is differentiable in $(-\infty,\infty)$.

185. $$y'=\frac{1}{\sqrt{1-\left(\dfrac{2x}{1+x^2}\right)^2}}\cdot\frac{2(1+x^2)-4x^2}{(1+x^2)^2}=\frac{2(1-x^2)}{\sqrt{(1-x^2)^2(1+x^2)}}$$

$$\Rightarrow y' = \begin{cases} \dfrac{2}{1+x^2} & \text{for } |x|<1 \\ \dfrac{-2}{1+x^2} & \text{for } |x|>1 \end{cases}$$

Hence for $|x| = 1$, the derivative does not exist.

186. Since the function is defined for $x \geq 0$, that is, not defined for $x < 0$. Hence, the function neither continuous nor differentiable at $x = 0$.

187. Function $f(x) = |x - 0.5| + |x - 1| + \tan x$ does not have a derivative at the points $x = 0.5, 1, \dfrac{\pi}{2} \in (0, 2)$.

188. Applying L'Hospital's Rule,

$$\lim_{x\to 5} \frac{\lambda x^{\lambda-1}}{1} = 500$$

$$\Rightarrow \lambda \cdot 5^{\lambda-1} = 500$$

$$\Rightarrow \lambda = 4$$

189. L.H.L. $= \lim_{h\to 0} f(0-h) = \lim_{h\to 0}(-h) = 0$

R.H.L. $= \lim_{h\to 0} f(0+h) = \lim_{h\to 0}(h^2) = 0$

But $f(0) = 1$

So, limit does not exist.

190. $f(x) = [\alpha + \beta \sin x]\ x \in (0, \pi)$

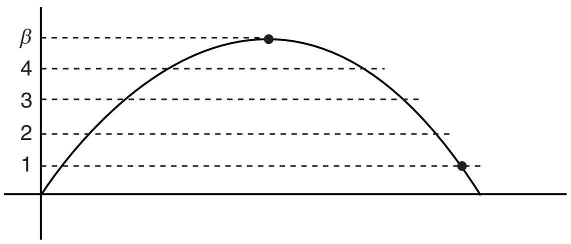


Figure 19.31

Total number of points of discontinuous are $2\beta - 1$ (Fig. 19.31).

191. $[x + [x]] = [x] + [x] = 2[x]$

192. $$A = \lim_{x\to 0}\left(\sin\frac{x}{m} + \cos\frac{3x}{m}\right)^{2m/x}$$

$$\ln A = \lim_{x\to 0} \frac{2m}{x} \ln\left(\sin\frac{x}{m} + \cos\frac{3x}{m}\right) \qquad \left(\frac{0}{0}\text{ form}\right)$$

Applying L'Hospital's rule

$$\lim_{x\to 0} \frac{2m}{\left(\sin\dfrac{x}{m} + \cos\dfrac{3x}{m}\right)} \cdot \frac{1}{m}\left(\cos\frac{x}{m} - 3\sin\frac{3x}{m}\right) = 2\frac{1}{1+0}(1-0) = 2$$

Therefore, $A = e^2$.

193. Drawing the graph, we can see that function is continuous at all points. (See Fig. 19.32) Hence, the correct answer is option (A).

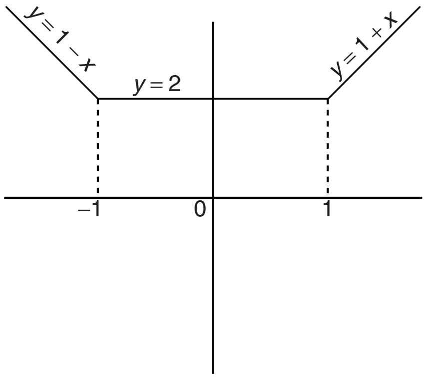


Figure 19.32

194. $$\lim_{x\to 0} \frac{1}{x} \log\left(\frac{1+ax}{1-bx}\right) = \lim_{x\to 0} \frac{1}{x}[\log(1+ax) - \log(1-bx)]$$

$$= \lim_{x\to 0} \frac{1}{x}\left[\left(ax - \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \cdots\right)\right] - \left[\left(-bx - \frac{(bx)^2}{2!} - \frac{(bx)^3}{3!} - \cdots\right)\right]$$

$$\lim_{x\to 0} \frac{1}{x}\left[(a+b)x - \frac{(a^2-b^2)x^2}{2!} + \frac{(a^3+b^3)x^3}{3!} + \cdots\right]$$

$$\lim_{x\to 0}\left[(a+b) - \frac{(a^2-b^2)x}{2!} + \frac{(a^3+b^3)}{3!}x^2 + \cdots\right] = (a+b)$$

195. $$\lim_{n\to\infty}(3^n + 5^n + 7^n)^{1/n} = \lim_{n\to\infty} 7\left[1 + \left(\frac{3}{7}\right)^n + \left(\frac{5}{7}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} 7\left[1 + \left(\frac{3}{7}\right)^n + \left(\frac{5}{7}\right)^n\right]^{1/n} = 7 \times e^0 = 7$$

196. $$f(x) = \frac{|x|}{x}$$

$$\Rightarrow f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

So, this function is discontinuous at only 1 point.

197. $\lim_{x\to 2^+}(-1)^{[x]} = (-1)^2 = 1,\ \lim_{x\to 2^-}(-1)^{[x]} = (-1)^1 = -1$

$\lim_{x\to 2}(-1)^{[x]}$ does not exist.

198. $$\lim_{x\to 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \qquad \left(\frac{0}{0}\right)\text{form}$$

Applying L'Hospital's form, we get

$$\lim_{x\to 0} \frac{\cos x + \sin x + \dfrac{1}{1-x}(-1)}{3x^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-\sin x + \cos x + \dfrac{1}{(x-1)^2}}{6x}\left(\frac{0}{0}\right)\text{form}$$

$$= \lim_{x\to 0}\frac{-\cos x-\sin x-\dfrac{2}{(x-1)^3}}{6}$$

$$= \frac{-3}{6} = \frac{-1}{2}$$

199. $\lim_{x\to 2}\dfrac{\sin(e^{x-2}-1)}{\log(x-1)}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 2}\frac{\cos(e^{x-2}-1)\cdot e^{x-2}}{\dfrac{1}{x-1}} = 1$$

200. $\lim_{x\to 0}\dfrac{x\cos x-\log(1+x)}{x^2}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{\cos x - x\sin x-\dfrac{1}{1+x}}{2x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-\sin x-\sin x - x\cos x+\dfrac{1}{(1+x)^2}}{2} = \frac{1}{2}$$

201. $\lim_{x\to 0}\dfrac{|x+1|+|x-1|-2}{x}$

$$= \lim_{x\to 0}\frac{x+1-(x-1)-2}{x}$$

$$= \lim_{x\to 0}\frac{2-2}{x} = 0$$

202. $\lim_{x\to 0}\dfrac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$

$$= \lim_{x\to 0}\frac{a^{\sin x}(a^{\tan x-\sin x}-1)}{\tan x-\sin x}$$

$$= a^0\cdot\log a$$

$$= \log a$$

203. $\lim_{n\to\infty}\dfrac{1+e^{\frac{1}{n}}+e^{\frac{2}{n}}+\cdots+e^{\frac{n-1}{n}}}{n}$

$$= \sum_{r=0}^{n-1} e^{\frac{r}{n}}\cdot\frac{1}{n}$$

$$= \int_0^1 e^x\,dx$$

$$= [e^x]_0^1$$

$$= e-1$$

204. $\lim_{x\to -1^+}\dfrac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}}$

$$= \lim_{h\to 0}\frac{\sqrt{\pi}-\sqrt{\cos^{-1}(-1+h)}}{\sqrt{(-1+h)+1}}$$

Applying L'Hospital's rule

$$\lim_{h\to 0}\frac{\dfrac{1}{2\sqrt{\cos^{-1}(-1+h)}\cdot\sqrt{h(2-h)}}}{\dfrac{1}{2\sqrt{h}}} = \frac{1}{\sqrt{2\pi}}$$

205. The form of the limit is 1^∞, so

$$\lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{1/x} = \lim_{x\to 0} e^{\left[\frac{f(1+x)}{f(1)}-1\right]\frac{1}{x}} = e^{\lim\limits_{x\to 0}\left[\frac{f(1+x)-f(1)}{f(1)\cdot(x)}\right]}$$

$$= e^{\frac{f'(1)}{f(1)}} = e^{\frac{6}{3}} = e^2$$

206. LHL

$\lim_{x\to 0-} g[f(x)]$

$\Rightarrow \lim_{h\to 0} g[f(0-h)]$

$\Rightarrow \lim_{h\to 0} g[\sin(0-h)]$

$\Rightarrow \lim_{h\to 0}[\sin(0-h)]^2+1$

$= 1$

RHL

$\lim_{x\to 0+} g[f(x)]$

$= \lim_{h\to 0} g[f(0+h)]$

$= \lim_{h\to 0} g[\sin(0+h)]$

$= \lim_{h\to 0}[\sin(0+h)]^2+1$

$= 1$

Therefore, LHL = RHL = 1.

207. LHL

$\lim_{h\to 0}\sin\dfrac{\pi}{2}(1-h)$

$= 1$

RHL

$\lim_{h\to 0}|2(1+h)-3|[1+h]$

$= |-1|\times 1$

$= 1$

Function is continuous but not differentiable at $x = 1$.

208. $\lim_{x\to 0}\dfrac{(\cos x-1)(\cos x-e^x)}{x^n} = \lim_{x\to 0}\dfrac{2\sin^2(x/2)(e^x-1+1-\cos x)}{x^n}$

$$\lim_{x\to 0}\frac{2\sin^2(x/2)(e^x-1+2\sin^2(x/2)]}{x^n}$$

$$= \lim_{x\to 0}\frac{1}{2}\left[\frac{\sin(x/2)}{x/2}\right]^2\frac{(e^x-1+2\sin^2(x/2)]}{x^{n-2}}$$

It is non-zero, if

$$n-2 = 1 \Rightarrow n = 3$$

Alternative method:

$$\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$$

$$= \lim_{x\to 0}\frac{\left(-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\cdots\right)(-x-x^2+\cdots)}{x^n} \text{ is non-zero}$$

$\Rightarrow n = 3$

209. $\lim_{n\to\infty}\dfrac{\log n}{\log(n-1)}\cdot\dfrac{\log(n+1)}{\log n}\dfrac{\log(n+2)}{\log(n+1)}\cdots\dfrac{\log n^k}{\log n^k-1}$

$$\Rightarrow \lim_{n\to\infty}\frac{\log n^k}{\log(n-1)} = k\lim_{n\to\infty}\frac{\log n}{\log(n-1)} = k$$

210. Here,

$$f(x)=\int_{-1}^{x}|t|\,dt=\left[\frac{t|t|}{2}\right]_{-1}^{x}=\frac{x|x|}{2}+\frac{1}{2}$$

$$\Rightarrow f'(x)=\frac{1}{2}(|x|+|x|)=|x|$$

Both f and f' are continuous for $x+1>0$.

211. $f'(x_0)=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}$

$$=\lim_{x\to x_0}\frac{(x-x_0)\phi(x)-0}{x-x_0}\quad\begin{bmatrix}f(x)=(x-x_0)\,\phi(x)\\ f(x_0)=0\end{bmatrix}$$

212. $f(x)=\{|x|-|x-1|\}^2$

$\Rightarrow f(x)=x^2+(x-1)^2-2|x|\,|x-1|$

$$\Rightarrow f(x)=\begin{cases}x^2+(x-1)^2-2(x)(x-1), & x\le 0\\ x^2+(x-1)^2+2x(x-1), & 0\le x\le 1\\ x^2+(x-1)^2-2x(x-1), & x\ge 1\end{cases}$$

$$\Rightarrow f(x)=\begin{cases}\cancel{x^2}+\cancel{x^2}-\cancel{2x}+1-\cancel{2x^2}+\cancel{2x}, & x\le 0\\ x^2+(x-1)^2+2x^2-2x, & 0\le x\le 1\\ \cancel{x^2}+\cancel{x^2}-\cancel{2x}+1-\cancel{2x^2}+\cancel{2x}, & x\ge 1\end{cases}$$

$$f(x)=\begin{cases}1 & x\le 0\\ 4x^2-4x+1 & 0\le x\le 1\\ 1 & x\ge 1\end{cases}$$

So, $f'(x)=\begin{cases}0 \text{ for } x<0 \text{ and for } x>1\\ 4(2x-1) \text{ for } 0<x<1\end{cases}$

213. $\lim\limits_{x\to\alpha}(1+ax^2+bx+c)^{1/(x-\alpha)}$ [Since $ax^2+bx+c=a(x-\alpha)(x-\beta)$]

$$=\lim_{x\to\alpha}[1+a(x-\alpha)(x-\beta)]^{\frac{a(x-\beta)}{a(x-\alpha)(x-\beta)}}=e^{\lim\limits_{x\to\alpha}a(x-\beta)}=e^{a(\alpha-\beta)}$$

214. $\lim\limits_{x\to 0}\dfrac{f(x^2)-f(x)}{f(x)-f(0)}\left(\dfrac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 0}\frac{2xf'(x^2)-f'(x)}{f'(x)}=0-1=-1$$

215. The function f is clearly continuous for $|x|>1$

$$\lim_{x\to 1^+}f(x)=1,\ \lim_{x\to 1^-}f(x)=\frac{1}{2^2}=\frac{1}{4},\ \lim_{x\to -1^+}f(x)=1,\ \lim_{x\to(-1)^-}f(x)=\frac{1}{4}$$

Also, $\lim\limits_{x\to\frac{1}{n}^+}f(x)=\dfrac{1}{n^2},\ \lim\limits_{x\to\frac{1}{n}^-}f(x)=\dfrac{1}{(n+1)^2}$.

Thus, f is discontinuous for $x=\pm\dfrac{1}{n}$, $n=1,2,3,\ldots$

216. $h(x)=g[f(x)]$

$$h(x)=\frac{|1+x-[x]|}{1+x-[x]}$$

$$\Rightarrow h(x)=\frac{|1+\{x\}|}{1+\{x\}}\qquad \{x^2\}\ge 0$$

$$\Rightarrow h(x)=\frac{1+\{x\}}{1+\{x\}}=1$$

$h(x)=1$

$\Rightarrow h'(x)=0$ for both $x=1$ and -1.

217. 1^∞ form

$$L=e^{\lim\limits_{x\to\infty}\left[\frac{x^2+5x+3}{x^2+x+2}-1\right]x}$$

$$L=e^{\lim\limits_{x\to\infty}\frac{x^2+5x+3-x^2-x-2}{x^2+x+2}\cdot x}$$

$$=e^{\lim\limits_{x\to\infty}\frac{(4x+1)x}{x^2+x+2}}$$

$$=e^{\lim\limits_{x\to\infty}\frac{x^2\left(4+\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x}+\frac{2}{x^2}\right)}}$$

$=e^4$

218. $g(x)=\sin x+\cos x$

$$=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$

$$\begin{array}{ccc} + \quad - & & + \\ \hline \frac{3\pi}{4} & \frac{7\pi}{4} & 2\pi\end{array}$$

$f[g(x)]$ is discontinuous at $\dfrac{3\pi}{4}$ and $\dfrac{7\pi}{4}$.

$$f[g(x)]=\begin{cases}+1, & x<\frac{3\pi}{4}\\ 0, & x=\frac{3\pi}{4}\\ -1, & \frac{3\pi}{4}<x<\frac{7\pi}{4}\\ 1, & x>\frac{7\pi}{4}\end{cases}$$

219. $f(x)=|x-1|+|x-3|$

$$\Rightarrow f(x)=\begin{cases}-2x+4, & x\le 1\\ 2, & 1<x<3\\ 2x-4, & x\ge 3\end{cases}$$

$\Rightarrow f'(x)$ at $x=2$ is zero

220. Obviously function must be discontinuous. Hence, the correct answer is (A).

221. $\lim\limits_{x\to a}\dfrac{\cos x-\cos a}{\cot x-\cot a}\left(\dfrac{0}{0}\right)$

$$\Rightarrow \lim_{x\to a}\frac{-\sin x}{-\operatorname{cosec}^2 x}=\sin^3 a$$

222. $$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$
$$= \lim_{h\to 0} \frac{f(3)\cdot f(h)-f(3)}{h}$$
$$= \lim_{h\to 0} f(3)\frac{[f(h)-f(0)]}{h}$$
$$= f(3)\cdot f'(0) = 3\times 11 = 33$$

223. $$L = \lim_{x\to 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\operatorname{cosec} x}$$
$$L = e^{\lim_{x\to 0}\left(\frac{1+\tan x - 1 - \sin x}{1+\sin x}\times\frac{1}{\sin x}\right)}$$
$$L = e^{\lim_{x\to 0}\frac{\tan x-\sin x}{1+\sin x}\times\frac{1}{\sin x}}$$
$$L = e^{\lim_{x\to 0}\left(\frac{\frac{1}{\cos x}-1}{1+\sin x}\right)}$$
$$= e^0 = 1$$

224. $$\lim_{x\to 1}\frac{\int_4^{f(x)} 2t\,dt}{x-1}\left(\frac{0}{0}\right)\text{form}$$
$$= \lim_{x\to 1}\frac{2f(x)\cdot f'(x)}{1}$$
$$= 2f(1)\,f'(1)$$
$$= 8f'(1)$$

225. $$\lim_{x\to 0}\frac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$$

On rationalising, we get
$$= \lim_{x\to 0}\frac{\cancel{a^2}-ax+\cancel{x^2}-(\cancel{a^2}+ax+\cancel{x^2})}{\cancel{a}+x-(\cancel{a}-x)}\times\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2}}$$
$$= \lim_{x\to 0}\frac{-2a\cancel{x}}{2\cancel{x}}\times\frac{\sqrt{a}+\sqrt{a}}{\sqrt{a^2}+\sqrt{a^2}}$$
$$= -\frac{a}{\sqrt{a}} = -\sqrt{a}$$

226. $$y = \lim_{x\to 0}\left(2\sin^2\frac{x}{2}\right)^{\ln\cos x} = \lim_{x\to 0}(1-\cos x)^{\ln\cos x} = \lim_{x\to 1}(1-x)^{\ln x}$$
$$\Rightarrow \ln y = \lim_{x\to 1}\ln x\ln(1-x) = \lim_{x\to 0}\frac{\ln(1-x)}{\frac{1}{\ln x}} = 0 \text{ (using L'Hospital rule)}$$

Practice exercise 2

1. $$S_n = \lim_{n\to\infty}\left(\frac{1}{n^6}+\frac{32}{n^6}+\frac{243}{n^6}+\cdots+\frac{1}{n^6}\right) \text{ (see Fig. 19.33)}$$
$$T_n = \lim_{n\to\infty}\left(\frac{1}{n^6}+\frac{32}{n^6}+\frac{243}{n^6}+\cdots+\frac{(n-1)^5}{n^6}\right)$$
$$S_n = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^5}{n^6} = \int_0^1 x^5\,dx = \left.\frac{x^6}{6}\right|_0^1 = \frac{1}{6}$$

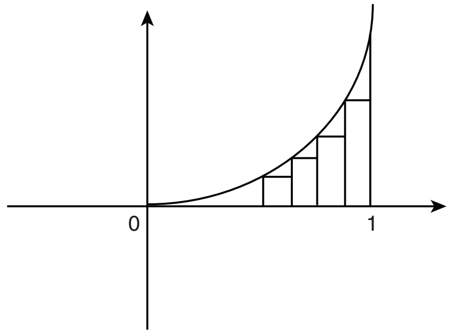


Figure 19.33

$$\Rightarrow S_n = \left(\frac{1}{6}\right)^+;\ T_n = \left(\frac{1}{6}\right)^-$$

But since x^5 is concave upward (Fig. 19.34) the area included in S_n for any two consecutive values of r is more than area excluded in T_n for same values of r. So,
$$S_n - \frac{1}{6} > \frac{1}{6} - T_n \Rightarrow S_n + T_n > \frac{1}{3}$$

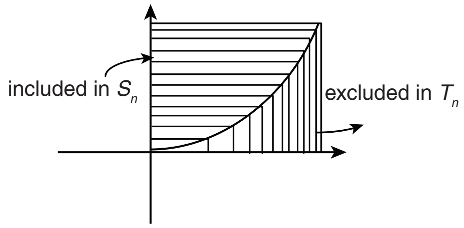


Figure 19.34

2. $$f\{g[n,h(x)]\} = \left|\sin\pi([n]+[n]^2)^{\frac{x}{\ln\left(\frac{e^x+e^{2x}+e^{3x}}{3}\right)}}\right|$$
$$\lim_{x\to 0} f\{g[n,h(x)]\} = \left|\sin\pi([n]+[n]^2)^{\frac{1}{2}}\right|$$
$$\Rightarrow \lim_{n\to\infty}\sin\left(\pi[n]\left(1+\frac{1}{[n]}\right)^{1/2}\right) = \lim_{n\to\infty}\left|\sin\left(\pi[n]+\frac{\pi}{2}-\frac{1}{[n]}\times\frac{\pi}{8}\cdots\right)\right| = 1$$
$$\lim_{n\to\infty}\lim_{x\to 0} g[n,\,h(x)] = \lim_{n\to\infty}\sin\left(\pi[n]+\frac{\pi}{2}-\frac{1}{8[n]}+\cdots\right)$$

Does not exist as value can be 1 or –1.

3. $$\lim_{n\to\infty}\tan\left(\frac{1}{n}\right)\ln\left(\frac{1}{n}\right) = 0 \Rightarrow f(x) = 1$$
$$\Rightarrow \int\frac{dx}{\sin^{11/3}x\cos^{1/3}x} = \frac{-3}{8}(\tan x)^{-8/3} - \frac{3}{2}(\tan x)^{-2/3} + C$$
$$g\left(\frac{\pi}{4}\right) = -\frac{15}{8}$$

4. $y = \lim_{x\to 0} x^5\left[\frac{1}{x^3}\right]$

$$x^5\left(\frac{1}{x^3}-1\right) < x^5\left[\frac{1}{x^3}\right] < x^5\frac{1}{x^3}$$

$$x^2 - x^5 < y < x^2 \Rightarrow y \to 0$$

5. RHL $= \lim_{x\to 0} f(x) = \lim_{h\to 0}\left\{1+\left(\frac{ch+dh^3}{h^2}\right)\right\}^{1/h} = e^d$ to exist the limit

$c = 0$,

LHL $= \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0}\frac{a(1-h\sin h)+b\cos h+5}{h^2} = \lim_{h\to 0}\frac{a+b\cos h+5}{h^2} - \frac{ah\sin h}{h^2}$$

Limit is possible if $a+b+5=0$ and $-a-\frac{b}{2}=3$

On solving,

$$a=-1, d=\log_e 3, c=0, b=-4$$

6. $x\cos\alpha + y\sin\alpha = x\cos\beta + y\sin\beta$

$$= 2\cdot\lim_{n\to\infty} a\left\{1+\left(\frac{b}{a}\right)^n\right\}^{1/n} = 2a \qquad (1)$$

$$\left[\text{As } 0<\frac{b}{a}<1 \Rightarrow \left(\frac{b}{a}\right)^n \to 0 \text{ as } n\to\infty\right]$$

Equation (1) shows that α and β are the roots of $x\cos\theta + y\sin\theta = 2a$ (2)

Therefore,

$$x^2\cos^2\theta = (2a - y\sin\theta)^2$$

$$\Rightarrow x^2(1-\sin^2\theta) = 4a^2 - 4ay\sin\theta + y^2\sin^2\theta$$

$$\Rightarrow (x^2+y^2)\sin^2\theta - 4ay\sin\theta + 4a^2 - x^2 = 0$$

$$\Rightarrow \sin\alpha + \sin\beta = \frac{4ay}{x^2+y^2} \;\&\; \sin\alpha\cdot\sin\beta = \frac{4a^2-x^2}{x^2+y^2}$$

(A) $\cos\alpha+\cos\beta = \cos\alpha\cos\beta \Rightarrow 4ax = 4a^2 - y^2$
$\Rightarrow y^2 = -4a(x-a)$ is a parabola.

(B) $\cos\alpha+\cos\beta = \sin\alpha+\sin\beta \Rightarrow 4ax = 4ay$
$\Rightarrow x = y$ is a straight line.

(C) $\cos\alpha+\cos\beta = \sin\alpha\sin\beta \Rightarrow 4ax = 4a^2 - x^2$
$\Rightarrow x^2+4ax+4a^2 = 8a^2$
$\Rightarrow\; x+2 = \pm 2\sqrt{2a}$ are pair of lines.

(D) $\cos\alpha\cos\beta + \sin\alpha\sin\beta = 0$
$\Rightarrow 4a^2 - y^2 + 4a^2 - x^2 = 0 \Rightarrow x^2+y^2 = 8a^2$ is a circle.

7. $f(x) = \left(\frac{x}{2+x}\right)^{2x}$

$$\Rightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{x}{2+x}\right)^{2x} = \lim_{x\to\infty}\left(1+\frac{x}{2+x}-1\right)^{2x}$$

$$= e^{\lim_{x\to\infty} 2x\left(-\frac{2}{2+x}\right)} = e^{\lim_{x\to\infty} -4\left(\frac{x}{2+x}\right)} = e^{-4}$$

Also $\lim_{x\to 1} f(x) = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

8. By standard results, the correct answers are options (B) and (C).

9. If $x \in Q$, then $n!\pi x \to$ multiple of π

$$\cos(n!\pi x) \to \pm 1$$

$$1+1=2$$

If $x \notin Q$ then $\cos(n!\,\pi x)$ be any number between -1 and 1.

$\lim_{m\to\infty}$ [1 + [any no between -1 & 1]2m]

$$[1] = 1$$

10. $\lim_{t\to 0}\frac{\sin(\tan t)}{\sin t} = \lim_{t\to 0}\frac{\tan t}{t} = 1$

$$\lim_{x\to\pi/2}\frac{\sin(\cos x)}{\cos x} = 1$$

$$\lim_{x\to 0}\frac{|x|}{x} = \text{DNE}$$

$$\lim_{x\to\pi/2}\frac{1-\cos x}{x^2} = \frac{4}{\pi^2}$$

11. (A) $\lim_{x\to 0-}\left[x+\frac{1}{2}\right] = 0,\ \lim_{x\to 0+}\left[2x+\frac{1}{3}\right] = 0$

Hence, $\lim_{x\to 0}\ [f(x)] = 0$.

(B) $\lim_{x\to 0-}\left(x+\frac{1}{2}\right) = \frac{1}{2},\ \lim_{x\to 0+}\left(2x+\frac{1}{3}\right) = \frac{1}{3}$

Since, $f(0^-) \neq f(0^+)$.

Therefore, $\lim_{x\to 0}\ f(x)$ does not exist.

(C) Since, $\lim_{x\to 0}\ f(x)$ does not exist so $\left[\lim_{x\to 0} f(x)\right]$ does not exist.

(D) $\lim_{x\to 0-}\frac{\left[x+\frac{1}{2}\right]}{x} = 0,\ \lim_{x\to 0+}\frac{\left[2x+\frac{1}{3}\right]}{x} = 0$

Therefore,

$$\lim_{x\to 0}\frac{[f(x)]}{x} = 0$$

12. (A) $\lim_{x\to 1}\ f(x)$ does not exist

(B) $\lim_{x\to 1} f(x) = \frac{2}{3}$

Hence, $f(x)$ has removable discontinuity at $x = 1$.

(C) $\lim_{x\to 1}\ f(x)$ does not exist

(D) $\lim_{x\to 1} f(x) = \frac{-1}{2\sqrt{2}}$

Hence, $f(x)$ has removable discontinuity at $x = 1$.

13.

$$f(x+y) = f(x) + f(y) + xy\,(x+y)$$

$$f(0) = 0$$

$$\Rightarrow \lim_{h\to 0}\frac{f(h)}{h}=-1$$

$$\Rightarrow \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)+f(h)+xh(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(h)}{h}+\lim_{h\to 0} x(x+h)=-1+x^2$$

$$\Rightarrow f'(x)=-1+x^2$$

$$\Rightarrow f(x)=\frac{x^3}{3}-x+c$$

Hence, $f(x)$ is a polynomial function, $f(x)$ is twice differentiable for all $x\in R$ and $f'(3)=3^2-1=8$.

14. $$f(x)=\int_{-2}^{x}|t+1|\,dt=-\int_{-2}^{-1}(t+1)\,dt+\int_{-1}^{x}(t+1)\,dt=\frac{1}{2}+\left(\frac{t^2}{2}+t\right)_{-1}^{x}$$

$$=\frac{x^2}{2}+x+1 \text{ for } x\geq -1$$

$f(x)$ is a quadratic polynomial.

Therefore, $f(x)$ is continuous as well as differentiable in $[-1, 1]$.

Also $f'(x)$ is continuous as well as differentiable in $[-1, 1]$.

15. $$f(x)=\begin{cases}\dfrac{1}{x+1}, & 0\leq x<1\\ \dfrac{2}{x}, & 1\leq x<2\\ \dfrac{3}{x-1}, & 2\leq x<\dfrac{5}{2}\end{cases}$$

Clearly, $f(x)$ is discontinuous and bijective function (see Fig. 19.35).

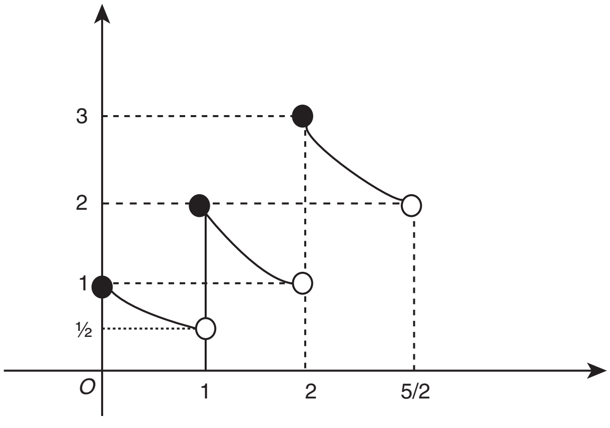


Figure 19.35

$$\lim_{x\to 1^-}f(x)=\frac{1}{2}$$

$$\lim_{x\to 1^+}f(x)=2$$

$$\min\left[\lim_{x\to 1^-}f(x),\ \lim_{x\to 1^+}f(x)\right]=\frac{1}{2}\neq f(1)$$

$$\max(1, 2)=2=f(1)$$

16. Both $x^2, -x^{3/2}$ have their RHL = 0 and RHD = 0.
Hence, the correct answers are options (A) and (D).

17. $$\frac{AB}{DC}=\frac{PA}{PD}\Rightarrow\frac{\theta}{\sin\theta}=\frac{PA}{PA-DA}=\frac{PA}{PA-(1-\cos\theta)}$$

$$\Rightarrow \theta\cdot PA-\theta(1-\cos\theta)=PA\cdot\sin\theta$$

$$\Rightarrow PA=\frac{\theta(1-\cos\theta)}{(\theta-\sin\theta)}$$

$$\lim_{\theta\to 0^+}\frac{\theta(1-\cos\theta)}{\theta-\sin\theta}=3$$

18. In ΔOCQ, $\cos\theta=\dfrac{OC}{OQ}=\dfrac{1}{OQ}$ (Fig. 19.36)

$$OQ=\frac{1}{\cos\theta}$$

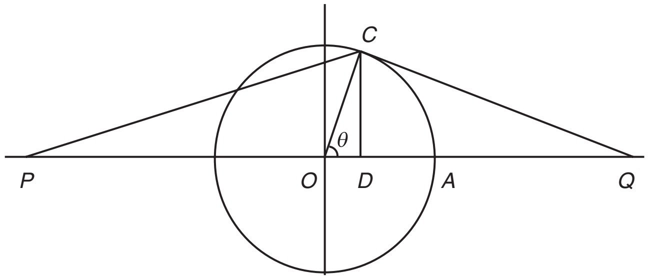


Figure 19.36

$$\text{Area of }\Delta CPQ=\frac{1}{2}\cdot PQ\cdot DC=\frac{1}{2}(PO+OQ)\cdot DC$$

$$=\frac{1}{2}\{PA-OA+OQ\}\cdot DC$$

$$=\frac{1}{2}\left\{\frac{\theta(1-\cos\theta)}{\theta-\sin\theta}-1+\frac{1}{\cos\theta}\right\}\cdot\sin\theta$$

$$=\frac{1}{2}\left\{\tan\theta+\frac{\sin^2\theta(1-\theta\cot\theta)}{(\theta-\sin\theta)}\right\}$$

19. $$\lim_{\theta\to 0^+}\frac{\Delta CPQ}{\sin^2\theta}=\lim_{\theta\to 0^+}\frac{1}{2}\left\{\frac{\theta\sin\theta-1+\cos\theta}{\cos\theta(\theta-\sin\theta)}\right\}$$

$$=\frac{1}{2}\lim_{h\to 0}\frac{1}{\cos h}\lim_{h\to 0}\left\{\frac{h\sin h-1+\cos h}{h-\sin h}\right\}$$

$$=\frac{1}{2}\cdot 1\cdot\lim_{h\to 0}\left\{\frac{h\cos h+\sin h-\sin h}{1-\cos h}\right\} \quad \text{L-Hospital rule}$$

$$=\frac{1}{2}\lim_{h\to 0}\left\{\frac{h\cos h}{1-\cos h}\right\}$$

$$=\frac{1}{2}h\lim_{h\to 0}\left\{\frac{-h\sin h+\cos h}{\sin h}\right\} \quad \text{L-Hospital rule}$$

$$\Rightarrow \frac{1}{0}=\text{not defined}$$

Common Solution for Questions 21–23:

$$f(x)=\lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}\right)^n=\lim_{n\to\infty}\left[1+\left(\cos\sqrt{\frac{x}{n}}-1\right)\right]^n$$

$$=e^{\lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}-1\right)n}=e^{-\lim_{n\to\infty}2\sin^2\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)n}$$

$$=e^{-2\lim_{n\to\infty}\frac{\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}{\frac{1}{n}}}=e^{-2\lim_{n\to\infty}\frac{1}{4}\frac{\frac{x}{n}}{\frac{1}{n}}}=e^{-\frac{x}{2}}$$

$y=f(x)=e^{-x/2}, x\geq 0$ (range = (0, 1])

$$g(x)=\lim_{n\to\infty}(1-x+x\sqrt[n]{e})^n$$

$$=e^{\lim_{n\to\infty}x\frac{(e^{1/n}-1)}{1/n}}=e^x \quad \forall x\in R$$

$h(x)=\tan^{-1}(g^{-1}[f^{-1}(x)])$

$$-\frac{x}{2}=\ln y\Rightarrow x=2\ln\frac{1}{y}\Rightarrow f^{-1}(x)=2\ln\frac{1}{x};\ 0<x\leq 1$$

$y=g(x)=e^x$

$x=\ln y\Rightarrow g^{-1}(x)=\ln x$

$$\Rightarrow g^{-1}\left(2\ln\frac{1}{x}\right)=\ln\left[2\ln\left(\frac{1}{x}\right)\right] \text{ for } 0<x<1$$

$$\Rightarrow h(x)=\tan^{-1}\left[\ln\left(\ln\frac{1}{x^2}\right)\right] \text{ for } 0<x<1$$

20. $$\lim_{x\to 0^+}\frac{\ln f(x)}{\ln g(x)}=\lim_{x\to 0^+}\frac{-x/2}{x}=-\frac{1}{2}$$

21. Domain of $h(x)$ is (0, 1).

22. $$h(x)=\tan^{-1}[\ln(\ln 1/x^2)] \quad (0<x<1)$$

$$1<\frac{1}{x^2}<\infty\Rightarrow 0<\ln\frac{1}{x^2}<\infty$$

Hence,

$$-\infty<\ln[\ln(1/x^2)]<\infty$$

Therefore, range of $h(x)$ is $(-\pi/2, \pi/2)$.

23. $$a=\lim_{n\to\infty}\lim_{\alpha\to 1^+}\frac{\alpha^n|\sin x|+\alpha^{-n}|\cos x|}{\alpha^n+\alpha^{-n}}$$

$$=\lim_{n\to\infty}\lim_{\alpha\to 1^+}\frac{|\sin x|+\alpha^{-2n}|\cos x|}{1+\alpha^{-2n}}=|\sin x|$$

24. $$b=\lim_{n\to\infty}\lim_{\alpha\to 1^-}\frac{\alpha^{-n}|\sin x|+\alpha^n|\cos x|}{\alpha^n+\alpha^{-n}}$$

$$=\lim_{n\to\infty}\lim_{\alpha\to 1^-}\frac{\alpha^{-2n}|\sin x|+|\cos x|}{\alpha^{2n}+1}=|\cos x|$$

$$c=\lim_{n\to\infty}\frac{\pi}{4n}\left[1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\cdots+\cos\frac{(n-1)\pi}{2n}\right]$$

$$=\lim_{n\to\infty}\frac{\pi}{4n}\left(\frac{\sin\frac{\pi}{4}\cos\frac{(n-1)\pi}{4n}}{\sin\frac{\pi}{4n}}\right)=\sin\frac{\pi}{4}\cos\frac{\pi}{4}=\frac{1}{2}$$

Hence, $b+c-\frac{1}{2}=|\cos x|$

25. $f(x)=\max\left\{|\sin x|, |\cos x|, \frac{1}{2}\right\}$

Hence, range of $f(x)$ is $\left[\frac{1}{\sqrt{2}}, 1\right]$.

26.

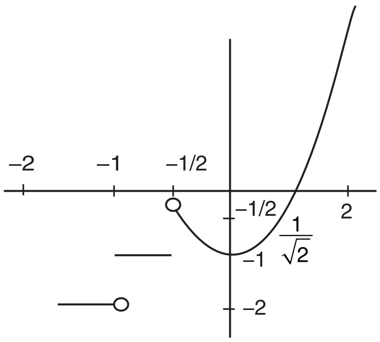


Figure 19.37

−1, −1/2 are two points of discontinuity (see Fig. 19.37)

27.

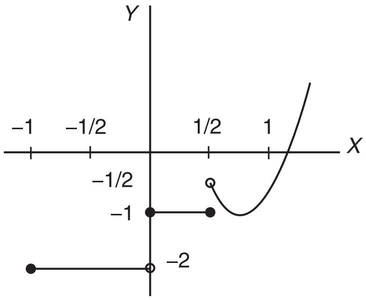


Figure 19.38

Discontinuous at 0, 1/2 (see Fig. 19.38)

28.

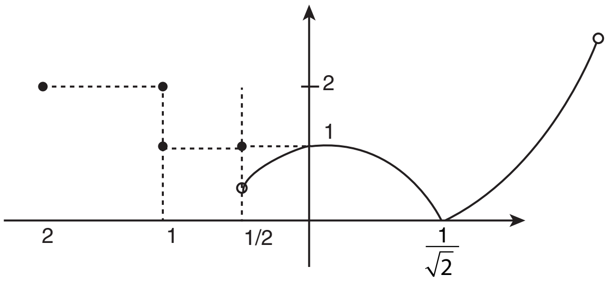


Figure 19.39

At −1, −1/2, $1/\sqrt{2}$ the function is not differentiable (see Fig. 19.39).

29. (A) $$\lim_{n\to\infty}\frac{\left(\sum_{x=1}^{n}x^4\right)\left(\sum_{x=1}^{n}x^5\right)}{\left(\sum_{x=1}^{n}x^t\right)\left(\sum_{x=1}^{n}x^{9-t}\right)}=\frac{\int_0^1 y^4dy\int_0^1 y^5dy}{\int_0^1 y^tdy\int_0^1 y^{9-t}dy}=\frac{(t+1)(10-t)}{30}=\frac{4}{5}$$

$\Rightarrow t^2-9t+14=0\Rightarrow(t-2)(t-7)=0\Rightarrow t=2, 7$

(B) $x^y = y^x \Rightarrow y \ln x = x \ln y$

Differentiating with respect to x, we get

$$y' = \frac{(x\ln y - y)}{(y\ln x - x)} = \frac{y^2(\ln x - 1)}{x^2(\ln y - 1)}$$

$$y' = \lim_{\substack{x \to e \\ y \to e}} \frac{y^2(\ln x - 1)}{x^2(\ln y - 1)} = \lim_{\substack{x \to e \\ y \to e}} \frac{y}{x\, y'}$$

$$\Rightarrow y' = \pm 1$$

So, $m = \pm 1$ and $3 - m$ is equal to 2 or 4.

(C) Put $x = y = 1$, then

$$[f(1)]^2 = f(1) + 6 \Rightarrow f(1) = 3, -2$$

Put $y = 1$, we get $f(x) = \frac{3}{2}\left(\frac{1+x}{x}\right)$ or $f(x) = -\left(\frac{1+x}{x}\right)$.

(D) If m is even $(\sqrt{2}+1)^m + (\sqrt{2}-1)^m = 2I$, then

$$\lim_{n\to\infty} \{(\sqrt{2}+1)^{n!+k}\} = \lim_{n\to\infty} \{2I - (\sqrt{2}-1)^{n!+k}\}$$

$$= \lim_{n\to\infty} \{-(\sqrt{2}-1)^{n!+k}\} = 1$$

$$\Rightarrow n! + k \text{ must be even integer} \Rightarrow k \text{ is even}$$

30. After rationalizing, we get

$$\lim_{x\to\infty} \frac{(a-2)x^3 + (3+c)x^2 + (b-3)x + (2+d)}{\sqrt{x^4 + ax^3 + 3x^2 + bx + 2} + \sqrt{x^4 + 2x^3 - cx^2 + 3x - d}} = 4$$

Since limit is finite, so degree of numerator must be 2. So,

$$a - 2 = 0 \Rightarrow a = 2$$

Now, dividing numerator and denominator by x^2, we get

$$\lim_{x\to\infty} \frac{(3+c) + \frac{b-3}{x} + \frac{2+d}{x^2}}{\sqrt{1 + \frac{a}{x} + \frac{3}{x^2} + \frac{b}{x^3} + \frac{2}{x^4}} + \sqrt{1 + \frac{2}{x} - \frac{c}{x^2} + \frac{3}{x^3} - \frac{d}{x^4}}} = 4$$

$$\Rightarrow \frac{3+c}{2} = 4 \Rightarrow c = 5$$

Hence, $a = 2$, $c = 5$ and b, d can be any real number.

Hence, the correct answer is (A) $\to$ (s); (B) $\to$ (p), (q), (r), (s), (t); (C) $\to$ (p); (D) $\to$ (p), (q), (r), (s), (t).

31. (A) $\displaystyle \lim_{x\to 1} \int_1^{x^2} \frac{f(t) - t}{(x-1)^2} dt = \lim_{x\to 1} \frac{\int_1^{x^2} (f(t) - t)\, dt}{(x-1)^2}$

$$= \lim_{x\to 1} \frac{2x(f(x^2) - x^2)}{2(x-1)} = \lim_{x\to 1} \frac{f(x^2) - x^2 + 2x^2 f'(x^2) - 2x^2}{1} = 4$$

(B) $\displaystyle \lim_{n\to\infty} \left(\frac{1 + \sqrt[n]{4}}{2}\right)^n = \lim_{n\to\infty} \left(1 + \frac{\sqrt[n]{4} - 1}{2}\right)^n$

$$= e^{\lim_{n\to\infty} \frac{4^{1/n} - 1}{2} \cdot n} = e^{\lim_{n\to\infty} \frac{1}{2} \frac{4^{1/n} - 1}{1/n}} = e^{\frac{1}{2}\ln 4} = 2$$

(C) $\displaystyle f(x) = \lim_{n\to\infty} \frac{2x}{\pi} \tan^{-1}(nx) = x, x > 0$

Hence,

$$\lim_{x\to 0^+} [f(x) - 1] = \lim_{x\to 0^+} [x - 1] = -1$$

(D) $\displaystyle \lim_{n\to\infty} \left[\sum_{r=1}^{n} \frac{1}{2^r}\right] = \lim_{n\to\infty} \left[1 - \frac{1}{2^n}\right] = 0$

32. (A) $\displaystyle \lim_{x\to 0} \left[\frac{f^2(a+x)}{f(a)}\right]^{1/x} = e^{\lim_{x\to 0}\left(\frac{f^2(a+x) - f(a)}{xf(a)}\right)}$

$$= e^{\lim_{x\to 0} \frac{2f(a+x) - f(a+x)}{f(a)}} = e^4$$

Hence, $k = 4$.

(B) $\displaystyle \lim_{x\to \frac{\pi}{2}^+} \frac{\cos[\tan^{-1}(\tan x)]}{x - \frac{\pi}{2}}$

$$\lim_{h\to 0^+} \frac{\cos \tan^{-1}\left[\tan\left(\frac{\pi}{2} + h\right)\right]}{h}$$

$$\lim_{h\to 0^+} \frac{\cos\left(h - \frac{\pi}{2}\right)}{h} = \lim_{h\to 0^+} \frac{\sin h}{h} = 1$$

(C) $\displaystyle \lim_{x\to \pi} \frac{\sin(\cos x + 1)}{\cos\left(\frac{x}{2}\right)} = \lim_{x\to\pi} \frac{\sin\left[2\cos^2\left(\frac{x}{2}\right)\right]}{\left(2\cos^2\frac{x}{2}\right)} \cdot \left(2\cos\frac{x}{2}\right)$

$$= 1 \times 0 = 0$$

(D) $\displaystyle \lim_{x\to 0} \frac{xe^{\sin x} - e^x \sin^{-1}(\sin x)}{\sin^2 x - x\sin x} = \lim_{x\to 0} \frac{xe^x}{\sin x} \cdot \frac{e^{\sin x - x} - 1}{\sin x - x} = 1$

33. Given inequality can be written as

$$f''(x) - 2f'(x) \geq 3[f'(x) - 2f(x)]$$

Let $f'(x) - 2f(x) = g(x)$. Then

$$g'(x) - 3g(x) \geq 0 \qquad \text{(Multiply } e^{-3x}\text{)}$$

$$\Rightarrow [g(x)e^{-3x}] \geq 0 \Rightarrow g(x)e^{-3x} \text{ is non-decreasing}$$

Now

$$g(0) = f'(0) - 2f(0) = -2$$

$$g(x)e^{-3x} \geq -2, \forall x \geq 0$$

$$f'(x) - 2f(x) \geq -2e^{3x}, \forall x \geq 0 \qquad \text{(Multiply } e^{-2x}\text{)}$$

$$\Rightarrow [f(x)e^{-2x}] \geq -2e^x, \forall x \geq 0$$

$$\Rightarrow [f(x)e^{-2x} + 2e^x] \geq 0$$

$$\Rightarrow f(x)e^{-2x} + 2e^x \geq 3$$

$$\Rightarrow f(x) \geq 3e^{2x} - 2e^{3x}, \forall x \geq 0$$

Comparing $ah(bx) - bh(ax)$ with $3e^{2x} - 2e^{3x}$, we get

$$h(x) = e^x, a = 3, b = 2$$

$$\Rightarrow (a+b)\, h(0) = 5$$

34. $\displaystyle \text{RHL} = \lim_{x\to 0^+} \frac{(1+a^3) + 8e^{1/x}}{1 + (2+b+b^2)e^{1/x}} = 2$

$$\Rightarrow 2 + b + b^2 = 4 \Rightarrow b^2 + b - 2 = 0 \Rightarrow b = 1$$

$$\text{LHL} = \lim_{x\to 0^-} \frac{(1+a^3) + 8e^{1/x}}{1 + (2+b+b^2)e^{1/x}} = 2$$

$$\Rightarrow 1 + a^3 = 2 \Rightarrow a = 1$$

35. Let $f(x) = y$. Then

$$\frac{dy}{dx} + y = 4xe^{-x} \cdot \sin 2x$$

$$\text{I.F} = e^x$$

$$ye^x = 4\int \underset{\text{I}}{x} \underset{\text{II}}{\sin 2x}\, dx$$

$$ye^x = 4\left[x\left(-\frac{\cos 2x}{2}\right) + \frac{1}{2}\int \cos 2x dx\right]$$

$$= ye^x = 4\left(-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right) + c$$

$$\Rightarrow ye^x = (\sin 2x - 2x\cos 2x) + c$$

$$f(0) = 0 \Rightarrow c = 0$$

Therefore, $y = e^{-x}(\sin 2x - 2x\cos 2x)$

Now,

$$f(k\pi) = e^{-k\pi}(\sin 2k\pi - 2k\pi\cos 2k\pi) = e^{-k\pi}(0 - 2k\pi)$$

$$f(k\pi) = -2\pi(ke^{-k\pi})$$

$$\sum f(k\pi) = 2\pi\sum_{k=1}^{\infty} ke^{-k\pi}$$

$$S = 1\cdot e^{-\pi} + 2e^{-2\pi} + 3e^{-3\pi} + \cdots\infty$$

$$S = e^{-2\pi} + 2e^{-3\pi} + \cdots\infty$$

$$S(1-e^{-\pi}) = e^{-\pi} + e^{-2\pi} + e^{-3\pi} + \cdots\infty$$

$$S(1-e^{-\pi}) = \frac{e^{-\pi}}{1-e^{-\pi}} = \frac{1}{e^{\pi}-1}$$

$$S = \frac{1}{(e^{\pi}-1)(1-e^{-\pi})} = \frac{e^{\pi}}{(e^{\pi}-1)^2}$$

Therefore,

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(k\pi) = \frac{-2\pi e^{\pi}}{(e^{\pi}-1)^2}$$

Hence, $p = 2$.

36. $\lim\limits_{x\to 0}([f(x)] + x^2)^{\frac{1}{\{f(x)\}}} = \lim\limits_{x\to 0}(1+x^2)^{\frac{1}{\frac{\tan x}{x}-1}} = \lim\limits_{x\to 0}(1+x^2)^{\frac{x}{\tan x - x}}$

$$= e^{\lim_{x\to 0}\frac{x^3}{\tan x - x}} = e^{\lim_{x\to 0}\frac{3x^2}{\sec^2 x - 1}} = e^{\lim_{x\to 0}\frac{6x}{2\sec^2 x\tan x}} = e^3$$

Therefore, $\lambda = 3$.

Solved JEE 2017 Questions

JEE Main 2017

1. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\cot x-\cos x}{(\pi-2x)^3}$ equals

(A) $\dfrac{1}{16}$ **(B)** $\dfrac{1}{8}$

(C) $\dfrac{1}{4}$ **(D)** $\dfrac{1}{24}$

(OFFLINE)

Solution: We have

$$\lim_{x\to\frac{\pi}{2}} \frac{\cot x(1-\sin x)}{-8\left(x-\frac{\pi}{2}\right)^3}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2}-x\right)}{8\left(\frac{\pi}{2}-x\right)}\frac{\left(1-\cos\left(\frac{\pi}{2}-x\right)\right)}{\left(\frac{\pi}{2}-x\right)^2}=\lim_{x\to\frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2}-x\right)}{8\left(\frac{\pi}{2}-x\right)}\frac{\left(\sin^2\left(\frac{\pi}{2}-x\right)\right)}{2\left(\frac{\pi}{2}-x\right)^2}$$

$$=\frac{1}{8}\times 1\times\frac{1}{2}=\frac{1}{16}$$

Hence, the correct answer is option (A).

2. If for $x\in\left(0,\frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\dfrac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\cdot g(x)$, then $g(x)$ equals

(A) $\dfrac{3x\sqrt{x}}{1-9x^3}$ **(B)** $\dfrac{3x}{1-9x^3}$

(C) $\dfrac{3}{1+9x^3}$ **(D)** $\dfrac{9}{1+9x^3}$

(OFFLINE)

Solution: We have

$$y=\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)=\tan^{-1}\left(\frac{2(3x\sqrt{x})}{1-(3x\sqrt{x})^2}\right)$$
$$=2\tan^{-1}(3x\sqrt{x})$$

Now, differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{2}{1+(3x\sqrt{x})^2}3\left((x)\frac{1}{2\sqrt{x}}+\sqrt{x}(1)\right)$$

$$=\frac{6}{1+9x^3}\left(\frac{\sqrt{x}}{2}+\sqrt{x}\right)=\frac{9\sqrt{x}}{1+9x^3}=\sqrt{x}\left(\frac{9}{1+9x^3}\right)=\sqrt{x}\cdot g(x)$$

Therefore, $g(x)=\dfrac{9}{1+9x^3}$.

Hence, the correct answer is option (D).

3. $\lim\limits_{x\to 3}\dfrac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$ is equal to

(A) $\sqrt{3}$ **(B)** $\dfrac{\sqrt{3}}{2}$

(C) $\dfrac{1}{2\sqrt{2}}$ **(D)** $\dfrac{1}{\sqrt{2}}$

(ONLINE)

Solution: It is given that

$$\lim_{x\to 3}\frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$$

Rationalizing it, we get

$$\lim_{x\to 3}\sqrt{3x}-3\times\frac{\sqrt{3x}+3}{\sqrt{3x}+3}\times\frac{1}{\sqrt{2x-4}-\sqrt{2}}\times\frac{\sqrt{2x-4}+\sqrt{2}}{\sqrt{2x-4}+\sqrt{2}}$$

$$\Rightarrow \lim_{x\to 3}\frac{(\sqrt{3x})^2-3^2}{\sqrt{3x}+3}\times\frac{(\sqrt{2x-4}+\sqrt{2})}{(\sqrt{2x-4})^2-(\sqrt{2})^2}$$

$$\Rightarrow \lim_{x\to 3}\frac{3x-9}{\sqrt{3x}+3}\times\frac{(\sqrt{2x-4}+\sqrt{2})}{2x-4-2}$$

$$\Rightarrow \lim_{x\to 3}\frac{3(x-3)}{\sqrt{3x}+3}\times\frac{\sqrt{2x-4}+\sqrt{2}}{2x-6}$$

$$\Rightarrow \lim_{x\to 3}\frac{3(x-3)}{\sqrt{3x}+3}\times\frac{\sqrt{2x-4}+\sqrt{2}}{2(x-3)}$$

$$\Rightarrow \lim_{x\to 3}\frac{3}{2}\frac{\sqrt{2x-4}+\sqrt{2}}{\sqrt{3x}+3}=\frac{3}{2}\times\frac{\sqrt{2\times 3-4}+\sqrt{2}}{\sqrt{3\times 3}+3}$$

$$=\frac{3}{2}\times\frac{(\sqrt{2}+\sqrt{2})}{3+3}$$

$$=\frac{3}{2}\times\frac{2\sqrt{2}}{6}=\frac{2\sqrt{2}}{4}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

Therefore,

$$\lim_{x\to 3}\frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}=\frac{1}{\sqrt{2}}$$

Hence, the correct answer is option (D).

4. If $\lim\limits_{n\to\infty}\dfrac{1^a+2^a+\cdots+n^a}{[(n+1)^{a-1}[(na+1)+(na+2)+\cdots+(na+n)]}=\dfrac{1}{60}$ for some positive real number a, then a is equal to

(A) $\dfrac{17}{2}$ **(B)** 8

(C) 7 **(D)** $\dfrac{15}{2}$

(ONLINE)

Solution: The given limit is

$$\lim_{n\to\infty} \frac{1^a + 2^a + 3^a + \cdots + n^a}{(n+1)^{a-1}[(na+1)+(na+2)+\cdots+(na+n)]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1^a + 2^a + \cdots + n^a}{(n+1)^{a-1}\left[n^2a + \frac{n(n+1)}{2}\right]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2\left[n\sum_{k=1}^{n} k^a\right]}{(n+1)^{a-1}[2n^2a + n^2 + n]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2\sum_{k=1}^{n}\left(\frac{k}{n}\right)^a}{\left(1+\frac{1}{a}\right)^{a-1}[2na+n+1]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\frac{2}{a}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^a}{\left(1+\frac{1}{n}\right)^{n-1}\left[2a+1+\frac{1}{n}\right]} = \frac{1}{60}$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^a = \int_0^1 x^a dx \qquad \text{(derivative as limit of a sum)}$$

Therefore,

$$\frac{2\int_0^1 x^a dx}{2a+1} = \frac{1}{60}$$

$$\Rightarrow \frac{2(x^{a+1})]_0^1}{(a+1)(2a+1)} = \frac{1}{60}$$

$$\Rightarrow \frac{2}{(a+1)(2a+1)} = \frac{1}{60}$$

$\Rightarrow (a + 1)\,(2a + 1) = 120$

$\Rightarrow 2a^2 + 3a + 1 = 120$

$\Rightarrow 2a^2 - 3a = 119 = 0$

$$\Rightarrow a = \frac{-3 \pm \sqrt{9+952}}{4} = \frac{-3 \pm \sqrt{961}}{4} = \frac{-3 \pm 31}{4} \qquad (\text{since } a > 0)$$

Hence, the value of a is

$$\frac{-3+31}{4} = \frac{28}{4} = 7 \Rightarrow a = 7$$

Hence, the correct answer is option (C).

5. The value of k for which the function

$$f(x) = \begin{cases} \left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}}, & 0 < x < \frac{\pi}{2} \\ k + \frac{2}{5}, & x = \frac{\pi}{2} \end{cases}$$

is continuous at $x = \frac{\pi}{2}$ is

(A) $\frac{17}{20}$ **(B)** $\frac{3}{5}$

(C) $-\frac{2}{5}$ **(D)** $\frac{2}{5}$

(ONLINE)

Solution: We have

$$f(x) = \left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}}$$

$$\lim_{x\to\frac{x}{2}} \left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}} = k + \frac{2}{5}$$

$$\Rightarrow \lim_{x\to\frac{x}{2}} \left(\frac{4}{5}\right)^{\tan 4x \cot 5x} = k + \frac{2}{5}$$

$$\Rightarrow \left(\frac{4}{5}\right)^{\lim_{x\to\frac{x}{2}}(\tan 4x.\cot(5x))} = k + \frac{2}{5}$$

$$\Rightarrow \left(\frac{4}{5}\right)^{0\times\cos\left(2x+\frac{x}{2}\right)} = k + \frac{2}{5}$$

$$\Rightarrow \left(\frac{4}{5}\right)^{0} = k + \frac{2}{5}$$

$$\Rightarrow k + \frac{2}{5} = 1 \Rightarrow k = 1 - \frac{2}{5} \Rightarrow k = \frac{3}{5}$$

Hence, the correct answer is option (B).

JEE Advanced 2017

1. Let $[x]$ be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous?

(A) $x = -1$ **(B)** $x = 0$

(C) $x = 1$ **(D)** $x = 2$

Solution: It is given that

$$f(x) = x\cos\,(\pi(x + [x]))$$

and $$f(x) = (-1)^{[x]}\, x\cos \pi x$$

This function is discontinuous at all integral points except $x = 0$. At $x = z$; $z = 0, \pm1, \pm2, \ldots$

$$f(z) = z\cos \pi(2z) = z$$

$$f(z^+) = z\cos \pi(2z) = z$$

$$f(z^-) = z\cos \pi(2z - 1) = -z$$

$$f(0) = 0$$

Hence, the correct answers are options (A), (C) and (D).

2. Let f: $\mathbb{R} \to (0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)?

(A) $x^9 - f(x)$ **(B)** $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\, dt$

(C) $e^x - \int_0^x f(t)\sin t\, dt$ **(D)** $f(x) + \int_0^{\pi/2} f(t)\sin t\, dt$

Solution: We discuss the options as follows:

Option (A): Let $g(x) = x^9 - f(x)$.

$$g(0) = -f(0) < 0 \quad [\text{as } f \in (0,1)]$$

$$g(1) = 1^9 - f(1) = 1 - f(1) > 0 \quad [\text{as } f \in (0,1)]$$

Hence, option (A) is correct.

Option (B): Let $g(x) = x - \int_0^{\pi/2 - x} f(t)\cos t\, dt.$

$$g(0) = 0 - \int_0^{\pi/2-0} f(t)\cos t\, dt < 0$$

and $$g(1) = 1 - \int_0^{\pi/2-1} f(t)\cos t\, dt > 0$$

Hence, option (B) is correct.

Option (C): Let $g(x) = e^x - \int_0^x f(t)\sin t\, dt.$

Differentiating w.r.t. x, we get

$$g'(x) = e^x - f(x)\sin x \quad (1)$$

Now

$$g(0) = 1$$

Also, we know that $f(x) \in (0,1) \Rightarrow 0 < f(x) < 1 \Rightarrow 0 < f(x)\sin x < 1$

Therefore, from Eq. (1), we get

$$g'(x) > 0$$

Thus, $g(x)$ is strictly an increasing function.

Hence, option (C) is incorrect.

Option (D): Let $g(x) = f(x) + \int_0^{\pi/2} f(t)\sin t\, dt$

Since $f(x) \in (0, 1)$, we get $g(x) > 0$.

Hence, option (D) is incorrect.

Hence, the correct answers are options (A) and (B).

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$. If $g(x) = \int_x^{\pi/2} [f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]dt$ for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$ _______.

Solution: It is given that

$$g(x) = \int_x^{\pi/2} [f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]dt$$

We know that

$$\frac{d}{dt}(f(t)\operatorname{cosec} t) = f'(t)\operatorname{cosec} t - f(t)\operatorname{cosec} t \cot t$$

Therefore,

$$g(x) = \int_x^{\pi/2} \frac{d}{dt}(f(t)\operatorname{cosec} t)dt$$

$$\Rightarrow g(x) = f(t)\operatorname{cosec} t\Big|_x^{\pi/2}$$

$$g(x) = f\left(\frac{\pi}{2}\right)\operatorname{cosec}\left(\frac{\pi}{2}\right) - f(x)\operatorname{cosec} x$$

It is given that $f(\pi/2) = 3$ and $\operatorname{cosec}(\pi/2) = 1$. Therefore,

$$g(x) = 3 - f(x)\operatorname{cosec} x = \frac{3 - f(x)}{\sin x}$$

Now,

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\left(\frac{3 - f(x)}{\sin x}\right) = 3 - \lim_{x\to 0}\frac{f(x)}{\sin x}$$

Since $f(0) = 0$ and $\sin 0 = 0$, we get

$$\lim_{x\to 0}\frac{f(x)}{\sin x} = \frac{0}{0}$$

Taking derivative: Using $\frac{d}{dx}\sin x = \cos x$, we get

$$\lim_{x\to 0}\frac{f'(x)}{\cos x} = \frac{1}{1} \quad [\text{as } f'(0) = 1 \text{ given}]$$

Therefore, $\lim_{x\to 0} g(x) = 3 - 1 = 2.$

Hence, the correct answer is (2).

4. If $f\colon \mathrm{R} \to \mathrm{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathrm{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}, f(1) = 1$, then

(A) $f'(1) \le 0$ **(B)** $0 < f'(1) \le \frac{1}{2}$

(C) $\frac{1}{2} < f'(1) \le 1$ **(D)** $f'(1) > 1$

Solution: It is given that

$$f''(x) > 0 \text{ and } f\left(\frac{1}{2}\right) = \frac{1}{2},\ f(1) = 1$$

Using Lagrange's mean value theorem, let us consider that $f : \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ be continuous and differentiable on $\left(\frac{1}{2}, 1\right)$, then there exists $c \in \left(\frac{1}{2}, 1\right)$ such that

$$f'(c) = \frac{f(1) - f\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} = \frac{1 - \frac{1}{2}}{\frac{1}{2}} = 1 \Rightarrow f'(c) = 1$$

Since $f'(x)$ is increasing function for $x \in \mathrm{R}$, we get $f'(1) > 1$.

Hence, the correct answer is option (D).

5. If $g(x)=\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)dt$, then

(A) $g'\left(\frac{\pi}{2}\right)=-2\pi$ **(B)** $g'\left(-\frac{\pi}{2}\right)=-2\pi$

(C) $g'\left(\frac{\pi}{2}\right)=2\pi$ **(D)** $g'\left(-\frac{\pi}{2}\right)=-2\pi$

Solution: It is given that

$$g(x)=\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)\,dt$$

Differentiating this equation, we get

$$g'(x)=[\sin^{-1}(\sin 2x)]2\cos 2x-[\sin^{-1}(\sin x)]\cos x$$

Therefore,

$$g'\left(\frac{\pi}{2}\right)=[\sin^{-1}(\sin\pi)]2\cos\pi-\left[\sin^{-1}\left(\sin\frac{\pi}{2}\right)\right]\cos\frac{\pi}{2}$$

$$\Rightarrow g'\left(\frac{\pi}{2}\right)=0$$

and $g'\left(-\frac{\pi}{2}\right)=[\sin^{-1}(-\sin\pi)]2\cos\pi-\left[\sin^{-1}\left(-\sin\frac{\pi}{2}\right)\cos\frac{\pi}{2}\right]=0$

There is no correct option.

6. If $f(x)=\begin{vmatrix}\cos(2x) & \cos(2x) & \sin(2x)\\ -\cos x & \cos x & -\sin x\\ \sin x & \sin x & \cos x\end{vmatrix}$, then

(A) $f'(x)=0$ at exactly three points in $(-\pi,\pi)$.

(B) $f'(x)=0$ at more than three points in $(-\pi,\pi)$.

(C) $f(x)$ attains its maximum at $x=0$.

(D) $f(x)$ attains its minimum at $x=0$.

Solution: It is given that

$$f(x)=\begin{vmatrix}\cos 2x & \cos 2x & \sin 2x\\ -\cos x & \cos x & -\sin x\\ \sin x & \sin x & \cos x\end{vmatrix}$$

$$\Rightarrow f(x)=\cos 2x(\cos^2 x+\sin^2 x)-\cos 2x(-\cos^2 x+\sin^2 x)$$
$$+\sin 2x(-\sin x\cos x-\sin x\cos x)$$

Using $\sin^2 x+\cos^2 x=1$, we get

$$f(x)=\cos 2x-\cos 2x(-\cos^2 x+\sin^2 x)+\sin 2x(-2\sin x\cos x)$$
$$=\cos 2x-\cos 2x(-\cos 2x)+\sin 2x(-\sin 2x)$$
$$=\cos 2x+\cos^2 2x-\sin^2 2x$$

Using $\cos^2 x+\sin^2 x=1$ and $\sin^2 2x=1-\cos^2 2x$, we get

$$f(x)=\cos 2x+\cos^2 2x-(1-\cos^2 2x)$$
$$=\cos 2x+\cos^2 2x-1+\cos^2 2x$$

Therefore,

$$f(x)=2\cos^2 2x+\cos 2x-1=\cos 4x+\cos 2x$$

Differentiating this, we get

$$f'(x)=4(-\sin 4x)+2(-\sin 2x)$$
$$f'(x)=-4\sin 4x-2\sin 2x$$

Now $f'(x)=0$ gives

$$-4\sin 4x-2\sin 2x=0$$

Using $\sin 2x=2\sin x\cos x$, we get

$$-4(2\sin 2x\cos 2x)-2\sin 2x=0$$
$$-8\sin 2x\cos 2x-2\sin 2x=0$$
$$-2\sin 2x(4\cos 2x+1)=0$$

$$\Rightarrow 2\sin 2x=0\Rightarrow \sin 2x=0\Rightarrow 2x=0,\pi,-\pi\Rightarrow x=0,\frac{\pi}{2},\frac{-\pi}{2}$$

and

$$4\cos 2x+1=0\Rightarrow \cos 2x=\frac{-1}{4}\Rightarrow 2x=\cos^{-1}\left(\frac{-1}{4}\right)\Rightarrow 2x=1.8+2\pi n$$

where $n=\cdots,-2,-1,0,1,2,\cdots$, which gives 4 points in the range $(-\pi,\pi)$.

Thus, the total points in $(-\pi,\pi)$ range are 7.

Now,

$$f''(x)=[-4(\cos 4x)4]-[2\cos(2x)2]=-16\cos 4x-4\cos 2x$$

At $x=0$, we get

$$f''(x)=-16-4=-20 \quad \text{(maxima)}$$

Thus, at $x=0$, the function $f(x)$ attains maximum at $x=0$.

Hence, the correct answers are options (B) and (C).

20 Differentiation

20.1 Introduction

The rate of change of one quantity with respect to some another quantity has a great importance in mathematics. The rate of change of a quantity y with respect to another quantity x is called the derivative or differential coefficient of y with respect to x.

20.2 Differentiation from First Principle

When the derivative of a function is calculated directly by using the definition of derivative, it is called differentiation from first principle. This method is also known as ***ab-initio* method** or **delta method**.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Illustration 20.1 Differentiate $x^{-\frac{3}{2}}$, with respect to x, from the first principle.

Solution: Let $f(x)=x^{-\frac{3}{2}}$. Then

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^{-\frac{3}{2}}-x^{-\frac{3}{2}}}{h}$$

where h is small increment in x. Now,

$$f'(x)=\lim_{h\to 0}\frac{x^{-\frac{3}{2}}\left[\left(1+\frac{h}{x}\right)^{-\frac{3}{2}}-1\right]}{h}$$

$$=\lim_{h\to 0}\frac{x^{-\frac{3}{2}}\left[1-\frac{3}{2}\cdot\frac{h}{x}+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2\,!}\frac{h^2}{x^2}+\cdots-1\right]}{h}$$

$$=\lim_{h\to 0}x^{-\frac{3}{2}}\left[-\frac{3}{2x}+\frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2\,!}\cdot\frac{h}{x^2}+\cdots\right]=x^{-\frac{3}{2}}\times\left(-\frac{3}{2x}\right)=-\frac{3}{2}x^{-\frac{5}{2}}$$

Illustration 20.2 Differentiate $\sqrt{\tan 3x}$, with respect to x, from the first principle.

Solution: Let $f(x)=\sqrt{\tan 3x}$. Then

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{\tan 3(x+h)}-\sqrt{\tan 3x}}{h}$$

where h is small increment in x.

Multiplying and dividing by $\left[\sqrt{\tan(3x+3h)}+\sqrt{\tan 3x}\right]$, we get

$$f'(x)=\lim_{h\to 0}\frac{\tan(3x+3h)-\tan 3x}{h\left[\sqrt{\tan(3x+3h)}+\sqrt{\tan 3x}\right]}$$

$$=\lim_{h\to 0}\frac{\tan(3x+3h-3x)[1+\tan(3x+3h)\tan 3x]}{h\left[\sqrt{\tan(3x+3h)}+\sqrt{\tan 3x}\right]}$$

$$\left[\begin{aligned}&\because\ \tan A-\tan B\\&=\tan(A-B)(1+\tan A\tan B)\end{aligned}\right]$$

$$=\lim_{h\to 0}\frac{\tan 3h}{3h}\cdot 3\cdot\frac{[1+\tan(3x+3h)\tan 3x]}{\sqrt{\tan(3x+3h)}+\sqrt{\tan 3x}}$$

$$=1\times 3\times\frac{1+\tan^2 3x}{2\sqrt{\tan 3x}}=\frac{3\sec^2 3x}{2\sqrt{\tan 3x}}$$

Illustration 20.3 Differentiate $e^{\sqrt{x}}$, with respect to x, from the first principle.

Solution: Let $f(x)=e^{\sqrt{x}}$. Then

$$f'(x)=\lim_{h\to 0}\frac{e^{\sqrt{x+h}}-e^{\sqrt{x}}}{h}$$

where h is small increment in x.

$$f'(x)=\lim_{h\to 0}\frac{e^{\sqrt{x}}\left[e^{\sqrt{x+h}-\sqrt{x}}-1\right]}{h}$$

$$=\lim_{h\to 0}e^{\sqrt{x}}\cdot\frac{e^{\sqrt{x+h}-\sqrt{x}}-1}{\sqrt{x+h}-\sqrt{x}}\cdot\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=e^{\sqrt{x}}\cdot 1\cdot\lim_{h\to 0}\frac{x+h-x}{h[\sqrt{x+h}+\sqrt{x}]}=e^{\sqrt{x}}\cdot\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

Illustration 20.4 Find the derivative of $\tan^{-1}x$ with the first principle.

Solution: Let $\tan^{-1}x=\theta$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then

$$x=\tan\theta \tag{1}$$

and

$$\tan^{-1}(x+h)=\theta+\Delta\theta$$

$$\Rightarrow x+h=\tan(\theta+\Delta\theta) \tag{2}$$

Let $\lim_{h\to 0}\frac{\tan^{-1}(x+h)-\tan^{-1}x}{h}=L$. Then

$$L=\lim_{h\to 0}\frac{\theta+\Delta\theta-\theta}{h}=\lim_{h\to 0}\frac{\Delta\theta}{h}\qquad\text{[from Eqs. (1) and (2)]}$$

$$= \lim_{h\to 0} \frac{\Delta\theta}{\tan(\theta+\Delta\theta)-\tan\theta} = \lim_{\Delta\theta\to 0} \frac{\Delta\theta}{\tan(\theta+\Delta\theta)-\tan\theta}$$

$$= \lim_{\Delta\theta\to 0} \frac{(\Delta\theta)\cos(\theta+\Delta\theta)\cos\theta}{\sin\Delta\theta} = \cos^2\theta = \frac{1}{\sec^2\theta} = \frac{1}{1+x^2}$$

20.3 Derivatives of Some of the Frequently Used Functions

Function	Derivative
c (constant)	0
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
$\log_e x$	$1/x$
$\log_a x$	$(1/x)\log_a e$
x^n	nx^{n-1}
a^x	$a^x \log_e a$
e^x	e^x

The above written derivatives can be easily found by using the definition of differentiation.

20.4 Rules to Find Out Derivatives

Let u and v are differentiable functions of x. Then following are the rules to find derivatives:

1. **Sum rule:**

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(2e^x + 3\log_e x) = 2\frac{de^x}{dx} + 3\frac{d(\log_e x)}{dx} = 2e^x + \frac{3}{x}$$

In general

$$\frac{d}{dx}(u \pm v \pm w \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \cdots$$

Illustration 20.5 Differentiate $5\sin x - 2\log_e x$.

Solution:

$$\frac{d}{dx}(5\sin x - 2\log_e x) = \frac{d}{dx}(5\sin x) - \frac{d}{dx}(2\log_e x) = 5\cos x - \frac{2}{x}$$

2. **Product rule:**

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d[(\sin x)e^x]}{dx} = \sin x\frac{de^x}{dx} + e^x\frac{d(\sin x)}{dx} = (\sin x)e^x + (\cos x)e^x$$

In general

$$\frac{d}{dx}(u_1u_2u_3\ldots) = (u_1u_2u_3\ldots)\left[\frac{1}{u_1}\frac{du_1}{dx} + \frac{1}{u_2}\frac{du_2}{dx} + \frac{1}{u_3}\frac{du_3}{dx} + \cdots\right]$$

Illustration 20.6 Differentiate $x^2e^x \sin x$.

Solution: First we differentiate x^2e^x, that is,

$$\frac{d}{dx}(x^2e^x) = x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2) = x^2e^x + 2xe^x$$

Now,

$$\frac{d}{dx}(x^2e^x\sin x) = x^2e^x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2e^x)$$

$$= x^2e^x\cos x + \sin x(x^2+2x)e^x$$

$$= e^x(x^2\cos x + x^2\sin x + 2x\sin x) = xe^x(x\cos x + x\sin x + 2\sin x)$$

3. **Quotient rule:** Here $v(x) \neq 0$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{x\frac{d(\tan x)}{dx} - (\tan x)\frac{dx}{dx}}{x^2} = \frac{x\sec^2 x - \tan x}{x^2}$$

Illustration 20.7 Differentiate $\frac{e^x}{1+\sin x}$.

Solution:

$$\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right) = \frac{(1+\sin x)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)e^x - e^x\cos x}{(1+\sin x)^2} = \frac{e^x(1+\sin x - \cos x)}{(1+\sin x)^2}$$

4. **Chain rule:** The chain rule is probably the most widely used differentiation rule in mathematics. Chain rule says that the derivative of the composition of two differentiable functions is the product of their derivatives evaluated at appropriate points. The formula is

$$\{f[g(x)]\}' = f'[g(x)]\cdot g'(x)$$

Illustration 20.8 Differentiate $\sin x^2$.

Solution: Put $y = x^2$ and $z = \sin y$. Then

$$\frac{dy}{dx} = 2x \text{ and } \frac{dz}{dy} = \cos y$$

Therefore,

$$\frac{d}{dx}(\sin x^2) = \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} = (\cos y)\,(2x)$$
$$= (\cos x^2)(2x) = 2x\cos x^2$$

This solution can be rewritten using a more convenient notation in the following manner:

$$\frac{d}{dx}(\sin x^2) = \frac{d(\sin x^2)}{d(x^2)}\cdot\frac{d(x^2)}{dx} = \cos x^2\cdot 2x = 2x\cos x^2$$

5. Differentiation of parametrically defined functions:

- **Working rule:**

(a) If x and y are functions of parameter θ, then find $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ respectively.

(b) Now

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$$

For example,

$$x = \sin t + \cos t$$
$$y = \cos t$$
$$\frac{dx}{dt} = \cos t - \sin t;\ \frac{dy}{dt} = -\sin t$$
$$\Rightarrow \frac{dy}{dx} = \frac{-\sin t}{\cos t - \sin t}$$

Illustration 20.9 Find $\frac{dy}{dx}$, if $x = a(\theta - \sin\theta);\ y = a(1-\cos\theta)$.

Solution: Consider

$$x = a(\theta - \sin\theta);\ y = a(1-\cos\theta)$$
$$\Rightarrow \frac{dx}{d\theta} = a(1-\cos\theta);\ \frac{dy}{d\theta} = a(0+\sin\theta) = a\sin\theta$$

Therefore,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\div\frac{dx}{d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Illustration 20.10 If $y = e^{-ax^2}\sin(x\log_e x)$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\{e^{-ax^2}\sin(x\log_e x)\}$$
$$= e^{-ax^2}\frac{d\sin(x\log_e x)}{dx} + \sin(x\log_e x)\cdot\frac{d(e^{-ax^2})}{dx}$$
$$= e^{-ax^2}\frac{d\sin(x\log_e x)}{d(x\log_e x)}\cdot\frac{d}{dx}(x\log_e x)$$
$$+\sin(x\log_e x)\frac{d(e^{-ax^2})}{d(-ax^2)}\cdot\frac{d}{dx}(-ax^2)$$
$$= e^{-ax^2}\cos(x\log_e x)\cdot\left[1\cdot\log_e x + x\cdot\frac{1}{x}\right] + \sin(x\log_e x)e^{-ax^2}(-2ax)$$
$$= e^{-ax^2}\cos(x\log_e x)(\log_e x+1) - 2axe^{-ax^2}\sin(x\log_e x)$$

Illustration 20.11 If $y = \sqrt{1-x^2} + \frac{\cot x}{\sqrt{x}}$, find $\frac{dy}{dx}$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{1-x^2} + \frac{d}{dx}\left(\frac{\cot x}{\sqrt{x}}\right)$$
$$= \frac{d\sqrt{1-x^2}}{d(1-x^2)}\cdot\frac{d(1-x^2)}{dx} + \frac{\sqrt{x}\frac{d}{dx}(\cot x) - \cot x\cdot\frac{d}{dx}\sqrt{(x)}}{(\sqrt{x})^2}$$
$$= \frac{1}{2\sqrt{1-x^2}}(-2x) + \frac{\sqrt{x}(-\operatorname{cosec}^2 x) - \cot x\cdot\frac{1}{2\sqrt{x}}}{x}$$
$$= -\frac{x}{\sqrt{1-x^2}} - \frac{2x\operatorname{cosec}^2 x + \cot x}{2x^{3/2}}$$

Your Turn 1

1. Find $\frac{dy}{dx}$, if $y = \frac{x^2+1}{x}$.

Ans. $\frac{dy}{dx} = \frac{x^2-1}{x^2}$

2. Find $\frac{dy}{dx}$, if $y = \frac{x^3-1}{x}$.

Ans. $\frac{dy}{dx} = \frac{2x^2+1}{x^2}$

3. Find $\frac{dy}{dx}$ of

(a) $y = (x+2)(x+3)$

(b) $y = \frac{3x+4}{4x+5}$

(c) $y = \sqrt{ax^2+2bx+c}$

Ans. (a) $\frac{dy}{dx} = 2x+5$; (b) $\frac{dy}{dx} = -\frac{1}{(4x+5)^2}$;
(c) $\frac{dy}{dx} = (ax^2+2bx+c)^{-1/2}(ax+b)$

4. If $x = e^{-t^2}$ and $y = \tan^{-1}(2t+1)$, find $\frac{dy}{dx}$.

Ans. $\frac{dy}{dt} = \frac{-e^{t^2}}{2t(2t^2+2t+1)}$

5. Find dy/dx:

(a) $\sin\sqrt{\cos x}$ **(b)** $\sin\left(\log\sqrt{\frac{x}{x+1}}\right)$

(c) $\log\left(x+e^{\sqrt{x}}\right)$

Ans. (a) $\frac{dy}{dx} = -\frac{\sin x\cdot\cos x\sqrt{\cos x}}{2\sqrt{\cos x}}$;

(b) $\frac{dy}{dx} = \frac{1}{2}\cos\left(\log\sqrt{\frac{x}{x+1}}\right)\frac{1}{x(x+1)}$;

(c) $\frac{dy}{dx} = \frac{2\sqrt{x}+e^{\sqrt{x}}}{2\sqrt{x}\left(x+e^{\sqrt{x}}\right)}$

6. **Differentiation of implicit function:** If a relation between x and y is such that y cannot be expressed in terms of x, then y is called an implicit function of x. Here we will give method to find $\frac{dy}{dx}$ if y is an implicit function of x.
   - **Working rule:**

   (a) Differentiate the given relation between x and y with respect to x.

   (b) Bring all the terms containing $\frac{dy}{dx}$ on left-hand side and remaining terms on right-hand side and then find $\frac{dy}{dx}$.

   (c) Use the given relation between x and y to get the result in simplified form.

Illustration 20.12 If $y = \tan(x+y)$, find $\frac{dy}{dx}$.

Solution:

$$y = \tan(x+y) \qquad (1)$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan(x+y)] = \frac{d}{d(x+y)}\tan(x+y)\cdot\frac{d}{dx}(x+y)$$

$$= \sec^2(x+y)\cdot\left(1+\frac{dy}{dx}\right) = \sec^2(x+y)+\sec^2(x+y)\frac{dy}{dx}$$

or
$$[1-\sec^2(x+y)]\frac{dy}{dx} = \sec^2(x+y)$$

Hence,

$$\frac{dy}{dx} = \frac{\sec^2(x+y)}{1-\sec^2(x+y)} = \frac{1+\tan^2(x+y)}{1-\left\{1+\tan^2(x+y)\right\}} = \frac{1+y^2}{1-(1+y^2)}$$

$$= -\frac{1+y^2}{y^2} \qquad [\text{from Eq. (1), } y=\tan(x+y)]$$

Illustration 20.13 If $x = y + \cfrac{1}{y+\cfrac{1}{y+\cfrac{1}{y+\dots\text{to }\infty}}}$, prove that $\frac{dy}{dx} = 2x^2+y^2-3xy$.

Solution:

$$x = y + \cfrac{1}{y+\cfrac{1}{y+\cfrac{1}{y+\dots\text{to }\infty}}}$$

Hence,

$$x = y+\frac{1}{x} \qquad (1)$$

Differentiating with respect to x, we get

$$1 = \frac{dy}{dx}-\frac{1}{x^2} \text{ or } \frac{dy}{dx} = 1+\frac{1}{x^2}$$

or
$$\frac{dy}{dx} = 1+(x-y)^2 \qquad \left[\text{As from Eq. (1), } \frac{1}{x} = x-y\right]$$

$$= 1+x^2+y^2-2xy \qquad (2)$$

From Eq. (1),

$$x^2 = xy+1,\ \ 1 = x^2-xy$$

Putting in Eq. (2), we get

$$\frac{dy}{dx} = x^2-xy+x^2+y^2-2xy$$

Hence,

$$\frac{dy}{dx} = 2x^2+y^2-3xy$$

Illustration 20.14 If $x^3y^2 = \log_e(x+y)+\sin(e^x)$, find $\frac{dy}{dx}$.

Solution:

$$x^3y^2 = \log_e(x+y)+\sin(e^x) \qquad (1)$$

Differentiating with respect to x, we get

$$3x^2y^2+x^3\cdot 2y\frac{dy}{dx} = \frac{1}{x+y}\left(1+\frac{dy}{dx}\right)+\cos(e^x)\cdot e^x$$

or
$$3x^2y^2+2x^3y\frac{dy}{dx} = \frac{1}{x+y}+\frac{1}{x+y}\cdot\frac{dy}{dx}+e^x\cos(e^x)$$

or
$$\left[2x^3y-\frac{1}{x+y}\right]\frac{dy}{dx} = \frac{1+(x+y)e^x\cos(e^x)-3x^2y^2\cdot(x+y)}{x+y}$$

or
$$\frac{dy}{dx} = \frac{1+(x+y)e^x\cos(e^x)-3x^3y^2-3x^2y^3}{2x^4y+2x^3y^2-1}$$

7. **Logarithmic differentiation:** When u and v both are functions of x, then derivative of a function of the form u^v cannot be found directly by using standard formula and hence in such cases both sides are differentiated after taking logarithm. This process is called logarithmic differentiation.

Illustration 20.15 If $y = x^x$, then find $\frac{dy}{dx}$.

Solution: [Here power is variable]

Given,

$$y = x^x \qquad (1)$$

Taking logarithm we get,

$$\log_e y = \log_e(x^x)$$

or
$$\log_e y = x\log_e x$$

Differentiating with respect to x, we get

$$\frac{d}{dx}(\log_e y) = \frac{d}{dx}(x\log_e x)$$

or
$$\frac{d}{dy}(\log_e y) = \frac{d(x)}{dx}\cdot\log_e x + x\frac{d}{dx}(\log_e x)$$

or
$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\cdot\log_e x + x\cdot\frac{1}{x} = \log_e x+1$$

Hence,

$$\frac{dy}{dx} = y(1+\log_e x) = x^x(1+\log_e x) \ [\text{from Eq. (1), } y = x^x]$$

Illustration 20.16 If $x^my^n = (x+y)^{m+n}$, then find $\frac{dy}{dx}$.

Solution:

$$x^my^n = (x+y)^{m+n}$$

Taking logarithm, we get

$$\log_e(x^m)+\log_e(y^n) = (m+n)\log_e(x+y)$$

or
$$m\log_e x+n\log_e y = (m+n)\log_e(x+y)$$

Differentiating with respect to x, we get,

$$\frac{m}{x}+\frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right) = \frac{m+n}{x+y}+\frac{m+n}{x+y}\frac{dy}{dx}$$

or $$\left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx}=\frac{m+n}{x+y}-\frac{m}{x}$$

$$\frac{nx+ny-my-ny}{y(x+y)}\frac{dy}{dx}=\frac{mx+nx-mx-my}{x(x+y)}$$

or $$\frac{nx-my}{y(x+y)}\frac{dy}{dx}=\frac{nx-my}{x(x+y)}$$

Hence, $$\frac{dy}{dx}=\frac{y}{x}$$

Illustration 20.17 If $y=e^x\sin x^3+(\tan x)^x$, then find $\frac{dy}{dx}$.

Solution: Let $u=e^x\sin x^3$ and $v=(\tan x)^x$.

Now, $$u=e^x\sin x^3$$

Differentiating with respect to x, we get

$$\frac{du}{dx}=e^x\cdot\frac{d[\sin(x)^3]}{dx}+\sin x^3\cdot\frac{d}{dx}(e^x)$$

$$=e^x\cdot\cos x^3\cdot 3x^2+\sin x^3\cdot e^x$$

Hence, $$\frac{du}{dx}=3x^2e^x\cos x^3+e^x\sin x^3$$

and $$v=(\tan x)^x$$

Hence,

$$\log_e v=x\log_e(\tan x)$$

Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx}=1\cdot\log_e(\tan x)+x\cdot\frac{1}{\tan x}\sec^2 x$$

Hence, $$\frac{dv}{dx}=v[\log_e(\tan x)+x\cot x\cdot\sec^2 x]$$

$$=(\tan x)^x[\log_e(\tan x)+x\cot x\sec^2 x]$$

Now $$y=u+v$$

Hence, $$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$=3x^2e^x\cos(x^3)+e^x\sin(x^3)+(\tan x)^x[\log_e(\tan x)+x\cot x\sec^2 x]$$

8. **Differentiation by substitution:** Sometimes, it is easier to differentiate, by making substitutions. Usually these examples involve inverse trigonometric functions.

Illustration 20.18 If $f(x)=\cos^{-1}\left(\frac{5x+12\sqrt{1-x^2}}{13}\right)$, $\frac{5}{13}<x<1$, then find $f'\left(\frac{1}{2}\right)$.

Solution: Put $x=\cos\theta$, then

$$f(x)=\cos^{-1}\left(\frac{5\cos\theta+12\sin\theta}{13}\right)>0=\cos^{-1}[\cos(\theta-\phi)]>0,$$

where $$\phi=\cos^{-1}\left(\frac{5}{13}\right)$$

$$=\phi-\theta \text{ as } x\in\left(\frac{5}{13},1\right)=\cos^{-1}\left(\frac{5}{13}\right)-\cos^{-1}x$$

Hence,

$$f'(x)=\frac{1}{\sqrt{1-x^2}}\Rightarrow f'\left(\frac{1}{2}\right)=\frac{2}{\sqrt{3}}$$

Illustration 20.19 Differentiate $\cos^{-1}(4x^3-3x)$, where $0\le x\le 1$.

Solution: Put $x=\cos\theta$, then

$$\cos^{-1}(4x^3-3x)=\begin{cases}3\theta, & \theta\in\left[0,\frac{\pi}{3}\right]\\ 2\pi-3\theta, & \theta\in\left(\frac{\pi}{3},\frac{\pi}{2}\right]\end{cases}$$

$$\frac{d}{dx}[\cos^{-1}(4x^3-3x)]=-\frac{3}{\sqrt{1-x^2}} \text{ if } \frac{1}{2}<x\le 1$$

$$=\frac{3}{\sqrt{1-x^2}} \text{ if } 0\le x<\frac{1}{2}$$

20.5 Derivative of Second Order y'' or y_2

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the derivative of second order and is denoted by y'' or y_2

Illustration 20.20 If $y=\log_e(\log_e x)$, find y_2.

Solution: Let $y=\log_e(\log_e x)$. Then

$$y_1=\frac{1}{\log_e x}\cdot\frac{1}{x}=\frac{1}{x\log_e x}$$

$$y_2=\frac{d}{dx}\left(\frac{1}{x\log_e x}\right)=\frac{d}{dx}(x\log_e x)^{-1}=-1(x\log_e x)^{-2}\cdot\frac{d}{dx}(x\log_e x)$$

$$=-\frac{1}{(x\log_e x)^2}\cdot\left(x\cdot\frac{1}{x}+\log_e x\cdot 1\right)=\frac{-(1+\log_e x)}{(x\log_e x)^2}$$

Illustration 20.21 If $y=a\cos(\log_e x)+b\sin(\log_e x)$, prove that $x^2y''+xy'+y=0$.

Solution: Consider

$$y=a\cos(\log_e x)+b\sin(\log_e x) \quad (1)$$

Differentiating with respect to x, we get

$$y'=a\cdot\left[-\sin(\log_e x)\cdot\frac{1}{x}\right]+b\left[\cos(\log_e x)\cdot\frac{1}{x}\right]$$

$$\Rightarrow xy'=-a\sin(\log_e x)+b\cos(\log_e x)$$

Again differentiating with respect to x, we get

$$xy''+y'\cdot 1=-a\cdot\left\{\cos(\log_e x)\cdot\frac{1}{x}\right\}+b\cdot\left\{-\sin(\log_e x)\cdot\frac{1}{x}\right\}$$

$$\Rightarrow x^2y''+xy'=-[a\cos(\log_e x)+b\sin(\log_e x)]=-y \quad \text{[from Eq. (1)]}$$

$$\Rightarrow x^2y''+xy'+y=0$$

20.6 Differentiation of a Function with Respect to Another Function

Let $u = f(x)$ and $v = g(x)$ be two functions of x. Then to find derivative of $f(x)$ with respect to $g(x)$, that is, to find $\frac{du}{dv}$, we use the formula

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

Illustration 20.22 Differentiate $\log_e \sin x$ with respect to $\sqrt{\cos x}$.

Solution: Let $u = \log_e \sin x$ and $v = \sqrt{\cos x}$. Then

$$\frac{du}{dx} = \cot x \text{ and } \frac{dv}{dx} = \frac{-\sin x}{2\sqrt{\cos x}}$$

Hence,

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = -2\sqrt{\cos x}\cot x \cdot \text{cosecx}$$

Some Standard Differentiations:

1. **Differentiation of algebraic functions:** $\frac{d}{dx}x^n = nx^{n-1}$

   In particular

   (a) $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$

   (b) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

   (c) $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$

2. **Differentiation of trigonometric functions:**

   (a) $\frac{d}{dx}\sin x = \cos x$

   (b) $\frac{d}{dx}\cos x = -\sin x$

   (c) $\frac{d}{dx}\tan x = \sec^2 x$

   (d) $\frac{d}{dx}\sec x = \sec x \tan x$

   (e) $\frac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x \cot x$

   (f) $\frac{d}{dx}\cot x = -\text{cosec}^2 x$

3. **Differentiation of logarithmic and exponential functions:**

   (a) $\frac{d}{dx}\log x = \frac{1}{x}$, for $x > 0$

   (b) $\frac{d}{dx}e^x = e^x$

   (c) $\frac{d}{dx}a^x = a^x \log a$, for $a > 0$

   (d) $\frac{d}{dx}\log_a x = \frac{1}{x\log a}$, for $x > 0, a > 0, a \neq 1$

4. **Differentiation of inverse trigonometrical functions:**

   (a) $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

   (b) $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$, for $-1 < x < 1$

   (c) $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

   (d) $\frac{d}{dx}\text{cosec}^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

   (e) $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$, for $x \in R$

   (f) $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$, for $x \in R$

5. **Differentiation of hyperbolic functions:**

   (a) $\frac{d}{dx}\sinh x = \cosh x$

   (b) $\frac{d}{dx}\cosh x = \sinh x$

   (c) $\frac{d}{dx}\tanh x = \text{sech}^2 x$

   (d) $\frac{d}{dx}\coth x = -\text{cosech}^2 x$

   (e) $\frac{d}{dx}\text{sech}\, x = -\text{sech}\, x \tanh x$

   (f) $\frac{d}{dx}\text{cosech}\, x = -\text{cosech}\, x \coth x$

   (g) $\frac{d}{dx}\sinh^{-1} x = 1/\sqrt{(1+x^2)}$

   (h) $\frac{d}{dx}\cosh^{-1} x = 1/\sqrt{(x^2-1)}$

   (i) $\frac{d}{dx}\tanh^{-1} x = 1/(x^2-1)$

   (j) $\frac{d}{dx}\coth^{-1} x = 1/(1-x^2)$

   (k) $\frac{d}{dx}\text{sech}^{-1} x = -1/x\sqrt{(1-x^2)}$

   (l) $\frac{d}{dx}\text{cosech}^{-1} x = -1/x\sqrt{(1+x^2)}$

Euler's Theorem on Homogeneous Functions

If $f(x, y)$ is a homogeneous function in x, y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Deduction of Euler's theorem

If $f(x, y)$ is a homogeneous function in x, y of degree n, then

1. $x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} = (n-1)\frac{\partial f}{\partial x}$

2. $x\frac{\partial^2 f}{\partial y \partial x} + y\frac{\partial^2 f}{\partial y^2} = (n-1)\frac{\partial f}{\partial y}$

3. $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y)$

Your Turn 2

1. Find $\frac{dy}{dx}$ of $\ln(xy) = x^2 + y^2$.

   Ans. $\frac{dy}{dx} = \frac{y(2x^2-1)}{x(1-2y^2)}$

2. Find $\frac{dy}{dx}$ of $x + y = \sin(xy)$.

Ans. $\frac{dy}{dx} = \frac{y\cos(xy)-1}{1-x\cos(xy)}$

3. Differentiate $(\log x)^{\tan x}$ with respect to $\sin(m\cos^{-1}x)$.

Ans. $\frac{-(\log x)^{\tan x}\left(\sec^2 x\cdot\log\log x+\frac{\tan x}{x\log x}\right)\sqrt{1-x^2}}{m\cos(m\cos^{-1}x)}$

4. If $x^y \cdot y^x = 1$, then find $\frac{dy}{dx}$.

Ans. $\frac{dy}{dx} = -\frac{(y+x\log y)}{(x+y\log x)}\cdot\frac{y}{x}$

5. Differentiate $\sin^2 x$ with respect to $(\log x)^2$.

Ans. $\frac{x\sin x\cos x}{\log x}$

6. If $x^2 + y^2 + xy = 2$, then find $\frac{dy}{dx}$.

Ans. $\frac{dy}{dx} = -\frac{(2x+y)}{(2y+x)}$

7. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\cdots\infty}}}$, then find $\frac{dy}{dx}$.

Ans. $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

8. If $5f(x) + 3f\left(\frac{1}{x}\right) = x + 2$ and $y = xf(x)$, then find $\frac{dy}{dx}$ at $x = 1$.

Ans. $\left(\frac{dy}{dx}\right)_{\text{at } x=1} = \frac{7}{8}$

Additional Solved Examples

1. If $f(x) = \sin x$, $g(x) = x^2$, $h(x) = \log x$ and $F(x) = (hogof)(x)$, then $F''(x)$ is

(A) $2\text{cosec}^3 x$ **(B)** $2\cot x^2 - 4x^2\text{cosec}^2 x^2$
(C) $2x\cot x^2$ **(D)** $-2\text{cosec}^2 x$

Solution:

$$F(x) = (hogof)(x) = h\{g[f(x)]\}$$
$$= h[g(\sin x)] = h(\sin^2 x) = \log_e(\sin^2 x)$$
$$= 2\log_e(\sin x)$$

Therefore, $F'(x) = 2\cot x \Rightarrow F''(x) = -2\,\text{cosec}^2 x$

Hence, the correct answer is option (D).

2. $x = e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$, then $\frac{dy}{dx}$ is

(A) $x\left(2+\frac{y^2}{x^4}-\frac{2y}{x^2}\right)+\frac{2y}{x}$ **(B)** $x\left(2+\frac{y^2}{x^4}-\frac{2y}{x^2}\right)$
(C) $\left(2+\frac{y^2}{x^4}-\frac{2y}{x^2}\right)$ **(D)** None of these

Solution:

$$x = e^{\tan^{-1}((y-x^2)/x^2)}$$

Taking log on both sides, we get

$$\log x = \tan^{-1}\left(\frac{y}{x^2}-1\right)$$
$$\Rightarrow \frac{y}{x^2}-1 = \tan\log x$$
$$\Rightarrow \frac{y}{x^2} = 1 + \tan\log x$$
$$\Rightarrow y = x^2 + x^2\tan\log x$$

Now,

$$\frac{dy}{dx} = 2x + 2x\tan\log x + x\sec^2(\log x)\times\frac{1}{x}$$
$$= 2x + 2x\tan\log x + x\sec^2(\log x)$$
$$= \frac{2}{x}(x^2 + x^2\tan\log x) + x\sec^2(\log x) = \frac{2y}{x} + 1 + \tan^2(\log x)$$
$$\Rightarrow \frac{2y}{x}+x\left[1+\left(\frac{y}{x^2}-1\right)^2\right] = \frac{2y}{x}+1+\frac{y^2}{x^4}+1-\frac{2y}{x^2}$$
$$= \frac{xy^2}{x^4}-\frac{2yx}{x^2}+\frac{2y}{x}+2x$$
$$= \frac{y^2}{x^3}-\frac{2y}{x}+\frac{2y}{x}+2x = \frac{y^2}{x^3}+2x$$

Hence, the correct answer is option (A).

3. If $y=(x+\sqrt{1+x^2})^m$, then $(1+x^2)y_2+xy_1-m^2y =$ _____.

(A) 0 **(B)** 1
(C) -1 **(D)** 2

Solution: Substituting the value of y in the given equation, we have

$$y_1 = m\left[x+\sqrt{1+x^2}\right]^{m-1}\cdot\left(1+\frac{1\cdot 2x}{2\sqrt{1+x^2}}\right)$$
$$= \frac{m\left(x+\sqrt{1+x^2}\right)^m}{\sqrt{1+x^2}}$$
$$= \frac{my}{\sqrt{1+x^2}}$$
$$\Rightarrow y_1^2(1+x^2) = m^2y^2$$

Differentiating with respect to x, we get

$$2y_1y_2(1+x^2)+y_1^2(2x) = 2m^2yy_1$$

Cancelling $2y_1$, we get

$$(1+x^2)y_2+xy_1 = m^2y$$
$$\Rightarrow (1+x^2)y_2+xy_1-m^2y = 0$$

Hence, the correct answer is option (A).

4. If $\sqrt{x+y}+\sqrt{y-x}=c$, then $\frac{d^2y}{dx^2}$ is

(A) $\frac{2}{c}$ **(B)** $\frac{-2}{c^2}$
(C) $\frac{2}{c^2}$ **(D)** None of these

Solution: We are given that

$$\sqrt{x+y}+\sqrt{y-x} = c \qquad (1)$$

Also

$$(\sqrt{x+y})^2-(\sqrt{y-x})^2 = x+y-(y-x)$$
$$\Rightarrow (\sqrt{x+y}+\sqrt{y-x})(\sqrt{x+y}-\sqrt{y-x}) = 2x$$

By Eq. (1), we have

$$\sqrt{x+y}-\sqrt{y-x}=\frac{2x}{c} \qquad (2)$$

Adding Eqs. (1) and (2), we have

$$2\sqrt{x+y}=c+\frac{2x}{c}$$

Squaring both the sides, we get

$$4(x+y)=c^2+\frac{4x^2}{c^2}+4x$$

After cancelling $4x$ from both the sides, we get

$$4y=c^2+\frac{4x^2}{c^2}$$

$$\Rightarrow 4\frac{dy}{dx}=\frac{8x}{c^2}\Rightarrow\frac{d^2y}{dx^2}=\frac{2}{c^2}$$

Hence, the correct answer is option (C).

5. If $f(x)=(1+x)^n$, then the value of $f(0)+f'(0)+\frac{f''(0)}{2!}+\cdots+\frac{1}{n!}f^n(0)$ is

(A) n **(B)** 2^n
(C) 2^{n-1} **(D)** None of these

Solution:

$$f(0)=1, f'(x)=n(1+x)^{n-1}$$

$$f''(x)=n(n-1)(1+x)^{n-2},\ldots,$$

$$f^{(n)}(x)=n(n-1)\ldots1=n!$$

$$\Rightarrow f'(0)=n, f''(0)=n(n-1),\ldots,\ f^n(0)=n!$$

Therefore, given expression is

$$1+\frac{n}{1}+\frac{n(n-1)}{2!}+\cdots+\frac{n!}{n!}$$

$$={}^nC_0+{}^nC_1+{}^nC_2+\cdots+{}^nC_n=2^n$$

Hence, the correct answer is option (B).

6. If $y=\sin(\sin x)$, and $\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+f(x)=0$, then $f(x)$ is

(A) $\sin^2x\sin(\cos x)$ **(B)** $\sin^2x\cos(\sin x)$
(C) $\cos^2x\sin(\cos x)$ **(D)** $\cos^2x\sin(\sin x)$

Solution:

$$\frac{dy}{dx}=\cos(\sin x)\cdot\cos x$$

$$\Rightarrow\frac{d^2y}{dx^2}=-\cos(\sin x)\cdot\sin x+\cos x[-\sin(\sin x)\cos x]$$

$$\Rightarrow\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x=-\cos(\sin x)\cdot\sin x-\cos^2x\cdot\sin(\sin x)$$
$$+\cos(\sin x)\cdot\cos x\cdot\tan x$$
$$=-\cos^2x\cdot\sin(\sin x)$$
$$\Rightarrow f(x)=\cos^2x\cdot\sin(\sin x)$$

Hence, the correct answer is option (D).

7. If $x=2\cos t-\cos 2t$ and $y=2\sin t-\sin 2t$, then the value of $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{2}$ is

(A) 3/2 **(B)** −5/2
(C) 5/2 **(D)** −3/2

Solution:

$$\frac{dx}{dt}=-2\sin t+2\sin 2t$$

$$\frac{dy}{dt}=2\cos t-2\cos 2t$$

Now,

$$\frac{dy}{dx}=\frac{2\cos t-2\cos 2t}{-2\sin t+2\sin 2t}=\frac{\cos t-\cos 2t}{\sin 2t-\sin t}$$

$$=\frac{2\sin\frac{3t}{2}\sin\frac{t}{2}}{2\cos\frac{3t}{2}\sin\frac{t}{2}}=\tan\frac{3t}{2}$$

Therefore,

$$\frac{d^2y}{dx^2}=\sec^2\frac{3t}{2}\times\frac{3}{2}\times\frac{dt}{dx}$$

$$=\frac{3}{2}\sec^2\frac{3t}{2}\cdot\frac{1}{2\sin 2t-2\sin t}$$

$$\Rightarrow\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}}=-\frac{3}{2}$$

Hence, the correct answer is option (D).

8. If $x=\sec\theta-\cos\theta,\ y=\sec^n\theta-\cos^n\theta$, then $\left(\frac{dy}{dx}\right)^2$ is equal to

(A) $\frac{n^2(y^2+4)}{x^2+4}$ **(B)** $\frac{n^2(y^2-4)}{x^2}$
(C) $n\frac{y^2-4}{x^2-4}$ **(D)** $\left(\frac{ny}{x}\right)^2-4$

Solution:

$$\frac{dy}{d\theta}=n\sec^{n-1}\theta\cdot\sec\theta\cdot\tan\theta-n\cdot\cos^{n-1}\theta\cdot(-\sin\theta)$$

$$=n\left[\sec^n\theta\frac{\sin\theta}{\cos\theta}+\cos^{n-1}\theta\cdot\sin\theta\right]$$

$$=\frac{n\sin\theta}{\cos\theta}\left[\sec^n\theta+\cos^n\theta\right]$$

$$=n\tan\theta(\sec^n\theta+\cos^n\theta)$$

and

$$\frac{dx}{d\theta}=\sec\theta\tan\theta+\sin\theta=\sec\theta\frac{\sin\theta}{\cos\theta}+\sin\theta$$

$$=\frac{\sin\theta}{\cos\theta}(\sec\theta+\cos\theta)=\tan\theta(\sec\theta+\cos\theta)$$

Therefore,

$$\frac{dy}{dx}=\frac{n\tan\theta(\sec^n\theta+\cos^n\theta)}{\tan\theta(\sec\theta+\cos\theta)}$$

$$=\frac{n(\sec^n\theta+\cos^n\theta)}{\sec\theta+\cos\theta}$$

Hence,

$$\left(\frac{dy}{dx}\right)^2=\frac{n^2(\sec^n\theta+\cos^n\theta)^2}{(\sec\theta+\cos\theta)^2}$$

$$=\frac{n^2\left[(\sec^n\theta-\cos^n\theta)^2+4\right]}{(\sec\theta-\cos\theta)^2+4}$$

$$= \frac{n^2(y^2+4)}{x^2+4}$$

Hence, the correct answer is option (A).

9. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx}$ is

(A) $\cos x^2 \cdot f'(x)$ **(B)** $-\cos x^2 \cdot f'(x)$

(C) $\frac{2(1+x-x^2)}{(x^2+1)^2}\sin\left(\frac{2x-1}{x^2+1}\right)^2$ **(D)** None of these

Solution: Let $\frac{2x-1}{x^2+1} = z$. Then

$$y = f(z)$$

Therefore,

$$\frac{dy}{dx} = f'(z)\cdot\frac{dz}{dx}$$

$$= \sin z^2 \cdot \frac{dz}{dx} \qquad [\text{because } f'(z) = \sin z^2]$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2 \frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2 \frac{2(1+x-x^2)}{(x^2+1)^2}.$$

Hence, the correct answer is option (C).

10. If $y^2 = p(x)$, a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ is equal to

(A) $p'''(x) + p'(x)$ **(B)** $p'''(x) + p''(x)$

(C) $p(x)\,p'''(x)$ **(D)** a constant

Solution:

$$y^2 = p(x) \qquad (1)$$

$$\Rightarrow 2y\,y_1 = p'(x) \qquad (2)$$

$$\Rightarrow 2(y\,y_2 + y_1 y_1) = p''(x)$$

$$\Rightarrow y\,y_2 = \frac{1}{2}[p''(x) - 2y_1^{\,2}]$$

Multiplying both sides by y^2, we have

$$\Rightarrow y^3\,y_2 = \frac{1}{2}[p''(x)y^2 - 2(y\,y_1)^2]$$

$$= \frac{1}{2}p''(x)\cdot p(x) - \left[\frac{1}{2}p'(x)\right]^2 \quad [\text{using Eqs. (1) and (2)}]$$

Now,

$$\frac{d}{dx}(y^3\,y_2) = \frac{1}{2}[p'''(x)p(x) + p'(x)p''(x)] - 2\times\left[\frac{1}{2}p'(x)\right]\times\left[\frac{1}{2}p''(x)\right]$$

$$\Rightarrow 2\frac{d}{dx}(y^3\,y_2) = p(x)p'''(x)$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Main/AIEEE Questions

1. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

(A) -1 **(B)** 1

(C) log 2 **(D)** $-$log 2

[AIEEE 2009]

Solution:

$$x^{2x} - 2x^x \cot y - 1 = 0 \qquad (1)$$

Now at $x = 1$,

$$1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \frac{\pi}{2}$$

On differentiating Eq. (1) with respect to x, we get

$$2x^{2x}(1+\log x) - 2\left[x^x(-\operatorname{cosec}^2 y)\frac{dy}{dx} + \cot y\, x^x(1+\log x)\right] = 0$$

At $\left(1, \frac{\pi}{2}\right)$, we have

$$2(1+\log 1) - 2\left(1(-1)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} + 0\right) = 0$$

$$\Rightarrow 2 + 2\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = -1$$

Hence, the correct answer is option (A).

2. $\frac{d^2x}{dy^2}$ equals

(A) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ **(B)** $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

(C) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ **(D)** $\left(\frac{d^2y}{dx^2}\right)^{-1}$

[AIEEE 2011]

Solution:

$$\frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right) = -\frac{1}{\left(\frac{dy}{dx}\right)^2}\frac{d}{dy}\left(\frac{dy}{dx}\right)$$

$$= -\left(\frac{dy}{dx}\right)^{-2}\frac{1}{\left(\frac{dy}{dx}\right)}\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

Hence, the correct answer is option (C).

3. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(A) $\frac{1}{2}$ **(B)** 1

(C) $\sqrt{2}$ **(D)** $\frac{1}{\sqrt{2}}$

[AIEEE 2013]

Solution: We have,

$$y = \sec(\tan^{-1}x)$$

Therefore,

$$\frac{dy}{dx} = \sec(\tan^{-1}x)\tan(\tan^{-1}x)\cdot\frac{1}{1+x^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \sqrt{2}\times 1\times\frac{1}{2} = \frac{1}{\sqrt{2}}$$

Hence, the correct answer is option (D).

4. If $y = e^{nx}$, then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$ is equal to

(A) ne^{nx} **(B)** ne^{-nx}

(C) 1 **(D)** $-ne^{-nx}$

[JEE MAIN 2014 (ONLINE PAPER SET 1)]

Solution:

$$y = e^{nx}$$

$$\frac{dy}{dx} = e^{nx}(n) \Rightarrow \frac{d^2y}{dx^2} = n^2e^{nx} \quad (1)$$

Now

$$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{ne^{nx}}\right) = \frac{1}{n}\frac{d}{dy}\left(e^{-nx}\right)$$

$$= \frac{1}{n}e^{-nx}(-n)\frac{dx}{dy} = -e^{nx}\left(\frac{1}{ne^{nx}}\right) = -\frac{1}{n}e^{-2nx} \quad (2)$$

Now from Eqs. (1) and (2), we get

$$\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = n^2e^{nx} \times \left(-\frac{1}{n}\right)e^{-2nx} = -n\,e^{-nx}$$

Hence, the correct answer is option (D).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. $\frac{d^2x}{dy^2}$ equals

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ **(B)** $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ **(D)** $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

[IIT-JEE 2007]

Solution: Since $\frac{dy}{dx} \times \frac{dx}{dy} = 1$, we get

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \left(\frac{dy}{dx}\right)^{-1}$$

$$\Rightarrow \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)^{-1} \times \frac{dx}{dy}$$

$$\Rightarrow \frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-2} \times \frac{d^2y}{dx^2} \times \left(\frac{dx}{dy}\right)$$

$$= -\frac{d^2y}{dx^2} \times \left(\frac{dy}{dx}\right)^{-3}$$

Hence, the correct answer is option (D).

2. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is ______. **[IIT-JEE 2009]**

Solution:

$$f(0) = 1, f'(x) = 3x^2 + \frac{1}{2}e^{x/2}$$

$$\Rightarrow f'[g(x)]g'(x) = 1$$

Put $x = 0$, we get

$$g'(1) = \frac{1}{f'(0)} = 2.$$

Hence, the correct answer is (2).

3. Let $f(\theta) = \sin\left[\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right]$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}[f(\theta)]$ is ______. **[IIT-JEE 2011]**

Solution:

$$\sin\left[\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right], \text{ where } \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$\sin\left[\tan^{-1}\left(\frac{\sin\theta}{\sqrt{2\cos^2\theta - 1}}\right)\right]$$

$$= \sin[\sin^{-1}(\tan\theta)] = \tan\theta$$

$$\frac{d(\tan\theta)}{d(\tan\theta)} = 1$$

Hence, the correct answer is (1).

Practice Exercise 1

1. If $\cos(x+y) = y\sin x$, then $\frac{dy}{dx}$ is equal to

(A) $-\frac{\sin(x+y) + y\cos x}{\sin x + \sin(x+y)}$ **(B)** $\frac{\sin(x+y) + y\cos x}{\sin x + \sin(x+y)}$

(C) $\frac{y\cos x - \sin(x+y)}{\sin x - \sin(x+y)}$ **(D)** None of these

2. If $\sin^2 x + 2\cos y + xy = 0$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{y + 2\sin x}{2\sin y + x}$ **(B)** $\frac{y + \sin 2x}{2\sin y - x}$

(C) $\frac{y + 2\sin x}{\sin y + x}$ **(D)** None of these

3. If $x^3 + 8xy + y^3 = 64$, then $\frac{dy}{dx}$ is equal to

(A) $-\frac{3x^2 + 8y}{8x + 3y^2}$ **(B)** $\frac{3x^2 + 8y}{8x + 3y^2}$

(C) $\frac{3x + 8y^2}{8x^2 + 3y}$ **(D)** None of these

4. If $\sin(x+y) = \log(x+y)$, then $\frac{dy}{dx}$ is equal to

(A) 2 **(B)** -2

(C) 1 **(D)** -1

5. If $\ln(x+y) = 2xy$, then $y'(0) =$

(A) 1 **(B)** -1

(C) 2 **(D)** 0

6. If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is equal to

(A) $\log x \cdot [\log(ex)]^{-2}$ **(B)** $\log x \cdot [\log(ex)]^2$

(C) $\log x \cdot (\log x)^2$ **(D)** None of these

7. $y=(\tan x)^{(\tan x)^{\tan x}}$, then at $x=\frac{\pi}{4}$, the value of $\frac{dy}{dx}$ is equal to

(A) 0 **(B)** 1
(C) 2 **(D)** None of these

8. If $y=(\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to

(A) $(\sin x)^{\tan x}\cdot(1+\sec^2 x\cdot\log\sin x)$
(B) $\tan x\cdot(\sin x)^{\tan x-1}\cdot\cos x$
(C) $(\sin x)^{\tan x}\cdot\sec^2 x\cdot\log\sin x$
(D) $\tan x\cdot(\sin x)^{\tan x-1}$

9. If $y=2^{1/\log_x 4}$, then x is equal to

(A) $\sqrt{y}$ **(B)** y
(C) y^2 **(D)** y^4

10. The derivative of $y=x^{\ln x}$ is

(A) $x^{\ln x}\ln x$ **(B)** $x^{\ln x-1}\ln x$
(C) $2x^{\ln x-1}\ln x$ **(D)** $x^{\ln x-2}$

11. If $y=\sin(2\sin^{-1}x)$, then $\frac{dy}{dx}$ is equal to

(A) $\frac{2-4x^2}{\sqrt{1-x^2}}$ **(B)** $\frac{2+4x^2}{\sqrt{1-x^2}}$
(C) $\frac{2-4x^2}{\sqrt{1+x^2}}$ **(D)** $\frac{2+4x^2}{\sqrt{1+x^2}}$

12. If $y=\cos^{-1}\left(\frac{3\cos x+4\sin x}{5}\right)$, then $\frac{dy}{dx}$

(A) 0 **(B)** 1
(C) -1 **(D)** $\frac{1}{2}$

13. $\frac{d}{dx}\cos^{-1}\frac{x-x^{-1}}{x+x^{-1}}$ is equal to

(A) $\frac{1}{1+x^2}$ **(B)** $\frac{-1}{1+x^2}$
(C) $\frac{2}{1+x^2}$ **(D)** $\frac{-2}{1+x^2}$

14. $\frac{d}{dx}\cos^{-1}\sqrt{\frac{1+x^2}{2}}$ is equal to

(A) $\frac{-1}{2\sqrt{1-x^4}}$ **(B)** $\frac{1}{2\sqrt{1-x^4}}$
(C) $\frac{-x}{\sqrt{1-x^4}}$ **(D)** $\frac{x}{\sqrt{1-x^4}}$

15. Differential coefficient of $\sin^{-1}\frac{1-x}{1+x}$ with respect to $\sqrt{x}$ is

(A) $\frac{1}{2\sqrt{x}}$ **(B)** $\frac{\sqrt{x}}{\sqrt{1-x}}$
(C) 1 **(D)** None of these

16. The differential coefficient of $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$ is

(A) 1 **(B)** -1
(C) 0 **(D)** None of these

17. If $y=x\log\left(\frac{x}{a+bx}\right)$, then $x^3\frac{d^2y}{dx^2}$ is equal to

(A) $x\frac{dy}{dx}-y$ **(B)** $\left(x\frac{dy}{dx}-y\right)^2$
(C) $y\frac{dy}{dx}-x$ **(D)** $\left(y\frac{dy}{dx}-x\right)^2$

18. If $y=x^3\log\log_e(1+x)$, then $y''(0)$ equals

(A) 0 **(B)** -1
(C) $6\log_e 2$ **(D)** 6

19. $\frac{d^2x}{dy^2}$ is equal to

(A) $\frac{1}{(dy/dx)^2}$ **(B)** $\frac{(d^2y/dx^2)}{(dy/dx)^2}$
(C) $\frac{d^2y}{dx^2}$ **(D)** $\frac{(-d^2y/dx^2)}{(dy/dx)^2}$

20. If $f_n(x)$, $g_n(x)$, $h_n(x)$, $n=1,2,3$ are polynomials in x such that $f_n(a)=g_n(a)=h_n(a)$, $n=1,2,3$ and $F(x)=\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$, then $F'(a)$ is equal to

(A) 0 **(B)** $f_1(a)g_2(a)h_3(a)$
(C) 1 **(D)** None of these

21. Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}$, where p is a constant. Then $\frac{d^3}{dx^3}[f(x)]$ at $x=0$ is

(A) p **(B)** $p+p^2$
(C) $p+p^3$ **(D)** Independent of p

22. $f(x)=\begin{vmatrix} x^3 & x^2 & 3x^2\\ 1 & -6 & 4\\ p & p^2 & p^3\end{vmatrix}$, here p is a constant, then $\frac{d^3f(x)}{dx^3}$ is

(A) Proportional to x^2 **(B)** Proportional to x
(C) Proportional to x^3 **(D)** A constant

23. If $y=\sin px$ and y_n is the n^{th} derivative of y, then $\begin{vmatrix} y & y_1 & y_2\\ y_3 & y_4 & y_5\\ y_6 & y_7 & y_8\end{vmatrix}$ is equal to

(A) 1 **(B)** 0
(C) – 1 **(D)** None of these

24. If $y^2 = ax^2 + bx + c$, then $y^3 \dfrac{d^2y}{dx^2}$ is
(A) A constant
(B) A function of x only
(C) A function of y only
(D) A function of x and y

25. If $y = a^x \cdot b^{2x-1}$, then $\dfrac{d^2y}{dx^2}$ is
(A) $y^2 \cdot \log ab^2$ **(B)** $y \cdot \log ab^2$
(C) y^2 **(D)** $y \cdot (\log ab^2)^2$

26. $\dfrac{d}{dx}\log(\log x)$ is equal to
(A) $\dfrac{x}{\log x}$ **(B)** $\dfrac{\log x}{x}$
(C) $(x \log x)^{-1}$ **(D)** None of these

27. $\dfrac{d}{dx}\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$ is equal to
(A) $1 - \dfrac{1}{x^2}$ **(B)** $1 + \dfrac{1}{x^2}$
(C) $1 - \dfrac{1}{2x}$ **(D)** None of these

28. If $y = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \infty$, then $\dfrac{dy}{dx}$ is equal to
(A) y **(B)** $y - 1$
(C) $y + 1$ **(D)** None of these

29. $\dfrac{d}{dx}\left(\tan^{-1} \dfrac{\cos x}{1 + \sin x}\right)$ is equal to
(A) $-\dfrac{1}{2}$ **(B)** $\dfrac{1}{2}$
(C) -1 **(D)** 1

30. $\dfrac{d}{dx}[\cos(1 - x^2)^2]$ is equal to
(A) $-2x(1 - x^2)\sin(1 - x^2)^2$
(B) $-4x(1 - x^2)\sin(1 - x^2)^2$
(C) $4x(1 - x^2)\sin(1 - x^2)^2$
(D) $-2(1 - x^2)\sin(1 - x^2)^2$

31. If $y = \sin^{-1}(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2})$, then $\dfrac{dy}{dx}$ is equal to
(A) $\dfrac{-2x}{\sqrt{1-x^2}} + \dfrac{1}{2\sqrt{x - x^2}}$ **(B)** $\dfrac{-1}{\sqrt{1-x^2}} - \dfrac{1}{2\sqrt{x - x^2}}$
(C) $\dfrac{1}{\sqrt{1-x^2}} + \dfrac{1}{2\sqrt{x - x^2}}$ **(D)** None of these

32. $\dfrac{d}{dx}\left(\tan^{-1}\sqrt{\dfrac{1 + \cos\frac{x}{2}}{1 - \cos\frac{x}{2}}}\right)$ is equal to
(A) $-\dfrac{1}{4}$ **(B)** $\dfrac{1}{2}$
(C) $-\dfrac{1}{2}$ **(D)** $\dfrac{1}{4}$

33. $\dfrac{d}{dx}\sqrt{\dfrac{1 - \sin 2x}{1 + \sin 2x}}$ is equal to
(A) $\sec^2 x$ **(B)** $-\sec^2\left(\dfrac{\pi}{4} - x\right)$
(C) $\sec^2\left(\dfrac{\pi}{4} + x\right)$ **(D)** $\sec^2\left(\dfrac{\pi}{4} - x\right)$

34. If $f(x) = x\tan^{-1}x$, then $f'(1)$ is equal to
(A) $1 + \dfrac{\pi}{4}$ **(B)** $\dfrac{1}{2} + \dfrac{\pi}{4}$
(C) $\dfrac{1}{2} - \dfrac{\pi}{4}$ **(D)** 2

35. If $f(x) = \log_x(\log x)$, then $f'(x)$ at $x = e$ is
(A) e **(B)** $\dfrac{1}{e}$
(C) 1 **(D)** None of these

36. If $f(x) = \sqrt{1 + \cos^2(x^2)}$, then $f'\left(\dfrac{\sqrt{\pi}}{2}\right)$ is
(A) $\sqrt{\pi}/6$ **(B)** $-\sqrt{(\pi/6)}$
(C) $1/\sqrt{6}$ **(D)** $\pi/\sqrt{6}$

37. $\dfrac{d}{dx}\tan^{-1}(\sec x + \tan x)$ is equal to
(A) 1 **(B)** 1/2
(C) $\cos x$ **(D)** $\sec x$

38. $\dfrac{d}{dx}(e^x \log \sin 2x)$
(A) $e^x(\log \sin 2x + 2\cot 2x)$
(B) $e^x(\log \cos 2x + 2\cot 2x)$
(C) $e^x(\log \cos 2x + \cot 2x)$
(D) None of these

39. If $y = \sec^{-1}\left(\dfrac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\dfrac{dy}{dx}$ is equal to
(A) 0 **(B)** $\dfrac{1}{\sqrt{x}+1}$
(C) 1 **(D)** None of these

40. $\dfrac{d}{dx}\sin^{-1}(3x - 4x^3)$ is equal to
(A) $\dfrac{3}{\sqrt{1-x^2}}$ **(B)** $\dfrac{-3}{\sqrt{1-x^2}}$
(C) $\dfrac{1}{\sqrt{1-x^2}}$ **(D)** $\dfrac{-1}{\sqrt{1-x^2}}$

41. If $y=\sqrt{\dfrac{1+\tan x}{1-\tan x}}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{1}{2}\sqrt{\dfrac{1-\tan x}{1+\tan x}}\cdot\sec^2\left(\dfrac{\pi}{4}+x\right)$

(B) $\sqrt{\dfrac{1-\tan x}{1+\tan x}}\cdot\sec^2\left(\dfrac{\pi}{4}+x\right)$

(C) $\dfrac{1}{2}\sqrt{\dfrac{1-\tan x}{1+\tan x}}\cdot\sec\left(\dfrac{\pi}{4}+x\right)$

(D) None of these

42. $\dfrac{d}{dx}\left(\dfrac{\sec x+\tan x}{\sec x-\tan x}\right)$ is equal to

(A) $\dfrac{2\cos x}{(1-\sin x)^2}$ **(B)** $\dfrac{\cos x}{(1-\sin x)^2}$

(C) $\dfrac{2\cos x}{1-\sin x}$ **(D)** None of these

43. $\dfrac{d}{dx}\left[\log\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right]$ is equal to

(A) $\sec x$ **(B)** $\operatorname{cosec} x$

(C) $\operatorname{cosec}\dfrac{x}{2}$ **(D)** $\sec\dfrac{x}{2}$

44. $\dfrac{d}{dx}\left[\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right]$ is equal to

(A) $-\dfrac{1}{2}$ **(B)** 0

(C) $\dfrac{1}{2}$ **(D)** 1

45. If $f(x)=\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$, then $f'\left(\dfrac{\pi}{3}\right)$ is equal to

(A) $\dfrac{1}{2(1+\cos x)}$ **(B)** $\dfrac{1}{2}$

(C) $\dfrac{1}{4}$ **(D)** None of these

46. $\dfrac{d}{dx}\left[\dfrac{e^{ax}}{\sin(bx+c)}\right]$ is equal to

(A) $\dfrac{e^{ax}[a\sin(bx+c)+b\cos(bx+c)]}{\sin^2(bx+c)}$

(B) $\dfrac{e^{ax}[a\sin(bx+c)-b\cos(bx+c)]}{\sin(bx+c)}$

(C) $\dfrac{e^{ax}[a\sin(bx+c)-b\cos(bx+c)]}{\sin^2(bx+c)}$

(D) None of these

47. If $y=\dfrac{e^x\log x}{x^2}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{e^x[1+(x+2)\log x]}{x^3}$ **(B)** $\dfrac{e^x[1-(x-2)\log x]}{x^4}$

(C) $\dfrac{e^x[1-(x-2)\log x]}{x^3}$ **(D)** $\dfrac{e^x[1+(x-2)\log x]}{x^3}$

48. If $y=\dfrac{e^{2x}\cos x}{x\sin x}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{e^{2x}[(2x-1)\cot x-x\operatorname{cosec}^2 x]}{x^2}$

(B) $\dfrac{e^{2x}[(2x+1)\cot x-x\operatorname{cosec}^2 x]}{x^2}$

(C) $\dfrac{e^{2x}[(2x-1)\cot x+x\operatorname{cosec}^2 x]}{x^2}$

(D) None of these

49. $\dfrac{d}{dx}[e^{-ax^2}\log(\sin x)]$

(A) $e^{-ax^2}(\cot x+2ax\log\sin x)$

(B) $e^{-ax^2}(\cot x+ax\log\sin x)$

(C) $e^{-ax^2}(\cot x-2ax\log\sin x)$

(D) None of these

50. If $y=\log x\cdot e^{(\tan x+x^2)}$, then $\dfrac{dy}{dx}$ is equal to

(A) $e^{(\tan x+x^2)}\left[\dfrac{1}{x}+(\sec^2 x+x)\log x\right]$

(B) $e^{(\tan x+x^2)}\left[\dfrac{1}{x}+(\sec^2 x-x)\log x\right]$

(C) $e^{(\tan x+x^2)}\left[\dfrac{1}{x}+(\sec^2 x+2x)\log x\right]$

(D) $e^{(\tan x+x^2)}\left[\dfrac{1}{x}+(\sec^2 x-2x)\log x\right]$

51. If $y=\sqrt{\dfrac{1+e^x}{1-e^x}}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$ **(B)** $\dfrac{e^x}{(1-e^x)\sqrt{1-e^x}}$

(C) $\dfrac{e^x}{(1-e^x)\sqrt{1+e^{2x}}}$ **(D)** $\dfrac{e^x}{(1-e^x)\sqrt{1+e^x}}$

52. $\dfrac{d}{dx}[e^x\log(1+x^2)]$ is equal to

(A) $e^x\left[\log(1+x^2)+\dfrac{2x}{1+x^2}\right]$

(B) $e^x\left[\log(1+x^2)-\dfrac{2x}{1+x^2}\right]$

(C) $e^x\left[\log(1+x^2)+\dfrac{x}{1+x^2}\right]$

(D) $e^x\left[\log(1+x^2)-\dfrac{x}{1+x^2}\right]$

53. If $y=\dfrac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{-8}{(e^{2x}-e^{-2x})^2}$ **(B)** $\dfrac{8}{(e^{2x}-e^{-2x})^2}$

(C) $\dfrac{-4}{(e^{2x}-e^{-2x})^2}$ **(D)** $\dfrac{4}{(e^{2x}-e^{-2x})^2}$

54. If $y=\dfrac{2(x-\sin x)^{3/2}}{\sqrt{x}}$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{2(x-\sin x)^{3/2}}{\sqrt{x}}\left[\dfrac{3}{2}\cdot\dfrac{1-\cos x}{1-\sin x}-\dfrac{1}{2x}\right]$

(B) $\dfrac{2(x-\sin x)^{3/2}}{\sqrt{x}}\left[\dfrac{3}{2}\cdot\dfrac{1-\cos x}{x-\sin x}-\dfrac{1}{2x}\right]$

(C) $\dfrac{2(x-\sin x)^{1/2}}{\sqrt{x}}\left[\dfrac{3}{2}\cdot\dfrac{1-\cos x}{x-\sin x}-\dfrac{1}{2x}\right]$

(D) None of these

55. $\dfrac{d}{dx}\left(\cos^{-1}\sqrt{\dfrac{1+\cos x}{2}}\right)$ is equal to

(A) 1 **(B)** $\dfrac{1}{2}$

(C) $\dfrac{1}{3}$ **(D)** None of these

56. If $y=\tan^{-1}\sqrt{\dfrac{1+\cos x}{1-\cos x}}$, then $\dfrac{dy}{dx}$ is equal to

(A) 0 **(B)** $-\dfrac{1}{2}$

(C) 1/2 **(D)** 1

57. If $y\sqrt{x^2+1}=\log(\sqrt{x^2+1}-x)$, then $(x^2+1)\dfrac{dy}{dx}+xy+1$ is equal to

(A) 0 **(B)** 1

(C) 2 **(D)** None of these

58. The derivative of $f(x)=|x^2-x|$ at $x=2$ is

(A) -3 **(B)** 0

(C) 3 **(D)** Not defined

59. The derivative of $f(x)=3|2+x|$ at the point $x_0=-3$ is

(A) 3 **(B)** -3

(C) 0 **(D)** Does not exist

60. Derivative of the function $f(x)=\log_5(\log_7 x)$, $x>7$ is

(A) $\dfrac{1}{x(\ln 5)(\ln 7)(\log_7 x)}$ **(B)** $\dfrac{1}{x(\ln 5)(\ln 7)}$

(C) $\dfrac{1}{x(\ln x)}$ **(D)** None of these

61. If $x=a(t-\sin t)$ and $y=a(1-\cos t)$, then $\dfrac{dy}{dx}$ is equal to

(A) $\tan\left(\dfrac{t}{2}\right)$ **(B)** $-\tan\left(\dfrac{t}{2}\right)$

(C) $\cot\left(\dfrac{t}{2}\right)$ **(D)** $-\cot\left(\dfrac{t}{2}\right)$

62. If $x=a\left(\cos t+\log\tan\dfrac{t}{2}\right)$, $y=a\sin t$, then $\dfrac{dy}{dx}$ is equal to

(A) $\tan t$ **(B)** $-\tan t$

(C) $\cot t$ **(D)** $-\cot t$

63. If $y\sec x+\tan x+x^2y=0$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{2xy+\sec^2 x+y\sec x\tan x}{x^2+\sec x}$

(B) $-\dfrac{2xy+\sec^2 x+\sec x\tan x}{x^2+\sec x}$

(C) $-\dfrac{2xy+\sec^2 x+y\sec x\tan x}{x^2+\sec x}$

(D) None of these

64. If $\sin(xy)+\dfrac{x}{y}=x^2-y$, then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{y[2xy-y^2\cos(xy)-1]}{xy^2\cos(xy)+y^2-x}$ **(B)** $\dfrac{[2xy-y^2\cos(xy)-1]}{xy^2\cos(xy)+y^2-x}$

(C) $-\dfrac{y[2xy-y^2\cos(xy)-1]}{xy^2\cos(xy)+y^2-x}$ **(D)** None of these

65. If $y=\left(1+\dfrac{1}{x}\right)^x$, then $\dfrac{dy}{dx}$ is equal to

(A) $\left(1+\dfrac{1}{x}\right)^x\left[\log\left(1+\dfrac{1}{x}\right)-\dfrac{1}{1+x}\right]$

(B) $\left(1+\dfrac{1}{x}\right)^x\left[\log\left(1+\dfrac{1}{x}\right)\right]$

(C) $\left(x+\dfrac{1}{x}\right)^x\left[\log(x-1)-\dfrac{x}{x+1}\right]$

(D) $\left(1+\dfrac{1}{x}\right)^x\left[\log\left(1+\dfrac{1}{x}\right)+\dfrac{1}{1+x}\right]$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If $\dfrac{d^2}{dx^2}\left(\dfrac{\cos^4 x+\cos^2 x+1}{\cos^2 x+\cos x+1}\right)=a\cos^2 x+b\cos x+c$, then

(A) $a=-4$ **(B)** $b=-1$

(C) $b=1$ **(D)** $c=-2$

2. If $y=\sec(\tan^{-1}x)$, then $\dfrac{dy}{dx}$ at $x=1$ is equal to

(A) $\dfrac{1}{\sqrt{2}}$ **(B)** $\sin^{-1}\left(\sin\dfrac{1}{\sqrt{2}}\right)$

(C) 1 **(D)** None of these

3. If $y=10^{10^x}$ and $\dfrac{1}{y}\dfrac{dy}{dx}=10x\cdot\lambda$, then value of λ is

(A) ln 10 **(B)** $(\ln 10)^2$

(C) $e^{\ln(\ln 10)^2}$ **(D)** $(\log 10^e)^2$

4. Given $f(x)=-\dfrac{x^3}{3}+x^2\sin 1.5a-x\sin a\cdot\sin 2a-5\sin^{-1}(a^2-8a+17)$, then

(A) $f'(x)=-x^2+2x\sin 6-\sin 4\sin 8$

(B) $f'(\sin 8)>0$

(C) $f'(x)$ is not defined at $x=\sin 8$

(D) $f'(\sin 8)<0$

Comprehension Type Questions

Paragraph for Questions 5–7: The graph of $y=f(x)$ is given with six labelled points (see Fig. 20.1). Out of these points, answer the following questions.

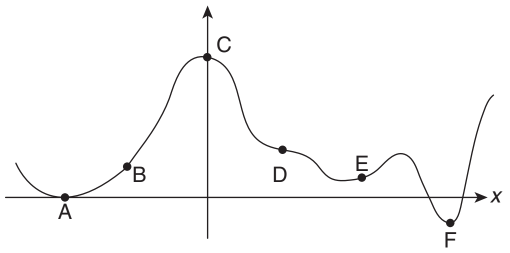


Figure 20.1

5. The point which has the greatest value of $f'(x)$ is
(A) B **(B)** C
(C) D **(D)** E

6. The point where f' and f'' are non-zero and of the same sign are
(A) B and D **(B)** D and E
(C) B and E **(D)** None of these

7. The points where at least two of f, f' and f'' are zero,
(A) C and D **(B)** A and D
(C) A and F **(D)** None of these

Paragraph for Questions 8–10: In certain problems, the differentiation of $\{f(x)\cdot g(x)\}$ appears. One student commits mistake and differentiates as $\frac{df}{dx}\cdot\frac{dg}{dx}$, but he gets correct result if $f(x) = x^3$ and $g(0) = \frac{1}{3}$.

8. The function $g(x)$ is
(A) $\frac{3}{|x-3|^3}$ **(B)** $\frac{4}{|x-3|^3}$
(C) $\frac{9}{|x-3|^3}$ **(D)** $\frac{27}{|x-3|^3}$

9. Derivative of $\{f(x-3)\cdot g(x)\}$ with respect to x at $x = 100$ is
(A) 0 **(B)** 1
(C) – 1 **(D)** 2

10. $\lim\limits_{x\to 0}\frac{f(x)\cdot g(x)}{x[1+g(x)]}$ will be
(A) 0 **(B)** – 1
(C) 1 **(D)** 2

Paragraph for Questions 11–13: Let $f(x)=\frac{1}{1+x^2}$. Let m be the slope, a be the x-intercept and b be the y-intercept of a tangent to $y = f(x)$, then

11. Abscissa of the point of contact of the tangent for which m is greatest
(A) $\frac{1}{\sqrt{3}}$ **(B)** 1
(C) – 1 **(D)** $-\frac{1}{\sqrt{3}}$

12. The greatest value of b is
(A) $\frac{9}{8}$ **(B)** $\frac{3}{8}$
(C) $\frac{1}{8}$ **(D)** $\frac{5}{8}$

13. The abscissa of the point of contact of tangent for which $\frac{1}{a}$ is greatest, is
(A) $\frac{1}{\sqrt{3}}$ **(B)** 1
(C) – 1 **(D)** $-\frac{1}{\sqrt{3}}$

Matrix Match Type Questions

14. **Match the following:**

Column I	Column II
(A) $\lim\limits_{x\to\frac{1}{\sqrt{2}}}\frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)}$ is equal to	**(p)** Does not exist
(B) If $f(x) = \log_{x^2}(\log x)$, then $f'\left(\frac{1}{2}\right)$ is equal to	**(q)** $-\frac{1}{\sqrt{2}}$
(C) For the function $f(x) = \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ if $\frac{dy}{dx} = \sec x + p$, then p is equal to	**(r)** 28
(D) $\lim\limits_{x\to 0}\frac{1}{x}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$ is equal to	**(s)** 1
	(t) 0

15. **Match the following:**

Column I	Column II
(A) If $y = \cos^{-1}(\cos x)$, then y' at $x = 5$ is equal to	**(p)** Does not exist
(B) For the function $f(x) = \ln|\tan x|$ $f'\left(-\frac{\pi}{4}\right)$ is equal to	**(q)** 2
(C) The derivative of $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ at $x = -1$ is	**(r)** $\frac{1}{2}$
(D) The derivative of $\frac{\log|x|}{x}$ at $x = -1$ is	**(s)** 1
	(t) –1

Integer Type Questions

16. If $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x}$, where $0 < x < \frac{2}{3}$ and $\frac{dy}{dx}=\frac{\lambda}{1+25x^2}$, then find λ.

17. The function $y = f(x)$ defined by the parametric equations $x = e^t\sin t$, $y = e^t\cos t$ satisfies the relation $y''(x + y)^2 = \lambda(xy' - y)$, then find λ.

Answer Key

Practice Exercise 1

1. (A)	**2.** (B)	**3.** (A)	**4.** (D)	**5.** (A)	**6.** (A)
7. (C)	**8.** (A)	**9.** (C)	**10.** (C)	**11.** (A)	**12.** (B)
13. (D)	**14.** (C)	**15.** (D)	**16.** (A)	**17.** (B)	**18.** (A)
19. (D)	**20.** (A)	**21.** (D)	**22.** (D)	**23.** (B)	**24.** (A)
25. (D)	**26.** (C)	**27.** (A)	**28.** (A)	**29.** (A)	**30.** (C)
31. (C)	**32.** (A)	**33.** (B)	**34.** (B)	**35.** (B)	**36.** (B)
37. (B)	**38.** (A)	**39.** (A)	**40.** (A)	**41.** (A)	**42.** (A)
43. (B)	**44.** (C)	**45.** (B)	**46.** (C)	**47.** (D)	**48.** (A)
49. (C)	**50.** (C)	**51.** (A)	**52.** (A)	**53.** (A)	**54.** (B)
55. (B)	**56.** (B)	**57.** (A)	**58.** (C)	**59.** (B)	**60.** (A)
61. (C)	**62.** (A)	**63.** (C)	**64.** (A)	**65.** (A)	

Practice Exercise 2

1. (A), (C)	**2.** (A), (B)	**3.** (B), (C)	**4.** (A), (D)	**5.** (A)	**6.** (C)
7. (B)	**8.** (C)	**9.** (A)	**10.** (A)	**11.** (D)	**12.** (A)
13. (A)	**14.** (A) → (q), (B) → (p), (C) → (t), (D) → (p)		**15.** (A) → (t), (B) → (q), (C) → (r), (D) → (s)		**16.** 5
17. 2					

Solutions

Practice Exercise 1

1. $$\cos(x+y) = (y\sin x)$$

$$\Rightarrow -\sin(x+y)\left(1+\frac{dy}{dx}\right) = y\cos x + \sin x\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y\cos x + \sin(x+y)}{\sin(x+y)+\sin x}$$

2. $$\sin^2 x + 2\cos y + xy = 0$$

$$\Rightarrow 2\sin x\cos x - 2\sin y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

Hence,
$$\frac{dy}{dx} = \frac{y+\sin 2x}{2\sin y - x}$$

3. $x^3 + 8xy + y^3 = 64 \Rightarrow 3x^2 + 8\left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$

Hence,
$$\frac{dy}{dx} = -\frac{3x^2+8y}{8x+3y^2}$$

4. It is an implicit function, so

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\cos(x+y) - \frac{1}{x+y}}{\cos(x+y) - \frac{1}{x+y}} = -1$$

5. $$\ln(x+y) = 2xy$$

Differentiating both sides with respect to x, we get

$$\left(\frac{1}{x+y}\right)\left(1+\frac{dy}{dx}\right) = 2\left(x\frac{dy}{dx}+y\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-2xy-2y^2}{2x^2+2xy-1}$$

At $x=0, y=1$ [from $\ln(x+y)=2xy$]

Hence,
$$y'(0) = \frac{1-2}{-1} = 1$$

6. $x^y = e^{x-y} \Rightarrow y\log x = x - y \Rightarrow y = \dfrac{x}{1+\log x}$

$$\Rightarrow \frac{dy}{dx} = \log x(1+\log x)^{-2} = \log x[\log ex]^{-2}$$

7. $$\log y = (\tan x)^{\tan x}\log\tan x \qquad (1)$$

Taking log again in Eq. (1), we get

$$\log(\log y) = \tan x\log\tan x + \log(\log\tan x)$$

Differentiating with respect to x, we get

$$\frac{1}{\log y}\cdot\frac{1}{y}\frac{dy}{dx}$$

$$= \sec^2 x\log\tan x + \tan x\cdot\frac{\sec^2 x}{\tan x} + \frac{1}{\log\tan x}\cdot\frac{1}{\tan x}\cdot\sec^2 x$$

Therefore,
$$\frac{dy}{dx} = y\log y\sec^2 x\cdot\left[\log\tan x + 1 + \frac{1}{\tan x\log\tan x}\right]$$

$$= y(\tan x)^{\tan x}\log\tan x\cdot\sec^2 x\left[(\log\tan x+1) + \frac{1}{\tan x\log\tan x}\right]$$

$$= y(\tan x)^{\tan x}\sec^2 x[\log\tan x(\log\tan x+1)+\cot x]$$

Now at $x=\dfrac{\pi}{4}$, $y=1$, $\log\tan\left(\dfrac{\pi}{4}\right) = \log 1 = 0$

Therefore,
$$\frac{dy}{dx} = 1\cdot 1\cdot 2[0+1] = 2$$

8. Given $y=(\sin x)^{\tan x}$; $\log y=\tan x\cdot\log\sin x$

Differentiating with respect to x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\tan x\cdot\cot x+\log\sin x\cdot\sec^2 x$$

$$\frac{dy}{dx}=(\sin x)^{\tan x}[1+\log\sin x\cdot\sec^2 x]$$

9. Given

$$y=2^{1/\log_x 4}\Rightarrow\log y=\frac{1}{\log_x 4}(\log 2)$$

$$\Rightarrow\log_x 4=\frac{\log 2}{\log y}\Rightarrow\frac{\log_e 4}{\log_e x}=\frac{\log_e 2}{\log_e y}\Rightarrow\frac{2\log 2}{\log x}=\frac{\log 2}{\log y}$$

$$\Rightarrow\log x=\log y^2\Rightarrow x=y^2$$

10. $$y=x^{\ln x}\Rightarrow\ln y=(\ln x)^2\Rightarrow\frac{1}{y}\frac{dy}{dx}=\frac{2\ln x}{x}$$

$$\Rightarrow\frac{dy}{dx}=y\frac{2\ln x}{x}=\frac{2(x^{\ln x})\ln x}{x}\Rightarrow\frac{dy}{dx}=2x^{\ln x-1}\ln x$$

11. Let $x=\sin\theta$. Then

$$2\sin^{-1}x=2\theta\Rightarrow y=\sin 2\theta$$

$$\Rightarrow\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2\cos 2\theta}{\cos\theta}=\frac{2(1-2\sin^2\theta)}{\sqrt{1-\sin^2\theta}}=\frac{2-4x^2}{\sqrt{1-x^2}}$$

12. $$y=\cos^{-1}\left[\frac{3}{5}\cos x-\frac{4}{5}\sin x\right]$$

Putting $\frac{3}{5}=r\cos\theta$, $\frac{4}{5}=r\sin\theta$, we get

$$r=1$$

$$\Rightarrow y=\cos^{-1}[\cos\theta\cos x-\sin\theta\sin x]=\theta+x\Rightarrow\frac{dy}{dx}=1$$

13. Putting $x=\cot\theta$, we have

$$y=\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)=\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$$

$$=\cos^{-1}(\cos 2\theta)=2\theta\Rightarrow\frac{dy}{dx}=\frac{-2}{1+x^2}$$

14. Putting $x^2=\cos 2\theta$, we have

$$\frac{d}{dx}\left[\cos^{-1}\sqrt{\frac{1+x^2}{2}}\right]=\frac{d}{dx}[\cos^{-1}\cos\theta]$$

$$=\frac{d}{dx}[\theta]=\frac{d}{dx}\left[\frac{1}{2}\cos^{-1}x^2\right]=\frac{-x}{\sqrt{1-x^4}}$$

15. Let $y=\sin^{-1}\frac{1-x}{1+x}$. Then

$$\frac{dy}{dx}=\frac{-1}{\sqrt{x}(1+x)}\tag{1}$$

and $$z=\sqrt{x}\Rightarrow\frac{dz}{dx}=\frac{1}{2\sqrt{x}}\tag{2}$$

Therefore, by Eqs. (1) and (2), we have

$$\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{-2}{1+x}$$

16. Let $y_1=\tan^{-1}\frac{2x}{1-x^2}$ and $y_2=\sin^{-1}\frac{2x}{1+x^2}$.

Differentiating y_1 and y_2 with respect to x, we get

$$\frac{dy_1}{dx}=\frac{d}{dx}\left(\tan^{-1}\frac{2x}{1-x^2}\right)$$

$$\frac{dy_2}{dx}=\frac{d}{dx}\left(\sin^{-1}\frac{2x}{1+x^2}\right)$$

Putting $x=\tan\theta$, we have

$$y_1=\tan^{-1}\tan 2\theta=2\theta=2\tan^{-1}x$$

and $$y_2=\sin^{-1}\sin 2\theta=2\tan^{-1}x$$

Again

$$\frac{dy_1}{dx}=\frac{d}{dx}(2\tan^{-1}x)=\frac{2}{1+x^2}\tag{1}$$

and $$\frac{dy_2}{dx}=\frac{d}{dx}(2\tan^{-1}x)=\frac{2}{1+x^2}\tag{2}$$

Hence,

$$\frac{dy_1}{dy_2}=1$$

17. From the given relation

$$\frac{y}{x}=\log x-\log(a+bx)$$

Differentiating, we get

$$\frac{\left(x\frac{dy}{dx}-y\right)}{x^2}=\frac{1}{x}-\frac{1}{a+bx}b=\frac{a}{x(a+bx)}$$

Hence,

$$x\frac{dy}{dx}-y=\frac{ax}{a+bx}\tag{1}$$

Differentiating again with respect to x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{dy}{dx}=\frac{(a+bx)a-ax.b}{(a+bx)^2}\Rightarrow x\frac{d^2y}{dx^2}=\frac{a^2}{(a+bx)^2}$$

$$\Rightarrow x^3\frac{d^2y}{dx^2}=\frac{a^2x^2}{(a+bx)^2}=\left(x\frac{dy}{dx}-y\right)^2 \quad \text{[by Eq. (1)]}$$

18. $$y=x^3\log\log_e(1+x)$$

$$\Rightarrow y'=3x^2\log\log_e(1+x)+\frac{x^3}{1+x}\cdot\frac{1}{\log_e(1+x)}$$

$$\Rightarrow y''=6x\log\log_e(1+x)+\frac{3x^2}{\log_e(1+x)}\cdot\frac{1}{(1+x)}$$

$$-\frac{x^3}{(1+x)^2\log_e(1+x)}-\frac{x^3}{(1+x)^2}\cdot\frac{1}{[\log_e(1+x)]^2}+\frac{3x^2}{(1+x)\log_e(1+x)}$$

$$\Rightarrow y''(0)=0$$

19. $$\frac{d^2x}{dy^2}=\frac{d}{dy}\left(\frac{dx}{dy}\right)=\frac{d}{dy}\left(\frac{1}{\frac{dy}{dx}}\right)=\frac{-1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}$$

20. We have

$$F(x)=\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$$

Hence,

$$F'(x)=\begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1'(x) & g_2'(x) & g_3'(x)\\ h_1(x) & h_2(x) & h_3(x)\end{vmatrix}$$

$$+\begin{vmatrix} f_1(x) & f_2(x) & f_3(x)\\ g_1(x) & g_2(x) & g_3(x)\\ h_1'(x) & h_2'(x) & h_3'(x)\end{vmatrix}$$

$\Rightarrow F'(a)=0$ (since $f_n(a)=g_n(a)=h_n(a), n=1,2,3$)
Therefore, two rows in each determinant become identical on putting $x=a$.

21. $$f'''(x)=\begin{vmatrix} \frac{d^3}{dx^3}x^3 & \frac{d^3}{dx^3}\sin x & \frac{d^3}{dx^3}\cos x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}=\begin{vmatrix} 6 & -\cos x & \sin x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}$$

Hence,

$$f'''(0)=\begin{vmatrix} 6 & -1 & 0\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}=0, \text{ which is independent of } p.$$

22. $$f(x)=\begin{vmatrix} x^3 & x^2 & 3x^2\\ 1 & -6 & 4\\ p & p^2 & p^3\end{vmatrix}$$

$$\Rightarrow f(x)=x^3(-6p^3-4p^2)-x^2(p^3-4p)+3x^2(p^2+6p)$$

$$\Rightarrow f(x)=-6p^3x^3-4p^2x^3-x^2p^3+4px^2+3p^2x^2+18px^2$$

Hence,

$$\frac{d}{dx}f(x)=-18p^3x^2-12p^2x^2-2xp^3+8px+6p^2x+36px$$

and $$\frac{d^2}{dx^2}f(x)=-36p^3x-24p^2x-2p^3+8p+6p^2+36p$$

and $$\frac{d^3f(x)}{dx^3}=-36p^3-24p^2=\text{a constant}$$

23. $$D=\begin{vmatrix} \sin px & p\cos px & -p^2\sin px\\ -p^3\cos px & p^4\sin px & p^5\cos px\\ -p^6\sin px & -p^7\cos px & p^8\sin px\end{vmatrix}$$

$$=p^9\begin{vmatrix} \sin px & p\cos px & -p^2\sin px\\ -\cos px & p\sin px & p^2\cos px\\ -\sin px & -p\cos px & p^2\sin px\end{vmatrix}$$

$$=-p^9\begin{vmatrix} \sin px & p\cos px & -p^2\sin px\\ \cos px & p\sin px & p^2\cos px\\ \sin px & p\cos px & -p^2\sin px\end{vmatrix}=0$$

24. $$y^2=ax^2+bx+c\Rightarrow 2y\frac{dy}{dx}=2ax+b$$

$$\Rightarrow 2\left(\frac{dy}{dx}\right)^2+2y\frac{d^2y}{dx^2}=2a\Rightarrow y\frac{d^2y}{dx^2}=a-\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y\frac{d^2y}{dx^2}=a-\left(\frac{2ax+b}{2y}\right)^2\Rightarrow y\frac{d^2y}{dx^2}=\frac{4ay^2-(2ax+b)^2}{4y^2}$$

$$\Rightarrow 4y^3\frac{d^2y}{dx^2}=4a(ax^2+bx+c)-(4a^2x^2+4abx+b^2)$$

$$\Rightarrow 4y^3\frac{d^2y}{dx^2}=4ac-b^2\Rightarrow y^3\frac{d^2y}{dx^2}=\frac{4ac-b^2}{4}=\text{a constant}$$

25. $$y=a^xb^{2x-1}$$

$$\frac{dy}{dx}=a^xb^{2x-1}\log a+2a^xb^{2x-1}\log b=a^xb^{2x-1}(\log a+2\log b)$$

$$\frac{d^2y}{dx^2}=a^xb^{2x-1}(\log a+2\log b)^2=a^xb^{2x-1}(\log ab^2)^2$$

$$=y(\log ab^2)^2$$

26. $$\frac{d}{dx}\log(\log x)=\frac{1}{x}\cdot\frac{1}{\log x}=(x\log x)^{-1}$$

27. $$\frac{d}{dx}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=\frac{d}{dx}\left[x+\frac{1}{x}+1\right]=1-\frac{1}{x^2}$$

28. $$y=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\infty\Rightarrow y=e^x$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=e^x=y$$

29. $$\frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right]$$

$$=\frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\right]$$

$$=\frac{d}{dx}\left\{\tan^{-1}\left[\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right]\right\}=\frac{d}{dx}\left[\tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]=-\frac{1}{2}$$

30. $$\frac{d}{dx}[\cos(1-x^2)^2]=-\sin(1-x^2)^2\frac{d}{dx}(1-x^2)^2$$

$$=4x(1-x^2)\sin(1-x^2)^2$$

31. Putting $x=\sin A$ and $\sqrt{x}=\sin B$, we have

$$y=\sin^{-1}(\sin A\sqrt{1-\sin^2 B}+\sin B\sqrt{1-\sin^2 A})$$

$$=\sin^{-1}[\sin(A+B)]=A+B=\sin^{-1}x+\sin^{-1}\sqrt{x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$$

32. Let

$$y=\tan^{-1}\sqrt{\frac{1+\cos\frac{x}{2}}{1-\cos\frac{x}{2}}}=\tan^{-1}\sqrt{\frac{2\cos^2\frac{x}{4}}{2\sin^2\frac{x}{4}}}$$

$$y=\tan^{-1}\cot\frac{x}{4}=\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{x}{4}\right)=\frac{\pi}{2}-\frac{x}{4}$$

Therefore,

$$\frac{dy}{dx}=-\frac{1}{4}$$

33. $y=\sqrt{\dfrac{1-\sin 2x}{1+\sin 2x}}=\dfrac{\cos x-\sin x}{\cos x+\sin x}$

$$=\frac{1-\tan x}{1+\tan x}=\tan\left(\frac{\pi}{4}-x\right)\Rightarrow\frac{dy}{dx}=-\sec^2\left(\frac{\pi}{4}-x\right)$$

34. $f(x)=x\tan^{-1}x$

Differentiating with respect to x, we get

$$f'(x)=x\frac{1}{1+x^2}+\tan^{-1}x$$

Now put $x=1$, then

$$f'(1)=\frac{1}{2}+\tan^{-1}(1)=\frac{\pi}{4}+\frac{1}{2}$$

35. $$f(x)=\log_x(\log x)=\frac{\log(\log x)}{\log x}$$

$$\Rightarrow f'(x)=\frac{\frac{1}{x}-\frac{1}{x}\log(\log x)}{(\log x)^2}\Rightarrow f'(e)=\frac{\frac{1}{e}-0}{1}=\frac{1}{e}$$

36. $$f(x)=\sqrt{1+\cos^2(x^2)}$$

$$f'(x)=\frac{1}{2\sqrt{1+\cos^2(x)^2}}\cdot(2\cos x^2)\cdot(-\sin x^2)\cdot(2x)$$

$$f'(x)=\frac{-x\sin 2x^2}{\sqrt{1+\cos^2(x^2)}}$$

At $x=\dfrac{\sqrt{\pi}}{2}$,

$$f'\left(\frac{\sqrt{\pi}}{2}\right)=\frac{-\frac{\sqrt{\pi}}{2}\cdot\sin\frac{2\pi}{4}}{\sqrt{1+\cos^2\frac{\pi}{4}}}=\frac{-\frac{\sqrt{\pi}}{2}\cdot 1}{\sqrt{\frac{3}{2}}}$$

Hence,

$$f'\left(\frac{\sqrt{\pi}}{2}\right)=-\sqrt{\frac{\pi}{6}}$$

37. $$\frac{d}{dx}\tan^{-1}(\sec x+\tan x)=\frac{d}{dx}\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$$

$$=\frac{d}{dx}\tan^{-1}\left(\frac{\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}\right)=\frac{d}{dx}\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$=\frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{1}{2}$$

38. $$\frac{d}{dx}(e^x\log\sin 2x)=e^x\log\sin 2x+2e^x\frac{1}{\sin 2x}\cos 2x$$

$$=e^x\log\sin 2x+e^x 2\cot 2x=e^x(\log\sin 2x+2\cot 2x)$$

39. $$y=\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$$

$$=\cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)=\frac{\pi}{2}$$

$$\Rightarrow\frac{dy}{dx}=0\qquad\left\{\text{As}\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right\}$$

40. Put $x=\sin\theta$, we get

$$\frac{d}{dx}\sin^{-1}(3x-4x^3)=\frac{d}{dx}\sin^{-1}(\sin 3\theta)=\frac{3}{\sqrt{1-x^2}}$$

41. $$y=\sqrt{\left(\frac{1+\tan x}{1-\tan x}\right)}\text{ or }y=\sqrt{\tan\left(\frac{\pi}{4}+x\right)}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan\left(\frac{\pi}{4}+x\right)}}\sec^2\left(\frac{\pi}{4}+x\right)$$

$$=\frac{1}{2}\sqrt{\left[\frac{1-\tan x}{1+\tan x}\right]}\sec^2\left(\frac{\pi}{4}+x\right)$$

42. $$\frac{d}{dx}\left(\frac{\sec x+\tan x}{\sec x-\tan x}\right)=\frac{d}{dx}\left(\frac{1+\sin x}{1-\sin x}\right)=\frac{2\cos x}{(1-\sin x)^2}$$

43. $$\frac{d}{dx}\left[\log\sqrt{\frac{1-\cos x}{1+\cos x}}\right]=\frac{d}{dx}\left[\log\left(\tan\frac{x}{2}\right)\right]=\operatorname{cosec}x$$

44. $$\frac{d}{dx}\left[\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}\right]=\frac{d}{dx}\left[\tan^{-1}\tan\frac{x}{2}\right]=\frac{1}{2}$$

45. $$f(x)=\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)=\tan^{-1}\left[\tan\frac{x}{2}\right]=\frac{x}{2}$$

$$\Rightarrow f'(x)=\frac{1}{2}$$

Hence,

$$f'\left(\frac{\pi}{3}\right)=\frac{1}{2}$$

46. $$\frac{d}{dx}\left[\frac{e^{ax}}{\sin(bx+c)}\right]=\frac{ae^{ax}\sin(bx+c)-be^{ax}\cos(bx+c)}{[\sin(bx+c)]^2}$$

$$=\frac{e^{ax}[a\sin(bx+c)-b\cos(bx+c)]}{\sin^2(bx+c)}$$

47. $$\frac{dy}{dx}=-2x^{-3}e^x\log x+x^{-2}\left(e^x\log x+\frac{e^x}{x}\right)=e^x\left[\frac{1+(x-2)\log x}{x^3}\right]$$

Aliter: Taking log we have,

$$\log y=x+\log\log x-2\log x$$

$$\Rightarrow\frac{1}{y}\frac{dy}{dx}=1+\frac{1}{x\log x}-\frac{2}{x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{e^x\log x}{x^2},\left[\frac{x\log x+1-2\log x}{x\log x}\right]$$

$$=\frac{e^x[(x-2)\log x+1]}{x^3}$$

48. $y=\dfrac{e^{2x}\cos x}{x\sin x}\Rightarrow \log y=2x+\log\cos x-\log x-\log\sin x$

$$\frac{1}{y}\frac{dy}{dx}=2+\left(\frac{-\sin x}{\cos x}\right)-\frac{1}{x}-\frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx}=e^{2x}\left[\frac{2}{x}\cot x-\frac{1}{x}-\frac{1}{x^2}\cot x-\frac{\cot^2 x}{x}\right]$$

$$=\frac{e^{2x}}{x^2}[(2x-1)\cot x-x\,\mathrm{cosec}^2 x]$$

49. $\dfrac{d}{dx}\{e^{-ax^2}\log(\sin x)\}=e^{-ax^2}(-2ax).\log(\sin x)+e^{-ax^2}\cot x$

$$=e^{-ax^2}[\cot x-2ax\log(\sin x)]$$

50. $$y=\log x\cdot e^{(\tan x+x^2)}$$

Therefore,

$$\frac{dy}{dx}=e^{(\tan x+x^2)}\cdot\frac{1}{x}+\log x\cdot e^{(\tan x+x^2)}(\sec^2 x+2x)$$

$$=e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x+2x)\log x\right]$$

51. $$y=\sqrt{\frac{1+e^x}{1-e^x}}\text{ or } y^2=\frac{1+e^x}{1-e^x}$$

$$2y\frac{dy}{dx}=\frac{(1-e^x)e^x+(1+e^x)e^x}{(1-e^x)^2}=\frac{2e^x}{(1-e^x)^2}$$

Hence,

$$\frac{dy}{dx}=\frac{e^x}{(1-e^x)^2}\sqrt{\left[\frac{1-e^x}{1+e^x}\right]\left[\frac{1-e^x}{1-e^x}\right]}=\frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$$

52. $\dfrac{d}{dx}[e^x\log(1+x^2)]=e^x\log(1+x^2)+e^x\dfrac{1}{(1+x^2)}2x$

$$=e^x\left[\log(1+x^2)+\frac{2x}{1+x^2}\right]$$

53. $$y=\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$$

Hence,

$$\frac{dy}{dx}=\frac{(e^{2x}-e^{-2x})2(e^{2x}-e^{-2x})-(e^{2x}+e^{-2x})2(e^{2x}+e^{-2x})}{(e^{2x}-e^{-2x})^2}$$

$$=\frac{-8}{(e^{2x}-e^{-2x})^2}$$

54. $$\log y=\log 2+\frac{3}{2}\log(x-\sin x)-\frac{1}{2}\log x$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{3}{2}\cdot\frac{1-\cos x}{x-\sin x}-\frac{1}{2x}\right]$$

55. $\dfrac{d}{dx}\left(\cos^{-1}\sqrt{\dfrac{1+\cos x}{2}}\right)=\dfrac{d}{dx}\left[\cos^{-1}\left(\cos\dfrac{x}{2}\right)\right]=\dfrac{1}{2}$

56. $$y=\tan^{-1}\sqrt{\frac{1+\cos x}{1-\cos x}}=\tan^{-1}\sqrt{\frac{2\cos^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}$$

$$=\tan^{-1}\cot\frac{x}{2}=\tan^{-1}\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)=\frac{\pi}{2}-\frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{2}$$

57. $$y\sqrt{x^2+1}=\log[\sqrt{x^2+1}-x]$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx}\sqrt{x^2+1}+y\cdot\frac{1}{2\sqrt{x^2+1}}\cdot 2x=\frac{1}{\sqrt{x^2+1}-x}\cdot\left\{\frac{1}{2}\frac{2x}{\sqrt{x^2+1}}-1\right\}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx}+xy=\sqrt{x^2+1}\cdot\frac{-1}{\sqrt{x^2+1}}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx}+xy+1=0$$

58. $f(x)=|x^2-x|\Rightarrow f'(x)=2x-1\Rightarrow f'(2)=3$

59. $f(x)=3|2+x|;\ f'(x)=-3,\qquad \left[\because |x|=\begin{cases}-x, & x<0\\ x, & x>0\end{cases}\right]$

60. $$f(x)=\log_5(\log_7 x)\Rightarrow f(x)=\log_5\left(\frac{\log_e x}{\log_e 7}\right)$$

$$\Rightarrow f(x)=\log_5\log_e x-\log_5\log_e 7$$

$$\Rightarrow f(x)=\frac{\log_e\log_e x}{\log_e 5}-\log_5\log_e 7$$

61. $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\sin t}{a(1-\cos t)}=\frac{\sin t}{1-\cos t}$$

$$=\frac{2\sin\left(\frac{t}{2}\right).\cos\left(\frac{t}{2}\right)}{2\sin^2\left(\frac{t}{2}\right)}=\cot\left(\frac{t}{2}\right)$$

62. Given that $x=a\left(\cos t+\log\tan\dfrac{t}{2}\right)$ and $y=a\sin t$.

Differentiating with respect to t, we get

$$\frac{dy}{dt}=a\cos t \qquad (1)$$

and $$\frac{dx}{dt}=a\left[-\sin t+\cot\left(\frac{t}{2}\right)\left(\frac{1}{2}\right)\sec^2\left(\frac{t}{2}\right)\right]$$

$$=a\left(-\sin t+\frac{1}{\sin t}\right)=a\frac{\cos^2 t}{\sin t}=a\cos t\cot t \qquad (2)$$

From Eqs. (1) and (2), we get

$$\frac{dy}{dx}=\tan t$$

63. $$y\sec x+\tan x+x^2y=0$$

$$\Rightarrow \sec x\frac{dy}{dx}+y\sec x\tan x+\sec^2 x+2xy+x^2\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{2xy+\sec^2 x+y\sec x\tan x}{x^2+\sec x}$$

64. $$\sin(xy)+\frac{x}{y}=x^2-y$$

Differentiating both sides, we get

$$\cos(xy)\frac{d}{dx}(xy)+x\left(-\frac{1}{y^2}\right)\frac{dy}{dx}+\frac{1}{y}=2x-\frac{dy}{dx}$$

$$\Rightarrow \left[x\cos(xy)-\frac{x}{y^2}+1\right]\frac{dy}{dx}=2x-\frac{1}{y}-y\cos(xy)$$

$$\Rightarrow \frac{dy}{dx}=\left[\frac{2xy^2-y-y^3\cos(xy)}{xy^2\cos(xy)-x+y^2}\right]$$

65. $$y=\left(1+\frac{1}{x}\right)^x \Rightarrow \log y = x\log\left(1+\frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=\log\left(1+\frac{1}{x}\right)-\frac{1}{1+x}$$

$$\Rightarrow \frac{dy}{dx}=\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]$$

Practice Exercise 2

1. $$\frac{d^2}{dx^2}(\cos^2 x-\cos x+1)=\frac{d}{dx}(-2\sin x\cos x+\sin x)$$

$$=\frac{d}{dx}(-\sin 2x+\sin x)=(-2\cos 2x+\cos x)$$

$$=-2(2\cos^2 x-1)+\cos x=-4\cos^2 x+2+\cos x$$

$$=-4\cos^2 x+\cos x+2$$

$$\Rightarrow a=-4, b=1, c=2$$

2. $y=\sec(\tan^{-1} x)$

Hence,

$$\frac{dy}{dx}=\frac{\sec(\tan^{-1}x)\cdot\tan(\tan^{-1}x)}{1+x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1}=\frac{1}{\sqrt{2}}=\sin^{-1}\left(\sin\frac{1}{\sqrt{2}}\right)$$

3. $$y=10^{10^x}$$

Hence,

$$\frac{dy}{dx}=10^{10^x}\ln 10\cdot 10^x\ln 10=y10^x(\ln 10)^2$$

$$\frac{1}{y}\frac{dy}{dx}=10^x(\ln 10)^2$$

Now,

$$\lambda=(\ln 10)^2=e^{\ln(\ln 10)^2}$$

4. $f(x)=-\dfrac{x^3}{3}+x^2\sin 1.5a-x\sin a\cdot\sin 2a-5\sin^{-1}(a^2-8a+17)$

$f(x)$ is defined when

$$-1\le a^2-8a+17\le 1$$

$$\Rightarrow -1\le(a-4)^2+1\le 1\Rightarrow a=4$$

Hence,

$$f(x)=-\frac{x^3}{3}+x^2\sin 6-x\sin 4\sin 8-\frac{5\pi}{2}$$

Hence,

$$f'(x)=-x^2+2x\sin 6-\sin 4\sin 8$$

$$f'(\sin 8)=-\sin^2 8+2\sin 8\sin 6-\sin 4\sin 8$$

$$=\sin 8\,[2\sin 6-(\sin 8+\sin 4)]$$

$$=\sin 8\,[2\sin 6-2\sin 6\cos 2]=2\sin 6\sin 8(1-\cos 2)$$

$$\sin 6<0, \sin 8>0, 1-\cos 2>0$$

Therefore,

$$f'(\sin 8)<0$$

5. $f'(x)$ (slope) is positive at B and E but $f'(x)$ has greatest value at B relative to E.

6. $f'(x)$ (slope) is positive at B and E and $f''(x)$ has also positive sign (concavity up) at E.

7. $f(x)$ has equal root at A, so $f(x)$ and $f'(x)$ is zero at A point. $f'(x)$ (slope) is zero at D and concavity of graph is not changing at D so $f''(x)$ is also at D.

8. $$f(x)\,g(x)=x^3 g(x)$$

$$\Rightarrow 3x^2 g'(x)=3x^2 g(x)+x^3 g'(x)$$

$$\Rightarrow 3g'(x)=3g(x)+xg'(x)$$

$$\Rightarrow (3-x)\,g'(x)=3g(x)$$

$$\Rightarrow \int\frac{g'(x)}{g(x)}dx=\int\frac{3}{3-x}dx$$

$$\Rightarrow \ln g(x)=-3\ln|3-x|+\ln c$$

Therefore,

$$g(x)=\frac{c}{|3-x|^3}$$

Now,

$$g(0)=\frac{c}{27}=\frac{1}{3}\Rightarrow c=9$$

Hence,

$$g(x)=\frac{9}{|3-x|^3}$$

9. $$f(x-3)\cdot g(x)=(x-3)^3\cdot g(x)=9$$

Therefore, derivative of $f(x-3)\cdot g(x)$ is 0.

10. $$\lim_{x\to 0}\frac{f(x)\cdot g(x)}{x[1+g(x)]}=\lim_{x\to 0}\frac{x^3\dfrac{9}{|3-x|^3}}{x\left(1+\dfrac{9}{|3-x|^3}\right)}=0$$

11. $$f'(x)=-\frac{2x}{(1+x^2)^2}$$

$$f''(x)=\frac{-2+6x^2}{(1+x^2)^3}$$

$$f''(x)=0 \text{ if } x=\pm\frac{1}{\sqrt{3}}$$

Hence, $f'(x)$ is greatest at $x=-\dfrac{1}{\sqrt{3}}$.

12. Equation of tangent at $x=\alpha$ is

$$y-\frac{1}{1+\alpha^2}=\frac{-2\alpha}{(1+\alpha^2)^2}(x-\alpha)$$

$$\Rightarrow b=\frac{1}{1+\alpha^2}+\frac{2\alpha^2}{(1+\alpha^2)^2}=\frac{1+3\alpha^2}{(1+\alpha^2)^2}$$

Hence,

$$\frac{db}{d\alpha}=\frac{(1+\alpha^2)^2\cdot 6\alpha-2(1+3\alpha^2)(1+\alpha^2)2\alpha}{(1+\alpha^2)^4}=\frac{2\alpha(1-3\alpha^2)}{(1+\alpha^2)^3}$$

$$\frac{db}{d\alpha}=0$$

Now,

$$\alpha = 0, \pm \frac{1}{\sqrt{3}}$$

At $\alpha = \pm \frac{1}{\sqrt{3}}, b = \frac{9}{8}$.

13.
$$a = \frac{1+3\alpha^2}{2\alpha}$$
$$\Rightarrow \frac{1}{a} = \frac{2\alpha}{1+3\alpha^2}$$

Hence, its greatest value is $\frac{1}{\sqrt{3}}$.

14. (A)

$$\lim_{x\to\frac{1}{\sqrt{2}}} \frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)} = \lim_{x\to\frac{1}{\sqrt{2}}} \frac{x-\sqrt{1-x^2}}{1-\frac{x}{\sqrt{1-x^2}}}$$
$$= \lim_{x\to\frac{1}{\sqrt{2}}} (-\sqrt{1-x^2}) = -\frac{1}{\sqrt{2}}$$

(B) $x = \frac{1}{2}$ is not in the domain. Hence, $f'\left(\frac{1}{2}\right)$ does not exist.

(C)
$$y = f(x) = \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

Therefore,

$$\frac{dy}{dx} = \frac{\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)}\cdot\frac{1}{2} = \frac{1}{2}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$$
$$= \frac{1}{\sin\left(\frac{\pi}{2}+x\right)} = \sec x$$

Thus, $p = 0$

(D) $\lim_{x\to 0}\frac{1}{x}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \lim_{x\to 0}\frac{|\tan x|}{x} = \text{Does not exist}$

15. (A)
$$y = \cos^{-1}(\cos x)$$
$$y' = \frac{-1}{\sqrt{1-\cos^2 x}}\cdot(-\sin x) = \frac{\sin x}{|\sin x|}$$

Therefore, y' at $x = 5$ is -1.

(B)
$$y = f(x) = \ln|\tan x|$$

Therefore,
$$f'(x) = (1/\tan x)(\sec^2 x)\cdot\left(\frac{|\tan x|}{\tan x}\right)$$
$$\Rightarrow f'\left(-\frac{\pi}{4}\right) = 2$$

(C) $\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$

$$= \frac{(1-x)^2}{2(1+x^2)}\cdot\frac{2}{(1-x)^2} = \frac{1}{1+x^2}$$

At $x = -1$, we have
$$\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{1}{2}$$

(D) $\frac{d}{dx}\frac{\ln|x|}{x} = \frac{x\cdot\frac{1}{x}-\ln|x|}{x^2} = \frac{1-\ln|x|}{x^2}\frac{\ln|x|}{x}$

$$\Rightarrow \frac{d}{dx}\frac{\ln|x|}{x} = 1 \qquad (\text{at } x = -1)$$

16. $y = \tan^{-1}\frac{4x}{1+5x^2} + \tan^{-1}\frac{2+3x}{3-2x} = \tan^{-1}\frac{5x-x}{1+5x\cdot x} + \tan^{-1}\frac{\frac{2}{3}+x}{1-\frac{2}{3}\cdot x}$

$$= \tan^{-1}5x - \tan^{-1}x + \tan^{-1}\frac{2}{3} + \tan^{-1}x$$
$$= \tan^{-1}5x + \tan^{-1}\frac{2}{3}$$

Therefore,
$$\frac{dy}{dx} = \frac{5}{1+25x^2}$$

Hence, $\lambda = 5$

17.
$$x = e^t\sin t \text{ and } y = e^t\cos t$$
$$\Rightarrow x^2+y^2 = e^{2t} \Rightarrow e^t = \sqrt{x^2+y^2} \qquad (1)$$

and
$$\tan t = \frac{x}{y} \Rightarrow t = \tan^{-1}\left(\frac{x}{y}\right) \qquad (2)$$

From Eqs. (1) and (2), we have
$$e^{\tan^{-1}\left(\frac{x}{y}\right)} = \sqrt{x^2+y^2} \qquad (3)$$

Taking log on both sides, we get
$$\tan^{-1}\left(\frac{x}{y}\right) = \frac{1}{2}\ln(x^2+y^2)$$

Differentiating both sides with respect to x, we have
$$\left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(\frac{y\cdot 1 - x\cdot y'}{y^2}\right) = \frac{1}{2}\cdot\frac{(2x+2yy')}{(x^2+y^2)}$$
$$\Rightarrow y' = \frac{y-x}{x+y} \qquad (4)$$

Again differentiating Eq. (4) with respect to 'x', we have
$$y'' = \frac{(x+y)(y'-1)-(y-x)(1+y')}{(x+y)^2}$$
$$\Rightarrow y''(x+y)^2 = y'(2x) - 2y$$
$$\Rightarrow y''(x+y)^2 = 2(xy'-y)$$

Hence proved and $\lambda = 2$.

Solved JEE 2017 Questions

JEE Main 2017

1. Let f be a polynomial function such that $f(3x)=f'(x)\cdot f''(x)$, for all $x\in\mathbb{R}$. Then

(A) $f(2)-f'(2)+f''(2)=10$ **(B)** $f''(2)-f(2)=4$

(C) $f''(2)-f'(2)=0$ **(D)** $f(2)-f'(2)=28$

(ONLINE)

Solution: Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-1} + \cdots + a_{n-1}x + a_n.$

$$f'(x)=a_0nx^{n-1}+a_1(n-1)x^{n-2}+\cdots+a_{n-1}$$

$$f''(x)=a_0n(n-1)x^{n-2}+a_1(n-1)(n-2)x^{n-3}+\cdots+a_{n-2}$$

Now,

$$f(3x) = 3^na_0x^n + 3^{n-1}a_1x^{n-1} + 3^{n-2}a_2x^{n-2} + \cdots + 3a_{n-1} + a_n$$

$$f'(x)\cdot f''(x)=[a_0nx^{n-1}+a_1(n-1)x^{n-2}+\cdots+a_{n-1}][a_0n(n-1)x^{n-2}$$
$$+a_1(n-1)(n-2)x^{n-3}\cdots+a_{n-2}]$$

Comparing highest powers of x, we get

$$3^n a_0x_n=a_0^2(n-1)x^{n-1+n-2}=a_0^2n^2(n-1)x^{2n-3}$$

Therefore,

$$2n-3=n$$

$$\Rightarrow n=3 \text{ and } 3^na_0=a_0^2n^2(n-1)$$

$$\Rightarrow a_0=27=\frac{3}{2}$$

Therefore,

$$f(x)=\frac{3}{2}x^3+a_1x^2+a_2x+a_3$$

$$f'(x)=\frac{9}{2}x^2+2a_1x+a_2$$

$$f''(x)=9x+2a_1$$

$$f(3x)=\frac{81}{2}x^3+9a_1x^2+3a_2x+a_3$$

Now,

$$f(3x)=f'(x)\cdot f''(x)$$

$$\Rightarrow \frac{81}{2}x^3+9a_1x^2+3a_2x+a_3=\left(\frac{9}{2}x^2+2a_1x+a_2\right)(9x+2a_1)$$

$$\Rightarrow \frac{81}{2}x^3+9a_1x^2+3a_2x+a_3=\frac{81}{2}x^3+[9a_1+18a_1]x^2$$
$$+[4a_1^2+9a_2]x+2a_1a_2$$

Comparing the coefficients, we get

$$9a_1=27a_1$$

$$\Rightarrow a_1=0,\ 3a_2=4a_1^2+9a_2=9a_2$$

$$\Rightarrow a_2=0$$

Therefore,

$$f(x)=\frac{3}{2}x^3$$

$$f'(x)=\frac{9}{2}x^2$$

$$f''(x)=9x$$

Hence, $f''(2)-f'(x)=18-18=0.$

Hence, the correct answer is option (C).

21 Applications of Derivatives

21.1 Geometrical Interpretation of Derivative

Let us consider a curve in the form $y = f(x)$ and two points P(x, y) and Q($x + \Delta x$, $y + \Delta y$) that lie on the curve (Fig. 21.1).

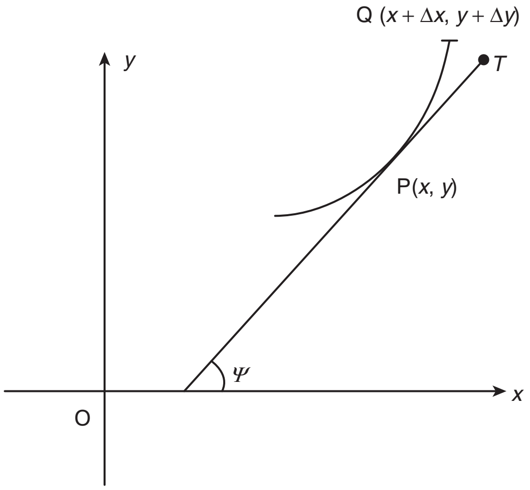


Figure 21.1

Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{y + \Delta y - y}{x + \Delta x - x} \right)$$

$$= \lim_{\Delta x \to 0} (\text{slope of the chord PQ})$$

Now, this is equal to the slope of the tangent PT at point P(x, y) which, in turn, is equal to $\tan \psi$. Here, ψ is the angle that the tangent at point P makes with the positive direction of x-axis.

Illustration 21.1 Find the slope of tangent at the point that has the ordinate –3 on the curve $x^3 = 3y^2$.

Solution: Differentiating the equation of the given curve w.r.t. x, we get

$$3x^2 = 3 \times \left(2y \frac{dy}{dx} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{2y}$$

Now, to obtain this value, we require abscissa as well. Substituting $y = -3$ in the equation of curve, we have

$$x^3 = 3\,(-3)^2 = 27$$

$$\Rightarrow x = 3$$

Therefore, the point of intersect is (3, –3). Hence, the slope of the tangent at this point is

$$\left.\frac{dy}{dx}\right|_{(3,-3)} = \frac{3^2}{2(-3)} = -\frac{3}{2}$$

21.2 Tangent and Normal

A tangent to a point is a line which touches the curve at that point. A normal to a point is the line which is perpendicular to the tangent at that point.

If the equation of a curve is $y = f(x)$ and a point A(x_1, y_1) lies on it, then the equation of the tangent at point A is

$$y - y_1 = \left.\left(\frac{dy}{dx} \right)\right|_A (x - x_1)$$

and the equation of the normal at point A is

$$y - y_1 = -\frac{1}{(dy/dx)|_A} (x - x_1)$$

Illustration 21.2 Find the equation of tangent and normal for illustration 21.1.

Solution: The equation of the tangent is

$$y - (-3) = -\frac{3}{2}(x - 3)$$

$$\Rightarrow 2y + 6 = -3x + 9$$

$$\Rightarrow 3x + 2y - 3 = 0$$

The equation of the normal is

$$y - (-3) = +\frac{2}{3}(x - 3)$$

$$\Rightarrow 3y + 9 = 2x - 6$$

$$\Rightarrow 2x - 3y - 15 = 0$$

Key Point:

When the curve is given in parametric form, that is, when $x = g(t)$ and $y = h(t)$, the equation of tangent at the point $t = t_1$ is

$$y - h(t_1) = \frac{h'(t_1)}{g'(t_1)} [x - g(t_1)]$$

and the equation of normal is

$$y - h(t_1) = -\frac{g'(t_1)}{h'(t_1)} [x - g(t_1)]$$

Illustration 21.3 Find the points on the curve $y = x^3 - x^2 - x + 3$ where the tangent is parallel to the x–axis.

Solution: Given curve is

$$y = x^3 - x^2 - x + 3$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 2x - 1$$

Since the tangent is parallel to the x–axis, the slope is

$$\tan 0° = 0$$

That is,

$$\frac{dy}{dx} = 0$$

Hence,

$$3x^2 - 2x - 1 = 0 \text{ or } (3x+1)(x-1) = 0$$

Therefore,

$$x = -\frac{1}{3} \text{ or } 1$$

For the first point, we have

$$x = -\frac{1}{3} \text{ and } y = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 3 = \frac{86}{27}$$

For the second point, we have

$$x = 1 \text{ and } y = 1^3 - 1^2 - 1 + 3 = 2$$

Hence, the points on the given curve are

$$\left(-\frac{1}{3}, \frac{86}{27}\right) \text{ and } (1, 2)$$

21.2.1 Length of Tangent, Normal, Subtangent and Subnormal

Let the tangent and the normal at point $P(x_1, y_1)$ meet x–axis at points T and N, respectively (Fig. 21.2). Here, PT is called the length of the tangent, which is equal to

$$(PM)\operatorname{cosec}\psi = \frac{y_1}{\sin\psi} \qquad \text{(from } \Delta TMP\text{)}$$

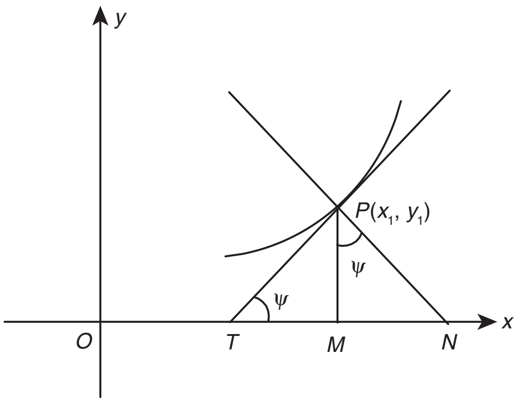


Figure 21.2

Hence, the length of tangent PT is

$$\left|\frac{y_1 \sec\psi}{\tan\psi}\right| = \left|\frac{y_1\sqrt{1+m^2}}{m}\right|$$

where $m = dy/dx$. Similarly, we can conclude with the following results:

1. PN is called the 'length of the normal', which is expressed as
$$(PM)\sec\psi = \left|y_1\sqrt{1+m^2}\right| \qquad \text{(from } \Delta MNP\text{)}$$
2. TM is called the 'subtangent' which is expressed as
$$(PM)\cot\psi = \left|\frac{y_1}{\tan\psi}\right| = \left|\frac{y_1}{m}\right| \qquad \text{(from } \Delta TMP\text{)}$$
3. MN is called the 'subnormal' which is expressed as
$$(PM)\tan\psi = |y_1 m| \qquad \text{(from } \Delta MNP\text{)}$$

Illustration 21.4 Find the length of tangent, normal, subtangent and subnormal to the curve $y = \dfrac{x}{1-x^2}$ at the point having abscissa $\sqrt{2}$.

Solution: At $x = \sqrt{2}$ and $y = -\sqrt{2}$, the point is $P(\sqrt{2}, -\sqrt{2})$. Now,

$$\frac{dy}{dx} = \frac{(1-x^2) - x(-2x)}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$$

Therefore,

$$\left.\frac{dy}{dx}\right|_P = \frac{1+2}{(1-2)^2} = 3 = m \qquad \text{(say)}$$

The equation of tangent is

$$y + \sqrt{2} = 3(x - \sqrt{2})$$

It intersects x–axis at point

$$T\left(\frac{4\sqrt{2}}{3}, 0\right)$$

The length of the tangent is

$$PT = \sqrt{\left(\frac{4\sqrt{2}}{3} - \sqrt{2}\right)^2 + (0+\sqrt{2})^2} = \frac{2\sqrt{5}}{3}$$

The perpendicular drawn from point P on x–axis meets at $M(\sqrt{2}, 0)$. The subtangent is

$$MT = \frac{4\sqrt{2}}{3} - \sqrt{2} = \frac{\sqrt{2}}{3}$$

The equation of normal is

$$y + \sqrt{2} = -\frac{1}{3}(x - \sqrt{2})$$

which intersects x–axis at point $N(-2\sqrt{2}, 0)$. The length of the normal is $PN = \sqrt{20}$ and of the subnormal is $MN = 3\sqrt{2}$.

Aliter: The length of the tangent is

$$\left|\frac{y\sqrt{1+m^2}}{m}\right| = \left|\frac{(-\sqrt{2})\sqrt{1+9}}{3}\right| = \frac{2\sqrt{5}}{3}$$

The length of the normal is

$$\left|y\sqrt{1+m^2}\right| = \left|-(\sqrt{2})\times\sqrt{10}\right| = 2\sqrt{5}$$

The length of the subtangent is

$$\left|\frac{y}{m}\right| = \left|\frac{-\sqrt{2}}{3}\right| = \frac{\sqrt{2}}{3}$$

The length of the subnormal is

$$|y \times m| = |-\sqrt{2}\times 3| = 3\sqrt{2}$$

Note (*Tangent and Normal*): Let $y = f(x)$ be the given curve. The equation of the tangent at (x_1, y_1) would be

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}(x - x_1)$$

or
$$y - f(x_1) = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}(x - x_1)$$

Similarly, the equation of the normal at (x_1, y_1) would be

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,\, y_1)}}(x - x_1)$$

provided that

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \neq 0$$

21.3 Angles Between Two Curves

Given two curves C_1: $y = f(x)$ and C_2: $y = g(x)$ intersecting at some point $P(x_1, y_1)$ (Fig. 21.3).

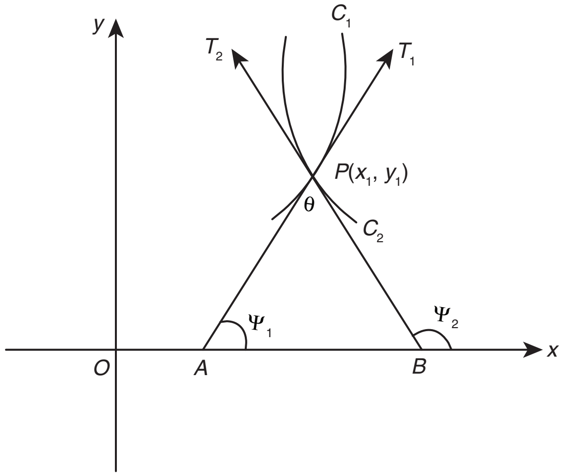


Figure 21.3

Let PT_1 be the tangent at point P to curve C_1 and let PT_1 make an angle ψ_1 with OX. Let PT_2 be the tangent at P to curve C_2 and let PT_2 make an angle ψ_2 with OX. The angle between two curves is defined to be the angle between the two tangents at the point of intersection. Therefore, from ΔABP, θ, the angle between the curves is

$$\angle APB = T_1PT_2 = \psi_2 - \psi_1$$

Now,

$$\tan\theta = \tan(\psi_2 - \psi_1) = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2 \tan\psi_1} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

If θ is the acute angle between the two curves, we have

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

where $m_1 = f'(x)$ at P and $m_2 = g'(x)$ at point P.

Remarks:

(i) The curves intersect orthogonally if $m_1 m_2 = -1$.

(ii) The curves touch each other if $m_1 = m_2$.

Illustration 21.5 Find the angle of intersection of the curves $y = x^3$ and $6y = 7 - x^2$.

Solution: The point of intersection is obtained by solving the equations simultaneously

$$y = x^3 \text{ and } y = \frac{7}{6} - \frac{x^2}{6}$$

That is,

$$6x^3 = 7 - x^2 \quad \text{or} \quad 6x^3 + x^2 - 7 = 0$$
$$\Rightarrow (x - 1)(6x^2 + 7x + 7) = 0$$

This gives $x = 1$ and the other factor gives complex roots. If $x = 1$, then $y = 1$ (by using equation $y = x^3$). Now, from curve C_1, we have

$$\frac{dy}{dx} = 3x^2 = 3\Big|_{x=1}$$

From curve C_2, we have

$$\frac{dy}{dx} = \frac{-2x}{6} = \frac{-1}{3}\Big|_{x=1}$$

Since the product of the slopes results to be -1, the curves intersect at right angles.

Illustration 21.6 Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Solution: First, we must find the common points. Solving the two equations simultaneously, we get

$$x^2 + \frac{16}{x^2} = 8 \quad \text{or} \quad x^4 - 8x^2 + 16 = 0$$

That is,

$$(x^2 - 4)^2 = 0 \quad \text{or} \quad x^2 = 4 \text{ or } x = \pm 2$$

Correspondingly, $y = \pm 2$. Hence, the common points are (2, 2) and (–2, –2).

1. For curve C_1: $\quad x\frac{dy}{dx} + y = 0$ and hence $\frac{dy}{dx} = \frac{-y}{x}$

2. For curve C_2: $\quad 2x + 2y\frac{dy}{dx} = 0$ and hence $\frac{dy}{dx} = \frac{-x}{y}$

At points (2, 2) and (–2, –2), $m_1 = -1$ and $m_2 = -1$, respectively. Hence, the curves touch each other at both points.

Note (*Angle of Intersection of Two Curves*): Let $y = f(x)$ and $y = g(x)$ be two given intersecting curves. The angle of intersection of these two curves is defined as the acute angle between the tangents that can be drawn to the given curves at the point of intersection.

21.4 *dy/dx* as Rate Measures

In this section, we discuss about how dy/dx is useful in determination of rates of change related to physical situations.

Illustration 21.7 A spherical balloon is pumped with air into it at the rate of 10 in^3/min. Find the rate of increase of radius of the balloon when its radius is 15 in.

Solution: Let y be the volume and x the radius of the balloon at any time t. It is given that $dy/dt = 10$ in^3/min. To find dx/dt when $x = 15$ in, since the balloon is spherical

$$y = \frac{4}{3}\pi x^3 \tag{1}$$

$$\frac{dy}{dt} = \frac{4}{3}\pi(3x^2)\frac{dx}{dt} = 4\pi x^2\frac{dx}{dt} \tag{2}$$

Therefore,

$$\frac{dx}{dt} = \frac{dy/dt}{4\pi x^2} = \frac{10}{4\pi x^2}$$

Therefore, when $x = 15$ in,

$$\frac{dx}{dt} = \frac{10}{4\pi(15)^2} = \left(\frac{1}{90\pi}\right)\text{ in/min}$$

Hence, the rate of increase of the radius of the balloon when its radius is 15 in is $(1/90\pi)$ in/min.

Illustration 21.8 The diameter of a cone is 10 in and its depth is 10 in. Water is poured into it at the rate of 4 in^3/min. At what rate is the water level rising at the instant when the depth is 6 in?

Solution: See Fig. 21.4. Let OAB be the cone and LM be the level of water at any time t.

Let $ON = y$, volume $OLM = V$ and radius $MN = r$.

Given $AB = 10$ in, $OC = 10$ in and $dV/dt = 4$ in^3/min.

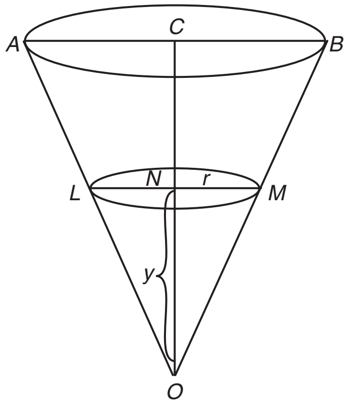


Figure 21.4

We should find dy/dt when $y = 6$ in. Now,

$$V = \frac{1}{3}(\pi r^2)(y) \qquad \left[\because \text{ volume of a cone} = \frac{1}{3}\pi r^2 h\right] \tag{1}$$

Now, from similar ΔONM and ΔOCB, we get

$$\frac{MN}{BC} = \frac{ON}{OC} \quad \text{or} \quad \frac{r}{5} = \frac{y}{10} \quad \text{or} \quad r = \frac{y}{2}$$

Substituting $r = y/2$ in Eq. (1), we get

$$V = \frac{1}{3}\pi\frac{y^2}{4}(y) = \frac{\pi}{12}y^3$$

Differentiating w.r.t. t, we get

$$\frac{dV}{dt} = \frac{\pi}{12}(3y^2)\frac{dy}{dt}$$

Therefore,

$$\frac{dy}{dt} = \frac{4}{\pi y^2}\frac{dV}{dt}$$

When $y = 6$ in, we get

$$\frac{dy}{dt} = \frac{4}{(\pi)6^2}4 = \left(\frac{4}{9\pi}\right)\text{ in/min}$$

Hence, when the depth of water is 6 in, the water level is rising at the rate $(4/9\pi)$ in/min.

21.5 Errors and Approximations

Let $y = f(x)$. Then we know that

$$\lim_{\delta x \to 0}\frac{f(x+\delta x) - f(x)}{\delta x} = f'(x)$$

Therefore,

$$\frac{f(x+\delta x) - f(x)}{\delta x} = f'(x) + \varepsilon$$

where $\varepsilon \to 0$, when $\delta x \to 0$. So,

$$f(x+\delta x) - f(x) = f'(x)\delta x + \varepsilon\,\delta x$$

or $\quad f(x+\delta x) - f(x) = f'(x)\delta x \quad$ (approximately)

or $\quad \delta y = f'(x)\delta x \quad$ [As $f(x+\delta x) - f(x) = \delta y$]

Thus, if δx is an error in x, then the corresponding error δy in y can be calculated.

Note:

1. δx and δy are known as differentials.
2. If $y = f(x)$ and δy is an increment in y, corresponding to an increment δx in x, then we have
$$\delta y = \frac{dy}{dx}(\delta x)$$
3. The error δx in x is called an absolute error.
4. $\dfrac{\delta x}{x}$ is called the relative error and $\left(\dfrac{\delta x}{x}\times 100\right)$ is called the percentage error.

Illustration 21.9 Find the approximate value of $\sqrt{0.037}$.

Solution: Let $f(x)=\sqrt{x}$. Then,

$$f(x+\delta x)-f(x)=f'(x)\delta x$$

or $$f(x+\delta x)-f(x)=\left(\frac{1}{2\sqrt{x}}\right)\delta x$$

We may write

$$0.037=(0.04-0.003)$$

Taking $x=0.04$ and $\delta x=-0.003$, we have

$$f(0.037)-f(0.04)=\left(\frac{1}{2\sqrt{0.04}}\right)(-0.003)$$

or $f(0.037)=f(0.04)-\dfrac{0.003}{2\times 0.2}=\left(\sqrt{0.04}-\dfrac{3}{400}\right)=\left(0.2-\dfrac{3}{400}\right)=\dfrac{77}{400}$

Therefore,

$$\sqrt{0.037}=\frac{77}{400}=0.1925$$

Illustration 21.10 Find the approximate value of tan 46° if it is given that $1°=0.01745$ rad.

Solution: Let $f(x)=\tan x$. Then $f'(x)=\sec^2 x$. Now,

$$f(x+\delta x)-f(x)=f'(x)\delta x$$

or $$f(x+\delta x)-f(x)=(\sec^2 x)\delta x$$

Taking $x=45°=(\pi/4)°$ and $\delta x=1°=0.01745$, we get

$$f(46°)-f(45°)=(\sec^2 45°)\times 0.01745$$

or $$\tan 46°=\tan 45°+(\sec^2 45°)\times 0.01745$$
$$=(1+2\times 0.01745)=1.03490$$

Illustration 21.11 The time T of oscillation of a simple pendulum of length l is given by $T=2\pi\sqrt{\dfrac{l}{g}}$. Find the percentage error in T corresponding to an error of 2% in the value of l.

Solution:

$$T=2\pi\sqrt{\frac{l}{g}}$$

$$\Rightarrow \log T=\log 2+\log\pi+\frac{1}{2}\log l-\frac{1}{2}\log g$$

$$\Rightarrow \frac{1}{T}\left(\frac{dT}{dl}\right)=\frac{1}{2l}$$

$$\Rightarrow \frac{1}{T}\left(\frac{dT}{dl}\right)\delta l=\left(\frac{1}{2l}\right)\delta l$$

$$\Rightarrow \left(\frac{1}{T}\right)\delta T=\left(\frac{1}{2l}\right)\delta l \qquad \left[\text{since } \delta T=\left(\frac{dT}{dl}\right)\delta l\right]$$

$$\Rightarrow \left(\frac{\delta T}{T}\times 100\right)=\frac{1}{2}\left(\frac{\delta l}{l}\times 100\right)=\frac{1}{2}\times 2=1$$

Therefore, percentage error in $T=1\%$.

Your Turn 1

1. Find the slopes of the curve $y=(x+2)(x-3)$ at the points where it meets x–axis. **Ans.** –5, 5
2. Find the points on the curve $y=x^3-2x^2+x-2$ when the gradient is zero.
   Ans. $(1,-2)$ and $\left(\dfrac{1}{3},-\dfrac{50}{27}\right)$
3. Find the equation of tangent and normal to the curve $x^3=y^2$ at the point (1, 1). Also find the length of tangent, normal, subtangent and subnormal.
   Ans. $3x-2y-1=0,\ 2x+3y-5=0,\ \dfrac{\sqrt{13}}{3},\dfrac{\sqrt{13}}{2},\dfrac{2}{3},\dfrac{3}{2}$
4. Find the angle of intersection of the curves $y=x^2$ and $y=x^3$.
   Ans. $\tan^{-1}\left(\dfrac{1}{7}\right), 0$
5. State true or false: The curves $x^2-y^2=16$ and $xy=25$ cut each other at right angles.
   Ans. True
6. If the radius of a circle is increasing at a constant rate of 2 ft/sec, then find the rate of increase of its area when the radius is 20 ft.
   Ans. (80π) sq.ft/sec
7. Water is poured at the rate of 1 ft^3/min into a cylindrical tub. If the tub has a circular base of radius a ft, then the rate at which water is rising in the tub is ________.
   Ans. $\left(\dfrac{1}{\pi a^2}\right)$ft/min

21.6 Monotonicity of Function

In this section, the behaviour of function is discussed. Generally, there are four types of behaviours shown in function in the intervals of its domains.

21.6.1 Increasing Behaviour of Function

If in an interval I, for any two points (x_1, y_1) and (x_2, y_2) (Fig. 21.5), we have

$$x_2>x_1 \Leftrightarrow y_2>y_1$$

The function is said to be monotonically increasing or simply increasing in I. If the function is differentiable in the required interval (which is, normally, true for most of the functions), it can be inferred that $(dy/dx)>0$ for all points in the interval.

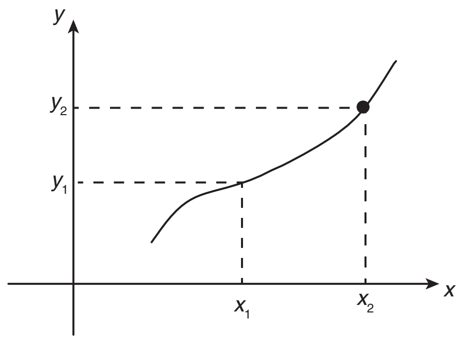


Figure 21.5

For example,

$$y = x^3$$

$$\Rightarrow \frac{dy}{dx} = 3x^2$$

Now, $(dy/dx) > 0$ for all real values of x except $x = 0$. Here, $(dy/dx) \geq 0$ for the entire domain, but still the function is increasing. For any two points such that $x_2 > x_1$, we have $y_2 > y_1$ certainly.

Note (*Increasing Function*): The function $f(x)$ is said to be increasing function in D_1 if for every $x_1, x_2 \in D_1$,

$$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$$

$f(x)$ is increasing function in $[a, b]$ if $f'(x) > 0 \quad \forall x \in (a, b)$.

21.6.2 Decreasing Behaviour of Function

If in an interval I, for any two points (x_1, y_1) and (x_2, y_2) (Fig. 21.6), we have

$$x_2 > x_1 \Leftrightarrow y_2 < y_1$$

The function is said to be monotonically decreasing function or simply decreasing function in I. Also, for a differentiable function, here $(dy/dx) < 0$ for all points in the interval.

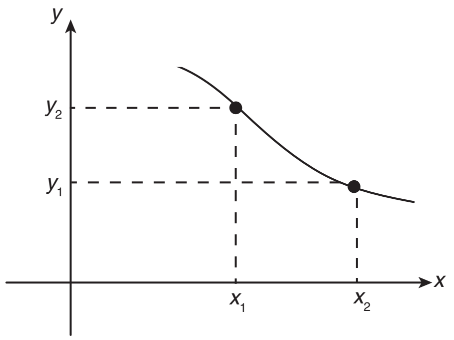


Figure 21.6

Note (*Decreasing Function*): The function $f(x)$ is said to be a decreasing function in D_1 if for every $x_1, x_2 \in D_1$,

$$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$$

$f(x)$ is decreasing in $[a, b]$ if $f'(x) < 0 \ \forall x \in (a, b)$.

21.6.3 Non-Decreasing Behaviour

See Fig. 21.7. In this case,

$$x_2 > x_1 \Leftrightarrow y_2 \geq y_1$$

for all points in that interval. This implies that

$$\frac{dy}{dx} \geq 0$$

where $\frac{dy}{dx} = 0$ for a continuous set of points in the interval.

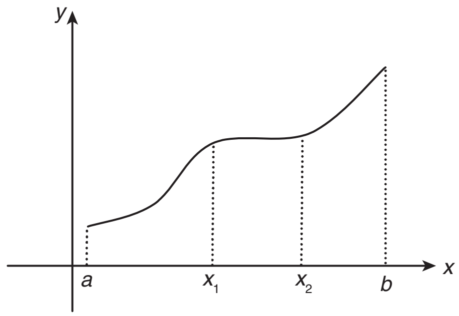


Figure 21.7

Note (*Non-Decreasing Function*): The function $f(x)$ is said to be non-decreasing in D_1 if for every $x_1, x_2 \in D_1$,

$$x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$$

The function $f(x)$ is non-decreasing in $[a, b]$ if $f'(x) \geq 0 \quad \forall x \in (a, b)$.

21.6.4 Non-Increasing Behaviour

See Fig. 21.8. In this case,

$$x_2 > x_1 \Leftrightarrow y_2 \leq y_1$$

for all points in that interval. This implies that

$$\frac{dy}{dx} \leq 0$$

where $\frac{dy}{dx} = 0$ for a continuous set of points in the interval.

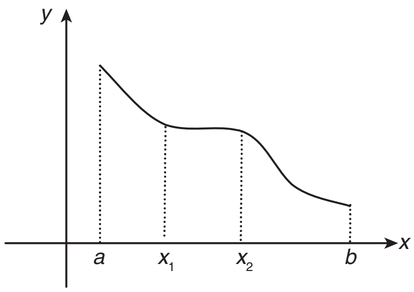


Figure 21.8

Note: These conditions normally are sufficient to find the intervals of increasing function. However, when the derivative may include zero also at specific points in the interval, the function may be still increasing.

***Non-Increasing Function*:** The function $f(x)$ is said to be a non-increasing function in D_1 if for every $x_1, x_2 \in D_1$,

$$x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$$

The function $f(x)$ is non-increasing in $[a, b]$ if $f'(x) \leq 0 \ \forall x \in (a, b)$.

Key Points:

1. If $f'(x) \geq 0 \ \forall x \in (a, b)$ and the points which make $f'(x) = 0$ [in between (a, b)], do not form an interval, then $f(x)$ would be increasing in $[a, b]$.
2. If $f'(x) \leq 0 \ \forall x \in (a, b)$ and the points which make $f'(x) = 0$ [in between (a, b)], do not form an interval, then $f(x)$ would be decreasing in [a, b].

3. If $f(0) = 0$ and $f'(x) \geq 0\ \forall x \in R$, then $f(x) \leq 0\ \forall\ x \in (-\infty, 0)$ and $f(x) \geq 0\ \forall\ x \in (0, \infty)$.
4. If $f(0) = 0$ and $f'(x) \leq 0\ \forall\ x \in R$, then $f(x) \geq 0\ \forall\ x \in (-\infty, 0)$ and $f(x) \leq 0\ \forall\ x \in (0, \infty)$.
5. A function is said to be monotonic if the function is either increasing or decreasing.
6. The points for which $f'(x) = 0$ (or the function does not exist) are called 'critical points'. Here, it should also be noted that the critical points are the interior points of an interval.
7. The stationary points are the points where $f'(x) = 0$ in the domain.

Illustration 21.12 Find the values of x for which the function $f(x) = 2x^3 - 21x^2 + 72x + 30$ is (a) increasing and (b) decreasing.

Solution:

$$f'(x) = 6x^2 - 42x + 72 = 6(x^2 - 7x + 12) = 6(x - 3)(x - 4)$$

(a) The function $f(x)$ is increasing if $f'(x) > 0$. That is, if $6(x - 3)(x - 4) > 0$ or if either $x > 4$ or $x < 3$ or if $x \in (-\infty, 3) \cup (4, \infty)$

(b) The function $f(x)$ is decreasing if $f'(x) < 0$. That is, if $6(x - 3)(x - 4) < 0$ or if $x \in (3, 4)$.

Illustration 21.13 Separate the intervals in which the function $f(x) = x - e^x$ is increasing or decreasing.

Solution:

$$f'(x) = 1 - e^x$$

Now, $\quad f'(x) > 0 \Rightarrow 1 - e^x > 0 \Rightarrow e^x < 1 \Rightarrow x < 0$

and $f'(x) < 0$ for all $x > 0$. Therefore, $f(x)$ is increasing in the interval $(-\infty, 0)$ and decreasing in the interval $(0, \infty)$.

Illustration 21.14 If $x > 0$, show that $\log(1 + x) < x$.

Solution: The method is to construct the function $f(x) = x - \log(1 + x)$ and show that (a) it is increasing and (b) $f(0) = 0$ or positive.

Let $f(x) = x - \log(1 + x)$. Then

$$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$$

is positive for all $x > 0$. Therefore, $f(x)$ is increasing for all $x > 0$ and

$f(0) = 0 - \log(1) = 0 - 0 = 0$

That is, $f(0) = 0$ and $f(x)$ is increasing for all $x > 0$ implies that $f(x)$ is positive for all $x > 0$. That is,

$$x - \log(1 + x) > 0$$

or $$\log(1 + x) < x$$

21.7 Maxima and Minima of Functions of a Single Variable

Let the function $f(x)$ be defined on an interval A. Let a and $b \in A$. Then

(i) $f(x)$ is said to have a maximum value at $x = a$ if $f(a) > f(a + h)$ and $f(a) > f(a - h)$ for all sufficiently small positive values of h. The point a is called the point at which the function is a maximum and $f(a)$ is the corresponding maximum value of the function.

(ii) $f(x)$ is said to have a minimum value at $x = b$ if $f(b) < f(b + h)$ and $f(b) < f(b - h)$ for all sufficiently small positive values of h. The point b is called the point at which the function is a minimum and $f(b)$ is the corresponding minimum value of the function.

(iii) The maximum and minimum values of $f(x)$ as defined above are not necessarily the greatest and least values of $f(x)$. They are maximum and minimum in the immediate neighbourhood of $x = a$ and $x = b$. Hence, these are also referred as 'local maximum' or 'local minimum'.

(iv) The points of maximum or minimum of a function are also called the 'points of extremum'.

(v) A necessary condition for the existence of an extremum (maximum or minimum) for a function $f(x)$ is either $f'(x) = 0$ or $f'(x)$ does not exist.

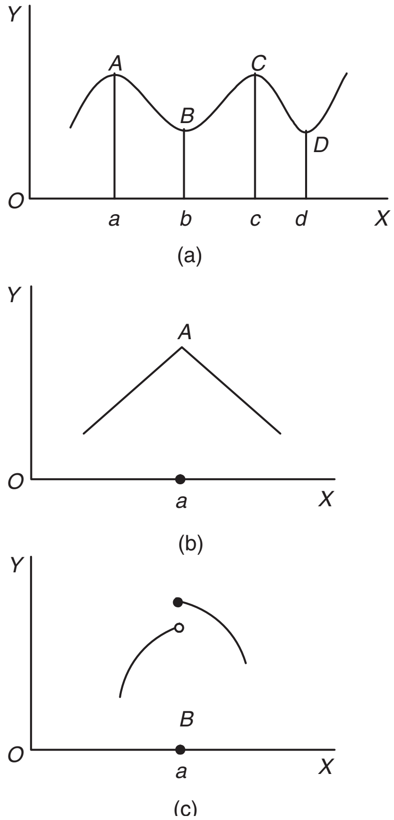


Figure 21.9

In Fig. 21.9(a), at the maximum and minimum points on the graph, the tangent is parallel to the x–axis and hence $f'(x) = 0$. In Fig. 21.9(b), the function is increasing as x approaches a from the left and the function is decreasing as x increases beyond a. The graph is not smooth and hence it has no tangent at $x = a$. In Fig. 21.9(c), $x = a$ is a point of local maxima ($f(x)$ is discontinuous at $x = a$).

Note: It is very important to note that for maxima and minima, we use the basic definition at all the critical points (i.e. the points where $f(x)$ is discontinuous).

1. Second Derivative Test (for Continuous Functions)

(i) If $f(x)$ has a maximum value at $x = a$ and $f'(a)$ exists, then $f'(a)$ must be zero. Similarly, if $f(x)$ has a minimum value at $x = b$ and $f'(b)$ exists, then $f'(b)$ must be zero.

(ii) If c be a point in the domain of $f(x)$ such that $f'(c) = 0$ and $f''(c) \neq 0$, then

(a) $x = c$ is a point of local maxima if $f''(c) < 0$

(b) $x = c$ is a point of local minima if $f''(c) > 0$

2. First Derivative Test (For Continuous Functions)

(i) If $f'(A) = 0$ (or it does not exist) and $f'(x)$ changes its sign from plus to minus as x passes through the point a from left to right, then $f(x)$ is maximum at $x = a$.

(ii) If $f'(B) = 0$ or does not exist and $f'(x)$ changes its sign from minus to plus as x passes through the point b from left to right, then $f(x)$ is minimum at $x = b$.

(iii) If the derivative does not change its sign in moving from left to right through the point a, then $f(x)$ has neither maximum nor minimum at $x = a$.

3. n^{th} Derivative Test

(i) It can be applied to $x = c$ only if $f'(c) = 0$ and $f''(c) = 0$.

(ii) By differentiation, find n^{th} derivative of $f(x)$ denoted by $f^n(x), n \in N$.

(iii) Step–by–step, find the earliest non–zero $f^n(c)$, $n = 3, 4, 5, 6, 7, \ldots$

(iv) In this process,

(a) if n is odd, then $x = c$ is neither local maximum nor local minimum point.

(b) if n is even, and if

$$f^n(c) = \begin{cases} \text{Positive}, & \Rightarrow x = c \text{ is local minimum point} \\ \text{Negative}, & \Rightarrow x = c \text{ is local maximum point} \end{cases}$$

4. Absolute Maximum/Minimum Points

(i) To find the absolute maximum/minimum values of $f(x)$ in open interval (a, b), we proceed as follows:

(a) Find all extremum points of $f(x)$ by using critical points. Let these extremum points be $c_1, c_2, c_3 \ldots$

(b) Compare the lengths of ordinates $f(c_1), f(c_2), f(c_3), \ldots$

(c) The greatest value of these ordinates is called absolute maximum value of $f(x)$.

(d) The least value of these ordinates is called absolute minimum value of $f(x)$.

Absolute maximum value/absolute minimum value can occur at more than one extremum point. Absolute maximum/minimum value is also called global maximum/minimum value of $f(x)$.

(ii) To find the absolute maximum/minimum value in closed interval $[a, b]$, include the values of ordinates at the end points, namely, $f(a)$ and $f(b)$ in the above procedure of comparison of the lengths of the ordinates at the extremum points.

21.7.1 Concept of Local Maximum and Local Minimum

Let $y = f(x)$ be a function defined at $x = a$ and also in the vicinity of the point $x = a$. Then, $f(x)$ is said to have a local maximum at $x = a$ if the value of the function at $x = a$ is greater than the value of the function at the neighbouring points of $x = a$. Similarly, $f(x)$ is said to have a local minimum at $x = a$ if the value of the function at $x = a$ is less than the value of the function at the neighbouring points of $x = a$.

21.7.1.1 Test for Local Maximum/Minimum

1. *Test for local maximum/minimum at $x = a$ if $f(x)$ is differentiable at $x = a$:* If $f(x)$ is differentiable at $x = a$ and if it is a critical point of the function; if $f'(a) = 0$ and $f'(x)$ changes its sign while passing through the point $x = a$, then

(i) $f(x)$ would have a local maximum at $x = a$ if $f'(a-0) > 0$ and $f'(a+0) < 0$. It means that $f'(x)$ should change its sign from positive to negative.

(ii) $f(x)$ would have local minimum at $x = a$ if $f'(a-0) < 0$ and $f'(a+0) > 0$. It means that $f'(x)$ should change its sign from negative to positive.

(iii) If $f(x)$ does not change its sign while passing through $x = a$, then $f(x)$ would have neither a maximum nor minimum at $x = a$.

Second–order derivative test for maxima and minima: Let $f(x)$ be a differentiable function on an interval I. Let $a \in$ I and $f''(x)$ is continuous at $x = a$. Then

(i) $x = a$ is a point of local maximum if $f'(a) = 0$ and $f''(a) < 0$.

(ii) $x = a$ is a point of local minimum if $f'(a) = 0$ and $f''(a) > 0$.

(iii) If $f'(a) = f''(a) = 0$ and $f'''(a) \neq 0$ exists, then $x = a$ is neither a point of local maximum nor a point of local minimum.

2. *Test for local maximum/minimum at $x = a$ if $f(x)$ is not differentiable at $x = a$:*

Case I: When $f(x)$ is continuous at $x = a$ and also $f'(a - h)$ and $f'(a + h)$ exist which are non–zero, then $f(x)$ has a local maximum or minimum at $x = a$ if $f'(a - h)$ and $f'(a + h)$ are of opposite signs.

(a) If $f'(a - h) > 0$ and $f'(a + h) < 0$, then $x = a$ will be a point of local maximum.

(b) If $f'(a - h) < 0$ and $f'(a + h) > 0$, then $x = a$ will be a point of local minimum.

Case II: When $f(x)$ is continuous and $f'(a - h)$ and $f'(a + h)$ exist, but one of them is zero, we should infer the information about the existence of local maximum/minimum from the basic definition of local maximum/minimum.

Case III: If $f(x)$ is not continuous at $x = a$ and $f'(a - h)$ and/or $f'(a + h)$ are not finite, then compare the values of $f(x)$ at the neighbouring points of $x = a$.

21.7.1.2 Concept of Global Maximum/Minimum

Let $y = f(x)$ be a given function with domain D. Let $[a, b] \subseteq D$. The global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in $[a, b]$. The global maximum and minimum in $[a, b]$ would always occur at critical points of $f(x)$ within $[a, b]$ or at the end points of the interval.

Illustration 21.15 Find the local maximum and minimum values of $f(x) = 2x^3 - 15x^2 + 36x + 11$.

Solution: Let $y = 2x^3 - 15x^2 + 36x + 11$. Therefore,

$$\frac{dy}{dx} = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$$

$$\frac{d^2y}{dx^2} = 12x - 30$$

For extremum,

$$dy/dx = 0$$
$$\Rightarrow x^2 - 5x + 6 = 0$$
$$\Rightarrow (x-2)(x-3) = 0$$

That is, $x = 2$ or $x = 3$. Now,

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 12(2) - 30 = -6 < 0$$

Therefore, y or $f(x)$ is a maximum when $x = 2$ and the maximum value of $f(x) = f(2)$. Therefore,

$$f(2) = 2(2^3) - 15(2^2) + 36(2) + 11 = 39$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=3} = 12 \times 3 - 30 = 6 > 0$$

Therefore, y or $f(x)$ is a minimum when $x = 3$ and the minimum value of

$$f(x) = f(3) = 2(3)^3 - 15(3)^2 + 36(3) + 11 = 38$$

Illustration 21.16 Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{\frac{1}{e}}$.

Solution: Let $y = \left(\frac{1}{x}\right)^x$. Then

$$\log y = x \log_e\left(\frac{1}{x}\right) = -x \log_e x \quad (1)$$

Therefore,

$$\frac{1}{y}\frac{dy}{dx} = -\left[x \cdot \frac{1}{x} + \log_e x\right] = -(1 + \log_e x) \quad (2)$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{1}{x}\right)^x (1 + \log_e x)$$

Differentiating, we get

$$\frac{1}{y}\frac{d^2y}{dx^2} - \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = -\frac{1}{x} \quad (3)$$

From Eq. (2), we get

$$\frac{dy}{dx} = -y(1 + \log_e x) = -\left(\frac{1}{x}\right)^x (1 + \log_e x)$$

For the maximum or minimum values of y, $dy/dx = 0$. Therefore,

$$\left(\frac{1}{x}\right)^x (1 + \log_e x) = 0$$

However, $(1/x)^x \neq 0$ for any value of x. Therefore,

$$1 + \log_e x = 0; \log_e x = -1; x = e^{-1} = \frac{1}{e} \quad (4)$$

When $x = 1/e$, Eq. (3) gives

$$\frac{1}{y}\frac{d^2y}{dx^2} - 0 = -e$$

Therefore,

$$\frac{d^2y}{dx^2} = -e(e)^{1/e} < 0$$

Hence, y is maximum when $x = 1/e$ and the maximum value of $y = e^{1/e}$.

Illustration 21.17 If $y = a\log|x| + bx^2 + x$ has its extreme values at $x = -1$ and $x = 2$, find a and b.

Solution:

$$y = a\log|x| + bx^2 + x$$

Therefore,

$$\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$$

For extreme values, $dy/dx = 0$.

$$\left.\frac{dy}{dx}\right|_{x=-1} = 0 \Rightarrow \frac{a}{-1} + 2b(-1) + 1 = 0$$

or

$$a + 2b - 1 = 0$$

and

$$\left.\frac{dy}{dx}\right|_{x=2} = 0 \Rightarrow \frac{a}{2} + 2b(2) + 1 = 0$$

or

$$a + 8b + 2 = 0$$

Solving these two equations simultaneously, $a = 2$ and $b = -(1/2)$.

Illustration 21.18 Find the maximum and minimum values of $f(x) = x + \sin 2x$ in $[0, 2\pi]$.

Solution:

$$f'(x) = 1 + 2\cos 2x; f''(x) = -4\sin 2x$$

$$f'(x) = 0 \Rightarrow \cos 2x = -\frac{1}{2} \Rightarrow 2x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

or

$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$f''\left(\frac{\pi}{3}\right) = -4\sin\frac{2\pi}{3} = -\frac{4\sqrt{3}}{2} = -2\sqrt{3} < 0$$

$$\Rightarrow x = \frac{\pi}{3}$$

which is local maximum point.

$$f''\left(\frac{2\pi}{3}\right) = -4\sin\frac{4\pi}{3} = 4\frac{\sqrt{3}}{2} = 2\sqrt{3} > 0$$

$$\Rightarrow x = \frac{2\pi}{3}$$

which is local minimum point.

Now, for absolute maximum/minimum, we compare the values of

$$f(0) = 0,\ f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2},\ f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2},\ f(2\pi) = 2\pi$$

Therefore, the absolute maximum is

$$f(x) = f(2\pi) = 2\pi$$

and the absolute minimum is

$$f(x) = f(0) = 0$$

Illustration 21.19 Find the points of local extremum of the function $f(x) = (2x-1)^{2/5}(x+2)$.

Solution:

$$f'(x) = (2x-1)^{2/5}(1) + (x+2)\,\frac{2}{5}\,(2x-1)^{-3/5}(2)$$

$$= (2x-1)^{2/5} + \frac{4(x+2)}{5(2x-1)^{3/5}}$$

$$= \frac{(2x-1)+4x+8}{5(2x-1)^{3/5}}$$

That is,

$$f'(x) = \frac{6x+7}{5(2x-1)^{3/5}}$$

For critical points, $f'(x) = 0$ or the function is not defined. Therefore,

$$x = -\frac{7}{6}, \frac{1}{2}$$

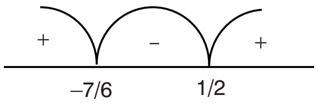


Figure 21.10

See Fig. 21.10. Near $x = -7/6$, the sign of $f'(x)$ changes from positive to negative. Therefore, the local maxima is at $x = -7/6$. Near $x = 1/2$, the sign of $f'(x)$ changes from negative to positive. Therefore, the local minima is at $x = 1/2$.

Illustration 21.20 Find the local maximum/minimum points of $f(x) = (x-2)^3(x-3)$.

Solution:

$$f'(x) = (x-2)^2(4x-11) = 0$$
$$\Rightarrow x = 2, 2, 11/4$$

are critical points.

$$f''(x) = 2(x-2)(4x-11) + 4(x-2)^2$$

$$f''(11/4) > 0 \Rightarrow x = 11/4$$

is local minimum point.

$$f''(2) = 0$$

Therefore, the test for second derivative fails in this case.

$$f'''(x) = 2(4x-11) + 16(x-2)$$

$$f'''(2) = -6 \neq 0$$

At the earliest non–zero derivative at $x = 2$ is of odd order, $x = 2$ is neither local maximum nor local minimum point. Hence, $x = 11/4$ is the one and only local minimum point of $f(x)$.

21.8 Mean Value Theorems

21.8.1 Rolle's Theorem

If $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b) and further $f(a) = f(b)$, then there is at least one point $x = c$ on the interval (a, b), where $f'(c) = 0$.

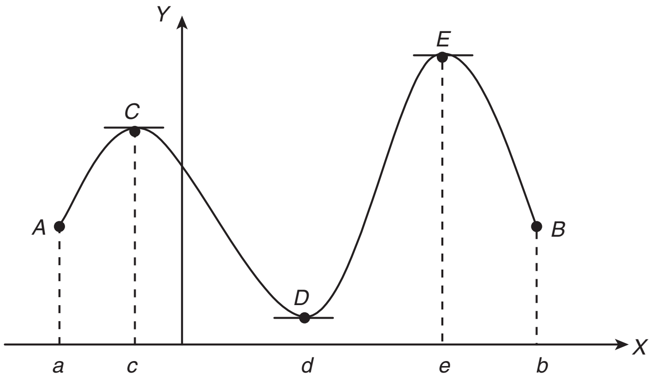


Figure 21.11

Fig. 21.11 shows that the graphical interpretation of Rolle's theorem. The slope of tangent is zero at points C, D and E.

Note (*Rolle's Theorem*): Let $y = f(x)$ be a given function and satisfies the following conditions:

1. $f(x)$ be continuous in $[a, b]$.
2. $f(x)$ be differentiable in (a, b).

If $f(a) = f(b)$, then $f'(c) = 0$ at least once for some $c \in (a, b)$.

Illustration 21.21 Verify Rolle's theorem for $f(x) = x^3(x-1)^2$ in the interval $0 \le x \le 1$. Also find the value of c in between a and b where $f'(x) = 0$

Solution: We have

$$f(x) = x^3(x-1)^2 \tag{1}$$

Therefore,

$$f'(x) = 3x^2(x-1)^2 + x^3[2(x-1)] = x^2(x-1)[3(x-1)+2x]$$

or

$$f'(x) = x^2(x-1)(5x-3) \tag{2}$$

Clearly, $f'(x)$ is finite for all x and hence $f(x)$ is differentiable at all x. Therefore,

(i) $f(x)$ is continuous at all x and hence also continuous in the closed interval $[0, 1]$.

(ii) $f(x)$ is differentiable in the open interval $(0, 1)$.

(iii) from Eq. (1), $f(0) = 0$ and $f(1) = 0$. Therefore, $f(0) = f(1)$.

Hence, all conditions of Rolle's theorem are satisfied. Now from Eq. (2),

$$f'(c) = 0 \Rightarrow c^2(c-1)(5c-3) = 0$$

or

$$c = 0, 1, \frac{3}{5}$$

However, $0 < c < 1$. Therefore, $c = 3/5$. Thus, there exists at least one c, that is, $c = 3/5$ between 0 and 1 such that $f'(c) = 0$. Hence, Rolle's theorem has been verified.

Illustration 21.22 Taking the functions $f(x) = (x-3)\log x$, prove that there is at least one value of x in $(1, 3)$ which satisfies $x\log x = 3 - x$.

Solution: We have

$$f(x) = (x-3)\log x \tag{1}$$

Therefore,

$$f'(x) = (x-3)\frac{1}{x} + 1(\log x) \tag{2}$$

Clearly, $f'(x)$ is finite for all positive values of x and hence $f(x)$ is differentiable for all $x > 0$. Therefore, $f(x)$ is differentiable in (1, 3); and therefore $f(x)$ is also continuous in [1, 3]. Also

$$f(1) = (1-3)(\log 1) = 0 \text{ and } f(3) = (3-3)(\log 3) = 0$$

Therefore,

$$f(1) = f(3)$$

Thus, by Rolle's theorem, there will be at least one value of x in (1, 3) such that $f'(x)=0$. Therefore, from Eq. (2), we get

$$\frac{x-3}{x} + \log x = 0$$

or $$x(\log x) = 3 - x$$

21.8.2 Lagrange's Mean Value Theorem

If $f(x)$ is continuous in the interval $[a, b]$ and differentiable in (a, b), then there exists at least one point $x = c$ in the interval (a, b), where

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

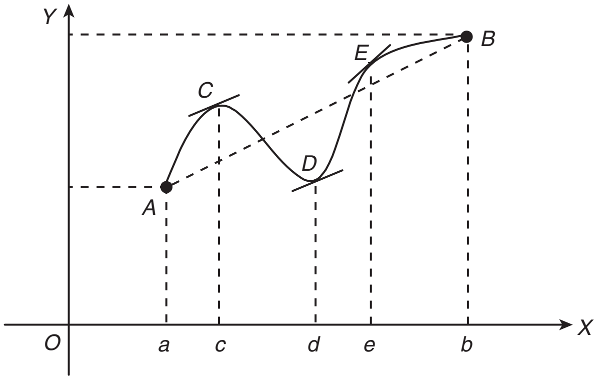


Figure 21.12

The geometrical meaning is clearly depicted in the graph shown in Fig. 21.12. Here,

$$\frac{f(b)-f(a)}{b-a}$$

is the slope of the chord AB. The tangents at C, D and E are parallel to this chord. Rolle's theorem is a special case of Lagrange's mean value theorem.

Note (*Lagrange's Mean Value Theorem*): If $y = f(x)$ be a given function, which is (i) continuous in $[a, b]$ and (ii) differentiable in (a, b), then

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

at least once for some $c \in (a, b)$.

Illustration 21.23 Find c of the Lagrange's mean value theorem for the function $f(x) = 3x^2 + 5x + 7$ in the interval [1, 3].

Solution: We have

$$f(x) = 3x^2 + 5x + 7 \tag{1}$$

Therefore,

$$f(1) = 3 + 5 + 7 = 15$$

and $$f(3) = 27 + 15 + 7 = 49$$

Now, $$f'(x) = 6x + 5 \tag{2}$$

Here $a = 1$ and $b = 3$. Now from Lagrange's mean value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Therefore,

$$6c + 5 = \frac{f(3)-f(1)}{3-1} = \frac{49-15}{2} = 17$$

or $$6c = 12 \text{ and thus } c = 2.$$

Illustration 21.24 Using mean value theorem, show that $|\cos a - \cos b| \le |a - b|$.

Solution:

Case I: When $a = b$,

$$|\cos a - \cos b| = 0 = |a - b| \tag{1}$$

Case II: When $a \ne b$, let $a < b$. Let

$$f(x) = \cos x$$

Then

$$f'(x) = -\sin x$$

Clearly, $f(x)$ is differentiable and continuous at all x. Therefore, by Lagrange's mean value theorem, there will be at least one c, $a < c < b$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

or $$\frac{\cos b - \cos a}{b-a} = -\sin c$$

Therefore,

$$\left|\frac{\cos a - \cos b}{a-b}\right| = |-\sin c|$$

or $$\frac{|\cos a - \cos b|}{|a-b|} = |-\sin c| \le 1 \qquad [\because |\sin\theta| \le 1]$$

or $$|\cos a - \cos b| \le |a - b| \tag{2}$$

From Eqs. (1) and (2), for all values of a and b,

$$|\cos a - \cos b| \le |a - b|$$

21.9 Geometrical Problems

In this section, we will use differential calculus in optimization problems.

Illustration 21.25 Find the cone of maximum volume that can be inscribed in a sphere of radius R.

Solution: Let ABC be the cone with radius R (Fig. 21.13).

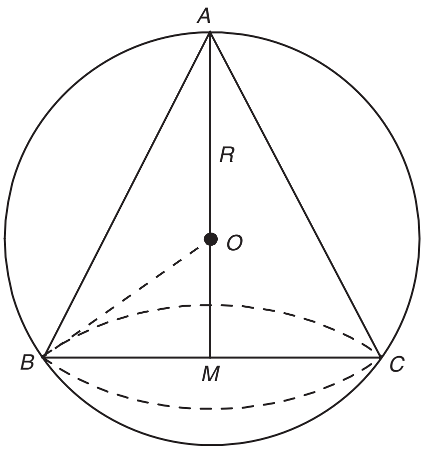


Figure 21.13

Here, $BM = MC = x$ and the height $AM = y$. In ΔOMB,

$$BM^2 + OM^2 = OB^2$$

$$x^2 + (y - R)^2 = R^2$$

Therefore,

$$x^2 = 2Ry - y^2$$

The volume (V) of the cone is

$$\frac{1}{3}\pi x^2 y = \frac{\pi}{3} y\,(2Ry - y^2)$$

$$= \frac{\pi}{3}(2Ry^2 - y^3)$$

Therefore,

$$\frac{dV}{dy} = \frac{\pi}{3}(4Ry - 3y^2) = \frac{\pi}{3}\,y\,(4R - 3y)$$

$$\frac{dV}{dy} = 0 \Rightarrow y = \frac{4R}{3} \qquad (y = 0 \text{ is meaningless in this context})$$

Now,

$$\frac{d^2V}{dy^2} = \frac{\pi}{3}(4R - 6y)$$

When $y = \frac{4R}{3}$, $\frac{d^2V}{dy^2}$ is $\frac{\pi}{3}(4R - 8R)$ which is negative. Therefore, V is maximum when $y = \frac{4R}{3}$. The cone has maximum volume when height is $\frac{4R}{3}$ and radius is $\frac{2\sqrt{2}\,R}{3}$.

Illustration 21.26 A rectangular sheet of metal has four equal square portions removed from the four corners and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is maximum, the depth will be $(1/6)[(a + b) - \sqrt{a^2 - ab + b^2}\,]$ where a and b, $(a > b)$ are the sides of the original rectangle.

Solution: Let $ABCD$ be the given rectangular sheet of metal with $AB = a$, $BC = b$ and x be the side of the four squares cut off (Fig. 21.14).

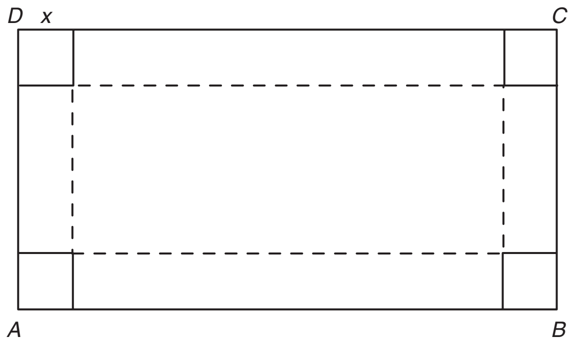


Figure 21.14

The volume (V) of the box is

$$(a - 2x)(b - 2x)x$$

That is,

$$V = abx - 2(a + b)\,x^2 + 4x^3$$

Therefore,

$$\frac{dV}{dx} = ab - 4x\,(a + b) + 12x^2$$

$$\Rightarrow \frac{d^2V}{dx^2} = 24x - 4\,(a + b)$$

Now, $dV/dx = 0$, when $12x^2 - 4(a + b)x + ab = 0$ or when

$$x = \frac{4\,(a + b) \pm \sqrt{16\,(a + b)^2 - 48ab}}{24}$$

$$= \frac{(a + b) \pm \sqrt{a^2 - ab + b^2}}{6}$$

The plus sign gives a value of x greater than $b/2$ and hence not admissible.

When

$$x = \frac{(a + b) - \sqrt{a^2 - ab + b^2}}{6}$$

d^2V/dx^2 is negative. Therefore, V is maximum when

$$x = \frac{(a + b) - \sqrt{a^2 - ab + b^2}}{6}$$

Your Turn 2

1. Separate the intervals in which the function $f(x) = |\sin x| + |\cos x|$ is increasing and decreasing.

   Ans. Increasing in $\left(0, \frac{\pi}{4}\right)$; decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

2. Prove that $x > \sin x$ for all $x \in (0, \infty)$.
3. State true or false: The function $f(x) = \sin x + \sqrt{3}\cos x$ has a maximum value at $x = \pi/6$.

   Ans. True

4. The global maximum value of $f(x) = x^2 - 4x + 20$ in the interval $[0, 5]$ is _______.

   Ans. 25

5. State true or false: The function $f(x) = x^{2/3}$ does not have any local extrema.

Ans. False

6. Find the minimum value of x^x.

Ans. $(1/e)^{1/e}$

7. Verify Rolle's theorem for $f(x) = (x - 1)^2 (x - 2)$ in the interval $[1, 2]$.

8. Find c of mean value theorem for $f(x) = \sqrt{x^2 - 4}$, $a = 2$ and $b = 3$.

Ans. $\sqrt{5}$

9. Prove that among all rectangles with the given perimeter, the square has maximum area.

10. Find the range of the function $f(x) = x^3 - 3x^2 + 6x - 2$ where $x \in [-1, 1]$.

Ans. $[-12, 2]$

Additional Solved Examples

1. Prove that $\sin x + 2x \geq \dfrac{3x(x+1)}{\pi}$ $\forall$ $x \in \left[0, \dfrac{\pi}{2}\right]$. (Justify the inequality, if any used).

Solution:

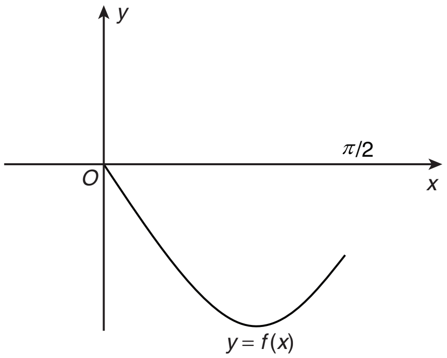


Figure 21.15

See Fig. 21.15. Let us consider

$$f(x) = 3x^2 + (3 - 2\pi)x - \pi \sin x$$

When $f(0) = 0$, $f(\pi/2)$ is negative. Therefore,

$$f'(x) = 6x + 3 - 2\pi - \pi \cos x$$

$$f''(x) = 6 + \pi \sin x > 0$$

It is clear that $f'(x)$ is increasing function in $\left[0, \dfrac{\pi}{2}\right]$ and there is no local maxima of $f(x)$ in $\left[0, \dfrac{\pi}{2}\right]$. The graph of $f(x)$ always lies below the x–axis in $\left[0, \dfrac{\pi}{2}\right]$. Therefore, $f(x) \leq 0$ in $x \in \left[0, \dfrac{\pi}{2}\right]$. Now,

$$3x^2 + 3x \leq 2\pi x + \pi \sin x$$

$$\Rightarrow \sin x + 2x \geq \frac{3x(x+1)}{\pi}$$

2. Find the coordinates of the points on the curve $y = (x^2 - 1)/(x^2 + 1)$, $x > 0$ such that tangent at these point(s) have the greatest slope.

Solution:

$$y = 1 - \frac{2}{x^2 + 1}$$

The slope(s) is(are)

$$s = \frac{dy}{dx} = \frac{4x}{(x^2+1)^2}$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{ds}{dx} = \frac{4\left[(x^2+1)^2 1 - (x)2(x^2+1)2x\right]}{(x^2+1)^4}$$

$$= 4\left(\frac{x^2 + 1 - 4x^2}{(x^2+1)^3}\right)$$

$$= \frac{4(1 - 3x^2)}{(1 + x^2)^3} = \frac{-12\left(x + \frac{1}{\sqrt{3}}\right)\left(x - \frac{1}{\sqrt{3}}\right)}{(1 + x^2)^3}$$

Therefore, for $\dfrac{ds}{dx} = 0, x = \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}$ and $\left(\dfrac{d^2s}{dx^2}\right)\Bigg|_{x = 1/\sqrt{3}} < 0$

Then the maximum is at $x = \dfrac{1}{\sqrt{3}}$ and $y = -\dfrac{1}{2}$. Hence, the point is $\left(\dfrac{1}{\sqrt{3}}, -\dfrac{1}{2}\right)$.

3. Prove that there exist exactly two non–similar isosceles triangles ABC such that $\tan A + \tan B + \tan C = 100$.

Solution: Let $A = B$. Then

$$2A + C = 180°$$

and $$2\tan A + \tan C = 100$$

Now,

$$2A + C = 180°$$

$$\Rightarrow \tan 2A = -\tan C \quad (1)$$

Also,

$$2\tan A + \tan C = 100$$

$$\Rightarrow 2\tan A - 100 = -\tan C \quad (2)$$

From Eqs. (1) and (2),

$$2\tan A - 100 = \frac{2\tan A}{1 - \tan^2 A}$$

Let $\tan A = x$. Then

$$\frac{2x}{1 - x^2} = 2x - 100$$

$$\Rightarrow x^3 - 50x^2 + 50 = 0$$

Let $f(x) = x^3 - 50x^2 + 50$. Then $f'(x) = 3x^2 - 100x$. Thus, $f'(x) = 0$ has roots $0, \dfrac{100}{3}$. Also $f(0), f\left(\dfrac{100}{3}\right) < 0$. Thus, $f(x) = 0$ has exactly three distinct real roots. Therefore, $\tan A$ has three distinct values; however, one of them will be obtuse angle. Hence, there exists exactly two non–similar isosceles triangles.

4. Let $A(p^2, -p)$, $B(q^2, q)$, $C(r^2, -r)$ be the vertices of a triangle ABC. A parallelogram $AFDE$ is drawn with D, E and F on the line segments BC, CA and AB, respectively. Show that the maximum area of the parallelogram is $(1/4)[(p+q)(q+r)(p-r)]$, given $p > r$.

Solution: See Fig. 21.16. Let $AF = \lambda AB$; $AE = \mu AC$. The area of the parallelogram $= AF\cdot AE \sin A$.

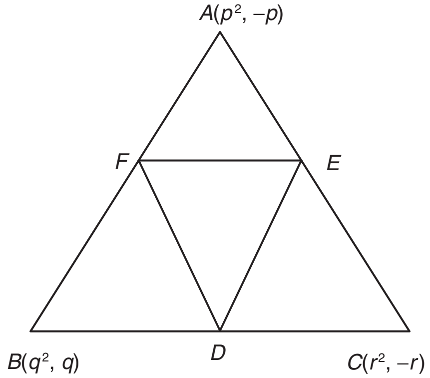


Figure 21.16

In similar triangles, namely, ABC and FBD, we have

$$\frac{FB}{AB}=\frac{BD}{BC}=\frac{FD}{AC} \Rightarrow 1-\lambda=\mu$$

The area is

$$\lambda\mu(AB)\cdot(AC)\sin A$$

That is,

$$\lambda(1-\lambda)AB\cdot(AC)\sin A$$

The area is maximum when λ is 1/2 (the vertex of parabola $y=\lambda-\lambda^2$), which implies that

$$\mu=\lambda=\frac{1}{2}$$

That is, F and E are the mid–points of AB and AC, respectively.

$$\text{Area}_{\text{max.}}=\frac{1}{4}AB\cdot AC\sin A$$

$$=\frac{1}{2}(\text{Area of }\Delta ABC)=\frac{1}{2}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & q & 1 \\ r^2 & -r & 1 \end{vmatrix}$$

$$=\frac{1}{2}(p+q)(q+r)(p-r)$$

5. Find the shortest distance between the curves $9x^2+9y^2-30y+16=0$ and $y^2=x^3$.

Solution: The equation $9x^2+9y^2-30y+16=0$ can be rewritten as

$$x^2+\left(y-\frac{5}{3}\right)^2=1$$

Any point on the curve $y^2=x^3$ can be taken to be (t^2, t^3). Let l be the distance between the centre of the given circle and the point (t^2, t^3). Then

$$L=l^2=t^4+\left(t^3-\frac{5}{3}\right)^2$$

Now, we calculate the minimum value of l. The required distance is

$$l-\text{Radius of given circle}$$

Now,

$$\frac{dL}{dt}=4t^3+2\left(t^3-\frac{5}{3}\right)\cdot 3t^2=0$$

For maximum or minimum, $t=0$ or 1. Now,

$$\frac{d^2L}{dt^2}=12t^2+30t^4-20t$$

$$\left.\frac{d^2L}{dt^2}\right|_{t=0}=0$$

However, $\left.\frac{d^3L}{dt^3}\right|_{t=0}\neq 0$ implies that there is neither maxima nor minima at $t=0$. Also, $\frac{d^2L}{dt^2}>0$ at $t=1$, which implies that L is minimum at $t=1$. So, the shortest distance is

$$(\text{Value of } l \text{ at } t=1)-(\text{Radius of circle})=\frac{\sqrt{13}}{3}-1$$

6. Find a polynomial $f(x)$ of degree 5 which increases in the interval $(-\infty, 2]$ and $[6, \infty)$ and decreases in the interval $[2, 6]$. Given that $f(0)=3$ and $f'(4)=0$.

Solution: See Fig. 21.17. The wavy curve of the derivative will be

$$f'(x)=k(x-2)(x-4)^2(x-6)$$

and

$$k>0.$$

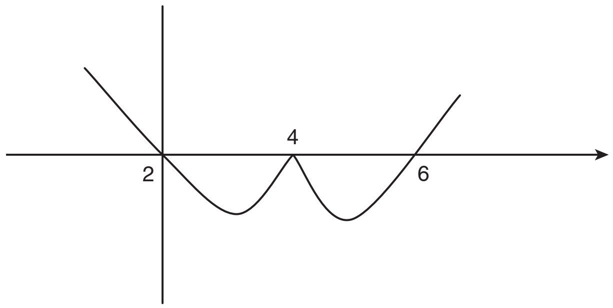


Figure 21.17

$$f(x)=k\int(x^2-8x+12)(x^2-8x+16)\,dx$$

$$=k\int[x^4-16x^3+64x^2+28(x^2-8x)+192]dx$$

$$=k\left(\frac{x^5}{5}-4x^4+92\frac{x^3}{3}-112x^2+192x\right)+c$$

Now, since $f(0)=3$, we have

$$f(x)=k\left[\frac{x^5}{5}-4x^4+\frac{92}{3}x^3-112x^2+192x\right]+3, \quad k>0$$

7. A conical vessel is to be prepared out of a circular sheet of gold of unit radius. How much sectorial area is to be removed from the sheet so that the vessel has maximum volume?

Solution: See Fig. 21.18.

Lateral height of the cone = Radius of the circle = 1

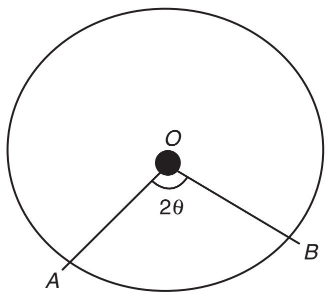


Figure 21.18

The lateral area of the cone is the area of the circle with sectorial area removed, that is,

$$\pi r(1) = \frac{\pi(1)^2}{2\pi}(2\pi - 2\theta)$$

That is,

$$r = \frac{\pi - \theta}{\pi} \qquad \text{(here } r \text{ is radius of the cone)}$$

Height h of the cone is

$$\sqrt{1^2 - r^2}$$

Volume of the cone is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{\pi-\theta}{\pi}\right)^2 \times \sqrt{1^2 - \left(\frac{\pi-\theta}{\pi}\right)^2}$$

Upon maximizing the volume V, we get

$$\frac{\pi-\theta}{\pi} = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \theta = \pi\left(1 - \sqrt{\frac{2}{3}}\right)$$

The area of the sectorial area removed is

$$\frac{1}{2}(1)^2(2\theta) = \pi\left(1 - \sqrt{\frac{2}{3}}\right)$$

8. Let $a + b = 4$, where $a < 2$, and let $g(x)$ be a differentiable function. If $(dg/dx) > 0$ for all x, prove that $\int_0^a g(x)dx + \int_0^b g(x)dx$ increases as $(b - a)$ increases.

Solution: Let $b - a = t$. Given that

$$a + b = 4$$

Thus,

$$a = 2 - \frac{t}{2};\ b = 2 + \frac{t}{2}$$

Let us consider

$$f(t) = \int_0^a g(x)dx + \int_0^b g(x)dx$$

Now,

$$f(t) = \int_0^{2-(t/2)} g(x)dx + \int_0^{2+(t/2)} g(x)dx$$

$$f'(t) = g\left(2 - \frac{t}{2}\right)\left(-\frac{1}{2}\right) + g\left(2 + \frac{t}{2}\right)\left(\frac{1}{2}\right)$$

$$f'(t) = \frac{1}{2}[g(b) - g(a)]$$

Since $\frac{dg}{dx} > 0$ for all values of x, $g(x)$ is increasing since $b > a$

$$g(b) > g(a)$$

Hence, $f'(t) > 0$, that is, $f(t)$ increasing as t increases. Therefore, $f(t)$ increases as $(b - a)$ increases.

9. Let $f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}, & 0 \le x < 1 \\ 2x - 3, & 1 \le x \le 3 \end{cases}$. Find all possible real values of b such that $f(x)$ has the smallest value at $x = 1$.

Solution: At $x = 1$, $f(x) = -1$.

Smallest value of $f(x) = -1$. At all other points of the interval, $f(x) > -1$.

Now, for $x \ge 1$, $f(x) = 2x - 3$. So,

$$f'(x) = 2 > 0$$

$\Rightarrow f(x)$ is an increasing function

$\Rightarrow$ Least value exists at $x = 1$

Now, for $x < 1$, $f'(x) = -3x^2 < 0$. Therefore,

$f(x)$ is decreasing function in the interval $0 \le x < 1$.

Therefore, $f(x)$ is the smallest at $x = 1$ and

$$f(1-0) = \lim_{h\to 0} -(1-h)^3 + \left(\frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}\right) \ge -1$$

$$\Rightarrow -1 + \frac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge -1$$

$$\Rightarrow \frac{(b^2+1)(b-1)}{(b+1)(b+2)} \ge 0$$

$$\Rightarrow \frac{(b-1)}{(b+1)(b+2)} \ge 0 \qquad \text{(since } b^2 + 1 \text{ is positive)}$$

$$\Rightarrow b \in (-2, -1) \cup [1, \infty)$$

10. Find all the possible values of the parameter a so that the function, $f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$, has a negative point of local minimum.

Solution: We have

$$f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$$

$$f'(x) = 3x^2 - 6(7 - a)x - 3(9 - a^2)$$

For the distinct real roots, $D > 0$

$$36(7 - a)^2 + (4 \times 3 \times 3)(9 - a^2) > 0$$

$$\Rightarrow 49 + a^2 - 14a + 9 - a^2 > 0$$

$$\Rightarrow 14a < 58 \Rightarrow a < \frac{29}{7}$$

For local minima:

$$f''(x) = 6x - 6(7 - a) > 0$$

$$\Rightarrow x - 7 + a > 0$$

Now, $7 - a < x$ as x must be negative. So,

$$7 - a < 0 \Rightarrow a > 7$$

Thus, by contradiction, that is, for real roots, $a < (29/7)$ and for negative point of local minimum $a > 7$. No possible value of a.

Previous Years' Solved JEE Main/AIEEE Questions

1. A value of C for which the conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is

(A) $2\log_3 e$ **(B)** $\frac{1}{2}\log_e 3$

(C) $\log_3 e$ **(D)** $\log_e 3$

Solution: Using mean value theorem, we get

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} \Rightarrow \frac{1}{c} = \frac{\log 3 - \log 1}{2} \Rightarrow c = \frac{2}{\log_e 3} = 2\log_3 e.$$

Hence, the correct answer is option (A).

2. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in

(A) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ **(B)** $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

(C) $\left(0, \frac{\pi}{2}\right)$ **(D)** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

[AIEEE 2007]

Solution:

$$f'(x) = \frac{1}{1+(\sin x + \cos x)^2}(\cos x - \sin x) = \frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

$f(x)$ is increasing if $f'(x) > 0$ for $\cos\left(\frac{\pi}{4}+x\right) > 0 \Rightarrow -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$

$$\Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$$

Hence, $f(x)$ is increasing when $x \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

Hence, the correct answer is option (B).

3. Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds true?

(A) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

(B) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(C) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(D) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

[AIEEE 2008]

Solution: Let $f(x) = x^3 - px + q$, from Fig. 21.19, for maxima and minima, $f'(x) = 0$. This implies that $3x^2 - p = 0 \Rightarrow x^2 = \frac{p}{3}$

Therefore, $x = \pm\sqrt{\frac{p}{3}}$. Now,

$$f''(x) = 6x \Rightarrow f''\left(-\sqrt{\frac{p}{3}}\right) < 0 \Rightarrow f''\left(\sqrt{\frac{p}{3}}\right) > 0$$

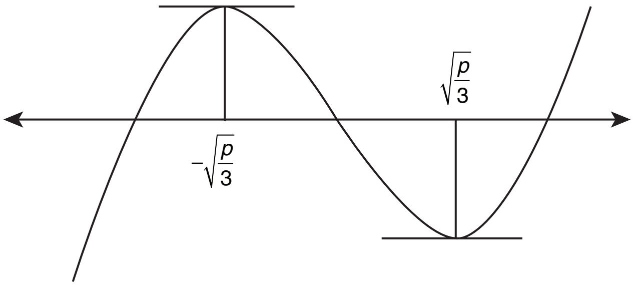


Figure 21.19

Therefore, there is a maxima at $-\sqrt{\frac{p}{3}}$ and minima at $\sqrt{\frac{p}{3}}$.

Hence, the correct answer is option (A).

4. How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have?

(A) 7 **(B)** 1

(C) 3 **(D)** 5

[AIEEE 2008]

Solution:

$$x^7 + 14x^5 + 16x^3 + 30z - 560 = 0$$

Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x$. Then

$$f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0 \,\forall x$$

Therefore, $f(x)$ is a strictly increasing function for all x.

So, it can have at the most one solution.

Hence, the correct answer is option (B).

5. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$

(A) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(B) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

(C) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P

(D) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

[AIEEE 2009]

Solution: $P(x) = x^4 + ax^3 + bx^2 + cx + d; P'(x) = 4x^3 + 3ax^2 + 2bx + c$

As $x = 0$ is a solution for $P'(x) = 0$, we have $c = 0$. Therefore, $P(x) = x^4 + ax^3 + bx^2 + d$

Further, we have $P(-1) < P(1)$, which implies that,

$$1 - a + b + d < 1 + a + b + d \Rightarrow a > 0$$

As $P'(x) = 0$, only when $x = 0$ and $P(x)$ is differentiable in $(-1, 1)$, we shall have the maximum and minimum at the points $x = -1$, 0 and 1 only. Also, we have, $P(-1) < P(1)$. Therefore,

Maximum value of $P(x)$ = Maximum value of $\{P(0), P(1)\}$;

Minimum value of $P(x)$ = Minimum value of $\{P(-1), P(0)\}$.

In the interval $[0, 1]$, we have,

$$P'(x) = 4x^3 + 3ax^2 + 2bx = x(4x^2 + 3ax + 2b)$$

As $P'(x)$ has only one root, that is, $x = 0$, $4x^2 + 3ax + 2b = 0$ has no real roots. Therefore, $(3a)^2 - 32b < 0 \Rightarrow \frac{3a^2}{32} < b$. Thus, $b > 0$.

Therefore, we have $a > 0$ and $b > 0$. That is,

$$P'(x) = 4x^3 + 3ax^2 + 2bx > 0 \qquad \forall x \in (0, 1)$$

Hence, $P(x)$ is increasing in $[0, 1]$. Therefore, maximum value of $P(x) = P(1)$

Similarly, $P(x)$ is decreasing in $[-1, 0]$. Therefore, minimum value of $P(x)$ does not occur at $x = -1$.

Hence, the correct answer is option (B).

6. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is

(A) $\frac{3\sqrt{2}}{8}$ **(B)** $\frac{2\sqrt{3}}{8}$

(C) $\frac{3\sqrt{2}}{5}$ **(D)** $\frac{\sqrt{3}}{4}$

[AIEEE 2009]

Solution: Let us consider that line (1) be $x - y + 1 = 0$ and line (2) be the tangent to the curve $x = y^2$. Therefore,

$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y} = \text{Slope of given line (2)}$$

In order to find the shortest distance, these two lines should be parallel, therefore equating their slopes,

$$\frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \Rightarrow (x, y) = \left(\frac{1}{4}, \frac{1}{2}\right)$$

This is the point on the curve from which if a perpendicular is drawn on to the given line, then the length of that perpendicular will be the shortest distance between the line and the curve.

Therefore, the shortest distance is $\dfrac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{1+1}} = \dfrac{3}{4\sqrt{2}} = \dfrac{3\sqrt{2}}{8}$

Hence, the correct answer is option (A).

7. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is, parallel to the x–axis is

(A) $y = 1$ **(B)** $y = 2$

(C) $y = 3$ **(D)** $y = 0$

[AIEEE 2010]

Solution: Since the equation is parallel to x–axis, we have,

$$\frac{dy}{dx} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0$$

Therefore,

$$x = 2 \Rightarrow y = 3$$

Thus, the equation of tangent is

$$y - 3 = 0(x - 2) \Rightarrow y - 3 = 0 \Rightarrow y = 3$$

Hence, the correct answer is option (C).

8. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\sin t\,dt$. Then f has

(A) local minimum at π and 2π

(B) local minimum at π and local maximum at 2π

(C) local maximum at π and local minimum at 2π

(D) local maximum at π and 2π

[AIEEE 2011]

Solution: We have,

$$f'(x) = \sqrt{x}\sin x \Rightarrow f'(x) = 0$$

This implies that $x = 0$ or $\sin x = 0$ and $x = 2\pi, \pi$. Therefore,

$$f''(x) = \sqrt{x}\cos x + \frac{1}{2\sqrt{x}}\sin x \Rightarrow \frac{1}{2\sqrt{x}}(2x\cos x + \sin x)$$

That is, at $x = \pi$,

$$f''(x) < 0 \Rightarrow \text{maxima;}$$

at $x = 2\pi$,

$$f''(x) > 0 \Rightarrow \text{minima.}$$

Hence, the correct answer is option (C).

9. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is

(A) $\frac{9}{7}$ **(B)** $\frac{7}{9}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{9}{2}$

[AIEEE 2012]

Solution: Volume of the sphere is given by: $v = \frac{4}{3}\pi r^3$

After 49 minutes of leakage, the volume is: $4500\pi - 49(72\pi) = 972\pi$. Therefore,

$$\frac{4}{3}\pi r^3 = 972\pi \Rightarrow r^3 = 729 \Rightarrow r = 9$$

Therefore, the rate (in meters per minute) at which the radius of the balloon decreases 49 min after the leakage began is

$$v = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} \Rightarrow 72\pi = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{72}{4\cdot 9\cdot 9} = \frac{2}{9}$$

Hence, the correct answer is option (C).

10. The population $p(t)$ at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is

(A) 2 ln 18 **(B)** ln 9 **(C)** $\frac{1}{2}\ln 18$ **(D)** ln 18

[AIEEE 2012]

Solution: We have,

$$\frac{d(p(t))}{dt} = \frac{1}{2}p(t) - 450$$

$$\Rightarrow \frac{d(p(t))}{dt} = \frac{p(t) - 900}{2} \Rightarrow 2\int\frac{d(p(t))}{p(t) - 900} = \int dt \Rightarrow 2\ln|p(t) - 900| = t + c$$

That is, $t = 0 \Rightarrow 2\ln 50 = 0 + c \Rightarrow c = 2\ln 50$

Therefore, $2\ln|p(t) - 900| = t + 2\ln 50$

$$\Rightarrow P(t) = 0 \Rightarrow 2\ln 900 = t + 2\ln 50$$

$$\Rightarrow t = 2(\ln 900 - \ln 50) = 2\ln\left(\frac{900}{50}\right) = 2\ln 18.$$

Hence, the correct answer is option (A).

11. Let $a, b \in R$ be such that the function f given by $f(x) = \ln|x| + bx^2 + ax,\ x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement 1: f has local maximum at $x = -1$ and at $x = 2$.

Statement 2: $a = \frac{1}{2}$ and $b = \frac{-1}{4}$

(A) Statement 1 is false, statement 2 is true.

(B) Statement 1 is true, statement 2 is true; statement 2 is a correct explanation for statement 1.

(C) Statement 1 is true, statement 2 is true; statement 2 is not a correct explanation for statement 1.
(D) Statement 1 is true, statement 2 is false.

[AIEEE 2012]

Solution:

$$f'(x) = \frac{1}{x} + 2bx + a$$

It is given that f has extreme values and hence differentiable. Therefore,

$$f'(-1) = 0 \Rightarrow a - 2b = 1; f'(2) = 0 \Rightarrow a + 4b = -\frac{1}{2} \Rightarrow a = \frac{1}{2}; b = -\frac{1}{4}$$

Therefore, $f''(-1), f''(2)$ are negative and f has local maxima at $x = -1$ and 2.

Hence, the correct answer is option (B).

12. Consider the function $f(x) = |x-2| + |x-5|$, $x \in R$.

Statement 1: $f'(4) = 0$.

Statement 2: f is continuous in [2, 5], differentiable in (2, 5) and $f(2) = f(5)$.

(A) Statement–1 is false, statement–2 is true.
(B) Statement–1 is true; statement–2 is true; statement 2 is a correct explanation for statement–1.
(C) Statement–1 is true; statement–2 is true; statement 2 is not a correct explanation for statement–1.
(D) Statement–1 is true, statement–2 is false.

[AIEEE 2012]

Solution:

$$\begin{aligned} f(x) &= 7 - 2x; \quad x > 2 \\ &= 3; \quad 2 \le x \le 5 \\ &= 2x - 7; \quad x > 5 \end{aligned}$$

$f(x)$ is constant function in [2, 5]. f is also continuous in [2, 5] and differentiable in (2, 5) and $f(2) = f(5)$; by Rolle's theorem $f'(4) = 0$. Therefore, both Statement 2 and Statement 1 are true and Statement 2 is correct explanation for Statement–1.

Hence, the correct answer is option (B).

13. The intercepts on x–axis made by tangents to the curve, $y = \int_0^x |t|\,dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to

(A) ± 2 **(B)** ± 3
(C) ± 4 **(D)** ± 1

[JEE MAIN 2013]

Solution: Slope of the tangent to the curve will be 2. So we can equate the slope as,

$$\frac{dy}{dx} = |x| = 2 \Rightarrow x = \pm 2$$

For $x = 2$, we have,

$$y = \int_0^2 |t|\,dt = 2$$

For $x = -2$, we have,

$$y = \int_0^{-2} |t|\,dt = -2$$

Therefore, one tangent passes through the point (2, 2) and has slope 2

$$y - 2 = 2(x - 2) \Rightarrow y = 2x - 2$$

The other tangent passes through the point (–2, –2) and has slope 2

$$y + 2 = 2(x + 2) \Rightarrow y = 2x + 2$$

Substituting $y = 0$, we get x–intercepts as, $x = 1$ and –1.

Hence, the correct answer is option (D).

14. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]

(A) lies between 2 and 3 **(B)** lies between –1 and 0
(C) does not exist **(D)** lies between 1 and 2

[JEE MAIN 2013]

Solution: When the given equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1], then $f'(x)$ will change sign, but $f'(x) = 6x^2 + 3 > 0$, for all values of $x \in$ R. Therefore, no value of k exists.

Hence, the correct answer is option (C).

15. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to

(A) $\frac{1}{1+\{g(x)\}^5}$ **(D)** $1+\{g(x)\}^5$
(C) $1 + x^5$ **(D)** $5x^4$

[JEE MAIN 2014 (OFFLINE)]

Solution: Given $g(x) = f^{-1}(x)$. Therefore,

$$f(g(x)) = x$$
$$f'(g(x))\, g'(x) = 1$$

Thus,
$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\frac{1}{1+\{g(x)\}^5}}$$

Thus, $g'(x) = 1 + \{g(x)\}^5$.

Hence, the correct answer is option (B).

16. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in [0, 1]$

(A) $f'(c) = g'(c)$ **(B)** $f'(c) = 2g'(c)$
(C) $2f'(c) = g'(c)$ **(D)** $2f'(c) = 3g'(c)$

[JEE MAIN 2014 (OFFLINE)]

Solution: Let $h(x) = f(x) - 2g(x)$. Then

$$h(0) = f(x) - 2\,g(x) = 2 - 2 \times 0 = 2$$

and
$$h(1) = f(1) - 2\,g(1) = 6 - 2 \times 2 = 2$$

Now $h(x)$ is a differentiable function in [0, 1] and $h(0) = h(1)$, so by Rolle's theorem $h'(c) = 0$ for some $c \in (0, 1)$. Therefore,

$$0 = f'(c) - 2\,g'(c) \Rightarrow f'(c) = 2\,g'(c)$$

Hence, the correct answer is option (B).

17. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x_2 + x$, then

(A) $\alpha = 2, \beta = -\frac{1}{2}$ **(B)** $\alpha = 2, \beta = \frac{1}{2}$
(C) $\alpha = -6, \beta = \frac{1}{2}$ **(D)** $\alpha = -6, \beta = -\frac{1}{2}$

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$f(x) = \alpha \log(x) + \beta x^2 + x$$

$$f'(x) = \frac{\alpha}{x} + 2\beta x + 1$$

Now $$f'(-1) = -\alpha - 2\beta + 1 = 0$$

$$f'(2) = \frac{\alpha}{2} + 4\beta x + 1 = 0$$

Thus, the equations are

$$\alpha + 2\beta - 1 = 0 \text{ and } \alpha + 8\beta + 2 = 0$$

On solving the above equations, we get

$$\beta = -\frac{1}{2}, \alpha = 2$$

Hence, the correct answer is option (A).

18. If the Rolle's theorem holds for the function $f(x) = 2x^3 + ax^2 + bx$ in the interval [–1, 1] for the point $c = \frac{1}{2}$, then the value of $2a + b$ is

(A) 1 **(B)** –1 **(C)** 2 **(D)** – 2

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: Since Rolle's theorem holds, so $f'\left(\frac{1}{2}\right) = 0$. Now

$$f'(x) = 6x^2 + 2ax + b$$

$$f'\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^2 + 2a \times \frac{1}{2} + b$$

or $$\frac{6}{4} + a + b = 0 \Rightarrow a + b = \frac{-3}{2} \quad (1)$$

Since, $$f(-1) = f(1)$$

$$\Rightarrow 2(-1)^3 + a(-1)^2 + b(-1) = 2 + a + b$$

$$\Rightarrow -2 + a - b = 2 + a + b$$

Therefore,

$$2b = -4 \Rightarrow b = -2$$

Therefore, from Eq. (1), we get $a = \frac{1}{2}$. Thus,

$$2a + b = 2\left(\frac{1}{2}\right) - 2 = -1$$

Hence, the correct answer is option (B).

19. For the curve $y = 3\sin\theta\cos\theta$, $x = e^{\theta}\sin\theta$, $0 \le \theta \le \pi$, the tangent is parallel to x–axis when θ is

(A) $\frac{3\pi}{4}$ **(B)** $\frac{\pi}{2}$

(C) $\frac{\pi}{4}$ **(D)** $\frac{\pi}{6}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{3}{2}\frac{d}{d\theta}\sin 2\theta}{e^{\theta}\cos\theta + (\sin\theta)e^{\theta}}$$

$$= \frac{\frac{3}{\cancel{2}}(\cos 2\theta)\cancel{2}}{e^{\theta}(\cos\theta + \sin\theta)} = \frac{3\{\cos^2\theta - \sin^2\theta\}}{e^{\theta}(\cos\theta + \sin\theta)}$$

$$= \frac{3}{e^{\theta}} \times (\cos\theta - \sin\theta)$$

Since tangent is parallel to x–axis, we have

$$\frac{dy}{dx} = 0 \Rightarrow \sin\theta = \cos\theta \quad (\text{since, } e^{\theta} > 0) \Rightarrow \theta = \frac{\pi}{4}$$

Hence, the correct answer is option (C).

20. The volume of the largest possible right circular cylinder that can be inscribed in a sphere of radius $= \sqrt{3}$ is

(A) $\frac{4}{3}\sqrt{3}\pi$ **(B)** $\frac{8}{3}\sqrt{3}\pi$

(C) 4π **(D)** 2π

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 21.20.

$$\text{Volume of required cylinder} = V = \pi r^2 h \quad (1)$$

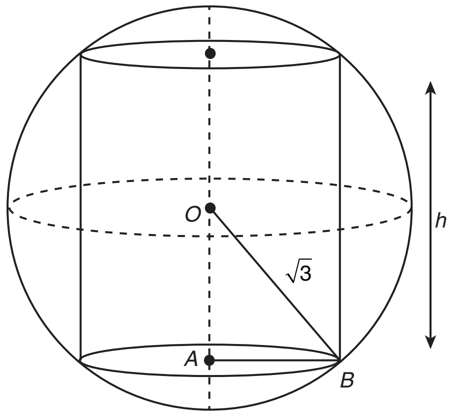


Figure 21.20

Now, by Pythagoras theorem, we have

$$\left(\sqrt{3}\right)^2 = r^2 + \left(\frac{h}{2}\right)^2$$

$$\Rightarrow r^2 = 3 - \frac{h^2}{4} \quad (2)$$

Therefore,

$$V = \pi\left(3 - \frac{h^2}{4}\right)h = \pi\left(3h - \frac{h^3}{4}\right) \quad (3)$$

$$V' = \pi\left(3 - \frac{1}{4} \times 3h^2\right) = \pi\left(3 - \frac{3h^2}{4}\right)$$

For Extreme V,

$$V' = 0 \Rightarrow 3 = \frac{3}{4}h^2 \Rightarrow h^2 = 4 \Rightarrow h = \pm 2$$

Now

$$V'' = -\frac{3\pi}{4} \times 2h = -\frac{3\pi}{4} \times 2 \times 2 = -3\pi < 0$$

Therefore, volume is maximum when $h = 2$. From Eq. (1), required volume is

$$V = \pi\left(3 \times 2 - \frac{1}{4} \times 2^3\right) = \pi(6 - 2) = 4\pi$$

Hence, the correct answer is option (C).

21. If $f(x)=x^2-x+5, x>\dfrac{1}{2}$ and $g(x)$ is its inverse function, then $g'(7)$ equals

(A) $-\dfrac{1}{3}$ **(B)** $\dfrac{1}{13}$ **(C)** $\dfrac{1}{3}$ **(D)** $-\dfrac{1}{13}$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$f(x)=x^2-x+5, x>\frac{1}{2}$$

Now

$$\left(f^{-1}(7)\right)'=\frac{1}{f'\left(f^{-1}(7)\right)}=\frac{1}{f'(2)}=\frac{1}{2(2)-1}=\frac{1}{3}$$

Since $f'(x) = 2x - 1$ and because if f is differentiable and non–zero at $x = a$, then f^{-1} is differentiable at $x = f(a) = b$ and we have

$$(f^{-1})(b)=\frac{1}{f'\left(f^{-1}(b)\right)}$$

Hence, the correct answer is option (C).

22. Let f and g be two differentiable functions on R such that $f'(x) > 0$ and $g'(x) < 0$, for all $x \in R$. Then for all x

(A) $f(g(x)) > f(g(x-1))$

(B) $f(g(x)) > f(g(x+1))$

(C) $g(f(x)) > g(f(x-1))$

(D) $g(f(x)) < g(f(x+1))$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$f'(x) > 0 \text{ and } g'(x) < 0 \text{ for all } x \in R$$

Now

$$x > x-1 \text{ and } x+1 > x$$

Therefore,

$$g(x) < g(x-1) \text{ and } g(x+1) < g(x)$$

Thus,

$$f(g(x)) < f(g(x-1) \text{ and } f(g(x+1) < f(g(x)) \text{ true}$$

Also

$$f(x) > f(x-1) \text{ and } f(x+1) > f(x)$$

Therefore,

$$g(f(x)) < g(f(x-1)) \text{ and } g(f(x+1) < g(f(x))$$

Hence, the correct answer is option (B).

23. If $f(\theta)=\begin{vmatrix} 1 & \cos\theta & 1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}\begin{matrix} 1 & 2 & x \\ 3 & -1 & 2 \end{matrix}$ and A and B are respectively the maximum and the minimum values of $f(\theta)$, then (A, B) is equal to

(A) $(3, -1)$ **(B)** $(4, 2-\sqrt{2})$

(C) $(2+\sqrt{2}, 2-\sqrt{2})$ **(D)** $(2+\sqrt{2}, -1)$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$f(\theta)=\begin{vmatrix} 1 & \cos\theta & 1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$$

Therefore,

$$f(\theta) = 1\,(1 + \sin\theta\cos\theta) - \cos\theta(-\sin\theta - \cos\theta) + 1\,(-\sin^2\theta + 1)$$
$$= 1 + \sin 2\theta + 2\cos^2\theta = 1 + \sin 2\theta + 1 + \cos 2\theta$$
$$=2+\sqrt{2}\left\{\sin\left(2\theta+\frac{\pi}{4}\right)\right\}$$

Therefore, max. $f(\theta)=2+\sqrt{2}$ and min. $f(\theta)=2-\sqrt{2}$ since,

$$-1\le\sin\left(2\theta+\frac{\pi}{4}\right)\le 1$$

Hence, the correct answer is option (C).

24. If the volume of a spherical ball is increasing at the rate of 4π cc/sec, then the rate of increase of its radius (in cm/sec), when the volume is 288π cc, is

(A) $\dfrac{1}{6}$ **(B)** $\dfrac{1}{9}$

(C) $\dfrac{1}{36}$ **(D)** $\dfrac{1}{24}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$\frac{dV}{dt}=4\pi \Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)=4\pi \Rightarrow \frac{\cancel{4\pi}}{\cancel{3}}\times\cancel{3}r^2\frac{dr}{dt}=\cancel{4\pi} \Rightarrow \frac{dr}{dt}=\frac{1}{r^2}$$

$$\left.\frac{dr}{dt}\right]_{\text{When vol.}=288}=\left.\frac{dr}{dt}\right]_{\text{When } r=6}$$

$$\left(\because \frac{4}{\cancel{3}}\pi r^3=\cancel{288}^{72} \Rightarrow r^3=216 \Rightarrow r=6\right)$$

$$\Rightarrow \frac{1}{6^2}=\frac{1}{36}$$

Hence, the correct answer is option (C).

25. The equation of a normal to the curve, $\sin y = x\sin\left(\dfrac{\pi}{3}+y\right)$ at $x = 0$, is

(A) $2x+\sqrt{3}y=0$ **(B)** $2y-\sqrt{3}x=0$

(C) $2y+\sqrt{3}x=0$ **(D)** $2x-\sqrt{3}y=0$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: Given value curve is

$$\sin y = x\sin\left(\frac{\pi}{3}+y\right) \Rightarrow \cos y\frac{dy}{dx}=x\cos\left(\frac{\pi}{3}+y\right)\frac{dy}{dx}+\sin\left(\frac{\pi}{3}+y\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0}=\frac{\sqrt{3}}{2} \Rightarrow \left(\frac{-dx}{dy}\right)_{x=0}=\frac{-2}{\sqrt{3}}$$

Therefore, Equation of normal at (0, 0) is

$$(y-0)=\frac{-2}{\sqrt{3}}(x-0) \Rightarrow y=\frac{-2}{\sqrt{3}}x$$

or

$$2x+\sqrt{3}y=0$$

Hence, the correct answer is option (A).

26. Let k and K be the minimum and the maximum values of the function $f(x)=\dfrac{(1+x)^{0.6}}{1+x^{0.6}}$ in [0, 1], respectively, then the ordered pair (k, K) is equal to

(A) $(1, 2^{0.6})$ **(B)** $(2^{-0.4}, 2^{0.6})$

(C) $(2^{-0.6}, 1)$ **(D)** $(2^{-0.4}, 1)$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: $f(x)=\dfrac{(1+x)^{0.6}}{1+x^{0.6}}, x\in[0,1]$; $(k, k) = ?$; where $k = f(x)$ min and $k = f(x)$ max.

$$f(x)=\frac{(1+x)^{3/5}}{(1+x^{3/5})}$$

$$\Rightarrow f'(x)=\frac{(1+x^{3/5})\cdot\frac{3}{5}(1-x)^{-2/5}-(1+x)^{3/5}\cdot\frac{3}{5}x^{-2/5}}{(1+x^{3/5})^2}$$

$$\Rightarrow f'(x)=\frac{3}{5}\left[\frac{(1+x^{3/5})x^{2/5}-(1+x)}{x^{2/5}(1+x)^{2/5}(1+x^{3/5})^2}\right]$$

$$\Rightarrow f'(x)=\frac{3}{5}\left[\frac{x^{2/5}+x-1-x}{x^{2/5}(1+x)^{2/5}(1+x^{3/5})^2}\right]$$

$\Rightarrow f'(x)=0$ for $x = 1, -1 \therefore f(-1) = 0$, $f(1)=\dfrac{(2)^{3/5}}{2}$

$\Rightarrow f'(x)=0$ for $x = 1, -1 \therefore f(-1) = 0$, $f(1)=\dfrac{(2)^{3/5}}{2}$

Also $f(0) = 1$, so in [0, 1], $f(x)$ has

$$\text{Minimum value} = f(1) = (2)^{-2/5} = (2)^{0.4} = k$$
$$\text{and maximum value} = f(0) = 1 = k$$

Therefore, $(k,k)\equiv(2^{-0.4},1)$.

Hence, the correct answer is option (D).

27. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side x units and a circle of radius r units. If the sum of the areas of the square and the circle so formed is minimum, then

(A) $2x = r$ **(B)** $2x = (\pi + 4)r$

(C) $(4 - \pi)x = \pi r$ **(D)** $x = 2r$

[JEE MAIN 2016 (OFFLINE)]

Solution: Side length of the square = x

Radius of circle = r

Perimeter of square + Perimeter of circle = 2

$$4x + 2\pi r = 2$$
$$\Rightarrow 2x + \pi r = 1$$

The sum of the area is

$$x^2 + \pi r^2 = A$$
$$\pi r = 1 - 2x$$
$$\Rightarrow r=\left(\frac{1-2x}{\pi}\right)$$

Now,
$$A=x^2+\frac{\pi(1-2x)^2}{\pi^2}$$
$$=x^2+\frac{(1-2x)^2}{\pi}$$

Therefore,
$$\frac{dA}{dx}=2x+\frac{1}{\pi}2(1-2x)(-2)$$
$$=2x-\frac{4}{\pi}(1-2x)=0$$

Hence,
$$\frac{d^2A}{dx^2}=2x+\frac{4}{\pi}(2)>0$$

Now,
$$x-\frac{2}{\pi}(1-2x)=0$$
$$\pi x - 2 + 4x = 0$$
$$\Rightarrow x=\left(\frac{2}{\pi+4}\right)$$

The minimum value occurs at
$$x=\frac{2}{\pi+4}$$

That is,
$$\pi r=1-\frac{4}{\pi+4}=\frac{\pi}{\pi+4}$$
$$\Rightarrow r=\frac{1}{(\pi+4)}$$
$$\Rightarrow x = 2r$$

Hence, the correct answer is option (D).

28. If m and M are the minimum and the maximum value of $4+\frac{1}{2}\sin^2 2x-2\cos^4 x,\ \ x\in R,$ then $M - m$ is equal to

(A) $\dfrac{9}{4}$ **(B)** $\dfrac{15}{4}$

(C) $\dfrac{7}{4}$ **(D)** $\dfrac{1}{4}$

[JEE MAIN 2016 (ONLINE SET–1)]

Solution: We have

$$y=4+\frac{1}{2}4(1-\cos^2 x)\cos^2 x-2\cos^4 x$$
$$= 4 + 2\cos^2 x - 4\cos^4 x$$
$$= 2(2 + \cos^2 x - 2\cos^4 x)$$
$$= 2(-2\cos^4 x + \cos^2 x + 2)$$
$$=-4\left(\cos^4 x-\frac{\cos^2 x}{2}-1\right)$$
$$=-4\left(\cos^4 x-\frac{\cos^2 x}{2}+\frac{1}{16}-\frac{1}{16}-1\right)$$
$$=\left[4\left(\cos^2 x-\frac{1}{4}\right)^2-\frac{17}{16}\right]$$
$$=\frac{17}{4}-4\left(\cos^2 x-\frac{1}{4}\right)^2$$

Therefore,
$$y_{\max}=M=\frac{17}{4}$$
$$y_{\min}=m=\frac{17}{4}-\left(4\times\frac{9}{16}\right)=\frac{17}{4}-\frac{9}{4}=\frac{8}{4}$$

Hence,

$$M-m=\frac{17}{4}-\frac{8}{4}=\frac{9}{4}$$

Hence, the correct answer is option (A).

29. The minimum distance of a point on the curve $y = x^2 - 4$ from the origin is

(A) $\frac{\sqrt{15}}{2}$ **(B)** $\sqrt{\frac{19}{2}}$ **(C)** $\sqrt{\frac{15}{2}}$ **(D)** $\sqrt{\frac{19}{2}}$

[JEE MAIN 2016 (ONLINE SET–1)]

Solution: See Fig. 21.21. We have

$$OP^2 = x^2 + (x^2 - 4)^2$$

$$\frac{d(OP)^2}{dx}=2x+2(x^2-4)(2x)=0$$

Now, $x = 0,\ \ 1 + 2(x^2 - 4) = 0$

$$2x^2-7=0 \Rightarrow x=\pm\sqrt{\frac{7}{2}}$$

Therefore, from the origin, the minimum distance of the point on the curve is

$$\sqrt{\frac{7}{2}+\left(\frac{7}{2}-4\right)^2}=\sqrt{\frac{7}{2}+\frac{1}{4}}=\frac{\sqrt{15}}{2}$$

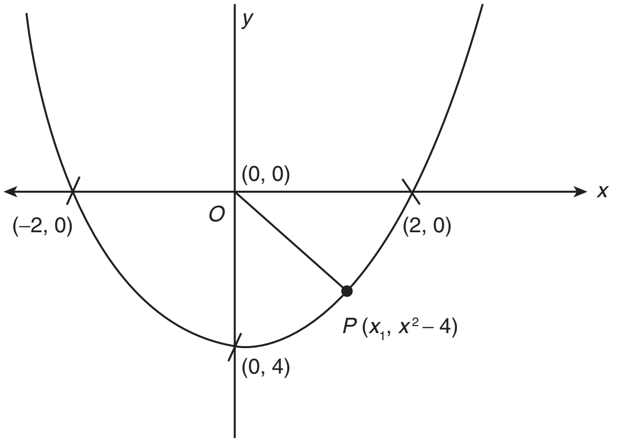


Figure 21.21

Hence, the correct answer is option (A).

30. Let $f(x) = \sin^4 x + \cos^4 x$. Then, f is an increasing function in the interval

(A) $\left[\frac{5\pi}{8},\frac{3\pi}{4}\right]$ **(B)** $\left[\frac{\pi}{2},\frac{5\pi}{8}\right]$

(C) $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ **(D)** $\left[0,\frac{\pi}{4}\right]$

[JEE MAIN 2016 (ONLINE SET–2)]

Solution: We have

$$f(x) = \sin^4 x + \cos^4 x = 1 - 2\sin^2 x \cos^2 x$$

$$f(x)=1-\frac{1}{2}\sin^2 2x$$

$$f'(x)=-\frac{2}{2}\sin 2x\cos 2x(2)=-\sin 4x$$

That is,

$$4x\in(\pi,2\pi)\cup(3\pi,4\pi)\cup(5\pi,6\pi)$$

$$x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{4},\pi\right)\cup\left(\frac{5\pi}{4},\frac{3\pi}{2}\right)$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

Paragraph for Questions 1–3: If a continuous function f defined on the real line R, assumes positive and negative value in R, then the equation $f(x) = 0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in R.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

1. The line $y = x$ meets $y = ke^x$ for $k \le 0$ at

(A) no point **(B)** one point

(C) two points **(D)** more than two points

[IIT–JEE 2007]

Solution: Line $y = x$ intersects the curve $y = ke^x (k \le 0)$ at exactly one point as shown in Fig. 21.22.

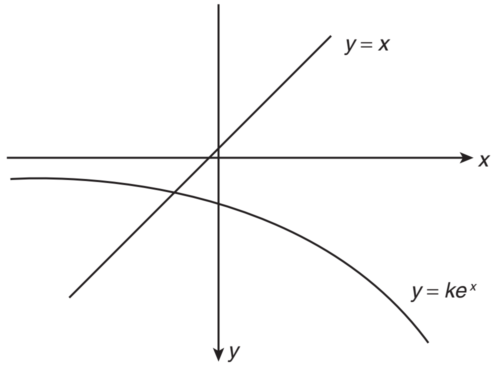


Figure 21.22

Hence, the correct answer is option (B).

2. The positive value of k for which $ke^x - x = 0$ has only one root is

(A) $\frac{1}{e}$ **(B)** 1

(C) e **(D)** $\log_e 2$

[IIT–JEE 2007]

Solution: Let $f(x) = ke^x - x$. Then

$$f'(x) = ke^x - 1$$

Substituting $f'(x) = 0 \Rightarrow x = -\log k$, we get

$$f''(x) = ke^x$$

$$f''(-\log k) = 1 > 0$$

which implies that $f(x)$ has one minima at point

$$x = -\log k$$

Since the equation has only one root, we get

$$f(-\log k) = 0$$

$$\Rightarrow 1 + \log k = 0 \Rightarrow k = \frac{1}{e}$$

Hence, the correct answer is option (A).

3. For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is

(A) $\left(0, \frac{1}{e}\right)$ **(B)** $\left(\frac{1}{e}, 1\right)$

(C) $\left(\frac{1}{e}, \infty\right)$ **(D)** (0, 1)

[IIT–JEE 2007]

Solution: We have $f(x) = ke^x - x$. As discussed in the Solution of Question 2, we can show that $f(x)$ has a minima at $x = -\log k$. Therefore, if $f(x)$ has two distinct roots, then $f(-\log k) < 0$. That is,

$$k < \frac{1}{e}$$

Hence,

$$k \in \left(0, \frac{1}{e}\right)$$

Hence, the correct answer is option (A).

4. The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is

(A) 0 **(B)** 1

(C) 2 **(D)** 3

[IIT–JEE 2008]

Solution: See Fig. 21.23. We have

$$f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$$

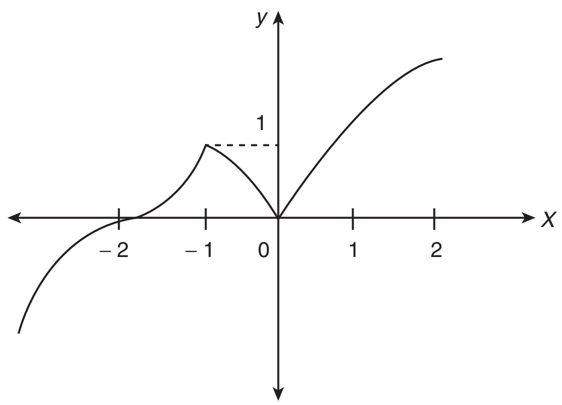


Figure 21.23

Clearly $x = -1$ is point of local maxima and $x = 0$ is a point of local minima. Therefore,

Total no. of local maxima and minima = 2

Hence, the correct answer is option (C).

5. Let $g(x) = \dfrac{(x-1)^n}{\log\cos^m(x-1)}; 0 < x < 2,$ m and n are integers, $m \ne 0$, $n > 0$, and let p be the left hand derivative of $|x - 3|$ at $x = 1$. If $\lim\limits_{x \to 1^+} g(x) = p,$ then

(A) $n = 1, m = 1$

(B) $n = 1, m = -1$

(C) $n = 2, m = 2$

(D) $n > 2, m = n$

[IIT–JEE 2008]

Solution: Let $f(x) = |x-1|$. Then

$$f(x) = \begin{cases} x-1, & x \ge 1 \\ 1-x, & x < 1 \end{cases}$$

$$f'(x) = \begin{cases} 1, & x \ge 1 \\ -1, & x < 1 \end{cases}$$

Therefore, $p = -1$.

Now, $$g(x) = \frac{(x-1)^n}{\log\cos^m(x-1)}$$

$$\lim_{x \to 1^+} g(x) = \lim_{h \to 0} \frac{h^n}{\log\cos h} \quad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{h \to 0} \frac{nh^{n-1}}{m \cdot (-\tan h)}$$

$$= \lim_{h \to 0} \frac{n \cdot h^{n-2}}{m \cdot \left(\dfrac{-\tan h}{h}\right)}$$

$$= \lim_{h \to 0} \frac{n \cdot h^{n-2}}{(-m)}$$

As $\lim\limits_{x \to 1^+} g(x) = -1$. Therefore,

$$\lim_{h \to 0} \frac{n.h^{n-2}}{(-m)} = -1$$

$$\Rightarrow n = 2 \text{ and } m = 2$$

Hence, the correct answer is option (C).

6. Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by

$g(u) = 2\tan^{-1}(e^u) - \dfrac{\pi}{2}$. Then, g is

(A) even and is strictly increasing in $(0, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) odd and is strictly increasing in $(-\infty, \infty)$

(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

[IIT–JEE 2008]

Solution: We have

$$g(u) = 2\tan^{-1}(e^4) - \frac{\pi}{2}$$

$$g(-u) = 2\tan^{-1}(e^{-4}) - \frac{\pi}{2}$$

$$= 2\tan^{-1}\left(\frac{1}{e^4}\right) - \frac{\pi}{2}$$
$$= 2\cot^{-1}(e^4) - \frac{\pi}{2}$$
$$= 2\left[\frac{\pi}{2} - \tan^{-1}(e^4)\right] - \frac{\pi}{2}$$
$$= \frac{\pi}{2} - 2\tan^{-1}(e^4)$$
$$= -\left(2\tan^{-1}e^4 - \frac{\pi}{2}\right)$$
$$= -g(4)$$

$\Rightarrow g(4)$ is an odd function

Now,

$$g'(4) = \frac{2e^4}{1+e^{24}} > 0$$

Therefore, $g(4)$ is strictly increasing function.

Hence, the correct answer is option (C).

Paragraph for Questions 7–9: Consider the function f: $(-\infty, \infty) \to (-\infty, \infty)$ defined by $f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2$.

7. Which of the following is true?

(A) $(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$

(B) $(2-a)^2 f''(1) - (2+a)^2 f''(-1) = 0$

(C) $f'(1)f'(-1) = (2-a)^2$

(D) $f'(1)f'(-1) = (2-a)^2$

[IIT–JEE 2008]

Solution: We have

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2$$
$$f'(x) = \frac{(x^2 + ax + 1)(2x - a) - (x^2 - ax + 1)(2x + a)}{(x^2 + ax + 1)^2}$$
$$= \frac{2a(x^2 - 1)}{(x^2 + ax + 1)^2}$$
$$f''(x) = \frac{4ax(x^2 + ax + 1)^2 - 4ax(x^2 - 1)(2x + a)(x^2 + ax + 1)}{(x^2 + ax + 1)^4}$$
$$f'(1) = \frac{4a}{(2+a)^2}, f''(-1) = \frac{-4a}{(2-a)^2}$$

Therefore,

$$(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$$

Hence, the correct answer is option (A).

8. Which of the following is true?

(A) $f(x)$ is decreasing on (–1, 1) and has a local minimum at $x = 1$

(B) $f(x)$ is increasing on (–1, 1) and has a local maximum at $x = 1$

(C) $f(x)$ is increasing on (–1, 1) but has neither a local maximum nor a local minimum at $x = 1$

(D) $f(x)$ is decreasing on (–1, 1) but has neither a local maximum nor a local minimum at $x = 1$

[IIT–JEE 2008]

Solution: We have

$$f'(x) = \frac{2a(x^2 - 1)}{(x^2 + ax + 1)^2}$$

Clearly,
$f(x)$ is Decreasing in (–1, 1)
$f(x)$ is Increasing in $(-\infty, -1) \cup (1, \infty)$
Therefore, $x = 1$ is point of minima.

Hence, the correct answer is option (A).

9. Let $g(x) = \int_0^{e^x} \frac{f''(t)}{1+t^2} dt$ which of the following is true?

(A) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$

(B) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$

(C) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$

(D) $g'(x)$ does not change sign on $(-\infty, \infty)$

[IIT–JEE 2008]

Solution: We have

$$g(x) = \int_0^{e^x} \frac{f''(t)}{1+t^2} dt$$
$$g'(x) = \frac{f'(e^x)}{1+e^{2x}} \cdot e^x$$
$$g'(x) > 0 \Rightarrow x \in (0, \infty)$$
$$g'(x) < 0 \Rightarrow x \in (-\infty, 0)$$

Hence, the correct answer is option (B).

10. Match the statements/expressions in Column I with the values given in Column II.

Column I	Column II
(A) The minimum value of $\frac{x^2 + 2x + 4}{x + 2}$ is	**(p)** 0
(B) Let A and B be 3 × 3 matrices of real numbers, where A is symmetric, B is skew–symmetric, and (A + B)(A – B) = (A – B)(A + B). If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible values of k are	**(q)** 1
(C) Let $a = \log_3 \log_3 2$. An integer k is satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	**(r)** 2
(D) If $\sin\theta = \cos\phi$, then the possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ are	**(s)** 3

[IIT–JEE 2008]

Solution:

$$(A) \to (r)$$

Let $y = \frac{x^2 + 2x + 4}{x + 2}$. Then

$$xy+2y=x^2+2x+4$$
$$x^2+(2-y)x+2(2-y)=0$$

As x is real, therefore,

$$D\ge 0$$
$$(2-y)^2-4\cdot 2(2-y)\ge 0$$
$$y^2+4y-12\ge 0$$
$$y\le -6 \text{ or } y\ge 2$$

Minimum value is 2.

$$(B)\to(q,s)$$
$$(A+B)(A-B)=(A-B)(A+B)$$
$$\Rightarrow AB=BA$$

As A is symmetric and B is skew-symmetric, so

$$(AB)^t=-AB$$
$$\Rightarrow k=1, \text{ and } k=3$$
$$(C)\to(r,s)$$
$$a=\log_3\log_3 2$$
$$\Rightarrow 3^{-a}=3^{-\log_3(\log_3 2)}=\log_2 3$$

Now,

$$1<2^{-k+\log_2 3}<2$$
$$1<3\cdot 2^{-k}<2$$
$$\Rightarrow \log_2\left(\frac{3}{2}\right)<k<\log_2 3$$
$$\Rightarrow k=1 \text{ or } k<2 \text{ and } k<3$$
$$(D)\to(p,r)$$

We have

$$\sin\theta=\cos\phi$$
$$\Rightarrow \cos\left(\frac{\pi}{2}-\theta\right)=\cos\phi$$
$$\Rightarrow \frac{\pi}{2}-\theta=2n\pi\pm\phi$$
$$\Rightarrow -2n\pi=\theta\pm\phi-\frac{\pi}{2}$$
$$\Rightarrow -2n=\frac{1}{\pi}\left(\theta\pm\phi-\frac{\pi}{2}\right)$$

Therefore, $\frac{1}{\pi}\left(\theta\pm\phi-\frac{\pi}{2}\right)$ is even number.

Hence, $\frac{1}{\pi}\left(\theta\pm\phi-\frac{\pi}{2}\right)=0,2$

Hence, the correct matches are (A)–(r); (B)–(q, s); (C)–(r, s); (D)–(p, r).

11. Match the statements/expressions in **Column I** with the open intervals in **Column II.**

Column I	Column II
(A) Interval contained in the domain of definition of non–zero solutions of the differential equation $(x-3)^2y'+y=0$	**(p)** $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
(B) Interval containing the value of the integral $\int_1^5(x-1)(x-2)(x-3)(x-4)(x-5)dx$	**(q)** $\left(0,\frac{\pi}{2}\right)$
(C) Interval in which at least one of the points of local maximum of $\cos^2 x+\sin x$ lies	**(r)** $\left(\frac{\pi}{8},\frac{5\pi}{4}\right)$
(D) Interval in which $\tan^{-1}(\sin x+\cos x)$ is increasing	**(s)** $\left(0,\frac{\pi}{8}\right)$
	(t) $(-\pi,\pi)$

[IIT–JEE 2009]

Solution:

(A) $(x-3)^2\frac{dy}{dx}+y=0$

$$\int\frac{dx}{(x-3)^2}=-\int\frac{dy}{y}$$
$$\Rightarrow \frac{1}{x-3}=\ln|y|+c$$

so domain is $R-\{3\}$.

(B) Put $x=t+3$

$\int_{-2}^{2}(t+2)(t+1)t(t-1)(t-2)dt=\int_{-2}^{2}t(t^2-1)(t^2-4)dt=0$ (being odd function)

(C) $f(x)=\frac{5}{4}-\left(\sin x-\frac{1}{2}\right)^2$

Maximum value occurs when $\sin x=\frac{1}{2}$

(D) $f'(x)>0$ if $\cos x>\sin x$.

Hence, the correct matches are (A)–(p, q, s); (B)–(p, t, s); (C)–(p, q, r, t); (D)–(s).

12. For function $f(x)=x\cos\frac{1}{x}, x\ge 1,$

(A) for atleast one x in interval $[1,\infty)$, $f(x+2)-f(x)<2$

(B) $\lim_{x\to\infty}f'(x)=1$

(C) for all x in the interval $[1,\infty)$, $f(x+2)-f(x)>2$

(D) $f'(x)$ is strictly decreasing in the interval $[1,\infty)$

[IIT–JEE 2009]

Solution: For $f(x)=x\cos\left(\frac{1}{x}\right), x\ge 1$

$$f'(x)=\cos\left(\frac{1}{x}\right)+\frac{1}{x}\sin\left(\frac{1}{x}\right)\to 1 \text{ for } x\to\infty$$

Also
$$f''(x)=\left(\frac{1}{x}\right)+\frac{1}{x}\sin\left(\frac{1}{x}\right)-\frac{1}{x^2}\sin\left(\frac{1}{x}\right)-\frac{1}{x^3}\cos\left(\frac{1}{x}\right)$$
$$=-\frac{1}{x^3}\cos\left(\frac{1}{x}\right)<0 \text{ for } x\ge 1$$

$$\Rightarrow f'(x) \text{ is decreasing for } [1,\infty)$$
$$\Rightarrow f'(x+2) < f'(x).$$

Also, $\lim_{x\to\infty} f(x+2)-f(x) = \lim_{x\to\infty}\left[(x+2)\cos\frac{1}{x+2} - x\cos\frac{1}{x}\right] = 2$

$$\therefore f(x+2)-f(x) > 2 \,\forall x \geq 1$$

Hence, the correct answers are options (B), (C) and (D).

13. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \left\{x \,\middle|\, x^2 + 20 \leq 9x\right\}$ is ______.

[IIT–JEE 2009]

Solution:

$$f'(x) = 6(x-2)(x-3)$$

So, $f(x)$ is increasing in $(3, \infty)$.

Also $A = \{4 \leq x \leq 5\}$. Therefore,

$$f_{\max} = f(5) = 7$$

Hence, the correct answer is (7).

14. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1$, 2 and $\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2.$ Then the value of $p(2)$ is ______.

[IIT–JEE 2009]

Solution: Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$. Then

$$P'(1) = P'(2) = 0$$
$$\lim_{x\to 0}\left(\frac{x^2+p(x)}{x^2}\right) = 2$$
$$\Rightarrow p(0) = 0 \Rightarrow e = 0$$
$$\lim_{x\to 0}\left(\frac{2x+p'(x)}{2x}\right) = 2$$
$$\Rightarrow p'(0) = 0 \Rightarrow d = 0$$
$$\lim_{x\to 0}\left(\frac{2+p''(x)}{2}\right) = 2$$
$$\Rightarrow c = 1$$

On solving, $a = 1/4$, $b = -1$

So,

$$p(x) = \frac{x^4}{4} - x^3 + x^2$$
$$\Rightarrow p(2) = 0$$

Hence, the correct answer is (0).

15. Let f, g and h be real–valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then

(A) $a = b$ and $c \neq b$ **(B)** $a = c$ and $a \neq b$

(C) $a \neq b$ and $c \neq b$ **(D)** $a = b = c$

[IIT–JEE 2010]

Solution:

$$f(x) = e^{x^2} + e^{-x^2} \Rightarrow f'(x) = 2x(e^{x^2} - e^{-x^2}) \geq 0 \;\forall\; x \in [0,1]$$

Clearly for $0 \leq x \leq 1$, $\quad f(x) \geq g(x) \geq h(x)$

As $f(1) = g(1) = h(1) = e + \frac{1}{e}$ and $f(1)$ is the greatest. Therefore,

$$a = b = c = e + \frac{1}{e} \Rightarrow a = b = c$$

Hence, the correct answer is option (D).

16. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010\,(x - 2009)\,(x - 2010)^2\,(x - 2011)^3\,(x - 2012)^4$, for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, for all $x \in R$, then the number of points in R at which g has a local maximum is

[IIT–JEE 2010]

Solution:

$$f(x) = \ln\{g(x)\}$$
$$g(x) = e^{f(x)}$$
$$g'(x) = e^{f(x)} \cdot f'(x)$$
$$g'(x) = 0 \Rightarrow f'(x) = 0 \text{ as } e^{f(x)} \neq 0$$
$$\Rightarrow 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4 = 0$$

So, there is only one point of local maxima.

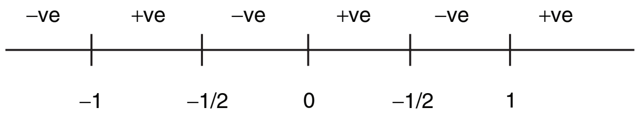
Hence, the correct answer is (1).

17. Let $f: R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is ______.

[IIT–JEE 2012]

Solution:

$$f'(x) = \frac{|x|}{x} + \frac{|x^2-1|}{x^2-1}\cdot(2x)$$
$$= \begin{cases} 2x-1, & x < -1 \\ -(2x+1), & -1 < x < 0 \\ 1-2x, & 0 < x < 1 \\ 2x+1, & x > 1 \end{cases}$$



So, $f'(x)$ changes sign at points

$$x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$$

So, total number of points of local maximum or minimum is 5.

Hence, the correct answer is (5).

18. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is ______.

[IIT–JEE 2012]

Solution: Let $p'(x) = k(x-1)(x-3)$. Then

$$p(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c$$

Now,

$$p(1) = 6 \Rightarrow \frac{4}{3}k + c = 6$$

also,

$$p(3)=2 \Rightarrow c=2$$

So, $k=3$, and

$$p'(0)=3k=9$$

Hence, the correct answer is (9).

Paragraph for Questions 19 and 20: Let $f(x)=(1-x)^2\sin^2 x+x^2$ for all $x\in R$, and let $g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)\,dt$ for all $x\in(1,\infty)$.

[IIT–JEE 2012]

19. Consider the statements:

P: There exists some $x\in R$, such that $f(x)+2x=2(1+x^2)$

Q: There exists some $x\in R$, such that $2f(x)+1=2x(1+x)$

Then

(A) both P and Q are true

(B) P is true and Q is false

(C) P is false and Q is true

(D) both P and Q are false.

Solution:

$$f(x)=(1-x)^2\sin^2 x+x^2 \quad \forall\ x\in R$$

$$g(x)=\int_1^x\left(\frac{2(t-1)}{t+1}-\ln t\right)f(t)\,dt \quad \forall\ x\in(1,\infty)$$

For statement P:

$$f(x)+2x=2(1+x^2) \qquad (1)$$

$$(1-x)^2\sin^2 x+x^2+2x=2+2x^2$$

$$(1-x)^2\sin^2 x=x^2-2x+2=(x-1)^2+1$$

$$(1-x)^2(\sin^2 x-1)=1$$

$$-(1-x)^2\cos^2 x=1$$

$$(1-x)^2\cdot\cos^2 x=-1$$

So, equation (1) will not have real solution.

Therefore, P is wrong.

For statement Q:

$$2(1-x)^2\sin^2 x+2x^2+1=2x+2x^2 \qquad (2)$$

$$2(1-x)^2\sin^2 x=2x-1$$

$$2\sin^2 x=\frac{2x-1}{(1-x)^2};\ \text{Let } h(x)=\frac{2x-1}{(1-x)^2}-2\sin^2 x$$

Clearly, $h(0)=-\text{ve},\ \lim_{x\to1^-}h(x)=+\infty$

So, by IVT, Eq. (2) will have solution.

Therefore, Q is correct.

Hence, the correct answer is option (C).

20. Which of the following is true?

(A) g is increasing on $(1,\infty)$

(B) g is decreasing on $(1,\infty)$

(C) g is increasing on (1, 2) and decreasing on $(2,\infty)$

(D) g is decreasing on (1, 2) and increasing on $(2,\infty)$

Solution:

$$g'(x)=\left(\frac{2(x-1)}{x+1}-\ln x\right)f(x)$$

For $x\in(1,\infty)$, $f(x)>0$

Let $h(x)=\left(\frac{2(x-1)}{x+1}-\ln x\right)$. Then

$$h'(x)=\left(\frac{4}{(x+1)^2}-\frac{1}{x}\right)=\frac{-(x-1)^2}{(x+1)^2x}<0$$

Also $h(1)=0$ so, $h(x)<0\ \forall x>1$

Therefore, $g(x)$ is decreasing on $(1,\infty)$.

Hence, the correct answer is option (B).

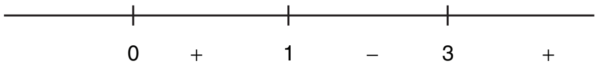
21. If $f(x)=\int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x\in(0,\infty)$, then

(A) f has a local maximum at $x=2$

(B) f is decreasing on (2, 3)

(C) there exists some $c\in(0,\infty)$ such that $f''(c)=0$

(D) f has a local minimum at $x=3$



[IIT–JEE 2012]

Solution:

$$f'(x)=e^{x^2}(x-2)(x-3)$$

Clearly, maxima at $x=2$, minima at $x=3$ and decreasing in $x\in(2,3)$.

$$f'(x)=0 \text{ for } x=2 \text{ and } x=3 \qquad \text{(Rolle's theorem)}$$

So, there exist $c\in(2,3)$ for which $f''(c)=0$.

Hence, the correct answers are options (A), (B), (C) and (D).

22. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are

(A) 24 **(B)** 32

(C) 45 **(D)** 60

[JEE ADVANCED 2013]

Solution: We have

$$V=(8\lambda-2x)(15\lambda-2x)x$$
$$=4x^3-46\lambda x^2+120\lambda^2x$$

Differentiating with respect to x, we get

$$\frac{dV}{dx}=12x^2-92\lambda x+120\lambda^2=0 \qquad \text{at } x=5$$

$$\Rightarrow\ 60\lambda^2-230\lambda+150=0$$

$$\Rightarrow\ 6\lambda^2-23\lambda+15=0$$

$$\Rightarrow\ (6\lambda-5)(\lambda-3)=0$$

For $\lambda = 3$, the lengths of sides (as shown in Fig. 21.24) are obtained as 45, 24.

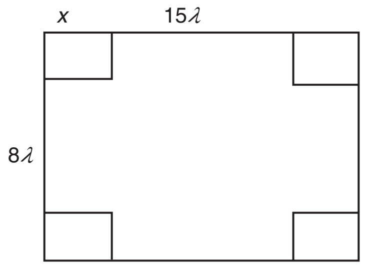


Figure 21.24

Hence, the correct answers are options (A) and (C).

23. The function $f(x) = 2|x| + |x+2| - ||x+2| - 2|x||$ has a local minimum or a local maximum at x equals

(A) -2 **(B)** $\frac{-2}{3}$

(C) 2 **(D)** $\frac{2}{3}$

[JEE ADVANCED 2013]

Solution: We have

$$
\begin{array}{ll}
x < -2, & f(x) = -2x - 4 \\
-2 \le x < -\frac{2}{3}, & f(x) = 2x + 4 \\
-\frac{2}{3} \le x \le 0, & f(x) = -4x \\
0 \le x < 2 & f(x) = 4x \\
x \ge 2, & f(x) = 2x + 4
\end{array}
$$

It is clear from Fig. 21.25 that $x = -2$ and $x = 0$ are the points of minima. Therefore, $x = -\frac{2}{3}$ is point of maxima.

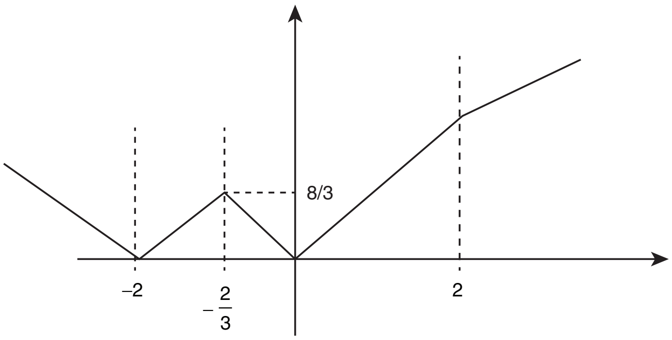


Figure 21.25

Hence, the correct answers are options (A) and (B).

Paragraph for Questions 24 and 25: Let f: [0, 1] → R (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \ge e^x$, $x \in [0, 1]$.

24. Which of the following is true for $0 < x < 1$?

(A) $0 < f(x) < \infty$ **(B)** $-\frac{1}{2} < f(x) < \frac{1}{2}$

(C) $-\frac{1}{4} < f(x) < 1$ **(D)** $-\infty < f(x) < 0$

[JEE ADVANCED 2013]

Solution: We have

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y \ge e^x$$

$$\Rightarrow e^{-x}\frac{d^2y}{dx^2} - 2e^{-x}\frac{dy}{dx} + e^{-x}y \ge 1$$

$$\Rightarrow \frac{d^2}{dx^2}(ye^{-x}) \ge 1 \quad (1)$$

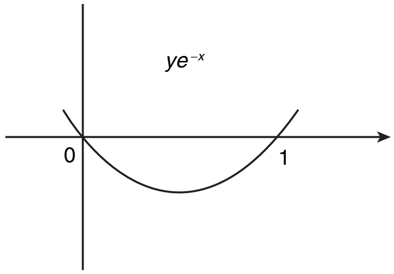


Figure 21.26

From Eq. (1) and Fig. 21.26, ye^{-x} is concave up. Hence, $-\infty < f(x) < 0$.

Hence, the correct answer is option (D).

25. If the function $e^{-x} f(x)$ assumes its minimum in the interval [0, 1] at $x = \frac{1}{4}$, which of the following is true?

(A) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$ **(B)** $f'(x) > f(x), 0 < x < \frac{1}{4}$

(C) $f'(x) < f(x), 0 < x < \frac{1}{4}$ **(D)** $f'(x) < f(x), \frac{3}{4} < x < 1$

[JEE ADVANCED 2013]

Solution: See Fig. 21.27. We know that $\frac{d}{dx}(ye^{-x})$ is an increasing function. Therefore,

$$
\begin{array}{ll}
0 < x < \frac{1}{4} & x > \frac{1}{4} \\
\frac{d}{dx}(ye^{-x}) < 0 & \frac{d}{dx}(ye^{-x}) > 0 \\
e^{-x}\frac{dy}{dx} - e^{-x}y < 0 & e^{-x}\frac{dy}{dx} - e^{-x}y > 0 \\
\frac{dy}{dx} < y & \frac{dy}{dx} > y \\
f'(x) < f(x) & f'(x) > f(x)
\end{array}
$$

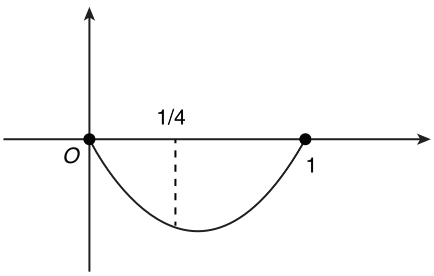


Figure 21.27

Hence, the correct answer is option (C).

26. Let $f:(0,\infty)\to R$ be given by $f(x)=\int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$. Then

(A) $f(x)$ is monotonically increasing on $[1, \infty)$
(B) $f(x)$ is monotonically decreasing on $(0, 1)$
(C) $f(x)+f\left(\frac{1}{x}\right)=0$, for all $x \in (0, \infty)$
(D) $f(2^x)$ is an odd function of x on R

[JEE ADVANCED 2014]

Solution:

$$f(x)=\int_{\frac{1}{x}}^{x}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$$

$$\Rightarrow \frac{d}{dx}f(x)=\frac{e^{-\left(x+\frac{1}{x}\right)}}{x}\frac{d}{dx}x-\frac{e^{-\left(\frac{1}{x}+x\right)}}{\frac{1}{x}}\times\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$=\frac{e^{-\left(x+\frac{1}{x}\right)}}{x}+xe^{-\left(x+\frac{1}{x}\right)}\times\left(-\frac{1}{x^2}\right)$$

$$=\frac{e^{-\left(x+\frac{1}{x}\right)}}{x}+\frac{1}{x}e^{-\left(x+\frac{1}{x}\right)}$$

$$=\frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}>0$$

Therefore, $f(x)$ is strictly increasing in $(0, \infty)$. (1)

Now $$f(x)+f\left(\frac{1}{x}\right)=\int_{\frac{1}{x}}^{x}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt+\int_{x}^{\frac{1}{x}}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$$

$$=\int_{\frac{1}{x}}^{\frac{1}{x}}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt=0 \quad (2)$$

Now $$f(2^x)+f\left(\frac{1}{2^x}\right)=f(2^x)+f(2^{-x})\ =0$$

Therefore, $f(2^x)$ is an odd function. (3)

Note: Let $2^x=\mu$ as $\log_2\mu=x$. For $\mu\in(0,\infty)$, $x\in(-\infty,\infty)$

We can say $f(2^x) = h(x)$ an odd function. Then $h(-x) = -h(x)$. Therefore, from Eqs. (1), (2) and (3), we can conclude that the correct options are (A), (C) and (D).

Hence, the correct answers are options (A), (C) and (D).

27. Let $f:R\to R$ and $g:R\to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h:R\to R$ by

$$h(x)=\begin{cases}\max\ \{f(x),g(x)\} & \text{if } x\le 0,\\ \min\ \{f(x),g(x)\} & \text{if } x>0.\end{cases}$$

The number of points at which $h(x)$ is not differentiable is ______.

[JEE ADVANCED 2014]

Solution:

$f:R\to R$ $\qquad$ $g:R\to R$

$f(x)=|x|+1$ $\qquad$ $g(x)=x^2+1$

$h:R\to R$

$$h(x)=\begin{cases}\max\{f(x),g(x)\} & \text{if } x\le 0,\\ \min\{f(x),g(x)\} & \text{if } x>0.\end{cases}$$

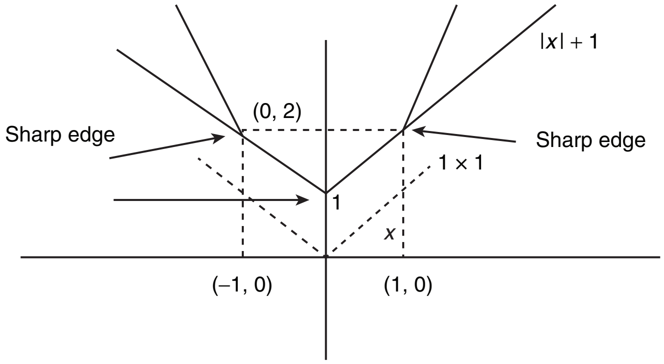


Figure 21.28

Points of Intersection are, (1, 2) and (–1, 2). [See Fig. 21.28.]

$h(x) = x^2 + 1$, in $(-\infty, -1)$
$x^2 + 1 = |x| + 1$ at -1
$|x| + 1$, in $(-1, 0)$
$|x| + 1 = x^2 + 1$ at 0
$= x^2 + 1$, in $(0, 1)$
$x^2 + 1 = |x| + 1$ at 1
$= |x| + 1$, in $(1, \infty)$

At sharp edges i.e. at –1, 0 and 1, there is no smooth turn, so no derivative exits there. Elsewhere, function is continuous and differentiable. Hence, there are three such points.

Hence, the correct answer is (3).

28. A cylindrical container is to be made from certain solid material with the following constraints: It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.
If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is ______.

[JEE ADVANCED 2015]

Solution:

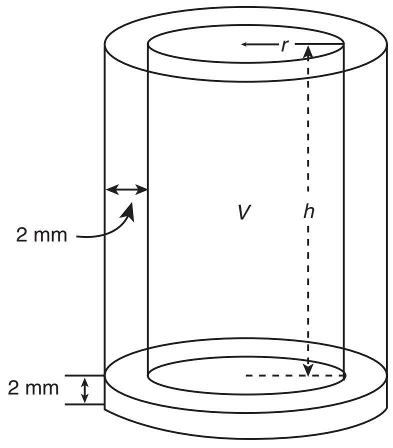


Figure 21.29

V_m = Volume of material used is minimum when $r = 10$ mm,

$$\frac{V}{250\pi} = ?$$

Here V is constant but r is variable and h is variable such that

$$V = \pi r^2 h \quad (1)$$

Volume of material used is

$$V_m = \underbrace{\pi(r+2)^2 h}_{\substack{\text{volume of}\\ \text{outer cylinder}\\ \text{without base}}} - \underbrace{\pi r^2 h}_{\substack{\text{volume of}\\ \text{inner cylinder}\\ \text{without base}}} + \underbrace{(r+2)^2(2)}_{\substack{\text{Volume of}\\ \text{Base disc}}}$$

$$= 4\pi h + 4r\pi h + 2\pi r^2 + 8\pi r + 8\pi$$

$$V_m = \frac{4\pi v}{\pi r^2} + \frac{4r\pi . v}{\pi r^2} + 2\pi r^2 + 8\pi^2 + 8\pi$$

Now, $\left.\frac{V_m}{dr}\right|_{r=10} = 4V\left(\frac{-2}{r^3}\right) + 4V\left(\frac{-1}{r^2}\right) + 4\pi r + 8\pi = 0$

$$\Rightarrow -\frac{8V}{1000} - \frac{4V}{100} + 40\pi + 8\pi = 0$$

$$\Rightarrow -\frac{48V}{1000} = -48\pi$$

$$\Rightarrow \frac{V}{250\pi} = 4$$

Hence, the correct answer is (4).

29. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are

(A) $m = 13, M = 24$

(B) $m = \frac{1}{4}, M = \frac{1}{2}$

(C) $m = -11, M = 0$

(D) $m = 1, M = 12$

[JEE ADVANCED 2015]

Solution:

$$f'(x) = \frac{192x^3}{2+\sin^4 \pi x} \; \forall x \in R,$$

Here $\int_{1/2}^{1} f(x) = \int_{1/2}^{1} \left(\int f'(x)dx\right) \quad (1)$

Here, $\frac{192x^3}{3} \le \frac{192x^3}{2+\sin^4 \pi x} \le \frac{192x^3}{2}$

$$\Rightarrow \frac{192}{3}\int_{1/2}^{x} x^3 dx \le \int_{1/2}^{x} \frac{192x^3}{2+\sin^4 \pi x}dx \le \frac{192}{2}\int_{1/2}^{x} x^3\, dx$$

$$\Rightarrow \frac{192}{3}\left(\frac{x^4}{4} - \frac{1}{64}\right) \le \int_{1/2}^{x} f'(x)dx \le \frac{192}{2}\left(\frac{x^4}{4} - \frac{1}{64}\right)$$

$$\Rightarrow 64\left[\frac{x^4}{4} - \frac{1}{64}\right] \le f(x) - f\left(\frac{1}{2}\right) \le 96\left[\frac{x^4}{4} - \frac{1}{64}\right]$$

$$\Rightarrow \left[\frac{16x^5}{5} - x\right]_{1/2}^{1} \le \int_{1/2}^{1} f(x)\, dx \le \left[\frac{24x^5}{5} - \frac{3}{2}x\right]_{1/2}^{1}$$

$$\Rightarrow \frac{26}{10} \le \int_{1/2}^{1} f(x)\, dx \le \frac{39}{10}$$

$$\Rightarrow 64\left[\frac{x^4}{4} - \frac{1}{64}\right] \le f(x) \le 96\left[\frac{x^4}{4} - \frac{1}{64}\right]$$

$$\Rightarrow \int_{1/2}^{1} (16x^4 - 1)dx \le \int_{1/2}^{1} f(x)dx \le \int_{1/2}^{1}\left(24x^4 - \frac{3}{2}\right)dx$$

Clearly, $\left(\frac{26}{10}, \frac{39}{10}\right) \subset (1,12)$

Hence, the correct answer is option (D).

30. Let $f: R \to (0,\infty)$ and $g: R \to R$ be twice differentiable function such that f'' and g'' are continuous functions on R. Suppose $f'(2) = g(2) = 0$, $f''(2) \ne g'(2) \ne 0$. If $\lim\limits_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

(A) f has a local minimum at $x = 2$.

(B) f has a local maximum at $x = 2$.

(C) $f''(2) > f(2)$.

(D) $f(x) - f''(x) = 0$ for at least one $x \in R$.

[JEE ADVANCED 2016]

Solution: Let $f: R \to (0,\infty)$ and $g: R \to R$

$$f(x) > 0 \;\; \forall x \in R$$

It is given that $f'(2) = 0$, $g(2) = 0$, $f''(2) \ne 0$ and $g'(2) \ne 0$.

It is also given that

$$\lim_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)} = 1 \qquad \left(\frac{0}{0} \text{ form}\right)$$

Applying L'Hospital rule, we get

$$\lim_{x\to 2}\frac{f'(x)g(x) + g'(x)f(x)}{f''(x)g'(x) + f'(x)g''(x)} = 1$$

For finite limit, we get

$$\frac{f'(2)g(2) + g'(2)f(2)}{f''(2)g'(2) + f'(2)g''(2)} = 1$$

$$\frac{g'(2)f(2)}{f''(2)g'(2)} = 1$$

$$\frac{f(2)}{f''(2)} = 1 \;\Rightarrow\; f''(2) = f(2) > 0 \text{ and } f'(2) = 0$$

which means that $f(x)$ has local minima at $x = 2$.

Hence, the correct answer is option (A).

$$f(2) - f''(2) = 0$$

Therefore, we can say that $f(x)-f''(x)=0$ has at least one solution in $x \in R$.
Hence, option (D) is correct.
Hence, the correct answers are options (A) and (D).

Practice Exercise 1

1. The points on the curve $y = 12x - x^3$ at which the gradient is zero are
   (A) (0, 2), (2, 16) **(B)** (0, –2), (2, –16)
   (C) (2, –16), (–2, 16) **(D)** (2, 16), (–2, –16)
2. The area of the triangle formed by the coordinate axes and a tangent to the curve $xy = a^2$ at the point (x_1, y_1) on it is
   (A) $\frac{a^2x_1}{y_1}$ **(B)** $\frac{a^2y_1}{x_1}$ **(C)** $2a^2$ **(D)** $4a^2$
3. The slope of tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, –1) is
   (A) $22/7$ **(B)** $6/7$ **(C)** -6 **(D)** None of these
4. The point of the curve $y^2 = 2(x-3)$ at which the normal is parallel to the line $y - 2x + 1 = 0$ is
   (A) (5, 2) **(B)** $\left(-\frac{1}{2}, -2\right)$
   (C) (5, –2) **(D)** $\left(\frac{3}{2}, 2\right)$
5. The line $x + y = 2$ is tangent to the curve $x^2 = 3 - 2y$ at its point
   (A) (1, 1) **(B)** (–1, 1)
   (C) $(\sqrt{3}, 0)$ **(D)** (3, –3)
6. If $x = t^2$ and $y = 2t$, then the equation of the normal at $t = 1$ is
   (A) $x + y - 3 = 0$ **(B)** $x + y - 1 = 0$
   (C) $x + y + 1 = 0$ **(D)** $x + y + 3 = 0$
7. The equation of the normal to the curve y = $\sin(\pi x/2)$ at the point (1, 1) is
   (A) $y = 1$ **(B)** $x = 1$
   (C) $y = x$ **(D)** $y - 1 = \frac{-2}{\pi}(x-1)$
8. The equation of tangent to the curve $y = 2\cos x$ at $x = \pi/4$ is
   (A) $y - \sqrt{2} = 2\sqrt{2}\left(x - \frac{\pi}{4}\right)$ **(B)** $y + \sqrt{2} = \sqrt{2}\left(x + \frac{\pi}{4}\right)$
   (C) $y - \sqrt{2} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)$ **(D)** $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$
9. At which point the line $\frac{x}{a} + \frac{y}{b} = 1$, touches the curve $y = be^{-x/a}$
   (A) (0, 0) **(B)** (0, a)
   (C) (0, b) **(D)** (b, 0)
10. The angle between curves $y^2 = 4x$ and $x^2 + y^2 = 5$ at (1, 2) is
    (A) $\tan^{-1}(3)$ **(B)** $\tan^{-1}(2)$ **(C)** $\frac{\pi}{2}$ **(D)** $\pi/4$
11. For the curve $by^2 = (x+a)^3$, the square of the subtangent is proportional to
    (A) $(\text{Subnormal})^{1/2}$ **(B)** Subnormal
    (C) $(\text{Subnormal})^{3/2}$ **(D)** None of these
12. The tangent to the curve $y = ax^2 + bx$ at $(2, -8)$ is parallel to x–axis. Then
    (A) $a = 2, b = -2$ **(B)** $a = 2, b = -4$
    (C) $a = 2, b = -8$ **(D)** $a = 4, b = -4$
13. The sum of intercepts on the coordinate axes made by the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is
    (A) a **(B)** $2a$ **(C)** $2\sqrt{a}$ **(D)** None of these
14. The coordinates of a point on the curve $y = x\log x$ at which the normal is parallel to the line $2x - 2y = 3$ are
    (A) (0, 0) **(B)** (e, e)
    (C) $(e^2, 2e^2)$ **(D)** $(e^{-2}, -2e^{-2})$
15. If normal to the curve $y = f(x)$ is parallel to x–axis, then the correct statement is
    (A) $\frac{dy}{dx} = 0$ **(B)** $\frac{dy}{dx} = 1$
    (C) $\frac{dx}{dy} = 0$ **(D)** None of these
16. The length of normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ at the point $\theta = \pi/2$ is
    (A) $2a$ **(B)** $a/2$ **(C)** $\sqrt{2}a$ **(D)** $a/\sqrt{2}$
17. The normal of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any θ is such that
    (A) It makes a constant angle with x–axis
    (B) It passes through the origin
    (C) It is at a constant distance from the origin
    (D) None of these
18. The slope of the tangent to the curve $x = 3t^2 + 1, y = t^3 - 1$ at $x = 1$ is
    (A) 0 **(B)** $1/2$ **(C)** ∞ **(D)** -2
19. An equation of the tangent to the curve $y = x^4$ from the point (2, 0) not on the curve is
    (A) $y = 0$ **(B)** $x = 0$
    (C) $x + y = 0$ **(D)** None of these
20. The angle of intersection of the curves $y = x^2$ and $x = y^2$ at (1, 1) is
    (A) $\tan^{-1}\left(\frac{4}{3}\right)$ **(B)** $\tan^{-1}(1)$
    (C) $90°$ **(D)** $\tan^{-1}\left(\frac{3}{4}\right)$

21. The abscissae of the points, where the tangent to curve $y = x^3 - 3x^2 - 9x + 5$ is parallel to x–axis, are

(A) 0 and 0 **(B)** $x = 1$ and -1

(C) $x = 1$ and -3 **(D)** $x = -1$ and 3

22. If the curve $y = a^x$ and $y = b^x$ intersect at angle α, find the value of $\tan\alpha$.

(A) $\frac{a-b}{1+ab}$ **(B)** $\frac{\log a - \log b}{1 + \log a \log b}$

(C) $\frac{a+b}{1-ab}$ **(D)** $\frac{\log a + \log b}{1 - \log a \log b}$

23. The equation of tangent at $(-4, -4)$ on the curve $x^2 = -4y$ is

(A) $2x + y + 4 = 0$ **(B)** $2x - y - 12 = 0$

(C) $2x + y - 4 = 0$ **(D)** $2x - y + 4 = 0$

24. The point at which the tangent to the curve $y = 2x^2 - x + 1$ is parallel to $y = 3x + 9$ will be

(A) (2, 1) **(B)** (1, 2) **(C)** (3, 9) **(D)** (–2, 1)

25. At what point on the curve $x^3 - 8a^2y = 0$, the slope of the normal is $-2/3$?

(A) (a, a) **(B)** $(2a, -a)$

(C) $(2a, a)$ **(D)** None of these

26. The length of the normal at point t of the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$, is

(A) $a \sin t$ **(B)** $2a\sin^3(t/2)\sec(t/2)$

(C) $2a\sin(t/2)\tan(t/2)$ **(D)** $2a\sin(t/2)$

27. The tangent drawn at the point (0, 1) on the curve $y = e^{2x}$ meets x–axis at the point

(A) $(1/2, 0)$ **(B)** $(-1/2, 0)$ **(C)** $(2, 0)$ **(D)** $(0, 0)$

28. The equation of the tangent to the curve $(1 + x^2)y = 2 - x$, where it crosses the x–axis, is

(A) $x + 5y = 2$ **(B)** $x - 5y = 2$

(C) $5x - y = 2$ **(D)** $5x + y - 2 = 0$

29. The equation of the tangent to curve $y = be^{-x/a}$ at the point where it crosses y–axis is

(A) $ax + by = 1$ **(B)** $ax - by = 1$

(C) $\frac{x}{a} - \frac{y}{b} = 1$ **(D)** $\frac{x}{a} + \frac{y}{b} = 1$

30. The angle of intersection of curves $y = x^2$ and $6y = 7 - x^3$ at (1, 1) is

(A) $\pi/4$ **(B)** $\pi/3$ **(C)** $\pi/2$ **(D)** π

31. The tangent to the curve $y = 2x^2 - x + 1$ at point P is parallel to $y = 3x + 4$, the coordinates of point P are

(A) (2, 1) **(B)** (1, 2) **(C)** (– 1, 2) **(D)** (2, – 1)

32. For the curve $xy = c^2$, the subnormal at any point varies as

(A) x^2 **(B)** x^3 **(C)** y^2 **(D)** y^3

33. The angle between the curves $y = \sin x$ and $y = \cos x$ is

(A) $\tan^{-1}(2\sqrt{2})$ **(B)** $\tan^{-1}(3\sqrt{2})$

(C) $\tan^{-1}(3\sqrt{3})$ **(D)** $\tan^{-1}(5\sqrt{2})$

34. If the normal to the curve $y^2 = 5x - 1$, at the point (1, –2) is of the form $ax - 5y + b = 0$, then a and b are

(A) 4, – 14 **(B)** 4, 14 **(C)** –4, 14 **(D)** –4, –14

35. If a tangent to the curve $y = 6x - x^2$ is parallel to the line $4x - 2y - 1 = 0$, then the point of tangency on the curve is

(A) (2, 8) **(B)** (8, 2) **(C)** (6, 1) **(D)** (4, 2)

36. The normal to the curves $x = a(1 + \cos\theta)$ and $y = a\sin\theta$ at angle θ always passes through the fixed point

(A) (a, a) **(B)** $(0, a)$ **(C)** $(0, 0)$ **(D)** $(a, 0)$

37. If ST and SN are the lengths of the subtangent and the subnormal at the point $\theta = \pi/2$ on the curve $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$, where $a \neq 1$, then

(A) $ST = SN$ **(B)** $ST = 2SN$

(C) $ST^2 = aSN^3$ **(D)** $ST^3 = aSN$

38. The equation of the tangent to the curves $x = 2\cos^3\theta$ and $y = 3\sin^3\theta$ at the point $\theta = \pi/4$ is

(A) $2x + 3y = 3\sqrt{2}$ **(B)** $2x - 3y = 3\sqrt{2}$

(C) $3x + 2y = 3\sqrt{2}$ **(D)** $3x - 2y = 3\sqrt{2}$

39. The curve given by $x + y = e^{xy}$ has a tangent parallel to the y-axis at the point

(A) (0, 1) **(B)** (1, 0) **(C)** (1, 1) **(D)** (–1, –1)

40. For which of the following intervals, the given function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is decreasing?

(A) $(-2, \infty)$ **(B)** $(-2, -1)$

(C) $(-\infty, -1)$ **(D)** $(-\infty, -2)$ and $(-1, \infty)$

41. $f(x) = x^3 - 27x + 5$ is an increasing function, when

(A) $x < -3$ **(B)** $|x| > 3$

(C) $x \leq -3$ **(D)** $|x| < 3$

42. If $f(x) = \sin x - \frac{x}{2}$ is increasing function, then

(A) $0 < x < \frac{\pi}{3}$ **(B)** $-\frac{\pi}{3} < x < 0$

(C) $-\frac{\pi}{3} < x < \frac{\pi}{3}$ **(D)** $x = \frac{\pi}{2}$

43. If x tends 0 to π, then the given function $f(x) = x\sin x + \cos x + \cos^2 x$ is

(A) Increasing
(B) Decreasing
(C) Neither increasing nor decreasing
(D) None of these

44. Let $y = x^2e^{-x}$. Then the interval in which y increases with respect to x is

(A) $(-\infty, \infty)$ **(B)** $(-2, 0)$ **(C)** $(2, \infty)$ **(D)** $(0, 2)$

45. The function $y = 2x^3 - 9x^2 + 12x - 6$ is monotonically decreasing, when

(A) $1 < x < 2$ **(B)** $x > 2$

(C) $x < 1$ **(D)** None of these

46. For which value of x, the function $f(x)=x^2-2x$ is decreasing?
(A) $x>1$ **(B)** $x>2$ **(C)** $x<1$ **(D)** $x<2$

47. The function $f(x)=\cos x-2px$ is monotonically decreasing for
(A) $p<\frac{1}{2}$ **(B)** $p>\frac{1}{2}$ **(C)** $p<2$ **(D)** $p>2$

48. If $f(x)=kx^3-9x^2+9x+3$ is monotonically increasing in each interval, then
(A) $k<3$ **(B)** $k\le 3$
(C) $k>3$ **(D)** None of these

49. In which interval is the given function $f(x)=2x^3-15x^2+36x+1$ is monotonically decreasing?
(A) [2, 3] **(B)** (2, 3) **(C)** $(-\infty,2)$ **(D)** $(3,\infty)$

50. The function $f(x)=\tan x-x$
(A) Always increases
(B) Always decreases
(C) Never decreases
(D) Sometimes increases and sometimes decreases

51. The function $f(x)=\log(1+x)-\frac{2x}{2+x}$ is increasing on
(A) $(0,\infty)$ **(B)** $(-\infty,0)$
(C) $(-\infty,\infty)$ **(D)** None of these

52. The values of a, for which the function $(a+2)x^3-3ax^2+9ax-1$ decreases monotonically throughout for all real x, are
(A) $a<-2$ **(B)** $a>-2$
(C) $-3<a<0$ **(D)** $-\infty<a\le -3$

53. If $f(x)=2x+\cot^{-1}x+\log(\sqrt{1+x^2}-x)$, then $f(x)$
(A) Increases in $[0,\infty)$
(B) Decreases in $[0,\infty)$
(C) Neither increases nor decreases in $(0,\infty)$
(D) Increases in $(-\infty,\infty)$

54. For all real values of x, the increasing function $f(x)$ is
(A) x^{-1} **(B)** x^2 **(C)** x^3 **(D)** x^4

55. The function $f(x)=x^3-3x^2-24x+5$ is an increasing function in the interval
(A) $(-\infty,-2)\cup(4,\infty)$ **(B)** $(-2,\infty)$
(C) $(-2,4)$ **(D)** $(-\infty,4)$

56. Which one is the correct statement about the function $f(x)=\sin 2x$?
(A) $f(x)$ is increasing in $\left(0,\frac{\pi}{2}\right)$ and decreasing in $\left(\frac{\pi}{2},\pi\right)$
(B) $f(x)$ is decreasing in $\left(0,\frac{\pi}{2}\right)$ and increasing in $\left(\frac{\pi}{2},\pi\right)$
(C) $f(x)$ is increasing in $\left(0,\frac{\pi}{4}\right)$ and decreasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$
(D) The statements (A), (B) and (C) are correct.

57. The function f defined by $f(x)=(x+2)e^{-x}$ is
(A) Decreasing for all x
(B) Decreasing in $(-\infty,-1)$ and increasing in $(-1,\infty)$
(C) Increasing for all x
(D) Decreasing in $(-1,\infty)$ and increasing in $(-\infty,-1)$

58. If $f(x)=x^3-10x^2+200x-10$, then
(A) $f(x)$ is decreasing in $[-\infty,10]$ and increasing in $[10,\infty]$
(B) $f(x)$ is increasing in $[-\infty,10]$ and decreasing in $[10,\infty]$
(C) $f(x)$ is increasing throughout real line
(D) $f(x)$ is decreasing throughout real line

59. If $f(x)=\frac{x}{\sin x}$ and $g(x)=\frac{x}{\tan x}$, where $0<x\le 1$, then in this interval
(A) Both $f(x)$ and $g(x)$ are increasing functions
(B) Both $f(x)$ and $g(x)$ are decreasing functions
(C) $f(x)$ is an increasing function
(D) $g(x)$ is an increasing function

60. Function $f(x)=2x^3-9x^2+12x+29$ is monotonically decreasing, when
(A) $x<2$ **(B)** $x>2$ **(C)** $x>1$ **(D)** $1<x<2$

61. $2x^3+18x^2-96x+45=0$ is an increasing function when
(A) $x\le -8, x\ge 2$ **(B)** $x<-2, x\ge 8$
(C) $x\le -2, x\ge 8$ **(D)** $0\le x\le -2$

62. The function $\frac{a\sin x+b\cos x}{c\sin x+d\cos x}$ is decreasing if
(A) $ad-bc>0$ **(B)** $ad-bc<0$
(C) $ab-cd>0$ **(D)** $ab-cd<0$

63. The function $f(x)=1-e^{-x^2/2}$ is
(A) Decreasing for all x
(B) Increasing for all x
(C) Decreasing for $x<0$ and increasing for $x>0$
(D) Increasing for $x<0$ and decreasing for $x>0$

64. Consider the following statements:

S: Both $\sin x$ and $\cos x$ are decreasing functions in $\left(\frac{\pi}{2},\pi\right)$

R: If a differentiable function decreases in (a, b), then its derivative also decreases in (a, b).

Which of the following is true?
(A) Both S and R are wrong
(B) Both S and R are correct but R is not the correct explanation for S
(C) S is correct and R is the correct explanation for S
(D) S is correct and R is wrong.

65. The function which is neither decreasing nor increasing in $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ is

(A) $\operatorname{cosec} x$ **(B)** $\tan x$ **(C)** x^2 **(D)** $|x-1|$

66. Function $f(x)=\dfrac{\lambda\sin x+6\cos x}{2\sin x+3\cos x}$ is monotonically increasing, if

(A) $\lambda>1$ **(B)** $\lambda<1$ **(C)** $\lambda<4$ **(D)** $\lambda>4$

67. On the interval (1, 3), the function $f(x)=3x+\dfrac{2}{x}$ is

(A) Strictly decreasing
(B) Strictly increasing
(C) Decreasing in (2, 3) only
(D) Neither increasing nor decreasing

68. If $f(x)=\sin x-\cos x$, the function decreasing in $0\le x\le 2\pi$ is

(A) $[5\pi/6, 3\pi/4]$ **(B)** $[\pi/4, \pi/2]$
(C) $[3\pi/2, 5\pi/2]$ **(D)** None of these

69. The function $f(x)=\dfrac{\log x}{x}$ is increasing in the interval

(A) $(1, 2e)$ **(B)** $(0, e)$
(C) $(2, 2e)$ **(D)** $(1/e, 2e)$

70. If $f(x)=xe^{x(1-x)}$, then $f(x)$ is

(A) Increasing on $\left[-\frac{1}{2},1\right]$
(B) Decreasing on R
(C) Increasing on R
(D) Decreasing on $\left[-\frac{1}{2},1\right]$

71. If $f(x)=x^3-6x^2+9x+3$ be a decreasing function, then x lies in

(A) $(-\infty,-1)\cap(3,\infty)$ **(B)** $(1, 3)$
(C) $(3, \infty)$ **(D)** None of these

72. If $f(x)=\dfrac{1}{x+1}-\log(1+x), x>0$, then $f(x)$ is

(A) An increasing function
(B) A decreasing function
(C) Both increasing and decreasing functions
(D) None of these

73. The function $f(x)=x+\cos x$ is

(A) Always increasing
(B) Always decreasing
(C) Increasing for certain range of x
(D) None of these

74. The function $f(x)=x^{1/x}$ is

(A) Increasing in $(1, \infty)$
(B) Decreasing in $(1, \infty)$
(C) Increasing in $(1, e)$, decreasing in (e, ∞)
(D) Decreasing in $(1, e)$, increasing in (e, ∞)

75. The function $f(x)=1-x^3-x^5$ is decreasing for

(A) $1\le x\le 5$ **(B)** $x\le 1$
(C) $x\ge 1$ **(D)** All values of x

76. The function x^x is increasing, when

(A) $x>\dfrac{1}{e}$ **(B)** $x<\dfrac{1}{e}$
(C) $x<0$ **(D)** For all real x

77. $2x^3-6x+5$ is an increasing function if

(A) $0<x<1$ **(B)** $-1<x<1$
(C) $x<-1$ or $x>1$ **(D)** $-1<x<-1/2$

78. The length of the longest interval, in which the function $3\sin x-4\sin^3 x$ is increasing, is

(A) $\dfrac{\pi}{3}$ **(B)** $\dfrac{\pi}{2}$ **(C)** $\dfrac{3\pi}{2}$ **(D)** π

79. Let $f(x)=x^3+bx^2+cx+d, 0<b^2<c$. Then $f(x)$

(A) is bounded **(B)** has a local maxima
(C) has a local minima **(D)** is strictly increasing

80. If $f(x)=x, -1\le x\le 1$, then function $f(x)$ is

(A) Increasing **(B)** Decreasing
(C) Stationary **(D)** Discontinuous

81. For all $x\in(0,1)$, which is the correct one?

(A) $e^x<1+x$ **(B)** $\log_e(1+x)<x$
(C) $\sin x>x$ **(D)** $\log_e x>x$

82. The function $f(x)=2x^3-3x^2+90x+174$ is increasing in the interval

(A) $\dfrac{1}{2}<x<1$ **(B)** $\dfrac{1}{2}<x<2$
(C) $3<x<\dfrac{59}{4}$ **(D)** $-\infty<x<\infty$

83. The function $f(x)=\tan^{-1}(\sin x+\cos x)$, $x>0$ is always an increasing function on the interval

(A) $(0,\pi)$ **(B)** $(0,\pi/2)$
(C) $(0,\pi/4)$ **(D)** $(0,3\pi/4)$

84. The function $f(x)=\left(\dfrac{e^{2x}-1}{e^{2x}+1}\right)$ is

(A) Increasing **(B)** Decreasing
(C) Even **(D)** None of these

85. What is the value of x if the function $x^5-5x^4+5x^3-10$ has a maximum?

(A) 3 **(B)** 2 **(C)** 1 **(D)** 0

86. The local maximum value of the function $\dfrac{\log x}{x}$ is

(A) e **(B)** 1 **(C)** $\dfrac{1}{e}$ **(D)** $2e$

87. The function $x^5-5x^4+5x^3-1$ is

(A) Maximum at $x=3$ and minimum at $x=1$
(B) Minimum at $x=1$
(C) Neither maximum nor minimum at $x=0$
(D) Maximum at $x=0$

88. The adjacent sides of a rectangle with given perimeter as 100 cm and enclosing maximum area are

(A) 10 cm and 40 cm **(B)** 20 cm and 30 cm
(C) 25 cm and 25 cm **(D)** 15 cm and 35 cm

89. The area of a rectangle will be maximum for the given perimeter, when rectangle is a

(A) Parallelogram **(B)** Trapezium
(C) Square **(D)** None of these

90. When 36 factorises into two factors in such a way that the sum of factors is minimum, then the factors are

(A) 2, 18 **(B)** 9, 4
(C) 3, 12 **(D)** None of these

91. If $f(x)=2x^3-3x^2-12x+5$ and $x\in[-2,4]$, then the maximum value of function is at the following value of *x*

(A) 2 **(B)** –1 **(C)** –2 **(D)** 4

92. The point for the curve $y=xe^x$

(A) $x=-1$ is minimum **(B)** $x=0$ is minimum
(C) $x=-1$ is maximum **(D)** $x=0$ is maximum

93. The maximum value of $(1/x)^x$ is

(A) $(e)^e$ **(B)** $(e)^e$ **(C)** $(e)^{-e}$ **(D)** $(1/e)^e$

94. The number that exceeds its square by the greatest amount is

(A) –1 **(B)** 0 **(C)** 1/2 **(D)** 1

95. If for a function $f(x), f'(a)=0, f''(a)=0, f'''(a)>0$, then at $x=a$, $f(x)$ is

(A) Minimum **(B)** Maximum
(C) Not an extreme point **(D)** Extreme point

96. The least value of the sum of any positive real number and its reciprocal is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

97. x^x has a stationary point at

(A) $x=e$ **(B)** $x=1/e$
(C) $x=1$ **(D)** $x=\sqrt{e}$

98. When *x* is positive, the minimum value of x^x is

(A) e^{-1} **(B)** $e^{-1/e}$ **(C)** $e^{1/e}$ **(D)** e^e

99. The maximum value of *xy* subject to $x+y=8$ is

(A) 8 **(B)** 16 **(C)** 20 **(D)** 24

100. A minimum value of $\int_0^x te^{-t^2}\,dt$ is

(A) 1 **(B)** 2 **(C)** 3 **(D)** 0

101. If the sum of two numbers is 3, then the maximum value of the product of the first number and the square of the second number is

(A) 4 **(B)** 3 **(C)** 2 **(D)** 1

102. Maximum value of $x(1-x)^2$, when $0\le x\le 2$, is

(A) 2/27 **(B)** 4/27 **(C)** 5 **(D)** 0

103. If from a wire of length 36 m, a rectangle of greatest area is made, then its two adjacent sides (in metre) are

(A) 6, 12 **(B)** 9, 9 **(C)** 10, 8 **(D)** 13, 5

104. The minimum value of $2x^2+x-1$ is

(A) $-1/4$ **(B)** $3/2$ **(C)** $-9/8$ **(D)** $\frac{9}{4}$

105. The minimum value of the function $y=2x^3-21x^2+36x-20$ is

(A) –128 **(B)** –126
(C) –120 **(D)** None of these

106. The sum of two non–zero numbers is 4. The minimum value of the sum of their reciprocals is

(A) 3/4 **(B)** 6/5
(C) 1 **(D)** None of these

107. The minimum value of $[(5+x)(2+x)]/[1+x]$ for non-negative real *x* is

(A) 12 **(B)** 1 **(C)** 9 **(D)** 8

108. One maximum point of $\sin^p x\cos^q x$ is

(A) $x=\tan^{-1}\sqrt{(p/q)}$ **(B)** $x=\tan^{-1}\sqrt{(q/p)}$
(C) $x=\tan^{-1}(p/q)$ **(D)** $x=\tan^{-1}(q/p)$

109. When 20 is divided into two parts so that the product of the cube of one quantity and the square of the other quantity is maximum. The parts are

(A) 10, 10 **(B)** 16, 4 **(C)** 8, 12 **D)** 12, 8

110. If $f(x)=(x^2-1)/(x^2+1)$, for every real number *x*, then the minimum value of *f*

(A) does not exist because *f* is unbounded
(B) is not attained even though *f* is bounded
(C) is equal to 1
(D) is equal to –1

111. The number of values of *x* where the function $f(x)=\cos x+\cos(\sqrt{2}x)$ attains its maximum is

(A) 0 **(B)** 1 **(C)** 2 **(D)** Infinite

112. The minimum value of $e^{(2x^2-2x+1)\sin^2 x}$ is

(A) e **(B)** $1/e$ **(C)** 1 **(D)** 0

113. If *x* and *y* be two variables such that $x>0$ and $xy=1$, then the minimum value of $x+y$ is

(A) 2 **(B)** 3 **(C)** 4 **(D)** 0

114. $x/(1+x\tan x)$ is maxima at

(A) $x=\sin x$ **(B)** $x=\cos x$
(C) $x=\pi/3$ **(D)** $x=\tan x$

115. If *x* is real, then the greatest and the least values of $(x^2-x+1)/(x^2+x+1)$ are

(A) $3, -\frac{1}{2}$ **(B)** $3, \frac{1}{3}$

(C) $-3, -\frac{1}{3}$ **(D)** None of these

116. The minimum value of $\frac{\log x}{x}$ in the interval $[2, \infty)$ is

(A) $\frac{\log 2}{2}$ **(B)** Zero

(C) $\frac{1}{e}$ **(D)** Does not exist

117. The maximum value of $x^4 e^{-x^2}$ is

(A) e^2 **(B)** e^{-2}

(C) $12e^{-2}$ **(D)** $4e^{-2}$

118. If $A + B = \pi/2$, the maximum value of $\cos A \cos B$ is

(A) 1/2 **(B)** 3/4

(C) 1 **(D)** 4/3

119. A cone of maximum volume is inscribed in a given sphere, then ratio of the height of the cone to diameter of the sphere is

(A) 2/3 **(B)** 3/4

(C) 1/3 **(D)** 1/4

120. The ratio of height of cone of maximum volume inscribed in a sphere to its radius is

(A) 3/4 **(B)** 4/3

(C) 1/2 **(D)** 2/3

121. The function $f(x) = x + \sin x$ has

(A) A minimum but no maximum
(B) A maximum but no minimum
(C) Neither maximum nor minimum
(D) Both maximum and minimum

122. The function $f(x) = ax + (b/x)$; $a, b, x > 0$ takes on the least value at x equal to

(A) B **(B)** $\sqrt{a}$ **(C)** $\sqrt{b}$ **(D)** $\sqrt{b/a}$

123. If $xy = c^2$, then the minimum value of $ax + by$ is

(A) $c\sqrt{ab}$ **(B)** $2c\sqrt{ab}$

(C) $-c\sqrt{ab}$ **(D)** $-2c\sqrt{ab}$

124. If $a^2x^4 + b^2y^4 = c^6$, then the maximum value of xy is

(A) $c^2/\sqrt{ab}$ **(B)** c^3/ab

(C) $c^3/\sqrt{2ab}$ **(D)** $c^3/2ab$

125. The function $f(x) = 2x^3 - 15x^2 + 36x + 4$ is maximum at

(A) $x = 2$ **(B)** $x = 4$

(C) $x = 0$ **(D)** $x = 3$

126. The maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is

(A) 0 **(B)** 12 **(C)** 16 **(D)** 32

127. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$ has

(A) No maxima and minima
(B) One maximum and one minimum
(C) Two maxima
(D) Two minima

128. If $f(x) = 1/(4x^2 + 2x + 1)$, then its maximum value is

(A) 4/3 **(B)** 2/3 **(C)** 1 **(D)** 3/4

129. If $f(x) = x + \frac{1}{x}$, $x > 0$, then its greatest value is

(A) -2 **(B)** 0

(C) 3 **(D)** None of these

130. The perimeter of a sector is p. The area of the sector is maximum when its radius is

(A) $\sqrt{p}$ **(B)** $1/\sqrt{p}$ **(C)** $p/2$ **(D)** $p/4$

131. If $y = a \log x + bx^2 + x$ has its extremum value at $x = 1$ and $x = 2$, then what is (a, b)?

(A) $\left(1, \frac{1}{2}\right)$ **(B)** $\left(\frac{1}{2}, 2\right)$

(C) $\left(2, \frac{-1}{2}\right)$ **(D)** $\left(\frac{-2}{3}, \frac{-1}{6}\right)$

132. In $(-4, 4)$ the function $f(x) = \int_{-10}^{x} (t^4 - 4)e^{-4t} dt$ has

(A) No extrema **(B)** One extremum

(C) Two extrema **(D)** Four extrema

133. On $[1, e]$, the greatest value of $x^2 \log x$ is

(A) e^2 **(B)** $\frac{1}{e} \log \frac{1}{\sqrt{e}}$

(C) $e^2 \log \sqrt{e}$ **(D)** None of these

134. At what value of x, the function $f(x) = x^{-x}$, $(x \in R)$ attains a maximum value?

(A) 2 **(B)** 3 **(C)** $1/e$ **(D)** 1

135. If $ab = 2a + 3b, a > 0, b > 0$ then the minimum value of ab is

(A) 12 **(B)** 24

(C) 1/4 **(D)** None of these

136. If PQ and PR are the two sides of a triangle, then the angle between them which gives maximum area of the triangle is

(A) π **(B)** $/3$ **(C)** $\pi/4$ **(D)** $\pi/2$

137. For what value of x, the function $y = a(1 - \cos x)$ is maximum?

(A) π **(B)** $\pi/2$

(C) $-\pi/2$ **(D)** $-\pi/6$

138. The minimum value of $[x^2 + (250/x)]$ is

(A) 75 **(B)** 50

(C) 25 **(D)** 55

139. The maximum value of $x^{1/x}$ is

(A) $1/e$ **(B)** $e^{1/e}$

(C) E **(D)** $1/e^e$

140. The minimum value of $4e^{2x}+9e^{-2x}$ is

(A) 11 **(B)** 12
(C) 10 **(D)** 14

141. The point $(0,5)$ is closest to the curve $x^2=2y$ at

(A) $(2\sqrt{2},0)$ **(B)** (0, 0)
(C) $(2,2)$ **(D)** None of these

142. The maximum value of xy when $x+2y=8$ is

(A) 20 **(B)** 16 **(C)** 24 **(D)** 8

143. The minimum value of P(1, 1) is

(A) $15/2$ **(B)** $11/2$ **(C)** $-13/2$ **(D)** $71/8$

144. If $P=(1,1)$, $Q=(3,2)$ and R is a point on x-axis, then the value of $PR+RQ$ will be minimum at

(A) $\left(\frac{5}{3},0\right)$ **(B)** $\left(\frac{1}{3},0\right)$ **(C)** (3, 0) **(D)** (1, 0)

145. Let $f(x)=1+2x^2+2^2x^4+\cdots+2^{10}x^{20}$. Then $f(x)$ has

(A) More than one minimum
(B) Exactly one minimum
(C) At least one maximum
(D) None of these

146. For which interval, the function $(x^2-3x)/(x-1)$ satisfies all the conditions of Rolle's theorem?

(A) [0, 3] **(B)** [– 3, 0]
(C) [1.5, 3] **(D)** For no interval

147. For the function $f(x)=e^x, a=0, b=1$, the value of c in mean value theorem will be

(A) $\log x$ **(B)** $\log(e-1)$ **(C)** 0 **(D)** 1

148. Rolle's theorem is not applicable to the function $f(x)=|x|$ defined on [–1, 1] because

(A) f is not continuous on $[-1, 1]$
(B) f is not differentiable on $(-1,1)$
(C) $f(-1)\neq f(1)$
(D) $f(-1)=f(1)\neq 0$

149. If $f(x)=\cos x, 0\le x\le(\pi/2)$, then the real number c of the mean value theorem is

(A) $\pi/6$ **(B)** $\pi/4$
(C) $\sin^{-1}(2/\pi)$ **(D)** $\cos^{-1}(2/\pi)$

150. From mean value theorem $f(b)-f(a)=(b-a)f'(x_1)$; $a<x_1<b$ if $f(x)=1/x$, then what is the value of x_1?

(A) $\sqrt{ab}$ **(B)** $(a+b)/2$
(C) $2ab/(a+b)$ **(D)** $(b-a)/(b+a)$

151. For the function $x+(1/x)$, $x\in[1,3]$, the value of c for the mean value theorem is

(A) 1 **(B)** $\sqrt{3}$
(C) 2 **(D)** None of these

152. If from mean value theorem, $f'(x_1)=\dfrac{f(b)-f(a)}{b-a}$, then

(A) $a<x_1\le b$ **(B)** $a\le x_1<b$
(C) $a<x_1<b$ **(D)** $a\le x_1\le b$

153. Consider the function $f(x)=e^{-2x}\sin 2x$ over the interval $\left(0,\frac{\pi}{2}\right)$. A real number $c\in\left(0,\frac{\pi}{2}\right)$, as guaranteed by Rolle's theorem, such that $f'(c)=0$ is

(A) $\pi/8$ **(B)** $\pi/6$
(C) $\pi/4$ **(D)** $\pi/3$

154. Let $f(x)=\sqrt{x-1}+\sqrt{x+24-10\sqrt{x-1}}$; $1<x<26$ be real-valued function. Then $f'(x)$ for $1<x<26$ is

(A) 0 **(B)** $1/\sqrt{x-1}$
(C) $2\sqrt{x-1}-5$ **(D)** None of these

155. If $f(x)$ satisfies the conditions of Rolle's theorem in [1,2] and $f(x)$ is continuous in [1, 2], then $\int_1^2 f'(x)dx$ is equal to

(A) 3 **(B)** 0 **(C)** 1 **(D)** 2

156. If the function $f(x)=x^3-6x^2+ax+b$ satisfies Rolle's theorem in the interval $[1,3]$ and $f'\left(\frac{2\sqrt{3}+1}{\sqrt{3}}\right)=0$, then

(A) $a=-11$ **(B)** $a=-6$
(C) $a=6$ **(D)** $a=11$

157. In mean value theorem $\dfrac{f(b)-f(a)}{b-a}=f'(c)$, if $a=0, b=\frac{1}{2}$ and $f(x)=x(x-1)(x-2)$, the value of c is

(A) $1-(\sqrt{15}/6)$ **(B)** $1+\sqrt{15}$
(C) $1-(\sqrt{21}/6)$ **(D)** $1+\sqrt{21}$

158. The abscissa of the points of the curve $y=x^3$ in the interval [–2, 2], where the slope of the tangents can be obtained by mean value theorem for the interval [–2, 2], are

(A) $\pm(2/\sqrt{3})$ **(B)** $\pm\sqrt{3}$
(C) $\pm(\sqrt{3}/2)$ **(D)** 0

159. In mean value theorem, $f(b)-f(a)=(b-a)f'(c)$ if $a=4$, $b=9$ and $f(x)=\sqrt{x}$, the value of c is

(A) 8.00 **(B)** 5.25 **(C)** 4.00 **(D)** 6.25

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. For the curve represented parametrically by the equations, $x=2\ln\cot t+1$ and $y=\tan t+\cot t$

(A) tangent at $t=\pi/4$ is parallel to x-axis
(B) normal at $t=\pi/4$ is parallel to y-axis
(C) tangent at $t=\pi/4$ is parallel to the line $y=x$
(D) tangent and normal intersect at the point (2, 1)

2. Let $g'(x)>0$ and $f'(x)<0$, $\forall x\in R$. Then

(A) $g(f(x+1))>g(f(x-1))$
(B) $f(g(x-1))>f(g(x+1))$
(C) $g(f(x+1))<g(f(x-1))$
(D) $g(g(x+1))<g(g(x-1))$

3. If $f(x) = x^3 - x^2 + 100x + 1001$, then

(A) $f(2000) > f(2001)$ **(B)** $f\left(\frac{1}{1999}\right) > f\left(\frac{1}{2000}\right)$

(C) $f(x+1) > f(x-1)$ **(D)** $f(3x-5) > f(3x)$

4. The abscissa of a point on the curve $xy = (a + x)2$, the normal at which the cuts off numerically equal intercepts from the coordinate axes is

(A) $-\frac{a}{\sqrt{2}}$ **(B)** $\sqrt{2}a$

(C) $\frac{\sqrt{2}a}{2}$ **(D)** $-\sqrt{2}a$

5. For the function $f(x) = \frac{\ln x}{x}$, which of the following statements are true?

(A) $f(x)$ has the horizontal tangent at $x = e$

(B) $f(x)$ cuts the x–axis only at the one point

(C) $f(x)$ is the many–one function

(D) $f(x)$ has the one vertical tangent

6. The equation of the tangent drawn to the curve $y = (x + 1)^3$ from the origin is

(A) $y = 3x$ **(B)** $y = -3x$

(C) $4y = 27x$ **(D)** $y = 0$

7. If the derivative of an odd cubic polynomial vanishes at two different values of 'x', then

(A) the coefficient of x^3 and x in the polynomial must be the same in sign

(B) the coefficient of x^3 and x in the polynomial must be the different in sign

(C) the values of 'x' where derivative vanishes are closer to the origin as compared to the respective roots on the either side of origin.

(D) the values of 'x' where derivative vanishes are far from the origin as compared to the respective roots on the either side of the origin.

8. Let $f(x) = (x-1)^4 (x-2)^n$, $n \in N$. Then $f(x)$ has

(A) Local minimum at $x = 2$ if n is even

(B) Local minimum at $x = 1$ if n is odd

(C) Local maximum at $x = 1$ if n is odd

(D) Local minimum at $x = 1$ if n is even

Comprehension Type Questions

Paragraph for Questions 9–11: Let $a(t)$ is a function of t such that $\frac{da}{dt} = 2$ for all the values of t and $a = 0$ when $t = 0$. Further $y = m(t)x + c(t)$ is the tangent to the curve $y = x^2 - 2ax + a^2 + a$ at the point whose abscissa is 0. Then

9. If the rate of change of the distance of the vertex of $y = x^2 - 2ax + a^2 + a$ from the origin with respect to t is k, then $k =$

(A) 2 **(B)** $2\sqrt{2}$ **(C)** $\sqrt{2}$ **(D)** $4\sqrt{2}$

10. If the rate of change of $c(t)$ with respect to t, when $t = k$, is ℓ, then

(A) $16 - 2\sqrt{2}$ **(B)** $8\sqrt{2} + 2$

(C) $10\sqrt{2} + 2$ **(D)** $16\sqrt{2} + 2$

11. The rate of change of $m(t)$, with respect to t, at $t = \ell$ is

(A) –2 **(B)** 2 **(C)** – 4 **(D)** 4

Paragraph for Questions 12–14: If a continuous function f defined on the real line R, assumes positive and negative values in R then the equation $f(x) = 0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in R.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

12. The line $y = x$ meets $y = ke^x$ for $k \le 0$ at

(A) no point **(B)** one point

(C) two points **(D)** more than two points

13. The positive value of k for which $ke^x - x = 0$ has only one root is

(A) $1/e$ **(B)** 1

(C) e **(D)** $\log_e 2$

14. For $k > 0$, the set of all the values of 'k' for which $ke^x - x = 0$ has two distinct roots is

(A) $\left(0, \frac{1}{e}\right)$ **(B)** $\left(\frac{1}{e}, 1\right)$ **(C)** $\left(\frac{1}{e}, \infty\right)$ **(D)** $(0, 1)$

Paragraph for Questions 15–17: Let f and g are the two functions such that $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b). Then at least one $c \in (a, b)$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(i) If $f(a) = f(b)$, then $f'(c) = 0$ (RMVT)

(ii) If $f(a) \ne f(b)$ and $a \ne b$ (LMVT)

(iii) If $g'(x) \ne 0$, then $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ (Cauchy's theorem)

15. The set of the values of k, for which the equation $x^3 - 3x + k = 0$ has two distinct roots in $(0, 1)$ is

(A) $(1, 4)$ **(B)** $(0, \infty)$ **(C)** $(0, 1)$ **(D)** ϕ

16. Which of the following is true?

(A) $|\tan^{-1}x - \tan^{-1}y| \le |x - y|\ \forall\, x, y \in R$

(B) $|\tan^{-1}x - \tan^{-1}y| \ge |x - y|\ \forall\, x, y \in R$

(C) $|\sin x - \sin y| \ge |x - y|\ \forall\, x, y \in R$

(D) None of these

17. If $0 < \alpha < \theta < \beta < \frac{\pi}{2}$, then $\frac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta}$ is equal to

(A) $\tan\theta$ **(B)** $-\tan\theta$ **(C)** $\cot\theta$ **(D)** $-\cot\theta$

Paragraph for Questions 18–20: Let $y = a\sqrt{x} + bx$ be curve, $(2x - y) + \lambda(2x + y - 4) = 0$ be family of lines.

18. If the curve has slope $-\frac{1}{2}$ at $(9, 0)$, then a tangent belonging to the family of lines is

(A) $x + 2y - 5 = 0$ **(B)** $x - 2y + 3 = 0$

(C) $3x - y - 1 = 0$ **(D)** $3x + y - 5 = 0$

19. A line of the family cutting positive intercepts on the axes and forming the triangle with the coordinate axes, then the minimum length of the line segment between the axes is

(A) $(2^{2/3} - 1)^{3/2}$ **(B)** $(2^{2/3} + 1)^{3/2}$

(C) $7^{3/2}$ **(D)** 27

20. Two perpendicular focal chords of the curve $y^2 - 4x - 4y + 4 = 0$ form the diagonals of a quadrilateral. Minimum area of a quadrilateral is

(A) 16 **(B)** 32
(C) 64 **(D)** 50

Paragraph for Questions 21–23: A function $f(x)$ having the following properties:

(i) $f(x)$ is the continuous except at $x = 3$

(ii) $f(x)$ is the differentiable except at $x = -2$ and $x = 3$

(iii) $f(0) = 0$, $\lim_{x\to 3} f(x) \to -\infty$, $\lim_{x\to -\infty} f(x) = 3$, $\lim_{x\to\infty} f(x) = 0$

(iv) $f'(x) > 0\ \forall\, x \in (-\infty, -2) \cup (3, \infty)$ and $f'(x) \le 0\ \forall\, x \in (-2, 3)$

(v) $f''(x) > 0\ \forall\, x \in (-\infty, -2) \cup (-2, 0)$ and $f''(x) < 0\ \forall\, x \in (0, 3) \cup (3, \infty)$

Then, answer the following questions:

21. Maximum possible number of solutions of $f(x) = |x|$ is

(A) 2 **(B)** 1 **(C)** 3 **(D)** 4

22. Graph of function $y = f(-|x|)$ is

(A) differentiable for all x, if $f'(0) = 0$

(B) continuous but not differentiable at two points, if $f'(0) = 0$

(C) continuous but not differentiable at one points, if $f'(0) = 0$

(D) discontinuous at two points, if $f'(0) = 0$

23. $f(x) + 3x = 0$ has five solutions if

(A) $f(-2) > 6$ **(B)** $f'(0) < -3$ and $f(-2) > 6$
(C) $f'(0) > -3$ **(D)** $f'(0) > -3$ and $f(-2) > 6$

Matrix Match Type Questions

24. Match the following:

List I	List II
(A) Circular plate is expanded by the heat from the radius 5 cm to 5.06 cm. Approximate increase in the area is	**(p)** 2
(B) If an edge of a cube increases by 1%, then the percentage increase in the volume is	**(q)** 0.6π
(C) If the rate of decrease of $y = \frac{x^2}{2} - 2x + 5$ is twice the rate of decrease of x, then x is equal to (given that the rate of decrease is non–zero)	**(r)** 3
(D) Rate of increase in the area of the equilateral triangle of side 15 cm, when each side is increasing at the rate of 0.1 cm/sec; is	**(s)** $\frac{3\sqrt{3}}{4}$
	(t) 4

25. Match the following:

List I	List II
(A) If a portion of the tangent at any point on the curve $x = at^3$, $y = at^4$ between the axes is divided by the point of contact in the ratio $m:n$ externally, then $\lvert n + m \rvert$ is equal to (m and n are coprime)	**(p)** 1
(B) The area of the triangle formed by the normal at the point (1, 0) on the curve $x = e^{\sin y}$ with the axes is	**(q)** $\frac{1}{2}$
(C) If the angle between the curves $x^2y = 1$ and $y = e^{2(1-x)}$ at the point (1, 1) is θ, then $\tan\theta$ is equal to	**(r)** 7
(D) The length of the sub–tangent at any point on the curve $y = be^{x/3}$ is equal to	**(s)** 3
	(t) 0

26. Match the following:

List I	List II
(A) A function f is differentiable in [0, 5] and $f(0) = 4$ and $f(5) = -1$. If $g(x) = \frac{f(x)}{x+1}$ and $c \in (0, 5)$, then $g'(c)$ is equal to	**(p)** 3
(B) Let $f(x)$ and $g(x)$ be differentiable for $0 \le x \le 1$, $f(0) = 2$, $g(0) = 0$, $f(1) = 6$. Let there exists a the real number $c \in (0, 1)$ such that $f'(c) = 2g'(c)$. Then $g(1)$ is equal to	**(q)** $-5/6$
(C) The length of the longest interval in which $f(x) = 3\sin x - 4\sin^3 x$ is increasing, is $\frac{\pi}{\lambda}$, then λ is	**(r)** 15
(D) If Lagrange's mean value theorem is satisfied for $f(x) = \sqrt{25 - x^2}$ and $c \in (1, 5)$, then the value of c^2 is	**(s)** 2
	(t) 10

27. Match the following:

List I	List II
(A) Number of values of 'x' lying in $\left(0, \frac{\pi}{2}\right)$, at which $f(x) = \ln(\sin x)$ is not monotonic, is	**(p)** 0
(B) If the greatest interval in which the function $f(x) = x^3 - 3x + 2$ is decreasing is $[a, b]$, then $a + b =$	**(q)** 2
(C) If $f(x) = \frac{x^2 + 2}{[x]}$, $1 \le x \le 3$ (where [·] greatest integer function), then the least value of $f(x)$ is	**(r)** -3
(D) Set of all possible values of 'a' such that $f(x) = e^2x - (a + 1)e^x + 2x$ is monotonically increasing for all $x \in R$ is $(-\infty, k]$, then k eqvals	**(s)** 3
	(t) -2

28. Match the following:

List I	List II
(A) Number of points which are the local extrema of $f(x) = \begin{cases} (2+x)^3; & -3 \le x \le -1 \\ x^{2/3}; & -1 < x < 2 \end{cases}$	**(p)** 1

List I	List II
(B) If $a+b=1; a>0, b>0$, then the minimum value of $\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}$ is	**(q)** 2
(C) The maximum value attained by $y=10-\|x-10\|, -1\le x\le 3$, is	**(r)** 3
(D) If $P(t^2, 2t)$, $t\in[0, 2]$ is an arbitrary point on the parabola $y^2=4x$ and Q is foot of the perpendicular from focus S on the tangent at P, then maximum area of the triangle PQS is	**(s)** 4
	(t) 5

Integer Type Questions

29. Let α be the angle in radians between $\frac{x^2}{36}+\frac{y^2}{4}=1$ and the circle $x^2+y^2=12$ at their points of the intersection. If $\alpha=\tan^{-1}\frac{k}{2\sqrt{3}}$, then find the value of k^2.

30. Find the minimum value of $(x_1-x_2)^2+\left(\sqrt{2-x_1^2}-\frac{9}{x_2}\right)^2$, where $x_1\in(0,\sqrt{2})$ and $x_2\in R^+$.

31. The values of 'a' for which the function $f(x)=\sin x-a\sin^2 x-\frac{1}{3}\sin 3x+2ax$ increases throughout the number line is $[\lambda,\infty)$, then find λ.

32. A cone is made from a circular sheet of the radius $\sqrt{3}$ by cutting out a sector and keeping the cut edges of the remaining piece together. The maximum volume attainable for the cone is $\frac{\lambda\pi}{3}$, then find λ.

33. If the possible values of 'a' such that the inequality $3-x^2>|x-a|$ has at least one negative solution is $a\in\left(-\frac{13}{4},\lambda\right)$, then find λ.

34. Let $f(x)=\begin{cases} xe^{ax} & , x\le 0\\ x+ax^2-x^3, & x>0\end{cases}$, where a is positive constant and the interval in which $f'(x)$ is increasing is $\left(-\frac{\lambda_1}{a},\frac{a}{\lambda_2}\right)$, then find $(\lambda_1+\lambda_2)$.

35. A cubic $f(x)$ vanishes at $x=-2$ and has relative minimum/maximum at $x=-1$ and $x=\frac{1}{3}$. If $\int_{-1}^{1} f(x)\,dx=\frac{14}{3}$, the cubic $f(x)=\lambda_1x^3+\lambda_2x^2-x+2$, then find $(\lambda_1+\lambda_2)$.

36. If $f(x)$ is a twice differentiable function such that $f(a)=0, f(b)=2, f(c)=-1, f(d)=2, f(e)=0$, where $a<b<c<d<e$, then the minimum number of the zeroes of $g(x)=(f'(x))^2+f''(x)f(x)$ in the interval $[a, e]$ is ______.

37. The chord of the parabola $y=-p^2x^2+5px-4$ touches the curve $y=\frac{1}{(1-x)}$ at the point $x=2$ and is bisected by that point. Find the number of the values of 'p'.

Answer Key

Practice Exercise 1

1. (D)	**2.** (C)	**3.** (B)	**4.** (C)	**5.** (A)	**6.** (A)
7. (B)	**8.** (C)	**9.** (C)	**10.** (A)	**11.** (B)	**12.** (C)
13. (A)	**14.** (D)	**15.** (C)	**16.** (C)	**17.** (C)	**18.** (A)
19. (A)	**20.** (D)	**21.** (D)	**22.** (B)	**23.** (D)	**24.** (B)
25. (C)	**26.** (C)	**27.** (B)	**28.** (A)	**29.** (D)	**30.** (C)
31. (B)	**32.** (D)	**33.** (A)	**34.** (A)	**35.** (A)	**36.** (D)
37. (A)	**38.** (C)	**39.** (B)	**40.** (D)	**41.** (B)	**42.** (C)
43. (B)	**44.** (D)	**45.** (A)	**46.** (C)	**47.** (B)	**48.** (C)
49. (B)	**50.** (A)	**51.** (A)	**52.** (D)	**53.** (A)	**54.** (C)
55. (A)	**56.** (C)	**57.** (D)	**58.** (C)	**59.** (C), (D)	**60.** (D)
61. (A)	**62.** (B)	**63.** (C)	**64.** (D)	**65.** (A)	**66.** (D)
67. (B)	**68.** (D)	**69.** (B)	**70.** (A)	**71.** (B)	**72.** (B)
73. (A)	**74.** (C)	**75.** (D)	**76.** (A)	**77.** (C)	**78.** (A)
79. (D)	**80.** (A)	**81.** (B)	**82.** (D)	**83.** (C)	**84.** (A)
85. (C)	**86.** (C)	**87.** (C)	**88.** (C)	**89.** (C)	**90.** (D)
91. (D)	**92.** (A)	**93.** (C)	**94.** (C)	**95.** (C)	**96.** (B)
97. (B)	**98.** (B)	**99.** (B)	**100.** (D)	**101.** (A)	**102.** (B)

103. (B)	**104.** (C)	**105.** (A)	**106.** (C)	**107.** (C)	**108.** (A)
109. (D)	**110.** (D)	**111.** (A)	**112.** (C)	**113.** (A)	**114.** (B)
115. (B)	**116.** (D)	**117.** (D)	**118.** (A)	**119.** (A)	**120.** (B)
121. (C)	**122.** (D)	**123.** (B)	**124.** (C)	**125.** (A)	**126.** (B)
127. (B)	**128.** (A)	**129.** (D)	**130.** (D)	**131.** (D)	**132.** (C)
133. (A)	**134.** (C)	**135.** (B)	**136.** (D)	**137.** (A)	**138.** (A)
139. (B)	**140.** (B)	**141.** (D)	**142.** (D)	**143.** (D)	**144.** (A)
145. (D)	**146.** (D)	**147.** (B)	**148.** (B)	**149.** (C)	**150.** (A)
151. (B)	**152.** (A)	**153.** (A)	**154.** (A)	**155.** (B)	**156.** (D)
157. (C)	**158.** (A)	**159.** (D)			

Practice Exercise 2

1. (A), (B) **2.** (B), (C) **3.** (B), (C) **4.** (A) **5.** (A), (B), (C) **6.** (C), (D)
7. (B), (C) **8.** (A), (C), (D) **9.** (B) **10.** (D) **11.** (C) **12.** (B)
13. (A) **14.** (A) **15.** (D) **16.** (A) **17.** (D) **18.** (B)
19. (B) **20.** (B) **21.** (C) **22.** (B) **23.** (D) **24.**(A)→(q),(B)→(r), (C)→ (t), (D)→ (s)
25. (A)→(r), (B)→ (q), (C)→(t), (D)→ (s) **26.** (A)→ (q), (B)→ (s), (C)→ (p), (D)→ (r)
27. (A) → (p), (B) → (p), (C) → (s), (D) →(s) **28.** (A) →(q), (B) →(r), (C) → (r), (D) → (t) **29.** 16 **30.** 8
31. 1 **32.** 2 **33.** 3 **34.** 5 **35.** 2 **36.** 6 **37.** 2

Solutions

Practice Exercise 1

1. We have

$$\frac{dy}{dx}=12-3x^2=0\Rightarrow x=\pm 2$$

Hence, the points are (2, 16) and (–2, –16).

2. We have

$$y=\frac{a^2}{x}$$

Therefore,

$$\frac{dy}{dx}=-\frac{a^2}{x^2}$$

Now, at (x_1, y_1).

At the point (x_1, y_1), we have

$$\frac{dy}{dx}=\frac{-a^2}{x_1^2}$$

Thus, the tangent to the curve is

$$y-y_1=\frac{-a^2}{x_1^2}(x-x_1)$$

$$\Rightarrow yx_1^2-y_1x_1^2=-a^2x+a^2x_1$$

$$\Rightarrow y'\frac{1}{\sqrt{1-[(2x)/(1+x^2)]^2}}\cdot\frac{2(1+x^2)-4x^2}{(1+x^2)^2} \qquad (\because x_1y_1=a^2)$$

This meets the x-axis where $y=0$. Therefore,

$$a^2x=2a^2x_1$$

or $$x=2x_1$$

Therefore, the point on x-axis is $(2x_1, 0)$. Now, the tangent meets y-axis where $x=0$. Since

$$x_1^2y=2a^2x_1$$

we have $$y=\frac{2a^2}{x_1}$$

So, the point on the y-axis is

$$\left(0,\frac{2a^2}{x_1}\right)$$

The required area is

$$\frac{1}{2}(2x_1)\left(\frac{2a^2}{x_1}\right)=2a^2$$

3. We have $t=2$ for the point (2,–1). Therefore, for $t=2$, we get

$$\frac{dy}{dx}=\frac{4t-2}{2t+3}=\frac{6}{7}$$

4. It is given that

$$y^2=2(x-3) \qquad (1)$$

Differentiating w.r.t. x, we get

$$2y\left(\frac{dy}{dx}\right)=2\Rightarrow\frac{dy}{dx}=\frac{1}{y}$$

The slope of the normal is

$$\frac{-1}{dy/dx}=-y$$

The slope of the given line is 2. Therefore,

$$y=-2$$

From Eq. (1), we get

$$x = 5$$

Thus, the required point is $(5, -2)$.

5. The given curve is

$$x^2 = 3 - 2y \quad (1)$$

Differentiating w.r.t. x, we get

$$2x = 0 - 2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -x$$

The slope of the tangent of the curve is $-x$. From the given line, the slope is -1 and hence $x = 1$ and from Eq. (1), $y = 1$. Therefore, the coordinate of the point is (1, 1).

6. We have $x = t^2$ and $y = 2t$. At $t = 1$, $x = 1$ and $y = 2$, we have

$$\left(\frac{dy}{dx}\right) = \frac{dy/dt}{dx/dt} = \frac{2}{2t} = \frac{1}{t} \Rightarrow \left(\frac{dy}{dx}\right)_{t=1} = 1$$

Therefore, the equation of the normal at (1, 2) is

$$y - 2 = -\left(\frac{1}{dy/dx}\right)(x-1)$$

$$\Rightarrow y - 2 = -1(x-1)$$

$$\Rightarrow x + y - 3 = 0$$

7. We have

$$y = \sin\frac{\pi x}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{2}\cos\frac{\pi}{2}x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,\,1)} = 0$$

Therefore, the equation of the normal is

$$y - 1 = \frac{1}{0}(x-1)$$

$$\Rightarrow x = 1$$

8. We have

$$y = 2\cos x$$

At $x = \pi/4$, $y = 2/\sqrt{2} = \sqrt{2}$ and therefore,

$$\frac{dy}{dx} = -2(\sin x)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\pi/4} = -\sqrt{2}$$

Therefore, the equation of the tangent at $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is

$$y - \sqrt{2} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

9. Let the point be (x_1, y_1). Therefore,

$$y_1 = be^{-x_1/a} \quad (1)$$

Also, the curve is

$$y = be^{-x/a}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b}{a}e^{-x/a}$$

Therefore,

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-b}{a}e^{-x_1/a} = \frac{-y_1}{a} \quad \text{[by Eq. (1)]}$$

Now, the equation of tangent of given curve at point (x_1, y_1) is

$$y - y_1 = \frac{-y_1}{a}(x - x_1) \Rightarrow \frac{x}{a} + \frac{y}{y_1} = \frac{x_1}{a} + 1$$

Comparing with $\frac{x}{a} + \frac{y}{b} = 1$, we get $y_1 = b$ and

$$1 + \frac{x_1}{a} = 1 \Rightarrow x_1 = 0$$

Hence, the point is (0, b).

10. For the curve

$$y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{4}{2y}$$

we have

$$\left.\left(\frac{dy}{dx}\right)\right|_{(1,2)} = 1$$

and for the curve

$$x^2 + y^2 = 5 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Therefore,

$$\left.\left(\frac{dy}{dx}\right)\right|_{(1,2)} = \frac{-1}{2}$$

The angle between the curves is

$$\theta = \tan^{-1}\left|\frac{(-1/2) - 1}{1 + (-1/2)}\right| = \tan^{-1}(3)$$

11. We have

$$by^2 = (x+a)^3$$

$$\Rightarrow 2by\left(\frac{dy}{dx}\right) = 3(x+a)^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{2by}(x+a)^2$$

Therefore, the subnormal is

$$y\frac{dy}{dx} = \frac{3}{2b}(x+a)^2$$

Therefore, the subtangent is

$$\frac{y}{dy/dx} = \frac{y}{3(x+a)^2/2by} = \frac{2by^2}{3(x+a)^2}$$

$$= \frac{2b[(x+a)^3/b]}{3(x+a)^2} = \frac{2}{3}(x+a)$$

or $\quad (\text{Subtangent})^2 = \frac{4}{9}(x+a)^2$

Also $\frac{(\text{Subtangent})^2}{\text{Subnormal}} = \frac{(4/9)(x+a)^2}{(3/2b)(x+a)^2} = \frac{8b}{27}$

$\Rightarrow (\text{Subtangent})^2 = \text{Constant} \times (\text{Subnormal})$

Therefore, $(\text{Subtangent})^2 \propto (\text{Subnormal})$

12. Now,

$$y = ax^2 + bx$$

That is,

$$\frac{dy}{dx}=2ax+b \Rightarrow \left(\frac{dy}{dx}\right)_{(2,-8)}=4a+b$$

Since the tangent is parallel to x–axis, we have

$$\frac{dy}{dx}=0 \Rightarrow b=-4a \qquad (1)$$

Now, the point (2, –8) is on the curve of $y=ax^2+bx$. Therefore,

$$-8=4a+2b \qquad (2)$$

From Eqs. (1) and (2), we get $a=2, b=-8$.

13. We have

$$\sqrt{x}+\sqrt{y}=\sqrt{a};\ \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

Therefore,

$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

Hence, the tangent at (x, y) is

$$Y-y=-\frac{\sqrt{y}}{\sqrt{x}}(X-x)$$

or $$X\sqrt{y}+Y\sqrt{x}=\sqrt{xy}\ (\sqrt{x}+\sqrt{y})=\sqrt{axy}$$

or $$\frac{X}{\sqrt{a}\sqrt{x}}+\frac{Y}{\sqrt{a}\sqrt{y}}=1$$

Clearly, its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

Sum of the intercepts is

$$\sqrt{a}(\sqrt{x}+\sqrt{y})=\sqrt{a}\cdot\sqrt{a}=a$$

14. We have

$$y=x\log x \Rightarrow \frac{dy}{dx}=1+\log x$$

The slope of the normal is

$$-\frac{1}{(dy/dx)}=\frac{-1}{1+\log x}$$

The slope of the line $2x-2y=3$ is 1. Therefore,

$$\frac{-1}{1+\log x}=1 \Rightarrow \log x=-2 \Rightarrow x=e^{-2}$$

$$\Rightarrow y=-2e^{-2}$$

Therefore, the coordinate of the point is $(e^{-2},-2e^{-2})$.

15. The slope of the normal is

$$-\frac{1}{(dy/dx)}$$

This is parallel to x–axis. Therefore,

$$-\frac{1}{(dy/dx)}=0 \Rightarrow \frac{dx}{dy}=0$$

16. The length of the normal is

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Now,

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1+\cos\theta)}=\frac{\sin\theta}{1+\cos\theta}=\frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

Therefore,

$$\left(\frac{dy}{dx}\right)_{\left(\theta=\frac{\pi}{2}\right)}=\left[\tan\frac{\theta}{2}\right]_{\left(\theta=\frac{\pi}{2}\right)}=1[y]_{\left(\theta=\frac{\pi}{2}\right)}=a\left(1-\cos\frac{\pi}{2}\right)=a$$

Hence, the length of the normal is

$$a\sqrt{1+(1)^2}=\sqrt{2}a$$

17. We have

$$y=a(\sin\theta-\theta\cos\theta), x=a(\cos\theta+\theta\sin\theta)$$

Therefore,

$$\frac{dy}{d\theta}=a[\cos\theta-\cos\theta+\theta\sin\theta]=a\theta\sin\theta$$

$$\frac{dx}{d\theta}=a(-\sin\theta+\sin\theta+\theta\cos\theta)=a\theta\cos\theta$$

That is,

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\theta\sin\theta}{a\theta\cos\theta}=\tan\theta$$

Therefore, slope of the tangent is $\tan\theta$ and the slope of the normal is $-\cot\theta$. Hence, the equation of normal is

$$[y-a\sin\theta+a\theta\cos\theta]=-\frac{\cos\theta}{\sin\theta}[x-a\cos\theta-a\theta\sin\theta]$$

$$\Rightarrow y\sin\theta-a\sin^2\theta+a\theta\sin\theta\cos\theta$$

$$=-x\cos\theta+a\cos^2\theta+a\theta\sin\theta\cos\theta$$

$$\Rightarrow x\cos\theta+y\sin\theta=a(\sin^2\theta+\cos^2\theta)$$

$$\Rightarrow x\cos\theta+y\sin\theta=a$$

Therefore, the distance from the origin is

$$\frac{a}{\sqrt{\sin^2\theta+\cos^2\theta}}=\text{Constant}$$

18. We have

$$x=3t^2+1, y=t^3-1$$

Therefore,

$$\frac{dy}{dt}=3t^2$$

Now, $$\frac{dy}{dx}=\left(\frac{dy/dt}{dx/dt}\right)=\frac{3t^2}{6t}=\frac{t}{2}$$

For $x=1$,

$$3t^2+1=1 \Rightarrow t=0 \Rightarrow \text{Slope}=\frac{0}{2}=0$$

19. Let the point of contact be (h, k), where $k=h^4$. The tangent is

$$y-k=4h^3(x-h) \qquad \left[\text{As}\ \frac{dy}{dx}=4x^3\right]$$

It passes through (2, 0). Therefore,

$$-k=4h^3(2-h)$$

$$\Rightarrow h = 0 \text{ or } \frac{8}{3}$$

$$\Rightarrow k = 0 \text{ or } \left(\frac{8}{3}\right)^4$$

Since points of contact are (0, 0) and $\left(\frac{8}{3}, \left(\frac{8}{3}\right)^4\right)$.

The equation of tangents are

$$y = 0 \text{ and } y - \left(\frac{8}{3}\right)^4 = 4\left(\frac{8}{3}\right)^3\left(x - \frac{8}{3}\right)$$

20. We have

$$y = x^2$$

$$\Rightarrow \frac{dy}{dx} = m_1 = 2x$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 2 = m_1 \text{ and } x = y^2$$

$$\Rightarrow 1 = 2y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = m_2 = \frac{1}{2y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{1}{2}$$

Therefore, the angle of intersection is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{2 - (1/2)}{1 + 2 \times (1/2)} = \frac{3}{4}$$

$$\Rightarrow \theta = \tan^{-1}(3/4)$$

21. We have

$$y = x^3 - 3x^2 - 9x + 5$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 6x - 9$$

We know that this equation gives the slope of the tangent to the curve. The tangent is parallel to the x-axis. Therefore,

$$\frac{dy}{dx} = 0$$

Therefore,

$$3x^2 - 6x - 9 = 0$$

$$\Rightarrow x = -1, 3$$

22. Clearly, the point of intersection of curves is (0, 1). Now, the slope of tangent of the first curve is

$$m_1 = \frac{dy}{dx} = a^x \log a$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = m_1 = \log a$$

The slope of the tangent of the second curve is

$$m_2 = \frac{dy}{dx} = b^x \log b$$

$$\Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(0,1)} = \log b$$

Therefore,

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\log a - \log b}{1 + \log a \log b}$$

23. We have

$$x^2 = -4y$$

$$\Rightarrow 2x = -4\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(-4,-4)} = 2$$

We know that the equation of tangent is

$$(y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x - x_1) \Rightarrow y + 4 = 2(x + 4)$$

$$\Rightarrow 2x - y + 4 = 0$$

24. We have

$$y = 2x^2 - x + 1 \Rightarrow \frac{dy}{dx} = 4x - 1$$

We know that this equation gives the slope of tangent to the curve. Since this tangent is parallel to $y = 3x + 9$, the slope of the tangent is 3 and so $4x - 1 = 3$ or $x = 1$. Therefore $y = 2x^2 - x + 1 = 2 - 1 + 1 = 2$. Thus, the point (x, y) is (1, 2).

25. We have

$$x^3 - 8a^2 y = 0$$

$$\Rightarrow 3x^2 - 8a^2\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow 3x^2 = 8a^2\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{8a^2}$$

Therefore, the slope of the normal is

$$-\frac{1}{(dy/dx)} = -\frac{1}{3x^2/8a^2} = -\frac{8a^2}{3x^2}$$

Given that

$$\frac{-8a^2}{3x^2} = \frac{-2}{3}$$

Therefore,

$$(x, y) = (2a, a)$$

26. We have

$$x = a(t + \sin t), \quad y = a(1 - \cos t)$$

Therefore,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(\sin t)}{a(1+\cos t)} = \tan\frac{t}{2}$$

The length of the normal is

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= a(1-\cos t)\sqrt{1+\tan^2(t/2)} = a(1-\cos t)\sec(t/2)$$
$$= 2a\sin^2(t/2)\sec(t/2) = 2a\sin(t/2)\tan(t/2)$$

27. We have

$$y = e^{2x}$$

$$\Rightarrow \frac{dy}{dx} = 2e^{2x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = 2$$

Therefore, the equation of tangent is

$$y - 1 = 2(x - 0) \Rightarrow y = 2x + 1$$

This tangent meets x–axis. Therefore,

$$y = 0$$

$$\Rightarrow 0 = 2x + 1 \Rightarrow x = -1/2$$

Therefore, the coordinates of the point are $\left(-\frac{1}{2}, 0\right)$.

28. We have

$$(1+x^2)y = 2 - x \qquad (1)$$

It meets x-axis, where $y = 0$. That is,

$$0 = 2 - x \Rightarrow x = 2$$

So, Eq. (1) meets x-axis at the point (2, 0). Also from Eq. (1),

$$y = \frac{2-x}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x^2)(-1)-(2-x)(2x)}{(1+x^2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2-4x-1}{(1+x^2)^2}$$

The slope of tangent at (2, 0) is

$$\frac{4-8-1}{(1+4)^2} = \frac{-5}{25} = \frac{-1}{5}$$

Therefore, the equation of the tangent at (2, 0) is

$$y - 0 = -\frac{1}{5}(x-2) \Rightarrow x + 5y = 2$$

29. The curve is $y = be^{-x/a}$.

Since the curve crosses y–axis (i.e. $x = 0$), $y = b$. Now,

$$\frac{dy}{dx} = \frac{-b}{a}e^{-x/a}$$

At point (0, b), we have

$$\left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a}$$

Therefore, the equation of tangent is

$$y - b = \frac{-b}{a}(x - 0)$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

30. We have

$$y = x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = m_1 = 2x = 2$$

and $$6y = 7 - x^3 \Rightarrow 6\cdot\frac{dy}{dx} = -3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = m_2 = -\frac{1}{2}$$

Clearly, $m_1 m_2 = -1$. Therefore, the angle of intersection is $\pi/2$.

31. We have

$$y = 2x^2 - x + 1$$

Let the coordinates of P be (h, k). Then

$$\left(\frac{dy}{dx}\right)_{(h,\,k)} = 4h - 1$$

Clearly, P is parallel to $y = 3x + 4$. Therefore, slopes are equal

$$4h - 1 = 3 \Rightarrow h = 1$$

Therefore, P is (1, 2).

32. We have

$$xy = c^2 \qquad (1)$$

Subnormal is $y(dy/dx)$. Therefore, from Eq. (1), we get

$$y = \frac{c^2}{x} \Rightarrow \frac{dy}{dx} = \frac{-c^2}{x^2}$$

Thus, the subnormal is

$$\frac{y(-c^2)}{x^2} = \frac{-yc^2}{(c^2/y)^2} = \frac{-yc^2y^2}{c^4} = \frac{-y^3}{c^2}$$

Therefore, the subnormal varies as y^3.

33. If $\sin x = \cos x$, then $x = \pi/4$.

If $y = \sin x$, then $\left(\frac{dy}{dx}\right)_{x=\pi/4} = \frac{1}{\sqrt{2}}$.

If $y = \cos x$, then $\left(\frac{dy}{dx}\right)_{x=\pi/4} = \frac{-1}{\sqrt{2}}$.

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = 2\sqrt{2} \Rightarrow \theta = \tan^{-1}(2\sqrt{2})$$

34. We have

$$y^2 = 5x - 1 \qquad (1)$$

At $(1,-2)$, we have

$$\frac{dy}{dx}=\left[\frac{5}{2y}\right]_{(1,-2)}=\frac{-5}{4}$$

Therefore, the equation of normal at the point (1, –2) is

$$[y-(-2)]\left[\frac{-5}{4}\right]+x-1=0$$

Therefore,

$$4x-5y-14=0 \qquad (2)$$

As the normal is of the form $ax-5y+b=0$ and on comparing this with Eq. (2), we get $a=4$ and $b=-14$.

35. Given that

$$y=6x-x^2 \qquad (1)$$

$$\frac{dy}{dx}=6-2x$$

Since, the tangent is parallel to the line

$$4x-2y-1=0$$

Therefore,

$$\frac{dy}{dx}=6-2x=\frac{-4}{-2}\Rightarrow 6-2x=2\Rightarrow x=2$$

Substituting the value of x in Eq. (1), we get $y=8$. Hence, the required point of tangency will be $(2,8)$.

36. The slope of the normal is

$$\frac{-dx}{dy}=\frac{-d[a(1+\cos\theta)]/d\theta}{d(a\sin\theta)/d\theta}=\frac{a\sin\theta}{a\cos\theta}=\tan\theta$$

Now, the equation of normal at θ is

$$y-a\sin\theta=\tan\theta\,[x-a(1+\cos\theta)]$$

Clearly, this line passes through $(a, 0)$.

37. We have

$$\frac{dx}{d\theta}=a(1+\cos\theta),\ \frac{dy}{d\theta}=a(\sin\theta)$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/2}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1+\cos\theta)}=1=\left.y\right|_{\theta=\pi/2}=a$$

The length of the subtangent is

$$ST=\frac{y}{dy/dx}=\frac{a}{1}=a$$

and the length of the subnormal is

$$SN=y\frac{dy}{dx}=a(1)=a$$

Hence, $ST=SN$

38. We have

$$\left.x\right|_{\theta=\pi/4}=\frac{2}{2\sqrt{2}}=\frac{1}{\sqrt{2}}$$

$$\left.y\right|_{\theta=\pi/4}=\frac{3}{2\sqrt{2}},\ \left.\frac{dy}{dx}\right|_{\theta=\pi/4}\left.\frac{9\sin^2\theta\cos\theta}{-6\cos^2\theta\sin\theta}\right|_{\theta=\pi/4}=\frac{-3}{2}$$

Therefore, the equation of the tangent is

$$\left(y-\frac{3}{2\sqrt{2}}\right)=\frac{-3}{2}\left(x-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow 3\sqrt{2}x+2\sqrt{2}y=6$$

$$\Rightarrow 3x+2y=3\sqrt{2}$$

39. The curve is

$$x+y=e^{xy}$$

Differentiating with respect to x, we get

$$1+\frac{dy}{dx}=e^{xy}\left(y+x\frac{dy}{dx}\right)$$

or

$$\frac{dy}{dx}=\frac{ye^{xy}-1}{1-xe^{xy}}$$

As tangent is parallel to y–axis.

Thus,

$$\frac{dy}{dx}=\infty\Rightarrow 1-xe^{xy}=0\Rightarrow 1-x(x+y)=0$$

This holds for $x=1,\ y=0$.

40. We have

$$f(x)=-2x^3-9x^2-12x+1$$

$$\Rightarrow f'(x)=-6x^2-18x-12$$

To be decreasing, we need to have $f'(x)<0$, that is,

$$-6x^2-18x-12<0$$

$$\Rightarrow x^2+3x+2>0$$

$$\Rightarrow (x+2)(x+1)>0$$

Therefore, either $x<-2$ or $x>-1$. So,

$$x\in(-1,\infty)\ \text{or}\ (-\infty,-2)$$

41. To be increasing, we need to have

$$f'(x)=3x^2-27>0$$

$$\Rightarrow x^2>9\Rightarrow |x|>3$$

42. We have

$$f(x)=\sin x-\frac{x}{2}\Rightarrow f'(x)=\cos x-\frac{1}{2}$$

We know that $f'(x)>0$ for increasing function. Obviously, it is increasing for

$$-\frac{\pi}{3}<x<\frac{\pi}{3}$$

43. We have

$$f(x)=x\sin x+\cos x+\cos^2 x$$

Therefore,

$$f'(x)=\sin x+x\cos x-\sin x-2\cos x\sin x\ =\cos x(x-2\sin x)$$

Hence, $x\to 0$ to π, then $f'(x)<0$. That is, $f(x)$ is decreasing function.

44. We know that

$$\frac{dy}{dx}=\frac{-1}{e^x}x(x-2)$$

is positive when $x(x-2)$ is negative, that is, x lies in the interval (0, 2). An exponential function, as we know, is always positive.

45. Here,

$$f(x) = y = 2x^3 - 9x^2 + 12x - 6$$

$$\Rightarrow f'(x) = 6x^2 - 18x + 12$$

Since $f(x)$ is increasing or decreasing in (a, b) according as $f'(x) > 0$ or < 0 for every $x \in (a, b)$. Hence, $f'(x) = 6(x-2)(x-1)$ which is obviously decreasing if $x \in (1, 2)$, that is,

$$1 < x < 2$$

46. We have

$$f(x) = (x-1)^2 - 1$$

Hence, decreasing in $x < 1$ (Fig. 21.30).

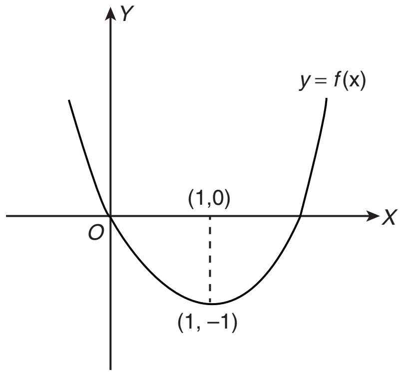


Figure 21.30

Aliter: $f'(x) = 2x - 2 = 2(x-1)$.

To be decreasing, we need to have

$$2(x-1) < 0 \Rightarrow (x-1) < 0 \Rightarrow x < 1$$

47. We know that $f(x)$ is monotonically decreasing if $f'(x) < 0$.

$$f'(x) = -\sin x - 2p < 0 \Rightarrow \frac{1}{2}\sin x + p > 0$$

$$\Rightarrow p > \frac{1}{2} \qquad [\because -1 \le \sin x \le 1].$$

48. We have

$$f'(x) = 3kx^2 - 18x + 9 = 3\,[kx^2 - 6x + 3] > 0,\ \forall x \in R$$

Therefore,

$$\Delta = b^2 - 4ac < 0,\ k > 0$$

That is,

$$36 - 12k < 0 \text{ or } k > 3$$

49. We have

$$y = f(x) = 2x^3 - 15x^2 + 36x + 1$$

Therefore,

$$\frac{dy}{dx} = f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$$

or $\qquad f'(x) = 6(x-2)(x-3)$

To be monotonically decreasing, we need to have $f'(x) < 0$, that is,

$$\Rightarrow (x-2)(x-3) < 0 \Rightarrow x \in (2, 3)$$

50. We have

$$y = \tan x - x \Rightarrow \frac{dy}{dx} = \sec^2 x - 1 = \tan^2 x \ge 0$$

51. We have

$$f(x) = \log(1+x) - \frac{2x}{2+x}$$

$$\Rightarrow f'(x) = \frac{1}{1+x} - \frac{(2+x)\cdot(2) - (2x)}{(2+x)^2}$$

$$\Rightarrow f'(x) = \frac{x^2}{(x+1)(x+2)^2}$$

Obviously,

$$f'(x) > 0 \text{ for all } x > 0$$

Hence, $f(x)$ is increasing on $(0, \infty)$.

52. If $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \le 0$ for all $x \in R$.

$$3(a+2)x^2 - 6ax + 9a \le 0 \text{ for all } x \in R$$

$$\Rightarrow (a+2)x^2 - 2ax + 3a \le 0 \text{ for all } x \in R$$

$$\Rightarrow a + 2 < 0 \text{ and discriminant } \le 0$$

$$\Rightarrow a < -2,\ -8a^2 - 24a \le 0 \Rightarrow a < -2 \text{ and } a(a+3) \ge 0$$

$$\Rightarrow a < -2,\ a \le -3 \text{ or } a \ge 0 \Rightarrow a \le -3 \Rightarrow -\infty < a \le -3$$

53. We have

$$f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$$

Therefore,

$$f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x}\left(\frac{x}{\sqrt{1+x^2}} - 1\right)$$

$$= \frac{1+2x^2}{1+x^2} - \frac{1}{\sqrt{1+x^2}} = \frac{1+2x^2}{1+x^2} - \frac{\sqrt{(1+x^2)}}{1+x^2}$$

$$= \frac{x^2 + \sqrt{1+x^2}\left(\sqrt{1+x^2} - 1\right)}{1+x^2} \ge 0 \text{ for all } x$$

Hence, $f(x)$ is an increasing function on $(-\infty, \infty)$ and in particular on $[0, \infty)$.

54. Since $f(x) = x^3 \Rightarrow f'(x) = 3x^2$, which is non–negative for all real values of x.

55. We have

$$f(x) = x^3 - 3x^2 - 24x + 5$$

For increasing function, we need to have $f'(x) > 0$, that is,

$$3x^2 - 6x - 24 > 0$$

$$\Rightarrow x^2 - 2x - 8 > 0$$

That is,

$$x^2 - 4x + 2x - 8 > 0 \Rightarrow (x+2)(x-4) > 0$$

$$x \in (-\infty, -2) \cup (4, \infty)$$

56. Since

$$f(x) = \sin 2x \Rightarrow f'(x) = 2\cos 2x$$

obviously,

$$f'(x) > 0 \text{ in } \left(0, \frac{\pi}{4}\right)$$

and $$f'(x) < 0 \text{ in } \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

Hence, the result.

57. We have
$$f(x) = (x+2)e^{-x}$$
$$f'(x) = e^{-x} - e^{-x}(x+2)$$
$$f'(x) = -e^{-x}[x+1]$$

For increasing function,
$$-e^{-x}(x+1) > 0 \text{ or } e^{-x}(x+1) < 0$$
$$e^{-x} > 0 \text{ or } (x+1) < 0$$
$$x \in (-\infty, \infty) \text{ and } x \in (-\infty, -1)$$

Therefore,
$$x \in (-\infty, -1)$$

Hence, the function is increasing in $(-\infty, -1)$.
For decreasing function,
$$-e^{-x}(x+1) < 0 \text{ or } e^{-x}(x+1) > 0,\ x \in (-1, \infty)$$

Hence, the function is decreasing in $(-1, \infty)$.

58. We have
$$f(x) = x^3 - 10x^2 + 200x - 10$$

That is,
$$f'(x) = 3x^2 - 20x + 200$$

For increasing function, we need to have
$$f'(x) > 0 \Rightarrow 3x^2 - 20x + 200 > 0$$
$$3\left[x^2 - \frac{20}{3}x + \frac{200}{3} + \frac{100}{9} - \frac{100}{9}\right] > 0$$
$$\Rightarrow 3\left[\left(x - \frac{10}{3}\right)^2 + \frac{500}{9}\right] > 0$$
$$\Rightarrow 3\left(x - \frac{10}{3}\right)^2 + \frac{500}{3} > 0$$

This is always increasing throughout the real line.

59. We have
$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = \frac{\cos x(\tan x - x)}{\sin^2 x}$$
$$0 < x \le 1 \Rightarrow x \in Q_1 \Rightarrow \tan x > x,\ \cos x > 0$$

Therefore,
$$f'(x) > 0 \text{ for } 0 < x \le 1$$

Thus, $f(x)$ is an increasing function. Now,
$$g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x} = \frac{\sin x\cos x - x}{\sin^2 x} = \frac{\sin 2x - 2x}{2\sin^2 x}$$
$$(\sin 2x - 2x)' = 2\cos 2x - 2 = 2[\cos 2x - 1] < 0$$
$$\Rightarrow \sin 2x - 2x \text{ is decreasing} \Rightarrow \sin 2x - 2x < 0$$

Therefore, $g'(x) < 0 \Rightarrow g(x)$ is decreasing.

60. A function is monotonically decreasing, when $f'(x) < 0$
$$6x^2 - 18x + 12 < 0$$
$$\Rightarrow x^2 - 3x + 2 < 0$$
$$\Rightarrow x^2 - 2x - x + 2 < 0$$
$$\Rightarrow (x-2)(x-1) < 0$$

Therefore, $x \in 1 < x < 2$.

61. We have $f'(x) = 6x^2 + 36x - 96 > 0$ for increasing function
$$f'(x) = (x+8)(x-2) \ge 0$$
$$\Rightarrow x \ge 2,\ x \le -8$$

62. Let us consider
$$y = \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$$

The function will be decreasing when $dy/dx < 0$.
$$\frac{(c\sin x + d\cos x)(a\cos x - b\sin x) - (a\sin x + b\cos x)(c\cos x - d\sin x)}{(c\sin x + d\cos x)^2} < 0$$
$$\Rightarrow ac\sin x\cos x - bc\sin^2 x + ad\cos^2 x$$
$$-bd\sin x\cos x - ac\sin x\cos x + ad\sin^2 x$$
$$-bc\cos^2 x + bd\sin x\cos x < 0$$
$$\Rightarrow ad(\sin^2 x + \cos^2 x) - bc(\sin^2 x + \cos^2 x) < 0$$
$$\Rightarrow (ad - bc) < 0$$

63. We have
$$f(x) = 1 - e^{-x^2/2}$$

Therefore,
$$f'(x) = -e^{-x^2/2}(-x) = xe^{-x^2/2}$$

For $f(x)$ to be increasing, then $f'(x) > 0$. So,
$$xe^{-x^2/2} > 0 \Rightarrow x > 0$$

and $f(x)$ to be decreasing for $x < 0$.

64. See Fig. 21.31. From the trend of value of $\sin x$ and $\cos x$, we know $\sin x$ and $\cos x$ decrease in $(\pi/2) < x < \pi$. So, statement S is correct.

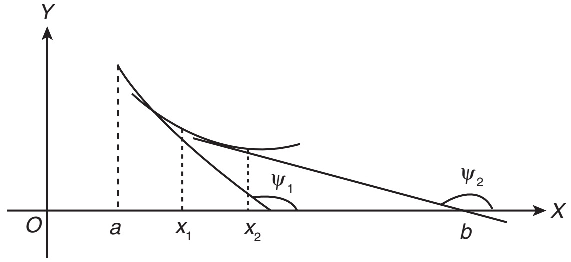


Figure 21.31

Statement R is incorrect which is depicted in the graph (Fig. 21.31). Clearly, $f(x)$ is differentiable in (a, b). Also, $a < x_1 < x_2 < b$. However,
$$f'(x_1) = \tan\theta_1 < \tan\theta_2 = f'(x_2)$$

65. The graph of cosec x is opposite in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ (Fig. 21.32).

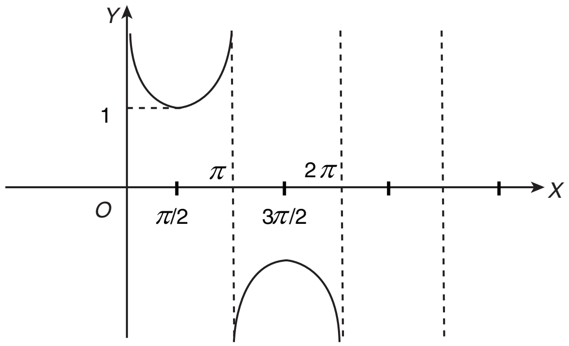


Figure 21.32

66. The function is monotonically increasing if

$$f'(x)>0$$

$$\Rightarrow \frac{(2\sin x+3\cos x)(\lambda\cos x-6\sin x)}{(2\sin x+3\cos x)^2} - \frac{(\lambda\sin x+6\cos x)(2\cos x-3\sin x)}{(2\sin x+3\cos x)^2}>0$$

$$\Rightarrow 3\lambda(\sin^2 x+\cos^2 x)-12(\sin^2 x+\cos^2 x)>0$$

$$\Rightarrow 3\lambda-12>0 \Rightarrow \lambda>4$$

67. We have

$$f(x)=3x+\frac{2}{x}$$

$$\Rightarrow f'(x)=3-\frac{2}{x^2}$$

Clearly, $f'(x)>0$ on the interval (1, 3); therefore, $f(x)$ is strictly increasing.

68. We have

$$f(x)=\sin x-\cos x$$

Therefore,

$$f'(x)=\cos x+\sin x=\sqrt{2}\left[\cos\left(x-\frac{\pi}{4}\right)\right] = \sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$$

For $f(x)$ to be decreasing,

$$f'(x)<0$$

$$\frac{\pi}{2}<\left(x-\frac{\pi}{4}\right)<\frac{3\pi}{2}\ (\text{within } 0\le x\le 2\pi)$$

$$\Rightarrow \frac{3\pi}{4}<x\le\frac{7\pi}{4}$$

69. We have

$$f(x)=\frac{\log x}{x}$$

Therefore,

$$f'(x)=\frac{1}{x^2}-\frac{\log x}{x^2}=\frac{1-\log x}{x^2}$$

For $f(x)$ to be increasing,

$$f'(x)>0$$

$$\Rightarrow 1-\log x>0$$

$$\Rightarrow 1>\log x$$

$$\Rightarrow e>x$$

Therefore, $f(x)$ is increasing in the interval $(0,e)$.

70. We have

$$f'(x)=e^{x(1-x)}+x(e^{x(1-x)})(1-2x)$$

$$=e^{x(1-x)}[1+x(1-2x)]=e^{x(1-x)}(-2x^2+x+1)$$

Now, by the sign–scheme (Fig. 21.33) for $-2x^2+x+1$, $f'(x)\ge 0$, if $x\in\left[-\frac{1}{2},1\right]$ because $e^x(1-x)$ is always positive.

So, $f(x)$ is increasing on $\left[-\frac{1}{2},1\right]$.

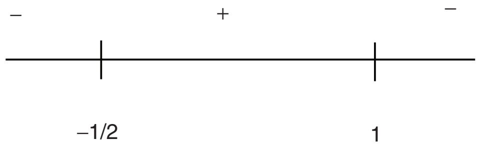


Figure 21.33

71. We have

$$f(x)=x^3-6x^2+9x+3$$

For the function to be decreasing, we need to have

$$f'(x)<0$$

$$\Rightarrow 3x^2-12x+9<0$$

$$\Rightarrow x^2-4x+3<0$$

$$\Rightarrow (x-3)\,(x-1)<0$$

Therefore, $x\in(1,3)$.

72. We have

$$f(x)=\frac{1}{x+1}-\log(1+x)$$

$$\Rightarrow f'(x)=-\frac{1}{(x+1)^2}-\frac{1}{1+x}$$

That is,

$$f'(x)=-\left[\frac{1}{x+1}+\frac{1}{(x+1)^2}\right]$$

Now, $f'(x)$ is negative when $x>0$ or $f'(x)<0$, $\forall x>0$. Therefore, $f(x)$ is decreasing function.

73. We have

$$f(x)=x+\cos x$$

$$\Rightarrow f'(x)=1-\sin x$$

Now, $f'(x)>0$ for all values of x. Therefore, $f(x)$ is an increasing function.

74. Let us consider

$$y=x^{1/x}$$

Taking log both sides, we have

$$\Rightarrow \log y=\left(\frac{1}{x}\right)\log x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{\log x}{x^2} = \frac{1-\log x}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = x^{1/x}\left(\frac{1-\log x}{x^2}\right)$$

Now, $x^{1/x} > 0$ for all x and $\frac{1-\log x}{x^2} > 0$ in (1, e) and $\frac{1-\log x}{x^2} < 0$ in (e,∞). Therefore, $f(x)$ is increasing in (1, e) and decreasing in (e,∞).

75. We have

$$f(x) = 1 - x^3 - x^5$$

$$\Rightarrow f'(x) = -3x^2 - 5x^4$$

That is, $f'(x) < 0$ for all values of x.

76. Let us consider $y = x^x$. Then

$$\frac{dy}{dx} = x^x(1+\log x)$$

For $(dy/dx) > 0$;

$$x^x(1+\log x) > 0$$

$$\Rightarrow 1 + \log x > 0 \Rightarrow \log_e x > \log_e \frac{1}{e}$$

For this to be positive, x should be greater than $1/e$.

77. Let us consider $f(x) = 2x^3 - 6x + 5$. Then

$$f'(x) = 6x^2 - 6 > 0$$
$$\Rightarrow x^2 - 1 > 0$$
$$\Rightarrow (x-1)(x+1) > 0$$
$$\Rightarrow x > 1 \text{ or } x < -1$$

78. We have

$$3\sin x - 4\sin^3 x = \sin 3x$$

which is increasing when $-\pi/2 \le 3x \le \pi/2$, that is, $-\pi/6 \le x \le \pi/6$.

The length of the interval is

$$\left|\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right| = \frac{\pi}{3}$$

79. Given that

$$f(x) = x^3 + bx^2 + cx + d$$

Therefore,

$$f'(x) = 3x^2 + 2bx + c$$

Now, its discriminant is $4(b^2 - 3c)$, so

$$4(b^2 - c) - 8c < 0$$

since $b^2 < c$ and $c > 0$. Therefore, $f'(x) > 0$ for all $x \in R$.

Hence, f is strictly increasing.

80. It is always increasing (Fig. 21.34).

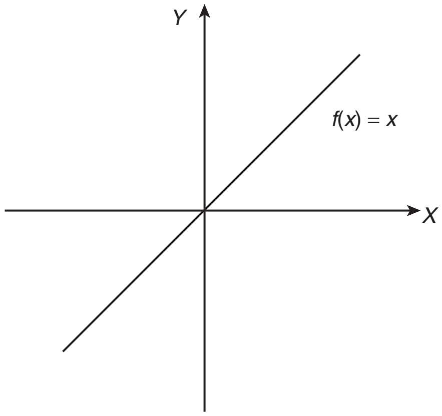


Figure 21.34

81. Both e^x and $1 + x$ are increasing and $\sqrt{e} \ge 1 + (1/2)$, because $\sqrt{e} = 1.65$ (approximately) and so option (A) is not correct. Since

$$\sin\frac{\pi}{6} < \frac{\pi}{6} \qquad \left(\because \frac{1}{2} < \frac{22}{42}\right)$$

option (C) is not correct. Now,

$$\log\frac{1}{2} < \frac{1}{2} \qquad \left(\because \log\frac{1}{2} \text{ is negative}\right)$$

Therefore, Option (D) is not correct. Thus, by elimination, option (B) is correct.

82. $f'(x) = 6(x^2 - x + 15) > 0\,\forall x$.

83. We have

$$f(x) = y = \tan^{-1}\left(\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right)$$

$$\Rightarrow \tan y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \Rightarrow \sec^2 y\frac{dy}{dx} = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$$

Now, $\frac{dy}{dx} > 0 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) > 0$

Therefore, $x \in \left(0, \frac{\pi}{4}\right)$.

84. We have

$$f(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\Rightarrow f(-x) = \frac{e^{-2x} - 1}{e^{-2x} + 1} = \frac{1 - e^{2x}}{1 + e^{2x}}$$

$$\Rightarrow f(x) = -\frac{e^{2x} - 1}{e^{2x} + 1} = -f(x)$$

Thus, $f(x)$ is an odd function. Now,

$$f(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\Rightarrow f'(x) = \frac{4e^{2x}}{(1 + e^{2x})^2} > 0\,\forall n \in R$$

which implies that $f(x)$ is an increasing function.

85. Obviously, it has a maximum at $x = 1$.

86. Let us consider

$$f(x) = \frac{\log x}{x}$$
$$\Rightarrow f'(x) = \frac{1}{x^2} - \frac{\log x}{x^2}$$

For maximum or minimum value of $f(x)$

$$f'(x) = 0$$
$$\Rightarrow f'(x) = \frac{1 - \log_e x}{x^2} = 0$$

or
$$\frac{1 - \log_e x}{x^2} = 0$$

Therefore, $\log_e x = 1$ or $x = e$, which lie in $(0, \infty)$. For $x = e$,

$$\frac{d^2y}{dx^2} = -\frac{1}{e^3}$$

which is negative. Hence, y is maximum at $x = e$ and its maximum value is

$$\frac{\log e}{e} = \frac{1}{e}$$

87. Let us consider

$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$
$$\Rightarrow f'(x) = 5x^4 - 20x^3 + 15x^2 = 0$$

Therefore,

$$(x-3)(x-1) = 0 \text{ or } x = 3, 1$$

Now, $f''(x) = 20x^3 - 60x^2 + 30x$

Substituting $x = 3$ and 1, we get $f''(3)$ is positive, $f''(1)$ is negative and $f''(0) = 0$. Hence, $f(x)$ is neither maximum nor minimum at $x = 0$.

88. We have

$$2x + 2y = 100 \Rightarrow x + y = 50 \quad (1)$$

Let the area of rectangle be A. Therefore,

$$A = xy \Rightarrow y = \frac{A}{x}$$

Substituting this in Eq. (1), we have

$$x + \frac{A}{x} = 50$$
$$\Rightarrow A = 50x - x^2$$
$$\Rightarrow \frac{dA}{dx} = 50 - 2x$$

For maximum area, $dA/dx = 0$. Therefore,

$$50 - 2x = 0$$

Therefore, $x = 25$ and $y = 25$. Hence, the adjacent sides are 25 and 25 cm.

89. We know that the perimeter of a rectangle is

$$S = 2(x + y)$$

where x and y are adjacent sides.

$$\Rightarrow y = \frac{S - 2x}{2}$$

Now, the area of a rectangle is

$$A = xy = \frac{x}{2}(S - 2x) = \frac{1}{2}(Sx - 2x^2)$$

Differentiating A w.r.t. x, we get

$$\frac{dA}{dx} = \frac{1}{2}(S - 4x) = 0$$

Therefore,

$$x = \frac{S}{4} \text{ and } y = \frac{S}{4}$$

Now, d^2A/dx^2 is negative. Hence, the area of rectangle is maximum when the rectangle is a square.

90. Let the two factors of 36 be x and $36/x$. Now, solving $x + (36/x)$ to be minimum. The factors will be 6, 6.

91. We have

$$f'(x) = 6x^2 - 6x - 12$$

That is,

$$f'(x) = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = -1, 2$$

Here, $f(4) = 128 - 48 - 48 + 5 = 37$. That is,

$$f(-1) = -2 - 3 + 12 + 5 = 12$$
$$f(2) = 16 - 12 - 24 + 5 = -15$$
$$f(-2) = -16 - 12 + 24 + 5 = 1$$

Therefore, the maximum value of the function is 37 at $x = 4$.

92. Given equation [of the curve (Fig. 21.35)] is $y = xe^x$. Therefore,

$$\frac{dy}{dx} = xe^x + e^x = e^x(1 + x) \text{ and } \frac{d^2y}{dx^2} = (x+2)e^x$$

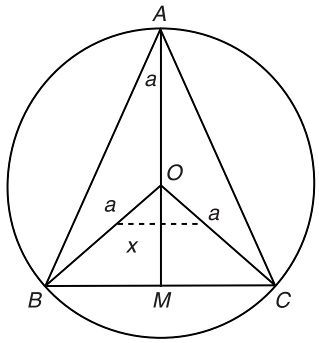


Figure 21.35

For the maximum or minimum value of $f(x)$, we have

$$\frac{dy}{dx} = 0 \Rightarrow x = -1$$

Therefore, $\{f''(x)\}_{x=-1}$ is positive. Hence, $f(x)$ is minimum at $x = -1$.

93. We have

$$f(x)=\left(\frac{1}{x}\right)^x$$

Since, $\left(\frac{1}{x}\right)^x = e^{x\log\left(\frac{1}{x}\right)}$

Thus, $\frac{dy}{du}=e^u$ where, $u=x\cdot\log\left(\frac{1}{x}\right)$

On solving this equation, we have

$$\Rightarrow f'(x)=\left(\frac{1}{x}\right)^x\left(\log\frac{1}{x}-1\right)$$

Now, $f'(x)=0\Rightarrow\log\frac{1}{x}=1=\log e\Rightarrow\frac{1}{x}=e\Rightarrow x=\frac{1}{e}$

Therefore, the maximum value of function is $e^{1/e}$.

94. Let the number be x. Then

$$y=x-x^2$$

$$\Rightarrow\frac{dy}{dx}=1-2x \text{ and } \frac{d^2y}{dx^2}=-2(<0)$$

$$\Rightarrow 1-2x=0\Rightarrow x=\frac{1}{2}$$

95. This is a fundamental property.

96. We have

$$f(x)=x+\frac{1}{x}\Rightarrow f'(x)=1-\frac{1}{x^2}$$

$$\Rightarrow f'(x)=0\Rightarrow x^2-1=0\Rightarrow x=1,-1$$

However, it is given that x is positive; hence, at $x=1$, we have

$$f(x)=1+\frac{1}{1}=2$$

97. Let us consider

$$y=x^x\Rightarrow\log y=x(\log x),\quad(x>0)$$

On differentiating, we get

$$\frac{dy}{dx}=x^x(1+\log x)$$

Therefore,

$$\frac{dy}{dx}=0$$

$$\Rightarrow\log x=-1\Rightarrow x=e^{-1}=\frac{1}{e}$$

Therefore, the stationary point is $x=1/e$.

98. We have,

$$\frac{d^2y}{dx^2}=x^x(1+\log x)^2+(x^x)\frac{1}{x}$$

When $x=1/e$

$$\frac{d^2y}{dx^2}=\left(\frac{1}{e}\right)^{(1/e)-1}>0$$

Therefore, y is minimum at $x=\frac{1}{e}$ and minimum value $=\left(\frac{1}{e}\right)^{1/e}=e^{-1/e}$.

99. We have

$$x+y=8$$

Therefore,

$$y=8-x \tag{1}$$

Now, $f(x)=xy=x(8-x)=8x-x^2$

Therefore,

$$f'(x)=8-2x$$

For maximum value of $f(x), f'(x)=0$. Therefore, $x=4$ and $y=4$ and hence the maximum value of xy is $4\times4=16$.

100. We have

$$f(x)=\int_0^x te^{-t^2}dt\Rightarrow f'(x)=xe^{-x^2}=0\Rightarrow x=0$$

$$f''(x)=e^{-x^2}(1-2x^2);\quad f''(0)=1>0$$

Therefore, the minimum value is $f(0)=0$.

101. Let the first number be $3-x$ and the second number be x. Accordingly, we have to maximize $(3-x)x^2$. Let us consider

$$f(x)=(3-x)x^2=3x^2-x^3\Rightarrow f'(x)=6x-3x^2$$

Therefore,

$$f'(x)=0\Rightarrow x=0,2$$

Also

$$f''(x)=6-6x$$

Obviously, $f''(2)=-6<0$. Therefore, the required maximum value is $(3-2)2^2=4$.

102. Given that

$$f(x)=x(1-x)^2$$

That is,

$$f(x)=x^3-2x^2+x$$

Now,

$$f'(x)=3x^2-4x+1$$

Substituting $f'(x)=0$, we have

$$3x^2-4x+1=0$$

$$3x^2-3x-x+1=0\Rightarrow x=1,1/3$$

Now, $f''(x)=6x-4$

Therefore, $f''(1)=2$ is positive and $f''(1/3)=-2$ is negative. Hence, the maximum value is $x=1/3$. The maximum is

$$f\left(\frac{1}{3}\right)=\frac{4}{27}$$

103. Given that

$$2(a+b)=36$$

$$a+b=18$$

The area of the rectangle is

$$ab=a(18-a)$$

Now, $A=18a-a^2$

That is,

$$\frac{dA}{da}=18-2a$$

Substituting $dA/da=0$, we get

$$18-2a=0 \Rightarrow a=9;\ b=9$$

104. We have

$$f(x)=2x^2+x-1$$

$$\Rightarrow f'(x)=4x+1 \Rightarrow f'(x)=0 \Rightarrow x=-\frac{1}{4}$$

Therefore, $f''(x)=4$ is positive. Therefore,

$$[f(-1/4)]_{\min}=\frac{2}{16}-\frac{1}{4}-1=\frac{-9}{8}$$

105. Given that

$$f(x)=2x^3-21x^2+36x-20$$

and $f'(x)=6x^2-42x+36$

On substituting, we get

$$f'(x)=0 \Rightarrow 6x^2-42x+36=0 \Rightarrow x^2-7x+6=0$$

$$\Rightarrow x^2-6x-x+6=0 \Rightarrow (x-1)(x-6)=0 \Rightarrow x=1,6$$

Now, $f''(x)=12x-42$

That is, $f''(1)=-30$ is negative and $f''(6)=30$ is positive. Hence, $x=6$ is the point of minima and the minimum value is

$$f(6)=2(6)^3-21(6)^2+36\times 6-20$$

$$f(6)=-128$$

106. Let us consider

$$x+y=4 \text{ or } y=4-x$$

That is,

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy} \text{ or } f(x)=\frac{4}{xy}=\frac{4}{x(4-x)}$$

Now, $f(x)=\dfrac{4}{4x-x^2}$

and $f'(x)=\dfrac{-4}{(4x-x^2)^2}(4-2x)$

Substituting $f'(x)=0$, we get $4-2x=0$. Therefore,

$$x=2 \text{ and } y=2$$

Therefore,

$$\min\left(\frac{1}{x}+\frac{1}{y}\right)=\frac{1}{2}+\frac{1}{2}=1$$

107. Given that

$$f(x)=\frac{[(5+x)(2+x)]}{[1+x]}$$

$$f(x)=1+\frac{4}{1+x}+(5+x)=(6+x)+\frac{4}{(1+x)}$$

$$\Rightarrow f'(x)=1-\frac{4}{(1+x)^2}=0;\ x^2+2x-3=0 \Rightarrow x=-3,1$$

Now, $f''(x)=\dfrac{8}{(1+x)^3}$

That is, $f''(-3)$ is negative and $f''(1)$ is positive. Hence, the minimum value at $x=1$ is

$$f(1)=\frac{(5+1)(2+1)}{(1+1)}=\frac{6\times 3}{2}=9.$$

108. Let us consider that

$$y=\sin^p x\,(\cos^q x)$$

Now, $\dfrac{dy}{dx}=p\sin^{p-1}x(\cos x)(\cos^q x)+q\cos^{q-1}x(-\sin x)\sin^p x$

$$\frac{dy}{dx}=p\sin^{p-1}x(\cos^{q+1}x)-q\cos^{q-1}x(\sin^{p+1}x)$$

Substituting $dy/dx=0$, we get

$$\tan^2 x=\frac{p}{q} \Rightarrow \tan x=\pm\sqrt{\frac{p}{q}}$$

Therefore, the point of maxima is

$$x=\tan^{-1}\sqrt{\frac{p}{q}}$$

109. Let us consider

$$x+y=20 \Rightarrow y=20-x$$

and $x^3(y^2)=z \Rightarrow z=x^3(y^2)$

$$z=x^3(20-x)^2 \Rightarrow z=400x^3+x^5-40x^4$$

Now, $\dfrac{dz}{dx}=1200x^2+5x^4-160x^3$

Now, $dz/dx=0$, then $x=12, 20.$ Also

$$\frac{d^2z}{dx^2}=2400x+16x^3-480x^2$$

and $\left(\dfrac{d^2z}{dx^2}\right)_{x=12}$

is negative. Hence, $x=12$ is the point of maxima. Therefore,

$$x=12,\ y=8$$

110. We have

$$f(x)=\frac{x^2-1}{x^2+1}=\frac{x^2+1-2}{x^2+1}=1-\frac{2}{x^2+1}$$

Therefore,

$$f(x)<1\ \forall x \text{ and } \geq -1 \qquad \left(\because \frac{2}{x^2+1}\leq 2\right)$$

Therefore,

$$-1 \le f(x) < 1$$

Hence, $f(x)$ has minimum value -1 and also there is no maximum value.

Aliter: We have

$$f'(x) = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

$$f'(x) = 0 \Rightarrow x = 0$$

Now, $$f''(x) = \frac{(x^2+1)^2 4 - 4x.2(x^2+1)2x}{(x^2+1)^4}$$

$$= \frac{(x^2+1)4 - 16x(x)}{(x^2+1)^3} = \frac{-12x^2+4}{(x^2+1)^3}$$

Therefore, $f''(0) > 0$ and there is only one critical point that has minima. Hence, $f(x)$ has the least value at $x = 0$.

$$f_{\min} = f(0) = \frac{-1}{1} = -1$$

111. We have

$$f(x) = \cos x + \cos(\sqrt{2}x)$$

$$f'(x) = -\sin x - \sqrt{2}\sin(\sqrt{2}x) = 0$$

Hence, $x = 0$ is the only solution.

$$f''(x) = -\cos x - 2\cos(\sqrt{2}x) < 0 \text{ at } x = 0$$

Hence, the maxima occurs at $x = 0$.

112. Given that

$$y = e^{(2x^2-2x+1)\sin^2 x}$$

For minima or maxima,

$$\frac{dy}{dx} = 0$$

Therefore,

$$e^{(2x^2-2x+1)\sin^2 x}[(4x-2)\sin^2 x + 2(2x^2-2x+1)\sin x \cos x] = 0$$

$$\Rightarrow [(4x-2)\sin^2 x + 2(2x^2-2x+1)\sin x \cos x] = 0$$

$$\Rightarrow 2\sin x[(2x-1)\sin x + (2x^2-2x+1)\cos x] = 0$$

$$\Rightarrow \sin x = 0$$

Therefore, y is minimum for $\sin x = 0$. Thus, the minimum value of y is

$$y = e^{(2x^2-2x+1)(0)} = e^0 = 1$$

113. We have

$$xy = 1 \Rightarrow y = \frac{1}{x}$$

and let $z = x + y$. Then

$$\frac{dz}{dx} = 1 - \frac{1}{x^2}$$

Now, $$\frac{dz}{dx} = 0 \Rightarrow 1 - \frac{1}{x^2} = 0$$

Thus,

$$x = -1, +1 \text{ and } \frac{d^2z}{dx^2} = \frac{2}{x^3}$$

Now, $$\left(\frac{d^2z}{dx^2}\right)_{x=1} = \frac{2}{1} = 2$$

which is positive. Hence, $x = 1$ is the point of minima and $x = 1$ and $y = 1$. Therefore, minimum value is

$$x + y = 2$$

114. If $x/(1 + x\tan x)$ is maxima, then its reciprocal

$$\frac{1 + x\tan x}{x}$$

is a minima. Let us consider

$$y = \frac{1 + x\tan x}{x} = \frac{1}{x} + \tan x$$

Therefore,

$$\frac{dy}{dx} = -\frac{1}{x^2} + \sec^2 x$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3} + 2\sec x \sec x \tan x$$

On substituting $dy/dx = 0$, we get

$$-\frac{1}{x^2} + \sec^2 x = 0$$

$$\Rightarrow \sec^2 x = \frac{1}{x^2}$$

$$\Rightarrow x^2 = \cos^2 x$$

$$\Rightarrow x = \cos x$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{2}{\cos^3 x} + 2\sec^2 x \tan x = 2\sec^2 x\,(\sec x + \tan x)$$

which is positive. At $x = \cos x$,

$$\frac{1 + x\tan x}{x}$$

is minimum and hence, $x/(1 + x\tan x)$ is maximum.

115. Let us consider

$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+x+1)(2x-1) - (x^2-x+1)(2x+1)}{(x^2+x+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2-2}{(x^2+x+1)^2} = 0 \Rightarrow 2x^2 - 2 = 0 \Rightarrow x = -1, +1$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{4(-x^3+3x+1)}{(x^2+x+1)^3}$$

At $x = -1$, $\frac{d^2y}{dx^2} < 0$, the function occupies the maximum value; therefore, $f(-1) = 3$ and at $x = 1$, $\frac{d^2y}{dx^2} > 0$, the function occupies the minimum value. Therefore,

$$f(1) = \frac{1}{3}$$

116. Let us consider

$$y=\frac{\log x}{x} \Rightarrow \frac{dy}{dx}=\frac{x(1/x)-\log x}{x^2}=\frac{1-\log x}{x^2}$$

Substituting $dy/dx=0$, we get

$$\frac{1-\log x}{x^2}=0$$

$$\Rightarrow 1-\log x=0 \Rightarrow x=e \text{ and } \frac{d^2y}{dx^2}=\frac{-3x+2x\log x}{x^4}$$

At $x=e$,

$$\frac{d^2y}{dx^2}=\frac{1}{-e^3}<0$$

Therefore, in $[2,\infty)$, the function $p^2=q$ is maximum and the minimum value does not exist.

117. We have

$$f(x)=x^4e^{-x^2} \Rightarrow f'(x)=4x^3e^{-x^2}+x^4e^{-x^2}(-2x)$$

For maximum value

$$f'(x)=0 \Rightarrow 4x^3e^{-x^2}-2x^5e^{-x^2}=0$$

$$\Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$$

Now,

$$f''(x)=12x^2e^{-x^2}+4x^2e^{-x^2}(-2x)-10x^4e^{-x^2}-2x^5e^{-x^2}(-2x)$$

$$\Rightarrow f''(\sqrt{2})=24e^{-2}-32e^{-2}-40e^{-2}+32e^{-2}$$

which is negative. Hence, $f(x)$ is maximum at $x=\sqrt{2}$ and therefore the maximum value is $4e^{-2}$.

118. Let us consider

$$f(A)=\cos A\cos B=\cos A\cos\left(\frac{\pi}{2}-A\right)=\cos A\sin A$$

Therefore,

$$f'(A)=\cos^2 A-\sin^2 A=\cos 2A$$

Now, $$f'(A)=0 \Rightarrow \cos 2A=0 \Rightarrow 2A=\frac{\pi}{2} \Rightarrow A=\frac{\pi}{4}$$

and $$f''(A)=-2\sin 2A=-2\sin\frac{\pi}{2}=-2 \quad \text{(negative)}$$

Hence, $f(A)$ is maximum at $\pi/4$. Therefore, maximum value is

$$\cos\frac{\pi}{4}\sin\frac{\pi}{4}=\frac{1}{2}$$

119. Let the diameter of the sphere be

$$AE=2r$$

Let radius of cone (Fig. 21.36) be x and height be y. Therefore, $AD=y$ since

$$BD^2=AD(DE)$$

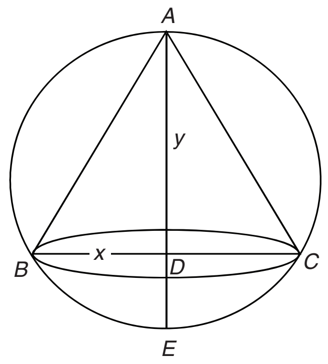


Figure 21.36

or $$x^2=y(2r-y) \qquad (1)$$

The volume of the cone is

$$V=\frac{1}{3}\pi x^2y=\frac{1}{3}\pi y(2r-y)y=\frac{1}{3}\pi(2ry^2-y^3)$$

$$\Rightarrow \frac{dV}{dy}=\frac{1}{3}\pi(4ry-3y^2) \Rightarrow \frac{dV}{dy}=0$$

$$\Rightarrow \frac{1}{3}\pi(4ry-3y^2)=0 \Rightarrow y(4r-3y)=0 \Rightarrow y=\frac{4}{3}r,0$$

Now, $$\frac{d^2V}{dy^2}=\frac{1}{3}\pi(4r-6y)$$

Substituting $y=(4/3)r$, we get

$$\frac{d^2V}{dy^2}=\frac{1}{3}\pi\left(4r-6\times\frac{4}{3}r\right)$$

which holds a negative value. So, the volume of the cone is maximum at

$$y=\frac{4}{3}r$$

$$\Rightarrow \frac{\text{Height}}{\text{Diameter}}=\frac{y}{2r}=\frac{2}{3}$$

120. See Fig. 21.37. Let the diameter of sphere be $AE=2r$. Let the radius of the cone be x and its height be y. Therefore, $AD=y$ since $BD^2=AD(DE)$.

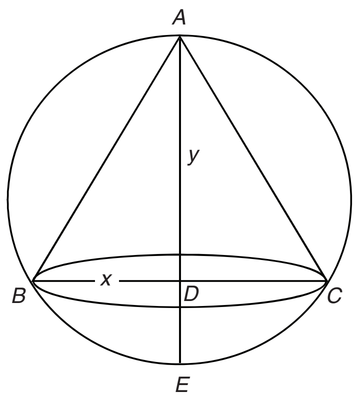


Figure 21.37

or $$x^2 = y(2r-y) \quad (1)$$

The volume of the cone is

$$V = \frac{1}{3}\pi x^2 y = \frac{1}{3}\pi y(2r-y)y = \frac{1}{3}\pi(2ry^2 - y^3)$$

$$\Rightarrow \frac{dV}{dy} = \frac{1}{3}\pi(4ry - 3y^2) \Rightarrow \frac{dV}{dy} = 0$$

$$\Rightarrow \frac{1}{3}\pi(4ry - 3y^2) = 0 \Rightarrow y(4r-3y) = 0 \Rightarrow y = \frac{4}{3}r, 0$$

Now, $\dfrac{d^2V}{dy^2} = \dfrac{1}{3}\pi(4r - 6y)$

Substituting $y = (4/3)r$, we get

$$\frac{d^2V}{dy^2} = \frac{1}{3}\pi\left(4r - 6\times\frac{4}{3}r\right)$$

which is of negative value and hence the volume of the cone is maximum at

$$y = \frac{4}{3}r$$

$$\Rightarrow \frac{\text{Height}}{\text{Radius}} = \frac{y}{r} = \frac{4}{3}$$

121. We have

$$f(x) = x + \sin x \Rightarrow f'(x) = 1 + \cos x$$

Now, $f'(x) = 0 \Rightarrow 1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow x = \pi$

and $f''(x) = -\sin x$, $f''(\pi) = 0$, $f'''(x) = -\cos x$

$$f'''(\pi) = 1 \neq 0$$

Therefore, neither maximum nor minimum.

122. We have

$$f(x) = ax + \frac{b}{x} \Rightarrow f'(x) = a - \frac{b}{x^2}$$

$$\Rightarrow f'(x) = 0 \Rightarrow x = \sqrt{\frac{b}{a}}$$

Now, $f''(x) = \dfrac{2b}{x^3}$ which implies that at $x = \sqrt{\dfrac{b}{a}}$, $f''(x)$ is positive. Therefore, $f(x)$ has the least value at

$$x = \sqrt{\frac{b}{a}}$$

123. We have

$$xy = c^2 \Rightarrow y = \frac{c^2}{x} \Rightarrow f(x) = ax + by = ax + \frac{bc^2}{x}$$

Differentiating w.r.t. x, we get

$$f'(x) = a - \frac{bc^2}{x^2}$$

Substituting $f'(x) = 0$, we get

$$ax^2 - bc^2 = 0$$

$$\Rightarrow x^2 = \frac{bc^2}{a} \Rightarrow x = \pm c\sqrt{b/a}$$

At $x = +c\sqrt{b/a}$, $ax + by$ will be minimum. The minimum value is

$$f\left(c\sqrt{\frac{a}{b}}\right) = (a)c\sqrt{\frac{a}{b}} + \frac{bc^2}{c}\sqrt{\frac{b}{a}} = 2c\sqrt{ab}$$

124. We have

$$a^2x^4 + b^2y^4 = c^6 \Rightarrow y = \left(\frac{c^6 - a^2x^4}{b^2}\right)^{1/4}$$

Hence,

$$f(x) = xy = x\left(\frac{c^6 - a^2x^4}{b^2}\right)^{1/4}$$

$$\Rightarrow f(x) = \left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{1/4}$$

On differentiating $f(x)$ w.r.t. x, we get

$$f'(x) = \frac{1}{4}\left(\frac{c^6x^4 - a^2x^8}{b^2}\right)^{-3/4}\left(\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2}\right)$$

Substituting $f'(x) = 0$, we get

$$\frac{4x^3c^6}{b^2} - \frac{8x^7a^2}{b^2} = 0$$

$$\Rightarrow x^4 = \frac{c^6}{2a^2} \Rightarrow x = \pm\frac{c^{3/2}}{2^{1/4}\sqrt{a}}$$

At $x = \dfrac{c^{3/2}}{2^{1/4}\sqrt{a}}$, $f(x)$ is maximum and hence

$$f\left(\frac{c^{3/2}}{2^{1/4}\sqrt{a}}\right) = \left(\frac{c^{12}}{2a^2b^2} - \frac{c^{12}}{4a^2b^2}\right)^{1/4} = \left(\frac{c^{12}}{4a^2b^2}\right)^{1/4} = \frac{c^3}{\sqrt{2ab}}$$

125. We have

$$f(x) = 2x^3 - 15x^2 + 36x + 4 \Rightarrow f'(x) = 6x^2 - 30x + 36 \quad (1)$$

We know that for its maximum value, $f'(x) = 0$.

$$6x^2 - 30x + 36 = 0 \Rightarrow (x-2)(x-3) = 0 \Rightarrow x = 2, 3$$

On differentiating Eq. (1), we get

$$f''(x) = 12x - 30$$

$$\Rightarrow f''(2) = 24 - 30 = -6 < 0$$

Therefore, $f(x)$ is maximum at $x = 2$.

126. We have

$$y = f(x) = -x^3 + 3x^2 + 9x - 27$$

The slope of this curve is

$$f'(x) = -3x^2 + 6x + 9$$

Let us consider

$$g(x) = f'(x) = -3x^2 + 6x + 9$$

On differentiating w.r.t. x, we get

$$g'(x) = -6x + 6$$

Substituting

$$g'(x) = 0 \Rightarrow x = 1$$

Now, $g''(x) = -6 < 0$

and hence at $x = 1$, $g(x)$, that is, the slope will have its maximum value. Therefore,

$$[g(1)]_{\max.} = -3 \times 1 + 6 + 9 = 12$$

127. We have

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$

and $\quad f'(x) = 6x^2 - 6x - 12$

Now, $\quad f'(x) = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = 2, -1$

and $\quad f''(x) = 12x - 6$

That is $f''(2)$, which is positive and $f''(-1)$ is negative.

Therefore, the given function has one maximum and one minimum.

128. We have

$$f(x) = \frac{1}{4x^2 + 2x + 1}$$

$$\Rightarrow f'(x) = \frac{-(8x+2)}{(4x^2 + 2x + 1)^2}$$

Substituting $f'(x) = 0$, we get

$$8x + 2 = 0 \Rightarrow x = \frac{-1}{4}$$

$$f''(x) = \frac{-[(4x^2 + 2x + 1)^2 8 - (8x+2)2(4x^2 + 2x + 1)(8x+2)]}{(4x^2 + 2x + 1)^4}$$

Now, $f''(-1/4)$ is negative (i.e. point of maxima). Therefore,

$$f(-1/4)_{\max.} = \frac{1}{4 \times (1/16) - 2 \times (1/4) + 1} = \frac{4}{3}$$

129. We have

$$f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - \frac{1}{x^2}$$

Substituting $f'(x) = 0$, we get $x = -1, 1$. Since $x > 0$, no maximum value can be found.

130. See Fig. 21.38.

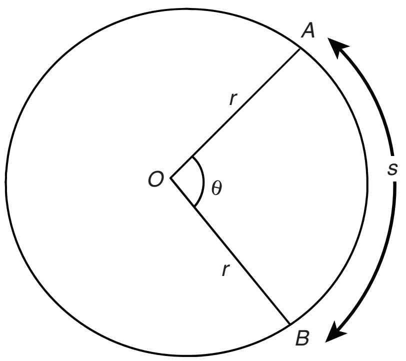


Figure 21.38

The perimeter of a sector is p. Let AOB be the sector with radius r. If the angle of the sector be θ radians, then the area of the sector is

$$A = \frac{1}{2} r^2 \theta \qquad (1)$$

Length of the arc is

$$s = r\theta \text{ or } \theta = \frac{s}{r}$$

Therefore, the perimeter of the sector is

$$p = r + s + r = 2r + s \qquad (2)$$

Substituting $\theta = \frac{s}{r}$ in Eq. (1), we have

$$A = \left(\frac{1}{2} r^2\right)\left(\frac{s}{r}\right) = \frac{1}{2} rs \Rightarrow s = \frac{2A}{r}$$

Now, substituting the value of s in Eq. (2), we get

$$p = 2r + \left(\frac{2A}{r}\right) \text{ or } 2A = pr - 2r^2$$

Differentiating w.r.t. r, we get

$$2\frac{dA}{dr} = p - 4r$$

We know that for the maximum value of area is

$$\frac{dA}{dr} = 0 \text{ or } p - 4r = 0 \text{ or } r = \frac{p}{4}$$

131. We have

$$\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = a + 2b + 1 = 0$$

$$\Rightarrow a = -2b - 1$$

and $\quad \left(\frac{dy}{dx}\right)_{x=2} = \frac{a}{2} + 4b + 1 = 0$

$$\Rightarrow \frac{-2b-1}{2} + 4b + 1 = 0$$

$$\Rightarrow -b + 4b + \frac{1}{2} = 0$$

$$\Rightarrow 3b = \frac{-1}{2} \Rightarrow b = \frac{-1}{6}$$

and $\quad a = \frac{1}{3} - 1 = \frac{-2}{3}$

132. We have

$$f(x) = \int_{-10}^{x} (t^4 - 4)e^{-4t} dt \Rightarrow f'(x) = (x^4 - 4)e^{-4x}$$

Now, $\quad f'(x) = 0 \Rightarrow x = \pm\sqrt{2}, \pm\sqrt{2}$

and $\quad f''(x) = -4(x^4 - 4)e^{-4x} + 4x^3 e^{-4x}$

At $x = \sqrt{2}$ and $x = -\sqrt{2}$, the given function has extreme value.

133. We have

$$f(x) = x^2 \log x \Rightarrow f'(x) = (2\log x + 1)x$$

Now, $\quad f'(x) = 0 \Rightarrow x = e^{-1/2}, 0$

Since $0 < e^{-1/2} < 1$ and none of these critical points lies in the interval [1, e], we only complete the value of $f(x)$ at the end points 1 and e. We have $f(1)=0, f(e)=e^2$. Therefore, the greatest value is e^2.

134. We have

$$f(x)=y=x^{-x} \Rightarrow \log y=-x\log x$$

Differentiating w.r.t. x,

$$\frac{1}{y}\cdot\frac{dy}{dx}=-\left[x.\frac{1}{x}+\log x\right]$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx}=-[1+\log x]$$

$$\Rightarrow \frac{dy}{dx}=-x^{-x}[1+\log x]$$

$$\Rightarrow \frac{dy}{dx}=x^{-x}\left[\log\frac{1}{x}-1\right]$$

Substituting $dy/dx=0$,

$$\log_e\frac{1}{x}=\log_e e$$

$$\Rightarrow \frac{1}{x}=e \Rightarrow x=\frac{1}{e}$$

135. We have

$$ab=2a+3b \Rightarrow (a-3)b=2a \Rightarrow b=\frac{2a}{a-3}$$

Now, $$z=ab=\frac{2a^2}{a-3}$$

$$\Rightarrow \frac{dz}{da}=\frac{2[(a-3)2a-a^2]}{(a-3)^2}=\frac{2[a^2-6a]}{(a-3)^2}$$

Substituting $dz/da=0$, we get

$$a^2-6a=0$$

That is, $a=0,6$. Now, at $a=6$, $\frac{d^2z}{da^2}$ is positive and when $a=6, b=4$

$$(ab)_{\min}=6\times 4=24$$

136. Let $PQ=a$ and $PR=b$. Then

$$\Delta=\frac{1}{2}ab\sin\theta$$

(As $-1\le \sin\theta \le 1$). Since, the area is maximum when $\sin\theta=1$, we have

$$\theta=\frac{\pi}{2}$$

137. We have

$$y=a(1-\cos x) \Rightarrow y'=a\sin x$$

$$\Rightarrow y'=0 \Rightarrow \sin x=0 \Rightarrow x=0,\pi$$

Now, $y''=a\cos x \Rightarrow y''(0)=a$ and $y''(\pi)=-a$

Hence, y is maximum when $x=\pi$.

138. Let us consider

$$y=f(x)=\left(x^2+\frac{250}{x}\right)$$

and $$\frac{dy}{dx}=f'(x)=2x-\frac{250}{x^2}$$

Substituting $f'(x)=0$, we get

$$2x^3-250=0 \Rightarrow x^3=125 \Rightarrow x=5$$

Also

$$\frac{d^2y}{dx^2}=f''(x)=2+\frac{500}{x^3}$$

Now,

$$f''(5)=2+\frac{500}{125}>0$$

Hence, at $x=5$, the function will be minimum. The minimum value is

$$f(5)=25+50=75$$

139. We have $y=x^{1/x}$. Taking log on both side, we have

$$\log y=\frac{1}{x}\log x$$

Differentiating on both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2}-\frac{\log x}{x^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x^2}(1-\log x)x^{1/x}$$

For maximum, we need to have

$$\frac{dy}{dx}=0 \Rightarrow x=e$$

Therefore,

$$y_{\max}=e^{1/e}$$

140. Let us consider

$$f(x)=4e^{2x}+9e^{-2x}$$

Therefore,

$$f'(x)=8e^{2x}-18e^{-2x}$$

Substituting $f'(x)=0$, we get

$$8e^{2x}-18e^{-2x}=0$$

Taking log both sides and solving, we have

$$e^{2x}=3/2 \Rightarrow x=\log(3/2)^{1/2}$$

Also

$$f''(x)=16e^{2x}+36e^{-2x}>0$$

Now,

$$f(\log(3/2)^{1/2})=4e^{2.(\log(3/2)^{1/2})}+9e^{-2(\log(3/2)^{1/2})}=4\times\frac{3}{2}+9\times\frac{2}{3}$$

$$=6+6=12$$

Hence, the minimum value is 12.

141. Let a point on the curve be (h, k). Then

$$h^2=2k \quad (1)$$

The distance is

$$D=\sqrt{h^2+(k-5)^2}$$

From Eq. (1),

$$D=\sqrt{2k+(k-5)^2}$$

$$\frac{dD}{dk}=\frac{1}{2\sqrt{2k+(k-5)^2}}\times 2(k-5)+2=0 \quad x\in(-1,\infty)$$

So, at $k=4$, the function D must be minimum. Then, the point is $(\pm 2\sqrt{2},4)$.

142. We have

$$x+2y=8,\quad y=\frac{8-x}{2}$$

Now, $f(x)=xy=x.\frac{(8-x)}{2}=4x-\frac{x^2}{2}$

And $f'(x)=4-x$

For extremum, we need to have $f'(x)=0$. Therefore, $x=4$ and $y=2$. Also,

$$f''(x)=-1<0$$

So, the maximum value of xy is $4\times 2=8$.

143. We have

$$f(a)=2a^2-3a+10$$
$$f'(a)=4a-3$$
$$f''(a)=4>0$$

For extremum,

$$f'(a)=0\Rightarrow a=\frac{3}{4}$$

Therefore, $f(a)$ is minimum at $a=3/4$.

$$f(a)_{\min}=2\times\left(\frac{3}{4}\right)^2-3\times\left(\frac{3}{4}\right)+10=\frac{71}{8}$$

144. Let the coordinates of R be $(x, 0)$. Given that $P(1,1)$ and $Q(3,2)$.

$$PR+RQ=\sqrt{(x-1)^2+(0-1)^2}+\sqrt{(x-3)^2+(0-2)^2}$$
$$=\sqrt{x^2-2x+2}+\sqrt{x^2-6x+13}$$

For minimum value of $PR+RQ$, we get

$$\frac{d}{dx}(PR+RQ)=0$$

$$\Rightarrow \frac{d}{dx}(\sqrt{x^2-2x+2})+\frac{d}{dx}(\sqrt{x^2-6x+13})=0$$

$$\Rightarrow \frac{(x-1)}{\sqrt{x^2-2x+2}}=-\frac{(x-3)}{\sqrt{x^2-6x+13}}$$

Squaring both sides, we get

$$\frac{(x-1)^2}{(x^2-2x+2)}=\frac{(x-3)^2}{x^2-6x+13}$$

$$\Rightarrow 3x^2-2x-5=0$$
$$\Rightarrow (3x-5)(x+1)=0$$

That is,

$$x=\frac{5}{3},-1$$

Also, $1<x<3$. Therefore, $R=(5/3,0)$.

145. We have

$$f(x)=1+2x^2+2^2x^4+2^3x^6+...2^{10}x^{20}$$
$$f'(x)=x[4+4(2)^2x^2+\cdots+20(2)^{10}x^{18}]$$

Therefore,

$$f'(x)=0\ \Rightarrow\ x=0$$

Also, $f''(0)>0$.

146. Here,

$$f(x)=\frac{x^2-3x}{x-1}\ \Rightarrow -\sin c=-\frac{2}{\pi}\Rightarrow c=\sin^{-1}\left(\frac{2}{\pi}\right)$$

Obviously, it is not derivable at $x=1$, that is, in $(0,3)$. Also, $f(a)=f(b)$ does not hold for $[-3,0]$ and $[1.5, 3]$. Hence the answer is (D).

147. Here,

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

$$\Rightarrow \frac{e^b-e^a}{b-a}=f'(c)$$

$$\Rightarrow \frac{e-1}{1-0}=e^c\Rightarrow c=\log(e-1).$$

148. We have

$$f(x)=\begin{cases}-x, \text{when } -1\le x<0\\ x, \text{when } 0\le x\le 1\end{cases}$$

Clearly, $f(-1)=|-1|=1=f(1)$. However,

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{h}=\lim_{h\to 0}\frac{|-h|}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$$

Therefore,

$$Rf'(0)\neq Lf'(0)$$

Hence, it is not differentiable on $(-1, 1)$.

149. We know that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow f'(c)=\frac{0-1}{\pi/2}=-\frac{2}{\pi} \tag{1}$$

However,

$$f'(x)=-\sin x\Rightarrow f'(c)=-\sin c \tag{2}$$

From Eqs. (1) and (2), we get

$$-\sin c=-\frac{2}{\pi}\Rightarrow c=\sin^{-1}\left(\frac{2}{\pi}\right)$$

150. We have

$$f'(x_1)=\frac{-1}{x_1^2}$$

Therefore,

$$\frac{-1}{x_1^2}=\frac{(1/b)-(1/a)}{b-a}=-\frac{1}{ab}\Rightarrow x_1=\sqrt{ab}$$

151. We have

$$f'(x)=1-\frac{1}{x^2}\Rightarrow f'(c)=1-\frac{1}{c^2}$$

$$1-\frac{1}{c^2}=\frac{(10/3)-2}{2}\Rightarrow 1-\frac{1}{c^2}=\frac{2}{3}\Rightarrow c^2=3$$

$$\Rightarrow c=\sqrt{3}$$

152. According to mean value theorem, in an interval $[a, b]$ for $f(x)$, we have

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

where $a<c<b$. Therefore, $a<x_1<b$.

153. We have

$$f(x)=e^{-2x}\sin 2x$$

$$\Rightarrow f'(x)=2e^{-2x}(\cos 2x-\sin 2x)$$

Now,

$$f'(c)=0$$

$$\Rightarrow \cos 2c-\sin 2c=0\Rightarrow \tan 2c=1\Rightarrow c=\frac{\pi}{8}$$

154. From Rolle's theorem in (1, 26), $f(1)=f(26)=5$. In the given interval, the function satisfies all conditions of Rolle's theorem. Therefore, in [1, 26], at least, there is a point for which $f'(x)=0$.

155. We have $\int_1^2 f'(x)dx=[f(x)]_1^2=f(2)-f(1)=0$ because $f(x)$ satisfies the conditions of Rolle's theorem. Therefore, $f(2)=f(1)]$).

156. We have

$$f(x)=x^3-6x^2+ax+b$$

$$\Rightarrow f'(x)=3x^2-12x+a$$

$$\Rightarrow f'(c)=0\Rightarrow f'\left(2+\frac{1}{\sqrt{3}}\right)=0$$

$$\Rightarrow 3\left(2+\frac{1}{\sqrt{3}}\right)^2-12\left(2+\frac{1}{\sqrt{3}}\right)+a=0$$

$$\Rightarrow 3\left(4+\frac{1}{3}+\frac{4}{\sqrt{3}}\right)-12\left(2+\frac{1}{\sqrt{3}}\right)+a=0$$

That is,

$$12+1+4\sqrt{3}-24-4\sqrt{3}+a=0\Rightarrow a=11$$

157. From mean value theorem,

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$a=0, f(a)=0\Rightarrow b=\frac{1}{2}, f(b)=\frac{3}{8}$$

$$f'(x)=(x-1)(x-2)+x(x-2)+x(x-1)$$

$$f'(c)=(c-1)(c-2)+c(c-2)+c(c-1)$$

$$=c^2-3c+2+c^2-2c+c^2-c$$

That is,

$$f'(c)=3c^2-6c+2$$

According to mean value theorem, we have

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow 3c^2-6c+2=\frac{(3/8)-0}{(1/2)-0}=\frac{3}{4}$$

$$\Rightarrow 3c^2-6c+\frac{5}{4}=0$$

$$c=\frac{6\pm\sqrt{36-15}}{2\times 3}=\frac{6\pm\sqrt{21}}{6}=1\pm\frac{\sqrt{21}}{6}$$

158. Given that the equation of curve is

$$y=x^3=f(x)$$

So, $f(2)=8$ and $f(-2)=-8$. Now,

$$f'(x)=3x^2\Rightarrow f'(x)=\frac{f(2)-f(-2)}{2-(-2)}$$

$$\Rightarrow\frac{8-(-8)}{4}=3x^2$$

Therefore,

$$x=\pm\frac{2}{\sqrt{3}}$$

159. We have

$$f(x)=\sqrt{x}$$

Therefore,

$$f(a)=\sqrt{4}=2,\quad f(b)=\sqrt{9}=3$$

Now, $$f'(x)=\frac{1}{2\sqrt{x}}$$

Also, $$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{3-2}{9-4}=\frac{1}{5}$$

Therefore,

$$\frac{1}{2\sqrt{c}}=\frac{1}{5}\Rightarrow c=\frac{25}{4}=6.25$$

Practice Exercise 2

1. $$\frac{dx}{dt}=\frac{2}{\cot t}(-\operatorname{cosec}^2 t)=\frac{-2}{\sin t\cos t}$$

$$\frac{dy}{dt}=\sec^2 t-\operatorname{cosec}^2 t=\frac{\sin^2 t-\cos^2 t}{\sin^2 t\cos^2 t}$$

Now,

$$\frac{dx}{dt}\text{ at }\left(t=\frac{\pi}{4}\right)=\frac{-2}{1/2}=-4$$

$$\frac{dy}{dt} \text{ at } \left(t = \frac{\pi}{4}\right) = 2 - 2 = 0$$

Here, $$\frac{dy}{dx} = 0 \Rightarrow -\frac{dx}{dy} = \infty$$

2. $g(x)$ is increasing and $f(x)$ is decreasing. So,
$g(x+1) > g(x-1)$ and $f(x+1) < f(x-1)$
$\Rightarrow f\{g(x+1)\} < f\{g(x-1)\}$ and $g\{f(x+1)\} < g\{f(x-1)\}$

3. $f(x) = x^3 - x^2 + 100x + 1001$
$$f'(x) = 3x^2 - 2x + 100 > 0 \ \forall \ x \in R$$
Therefore, $f(x)$ is increasing (strictly).

Therefore,
$$f\left(\frac{1}{1999}\right) > f\left(\frac{1}{2000}\right)$$
$$\Rightarrow f(x+1) > f(x-1)$$

4.
$$xy = (a+x)^2$$
$$y + xy' = 2(a+x)$$
$$y' = \pm 1$$
$$y \pm x = 2(a+x)$$
$$\frac{(a+x)^2}{x} \pm x = 2(a+x) \Rightarrow \pm x = 2(a+x) - \frac{(a+x)^2}{x}$$
$$\pm x^2 = (2+x)[x-a] \Rightarrow \pm x^2 = x^2 - a^2$$
$$2x^2 = a^2 \Rightarrow x = \pm \frac{a}{\sqrt{2}}$$

5.
$$f(x) = \frac{\ln x}{x} \quad (1)$$
Since domain is R^+, we have
$$f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

(A) For horizontal tangent
$$f'(x) = 0 \Rightarrow \ln x = 1 \Rightarrow x = e \quad \text{(True)}$$

(B) If Eq. (1) cuts the x–axis, then
$$\frac{\ln x}{x} = 0 \Rightarrow x = 1 \quad \text{(True)}$$

(C) $f'(x)$ is +ve if $x \in (0, e)$ and $f'(x)$ is –ve if $x \in (e, \infty)$. Therefore, $f(x)$ is not monotonic.
Hence, $f(x)$ is many–one. (True)

(D) For vertical tangent $f'(x) = \infty$, so
$$\frac{1 - \ln x}{x^2} = \infty \Rightarrow \frac{x^2}{1 - \ln x} = 0 \Rightarrow x = 0$$
which is not in the domain of $f(x)$. (False)

6. See Fig. 21.39.
Note: Curve is not passing through origin.
Let (α, β) be the point of contact. Then
$$\left(\frac{dy}{dx}\right)_P = \frac{\beta - 0}{\alpha - 0} \Rightarrow 3(\alpha+1)^2 = \frac{\beta}{\alpha} \quad (1)$$
Also (α, β) lies on the curve. Therefore,
$$3(\alpha+1)^2 = \frac{(\alpha+1)^3}{\alpha}$$
$$\Rightarrow (\alpha+1)^2 \left\{3 - \frac{\alpha+1}{\alpha}\right\} = 0$$

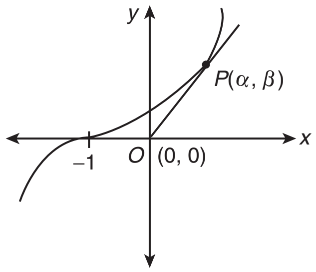


Figure 21.39

$$\Rightarrow \alpha = -1$$
or $$\alpha = \frac{1}{2}$$
Therefore, equation of tangent is
$$y = 0$$
or $$y = \frac{27}{4}x$$

7. See Fig. 21.40.
$$f(x) = ax^3 + bx^2 + cx + d$$
Now, $f(x)$ is odd. Therefore,
$$f(-x) = -f(x)$$
$$\Rightarrow -ax^3 - bx^2 - cx - d = -ax^3 + bx^2 - cx + d$$
It gives $b = 0 = d$
$$f(x) = ax^3 + cx = x(ax^2 + c)$$
Therefore,
$$f'(x) = 3ax^2 + c = 0$$
Only when $x^2 = -\dfrac{c}{3a}$ is positive.

Therefore, c and a are of different signs.

Let $-\dfrac{c}{a} = k$.

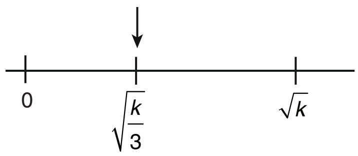


Figure 21.40

So, non–zero root of $f(x)$ is $\pm\sqrt{k}$.

Also $\pm\sqrt{\dfrac{k}{3}}$ is closer to origin than $\pm\sqrt{k}$.

8.
$$f(x) = (x-1)^4 (x-2)^n, \ n \in N \quad (1)$$
Therefore,
$$f'(x) = 4(x-1)^3 (x-2)^n + (x-1)^4 n (x-2)^{n-1}$$
$$= (x-1)^3 (x-2)^{n-1} (4x - 8 + nx - n)$$
$$= (x-1)^3 (x-2)^{n-1} [(n+4)x - (n+8)]$$

If n is odd, then $f'(x) > 0$ if $x < 1$ and sufficiently close to 1 and $f'(x) < 0$ if $x > 1$ and sufficiently close to 1. Therefore, $x = 1$ is point of local maximum.

Similarly, if n is even, then $x = 1$ is a point of local minimum.

Further if n is even, then $f'(x) < 0$ for $x < 2$ and sufficiently close to 2 and $f'(x) > 0$ for $x > 2$ and sufficiently close to 2.

Therefore, $x = 2$ is a point of local minimum.

9. $$\frac{da}{dt} = 2 \Rightarrow a = 2t + c$$

Since $c = 0$ {Because $a = 0$, when $t = 0$}

Therefore,

$$a = 2t$$

Therefore, the curve $y = x^2 - 2ax + a^2 + a$ becomes

$$y = x^2 - 4tx + 4t^2 + 2t$$

If $x = 0$, then

$$y = 4t^2 + 2t$$

Now,

$$\frac{dy}{dx} = 2x - 4t$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{at\ x=0} = -4t$$

Therefore, equation of the tangent is

$$y - (4t^2 + 2t) = -4t(x - 0)$$

That is,

$$y = -4tx + 4t^2 + 2t$$

Vertex of $y = x^2 - 4tx + 4t^2 + 2t$ is $(2t, 2t)$.

Therefore, distance of vertex from the origin $= 2\sqrt{2}\,t$.

Therefore, rate of change of distance of vertex from origin with respect to $t = 2\sqrt{2}$.

That is, $k = 2\sqrt{2}$

10. $$c(t) = 4t^2 + 2t$$

Therefore,

$$\frac{dc}{dt} = 8t + 2$$

$$\Rightarrow \left.\frac{dc}{dt}\right|_{at\ t=2\sqrt{2}} = 16\sqrt{2} + 2$$

$$\Rightarrow \ell = 16\sqrt{2} + 2$$

11. $$m(t) = -4t$$

Therefore,

$$\frac{dm}{dt} = -4$$

$$\Rightarrow \left.\frac{dm}{dt}\right|_{at\ t=\ell} = -4$$

12.

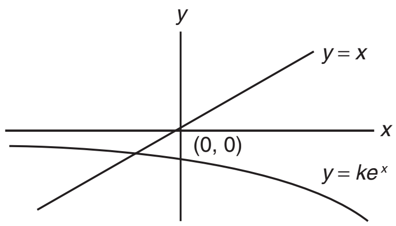


Figure 21.41

With the help of graph shown in Fig. 21.41, we can see that there is only one solution.

13. Consider $y = ke^x$ and $y = x$.

Let (α, ke^{α}) be a point on $y = ke^x$.

If it lies on $y = x$ also, then $\alpha = ke^{\alpha}$.

Now,

$$\frac{dy}{dx} = ke^x$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{x=\alpha} = ke^{\alpha} = \alpha = 1$$

$y = x$ is tangent to $y = ke^x$ at one point.

Therefore,

$$1 = ke$$

That is,

$$k = 1/e$$

14. See Fig. 21.42. Consider $y = ke^x$ and $y = x$.

From the above question, $e^x = \frac{x}{k}$

If we decrease the value of k from $\frac{1}{e}$, then slope of $y = \frac{x}{k}$ increases.

Therefore, $y = e^x$ and $y = \frac{x}{k}$ intersect at two distinct points (See Fig. 21.42).

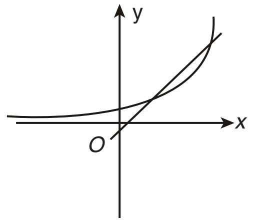


Figure 21.42

Therefore, $k \in \left(0, \frac{1}{e}\right)$

15. Let $0 < \alpha < \beta < 1$, and α, β are the roots of $f(x) = x^3 - 3x + k = 0$. Then

$$f(\alpha) = f(\beta) = 0$$
$$\Rightarrow f(x) \text{ satisfies RMVT}$$
$$\Rightarrow f'(c) = 0$$
$$\Rightarrow 3c^2 = 3$$
$$\Rightarrow c = \pm 1$$

But c must be lies between α and β.

Hence, $k \in \phi$.

16. Let $f(x) = \tan^{-1}x$. Then for some $\alpha \in (x, y)$, we have

$$f'(\alpha) = \frac{\tan^{-1}y - \tan^{-1}x}{y - x} \text{ (LMVT)}$$

$$\Rightarrow \left|\frac{1}{1+\alpha^2}\right| = \left|\frac{\tan^{-1}x - \tan^{-1}y}{x-y}\right| \left(\left|\frac{1}{1+\alpha^2}\right| \le 1\right)$$

$$\Rightarrow |\tan^{-1}x - \tan^{-1}y| \le |x - y|$$

17. Let $f(x) = \sin x$ and $g(x) = \cos x$.

Also, $\sin x \ne 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

Then, by Cauchy's theorem, we have

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

$$\Rightarrow \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

18. Putting x = 9, y = 0 in the given equation of curve, we have

$$0 = 3a + 9b - \frac{1}{2} = \frac{a}{2\times 3} + b$$

$$\Rightarrow a = -3b \quad (1)$$

$$\frac{dy}{dx} = \frac{a}{2\sqrt{x}} + b$$

$$\left.\frac{dy}{dx}\right|_{(9,0)} = \frac{a}{6} + b = -\frac{1}{2} \quad (2)$$

Using Eqs. (1) and (2), we get

$$b = -1 \text{ and } a = 3$$

Therefore,

$$y = 3\sqrt{x} - x$$

Point (1, 2) lies on curve as well as it is point of intersection of family of lines.

$$\frac{dy}{dx} = \frac{3}{2\sqrt{x}} - 1$$

$$\frac{dy}{dx} \text{ at } (1, 2) \text{ is } \frac{1}{2}$$

$$y - 2 = \frac{1}{2}(x - 1)$$

$$\Rightarrow x - 2y + 3 = 0$$

19. See Fig. 21.43.

$$\sin\theta = \frac{2}{PA}$$

$PA = 2\operatorname{cosec}\theta$

$$\cos\theta = \frac{1}{BP}$$

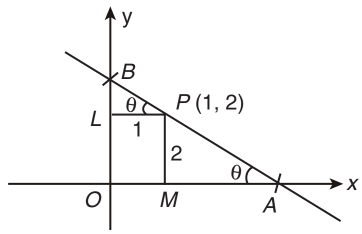


Figure 21.43

$BP = \sec\theta$

$AB = AP + BP = 2\operatorname{cosec}\theta + \sec\theta$

Therefore, minimum value of $AB = (2^{2/3} + 1)^{3/2}$.

20. Let $y - 2 = m(x - 1)$ be a focal chord. Then

$$y = mx + 2 - m$$

$$\Rightarrow (mx + 2 - m)^2 - 4x - 4(mx + 2 - m) + 4 = 0$$

That is, $(mx - m)^2 - 4x = 0$

$$\Rightarrow m^2x^2 - (2m^2 + 4)x + m^2 = 0$$

Now,

$$x_1 + x_2 = \frac{2(m^2+2)}{m^2};\ x_1x_2 = 1$$

$$|x_2 - x_1| = \sqrt{(x_1+x_2)^2 - 4x_1x_2}$$

$$= \sqrt{4\frac{(m^2+2)^2}{m^4} - 4} = \frac{2}{m^2}\sqrt{m^4 + 4m^2 + 4 - m^4}$$

$$|x_2 - x_1| = \frac{4}{m^2}\sqrt{m^2+1}$$

$$\text{Length of diagonal} = \sqrt{(x_2-x_1)^2 + m^2(x_2-x_1)^2} = \frac{4}{m^2}(m^2+1)$$

Length of diagonal of perpendicular chord is

$$4m^2\left(\frac{1}{m^2} + 1\right) = 4(1 + m^2)$$

$$\text{Area} = \frac{1}{2}\cdot\frac{4}{m^2}(1+m^2)\,4(1+m^2) = 8\left(m^2 + 2 + \frac{1}{m^2}\right)$$

$$\Rightarrow \text{Minimum area} = 16 + 8 \times 2 = 32$$

21. Graph of $y = f(x)$. See Fig. 21.44.

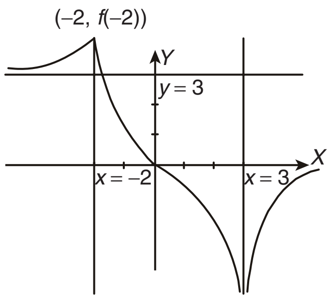


Figure 21.44

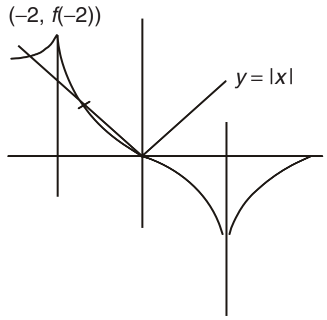


Figure 21.45

Graph of $f(x) = |x|$ has 3 points of intersection, so equation has 3 solutions (See Fig. 21.45.). Hence, (C) is the correct answer.

22. See Fig. 21.46.

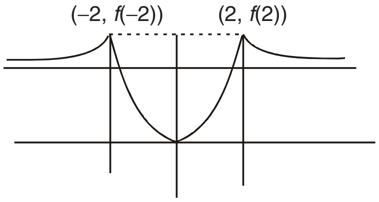


Figure 21.46

Hence, (B) is the correct answer.

23. See Fig. 21.47.

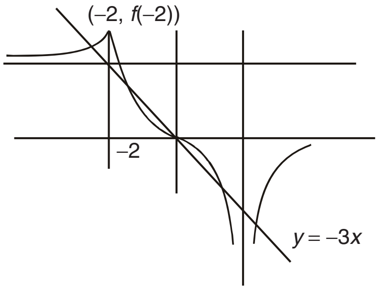


Figure 21.47

Hence, (D) is the correct answer.

24. **(A)** $r = 5$ cm, $\delta dr = 0.06$

$A = \pi r^2 \delta A = 2\pi r \delta r = 10\pi \times 0.06 = 0.6\pi$

(B) $v = x^3$, $\delta v = 3x^2 \delta x$

$$\frac{\delta v}{v} \times 100 = 3\frac{\delta x}{x} \times 100 = 3 \times 1 = 3$$

(C) $(x-2)\dfrac{dx}{dt} = 2\dfrac{dx}{dt} \Rightarrow x = 4$

(D) $A = \dfrac{\sqrt{3}}{4}x^2$

Now,

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} \times \frac{dx}{dt} = \frac{\sqrt{3}}{2} \cdot 15 \cdot \frac{1}{10} = \frac{3\sqrt{3}}{4}$$

25. See Fig. 21.48.

(A) $\dfrac{dy}{dx} = \dfrac{4t}{3}$

$$\text{Tangent is } y - at^4 = \frac{4t}{3}(x - at^3)$$

$$x\text{-intercept} = \frac{at^3}{4};\ y\text{-intercept} = -\frac{at^4}{3}$$

If P divides AB in the ratio λ:1, we have

$$at^3 = \frac{\lambda \cdot 0 + \frac{at^3}{4}}{\lambda + 1} \Rightarrow \lambda = \frac{-3}{4}$$

Therefore,

$$\frac{m}{n} = -\frac{3}{4}$$

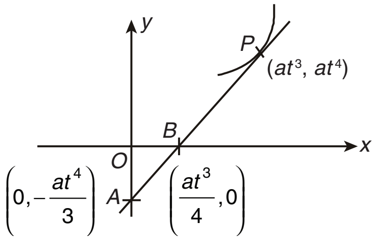


Figure 21.48

$\Rightarrow m = 3, n = 4$

$\Rightarrow m + n = 7$

(B) $\dfrac{dx}{dy} = e^{\sin y}\cos y$; slope of normal $= -1$

Equation of normal is $x + y = 1$.

Area $= \dfrac{1}{2}$

(C) $y = \dfrac{1}{x^2} : \dfrac{dy}{dx} = -\dfrac{2}{x^3}$: slope of tangent $= -2$

$y = e^{2-2x} : \dfrac{dy}{dx} = e^{2-2x} \cdot (-2)$: slope of tangent $= -2$

Therefore,

$$\tan\theta = 0$$

(D) Length of sub–tangent $= \left|\dfrac{y}{y'}\right| = \left|\dfrac{be^{x/3}}{b\frac{1}{3}e^{x/3}}\right| = 3$

26. **(A)** Using LMVT, we get

$$g'(c) = \frac{g(5) - g(0)}{5} = \frac{-1/6 - 4}{5} = -\frac{5}{6}$$

(B) Let $\phi(x) = f(x) - 2g(x)$, $x \in [0, 1]$. Then

$$\phi(0) = 2,\ \phi(1) = 6 - 2g(1)$$

Now, $\phi'(x) = f'(x) - 2g'(x) \Rightarrow \phi(x)$ satisfies condition of Rolle's theorem on [0, 1], so

$$\phi(0) = \phi(1) \Rightarrow 2 = 6 - 2g(1)$$

$$\Rightarrow g(1) = 2$$

(C) $f(x) = \sin 3x$

Clearly, longest length $= \frac{\pi}{6}e - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$

Therefore, $\lambda = 3$.

(D) $f'(c) = \frac{f(5)-f(1)}{5-1} \Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{-\sqrt{6}}{2} \Rightarrow c^2 = 15$

27. (A) $f(x) = \ln(\sin x)$

$$f'(x) = \frac{\cos x}{\sin x} > 0$$

Therefore, the required number of values of x is 0.

(B) $f'(x) = 3x^2 - 3 \le 0$ if $-1 \le x \le 1$

Therefore, $a = -1, b = 1$

Therefore, $a + b = 0$

(C) $f(x) = \begin{cases} x^2+2, & 1 \le x < 2 \\ \frac{x^2+2}{2}, & 2 \le x < 3 \\ \frac{x^2+2}{3}, & x = 3 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} 2x, & 1 < x < 2 \\ x, & 2 < x < 3 \end{cases}$$

Therefore, least value of $f(x)$ in [1, 2) is 3.

Least value of $f(x)$ in [2, 3) is 3. So,

$$f(3) = \frac{11}{3}$$

Therefore, the least value of $f(x)$ is 3.

(D) $f(x) = e^{2x} - (a+1)e^x + 2x$

$f'(x) = 2e^{2x} - (a+1)e^x + 2$

Now,

$$2e^{2x} - (a+1)e^x + 2 \ge 0 \text{ for all } x \in R$$

$$\Rightarrow 2\left(e^x + \frac{1}{e^x}\right) - (a+1) \ge 0 \text{ for all } x \in R$$

$$\Rightarrow 4 - (a+1) \in 0$$

$$\Rightarrow a \in 3$$

$$\Rightarrow a = 3$$

28. (A) See Fig. 21.49. By graph, it is clear that at x = −1 is local max. and x = 0 is local min.

(B) $a + b = 1$

$$\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)}e = \sqrt{1+\frac{1}{a}+\frac{1}{b}+\frac{1}{ab}} = \sqrt{1+\frac{2}{ab}}$$

$$\sqrt{ab} < \frac{a+b}{2} = \frac{1}{2}$$

Therefore,

$$ab < \frac{1}{4} \Rightarrow \frac{1}{ab} > 4$$

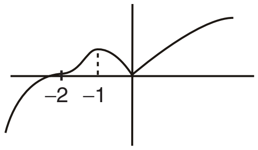


Figure 21.49

Therefore,

$$\sqrt{\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)} \ge \sqrt{1+8} = 3$$

(C) $y = 10 - (10 - x) = x$

Therefore, the maximum value is $y = 3$.

(D) Equation of tangent at P is $ty = x + t^2$.

It intersects the line $x = 0$ at Q. Therefore, coordinates of Q are $(0, t)$. Therefore,

$$\text{Area of } \Delta PQS = \frac{1}{2}\begin{vmatrix} 0 & t & 1 \\ 1 & 0 & 1 \\ t^2 & 2t & 1 \end{vmatrix} = \frac{1}{2}[-t(1-t^2)+2t]$$

$$= \frac{1}{2}(t + t^3)$$

Now,

$$\frac{dA}{dt} = \frac{1}{2}(3t^2+1) > 0 \ \forall t \in [0, 2]$$

Therefore, area is maximum for $t = 2$.

Hence,

$$\text{maximum area} = \frac{1}{2}[2+8] = 5$$

29. For the points of intersection, we have

$$\frac{12-y^2}{36} + \frac{y^2}{4} = 1$$

$$\Rightarrow y = \pm\sqrt{3} \text{ and } x = \pm 3$$

Consider the point $P(3, \sqrt{3})$. Equation of the tangent at P to the circle is $3x + \sqrt{3}\,y = 12$.

Therefore, slope of this tangent is $-\sqrt{3}$.

Equation of the tangent at P to the ellipse is

$$\frac{x}{12} + \frac{\sqrt{3}}{4}y = 1$$

Therefore, slope of this tangent is $-\frac{1}{3\sqrt{3}}$.

If α is angle between these tangents, then

$$\tan\alpha = \frac{2}{\sqrt{3}}$$

Therefore,

$$\alpha = \tan^{-1}\frac{2}{\sqrt{3}}$$

Therefore,

$$k = 4$$

Hence,

$$k^2 = 16$$

30. See Fig. 21.50.

$$d^2 = (x_1 - x_2)^2 + \left(\sqrt{2 - x_1^2} - \frac{9}{x_2}\right)^2$$

Let $y_1^2 = 2 - x_1^2$. Then

$$x_1^2 + y_1^2 = 2$$

$$y_2 = \frac{9}{x_2} \Rightarrow x_2 y_2 = 9$$

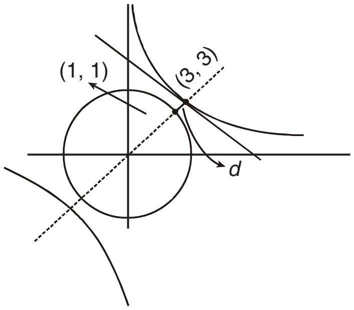


Figure 21.50

$d \rightarrow$ shortest distance between two curves will be along the common normal $y = x$

Therefore,

$$d^2 = 4 + 4 = 8$$

31. See Fig. 21.51.

$$f'(x) = \cos x - 2a \cos 2x - \cos 3x + 2a \geq 0 \ \forall\, x \in R$$

$$\Rightarrow \cos x - \cos 3x + 2a\,(1 - \cos 2x) \geq 0$$

$$\Rightarrow 2 \sin^2 x + \sin x + 4a \sin^2 x \geq 0$$

$$\Rightarrow 2 \sin^2 x\,(\cos x + a) \geq 0$$

$$\Rightarrow a \geq -\cos x \Rightarrow a \geq 1$$

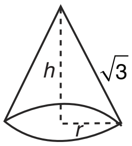


Figure 21.51

32. See Fig. 21.52.

$$3 = h^2 + r^2$$

$$\Rightarrow r^2 = 3 - h^2$$

Now,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(3 - h^2)\,h$$

Therefore,

$$\frac{dV}{dh} = \frac{1}{3}\pi(3 - 3h^2)$$

$$\frac{dV}{dh} = 0 \text{ at } h = 1$$

$$\frac{d^2V}{dh^2} < 0 \text{ at } h = 1$$

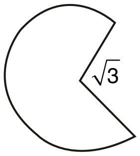


Figure 21.52

$$\Rightarrow \text{Vmax} = \frac{2\pi}{3}$$

Therefore, $\lambda = 2$.

33. See Fig. 21.53. If a < a_{min}, then the curve $y = |x - a|$ will not intersect the curve $y = -x^2 + 3$.

Similarly, if $a > a_{max}$, then the curve $y = |x - a|$ will not intersect the curve $y = 3 - x^2$ for any $x \in \left(-\sqrt{3}, 0\right)$.

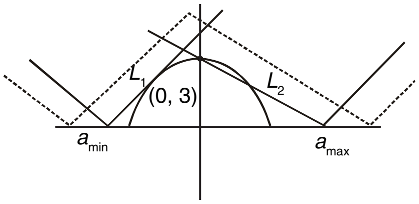


Figure 21.53

Case I:

L_1 is tangent to $y = -x^2 + 3$ and its equation is $y = x - a$

Therefore,

$$\frac{dy}{dx} = -2x = 1$$

That is,

$$x = -\frac{1}{2}$$

Therefore, $\left(-\frac{1}{2}, \frac{11}{4}\right)$ lies on $y = x - a$

Thus, point of contact is $\left(-\frac{1}{2}, \frac{11}{4}\right)$.

Since, it lies on $y = x - a$. Therefore

$$a_{min} = -\frac{13}{4}$$

So, the inequality has a negative solution if $-\frac{13}{4} < a < 0$. (1)

Case II:

Line L_2 is $y = a - x$ and passes through (0, 3) if $a = 3$.

Thus, the inequality has a negative solution if $-\sqrt{3} < a < 3$. (2)

From Eqs. (1) and (2), we get that the inequation has at least one negative solution if $-\frac{13}{4} < a < 3$.

34.

$$f(x) = \begin{cases} xe^{ax}, & x \leq 0 \\ x + ax^2 - x^3, & x > 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} e^{ax} + axe^{ax} & , x \le 0 \\ 1 + 2ax - 3x^2, & x > 0 \end{cases}$$

and

$$f''(x) = \begin{cases} (2a)e^{ax} + a^2xe^{ax}, & x \le 0 \\ 2a - 6x & , x > 0 \end{cases}$$

For $x < 0$,

$$f''(x) > 0$$
$$\Rightarrow (2a + a^2x)\, e^{ax} > 0$$
$$\Rightarrow a\,(2 + ax) > 0 \Rightarrow x > \frac{-2}{a}e$$

For $x > 0$, $2a - 6x > 0$, $x < \frac{a}{3}$.

Also f is continuous at 0.

Therefore, $f(x)$ is increasing in $\left(-\frac{2}{a}, \frac{a}{3}\right)$.

35. $x = -1$ and $x = \frac{1}{3}$ are roots of $f'(x) = 0$. Therefore,

$$f'(x) = a(3x - 1)(x + 1) = a(3x^2 + 2x - 1)$$
$$\Rightarrow f(x) = a\,(x^3 + x^2 - x + b)$$
$$f(-2) = 0 \Rightarrow b = 2 \Rightarrow f(x) = a\,(x^3 + x^2 - x + 2)$$

$$\int_{-1}^{1} f(x)\,dx = \frac{14}{3} \Rightarrow \int_{-1}^{1} a(x^3 + x^2 - x + 2) = \frac{14}{3}$$

$$\Rightarrow a\int_{-1}^{1} x^2 + 2 = \frac{14}{3} \Rightarrow 2a\left(\frac{1}{3} + 2\right) = \frac{14}{3} \Rightarrow a = 1$$

Therefore,

$$f(x) = x^3 + x^2 - x + 2$$

36. $$g(x) = \frac{d}{dx}\,(f(x)\cdot f'(x))$$

To get the zero of $g(x)$, we take function

$$h(x) = f(x)\cdot f'(x)$$

Between any two roots of $h(x)$, there lies at least one root of $h'(x) = 0$. That is,

$$g(x) = 0$$

Now,

$$h(x) = 0 \text{ and } f(x) = 0$$

or $$f'(x) = 0$$

As $f(x) = 0$ has 4 minimum solutions and $f'(x) = 0$ has minimum 3 solutions, $h(x) = 0$ has minimum 7 solutions and $h'(x) = g(x) = 0$ has minimum 6 solutions.

37. Given

$$y = -p^2x^2 + 5\,px - 4 \quad (1)$$
$$y = \frac{1}{(1-x)} \quad (2)$$

Chord touches curve (2) at $x = 2$ which gives $y = -1$.

Let (x_1, y_1) and (x_2, y_2) are ends of chord.

Touching point is middle point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$$\frac{x_1 + x_2}{2} = 2;\ x_1 + x_2 = 4$$

and $$y_1 + y_2 = -2$$

(x_1, y_1) and (x_2, y_2) satisfy the curve.

Therefore,

$$y_1 = -p^2x_1^2 + 5px_1 - 4 \quad (3)$$

and $$y_2 = -p^2x_2^2 + 5px_2 - 4 \quad (4)$$

Subtracting Eq. (4) from Eq. (3), we get

$$y_1 - y_2 = -p^2x_1^2 + 5px_1 - 4 + p^2x_2^2 - 5px_2 + 4$$
$$\Rightarrow y_1 - y_2 = -p^2(x_1^2 - x_2^2) + 5p(x_1 - x_2)$$

$$\Rightarrow \frac{y_1 - y_2}{x_1 - x_2} = -p^2\,(x_1 + x_2) + 5p = -4p^2 + 5p \quad (5)$$

Again, $$\left(\frac{dy}{dx}\right)_{at\,x=2} = \frac{1}{(1-x)^2} = \frac{1}{1}$$

Therefore, from Eq. (5), we get

$$1 = -4p^2 + 5p$$
$$\Rightarrow 4p^2 - 5p + 1 = 0$$
$$\Rightarrow p = 1, \frac{1}{4}$$

Solved JEE 2017 Questions

JEE Main 2017

1. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point:

(A) $\left(\frac{1}{2}, \frac{1}{2}\right)$ **(B)** $\left(\frac{1}{2}, -\frac{1}{3}\right)$

(C) $\left(\frac{1}{2}, \frac{1}{3}\right)$ **(D)** $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

(OFFLINE)

Solution: It is given that

$$y(x-2)(x-3) = x + 6$$

At y-axis, we know that $x = 0$. Therefore,

$$y(-2)(-3) = 0 + 6$$

Now,

$$y(x^2 - 5x + 6) = x + 6$$

$$\Rightarrow y = \frac{x+6}{x^2-5x+6}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{[(x^2-5x+6)(1)]-[(x+6)(2x-5)]}{(x^2-5x+6)^2}$$

At $x = 0$, we have $y = 1$ as follows:

$$y = \frac{6-[(6)(-5)]}{6^2} = 1$$

Therefore, the equation of normal is

$$y - 1 = -1(x - 0)$$

That is, $y + x - 1 = 0$ or $y + x = 1$.

Thus, the normal to the given curve line passes through $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Hence, the correct answer is option (A).

2. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then, the maximum area (in sq.m) of the flower-bed, is:

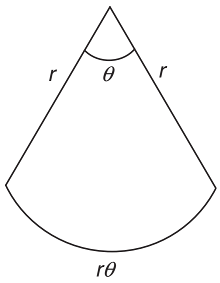
(A) 10 **(B)** 25

(C) 30 **(D)** 12.5

(OFFLINE)

Solution: It is given that $r + r + r\theta = 20$ meters. Therefore,

$$\theta = \frac{20-2r}{r}$$



Now, the area is

$$\frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{20-2r}{r}\right)$$

That is,

$$z = \frac{1}{2}(20r - 2r^2)$$

Differentiating w.r.t. r, we get

$$\frac{dz}{dr} = \frac{1}{2}(20-4r) = 0$$
$$\Rightarrow r = 5$$

At $r = 5$, we get $\theta = 2$; therefore, $\frac{d^2z}{dr^2} < 0$ (hence, it is maxima).

Therefore, the maximum area is

$$z = \frac{1}{2}r^2\theta = \frac{1}{2}\times 5^2 \times 2 = 25 \text{ m}^2$$

Hence, the correct answer is option (B).

3. The tangent at the point $(2, -2)$ to the curve, $x^2y^2 - 2x = 4(1-y)$ does not pass through the point

(A) $(-2, -7)$ **(B)** $(-4, -9)$

(C) $\left(4, \frac{1}{3}\right)$ **(D)** $(8, 5)$

(ONLINE)

Solution: The given curve is

$$x^2y^2 - 2x = 4(1-y)$$

Differentiating this equation, we get

$$2x^2y\frac{dy}{dx} + 2xy^2 - 2 = 4\left(\frac{-dy}{dx}\right)$$

$$\Rightarrow 2x^2y\frac{dy}{dx} + 4\frac{dy}{dx} = 2 - 2xy^2$$

$$\Rightarrow (2x^2y+4)\frac{dy}{dx} = 2 - 2xy^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2-2xy^2}{2x^2y+4}$$

The tangent at point $(2, -2)$ is

$$\left.\frac{dy}{dx}\right|_{(2,-2)} = \frac{2-2(2)(-2)^2}{2(2)^2(-2)+4} = \frac{2-2(2)(4)}{2(4)(-2)+4}\frac{2-16}{-16+4} = \frac{-14}{-12}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(2,-2)} = \frac{14}{12} = \frac{7}{6}$$

The equation of tangent is given by

$$(y - y_1) = \left.\frac{dy}{dx}\right|_{(x,y)}(x - x_1)$$

Here, $x_1 = 2$ and $y_1 = 2$. Therefore,

$$y - (-2) = \frac{7}{6}(x - 2)$$

$$\Rightarrow y+2=\frac{7}{6}(x-2)$$

$$\Rightarrow 6(y+2)=7(x-2)$$

$$\Rightarrow 6y+12=7x-14$$

$$\Rightarrow 7x-6y=26$$

All the given points (8, 5), (–4, –9) and $\left(4,\frac{1}{3}\right)$ satisfy the equation except the point (–2, –7):

$$7x-6y\big|_{(-2,-7)}=28$$

Therefore, the tangent at point (2, –2) of the given curve does not pass through the point (–2, –7).
Hence, the correct answer is option (A).

4. The curve satisfying the differential equation, $ydx-(x+3y^2)dy=0$ and passing through the point (1, 1) also passes through the point

(A) $\left(\frac{1}{4},\frac{1}{2}\right)$ **(B)** $\left(\frac{1}{4},\frac{-1}{2}\right)$

(C) $\left(\frac{1}{3},\frac{-1}{3}\right)$ **(D)** $\left(\frac{-1}{3},\frac{1}{3}\right)$

(ONLINE)

Solution: The given differential equation is

$$ydx-(x+3y^2)dy=0$$

$$\Rightarrow ydx=(x+3y^2)dy$$

$$\Rightarrow \frac{dx}{dy}=\frac{x+3y^2}{y}$$

$$\Rightarrow \frac{dx}{dy}=\frac{x}{y}+3y$$

$$\Rightarrow \frac{dx}{dy}-\left(\frac{1}{y}\right)x=3y$$

The above differential equation is of the form

$$\frac{dx}{dy}+P(y)x=Q(y)$$

with $P(y)=\frac{-1}{y}$ and $Q(y)=3y$.

To solve the differential equation of this form, let us find the integrating factor:

$$\text{I.F}=e^{\int P.dy}=e^{\int\frac{-1}{y}dy}=e^{-\log y}$$

Using standard integral $\int\frac{1}{y}dy=\log y$, we get

$$\text{I.F.}=e^{\log(1/y)} \qquad \left(\text{since } \log\left(\frac{a}{b}\right)=\log a-\log b\right)$$

$$\text{I.F.}=\frac{1}{y}$$

Solution of integral has the form

$$x(\text{I.F.})=\int Q\times(\text{I.F.})dx$$

$$\Rightarrow x\times\frac{1}{y}=\int 3y\times\frac{1}{y}dx$$

$$\Rightarrow x\times\frac{1}{y}=\int 3\,dy$$

$$\Rightarrow x\times\frac{1}{y}=3y+C \qquad \left(\text{since } \int x^n dx=\frac{x^{n+1}}{n+1}\right)$$

$$\Rightarrow x=3y^2+Cy$$

For point (1, 1): $1=3+C\Rightarrow C=-2$
Therefore, $x=3y^2-2y$.
This equation is satisfied by point $\left(\frac{-1}{3},\frac{1}{3}\right)$.

$$\frac{-1}{3}=3\times\left(\frac{1}{3}\right)^2=2\left(\frac{1}{3}\right)\Rightarrow\frac{-1}{3}=\frac{1}{3}-\frac{2}{3}\Rightarrow\frac{-1}{3}=\frac{-1}{3}$$

Hence, the correct answer is option (D).

5. If $2x=y^{1/5}+y^{-(1/5)}$ and $(x^2-1)\frac{d^2y}{dx^2}+\lambda x\frac{dy}{dx}+ky=0$, then $\lambda+k$ is equal to

(A) –23 **(B)** –24

(C) 26 **(D)** –26

(ONLINE)

Solution: It is given that

$$2x=y^{1/5}+y^{-1/5}$$

$$\Rightarrow 2x=y^{1/5}+1/y^{1/5}$$

Therefore,

$$2x=a+\frac{1}{a}\Rightarrow a^2+2ax+1=0$$

$$a=\frac{2x\pm\sqrt{4x^2-4}}{2} \qquad \left(\text{since } x=\frac{2b\pm\sqrt{b^2-4ac}}{2}\right)$$

$$\Rightarrow a=\frac{2x\pm2\sqrt{x^2-1}}{2}$$

$$\Rightarrow a=x\pm\sqrt{x^2-1}$$

$$\Rightarrow y^{1/5}=x\pm\sqrt{x^2-1}$$

$$\Rightarrow y=(x\pm\sqrt{x^2-1})^5$$

Therefore,

$$\frac{dy}{dx}=5(x\pm\sqrt{x^2-1})^4\left(1\pm\frac{2x}{2\sqrt{x^2-1}}\right)=5(x+\sqrt{x^2-1})^4\left(\frac{\sqrt{x^2-1}\pm x}{\sqrt{x^2-1}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-5y}{\sqrt{x^2-1}} \qquad (1)$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{\left[\sqrt{x^2-1}\left(-5\frac{dy}{dx}\right)-5(-5y)\frac{1}{2}\frac{2x}{\sqrt{x^2-1}}\right]}{(x^2-1)}$$

Therefore,

$$(x^2-1)\frac{d^2y}{dx^2} = -5\sqrt{x^2-1}\frac{dy}{dx} + 5y\frac{x}{\sqrt{x^2-1}}$$

$$\Rightarrow (x^2-1)\frac{d^2y}{dx^2} = 25y - x\frac{dy}{dx}$$

$$\Rightarrow (x^2-1)\frac{d^2y}{dx^2} + 1x\frac{dy}{dx} - 25y = 0$$

Therefore, $\lambda = 1$, $k = -25$; hence,

$$\lambda + k = -24$$

Hence, the correct answer is option (B).

6. A tangent to the curve, $y = f(x)$ at P(x, y) meets x-axis at A and y-axis at B. If AP : BP = 1 : 3 and $f(1) = 1$, then the curve also passes through the point

(A) $\left(\frac{1}{2}, 4\right)$ **(B)** $\left(\frac{1}{3}, 24\right)$

(C) $\left(2, \frac{1}{8}\right)$ **(D)** $\left(3, \frac{1}{28}\right)$

(ONLINE)

Solution: We have

$$\frac{(y-y_2)}{(x-x_1)} = f'(x_1) \Rightarrow y - y_1 = f'(x_1)(x - x_1)$$

When $y = 0$: $\frac{-y_1}{f'(x_1)} = x - x_1 \Rightarrow x = x_1 - \frac{y_1}{f'(x_1)}$.

Therefore, point A is $A\left(x_1 - \frac{y_1}{f'(x_1)}, 0\right)$,

When $x = 0$: $y - y_1 = f'(x)\cdot(-x_1) \Rightarrow y = y_1 - x_1 f'(x_1)$.

Therefore, point B is $B(0, y_1 - x_1 f'(x_1))$.

Point P divides AB in the ratio 1 : 3.

$$x_1 = \frac{\left[3\left(x_1 - \frac{y_1}{f'(x_1)}\right)\right]}{4}$$

$$y_1 = \frac{y_1 - x_1 f'(x_1)}{4}$$

Therefore,

$$4y_1 = y_1 - x_1 f'(x_1)$$

$$\Rightarrow f'(x_1) = \frac{-3y_1}{x_1} \Rightarrow f'(x) = \frac{-3y}{x}$$

Now,

$$\frac{dy}{dx} = \frac{-3y}{x} \Rightarrow \frac{dy}{y} = \frac{-3dx}{x}$$

On integrating, we get

$$\ln y = -3\ln x + C \Rightarrow y = kx^{-3}$$

$$y(1) = 1 \Rightarrow k = 1$$

$$y = \frac{1}{x^3}$$

When we substitute the values from the given options, only option (C) satisfies the above equation.

Hence, the correct answer is option (C).

7. The function f defined by $f(x) = x^3 - 3x^2 + 5x + 7$ is

(A) decreasing in R.

(B) increasing in R.

(C) decreasing in $(0,\infty)$ and increasing in $(-\infty,0)$.

(D) increasing in $(0,\infty)$ and decreasing in $(-\infty,0)$.

(ONLINE)

Solution: The given function is

$$f(x) = x^2 - 3x^2 + 5x + 7$$

$$f'(x) = 3x^2 - 6x + 5$$

The discriminant of the above quadratic equation is

$$\Delta = 36 - 4(3)(5) = 36 - 60 < 0$$

Therefore,

$$f'(x) > 0 \ \forall x \in R^+$$

Also,

$$f'(x) > 0 \ \forall x \in R^-$$

Therefore, the given function f is increasing in R.

Hence, the correct answer is option (B).

JEE Advanced 2017

Directions for Questions 1–3: Answer the questions by appropriately matching the information given in the three columns of the following table:

Let $f(x) = x + \log_e x - x\log_e x,\ x \in (0,\infty)$.

- Column 1 contains information about zeros of $f(x), f'(x)$ and $f''(x)$.
- Column 2 contains information about the limiting behaviour of $f(x), f'(x)$ and $f''(x)$ at infinity.
- Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column 1	Column 2	Column 3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x\to\infty} f(x) = 0$	(P) f is increasing in (0, 1)
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x\to\infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x\to\infty} f'(x) = -\infty$	(R) f' is increasing in (0, 1)
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x\to\infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

1. Which of the following options is the only CORRECT combination?

(A) (I) (i) (P) **(B)** (II) (ii) (Q)

(C) (III) (iii) (R) **(D)** (IV) (iv) (S)

Solution: It is given that

$$f(x) = x + \log_e x - x\log_e x, \quad x \in (0,\infty)$$

$$\Rightarrow f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}(x + \log x - x\log x) = 1 + \frac{1}{x} - \log x - x\cdot\frac{1}{x}$$

$$f'(x) = \frac{1}{x} - \log x$$

$$\Rightarrow f''(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{1}{x} - \log x\right) = \frac{-1}{x^2} - \frac{1}{x}$$

- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(x + \log x - x\log x) = -\infty$. Hence, option (ii) is correct.
- $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(\frac{1}{x} - \log x\right) = -\infty$. Hence, option (iii) is correct.
- $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty}\left(\frac{-1}{x^2} - \frac{1}{x}\right) = 0$. Hence, option (iv) is correct.

Also, we have

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}\left(\frac{1}{x} - \log x\right) = \infty$$

Thus, $\lim_{x\to 0^+} f'(x) = \infty$ and $\lim_{x\to\infty} f'(x) = -\infty$ implies $f'(x)$ is decreasing function.
So, options (Q) and (S) are correct. Therefore, from the given options, (II) (ii) (Q) is correct.
Hence, the correct answer is option (B).

2. Which of the following options is the only CORRECT combination?

(A) (I) (i) (R)	**(B)** (II) (iii) (S)
(C) (III) (iv) (P)	**(D)** (IV) (i) (S)

Solution: Following equations in the solution of Question 1, we have the following conclusions:

- In Column 2, option (i) is false and (ii) (iii) and (iv) are correct.
- In Column 3, options (Q) and (S) are correct and options (P) and (R) are false.

Thus, from the given options, only (II) (iii) (S) is correct.
Hence, the correct answer is option (B).

3. Which of the following options is the only INCORRECT combination?

(A) (I) (iii) (P)	**(B)** (II) (iv) (Q)
(C) (III) (i) (R)	**(D)** (II) (iii) (P)

Solution: Following explanation of Question 1, we have the following conclusions:

- In Column 2, option (i) is false.
- In Column 3, options (P) and (R) are false.

Thus, from given options, only (III) (i) (R) combination is INCORRECT.
Hence, the correct answer is option (C).

4. If f: R→R is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in$ R, and $f(0) = 1$, then
(A) $f(x)$ is increasing in $(0,\infty)$.
(B) $f(x)$ is decreasing in $(0,\infty)$.
(C) $f(x) > e^{2x}$ in $(0,\infty)$.
(D) $f'(x) < e^{2x}$ in $f'(x) < e^{2x}$.

Solution: It is given that f: R→R is differentiable function such that $f'(x) > 2f(x)$ for all $x \in$ R and $f(0) = 1$.

$$f'(x) > 2f(x) \Rightarrow f'(x) - 2f(x) > 0$$

Multiplying this with e^{-2x}, we get

$$e^{-2x}f'(x) - 2e^{-2x}f(x) > 0 \Rightarrow \frac{d}{dx}[e^{-2x}f(x)] > 0$$

Therefore, $e^{-2x}f(x)$ is an increasing function.

Let $e^{-2x}f(x) = g(x)$.

- for $x = 0$: $e^{-20}f(0) = 1 = g(0)$.
- for $x > 0$: $g(x) > g(0)$.

That is,

$$e^{-2x}f(x) > f(0) \;\Rightarrow e^{-2x}f(x) > 1. \;\Rightarrow f(x) > \frac{1}{e^{-2x}}$$

Thus,

$$f(x) > e^{2x} \text{ [in } (0, \infty] \qquad (1)$$

It is given that $f'(x) > 2f(x)$. Now, using (1), we get

$$f'(x) > 2f(x) > 2e^{2x}$$

Thus, $f(x)$ is an increasing function [in (0, ∞].
Hence, the correct answers are options (A) and (C).

22 Indefinite Integration

22.1 Primitive or Anti-Derivative of a Function

A function $\phi(x)$ is called a **primitive** or an **anti-derivative** of a function $f(x)$ if $\phi'(x) = f(x)$.

For example, $\frac{x^5}{5}$ is a primitive of x^4, because $\frac{d}{dx}\left(\frac{x^5}{5}\right) = x^4$.

Let $\phi(x)$ be a primitive of a function $f(x)$ and let c be any constant. Then

$$\frac{d}{dx}(\phi(x)+c) = \phi'(x) = f(x) \text{ [since } \phi'(x) = f(x)]$$

So, $\phi(x)+c$ is also a primitive of $f(x)$.

Thus, if a function $f(x)$ possesses a primitive, then it possesses infinitely many primitives that are contained in the expression $\phi(x)+c$ where c is a constant.

For example, $\frac{x^5}{5}, \frac{x^5}{5} - 2, \frac{x^5}{5} + 1$, etc. are primitives of x^4.

22.2 Indefinite Integral and Indefinite Integration

Let $f(x)$ be a function. Then the collection of all its primitives is called the **indefinite integral** of $f(x)$ and is denoted by $\int f(x)dx$.

Thus, $\frac{d}{dx}(\phi(x)+c) = f(x) \Rightarrow \int f(x)dx = \phi(x)+c$.

where $\phi(x)$ is the primitive of $f(x)$ and c is an arbitrary constant known as the **constant of integration**.

Here $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration and dx is the element of integration.

The process of finding an indefinite integral of a given function is called integration of the function.

It follows from the above discussion that integrating a function $f(x)$ means finding a function $\phi(x)$ such that $\frac{d}{dx}\phi(x) = f(x)$.

22.2.1 Fundamental Properties of Integration

1. $\int c \cdot f(x)dx = c\int f(x)dx$
2. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
3. $\int f(x)dx = g(x)+c \Rightarrow \int f(ax+b)dx = \frac{1}{a}g(ax+b)+c$

Note: Every continuous function is integrable.

22.2.2 Fundamental Formulas on Integration

1. $\int 1dx = x+c$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1}+c,\ n \neq -1 \quad \left[\text{since, } \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n\right]$
3. $\int \frac{1}{x}dx = \ln(x)+c \left[\text{since, } \frac{d}{dx}(\ln x) = \frac{1}{x}\right]$
4. $\int (ax+b)^n\,dx = \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1}+c,\ n \neq -1$
5. $\int \frac{1}{(ax+b)}dx = \frac{1}{a}\cdot \ln|ax+b|+c$
6. $\int e^x\,dx = e^x + c \quad \left[\text{since, } \frac{d}{dx}(e^x) = e^x\right]$
7. $\int a^x\,dx = \frac{a^x}{\ln a}+c \quad \left[\text{since, } \frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = a^x\right]$
8. $\int \sin x\,dx = -\cos x + c \quad \left[\text{since, } \frac{d}{dx}(-\cos x) = \sin x\right]$
9. $\int \cos x\,dx = \sin x + c \quad \left[\text{since, } \frac{d}{dx}(\sin x) = \cos x\right]$
10. $\int \sec^2 x\,dx = \tan x + c \quad \left[\text{since, } \frac{d}{dx}(\tan x) = \sec^2 x\right]$
11. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c \quad \left[\text{since, } \frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x\right]$
12. $\int \sec x \cdot \tan x\,dx = \sec x + c \quad \left[\text{since, } \frac{d}{dx}(\sec x) = \sec x \cdot \tan x\right]$
13. $\int \operatorname{cosec} x \cdot \cot x\,dx = -\operatorname{cosec} x + c$

    $\left[\text{since, } \frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cdot \cot x\right]$
14. $\int \tan x\,dx = \ln|\sec x| + c = -\ln|\cos x| + c$

    $\left[\text{since, } \frac{d}{dx}(\ln\cos x) = -\tan x\right]$
15. $\int \cot x\,dx = \ln|\sin x| + c = -\ln|\operatorname{cosec} x| + c$

    $\left[\text{since, } \frac{d}{dx}(\ln\sin x) = \cot x\right]$

16. $\int \sec x\,dx = \ln|\sec x + \tan x| + c = \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+c$

$\left[\text{since, } \frac{d}{dx}(\ln(\sec x + \tan x)) = \sec x\right]$

17. $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c = \ln\tan\left(\frac{x}{2}\right)+c$

$\left[\text{since, } \frac{d}{dx}(\ln(\operatorname{cosec} x - \cot x)) = \operatorname{cosec} x\right]$

18. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c = -\cos^{-1}x + c$

$\left[\text{since, } \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$

19. $\int \frac{dx}{1+x^2} = \tan^{-1}x + c = -\cot^{-1}x + c$

$\left[\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}\right]$

20. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + c = -\operatorname{cosec}^{-1}x + c$

$\left[\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}, \frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}\right]$

22.2.2.1 Some Standard Results on Integration

21. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + c$

$\left[\text{since } \frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}\right]$

22. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\cdot\tan^{-1}\frac{x}{a} + c = -\frac{1}{a}\cdot\cot^{-1}\frac{x}{a} + c$

$\left[\text{since } \frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2+x^2}\right]$

23. $\int \frac{dx}{x\cdot\sqrt{x^2-a^2}} = \frac{1}{a}\cdot\sec^{-1}\frac{x}{a} + c = -\frac{1}{a}\cdot\operatorname{cosec}^{-1}\frac{x}{a} + c$

$\left[\text{since } \frac{d}{dx}\left(\sec^{-1}\frac{x}{a}\right) = \frac{a}{x\cdot\sqrt{x^2-a^2}}\right]$

24. $\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\cdot\coth^{-1}\frac{x}{a} + c = \frac{1}{2a}\cdot\ln\left|\frac{x-a}{x+a}\right| + c,\ x > a$

25. $\int \frac{dx}{a^2-x^2} = -\frac{1}{a}\cdot\tanh^{-1}\frac{x}{a} + c = \frac{1}{2a}\cdot\ln\left|\frac{a+x}{a-x}\right| + c,\ x < a$

26. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c = \cosh^{-1}\frac{x}{a} + c$

27. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + c = \sinh^{-1}\frac{x}{a} + c$

28. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

29. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$

30. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

Key points:

1. The signum function has an anti-derivative on any interval which does not contain the point $x = 0$, and does not possess an anti-derivative on any interval which contains the point.
2. The anti-derivative of every odd function is an even function and vice versa.

Illustration 22.1 Evaluate $\int \frac{\sin x}{1+\sin x}dx$.

Solution:

$$\int \frac{\sin x}{1+\sin x}dx = \int \frac{\sin x}{(1+\sin x)}\cdot\frac{(1-\sin x)}{(1-\sin x)}dx = \int \frac{\sin x - \sin^2 x}{\cos^2 x}dx$$

$$= \int\left(\sec x\cdot\tan x - \tan^2 x\right)dx$$

$$= \int(1-\sec^2 x + \sec x\cdot\tan x)dx = x - \tan x + \sec x + c$$

Illustration 22.2 Evaluate $\int \frac{(x+1)^2}{x(x^2+1)}dx$.

Solution:

$$\int \frac{(x+1)^2}{x(x^2+1)}dx = \int \frac{(x^2+2x+1)}{x(x^2+1)}dx$$

$$= \int \frac{x^2+1}{x(x^2+1)}dx + \int \frac{2x}{x(x^2+1)}dx$$

$$= \int \frac{1}{x}dx + \int \frac{2}{(x^2+1)}dx = \ln x + 2\tan^{-1}x + c$$

Illustration 22.3 Evaluate $\int \frac{ax^3+bx^2+c}{x^4}dx$.

Solution:

$$\int \frac{ax^3+bx^2+c}{x^4}dx = \int\left(\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x^4}\right)dx = a\ln x - \frac{b}{x} - \frac{c}{3x^3} + k$$

Illustration 22.4 Evaluate $\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}}dx$.

Solution:

$$\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{x+1}}dx = \int \frac{\sqrt{x+1}(\sqrt{x}+\sqrt{x+1})}{\sqrt{x}+\sqrt{x+1}}dx$$

$$= \int \sqrt{x+1}\,dx = \frac{2}{3}(x+1)^{\frac{3}{2}} + c$$

Illustration 22.5 Evaluate $\int(\sin^4 x - \cos^4 x)dx$.

Solution:

$$\int(\sin^4 x - \cos^4 x)dx = \int(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)dx$$

$$= \int(\sin^2 x - \cos^2 x)\,dx = -\int\cos 2x\,dx = -\frac{1}{2}\sin 2x + c$$

Illustration 22.6 Evaluate $\int \sqrt{1+\sin\left(\frac{x}{4}\right)}\,dx$.

Solution:

$$\int \sqrt{1+\sin\left(\frac{x}{4}\right)}\,dx = \int \sqrt{\sin^2\left(\frac{x}{8}\right)+\cos^2\left(\frac{x}{8}\right)+2\sin\left(\frac{x}{8}\right)\cos\left(\frac{x}{8}\right)}\,dx$$

$$= \int \sqrt{\left[\sin\left(\frac{x}{8}\right)+\cos\left(\frac{x}{8}\right)\right]^2}\,dx$$

$$= \int \left[\sin\left(\frac{x}{8}\right)+\cos\left(\frac{x}{8}\right)\right]dx = -\frac{\cos(x/8)}{1/8}+\frac{\sin(x/8)}{1/8}+c$$

$$= 8[\sin(x/8)-\cos(x/8)]+c$$

Illustration 22.7 Evaluate $\int \frac{2x}{(2x+1)^2}\,dx$.

Solution:

$$\int \frac{2x}{(2x+1)^2}\,dx = \int \frac{2x+1-1}{(2x+1)^2}\,dx$$

$$= \int \frac{1}{(2x+1)}\,dx - \int \frac{1}{(2x+1)^2}\,dx$$

$$= \frac{1}{2}\ln|2x+1| + \frac{1}{2(2x+1)} + c$$

Illustration 22.8 Evaluate $\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cdot\cos^2 x}\,dx$.

Solution:

$$\int \frac{\sin^2 x-\cos^2 x}{\sin^2 x\cdot\cos^2 x}\,dx = \int \sec^2 x\,dx - \int \operatorname{cosec}^2 x\,dx$$

$$= \tan x + \cot x + c$$

Illustration 22.9 Evaluate $\int (3\operatorname{cosec}^2 x + 2\sin 3x)\,dx$.

Solution:

$$\int (3\operatorname{cosec}^2 x + 2\sin 3x)\,dx = -3\cot x - \frac{2}{3}\cos 3x + c$$

Illustration 22.10 Evaluate $\int \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$.

Solution:

$$\int \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx = \int \frac{(\sqrt{1+x}-\sqrt{x})}{(\sqrt{1+x}+\sqrt{x})\cdot(\sqrt{1+x}-\sqrt{x})}\,dx$$

$$= \int (\sqrt{1+x}-\sqrt{x})\,dx = \frac{(x+1)^{3/2}}{3/2} - \frac{(x)^{3/2}}{3/2} + c$$

$$= \frac{2}{3}[(x+1)^{3/2}-(x)^{3/2}]+c$$

Your Turn 1

1. $\int \frac{\sin x}{\sin(x-\alpha)}\,dx =$

(A) $x\cos\alpha - \sin\alpha \ln\sin(x-\alpha)+c$

(B) $x\cos\alpha + \sin\alpha \ln\sin(x-\alpha)+c$

(C) $x\sin x - \alpha - \sin\alpha \ln\sin(x-\alpha)+c$

(D) None of these

Ans. (B)

2. $\int \frac{\cos x - 1}{\cos x + 1}\,dx =$

(A) $2\tan\frac{x}{2} - x + c$ **(B)** $\frac{1}{2}\tan\frac{x}{2} - x + c$

(C) $-\frac{1}{2}\tan\frac{x}{2} + x + c$ **(D)** $-2\tan\frac{x}{2} + x + c$

Ans. (D)

3. $\int \frac{1}{1-\sin x}\,dx =$

(A) $x+\cos x+c$ **(B)** $1+\sin x+c$

(C) $\sec x - \tan x + c$ **(D)** $\sec x + \tan x + c$

Ans. (D)

4. If $\int (\sin 2x - \cos 2x)\,dx = \frac{1}{\sqrt{2}}\sin(2x-\alpha)+b$, then

(A) $a=\frac{\pi}{4}, b=0$

(B) $a=-\frac{\pi}{4}, b=0$

(C) $a=\frac{5\pi}{4}$, $b=$ any constant

(D) $a=-\frac{5\pi}{4}$, $b=$ any constant **Ans.** (D)

5. $\int \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)dx =$

(A) $-e^x+c$ **(B)** e^x+c

(C) $e^{-x}+c$ **(D)** $-e^{-x}+c$

Ans. (B)

6. $\int \frac{\cot x\cdot\tan x}{\sec^2 x - 1}\,dx =$

(A) $\cot x - x + c$ **(B)** $-\cot x + x + c$

(C) $\cot x + x + c$ **(D)** $-\cot x - x + c$

Ans. (D)

7. $\int (\sec x + \tan x)^2\,dx =$

(A) $2(\sec x + \tan x) - x + c$ **(B)** $\frac{1}{3}(\sec x+\tan x)^3+c$

(C) $\sec x(\sec x+\tan x)+c$ **(D)** $2(\sec x + \tan x)+c$

Ans. (A)

8. $\int x^{51}(\tan^{-1}x + \cot^{-1}x)\,dx =$

(A) $\frac{x^{52}}{52}(\tan^{-1}x+\cot^{-1}x)+c$ **(B)** $\frac{x^{52}}{52}(\tan^{-1}x-\cot^{-1}x)+c$

(C) $\frac{\pi x^{52}}{104}+\frac{\pi}{2}+c$ **(D)** $\frac{\pi x^{52}}{52}+\frac{\pi}{2}+c$

Ans. (A)

9. $\int 5\sin x\,dx =$

(A) $5\cos x + c$ **(B)** $-5\cos x + c$

(C) $5\sin x + c$ **(D)** $-5\sin x + c$

Ans. (B)

10. $\int \frac{\tan x}{\sec x + \tan x} dx =$

(A) $\sec x + \tan x - x + c$ (B) $\sec x - \tan x + x + c$
(C) $\sec x + \tan x + x + c$ (D) $-\sec x - \tan x + x + c$

Ans. (B)

22.3 Methods of Integration

22.3.1 Integration by Substitution

1. $I = \int f(\phi(x))\phi'(x)dx$: Here, we put $\phi(x) = t$, so that $\phi'(x)dx = dt$ and in that case

$$\int f(\phi(x))\phi'(x)dx = \int f(t)dt$$

Illustration 22.11 Evaluate $\int x^3 \sin x^4 \, dx$.

Solution: We have

$$I = \int x^3 \sin x^4 \, dx$$

Let $x^4 = t$. Then

$$4x^3 dx = dt \Rightarrow dx = \frac{dt}{4x^3}$$

$$\Rightarrow I = \int \frac{\sin t}{4} dt = -\frac{\cos t}{4} + c = \frac{\cos x^4}{4} + c$$

Illustration 22.12 Evaluate $\int \frac{\sin(\ln x)}{x} dx$.

Solution: We have

$$I = \int \frac{\sin(\ln x)}{x} dx$$

Let $\ln x = t$. Then

$$\frac{dx}{x} = dt$$

$$\Rightarrow I = \int \sin t \, dt = -\cos t + c - \cos(\ln x) + c$$

Illustration 22.13 Evaluate $\int \frac{x}{x^4 + x^2 + 1} dx$.

Solution: We have

$$I = \int \frac{x}{x^4 + x^2 + 1} dx$$

Let $x^2 = t$. Then $2x dx = dt$.

$$I = \int \frac{x}{x^4 + x^2 + 1} dx = \frac{1}{2} \int \frac{1}{t^2 + t + 1} dt$$

$$I = \frac{1}{2} \int \frac{1}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\left(t + \frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\right) + c = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\left(x^2 + \frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\right) + c$$

2. $I = \int f(x) \cdot f'(x) dx$: In this case, we put $f(x) = t \Rightarrow f'(x)dx = dt$

Illustration 22.14 Evaluate $\int \sin x \cdot \cos x \, dx$.

Solution: We have

$$I = \int \sin x \cdot \cos x \, dx$$

Let $\sin x = t$. Then $\cos x dx = dt$.

$$I = \int \sin x \cdot \cos x \, dx = \int t \, dt$$

$$\Rightarrow I = \frac{t^2}{2} + c = \frac{(\sin x)^2}{2} + c$$

Illustration 22.15 Evaluate

$$\int (x^{3m} + x^{2m} + x^m) \cdot (2x^{2m} + 3x^m + 6)^{1/m} dx, \ (x > 0)$$

Solution:

$$I = \int (x^{3m-1} + x^{2m-1} + x^{m-1}) \cdot (2x^{3m} + 3x^{2m} + 6x^m)^{1/m} dx$$

Put $(2x^{3m} + 3x^{2m} + 6x^m) = t$. Then

$$6m(x^{3m-1} + x^{2m-1} + x^{m-1})dx = dt$$

$$I = \int \frac{t^{\frac{1}{m}}}{6m} dt = \frac{t^{\frac{1}{m}+1}}{6m\left(\frac{1}{m}+1\right)} + c = \frac{t^{\frac{m+1}{m}}}{6(m+1)} + c$$

$$\Rightarrow I = \frac{(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m+1}{m}}}{6(m+1)} + c$$

Illustration 22.16 Evaluate $\int \frac{x^2 \tan^{-1} x^3}{1 + x^6} dx$.

Solution:

$$I = \int \frac{x^2 \tan^{-1} x^3}{1 + x^6} dx$$

Put $\tan^{-1} x^3 = t$. Then

$$\frac{3x^2}{1 + x^6} dx = dt$$

$$\Rightarrow I = \frac{1}{3} \int t dt = \frac{1}{3} \cdot \frac{t^2}{2} + c = \frac{(\tan^{-1} x^3)^2}{6} + c$$

3. $I = \int \frac{f'(x)}{f(x)} dx$: In this case, we put $f(x) = t$ and $f'(x)dx = dt$. So,

$$I = \int \frac{f'(x)}{f(x)} dx = \int \frac{1}{t} dt = \ln(f(x)) + c$$

Illustration 22.17 Evaluate $\int \frac{x^3}{1 + x^4} dx$.

Solution:

$$I = \int \frac{x^3}{1 + x^4} dx$$

Put $1+x^4=t$. Then $\Rightarrow 4x^3dx=dt$

$$\Rightarrow I=\frac{1}{4}\int\frac{1}{t}dt=\frac{1}{4}\ln t+c=\frac{1}{4}\ln(1+x^4)+c$$

Illustration 22.18 Evaluate $\int\frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}dx$.

Solution:

$$I=\int\frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}dx$$

Put $a^2\sin^2 x+b^2\cos^2 x=t$. Then $\sin 2x(a^2-b^2)dx=dt$

$$\Rightarrow I=\frac{1}{(a^2-b^2)}\int\frac{1}{t}dt=\frac{1}{(a^2-b^2)}\ln t+c$$
$$=\frac{1}{(a^2-b^2)}\ln(a^2\sin^2 x+b^2\cos^2 x)+c$$

Illustration 22.19 Evaluate $\int\frac{1}{1+e^x}dx$.

Solution:

$$I=\int\frac{1}{1+e^x}dx=\int\frac{e^{-x}}{1+e^{-x}}dx$$

Put $1+e^{-x}=t$. Then $-e^{-x}dx=dt$

$$\Rightarrow I=-\int\frac{1}{t}dt=-\ln t+c=-\ln(1+e^{-x})+c$$

4. $I=\int(f(x))^n\cdot f'(x)dx$: In this case, we put $f(x)=t$ and $f'(x)dx=dt$. So,

$$I=\int(f(x))^n\cdot f'(x)dx=\frac{(f(x))^{n+1}}{n+1}+c$$

Illustration 22.20 Evaluate $\int\frac{(\ln x)^5}{x}dx$.

Solution:

$$I=\int\frac{(\ln x)^5}{x}dx$$

Put $\ln x=t$. Then $\frac{dx}{x}=dt$

$$\Rightarrow I=\int t^5dt=\frac{t^6}{6}+c=\frac{(\ln x)^6}{6}+c$$

Illustration 22.21 Evaluate $\int\sin^{10}x\cdot\cos x\,dx$.

Solution:

$$I=\int\sin^{10}x\cdot\cos x\,dx$$

Put $\sin x=t$. Then $\cos x\,dx=dt$

$$\Rightarrow I=\int t^{10}dt=\frac{t^{11}}{11}+c=\frac{(\sin x)^{11}}{11}+c$$

Illustration 22.22 Evaluate $\int\cos 3x\cdot\sqrt{2+\sin 3x}\,dx$.

Solution:

$$I=\int\cos 3x\cdot\sqrt{2+\sin 3x}\,dx$$

Put $2+\sin 3x=t$. Then

$$3\cos 3x\,dx=dt$$

$$\Rightarrow I=\frac{1}{3}\int t^{\frac{1}{2}}dt=\frac{1}{3}\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c=\frac{2(2+\sin 3x)^{\frac{3}{2}}}{9}+c$$

5. **Standard substitutions:**

	Integrand form	Substitution
(a)	$\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$, a^2-x^2	$x=a\sin\theta$ or $a\cos\theta$
(b)	$\sqrt{a^2+x^2}$, $\frac{1}{\sqrt{a^2+x^2}}$, a^2+x^2	$x=a\tan\theta$ or $a\cot\theta$
(c)	$\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$, x^2-a^2	$x=a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$
(d)	$\sqrt{\frac{x}{x+a}}$, $\sqrt{\frac{x+a}{x}}$, $\sqrt{x(x+a)}$, $\sqrt{\frac{1}{x(x+a)}}$	$x=a\tan^2\theta$ or $a\cot^2\theta$
(e)	$\sqrt{\frac{x}{a-x}}$, $\sqrt{\frac{a-x}{x}}$, $\sqrt{x(a-x)}$, $\frac{1}{\sqrt{x(a-x)}}$	$x=a\sin^2\theta$ or $x=a\cos^2\theta$
(f)	$\sqrt{\frac{x}{x-a}}$, $\sqrt{\frac{x-a}{x}}$, $\sqrt{x(x-a)}$, $\sqrt{\frac{1}{x(x-a)}}$	$x=a\sec^2\theta$ or $a\,\mathrm{cosec}^2\theta$
(g)	$\sqrt{\frac{a+x}{a-x}}$, $\sqrt{\frac{a-x}{a+x}}$	$x=a\cos 2\theta$
(h)	$\sqrt{\frac{x-\alpha}{\beta-x}}$, $\sqrt{(x-\alpha)(\beta-x)}$, $(\beta>\alpha)$	$x=\beta\sin^2\theta+\alpha\cos^2\theta$

Illustration 22.23 Evaluate $\int\frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$.

Solution: Let $x=\sin^2\theta$. Then

$$dx=2\sin\theta\cdot\cos\theta\cdot d\theta$$

$$I=\int\frac{2\sin\theta\cdot\cos\theta\cdot d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta-\sin^4\theta}}=\int\frac{2d\theta}{(1+\sin\theta)}=\int\frac{2(1-\sin\theta)d\theta}{\cos^2\theta}$$

$$\Rightarrow I = 2\int(\sec^2\theta - \sec\theta\cdot\tan\theta)\,d\theta = 2(\tan\theta - \sec\theta) + c$$

$$\Rightarrow I = 2\left[\frac{\sin\theta - 1}{\cos\theta}\right] + c = \frac{2(\sqrt{x}-1)}{\sqrt{1-x}} + c$$

Illustration 22.24 Evaluate $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$.

Solution:

$$I = \int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx = \int\sqrt{\frac{(1-\sqrt{x})^2}{1-x}}\,dx = \int\frac{1}{\sqrt{1-x}}dx - \int\frac{\sqrt{x}}{\sqrt{1-x}}\,dx$$

Let $x = \sin^2\theta$. Then

$$dx = 2\sin\theta\cdot\cos\theta\cdot d\theta$$

$$I = \int 2\sin\theta\,d\theta - \int 2\sin^2\theta\,d\theta = 2\int\sin\theta\,d\theta - \int(1-\cos 2\theta)\,d\theta$$

$$= -2\cos\theta - \theta + \frac{\sin 2\theta}{2} + c$$

$$I = \sqrt{x}\cdot\sqrt{1-x} - 2\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$$

Illustration 22.25 Evaluate $\int t\sqrt{\frac{t^2+1}{t^2-1}}dt$.

Solution: Put $s = t^2$. Then $ds = 2t\,dt$.

Now,

$$I = \frac{1}{2}\int\sqrt{\frac{s+1}{s-1}}\,ds = \frac{1}{2}\int\frac{1+s}{\sqrt{s^2-1}}ds$$

$$= \frac{1}{2}\int\frac{1}{\sqrt{s^2-1}}ds + \frac{1}{2}\int\frac{s}{\sqrt{s^2-1}}ds = \frac{1}{2}\ln\left|s+\sqrt{s^2-1}\right| + \frac{1}{4}\int\frac{2s\,ds}{\sqrt{s^2-1}}$$

Let $s^2 = x \Rightarrow 2s\,ds = dx$. Then

$$\frac{1}{4}\int\frac{2s\,ds}{\sqrt{s^2-1}} = \frac{1}{4}\int\frac{dx}{\sqrt{x-1}} = \frac{1}{2}(\sqrt{x-1}) = \frac{1}{2}\sqrt{s^2-1}$$

So,

$$I = \frac{1}{2}\ln\left|s+\sqrt{s^2-1}\right| + \frac{1}{2}\sqrt{s^2-1}$$

$$= \frac{1}{2}\ln\left|t^2+\sqrt{t^4-1}\right| + \frac{1}{2}\sqrt{t^4-1} + c$$

Illustration 22.26 Evaluate $\int\frac{dx}{(a^2+x^2)^{3/2}}$.

Solution:

$$I = \int\frac{dx}{(a^2+x^2)^{3/2}}$$

Put $x = a\tan\theta$. Then

$$dx = a\sec^2\theta\,d\theta$$

Therefore,

$$I = \int\frac{a\sec^2\theta\,d\theta}{(a^2+(a\tan\theta)^2)^{3/2}}$$

$$\Rightarrow I = \int\frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta} = \int\frac{d\theta}{a^2\sec\theta}$$

$$\Rightarrow I = \frac{1}{a^2}\int\frac{d\theta}{\sec\theta} = \frac{1}{a^2}\int\cos\theta\,d\theta = \frac{1}{a^2}\sin\theta + c$$

$$\Rightarrow I = \frac{x}{a^2(x^2+a^2)^{1/2}} + c$$

Illustration 22.27 Evaluate $\int\sqrt{\frac{1-x}{1+x}}dx$.

Solution:

$$I = \int\sqrt{\frac{1-x}{1+x}}\,dx$$

Put $x = \cos 2\theta$. Then $dx = -2\sin 2\theta\cdot d\theta$.

$$I = -2\int\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\sin 2\theta\cdot d\theta$$

$$\Rightarrow I = -2\int\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\sin 2\theta\cdot d\theta = -4\int\tan\theta\cdot\sin\theta\cdot\cos\theta\cdot d\theta$$

$$\Rightarrow I = -4\int\sin^2\theta\,d\theta = -2\int(1-\cos 2\theta)d\theta$$

$$\Rightarrow I = -2\left(\theta - \frac{\sin 2\theta}{2}\right) + c = -2\theta - \sin 2\theta + c$$

$$\Rightarrow I = -\cos^{-1}x + \sqrt{1-x^2} + c$$

6. Some more substitution:

1. For the type $\left(\sqrt{x^2+a^2}\pm x\right)^n$ or $\left(x\pm\sqrt{x^2-a^2}\right)^n$, put the expression within the bracket $= t$.

2. For the type $(x+a)^{-1-\frac{1}{n}}\cdot(x+b)^{-1+\frac{1}{n}}$ or

$$\left(\frac{(x+b)}{(x+a)}\right)^{-1+\frac{1}{n}}\cdot\frac{1}{(x+a)^2}\ (n\in N, n>1),\text{ put }\frac{(x+b)}{(x+a)} = t.$$

3. For $\frac{1}{(x+a)^{n_1}(x+b)^{n_2}}$, $n_1, n_2 \in N$ (and >1), again put $(x+a) = t(x+b)$.

Illustration 22.28 Evaluate $\int\frac{dx}{(x+1)^{6/5}(x-3)^{4/5}}$

Solution:

$$I = \int\frac{dx}{(x+1)^{6/5}(x-3)^{4/5}} = \int\frac{dx}{(x+1)^2\left(\frac{(x-3)}{(x+1)}\right)^{4/5}}$$

Put $\left(\frac{(x-3)}{(x+1)}\right) = t$. Then

$$dt = \frac{4}{(x+1)^2}dx$$

Hence,

$$I = \int\frac{dt}{4t^{4/5}} = \frac{5}{4}t^{1/5} + c = \frac{5}{4}\left(\frac{(x-3)}{(x+1)}\right)^{1/5} + c$$

Illustration 22.29 Evaluate $\int \frac{dx}{(x+1)^2(x-3)^3}$.

Solution:

$$I = \int \frac{dx}{(x+1)^2(x-3)^3}$$

Put $\left(\frac{(x-3)}{(x+1)}\right) = t$. Then $dt = \frac{4}{(x+1)^2}dx$

$$\Rightarrow (x+1) = \frac{4}{(1-t)}$$

$$I = \int \frac{(1-t)^3 dt}{4^4 t^3} = \int \frac{(1-t^3+3t^2-3t)dt}{4^4 t^3} = \frac{1}{4^4}\int (t^{-3} - 1 + 3t^{-1} - 3t^{-2})dt$$

$$= \frac{1}{4^4}\left(-\frac{1}{2t^2} - t + 3\ln t + 3t^{-1}\right) + c$$

$$I = \frac{1}{4^4}\left(-\frac{1}{2}\left(\frac{(x-3)}{(x+1)}\right)^{-2} - \left(\frac{(x-3)}{(x+1)}\right) + 3\ln\left(\frac{(x-3)}{(x+1)}\right) + 3\left(\frac{(x-3)}{(x+1)}\right)^{-1}\right) + c$$

Illustration 22.30 Evaluate $\int \left(\sqrt{x^2+2^2}+x\right)^3 dx.$

Solution:

$$I = \int \left(\sqrt{x^2+2^2}+x\right)^3 dx$$

Put $\left(\sqrt{x^2+2^2}+x\right) = t$. Then

$$dx = \frac{\sqrt{x^2+2^2}}{\left(\sqrt{x^2+2^2}+x\right)}dt$$

$$\Rightarrow \sqrt{x^2+2^2} = \frac{t^2+2^2}{2t} \Rightarrow dx = \frac{t^2+2^2}{2t^2}dt$$

$$I = \int \frac{(t^2+2^2)\cdot t^3}{2t^2}dt = \frac{1}{2}\int (t^3+2^2 t)dt = \frac{1}{2}\left(\frac{t^4}{4} + 2^2\frac{t^2}{2}\right) + c$$

$$= \frac{1}{2}\left(\frac{\left(\sqrt{x^2+2^2}+x\right)^4}{4} + 2\left(\sqrt{x^2+2^2}+x\right)^2\right) + c$$

Your Turn 2

1. $\int \frac{dx}{x\sqrt{1-(\ln x)^2}} =$

   (A) $\cos^{-1}(\ln x)+c$ **(B)** $x\ln(1-x^2)+c$
   (C) $\sin^{-1}(\ln x)+c$ **(D)** $\frac{1}{2}\cos^{-1}(\ln x)+c$ **Ans.** (C)

2. $\int \frac{f'(x)dx}{(f(x))^2} =$

   (A) $-(f(x))^{-1}+c$ **(B)** $\ln(f(x))+c$
   (C) $e^{f(x)}+c$ **(D)** None of these **Ans.** (A)

3. For which of the following functions, the substitution $x^2 = t$ is applicable?

   (A) $\int x^6 \tan^{-1} x^3\, dx$ **(B)** $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$
   (C) $\int x^3 \cos x^2\, dx$ **(D)** None of these **Ans.** (C)

4. $\int \tan x \cdot \sec^2 x \cdot \sqrt{1-\tan^2 x}\, dx =$

   (A) $-\frac{1}{3}(1-\tan^2 x)^{3/2}+c$ **(B)** $\frac{1}{3}(1-\tan^2 x)^{3/2}+c$
   (C) $-\frac{2}{3}(1-\tan^2 x)^{2/3}+c$ **(D)** None of these **Ans.** (A)

5. $\int \frac{\sin 2x}{\sin 5x \cdot \sin 3x}dx =$

   (A) $\ln\sin 3x - \ln\sin 5x + c$
   (B) $\frac{1}{3}\ln\sin 3x + \frac{1}{5}\ln\sin 5x + c$
   (C) $\frac{1}{3}\ln\sin 3x - \frac{1}{5}\ln\sin 5x + c$
   (D) $3\ln\sin 3x - 5\ln\sin 5x + c$ **Ans.** (C)

6. $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx =$

   (A) $\cos^{-1}(e^x)+c$ **(B)** $-\cos^{-1}(e^x)+c$
   (C) $\cos^{-1}(e^{2x})+c$ **(D)** $\sqrt{1-e^{2x}}+c$ **Ans.** (B)

7. $\int \frac{a^x\cos(a^x)}{\ln a}dx =$

   (A) $\sin a^x + c$ **(B)** $a^x \sin a^x + c$
   (C) $\frac{\sin(a^x)}{\ln^2 a}+c$ **(D)** $\ln\sin a^x + c$ **Ans.** (C)

8. $\int \frac{\sin x}{(a+b\cos x)^2}dx =$

   (A) $\frac{1}{b}(a+b\cos x)+c$ **(B)** $\frac{1}{b(a+b\cos x)}+c$
   (C) $\frac{1}{b}\ln(a+b\cos x)+c$ **(D)** None of these **Ans.** (B)

9. $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx =$

   (A) $\ln\left(\tan x + \sqrt{\tan^2 x+4}\right)+c$
   (B) $\frac{1}{2}\ln\left(\tan x + \sqrt{\tan^2 x+4}\right)+c$
   (C) $\ln\left(\frac{1}{2}\tan x + \frac{1}{2}\sqrt{\tan^2 x+4}\right)+c$
   (D) None of these **Ans.** (A)

10. $\int \frac{2x\cdot\tan^{-1}x^2}{x^4+1}dx =$

(A) $(\tan^{-1}x^2)^2+c$ **(B)** $\frac{1}{2}(\tan^{-1}x^2)^2+c$

(C) $2(\tan^{-1}x^2)^2+c$ **(D)** None of these **Ans.** (B)

22.3.2 Integration by Parts

If F and G are two functions of x, then integral of the product of these two functions is given by

$$\int F\cdot G\,dx = F\int G\,dx - \int\left(\frac{dF}{dx}\int G\,dx\right)dx$$

Or we can say that the integral of the product of two functions = (First function) × (Integral of second function) – Integral of {(Differentiation of first function) × (Integral of second function)}.

Note: In applying the above rule, care has to be taken in the selection of the first function (F) and the second function (G).

Normally, we use the following methods:

1. In the product of two functions, one of the function is not directly integrable (that is, $\ln|x|$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, etc.), then we take it as the first function and the remaining function is taken as the second function.
2. If there is no other function, then unity is taken as the second function. For example, in the integration of $\int \ln|x|dx$, $\int \sin^{-1}x\,dx$, 1 is taken as the second function.
3. If both of the functions are directly integrable, then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable.

In the above stated order, the function on the left is always chosen as the first function. This rule is known as **ILATE (Inverse, Logarithmic, Algebraic, Trigonometric, Exponential)**.

For example, in the integration of $\int x\sin x\,dx$, x is taken as the first function and $\sin x$ is taken as the second function.

Illustration 22.31 Evaluate $\int \sec^3\theta\, d\theta$.

Solution:

$$I=\int \sec^3\theta\, d\theta = \sec\theta\int\sec^2\theta\,d\theta - \int \tan\theta(\sec\theta\tan\theta)\,d\theta$$

$$= \sec\theta\cdot\tan\theta - \int\sec\theta(\sec^2\theta-1)\,d\theta$$

$$= \sec\theta\cdot\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta$$

$$\Rightarrow I = \sec\theta\cdot\tan\theta - I + \int\sec\theta\,d\theta$$

$$\Rightarrow I = \frac{1}{2}[\sec\theta\cdot\tan\theta] + \frac{1}{2}\ln|\sec\theta+\tan\theta| + c$$

Illustration 22.32 Evaluate $\int x\cdot\sin x\,dx$.

Solution:

$$I=\int x\cdot\sin x\,dx = -x\cos x + \int\cos x\,dx = -x\cos x+\sin x + c$$

Illustration 22.33 Evaluate $\int x\sec^2 x\,dx$.

Solution:

$$I=\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln\cos x + c$$

Illustration 22.34 Evaluate $\int (f(x)g''(x)-g(x)f''(x))dx$.

Solution:

$$I=\int[(f(x)g''(x)-g(x)f''(x))]\,dx = \int f(x)g''(x)dx - \int g(x)f''(x)dx$$

$$I=[f(x)g'(x)-\int f'(x)g'(x)dx]-[g(x)f'(x)-\int f'(x)g'(x)dx]$$

$$= f(x)g'(x)-g(x)f'(x)$$

Illustration 22.35 Evaluate $\int\sqrt{x^2+a^2}\,dx$.

Solution:

$$I=\int\sqrt{x^2+a^2}\,dx = \sqrt{x^2+a^2}\int 1dx - \int\frac{2x^2}{2\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - \int\frac{x^2+a^2}{\sqrt{x^2+a^2}}dx + \int\frac{a^2}{\sqrt{x^2+a^2}}dx$$

$$\Rightarrow I = x\sqrt{x^2+a^2} - I + a^2\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$\Rightarrow 2I = x\sqrt{x^2+a^2} + a^2\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$\Rightarrow I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

Illustration 22.36 Evaluate $\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx$.

Solution:

$$I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx = \frac{2}{\pi}\int\left(\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}\right)dx$$

$$\left(\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}\right)$$

For first expression, $\int\sin^{-1}\sqrt{x}\,dx$

Put $x=\sin^2\theta$. Then

$$1-2x=\cos 2\theta \Rightarrow dx = \sin 2\theta\,d\theta$$

$$\int\theta\sin 2\theta\,d\theta = -\frac{\theta\cos 2\theta}{2}+\frac{1}{2}\int\cos 2\theta\,d\theta$$

$$= -\frac{\theta\cos 2\theta}{2}+\frac{\sin 2\theta}{4}+c$$

$$= \frac{(2x-1)}{2}\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x(1-x)}+c$$

For second expression, $\int\cos^{-1}\sqrt{x}\,dx$

Put $x=\cos^2\theta$. Then

$$dx=-2\sin\theta\cos\theta\,d\theta \Rightarrow dx = -\sin 2\theta\,d\theta$$

$$\int\cos^{-1}\sqrt{x}\,dx = -\int\theta\sin 2\theta\,d\theta = \frac{\theta\cos 2\theta}{2}-\frac{1}{2}\int\cos 2\theta\,d\theta$$

$$= \frac{\theta\cos 2\theta}{2}-\frac{\sin 2\theta}{4}+k$$

$$= \frac{(2x-1)}{2}\cos^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x(1-x)}+k$$

Therefore,

$$I=\frac{2}{\pi}\left((2x-1)\sin^{-1}\sqrt{x}+\sqrt{x(1-x)}\right)-x+a$$

Illustration 22.37 Evaluate $\int x^3 \ln x\, dx$.

Solution:

$$I=\int x^3\ln x\, dx=\frac{x^4}{4}\cdot\ln x-\int\frac{x^4}{4}\cdot\frac{1}{x}dx$$
$$=\frac{x^4}{4}\cdot\ln x-\frac{x^4}{16}+c$$

Illustration 22.38 Evaluate $\int\frac{x^2dx}{(x\sin x+\cos x)^2}$.

Solution:

$$I=\int\frac{x^2}{(x\sin x+\cos x)^2}\,dx=\int\frac{x\cos x}{(x\sin x+\cos x)^2}\cdot\frac{x}{\cos x}\,dx$$
$$I=\frac{-1}{(x\sin x+\cos x)}\cdot\frac{x}{\cos x}+\int\frac{1}{(x\sin x+\cos x)}\cdot\frac{\cos x+x\sin x}{\cos^2 x}\,dx$$
$$I=\frac{-1}{(x\sin x+\cos x)}\cdot\frac{x}{\cos x}+\int\sec^2 x\,dx$$
$$I=\frac{-1}{(x\sin x+\cos x)}\cdot\frac{x}{\cos x}+\tan x+c$$

22.3.2.1 Some Important Results

1. $\int e^x(f(x)+f'(x))dx=e^xf(x)+c$

Illustration 22.39 Evaluate $\int e^x(1-\cot x+\cot^2 x)dx$.

Solution:

$$\int e^x(1-\cot x+\cot^2 x)dx=\int e^x(-\cot x+\mathrm{cosec}^2x)dx=-e^x\cot x+c$$

Illustration 22.40 Evaluate $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$.

Solution:

$$\left(\frac{1-\sin x}{1-\cos x}\right)=\frac{1}{2\sin^2\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}=\mathrm{cosec}^2\frac{x}{2}-\cot\frac{x}{2}$$

$$\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx=\int e^x\left(\mathrm{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right)dx=-e^x\cot\frac{x}{2}+c$$

Illustration 22.41 Evaluate $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$.

Solution:

$$I=\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx\Rightarrow I=\frac{e^x}{x}+c$$

Illustration 22.42 Evaluate $\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$.

Solution:

$$I=\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$$
$$=\int e^x\frac{(1+x^2-2x)}{(1+x^2)^2}dx$$
$$=\int e^x\left(\frac{1}{(1+x^2)}-\frac{2x}{(1+x^2)^2}\right)dx$$
$$\Rightarrow I=\frac{e^x}{(1+x^2)}+c$$

Illustration 22.43 Evaluate $\int\frac{e^{2\tan^{-1}x}(1+x)^2}{1+x^2}\,dx$.

Solution: Put $\tan^{-1}x=t$. Then

$$\frac{1}{1+x^2}dx=dt$$

$$I=\int\frac{e^{2\tan^{-1}x}(1+x)^2}{1+x^2}dx=\int e^{2t}(1+\tan t)^2dt=\int e^{2t}(\sec^2 t+2\tan t)\,dt$$

$$\Rightarrow I=\int e^{2t}\sec^2 t\,dt+2\tan t\cdot\frac{e^{2t}}{2}-2\int\frac{e^{2t}}{2}\sec^2 t\,dt$$

$$I=\int e^{2t}\sec^2 t\,dt+\tan t\cdot e^{2t}-\int e^{2t}\sec^2 t\,dt+c$$

$$I=\tan t\cdot e^{2t}+c\Rightarrow I=x\cdot e^{2\tan^{-1}x}+c$$

Illustration 22.44 Evaluate $\int\left(\frac{x+3}{(x+4)^2}\right)e^x\,dx$.

Solution:

$$I=\int\left(\frac{x+3}{(x+4)^2}\right)e^x\,dx=\int\left(\frac{x+4-1}{(x+4)^2}\right)e^x\,dx$$
$$=\int\left(\frac{x+4}{(x+4)^2}\right)e^x\,dx-\int\left(\frac{1}{(x+4)^2}\right)e^x\,dx$$
$$I=\int\frac{e^x}{(x+4)}dx-\int\frac{e^x}{(x+4)^2}dx$$
$$I=\left(\frac{1}{(x+4)}\right)e^x+\int\left(\frac{1}{(x+4)^2}\right)e^x\,dx-\int\left(\frac{1}{(x+4)^2}\right)e^x\,dx+c$$
$$I=\left(\frac{1}{(x+4)}\right)e^x+c$$

2. $\int(f(x)+xf'(x))dx=xf(x)+c$

Illustration 22.45 Evaluate $\int(x\cos x+\sin x)dx$.

Solution:

$$I=\int(x\cos x+\sin x)dx=x\sin x+c$$

Illustration 22.46 Evaluate $\int\left(\frac{x+\sin x}{1+\cos x}\right)dx$.

Solution:

$$I=\int\left(\frac{x+\sin x}{1+\cos x}\right)dx=\frac{1}{2}\int\left(x\sec^2\frac{x}{2}\right)dx+\int\tan\frac{x}{2}dx$$

$$I=\frac{1}{2}\frac{x\tan\frac{x}{2}}{\frac{1}{2}}-\int\tan\frac{x}{2}dx+\int\tan\frac{x}{2}dx+c=x\tan\frac{x}{2}+c$$

3. $\int e^{ax}\sin bx\,dx, \int e^{ax}\cos bx\,dx$:

Working rule: To evaluate $\int e^{ax}\sin bx\,dx$ or $\int e^{ax}\cos bx\,dx$, proceed as follows:

(a) Put the given integral equal to I.
(b) Integrate by parts, taking e^{ax} as the first function.
(c) Again, integrate by parts taking e^{ax} as the first function. This will involve I.
(d) Transpose and collect terms involving I and then obtain the value of I.

Let $I=\int e^{ax}\sin bx\,dx$. Then

$$I=\int e^{ax}\sin bx\,dx=-e^{ax}\frac{\cos bx}{b}-\int ae^{ax}\left(-\frac{\cos bx}{b}\right)dx$$
$$=-e^{ax}\frac{\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bx\,dx$$
$$=-e^{ax}\frac{\cos bx}{b}+\frac{a}{b}\left(e^{ax}\frac{\sin bx}{b}-\frac{a}{b}\int e^{ax}\sin bx\,dx\right)$$
$$=-e^{ax}\frac{\cos bx}{b}+\frac{a}{b^2}e^{ax}\sin bx-\left(\frac{a}{b}\right)^2\int e^{ax}\sin bx\,dx$$
$$=-e^{ax}\frac{\cos bx}{b}+\frac{a}{b^2}e^{ax}\sin bx-\left(\frac{a}{b}\right)^2 I$$
$$\Rightarrow I+\left(\frac{a}{b}\right)^2 I=\frac{e^{ax}}{b^2}(-b\cos bx+a\sin bx)+c$$
$$\Rightarrow I=\frac{e^{ax}}{a^2+b^2}(-b\cos bx+a\sin bx)+c$$

Thus,

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(-b\cos bx+a\sin bx)+c$$
$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

Similarly,

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(b\sin bx+a\cos bx)+c$$
$$=\frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx-\tan^{-1}\frac{b}{a}\right)+c$$

Illustration 22.47 Evaluate $\int e^x\sin x\,dx$.

Solution:

$$\int e^x\sin x\,dx=\frac{e^x}{1^2+1^2}(1\cdot\sin x-1\cdot\cos x)+c=\frac{e^x}{2}(\sin x-\cos x)+c$$

Illustration 22.48 If $u=\int e^{ax}\cos bx\,dx$ and $v=\int e^{ax}\sin bx\,dx$, then find $(a^2+b^2)(u^2+v^2)$.

Solution:

$$u=\int e^{ax}\cos bx\,dx=e^{ax}\frac{\sin bx}{b}-\int ae^{ax}\left(\frac{\sin bx}{b}\right)dx=e^{ax}\frac{\sin bx}{b}-\frac{a}{b}v$$

$$\Rightarrow bu+av=e^{ax}\sin bx \qquad (1)$$

Similarly,

$$bv-au=-e^{ax}\cos bx \qquad (2)$$

Squaring Eqs. (1) and (2) and adding, we get

$$(a^2+b^2)(u^2+v^2)=e^{2ax}$$

Your Turn 3

1. $\int x\sec^2 x\,dx=$

(A) $\tan x+\ln\cos x+c$
(B) $\frac{x^2}{2}\sec^2 x+\ln\cos x+c$
(C) $x\tan x+\ln\sec x+c$
(D) $x\tan x+\ln\cos x+c$

Ans. (D)

2. $\int\sin(\ln x)\,dx=$

(A) $\frac{1}{2}x(\cos(\ln x)-\sin(\ln x))+c$
(B) $\cos(\ln x)-x+c$
(C) $\frac{1}{2}x(\sin(\ln x)-\cos(\ln x))+c$
(D) $-\cos(\ln x)+c$

Ans. (C)

3. If $\int x\sin x\,dx=-x\cos x+A$, then $A=$

(A) $\sin x+c$
(B) $\cos x+c$
(C) Constant
(D) None of these

Ans. (A)

4. $\int x\ln x\,dx=$

(A) $\frac{x^2}{2}\ln x-\frac{x^2}{2}+c$
(B) $\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$
(C) $\frac{x^2}{2}\ln x+\frac{x^2}{2}+c$
(D) None of these

Ans. (B)

5. $\int x\cos dx=$

(A) $x\sin x+\cos x+c$
(B) $x\sin x-\cos x+c$
(C) $x\cos x+\sin x+c$
(D) $x\cos x-\sin x+c$

Ans. (A)

6. $\int x\cos^2 x\,dx=$

(A) $\frac{x^2}{4}-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+c$
(B) $\frac{x^2}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+c$
(C) $\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+c$
(D) $\frac{x^2}{4}+\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+c$

Ans. (B)

7. $\int \tan^{-1} x\,dx =$

(A) $x\tan^{-1}x + \frac{1}{2}(\ln(1+x^2)) + c$

(B) $x\tan^{-1}x - \frac{1}{2}(\ln(1+x^2)) + c$

(C) $(x-1)\tan^{-1}x + c$

(D) $x\tan^{-1}x - x\ln(1+x^2) + c$ **Ans.** (B)

8. $\int x\tan^{-1} x\,dx =$

(A) $\frac{(1+x^2)}{2}\tan^{-1}x - \frac{x}{2} + c$

(B) $\frac{(x^2-1)}{2}\tan^{-1}x - \frac{x}{2} + c$

(C) $\frac{(x^2+1)}{2}\tan^{-1}x + \frac{x}{2} + c$

(D) $\frac{(x^2+1)}{2}\tan^{-1}x - x + c$ **Ans.** (A)

9. $\int \left(\ln(\ln x) + \frac{1}{(\ln x)^2}\right) dx =$

(A) $x\ln(\ln x) + \frac{x}{(\ln x)} + c$

(B) $x\ln(\ln x) - \frac{x}{(\ln x)} + c$

(C) $x\ln(\ln x) + \frac{\ln x}{x} + c$

(D) $x\ln(\ln x) - \frac{\ln x}{x} + c$ **Ans.** (B)

10. $\int (\sin(\ln x) + \cos(\ln x))\,dx =$

(A) $x\cos(\ln x) + c$ **(B)** $\sin(\ln x) + c$

(C) $\cos(\ln x) + c$ **(D)** $x\sin(\ln x) + c$ **Ans.** (D)

22.4 Integration by Partial Fractions

A function of the form $U(x)/V(x)$, where $U(x)$ and $V(x)$ are polynomials, is called a rational function. Consider the rational function

$$\frac{x+7}{(2x-3)(3x+4)} = \frac{1}{(2x-3)} - \frac{1}{(3x+4)}$$

The two fractions on the RHS are called partial fractions. To integrate the rational function on the LHS, it is enough to integrate the two fractions on the RHS, which are easily integrable. This is known as a method of partial fractions.

In case the degree of $U(x)$ (numerator) is not less than that of $V(x)$ (denominator), we carry out the division of $U(x)$ by $V(x)$ and reduce the degree of the numerator.

In order to write $U(x)/V(x)$ in partial fractions, first of all we write $V(x) = (x-\alpha)^k \ldots (x^2+ax+b)^r$ where binomials are different, and then set

$$\frac{U(x)}{V(x)} = \frac{P_1}{(x-\alpha)} + \frac{P_2}{(x-\alpha)^2} + \cdots + \frac{P_k}{(x-\alpha)^k} + \frac{A_1x+B_1}{x^2+ax+b} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \cdots + \frac{A_rx+B_r}{(x^2+ax+b)^r} + \cdots$$

where $P_1, P_2, \cdots, P_k, A_1, A_2, \cdots, A_r, B_1, B_2, \cdots, B_r$ are real constants to be determined. Reducing both sides of the above identity to the integral form and equating the coefficients of equal powers of x, which gives a system of linear equations in the coefficient. Determine these coefficient. (This method is called the method of comparison of coefficients.) The constants can also be obtained by substituting suitably chosen numerical values of x in both sides of the identity.

Key point:
Before proceeding to write a rational function as a sum of partial fractions, we should ascertain that it is either a proper rational fraction or is rewritten as one.

A rational function $U(x)/V(x)$ is proper if the degree of polynomial $V(x)$ is greater than the degree of the polynomial $U(x)$. In case the degree of $U(x)$ is greater than or equal to the degree of $V(x)$, we first write $\frac{U(x)}{V(x)} = h(x) + \frac{u(x)}{v(x)}$, where $h(x)$ is a polynomial and $u(x)$ is a polynomial of degree less than the degree of polynomial $V(x)$.

Illustration 22.49 Evaluate $\int \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$.

Solution: Put $\sin x = t$. Then

$$\cos x\,dx = dt$$

$$\int \frac{dt}{(1+t)(2+t)} = \int \frac{dt}{(1+t)} - \int \frac{dt}{(2+t)} = \ln(1+t) - \ln(2+t) + c$$

$$= \ln\frac{(1+\sin x)}{(2+\sin x)} + c$$

Illustration 22.50 Evaluate $\int \frac{dx}{\sin x(2+\cos x - 2\sin x)}$.

Solution: Put $\tan\frac{x}{2} = t$. Then

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt \Rightarrow dx = \frac{2dt}{\sec^2\frac{x}{2}} \Rightarrow dx = \frac{2dt}{1+t^2}$$

$$I = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}\left(2 + \frac{1-t^2}{1+t^2} - 2\frac{2t}{1+t^2}\right)} = \int \frac{1+t^2}{t(t^2-4t+3)}dt$$

Expands into simple fractions

$$\frac{1+t^2}{t(t-3)(t-1)} = \frac{A}{t} + \frac{B}{(t-3)} + \frac{C}{(t-1)}$$

$$\Rightarrow 1+t^2 = A(t-3)(t-1) + Bt(t-1) + C(t-3)t$$

After solve the coefficients, $A = \frac{1}{3}; B = \frac{5}{3}; C = -1.$

Hence,

$$I=\frac{1}{3}\int\frac{dt}{t}+\frac{5}{3}\int\frac{dt}{t-3}-\int\frac{dt}{t-1}$$

$$=\frac{1}{3}\ln|t|+\frac{5}{3}\ln|t-3|-\ln|t-1|+c$$

$$=\frac{1}{3}\ln\left|\tan\frac{x}{2}\right|+\frac{5}{3}\ln\left|\tan\frac{x}{2}-3\right|-\ln\left|\tan\frac{x}{2}-1\right|+c$$

Illustration 22.51 Evaluate $\int\frac{3x+1}{(x-2)^2(x+2)}dx$.

Solution:

$$\frac{3x+1}{(x-2)^2(x+2)}=\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x+2)}$$

$$3x+1=A(x-2)(x+2)+B(x+2)+C(x-2)^2 \qquad (1)$$

Putting $x=2$ and -2 successively in Eq. (1), we get

$$B=\frac{7}{4},\ C=-\frac{5}{16}$$

Now, we put $x=0$ and get $A=\frac{5}{16}$. Therefore,

$$I=\int\frac{3x+1}{(x-2)^2(x+2)}dx$$

$$I=\int\frac{5}{16(x-2)}dx+\int\frac{7}{4(x-2)^2}dx-\int\frac{5}{16(x+2)}dx$$

$$I=\frac{5}{16}\ln|x-2|-\frac{7}{4(x-2)}-\frac{5}{16}\ln|x+2|+c$$

$$I=\frac{5}{16}\ln\left|\frac{x-2}{x+2}\right|-\frac{7}{4(x-2)}+c$$

Illustration 22.52 Evaluate $\int\frac{2x^2+3}{(x^2-1)(x^2+4)}dx$.

Solution:

$$\frac{2x^2+3}{(x^2-1)(x^2+4)}=\frac{A}{(x^2-1)}+\frac{B}{(x^2+4)}$$

Therefore,

$$2x^2+3=A(x^2+4)+B(x^2-1)$$

Comparing the coefficient of x^2 and constant terms

$$\Rightarrow A+B=2,\ 4A-B=3$$

$$\Rightarrow A=1,\ B=1$$

$$I=\int\frac{1}{(x^2-1)}dx+\int\frac{1}{(x^2+4)}dx$$

$$\Rightarrow I=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+\frac{1}{2}\tan^{-1}\frac{x}{2}+c$$

Integrals of the form:

1. $\int\frac{dx}{ax^2+bx+c},\ \int\frac{dx}{\sqrt{ax^2+bx+c}},\ \int\sqrt{ax^2+bx+c}\,dx$

Convert $ax^2+bx+c=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}$

And then use formulas

(a) $\int\frac{dx}{x^2-a^2},\ \int\frac{dx}{x^2+a^2},\ \int\frac{dx}{a^2-x^2}$ for $\int\frac{dx}{ax^2+bx+c}$

(b) $\int\frac{dx}{\sqrt{x^2-a^2}},\ \int\frac{dx}{\sqrt{x^2+a^2}},\ \int\frac{dx}{\sqrt{a^2-x^2}}$ for $\int\frac{dx}{\sqrt{ax^2+bx+c}}$

(c) $\int\sqrt{x^2-a^2}\,dx,\ \int\sqrt{x^2+a^2}\,dx,\ \int\sqrt{a^2-x^2}\,dx$ for $\int\sqrt{ax^2+bx+c}\,dx$.

Illustration 22.53 Evaluate $\int\frac{1}{x^2+x+1}dx$.

Solution:

$$I=\int\frac{1}{x^2+x+1}dx=\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

Illustration 22.54 Evaluate $\int\frac{1}{\sqrt{2x^2+3x+2}}dx$.

Solution:

$$I=\int\frac{1}{\sqrt{2x^2+3x+2}}dx=\frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}}dx$$

$$=\frac{1}{\sqrt{2}}\ln\left(\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}\right)+c$$

Illustration 22.55 Evaluate $\int\sqrt{x^2-x}\,dx$.

Solution:

$$I=\int\sqrt{x^2-x}\,dx=\int\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx$$

$$I=\frac{\left(x-\frac{1}{2}\right)}{2}\sqrt{x^2-x}-\frac{1}{8}\ln\left(\left(x-\frac{1}{2}\right)+\sqrt{x^2-x}\right)+c$$

Illustration 22.56 Evaluate $\int\frac{1}{2x^2+x+1}dx$.

Solution:

$$I=\int\frac{1}{2x^2+x+1}dx=\frac{1}{2}\int\frac{1}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}dx$$

$$I=\frac{1}{2}\cdot\frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{\left(x+\frac{1}{4}\right)}{\left(\frac{\sqrt{7}}{4}\right)}\right)+c$$

$$I=\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{(4x+1)}{\sqrt{7}}\right)+c$$

Illustration 22.57 Evaluate $\int\frac{\cos x}{\sin^2 x+4\sin x+5}dx.$

Solution:

$$I=\int\frac{\cos x}{\sin^2 x+4\sin x+5}dx=\int\frac{\cos x}{(\sin x+2)^2+1}dx$$

Put $(\sin x+2)=t$. Then $\cos x\,dx=dt$.

$$I=\int\frac{1}{t^2+1}dt=\tan^{-1}t+c=\tan^{-1}(\sin x+2)+c$$

Illustration 22.58 Evaluate $\int\frac{1}{\sqrt{2-3x-x^2}}dx.$

Solution:

$$I=\int\frac{1}{\sqrt{2-3x-x^2}}dx=\int\frac{1}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x+\frac{3}{2}\right)^2}}dx$$

Put $\left(x+\frac{3}{2}\right)=t$. Then $dx=dt$.

$$I=\int\frac{1}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}}dt=\sin^{-1}\left(\frac{2t}{\sqrt{17}}\right)+c=\sin^{-1}\left(\frac{2x+3}{\sqrt{17}}\right)+c$$

Illustration 22.59 Evaluate $\int\frac{1}{\sqrt{x^2-4x+2}}dx.$

Solution:

$$I=\int\frac{1}{\sqrt{x^2-4x+2}}dx=\int\frac{1}{\sqrt{(x-2)^2-(\sqrt{2})^2}}dx$$

Put $(x-2)=t$. Then $dx=dt$.

$$I=\int\frac{1}{\sqrt{t^2-(\sqrt{2})^2}}dt$$

$$\Rightarrow I=\ln\left|t+\sqrt{t^2-(\sqrt{2})^2}\right|+c$$

$$\Rightarrow I=\ln\left|(x-2)+\sqrt{x^2-4x+2}\right|+c$$

Illustration 22.60 Evaluate $\int\sqrt{x^2+8x+12}\,dx.$

Solution:

$$I=\int\sqrt{x^2+8x+12}\,dx=\int\sqrt{(x+4)^2-(2)^2}\,dx$$

Put $(x+4)=t$. Then $dx=dt$.

$$I=\int\sqrt{t^2-2^2}\,dt=\frac{t}{2}\sqrt{t^2-2^2}-\frac{4}{2}\ln\left|t+\sqrt{t^2-2^2}\right|+c$$

$$I=\frac{(x+4)}{2}\sqrt{(x+4)^2-2^2}-\frac{4}{2}\ln\left|(x+4)+\sqrt{(x+4)^2-2^2}\right|+c$$

$$I=\frac{(x+4)}{2}\sqrt{x^2+8x+12}-2\ln\left|(x+4)+\sqrt{x^2+8x+12}\right|+c$$

Illustration 22.61 Evaluate $\int\sqrt{2ax-x^2}\,dx.$

Solution:

$$I=\int\sqrt{2ax-x^2}\,dx=\int\sqrt{a^2-a^2+2ax-x^2}\,dx$$

$$=\int\sqrt{a^2-(a^2-2ax+x^2)}\,dx=\int\sqrt{a^2-(x-a)^2}\,dx$$

Put $(x-a)=t$. Then $dx=dt$.

$$I=\int\sqrt{a^2-t^2}\,dt=\frac{t}{2}\sqrt{a^2-t^2}+\frac{a^2}{2}\sin^{-1}\frac{t}{a}+c$$

$$I=\frac{(x-a)}{2}\sqrt{a^2-(x-a)^2}+\frac{a^2}{2}\sin^{-1}\frac{(x-a)}{a}+c$$

$$I=\frac{(x-a)}{2}\sqrt{2ax-x^2}+\frac{a^2}{2}\sin^{-1}\frac{(x-a)}{a}+c$$

2. $\int\frac{px+q}{(ax^2+bx+c)}dx,\ \int\frac{px+q}{\sqrt{(ax^2+bx+c)}}dx,$

$\int(px+q)\sqrt{(ax^2+bx+c)}\,dx$

In these types of integrals, we write $px+q=\ell$ (differential coefficient of ax^2+bx+c) $+ m$.

Find ℓ and m by comparing the coefficient of x and constant terms on both sides of the identity. In this way, the question will reduce to the sum of two integrals, which can be integrated easily.

Illustration 22.62 Evaluate $\int\frac{x+1}{\sqrt{(2x^2+x-3)}}dx.$

Solution: Let $x+1=A$(differential coefficient of $2x^2+x-3$) $+B$. Then

$$x+1=A(4x+1)+B=4Ax+A+B$$

Equating the coefficients, $A=\frac{1}{4}, B=\frac{3}{4}$. We get

$$I=\int\frac{x+1}{\sqrt{(2x^2+x-3)}}dx$$

$$I=\frac{1}{4}\int\frac{4x+1}{\sqrt{(2x^2+x-3)}}dx+\frac{3}{4}\int\frac{1}{\sqrt{(2x^2+x-3)}}dx$$

Let $I_1=\frac{1}{4}\int\frac{4x+1}{\sqrt{(2x^2+x-3)}}dx$ and $I_2=\frac{3}{4}\int\frac{1}{\sqrt{(2x^2+x-3)}}dx$

Put $(2x^2+x-3)=t$. Then $(4x+1)dx=dt$.

$$I_1=\frac{1}{4}\int\frac{1}{\sqrt{t}}dt=\frac{1}{2}\sqrt{t}+k_1=\frac{1}{2}\sqrt{(2x^2+x-3)}+k_1$$

$$I_2=\frac{3}{4\sqrt{2}}\int\frac{1}{\sqrt{\left(x^2+\frac{x}{2}-\frac{3}{2}\right)}}dx=\frac{3}{4\sqrt{2}}\int\frac{1}{\sqrt{\left(\left(x+\frac{1}{4}\right)^2-\left(\frac{5}{4}\right)^2\right)}}dx$$

$$=\frac{3}{4\sqrt{2}}\ln\left(\left(x+\frac{1}{4}\right)+\sqrt{\left(\left(x+\frac{1}{4}\right)^2-\left(\frac{5}{4}\right)^2\right)}\right)+k_2$$

Hence,

$$I=\frac{1}{2}\sqrt{(2x^2+x-3)}+\frac{3}{4\sqrt{2}}\ln\left(\left(x+\frac{1}{4}\right)+\sqrt{\left(x^2+\frac{x}{2}-\frac{3}{2}\right)}\right)+c$$

Illustration 22.63 Evaluate $\int\frac{3x+2}{(4x^2+4x+5)}dx$.

Solution: Express

$3x + 2 = \ell$ (differential coefficient of $4x^2 + 4x + 5$) + m

$$\Rightarrow 3x+2=\ell(8x+4)+m=8\ell x+4\ell+m$$

Comparing the coefficients, we get

$$8\ell=3 \text{ and } 4\ell+m=2 \Rightarrow \ell=3/8 \text{ and } m=2-4\ell=1/2$$

$$\Rightarrow I=\frac{3}{8}\int\frac{8x+4}{(4x^2+4x+5)}dx+\frac{1}{2}\int\frac{1}{(4x^2+4x+5)}dx$$

$$=\frac{3}{8}\ln(4x^2+4x+5)+\frac{1}{8}\int\frac{1}{\left(x^2+x+\frac{5}{4}\right)}dx$$

$$=\frac{3}{8}\ln(4x^2+4x+5)+\frac{1}{8}\tan^{-1}\left(x+\frac{1}{2}\right)+c$$

Illustration 22.64 Evaluate $\int\frac{5-2x}{\sqrt{6+x-x^2}}dx$.

Solution:

$$I=\int\frac{5-2x}{\sqrt{6+x-x^2}}dx$$

Let $5-2x=M\frac{d}{dx}(6+x-x^2)+N=M(1-2x)+N$

Equating the coefficients of x and constant terms on both sides, we get

$$M=1,\ M+N=5\Rightarrow N=4$$

Therefore,

$$5-2x=1-2x+4$$

Hence,

$$I=\int\frac{(1-2x)+4}{\sqrt{6+x-x^2}}dx=\int\frac{(1-2x)}{\sqrt{6+x-x^2}}dx+\int\frac{4}{\sqrt{6+x-x^2}}dx=I_1+4I_2$$

Now,

$$I_1=\int\frac{(1-2x)}{\sqrt{6+x-x^2}}dx$$

Putting $6+x-x^2=t$. Then $(1-2x)dx=dt$, we have

$$I_1=\int\frac{1}{\sqrt{t}}dt=2\sqrt{t}+k_1=2\sqrt{6+x-x^2}+k_1$$

And

$$I_2=\int\frac{dx}{\sqrt{6+x-x^2}}=\frac{dx}{\sqrt{6-(x^2-x)}}$$

$$I_2=\int\frac{4}{\sqrt{6-(x^2-x)}}dx=\int\frac{1}{\sqrt{6-\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)}}dx$$

$$=\int\frac{1}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}}dx$$

$$=\sin^{-1}\left(\frac{2}{5}\left(x-\frac{1}{2}\right)\right)+k_2=\sin^{-1}\left(\frac{2x-1}{5}\right)+k_2$$

$$I_1+4I_2=2\sqrt{6+x-x^2}+4\sin^{-1}\left(\frac{2x-1}{5}\right)+k_1+4k_2$$

$$\Rightarrow I=2\sqrt{6+x-x^2}+4\sin^{-1}\left(\frac{2x-1}{5}\right)+c$$

Illustration 22.65 Evaluate $\int\frac{x+2}{\sqrt{x^2-2x+4}}dx$.

Solution:

$$I=\int\frac{x+2}{\sqrt{x^2-2x+4}}dx=\int\frac{x-1+3}{\sqrt{(x^2-2x+4)}}dx$$

$$I=\int\frac{x-1}{\sqrt{(x^2-2x+4)}}dx+\int\frac{3}{\sqrt{(x-1)^2+(\sqrt{3})^2}}dx$$

Put $x^2-2x+4=t^2$. Then

$(2x-2)dx=2tdt\Rightarrow(x-1)dx=tdt$

$$\Rightarrow I=\int\frac{t}{t}dt+\int\frac{3}{\sqrt{(x-1)^2+(\sqrt{3})^2}}dx$$

$$I=\sqrt{x^2-2x+4}+3\ln\left|x+\sqrt{x^2-2x+4}\right|+c$$

Illustration 22.66 Evaluate $\int(2x+3)\sqrt{x^2+4x+3}\,dx$.

Solution: Let $(2x+3)=M\frac{d}{dx}(x^2+4x+3)+N$. Then

$$(2x+3)=M(2x+4)+N$$

Equating the coefficients of x and constant terms on both sides, we get

$$M=1,\ N=3-4M=-1$$

Therefore,

$$(2x+3)=(2x+4)-1$$

$$I=\int((2x+4)-1)\sqrt{x^2+4x+3}\,dx$$

$$I=\int(2x+4)\sqrt{x^2+4x+3}\,dx-\int\sqrt{x^2+4x+3}\,dx=I_1-I_2$$

Putting $x^2+4x+3=t$. Then $(2x+4)dx=dt$, we have

$$I_1=\int t^{1/2}dt=\frac{t^{3/2}}{3/2}+c_1=\frac{2}{3}(x^2+4x+3)^{3/2}+c_1$$

$$I_2=\int\sqrt{x^2+4x+3}\,dx=\int\sqrt{(x+2)^2-1}\,dx$$

$$=\frac{(x+2)}{2}\sqrt{x^2+4x+3}-\frac{1}{2}\ln\left|(x+2)+\sqrt{x^2+4x+3}\right|+c_2$$

$$I=I_1-I_2$$

$$=\frac{(x^2+4x+3)^{3/2}}{3/2}-\frac{(x+2)}{2}\sqrt{x^2+4x+3}+\frac{1}{2}\ln\left|(x+2)+\sqrt{x^2+4x+3}\right|+c$$

Illustration 22.67 Evaluate $\int(2x-5)\sqrt{x^2-4x+3}\,dx$.

Solution:

$$I=\int(2x-5)\sqrt{x^2-4x+3}\,dx$$

Let $(2x-5)=M\frac{d}{dx}(x^2-4x+3)+N$. Then

$$(2x-5)=M(2x-4)+N$$

Equating the coefficients of x and constant terms on both sides, we get

$$M=1,\ N=4M-5=-1$$

Therefore,

$$(2x-5)=(2x-4)-1$$

$$I=\int((2x-4)-1)\sqrt{x^2-4x+3}\,dx$$

$$I=\int(2x-4)\sqrt{x^2-4x+3}\,dx-\int\sqrt{x^2-4x+3}\,dx$$

$$I=I_1-I_2$$

$$I_1=\int(2x-4)\sqrt{x^2-4x+3}\,dx=\frac{(x^2-4x+3)^{3/2}}{3/2}+c_1$$

$$I_2=\int\sqrt{x^2-4x+3}\,dx=\int\sqrt{x^2-4x+4-4+3}\,dx$$

$$=\int\sqrt{(x-2)^2-1^2}\,dx$$

$$=\frac{(x-2)}{2}\sqrt{x^2-4x+3}-\frac{1}{2}\ln\left|(x-2)+\sqrt{x^2-4x+3}\right|+c_2$$

Now,

$$I=I_1-I_2$$

$$\Rightarrow I=\frac{(x^2-4x+3)^{3/2}}{3/2}-\frac{(x-2)}{2}\sqrt{x^2-4x+3}+\frac{1}{2}\ln\left|(x-2)+\sqrt{x^2-4x+3}\right|+c$$

3. $\int\frac{f(x)}{ax^2+bx+c}dx$ where $f(x)$ is a polynomial of degree greater than 2.

To evaluate the integrals of the above form, divide the numerator by the denominator. Then, the integrals take the form given by

$$\frac{f(x)}{ax^2+bx+c}=Q(x)+\frac{R(x)}{ax^2+bx+c}$$

where $Q(x)$ is a polynomial and $R(x)$ is a linear polynomial in x.

Then, we have

$$\int\frac{f(x)}{ax^2+bx+c}dx=\int Q(x)dx+\int\frac{R(x)}{ax^2+bx+c}dx$$

The integrals on RHS can be obtained by the methods discussed earlier.

Illustration 22.68 Evaluate $\int\frac{(x^3+8)(x-1)}{x^2-2x+4}dx$.

Solution:

$$I=\int\frac{(x^3+8)(x-1)}{x^2-2x+4}dx=\int\frac{(x-1)(x^2-2x+4)(x-1)}{x^2-2x+4}dx$$

$$I=\int(x+2)(x-1)dx=\int(x^2+x-2)dx=\frac{x^3}{3}+\frac{x^2}{2}-2x+c$$

Illustration 22.69 Evaluate $\int\frac{2x^3-3x^2+5x+6}{x^2+3x+2}dx$.

Solution:

$$I=\int\frac{2x^3-3x^2+5x+6}{x^2+3x+2}dx$$

$$I=\int\left((2x+3)-\frac{8x}{x^2+3x+2}\right)dx$$

$$I=\int(2x+3)dx-4\int\frac{2x+3}{x^2+3x+2}dx+12\int\frac{1}{(x+1)(x+2)}dx$$

$$I=(x^2+3x)-4\ln\left|x^2+3x+2\right|+12\int\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx$$

$$I=(x^2+3x)-4\ln\left|(x+1)(x+2)\right|+12\ln\left|x+1\right|-12\ln\left|x+2\right|+c$$

$$I=(x^2+3x)+8\ln\left|x+1\right|-16\ln\left|x+2\right|+c=x^2-3x+8\ln(x+1)-16\ln(x+2)+c$$

4. $\int\frac{ax^2+bx+c}{px^2+qx+r}dx,\ \int\frac{ax^2+bx+c}{\sqrt{px^2+qx+r}}dx$

In this case, substitute

$$ax^2+bx+c=M(px^2+qx+r)+N\frac{d}{dx}(px^2+qx+r)+R$$

Find M, N and R by comparing the coefficient of x^2, x and constant terms on both sides of the identity.

Illustration 22.70 Evaluate $\int\frac{x^2+x+1}{x^2+3x+2}dx$.

Solution:

$$I=\int\frac{x^2+x+1}{x^2+3x+2}dx$$

$$x^2+x+1=M(x^2+3x+2)+N\frac{d}{dx}(x^2+3x+2)+R$$

$$x^2+x+1=Mx^2+(3M+2N)x+(2M+3N+R)$$

Comparing coefficients of variables and constant terms

$$M=1,\ (3M+2N)=1,(2M+3N+R)=1$$

$$M=1,N=-1,R=2$$

$$I=\int\frac{(x^2+3x+2)-\frac{d}{dx}(x^2+3x+2)+2}{x^2+3x+2}dx$$

$$=\int 1dx-\int\frac{2x+3}{x^2+3x+2}dx+\int\frac{2}{(x+1)(x+2)}dx$$

$$I=x-\ln\left|x^2+3x+2\right|+\ln\left|\frac{x+1}{x+2}\right|+c$$

Illustration 22.71 Evaluate $\int\frac{x^2-x+1}{\sqrt{x^2+4x+3}}dx.$

Solution:

$$I=\int\frac{x^2-x+1}{\sqrt{x^2+4x+3}}dx$$

$$x^2-x+1=M(x^2+4x+3)+N\frac{d}{dx}(x^2+4x+3)+R$$

$$x^2-x+1=Mx^2+(4M+2N)x+(3M+4N+R)$$

Comparing coefficients of variables and constant terms

$$M=1,\ (4M+2N)=-1,\ (3M+4N+R)=1$$

$$M=1,\ N=-\frac{5}{2},\ R=8$$

$$I=\int\frac{(x^2+4x+3)-\frac{5}{2}\frac{d}{dx}(x^2+4x+3)+8}{\sqrt{(x^2+4x+3)}}dx$$

$$=\int\sqrt{(x^2+4x+3)}\ dx-\frac{5}{2}\int\frac{(2x+4)}{\sqrt{(x^2+4x+3)}}dx+8\int\frac{1}{\sqrt{(x^2+4x+3)}}dx$$

$$I=I_1-\frac{5}{2}I_2+8I_3$$

$$I_1=\int\sqrt{x^2+4x+3}\ dx=\int\sqrt{(x+2)^2-1}\ dx$$

$$=\frac{(x+2)}{2}\sqrt{x^2+4x+3}-\frac{1}{2}\ln\left|(x+2)+\sqrt{x^2+4x+3}\right|+c_1$$

$$I_2=\int\frac{(2x+4)}{\sqrt{x^2+4x+3}}dx=2\sqrt{x^2+4x+3}+c_2$$

$$I_3=\int\frac{1}{\sqrt{(x+2)^2-1}}dx=\ln\left|(x+2)+\sqrt{x^2+4x+3}\right|+c_3$$

$$I=I_1-\frac{5}{2}I_2+8I_3$$

$$I=\frac{(x-8)}{2}\sqrt{x^2+4x+3}+\frac{15}{2}\ln\left|(x+2)+\sqrt{x^2+4x+3}\right|+c$$

5. $\int\frac{x^2}{x^4+1}dx,\int\frac{1}{x^4+1}dx,\int\frac{x^2+a^2}{x^4+kx^2+1}dx,\int\frac{x^2-a^2}{x^4+kx^2+1}dx,(k\in R)$

Rule for this form:

(a) To evaluate these types of integrals divide the numerator and denominator by x^2.

(b) Put $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$ and $x+\frac{a^2}{x}=t$ or $x-\frac{a^2}{x}=t$ as required.

Similar form is: $\int\sqrt{\tan x}\,dx,\int\sqrt{\cot x}\,dx,\int\frac{dx}{\sin^4 x+\cos^4 x},$

$\int\frac{dx}{\sin^6 x+\cos^6 x},\int\frac{(\pm\sin x\pm\cos x)}{a+b\sin x}dx$

Illustration 22.72 Evaluate $\int\frac{5}{x^4+1}dx.$

Solution:

$$I=\int\frac{5}{x^4+1}dx=\frac{5}{2}\left(\int\frac{(x^2+1)-(x^2-1)}{x^4+1}dx\right)$$

$$=\frac{5}{2}\left(\int\frac{\left(1+\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}dx-\int\frac{\left(1-\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}}dx\right)$$

$$=\frac{5}{2}\left(\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx-\int\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-2}dx\right)=\frac{5}{2}(I_1-I_2)$$

$$I_1=\int\frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+2}dx$$

For I_1, we write

$$x-\frac{1}{x}=t\Rightarrow\left(1+\frac{1}{x^2}\right)dx=dt$$

$$I_1=\int\frac{1}{t^2+2}dt$$

$$I_1=\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+c=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}}+c_1$$

For I_2, we write

$$x+\frac{1}{x}=t\Rightarrow\left(1-\frac{1}{x^2}\right)dx=dt$$

$$I_2=\int\frac{1}{t^2-2}dt=\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c=\frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+c_2$$

Combining the two integrals, we get

$$I=\frac{5}{2}\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|\right)+k$$

Illustration 22.73 Evaluate $\int\sqrt{\tan x}\,dx$.

Solution: Put $\tan x=t^2$. Then

$$\sec^2 x\,dx=2tdt\Rightarrow dx=\frac{2t}{1+t^4}dt$$

$$I=\int\sqrt{\tan x}\,dx=\int\frac{2t\cdot t}{1+t^4}dt=\int\frac{t^2+1}{1+t^4}dt+\int\frac{t^2-1}{1+t^4}dt$$

$$I=\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}dx+\int\frac{\left(1-\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2-2}dx$$

$$I=I_1+I_2$$

For I_1, we write $\left(t-\frac{1}{t}\right)=z\Rightarrow\left(1+\frac{1}{t^2}\right)dx=dz$

$$I_1=\int\frac{1}{z^2+2}dt$$

$$I_1=\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}+c=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}}+c$$

$$I_1=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(\sqrt{\tan x}-\frac{1}{\sqrt{\tan x}}\right)}{\sqrt{2}}+c$$

For I_2, we write $t+\frac{1}{t}=z\Rightarrow\left(1-\frac{1}{t^2}\right)dx=dz$

$$I_2=\int\frac{1}{z^2-2}dt=\frac{1}{2\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right|+c=\frac{1}{2\sqrt{2}}\ln\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|+c_1$$

$$I_2=\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}-\sqrt{2}}{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}+\sqrt{2}}\right|+c_2$$

Combining the two integrals, we get

$$I=\left(\frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(\sqrt{\tan x}-\frac{1}{\sqrt{\tan x}}\right)}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}-\sqrt{2}}{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}+\sqrt{2}}\right|\right)+k$$

Illustration 22.74 Evaluate $\int(\sqrt{\tan x}+\sqrt{\cot x})dx$.

Solution:

$$I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx$$

Put $\tan x=t^2$. Then

$$\sec^2 x\,dx=2tdt\Rightarrow dx=\frac{2t}{1+t^4}dt$$

$$I=\int\left(t+\frac{1}{t}\right)\frac{2t}{1+t^4}\,dt=2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}-2+2\right)}dt=2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}dt$$

Put $\left(t-\frac{1}{t}\right)=p$. Then $\left(1+\frac{1}{t^2}\right)dt=dp$.

$$I=2\int\frac{1}{p^2+2}dp=\frac{2}{\sqrt{2}}\tan^{-1}\frac{p}{\sqrt{2}}+c=\frac{2}{\sqrt{2}}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}}+c$$

$$\Rightarrow I=\sqrt{2}\tan^{-1}\frac{\left(\sqrt{\tan x}-\frac{1}{\sqrt{\tan x}}\right)}{\sqrt{2}}+c$$

6. **Substitution for some irrational functions:**

(a) $\int f(x,(ax+b)^{1/n})dx$, put $(ax+b)=t^n$

(b) $\int\frac{dx}{(px+q)\sqrt{(ax+b)}}$, put $(ax+b)=t^2$

(c) $\int\frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}$, put $(px+q)=\frac{1}{t}$

(d) $\int\frac{dx}{(px^2+q)\sqrt{(ax^2+b)}}$, first put $x=\frac{1}{t}$ and then $(a+bt^2)=z^2$

(e) $\int\frac{(ax^2+bx+c)dx}{(dx+e)\sqrt{(fx^2+gx+h)}}$, here, we write

$$ax^2+bx+c=A_1(dx+e)(2fx+g)+B_1(dx+e)+C_1$$

where A_1, B_1 and C_1 are constants which can be obtained by comparing the coefficient of like terms on both sides.

Illustration 22.75 Evaluate $\int\frac{x\,dx}{(x-3)\sqrt{x+1}}$.

Solution: Put $x+1=t^2$. Then $dx=2tdt$, we get

$$I=\int\frac{2t(t^2-1)}{(t^2-4)t}dt$$

$$=2\int\left(1+\frac{3}{(t^2-4)}\right)dt=2t+\frac{3}{2}\ln\left|\frac{t-2}{t+2}\right|+c$$

$$=2\sqrt{x+1}+\frac{3}{2}\ln\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+c$$

Illustration 22.76 Evaluate $\int\frac{dx}{(x-3)\sqrt{x+1}}$.

Solution: Put $x+1=t^2$. Then $dx=2tdt$, we get

$$I=\int\frac{2t}{(t^2-4)t}dt=2\int\frac{1}{(t^2-4)}dt=2\frac{1}{4}\ln\left|\frac{t-2}{t+2}\right|+c$$
$$=\frac{1}{2}\ln\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+c$$

Illustration 22.77 Evaluate $\int\frac{dx}{(x+1)\sqrt{(x^2-x+1)}}$.

Solution:

$$I=\int\frac{dx}{(x+1)\sqrt{(x^2-x+1)}}$$

Put $x+1=\frac{1}{t}$. Then $dx=-\frac{1}{t^2}dt$, we get

$$I=-\int\frac{1}{t^2.\frac{1}{t^2}\sqrt{(1-t)^2-t(1-t)+t^2}}dt=-\int\frac{1}{\sqrt{3t^2-3t+1}}dt$$

$$I=-\frac{1}{\sqrt{3}}\int\frac{1}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{12}}}dt=-\frac{1}{\sqrt{3}}\ln\left|\left(t-\frac{1}{2}\right)+\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{1}{12}}\right|+c$$

$$I=-\frac{1}{\sqrt{3}}\ln\left|\left(\frac{1-x}{2(1+x)}\right)+\sqrt{\left(\frac{1-x}{2(1+x)}\right)^2+\frac{1}{12}}\right|+c$$

Illustration 22.78 Evaluate $\int\frac{(2x^2+3x+2)dx}{(x+1)\sqrt{(x^2-x+1)}}$

Solution:

$$I=\int\frac{(2x^2+3x+2)dx}{(x+1)\sqrt{(x^2-x+1)}}$$

$$(2x^2+3x+2)=a(x+1)\frac{d}{dx}(x^2-x+1)+b(x+1)+c$$
$$a=1,b=2,c=1$$

$$I=\int\frac{(x+1)(2x-1)+2(x+1)+1}{(x+1)\sqrt{(x^2-x+1)}}dx$$

$$I=\int\frac{(2x-1)}{\sqrt{(x^2-x+1)}}dx+2\int\frac{1}{\sqrt{(x^2-x+1)}}dx+\int\frac{1}{(x+1)\sqrt{(x^2-x+1)}}dx$$

$$I=I_1+2I_2+I_3$$

$$I_1=\int\frac{(2x-1)}{\sqrt{(x^2-x+1)}}dx=2\sqrt{(x^2-x+1)}+c_1$$

$$I_2=\int\frac{1}{\sqrt{(x^2-x+1)}}dx=\int\frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}}$$

$$=\ln\left|\left(x-\frac{1}{2}\right)+\sqrt{(x^2-x+1)}\right|+c_2$$

$$I_3=\int\frac{1}{(x+1)\sqrt{(x^2-x+1)}}dx \text{ (Same as Illustration 22.77)}$$

$$=-\frac{1}{\sqrt{3}}\ln\left|\left(\frac{1-x}{2(1+x)}\right)+\sqrt{\left(\frac{1-x}{2(1+x)}\right)^2+\frac{1}{12}}\right|+c_3$$

$$I=2\sqrt{(x^2-x+1)}+2\ln\left|\left(x-\frac{1}{2}\right)+\sqrt{(x^2-x+1)}\right|-$$
$$\frac{1}{\sqrt{3}}\ln\left|\left(\frac{1-x}{2(1+x)}\right)+\sqrt{\left(\frac{1-x}{2(1+x)}\right)^2+\frac{1}{12}}\right|+c$$

Illustration 22.79 Evaluate $\int\frac{dx}{(x^2+2)\sqrt{x^2+1}}$.

Solution:

$$I=\int\frac{dx}{(x^2+2)\sqrt{x^2+1}}$$

Put $x=\frac{1}{t}$. Then $dx=-\frac{1}{t^2}dt$, we get

$$I=-\int\frac{dt}{t^2\left(\frac{1}{t^2}+2\right)\sqrt{\frac{1}{t^2}+1}}=-\int\frac{tdt}{(1+2t^2)\sqrt{t^2+1}}$$

Put $(1+t^2)=z^2$. Then $tdt=zdz$, we get

$$I=-\int\frac{z\,dz}{\left[1+2(z^2-1)\right]z}=-\int\frac{dz}{(2z^2-1)}$$

$$I=-\frac{1}{2}\int\frac{dz}{\left(z^2-\frac{1}{2}\right)}=-\frac{1}{2\sqrt{2}}\ln\left|\frac{z-\frac{1}{\sqrt{2}}}{z+\frac{1}{\sqrt{2}}}\right|+c$$

$$I=-\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}z-1}{\sqrt{2}z+1}\right|+c=-\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2(1+t^2)}-1}{\sqrt{2(1+t^2)}+1}\right|+c$$

$$I=-\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2(1+x^2)}-x}{\sqrt{2(1+x^2)}+x}\right|+c$$

Your Turn 4

1. $\int\frac{dx}{(x-x^2)}=$

(A) $\ln x-\ln(1-x)+c$ **(B)** $\ln(1-x^2)+c$

(C) $-\ln x+\ln(1-x)+c$ **(D)** $\ln(x-x^2)+c$ **Ans.** (A)

2. $\int\frac{dx}{(1+x+x^2+x^3)}=$

(A) $\ln\sqrt{1+x}-\frac{1}{2}\ln\sqrt{1+x^2}+\frac{1}{2}\tan^{-1}x+c$

(B) $\ln\sqrt{1+x}-\frac{1}{2}\ln\sqrt{1+x^2}+\tan^{-1}x+c$

(C) $\ln\sqrt{1+x^2}-\frac{1}{2}\ln\sqrt{1+x}+\frac{1}{2}\tan^{-1}x+c$

(D) $\ln\sqrt{1+x^2}+\ln\sqrt{1+x}+\tan^{-1}x+c$ **Ans.** (A)

3. $\int\frac{(x-1)dx}{(x-2)(x-3)}=$

(A) $\ln(x-3)-\ln(x-2)+c$

(B) $2\ln(x-3)-\ln(x-2)+c$

(C) $\ln(x-3)+\ln(x-2)+c$

(D) $2\ln(x-3)+\ln(x-2)+c$ **Ans.** (B)

4. $\int\frac{dx}{\cos x(1+\cos x)}=$

(A) $\ln(\sec x+\tan x)+2\tan\frac{x}{2}+c$

(B) $\ln(\sec x+\tan x)-2\tan\frac{x}{2}+c$

(C) $\ln(\sec x+\tan x)+\tan\frac{x}{2}+c$

(D) $\ln(\sec x+\tan x)-\tan\frac{x}{2}+c$ **Ans.** (D)

5. $\int\frac{dx}{(1+x)(2+x)}=$

(A) $\ln(x+2)-\ln(x+1)+c$ **(B)** $\ln(x+2)+\ln(x+1)+c$

(C) $\ln(x+1)-\ln(x+2)+c$ **(D)** None of these **Ans.** (C)

6. Correct evaluation of $\int\frac{x\,dx}{(x-1)(x-2)}$ is

(A) $2\ln(x-2)-\ln(x-1)+c$ **(B)** $\ln(x-1)-\ln(x-2)+c$

(C) $\frac{(x-1)}{(x-2)}+c$ **(D)** $2\ln\frac{(x-2)}{(x-1)}+c$ **Ans.** (A)

7. $\int\frac{dx}{(x-1)(x^2+1)}=$

(A) $\frac{1}{2}\ln(x-1)-\frac{1}{4}\ln(x^2+1)-\frac{1}{2}\tan^{-1}x+c$

(B) $\frac{1}{2}\ln(x-1)+\frac{1}{4}\ln(x^2+1)-\frac{1}{2}\tan^{-1}x+c$

(C) $\frac{1}{2}\ln(x-1)-\frac{1}{2}\ln(x^2+1)-\frac{1}{2}\tan^{-1}x+c$

(D) None of these **Ans.** (A)

8. $\int\frac{(x^2+x-1)dx}{(x^2+x-6)}=$

(A) $x+\ln(x+3)+\ln(x-2)+c$

(B) $x-\ln(x+3)+\ln(x-2)+c$

(C) $x-\ln(x+3)-\ln(x-2)+c$

(D) None of these **Ans.** (B)

9. $\int\frac{x^2\,dx}{(x^2+2)(x^2+3)}=$

(A) $-\sqrt{2}\tan^{-1}x+\sqrt{3}\tan^{-1}x+c$

(B) $-\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}+\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}+c$

(C) $\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}+\sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}}+c$

(D) None of these **Ans.** (B)

10. $\int\frac{dx}{(x^2+1)(x^2+4)}=$

(A) $\frac{1}{3}\tan^{-1}x-\frac{1}{3}\tan^{-1}\frac{x}{2}+c$

(B) $\frac{1}{3}\tan^{-1}x+\frac{1}{3}\tan^{-1}\frac{x}{2}+c$

(C) $\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+c$

(D) $\tan^{-1}x-2\tan^{-1}\frac{x}{2}+c$ **Ans.** (C)

7. Substitution for trigonometric functions:

(a) $\int\frac{dx}{(a+b\cos x)}, \int\frac{dx}{(a+b\sin x)}$

Use $\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and put $\tan\frac{x}{2}=t$.

Illustration 22.80 Evaluate $\int\frac{dx}{(2+\cos x)}$.

Solution:

$$I=\int\frac{dx}{(2+\cos x)}=\int\frac{dx}{\left(2+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}=\int\frac{\sec^2\frac{x}{2}}{3+\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t$. Then $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$.

$$I=2\int\frac{dt}{(3+t^2)}=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)+c$$

Illustration 22.81 Evaluate $\int\frac{dx}{(5+4\cos x)}$.

Solution:

$$I=\int\frac{dx}{(5+4\cos x)}$$

$$I=\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{\left(5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)\right)}=\int\frac{\sec^2\frac{x}{2}}{9-\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2}=t$. Then $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$.

$$I=2\int\frac{dt}{(9+t^2)}=\frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right)+c=\frac{2}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right)+c$$

(b) $\int\frac{dx}{(a\sin x+b\cos x+c)}, \int\frac{dx}{(a\cos x+b\sin x)}$

Use $\cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$, $\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and put $\tan\frac{x}{2}=t$

Or $a=r\cos\alpha$ and $b=r\sin\alpha\Rightarrow r=\sqrt{a^2+b^2}$ and $\alpha=\tan^{-1}\frac{b}{a}$.

Illustration 22.82 Evaluate $\int\frac{dx}{(\sin x+\cos x+2)}$.

Solution:

$$I=\int\frac{dx}{(\sin x+\cos x+2)}=\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{\left(2\left(1+\tan^2\frac{x}{2}\right)+\left(1-\tan^2\frac{x}{2}\right)+2\tan\frac{x}{2}\right)}$$

$$I=\int\frac{\sec^2\frac{x}{2}dx}{\left(\tan^2\frac{x}{2}+2\tan\frac{x}{2}+3\right)}$$

Put $\tan\frac{x}{2}=t$. Then $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$.

$$I=2\int\frac{dt}{(t^2+2t+3)}=2\int\frac{dt}{\left[(t+1)^2+2\right]}=\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right)+c$$

$$\Rightarrow I=\sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\sqrt{2}}\right)+c$$

Illustration 22.83 Evaluate $\int\frac{dx}{(4\sin x+3\cos x)}$.

Solution:

$$I=\int\frac{dx}{(4\sin x+3\cos x)}$$

$$3\cos x+4\sin x=5\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)=5\cos(x-\alpha),\ \tan\alpha=\frac{4}{3}$$

$$I=\frac{1}{5}\int\frac{dx}{\cos(x-\alpha)}=\frac{1}{5}\int\sec(x-\alpha)dx=\frac{1}{5}[\sec(x-\alpha)+\tan(x-\alpha)]+c$$

Illustration 22.84 Evaluate $\int\frac{dx}{(1-\sin x-\cos x)}$.

Solution:

$$I=\int\frac{dx}{(1-\sin x-\cos x)}=\int\frac{dx}{\left(1-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}-\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$I=\int\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{\left(1+\tan^2\frac{x}{2}-2\tan\frac{x}{2}-1+\tan^2\frac{x}{2}\right)}$$

$$I=\int\frac{\sec^2\frac{x}{2}dx}{\left(2\tan^2\frac{x}{2}-2\tan\frac{x}{2}\right)}$$

Put $\tan\frac{x}{2}=t$. Then $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$. Therefore,

$$I=\int\frac{dt}{(t^2-t)}=\int\frac{dt}{t(t-1)}=\int\left(\frac{1}{t-1}-\frac{1}{t}\right)dt$$

$$I=\ln(t-1)-\ln t+c=\ln\left|\frac{t-1}{t}\right|+c$$

$$\Rightarrow I=\ln\left|\frac{\tan\frac{x}{2}-1}{\tan\frac{x}{2}}\right|+c$$

(c) $\int\frac{dx}{(a+b\cos^2 x)}, \int\frac{dx}{(a+b\sin^2 x)}$,

$\int\frac{dx}{(a\sin^2 x+b\cos^2 x+c)}, \int\frac{dx}{(a\cos^2 x+b\sin^2 x)}, \int\frac{dx}{(a\cos x+b\sin x)^2}$

Divide the numerator and the denominator by $\sin^2 x$ or $\cos^2 x$ and put $\tan x=t$.

Illustration 22.85 Evaluate $\int\frac{dx}{(1+2\cos^2 x)}$.

Solution:

$$I=\int\frac{dx}{(1+2\cos^2 x)}=\int\frac{\sec^2 x\,dx}{(\sec^2 x+2)}=\int\frac{\sec^2 x\,dx}{(\tan^2 x+3)}$$

(Dividing the numerator and the denominator by $\cos^2 x$)

Put $\tan x=t$. Then $\sec^2 x\,dx=dt$.

$$I=\int\frac{dt}{(t^2+3)}$$

$$I=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)+c=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x}{\sqrt{3}}\right)+c$$

Illustration 22.86 Evaluate $\int\frac{dx}{(3\cos x+\sin x)^2}$.

Solution:

$$I=\int\frac{dx}{(3\cos x+\sin x)^2}$$

(Dividing the numerator and the denominator by $\cos^2 x$)

$$I=\int\frac{dx}{(3\cos x+\sin x)^2}=\int\frac{\sec^2 x\,dx}{(3+\tan x)^2}$$

Put $\tan x=t$. Then $\sec^2 x\,dx=dt$.

$$I=\int\frac{dt}{(3+t)^2}=-\frac{1}{(3+t)}+c$$

$$I=-\frac{1}{(3+\tan x)}+c$$

Illustration 22.87 Evaluate $\int\frac{\cos x}{\cos 3x}dx$.

Solution:

$$I=\int\frac{\cos x}{\cos 3x}dx=\int\frac{\cos x}{(4\cos^3 x-3\cos x)}dx=\int\frac{1}{(4\cos^2 x-3)}dx$$

(Dividing the numerator and the denominator by $\cos^2 x$)

$$I=\int\frac{\sec^2 x}{(4-3(1+\tan^2 x))}dx=\int\frac{\sec^2 x}{(1-3\tan^2 x)}dx$$

Put $\tan x=t$. Then $\sec^2 x\,dx=dt$.

$$I=\int\frac{1}{(1-3t^2)}dt=\frac{1}{3}\int\frac{1}{\left(\frac{1}{3}-t^2\right)}dt$$

$$I=\frac{1}{2\sqrt{3}}\ln\left|\frac{1+\sqrt{3}t}{1-\sqrt{3}t}\right|+c=\frac{1}{2\sqrt{3}}\ln\left|\frac{1+\sqrt{3}\tan x}{1-\sqrt{3}\tan x}\right|+c$$

Illustration 22.88 Evaluate $\int\frac{dx}{(4\sin^2 x+5\cos^2 x+4\sin x\cos x)}$

Solution:

$$I=\int\frac{dx}{(4\sin^2 x+5\cos^2 x+4\sin x\cos x)}$$

(Dividing the numerator and the denominator by $\cos^2 x$)

$$I=\int\frac{dx}{(4\sin^2 x+5\cos^2 x+4\sin x\cos x)}=\int\frac{\sec^2 x dx}{(4\tan^2 x+5+4\tan x)}$$

$$I=\frac{1}{4}\int\frac{dx}{\left(\left(\tan x+\frac{1}{2}\right)^2+1\right)}$$

Put $\tan x+\frac{1}{2}=t$. Then $\sec^2 x\,dx=dt$.

$$I=\frac{1}{4}\int\frac{dx}{(t^2+1)}=\frac{1}{4}\tan^{-1}t+c=\frac{1}{4}\tan^{-1}\left(\tan x+\frac{1}{2}\right)+c$$

(d) $\int\frac{(a\cos x+b\sin x)}{(p\cos x+q\sin x)}dx$

$$(a\cos x+b\sin x)=l(p\cos x+q\sin x)+m\frac{d}{dx}(p\cos x+q\sin x)$$

Compare both side coefficients of $\sin x$ and $\cos x$, and calculate the value of l and m.

$$\int\frac{(a\cos x+b\sin x)}{(p\cos x+q\sin x)}dx$$

$$(a\cos x+b\sin x+c)=l(p\cos x+q\sin x+r)+m\frac{d}{dx}(p\cos x+q\sin x+r)+n$$

Compare both side coefficients of $\sin x$, $\cos x$ and constant term, and calculate the value of l, m and n.

Illustration 22.89 Evaluate $\int\frac{(4\cos x+5\sin x)}{(2\cos x+3\sin x)}dx$.

Solution:

$$I=\int\frac{(4\cos x+5\sin x)}{(2\cos x+3\sin x)}dx$$

$$(4\cos x+5\sin x)=a(2\cos x+3\sin x)+b\frac{d}{dx}(2\cos x+3\sin x)$$
$$=a(2\cos x+3\sin x)+b(-2\sin x+3\cos x)$$

Comparing the coefficients of $\cos x$ and $\sin x$, we get

$$a=\frac{23}{13},\ b=\frac{2}{13}$$

$$I=\frac{23}{13}\int 1dx+\frac{2}{13}\int\frac{(3\cos x-2\sin x)}{(2\cos x+3\sin x)}dx$$
$$=\frac{23}{13}x+\frac{2}{13}\ln|2\cos x+3\sin x|+c$$

Illustration 22.90 Evaluate $\int\frac{(\cos x-3\sin x+4)}{(\cos x+\sin x+2)}dx$.

Solution:

$$I=\int\frac{(\cos x-3\sin x+4)}{(\cos x+\sin x+2)}dx$$

$$(\cos x-3\sin x+4)=a(\cos x+\sin x+2)+b\frac{d}{dx}(\cos x+\sin x+2)+c$$
$$=a(\cos x+\sin x+2)+b(-\sin x+\cos x)+c$$

Comparing the coefficients of $\cos x$, $\sin x$ and constant, we get

$$a=-1, b=2, c=6$$

$$I=-\int 1dx+2\int\frac{(\cos x-\sin x)}{(\cos x+\sin x+2)}dx+6\int\frac{dx}{(\cos x+\sin x+2)}$$

For IIIrd integral,

$$6\int\frac{dx}{(\cos x+\sin x+2)}=6\int\frac{\left(1+\tan^2\frac{x}{2}\right)}{1-\tan^2\frac{x}{2}+2\tan\frac{x}{2}+2+2\tan^2\frac{x}{2}}dx$$

$$=6\int\frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2}+2\tan\frac{x}{2}+3}dx=6\int\frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2+2}dx$$

$$\Rightarrow I=-\int 1dx+2\int\frac{(\cos x-\sin x)}{(\cos x+\sin x+2)}dx+6\int\frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2+2}dx$$

$$I=-x+2\ln|(\cos x+\sin x+2)|+3\sqrt{2}\tan^{-1}\left(\frac{\left(1+\tan\frac{x}{2}\right)}{\sqrt{2}}\right)+c$$

Illustration 22.91 Evaluate $\int\frac{dx}{1+\cot x}$.

Solution:

$$I=\int\frac{dx}{1+\cot x}=\int\frac{\sin x}{\sin x+\cos x}dx$$

$$\sin x=M\frac{d}{dx}(\sin x+\cos x)+N(\sin x+\cos x)$$
$$=M(-\sin x+\cos x)+N(\sin x+\cos x)$$

Comparing the coefficients of sin x and cos x, we have

$$-M+N=1,\ M+N=0$$

Solving these equations, we have $M=-\frac{1}{2}, N=\frac{1}{2}$.

$$\sin x=-\frac{1}{2}(-\sin x+\cos x)+\frac{1}{2}(\sin x+\cos x)$$

$$I=\int\frac{\sin x}{\sin x+\cos x}dx$$
$$=-\frac{1}{2}\int\frac{(-\sin x+\cos x)}{(\sin x+\cos x)}dx+\frac{1}{2}\int\frac{(\sin x+\cos x)}{(\sin x+\cos x)}dx$$
$$=-\frac{1}{2}\ln|(\sin x+\cos x)|+\frac{1}{2}x+c$$

(e) $\int\sin^m x\cdot\cos^n x\,dx$, $(m, n\in N)$

If one out of m and n is odd, then substitute for term of even power.
If both are odd, then substitute either of the term.
If both are even, then use trigonometric identities only.

The above substitution enables us to integrate any function of the form $R(\sin x, \cos x)$. However, in practice, it sometimes leads to extremely complex rational function. In some cases, the integral can be simplified by:

(i) Substituting $\sin x=t$, if the integral is of the form $\int R(\sin x)\cos x\,dx$.

(ii) Substituting $\cos x=t$, if the integral is of the form $\int R(\cos x)\sin x\,dx$.

(iii) Substituting $\tan x=t$, that is, $dx=\frac{dt}{1+t^2}$, if the integral is dependent only on $\tan x$.

(iv) Substituting $\cos x=t$, if $R(-\sin x, \cos x)=-R(\sin x, \cos x)$.

(v) Substituting $\sin x=t$, if $R(\sin x, -\cos x)=-R(\sin x, \cos x)$.

(vi) Substituting $\tan x=t$, if $R(-\sin x, -\cos x)=-R(\sin x, \cos x)$.

Illustration 22.92 Evaluate $\int\sin^3 x\cdot\cos^2 x\,dx$.

Solution:

$$I=\int\sin x\cdot(1-\cos^2 x)\cdot\cos^2 x\,dx$$

Put $\cos x=t$. Then $-\sin x\,dx=dt$.

$$I=-\int(t^2-t^4)dt=\frac{t^5}{5}-\frac{t^3}{3}+c$$
$$=\frac{\sin^5 x}{5}-\frac{\sin^3 x}{3}+c$$

Illustration 22.93 Evaluate $\int\frac{d\theta}{\sin\theta\cdot\cos^3\theta}$.

Solution:

$$I=\int\frac{d\theta}{\sin\theta\cdot\cos^3\theta}=\int\frac{\sec^2\theta}{\sin\theta\cdot\cos\theta}d\theta=\int\frac{\sec^2\theta}{\tan\theta\cdot\cos^2\theta}d\theta$$

Put $\tan\theta=t$. Then $\sec^2\theta\,d\theta=dt$.

$$I=\int\frac{1+t^2}{t}dt=\int\left(\frac{1}{t}+t\right)dt$$

$$I=\ln|t|+\frac{t^2}{2}+c=\ln|\tan\theta|+\frac{\tan^2\theta}{2}+c$$

Illustration 22.94 Evaluate $\int\frac{\sin^3 2x}{\cos^5 2x}dx$.

Solution:

$$I=\int\frac{\sin^3 2x}{\cos^5 2x}dx$$

The given equation may be written as

$$I=\int\frac{\sin^3 2x\cdot\sec^2 2x}{\cos^3 2x}dx=\int\tan^3 2x\cdot\sec^2 2x\,dx$$

Put $\tan 2x=t$. Then $2\sec^2 2x\,dx=dt$.

$$I=\frac{1}{2}\int t^3 dt=\frac{t^4}{8}+c=\frac{\tan^4 2x}{8}+c$$

Illustration 22.95 Evaluate $\int\frac{\sin nx}{\sin x}dx$.

Solution:

$$I_n=\int\frac{\sin nx}{\sin x}dx$$

$$I_{n-2}=\int\frac{\sin(n-2)x}{\sin x}dx$$

$$I_n-I_{n-2}=\int\frac{\sin nx-\sin(n-2)x}{\sin x}dx=\int\frac{2\cos(n-1)x\cdot\sin x}{\sin x}dx$$

$$I_n-I_{n-2}=\frac{2\sin(n-1)x}{(n-1)}$$

8. Integrals of the form: $\int x^m(a+bx^n)^p\,dx$

(a) If $p\in N$ (natural number). We expand the integral with the help of binomial theorem and integrate.

Illustration 22.96 Evaluate $\int x^{1/3}(2+x^{1/2})^2\,dx$.

Solution:

$$I=\int x^{1/3}(2+x^{1/2})^2\,dx$$

Since P is a natural number. So

$$I=\int x^{1/3}(4+x+4x^{1/2})dx$$

$$=\int (4x^{1/3}+x^{4/3}+4x^{5/6})dx$$

$$=\frac{4x^{4/3}}{4/3}+\frac{x^{7/3}}{7/3}+\frac{4x^{11/6}}{11/6}+c$$

$$I=3x^{4/3}+\frac{3}{7}x^{7/3}+\frac{24}{11}x^{11/6}+c$$

(b) If $p\in I^-$ (that is negative integer). Write $x=t^k$ where k is the LCM of the denominator of m and n.

Illustration 22.97 Evaluate $\int x^{-2/3}(2+x^{2/3})^{-1}dx$.

Solution: If we substitute $x=t^3$ (as we know $p\in$ negative integer), then

$$x=t^3 \Rightarrow dx=3t^2dt$$

$$I=\int\frac{3t^2}{t^2(1+t^2)}dt=3\int\frac{dt}{(1+t^2)}=3\tan^{-1}t+c$$

$$I=3\tan^{-1}(x^{1/3})+c$$

(c) If $\frac{m+1}{n}$ is an integer and $p\in$ fraction, then put $(a+bx^n)=t^k$, where k is the denominator of the fraction p.

(d) If $\left(\frac{m+1}{n}+p\right)$ is an integer and $p\in$ fraction, then put $(a+bx^n)=t^k\cdot x^n$, where k is the denominator of the fraction P.

Illustration 22.98 Evaluate $\int x^{-11}(1+x^4)^{-1/2}\,dx$.

Solution: Here,

$$\left(\frac{m+1}{n}+p\right)=\left(\frac{-11+1}{4}+\frac{1}{2}\right)=-3$$

If we substitute $1+\frac{1}{x^4}=t^2$, then

$$-\frac{4}{x^5}dx=2tdt$$

$$I=\int\frac{1}{x^{11}(1+x^4)^{1/2}}dx=\int\frac{1}{x^{11}\cdot x^2(1+x^{-4})^{1/2}}dx$$

$$I=-\frac{1}{4}\int\frac{2t}{x^8t}dt=-\frac{1}{2}\int(t^2-1)^2\,dt=-\frac{1}{2}\int(t^4-2t^2+1)dt$$

$$I=-\frac{1}{2}\left(\frac{t^5}{5}-2\frac{t^3}{3}+t\right)+c$$

$$=-\frac{1}{2}\left[\frac{\left(1+x^{-4}\right)^{5/2}}{5}-\frac{2}{3}\left(1+x^{-4}\right)^{3/2}+\left(1+x^{-4}\right)^{1/2}\right]+c$$

Your Turn 5

1. $\int\frac{dx}{5+4\cos x}=$

(A) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan x\right)+c$ **(B)** $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan x\right)+c$

(C) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$ **(D)** $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$

Ans. (C)

2. $\int\frac{dx}{(x^2+a^2)(x^2+b^2)}=$

(A) $\frac{1}{(a^2-b^2)}\left(\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)+c$

(B) $-\frac{1}{(a^2-b^2)}\left(\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)+c$

(C) $\left(\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)+c$

(D) $-\left(\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)-\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)+c$ **Ans.** (A)

3. $\int\frac{1}{1+\cos^2 x}dx=$

(A) $\frac{1}{\sqrt{2}}\tan^{-1}(\tan x)+c$ **(B)** $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{2}\tan x\right)+c$

(C) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)+c$ **(D)** None of these **Ans.** (C)

4. $\int\frac{1}{1+3\sin^2 x}dx=$

(A) $\frac{1}{3}\tan^{-1}(3\tan^2 x)+c$ **(B)** $\frac{1}{2}\tan^{-1}(2\tan x)+c$

(C) $\tan^{-1}(\tan x)+c$ **(D)** None of these **Ans.** (B)

5. $\int\frac{1}{2x^2+x+1}dx$ equals

(A) $\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$ **(B)** $\frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$

(C) $\frac{1}{2}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right)+c$ **(D)** None of these **Ans.** (D)

6. $\int\frac{dx}{7+5\cos x}=$

(A) $\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{1}{\sqrt{6}}\tan\frac{x}{2}\right)+c$ **(B)** $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)+c$

(C) $\frac{1}{4}\tan^{-1}\left(\tan\frac{x}{2}\right)+c$ **(D)** $\frac{1}{7}\tan^{-1}\left(\tan\frac{x}{2}\right)+c$

Ans. (A)

7. $\int\frac{1}{x^2+4x+13}dx$ is equal to

(A) $\ln|x^2+4x+13|+c$ **(B)** $\frac{1}{3}\tan^{-1}\left(\tan\frac{x+2}{3}\right)+c$

(C) $\ln|2x+4|+c$ **(D)** $\dfrac{(2x+4)}{(x^2+4x+13)^2}+c$

Ans. (B)

8. $\displaystyle\int\frac{dx}{\cos x-\sin x}$ is equal to

(A) $\dfrac{1}{\sqrt{2}}\ln\left|\tan\left(\dfrac{x}{2}+\dfrac{3\pi}{8}\right)\right|+c$

(B) $\dfrac{1}{\sqrt{2}}\ln\left|\cot\dfrac{x}{2}\right|+c$

(C) $\dfrac{1}{\sqrt{2}}\ln\left|\tan\left(\dfrac{x}{2}-\dfrac{3\pi}{8}\right)\right|+c$

(D) $\dfrac{1}{\sqrt{2}}\ln\left|\tan\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)\right|+c$ **Ans.** (A)

9. $\displaystyle\int\frac{1}{x^2+2x+2}dx=$

(A) $\sin^{-1}(x+1)+c$ **(B)** $\sin h^{-1}(x+1)+c$

(C) $\tan h^{-1}(x+1)+c$ **(D)** $\tan^{-1}(x+1)+c$ **Ans.** (D)

10. $\displaystyle\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx=$

(A) $\dfrac{12}{13}x-\dfrac{5}{13}\ln|3\cos x+2\sin x|+c$

(B) $\dfrac{12}{13}x+\dfrac{5}{13}\ln|3\cos x+2\sin x|+c$

(C) $\dfrac{13}{12}x+\dfrac{5}{13}\ln|3\cos x+2\sin x|+c$

(D) None of these **Ans.** (A)

9. Reduction formulae:

(a) $\displaystyle\int\sin^n x\,dx=-\frac{\cos x\cdot\sin^{n-1}x}{n}+\frac{n-1}{n}\int\sin^{n-2}x\,dx$

If $I_n=\int\sin^n x\,dx$, then

$$I_n=-\frac{\cos x\cdot\sin^{n-1}x}{n}+\frac{n-1}{n}I_{n-2}$$

(b) $\displaystyle\int\cos^n x\,dx=\frac{\sin x\cdot\cos^{n-1}x}{n}+\frac{n-1}{n}\int\cos^{n-2}x\,dx$

If $I_n=\int\cos^n x\,dx$, then

$$I_n=\frac{\sin x\cdot\cos^{n-1}x}{n}+\frac{n-1}{n}I_{n-2}$$

(c) $\displaystyle\int\tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int\tan^{n-2}x\,dx$

If $I_n=\int\tan^n x\,dx$, then

$$I_n=\frac{\tan^{n-1}x}{n-1}-I_{n-2}$$

(d) $\displaystyle\int\cot^n x\,dx=-\frac{\cot^{n-1}x}{n-1}-\int\cot^{n-2}x\,dx$

If $I_n=\int\cot^n x\,dx$, then

$$I_n=-\frac{\cot^{n-1}x}{n-1}-I_{n-2}$$

(e) $\displaystyle\int\sec^n x\,dx=\frac{1}{n-1}[\sec^{n-2}x\cdot\tan x+(n-2)\int\sec^{n-2}x\,dx]$

If $I_n=\int\sec^n x\,dx$, then

$$(n-1)I_n=\sec^{n-2}x\cdot\tan x+(n-2)I_{n-2}$$

(f) $\displaystyle\int\operatorname{cosec}^n x\,dx=\frac{1}{n-1}[-\operatorname{cosec}^{n-2}x\cdot\cot x+(n-2)\int\operatorname{cosec}^{n-2}x\,dx]$

If $I_n=\int\operatorname{cosec}^n x\,dx$, then

$$(n-1)I_n=-\operatorname{cosec}^{n-2}x\cdot\cot x+(n-2)I_{n-2}$$

(g) $\displaystyle\int\sin^p x\cdot\cos^q x\,dx=-\frac{\sin^{q+1}\cdot\cos^{p-1}}{p+q}+\frac{p-1}{p+q}\int\sin^{p-2}x\cdot\cos^q x\,dx$

If $I_{p,q}=\int\sin^p x\cdot\cos^q x\,dx$, then

$$(p+q)I_{(p,q)}=-\sin^{q+1}\cdot\cos^{p-1}x+(p-1)I_{(p-2,q)}$$

(h) $\displaystyle\int\frac{dx}{(x^2+k)^n}=\frac{x}{k(2n-2)(x^2+k)^{n-1}}+\frac{(2n-3)}{k(2n-2)}\int\frac{dx}{(x^2+k)^{n-1}}$

If $I_n=\displaystyle\int\frac{dx}{(x^2+k)^n}$, then

$$k(2n-2)I_n=\frac{x}{(x^2+k)^{n-1}}+(2n-3)I_{n-1}$$

(i) Reduction formulae for $I_{(n,m)}=\displaystyle\int\frac{\sin^n x}{\cos^m x}dx$ is

$$I_{(n,m)}=\frac{1}{m-1}\cdot\frac{\sin^{n-1}x}{\cos^{m-1}x}-\frac{n-1}{m-1}\cdot\frac{\sin^{n-1}x}{\cos^{m-1}x}\cdot I_{(n-2,m-2)}$$

Illustration 22.99 Evaluate $\int\tan^5 x\,dx$.

Solution: Using

$$I_n=\frac{\tan^{n-1}x}{n-1}-I_{n-2}$$

Put $n=5$, we get

$$I_5=\int\tan^5 x\,dx\Rightarrow I_5=\frac{\tan^4 x}{4}-I_3$$

$$=\frac{\tan^4 x}{4}-\frac{\tan^2 x}{2}+\ln|\sec x|+c$$

Illustration 22.100 Evaluate $\int\sin^8 x\,dx$.

Solution: Let $\cos x+i\sin x=y$. Then

$$2\cos x=y+\frac{1}{y}\text{ and }2\cos nx=y^n+\frac{1}{y^n}$$

$$2i\sin x=y-\frac{1}{y}\text{ and }2i\sin nx=y^n-\frac{1}{y^n}$$

(Remember as the standard results)

Thus,

$$(2i\sin x)^8 = \left(y-\frac{1}{y}\right)^8 = \left(y^8+\frac{1}{y^8}\right)-8\left(y^6+\frac{1}{y^6}\right)$$
$$+28\left(y^4+\frac{1}{y^4}\right)-56\left(y^2+\frac{1}{y^2}\right)+70$$
$$= 2\cos 8x - 16\cos 6x + 56\cos 4x - 112\cos 2x + 70$$

So

$$\sin^8 x = \frac{1}{2^7}(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35)$$

and

$$\int \sin^8 x\,dx = \int \frac{1}{2^7}(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35)\,dx$$

$$\int \sin^8 x\,dx = \frac{1}{2^7}\left(\frac{\sin 8x}{8} - 8\frac{\sin 6x}{6} + 28\frac{\sin 4x}{4} - 56\frac{\sin 2x}{2} + 35x\right) + c$$

$$= \frac{1}{2^7}\left(\frac{\sin 8x}{8} - \frac{4}{3}\sin 6x + 7\sin 4x - 28\sin 2x + 35x\right) + c$$

Illustration 22.101 Evaluate $\int \operatorname{cosec}^4 x\,dx$.

Solution:

$$I = \int \operatorname{cosec}^4 x\,dx = \int \operatorname{cosec}^2 x(1+\cot^2 x)\,dx$$

$$I = \int \operatorname{cosec}^2 x\,dx + \int \operatorname{cosec}^2 x \cot^2 x\,dx = -\cot x - \frac{\cot^3 x}{3} + c$$

10. Some non-integral functions:

(a) $\int \frac{dx}{\ln x}$ (b) $\int e^{x^2}\,dx$

(c) $\int \frac{x^2}{1+x^5}\,dx$ (d) $\int \sqrt[3]{1+x^2}\,dx$

(e) $\int \sqrt[2]{1+x^3}\,dx$ (f) $\int \sqrt[2]{1-k^2\sin^2 x}\,dx$

(g) $\int e^{-x^2}\,dx$ (h) $\int \frac{\sin x}{x}\,dx$

(i) $\int \sqrt{\sin x}\,dx$ (j) $\int \sin x^2\,dx$

(k) $\int \cos x^2\,dx$ (l) $\int x\tan x\,dx$

Additional Solved Examples

1. $\int \frac{dx}{x(\ln^2 x + 4\ln x - 1)}$ is equal to

(A) $\frac{1}{\sqrt{5}}\ln\left|\frac{\ln x + 2 - \sqrt{5}}{\ln x + 2 + \sqrt{5}}\right| + c$ **(B)** $\frac{1}{2}\ln\left|\frac{\ln x + 2 - \sqrt{5}}{\ln x + 2 + \sqrt{5}}\right| + c$

(C) $\frac{1}{2\sqrt{5}}\ln\left|\frac{\ln x + 2 - \sqrt{5}}{\ln x - 2 + \sqrt{5}}\right| + c$ **(D)** $\frac{1}{2\sqrt{5}}\ln\left|\frac{\ln x + 2 - \sqrt{5}}{\ln x + 2 + \sqrt{5}}\right| + c$

Solution:

$$I = \int \frac{dx}{x(\ln^2 x + 4\ln x - 1)}$$

Let $\ln x = t$. Then $\frac{1}{x}dx = dt$. Therefore,

$$I = \int \frac{dt}{(t^2+4t-1)} = \int \frac{dt}{((t+2)^2-5)} = \frac{1}{2\sqrt{5}}\ln\left|\frac{t+2-\sqrt{5}}{t+2+\sqrt{5}}\right| + c$$

$$\Rightarrow I = \frac{1}{2\sqrt{5}}\ln\left|\frac{\ln x + 2 - \sqrt{5}}{\ln x + 2 + \sqrt{5}}\right| + c$$

Hence, the correct answer is option (D).

2. $\int x^x(1+\ln x)\,dx$ is equal to

(A) $x^x \ln x + c$ **(B)** $x^x + c$

(C) $x\ln x + c$ **(D)** None of these

Solution:

$$I = \int x^x(1+\ln x)\,dx$$

Let $x^x = t$. Then

$$x\ln x = \ln t$$

$$\Rightarrow \left(1\cdot\ln x + x\cdot\frac{1}{x}\right)dx = \frac{dt}{t} \Rightarrow dx\,(1+\ln x)x^x = dt$$

Therefore,

$$I = \int dt \Rightarrow I = t + c = x^x + c$$

Hence, the correct answer is option (B).

3. $\int \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$ is equal to

(A) $\frac{1}{40}\ln\left|\frac{5+4(\sin x - \cos x)}{5-4(\sin x - \cos x)}\right| + c$

(B) $\frac{1}{40}\ln\left|\frac{5-4(\sin x - \cos x)}{5+4(\sin x - \cos x)}\right| + c$

(C) $\frac{1}{40}\ln\left|\frac{5+4(\sin x + \cos x)}{5-4(\sin x + \cos x)}\right| + c$

(D) $\frac{1}{40}\ln\left|\frac{5-4(\sin x + \cos x)}{5+4(\sin x + \cos x)}\right| + c$

Solution:

$$I = \int \frac{\sin x + \cos x}{9 + 16\sin 2x}\,dx$$

Let $t = \sin x - \cos x$. Then $t^2 = 1 - \sin 2x \Rightarrow \sin 2x = (1 - t^2)$

Also,

$$dt = \cos x + \sin x$$

Therefore,

$$= \int \frac{dt}{9+16-16t^2} = \int \frac{dt}{25-16t^2}$$

$$= \frac{1}{16}\int \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2} = \frac{1}{16}\cdot\frac{1}{2\cdot\frac{5}{4}}\ln\left|\left(\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right)\right| + c$$

$$= \frac{1}{40}\ln\left|\frac{5+4t}{5-4t}\right| + c = \frac{1}{40}\ln\left|\frac{5+4(\sin x - \cos x)}{5-4(\sin x - \cos x)}\right| + c$$

Hence, the correct answer is option (A).

4. For what value of a and b, the equation

$\int(\sin 2x - \cos 2x)\,dx = \frac{1}{\sqrt{2}}\sin(2x - a) + b$ holds good.

(A) $a=-\dfrac{5\pi}{4}$, b is any arbitrary constant

(B) $a=\dfrac{5\pi}{4}$, b is any arbitrary constant

(C) $a=-\dfrac{\pi}{4}$, b is any arbitrary constant

(D) $a=\dfrac{\pi}{4}$, b is any arbitrary constant

Solution:

$$\int(\sin 2x-\cos 2x)\,dx=\int\sqrt{2}\left(\frac{\sin 2x}{\sqrt{2}}-\frac{\cos 2x}{\sqrt{2}}\right)dx=$$

$$-\int\sqrt{2}\cos\left(2x+\frac{\pi}{4}\right)dx$$

$$=-\frac{\sqrt{2}}{2}\sin\left(2x+\frac{\pi}{4}\right)+c=\frac{1}{\sqrt{2}}\sin\left(2x+\frac{5\pi}{4}\right)+c$$

$a=-\dfrac{5\pi}{4}$, b is any arbitrary constant.

Hence, the correct answer is option (A).

5. For what value of a and b, the equation

$\displaystyle\int\frac{dx}{1+\sin x}=\tan\left(\frac{x}{2}+a\right)+b$ holds good.

(A) $a=-\dfrac{5\pi}{4}$, b is any arbitrary constant

(B) $a=\dfrac{5\pi}{4}$, b is any arbitrary constant

(C) $a=-\dfrac{\pi}{4}$, b is any arbitrary constant

(D) $a=\dfrac{\pi}{4}$, b is any arbitrary constant

Solution:

$$\int\frac{dx}{1+\sin x}=\int\frac{dx}{1+\cos\left(\frac{\pi}{2}-x\right)}=\frac{1}{2}\int\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}{-\frac{1}{2}}+c=-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+c=\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+c$$

$a=-\dfrac{\pi}{4}$, b any arbitrary constant.

Hence, the correct answer is option (C).

6. $\displaystyle\int\cos\left(\ln\frac{x}{a}\right)dx$ is equal to

(A) $x\left(\cos\left(\ln\dfrac{x}{a}\right)-\sin\left(\ln\dfrac{x}{a}\right)\right)+c$

(B) $\dfrac{x}{2}\left(\cos\left(\ln\dfrac{x}{a}\right)+\sin\left(\ln\dfrac{x}{a}\right)\right)+c$

(C) $\dfrac{x}{2}\left(\cos\left(\ln\dfrac{x}{a}\right)-\sin\left(\ln\dfrac{x}{a}\right)\right)+c$

(D) $x\left(\cos\left(\ln\dfrac{x}{a}\right)+\sin\left(\ln\dfrac{x}{a}\right)\right)+c$

Solution:

$$I=\int\cos\left(\ln\frac{x}{a}\right)dx$$

Let $\ln\dfrac{x}{a}=t$. Then

$$x=a\cdot e^t\Rightarrow dx=ae^t\,dt$$

$$I=a\int e^t\cos t\,dt=\frac{ae^t}{2}(\cos t+\sin t)+c$$

$$=\frac{x}{2}\left(\cos\left(\ln\frac{x}{a}\right)+\sin\left(\ln\frac{x}{a}\right)\right)+c$$

Hence, the correct answer is option (B).

7. $\displaystyle\int\frac{\ln(x+\sqrt{x^2+1})}{\sqrt{x^2+1}}dx$ is equal to

(A) $\dfrac{1}{2}\ln^2(x-\sqrt{x^2+1})+c$ **(B)** $\ln^2(x-\sqrt{x^2+1})+c$

(C) $\dfrac{1}{2}\ln^2(x+\sqrt{x^2+1})+c$ **(D)** $\ln^2(x+\sqrt{x^2+1})+c$

Solution:

$$I=\int\frac{\ln(x+\sqrt{x^2+1})}{\sqrt{x^2+1}}dx$$

Let $\ln(x+\sqrt{x^2+1})=t$. Then

$$\frac{1}{(x+\sqrt{x^2+1})}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)dx=dt\Rightarrow\frac{dx}{\sqrt{x^2+1}}=dt$$

$$\Rightarrow I=\int t\,dt=\frac{t^2}{2}+c$$

$$I=\frac{1}{2}\left[\ln\ (x+\sqrt{x^2+1})\right]^2+c$$

Hence, the correct answer is option (C).

8. $\displaystyle\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx$ is equal to

(A) $\dfrac{1}{(x\sin x+\cos x)^2}+k$ **(B)** $-\dfrac{1}{(x\sin x+\cos x)}+k$

(C) $\dfrac{1}{(x\sin x+\cos x)^3}+k$ **(D)** $\dfrac{1}{(x\sin x+\cos x)^4}+k$

Solution:

$$I=\int\frac{x\cos x}{(x\sin x+\cos x)^2}dx$$

Let $\dfrac{1}{x\sin x+\cos x}=t$. Then

$$\frac{(x\sin x+\cos x)\cdot 0-1(x\cos x+\sin x-\sin x)}{(x\sin x+\cos x)^2}=\frac{dt}{dx}$$

$$\Rightarrow\frac{-x\cos x}{(x\sin x+\cos x)^2}=\frac{dt}{dx}$$

Therefore,

$$I=-\int dt=-\frac{1}{x\sin x+\cos x}+c$$

Hence, the correct answer is option (B).

9. $\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx$ is equal to

(A) $\sqrt{2}\sin^{-1}(\sin x+\cos x)+c$ **(B)** $\frac{1}{\sqrt{2}}\sin^{-1}(\sin x-\cos x)+c$

(C) $\sqrt{2}\sin^{-1}(\sin x-\cos x)+c$ **(D)** $\frac{1}{\sqrt{2}}\sin^{-1}(\sin x+\cos x)+c$

Solution:

$$I=\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

$$=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{2\sin x\cdot\cos x}}\,dx$$

$$=\sqrt{2}\int\frac{(\sin x+\cos x)}{\sqrt{\sin 2x}}\,dx$$

Let $t=\sin x-\cos x$. Then

$$t^2=1-\sin 2x\Rightarrow dt=(\cos x+\sin x)\,dx$$

Therefore,

$$I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}=\sqrt{2}\sin^{-1}(t)+c$$

$$I=\sqrt{2}\sin^{-1}(\sin x-\cos x)+c.$$

Hence, the correct answer is option (C).

10. $\int\frac{1}{x^2(x^4+1)^{3/4}}dx$ is equal to

(A) $\left(1+\frac{1}{x^4}\right)^{1/4}+c$ **(B)** $-\left(1+\frac{1}{x^4}\right)^{1/4}+c$

(C) $-\left(1-\frac{1}{x^4}\right)^{1/4}+c$ **(D)** $\left(1-\frac{1}{x^4}\right)^{1/4}+c$

Solution:

$$I=\int\frac{1}{x^2(x^4+1)^{3/4}}dx=\int\frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

Let $1+x^{-4}=t$. Then

$$\frac{-4}{x^5}\,dx=dt\Rightarrow\frac{1}{x^5}\,dx=-\frac{1}{4}\,dt$$

$$\Rightarrow I=-\frac{1}{4}\int\frac{dt}{t^{3/4}}=-\frac{1}{4}\int t^{-3/4}dt=-\frac{1}{4}\times 4t^{1/4}+c$$

$$\Rightarrow I=-\left(1+\frac{1}{x^4}\right)^{1/4}+c$$

Hence, the correct answer is option (B).

11. $\int\frac{\sqrt{5+x^{10}}}{x^{16}}\,dx$ is equal to

(A) $-\frac{1}{75}\left(1+\frac{5}{x^{10}}\right)^{3/2}+c$ **(B)** $-\frac{1}{75}\left(1-\frac{5}{x^{10}}\right)^{3/2}+c$

(C) $\frac{1}{75}\left(1-\frac{5}{x^{10}}\right)^{3/2}+c$ **(D)** $\frac{1}{75}\left(1+\frac{5}{x^{10}}\right)^{3/2}+c$

Solution:

$$I=\int\frac{\sqrt{5+x^{10}}}{x^{16}}\,dx=\int\frac{x^5\sqrt{\frac{5}{x^{10}}+1}}{x^{16}}\,dx=\int\frac{\sqrt{\frac{5}{x^{10}}+1}}{x^{11}}\,dx$$

Let $1+\frac{5}{x^{10}}=t$. Then

$$5\left(\frac{-10}{x^{11}}\right)dx=dt\Rightarrow\frac{1}{x^{11}}dx=-\frac{1}{50}\,dt$$

$$\Rightarrow I=-\frac{1}{50}\int\sqrt{t}dt=-\frac{1}{50}\times\frac{2}{3}t^{3/2}=-\frac{1}{75}\left(1+\frac{5}{x^{10}}\right)^{3/2}+c$$

Hence, the correct answer is option (A).

12. $\int\text{cosec}^6 x dx$ is equal to

(A) $-\cot x-\frac{\cot^5 x}{5}-\frac{2\cot^5 x}{3}+k$

(B) $-\frac{\cot x}{3}+\frac{2\cot^5 x}{5}+2\cot^{-3}x+k$

(C) $\frac{\tan^3 x}{3}-\frac{\tan x}{5}+2\tan^3 x+k$

(D) None of these

Solution:

$$I=\int\text{cosec}^6 x dx=\int\text{cosec}^4 x\cdot\text{cosec}^2 x\,dx$$

$$=\int(1+\cot^2 x)^2\cdot\text{cosec}^2 x\,dx=\int(1+\cot^4 x+2\cot^2 x)\cdot\text{cosec}^2 x\,dx$$

Let $\cot x=t$. Then

$$-\text{cosec}^2 x\,dx=dt$$

$$\Rightarrow I=-\int(1+t^4+2t^2)\,dt=-t-\frac{t^5}{s}-\frac{2}{3}t^3+c$$

$$\Rightarrow I=-\cot x-\frac{\cot^5 x}{5}-\frac{2}{3}\cot^3 x+c$$

Hence, the correct answer is option (A).

13. $\int\frac{(\sqrt{x^2+1})\{\ln(x^2+1)-2\ln x\}}{x^4}dx$ is equal to

(A) $\frac{(x^2+1)\sqrt{x^2+1}}{x^3}\left[2-3\ln\left(\frac{x^2+1}{x^2}\right)\right]+c$

(B) $\frac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left[2+3\ln\left(\frac{x^2+1}{x^2}\right)\right]+c$

(C) $\frac{(x^2+1)\sqrt{x^2+1}}{x^3}\left[2+3\ln\left(\frac{x^2+1}{x^2}\right)\right]+c$

(D) $\frac{(x^2+1)\sqrt{x^2+1}}{9x^3}\left[2-3\ln\left(\frac{x^2+1}{x^2}\right)\right]+c$

Solution:

$$I=\int\frac{(\sqrt{x^2+1})\{\ln(x^2+1)-2\ln x\}}{x^4}\,dx=\int\frac{\left(\frac{\sqrt{x^2+1}}{x^2}\right)\left\{\ln\left(\frac{x^2+1}{x^2}\right)\right\}}{x^3}\,dx$$

Let $\dfrac{x^2+1}{x^2}=t$. Then

$$-\frac{2}{x^3}dx=dt$$
$$\Rightarrow I=-\frac{1}{2}\int\sqrt{t}\ln t\,dt$$
$$=-\frac{1}{2}\left[(\ln t)\cdot\frac{2t^{3/2}}{3}-\frac{2}{3}\int\frac{1}{t}\cdot t^{3/2}dt\right]$$
$$=\frac{1}{3}\left[\int t^{1/2}dt-t^{3/2}(\ln t)\right]$$
$$=\frac{1}{9}t^{3/2}\,[2-3\ln t]+c$$
$$=\frac{1}{9}\frac{(x^2+1)\sqrt{x^2+1}}{x^3}\left[2-3\ln\left(\frac{x^2+1}{x^2}\right)\right]+c$$

Hence, the correct answer is option (D).

14. $\displaystyle\int\frac{\sqrt{1+x^2}}{x^4}dx$ is equal to

(A) $\dfrac{1}{3}\left(1+\dfrac{1}{x^2}\right)^{3/2}+c$ **(B)** $\dfrac{1}{3}\left(1-\dfrac{1}{x^2}\right)^{3/2}+c$

(C) $-\dfrac{1}{3}\left(1+\dfrac{1}{x^2}\right)^{3/2}+c$ **(D)** $-\dfrac{1}{3}\left(1-\dfrac{1}{x^2}\right)^{3/2}+c$

Solution:

$$I=\int\frac{\sqrt{1+x^2}}{x^4}dx=\int\frac{\sqrt{1+\frac{1}{x^2}}}{x^3}dx$$

Let $1+\dfrac{1}{x^2}=t$. Then

$$-\frac{2}{x^3}dx=dt$$
$$\Rightarrow I=-\frac{1}{2}\int\sqrt{t}\,dt$$
$$=-\frac{1}{2}\cdot\frac{2}{3}t^{3/2}+c$$
$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}+c$$

Hence, the correct answer is option (C).

15. $\displaystyle\int\frac{dx}{5+4\cos x}$ is equal to

(A) $\dfrac{2}{3}\tan^{-1}\left(\dfrac{1}{3}\tan\dfrac{x}{2}\right)+c$ **(B)** $-\dfrac{2}{3}\tan^{-1}\left(\dfrac{1}{3}\tan\dfrac{x}{2}\right)+c$

(C) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{1}{3}\tan\dfrac{x}{2}\right)+c$ **(D)** None of these

Solution:

$$I=\int\frac{dx}{5+4\cos x}=\int\frac{dx}{5+4\left[\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right]}$$

Let $t=\tan\dfrac{x}{2}$. Then

$$dt=\frac{1}{2}\sec^2\frac{x}{2}dx$$
$$\Rightarrow dx=\frac{2dt}{1+\tan^2\frac{x}{2}}$$
$$dx=\frac{2dt}{1+t^2}$$
$$\Rightarrow I=\int\frac{\frac{2dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}=2\int\frac{dt}{t^2+9}$$
$$=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+c$$

Hence, the correct answer is option (A).

16. $\displaystyle\int\frac{dx}{(2x+3)\sqrt{4x+5}}$ is equal to

(A) $\tan^{-1}\sqrt{4x-5}+c$ **(B)** $\tan^{-1}\sqrt{4x+5}+c$

(C) $\tan^{-1}\sqrt{5x+4}+c$ **(D)** $\tan^{-1}\sqrt{5x-4}+c$

Solution:

$$I=\int\frac{dx}{(2x+3)\sqrt{4x+5}}$$

Put $4x+5=t$. Then

$$x=\frac{t-5}{4}\Rightarrow dx=\frac{dt}{4}$$
$$\Rightarrow I=\frac{1}{4}\int\frac{dt}{\left(\frac{2t-10}{4}+3\right)\sqrt{t}}=\frac{1}{2}\int\frac{dt}{(t+1)\sqrt{t}}$$

Let $\sqrt{t}=u$. Then $\dfrac{1}{2\sqrt{t}}dt=du$.

Therefore,

$$I=\int\frac{du}{u^2+1}=\tan^{-1}\sqrt{t}+c$$
$$\Rightarrow I=\tan^{-1}\sqrt{4x+5}+c.$$

Hence, the correct answer is option (B).

17. $\int e^{ax}\cos bx\,dx$ is equal to

(A) Real part of $\int e^{(a+bi)x}dx$

(B) Imaginary part of $\int e^{(a+bi)x}dx$

(C) Neither real nor imaginary part of $\int e^{(a+bi)x}dx$

(D) None of these

Solution:

$$I=\int e^{ax}\cos bx\,dx$$
$$=\text{real part of}\int e^{ax}\,e^{ibx}dx$$
$$=\text{real part of}\int e^{ax+ibx}dx$$

Hence, the correct answer is option (A).

18. $\int e^{2x^2+\ln x}\, dx$ is equal to

(A) $\frac{e^{2x^2}}{4}+c$ **(B)** $\frac{e^{2x^2}}{2}+c$

(C) $\frac{e^{2x^2}}{4}+\frac{x^2}{2}+c$ **(D)** $\frac{xe^{2x^2}}{4}+c$

Solution:

$$I=\int e^{2x^2+\ln x}\, dx \Rightarrow xe^{2x^2}\, dx$$

Let $x^2 = t$. Then

$$2x\, dx = dt$$

$$\Rightarrow I=\frac{1}{2}\int e^{2t}\, dt=\frac{e^{2t}}{4}+c=\frac{e^{2x^2}}{4}+c$$

Hence, the correct answer is option (A).

19. $\int \frac{2^{x+1}-5^{x-1}}{10^x} dx$ is equal to

(A) $\frac{2}{\ln 5}5^x+\frac{1}{5\ln 2}2^x+c$ **(B)** $\frac{-2}{\ln 5}5^{-x}+\frac{1}{5\ln 2}2^{-x}+c$

(C) $\frac{1}{2\ln 5}5^{-x}\frac{1}{5\ln 2}2^{-x}+c$ **(D)** None of these

Solution:

$$I=\int \frac{2^{x+1}-5^{x-1}}{10^x}dx = \int\left[2\left(\frac{1}{5}\right)^x-\frac{1}{5}\left(\frac{1}{2}\right)^x\right]dx$$

$$=\frac{2\left(\frac{1}{5}\right)^x}{\ln\frac{1}{5}}-\frac{\frac{1}{5}\left(\frac{1}{2}\right)^x}{\ln\frac{1}{2}}+c$$

$$=\frac{-2}{\ln 5}5^{-x}+\frac{1}{5\ln 2}2^{-x}+c$$

Hence, the correct answer is option (B).

20. If $I_n=\int (\ln x)^n dx$, then I_n+nI_{n-1} is equal to

(A) $x(\ln x)^{n-1}$ **(B)** $x(\ln x)^n$

(C) $nx(\ln x)^n$ **(D)** None of these

Solution:

$$I_n=\int (\ln x)^n dx$$

$$I_n=x(\ln x)^n-n\int(\ln x)^{n-1}dx$$

$$I_n=x(\ln x)^n-n\,I_{n-1}$$

$$I_n+nI_{n-1}=x(\ln x)^n$$

Hence, the correct answer is option (B).

21. If $\int\frac{dx}{x-x^3}=A\ln\left|\frac{x^2}{1-x^2}\right|+c$, then A is equal to

(A) 2 **(B)** 1/2

(C) 2/3 **(D)** 1/4

Solution:

$$I=\int\frac{dx}{x-x^3}=\int\frac{dx}{x^3\left(\frac{1}{x^2}-1\right)}$$

Let $\frac{1}{x^2}=t$. Then $-\frac{2}{x^3}\,dx=dt$. Therefore,

$$I=-\frac{1}{2}\int\frac{dt}{(t-1)}=\frac{1}{2}\ln|t-1|+c=-\frac{1}{2}\ln\left|\frac{1-x^2}{x^2}\right|+c$$

$$=\frac{1}{2}\ln\left|\frac{x^2}{1-x^2}\right|+c\Rightarrow A=\frac{1}{2}$$

Hence, the correct answer is option (B).

22. $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ is equal to

(A) $xe^{\tan^{-1}x}+c$ **(B)** $x^2e^{\tan^{-1}x}+c$

(C) $\frac{1}{xe^{\tan^{-1}x}}+c$ **(D)** $\frac{1}{x^2}e^{\tan^{-1}x}+c$

Solution:

$$I=\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$$

Let $p=\tan^{-1}x$. Then

$$x=\tan p\Rightarrow dx=\sec^2 p\, dp$$

$$\Rightarrow I=\int e^p(\sec^2 p+\tan p)\, dp$$

$$=e^p\tan p = x\,e^{\tan^{-1}x}+c$$

Hence, the correct answer is option (A).

23. $\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx$ is equal to

(A) $\frac{x^2}{2}+\log|x|+c$ **(B)** $\frac{x^2}{2}+\log|x|+2x+c$

(C) $\frac{1}{3}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3+c$ **(D)** $\frac{x^2}{2}+\log x-2x+c$

Solution:

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2dx=\int\left(x+\frac{1}{x}-2\right)dx=\frac{x^2}{2}+\log x-2x+c$$

Hence, the correct answer is option (D).

24. $\int\frac{\cos 2x}{\cos x+\sin x}dx$ is equal to

(A) $\sin x-\cos x+c$ **(B)** $-\sin x+\cos x+c$

(C) $\sin x+\cos x+c$ **(D)** None of these

Solution:

$$\int\frac{\cos 2x}{\cos x+\sin x}dx=\int\frac{(\cos x+\sin x)(\cos x-\sin x)}{\cos x+\sin x}dx$$

$$=\sin x+\cos x+c$$

Hence, the correct answer is option (C).

25. If $\int\frac{\sqrt{\cot x}}{\sin x\cos x}dx=A\sqrt{\cot x}+B$, then A is equal to

(A) 1 **(B)** 2

(C) -1 **(D)** -2

Solution:

$$\int \frac{\sqrt{\cot x}}{\sin x \cos x} dx = \int \frac{\sqrt{\cot x}}{\cot x} \cdot \text{cosec}^2 x\, dx$$

$$= \int \frac{\text{cosec}^2 x}{\sqrt{\cot x}} dx = -2\sqrt{\cot x} + B = A\sqrt{\cot x} + B$$

$$\Rightarrow A = -2$$

Hence, the correct answer is option (D).

26. The value of $\int \frac{\ln(x/e)}{(\ln x)^2} dx$ is

(A) $\frac{x+1}{(\ln x)^2} + c$ **(B)** $\frac{x-1}{(\ln x)^2} + c$

(C) $\frac{x}{\ln x} + c$ **(D)** $\frac{\ln x}{x} + c$

Solution:

$$I = \int \frac{\ln(x/e)}{(\ln x)^2} dx = \int \frac{\ln(x) - 1}{(\ln x)^2} dx$$

Put $\ln x = t$. Then

$$x = e^t \Rightarrow dx = e^t dt$$

$$I = \int e^t \left(\frac{t-1}{t^2}\right) dt = \int e^t \left(\frac{1}{t} - \frac{1}{t^2}\right) dt = \frac{e^t}{t} + c = \frac{x}{\ln x} + c$$

Hence, the correct answer is option (C).

27. $I = \int \frac{(10x^9 + 10^x \log_e 10)}{(x^{10} + 10^x)} dx$ is equal to

(A) $10^x + x^{10} + c$ **(B)** $10^x - x^{10} + c$

(C) $10^x + x^{10} + c$ **(D)** $\ln (10^x + x^{10}) + c$

Solution: If $p = x^{10} + 10^x$, then

$$(10x^9 + 10^x \log_e 10) dx = dp$$

$$\Rightarrow I = \int dp = p + c$$

$$\Rightarrow I = \ln(x^{10} + 10^x) + c$$

Hence, the correct answer is option (D).

28. The value of the integral $\int e^{\sin^2 x}(\cos x + \cos^3 x) \sin x\, dx$ is

(A) $\frac{1}{2} e^{\sin^2 x}(3 - \sin^2 x) + c$

(B) $e^{\sin^2 x}\left(1 + \frac{1}{2}\cos^2 x\right) + c$

(C) $e^{\sin^2 x}(3\cos^2 x + 2\sin^2 x) + c$

(D) $e^{\sin^2 x}(2\cos^2 x + 3\sin^2 x) + c$

Solution:

$$\text{I} = \int e^{\sin^2 x}(\cos x + \cos^3 x) \sin x dx = \int e^{\sin^2 x}(2 - \sin^2 x) \cos x \sin x dx$$

Put $t = \sin^2 x$. Then

$$dt = 2\sin x \cos x\, dx$$

The integral reduces to

$$\Rightarrow I = \frac{1}{2}\int e^t (2 - t) dt = \frac{3}{2} e^t - \frac{te^t}{2} + c$$

$$= \frac{1}{2} e^{\sin^2 x}(3 - \sin^2 x) + c \qquad \text{(Option A)}$$

$$= e^{\sin^2 x}\left(1 + \frac{1}{2}\cos^2 x\right) + c \qquad \text{(Option B)}$$

Hence, the correct answers are options (A) and (B).

29. $\int \frac{x^3 - 3x + 7}{x^2 + 4} dx$ is equal to

(A) $\frac{x^2}{2} + \frac{7}{2}\ln(x^2 + 4) + c$

(B) $\frac{x^2}{2} + \frac{7}{2}\tan^{-1}\frac{x}{2} - \frac{7}{2}\ln(x^2 + 4) + c$

(C) $-\frac{x^2}{2} + \frac{7}{2}\tan^{-1}\frac{x^2}{2} + \frac{7}{2}\ln\left(x^2 + 4\right) + c$

(D) $\frac{x}{2} + \frac{7}{2}\tan^{-1}\frac{x}{2} + c$

Solution:

$$\frac{x^3 - 3x + 7}{x^2 + 4} = x - \frac{7(x-1)}{x^2 + 4}$$

$$\int \frac{x^3 - 3x + 7}{x^2 + 4} dx = \int x dx - 7\int \frac{(x-1)}{x^2 + 4} dx$$

$$\int \frac{x^3 - 3x + 7}{x^2 + 4} dx = \frac{x^2}{2} - 7\int \frac{(x-1)}{x^2 + 4} dx$$

$$= \frac{x^2}{2} + \frac{7}{2}\tan^{-1}\frac{x}{2} - \frac{7}{2}\ln(x^2 + 4) + c$$

Hence, the correct answer is option (B).

30. $\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e} dx$ is equal to

(A) $\frac{1}{e}\ln (x^e + e^x) + c$ **(B)** $\frac{1}{e}\ln (x + e) + c$

(C) $\frac{1}{e}\ln (x^{-e} + e^{-x}) + c$ **(D)** None of these

Solution:

$$I = \int \frac{e^{x-1} + x^{e-1}}{e^x + x^e} dx$$

Let $e^x + x^e = t$. Then

$$(e^x + e \cdot x^{e-1})\, dx = dt \Rightarrow e(e^{x-1} + x^{e-1})\, dx = dt$$

Therefore,

$$I = \frac{1}{e}\int \frac{dt}{t} = \frac{1}{e}\ln(e^x + x^e) + c$$

Hence, the correct answer is option (A).

Previous Years' Solved JEE Main/AIEEE Questions

1. $\int \frac{dx}{\cos x + \sqrt{3}\sin x}$ equals to

(A) $\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+c$ **(B)** $\frac{1}{2}\log\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)+c$

(C) $\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+c$ **(D)** $\log\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)+c$

[AIEEE 2007]

Solution:

$$\int\frac{dx}{\cos x+\sqrt{3}\sin x}=\frac{1}{2}\int\frac{dx}{\frac{1}{\sqrt{4}}\cos x+\frac{\sqrt{3}}{\sqrt{4}}\sin x}$$
$$=\frac{1}{2}\int\frac{dx}{\cos\left(x-\frac{\pi}{3}\right)}$$
$$=\frac{1}{2}\int\sec\left(x-\frac{\pi}{3}\right)dx$$
$$=\frac{1}{2}\log\tan\left(\frac{x}{2}-\frac{\pi}{6}+\frac{\pi}{4}\right)+c$$
$$=\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+c$$

Hence, the correct answer is option (A).

2. The value of $\sqrt{2}\int\frac{\sin x\,dx}{\sin\left(x-\frac{\pi}{4}\right)}$ is

(A) $x+\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$ **(B)** $x-\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$

(C) $x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$ **(D)** $x-\log\left|\cos\left(x-\frac{\pi}{4}\right)\right|+c$

[AIEEE 2008]

Solution:

$$\sqrt{2}\int\frac{\sin x\,dx}{\sin\left(x-\frac{\pi}{4}\right)}=\sqrt{2}\int\frac{\sin\left(x-\frac{\pi}{4}+\frac{\pi}{4}\right)dx}{\sin\left(x-\frac{\pi}{4}\right)}$$
$$=\sqrt{2}\int\left(\cos\frac{\pi}{4}+\cot\left(x-\frac{\pi}{4}\right)\sin\frac{\pi}{4}\right)dx=\int dx+\int\cot\left(x-\frac{\pi}{4}\right)dx$$
$$=x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+c$$

Hence, the correct answer is option (C).

3. If the integral $\int\frac{5\tan x}{\tan x-2}dx=x+a\ln|\sin x-2\cos x|+k$, then a is equal to

(A) -1 **(B)** -2
(C) 1 **(D)** 2 **[AIEEE 2012]**

Solution:

$$\int\frac{5\tan x}{\tan x-2}\,dx=\int\frac{5\sin x}{\sin x-2\cos x}\,dx$$
$$\Rightarrow\int\left[\frac{2(\cos x+2\sin x)+(\sin x-2\cos x)}{\sin x-2\cos x}\right]dx$$
$$=2\int\left(\frac{\cos x+2\sin x}{\sin x-2\cos x}\right)dx+\int dx=2\log|\sin x-2\cos x|+x+k$$

Therefore, $a=2$.

Hence, the correct answer is option (D).

4. If $\int f(x)\,dx=\psi(x)$, then $\int x^5f(x^3)\,dx$ is equal to

(A) $\frac{1}{3}x^3\psi(x^3)-3\int x^3\psi(x^3)dx+C$

(B) $\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)dx+C$

(C) $\frac{1}{3}\left[x^3\psi(x^3)-\int x^3\psi(x^3)dx\right]+C$

(D) $\frac{1}{3}\left[x^3\psi(x^3)-\int x^2\psi(x^3)dx\right]+C$ **[JEE MAIN 2013]**

Solution: We have

$$\int f(x)\,dx=\psi(x)$$

Let $x^3=t$ and $x^2\,dx=dt/3$. Then

$$\int x^5f(x^3)\,dx=\frac{1}{3}\int tf(t)dt=\frac{1}{3}\left[t\int f(t)dt-\int\left[1\cdot\int f(t)dt\right]dt\right]$$
$$=\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)\,dx+C$$

Hence, the correct answer is option (B).

5. The integral $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx$ is equal to

(A) $(x+1)e^{x+\frac{1}{x}}+c$ **(B)** $-xe^{x+\frac{1}{x}}+c$

(C) $(x-1)e^{x+\frac{1}{x}}+c$ **(D)** $xe^{x+\frac{1}{x}}+c$

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx=\int\left\{e^{x+\frac{1}{x}}dx+x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\right\}dx=xe^{x+\frac{1}{x}}+c$$

$\left[\text{Since}, \int\{xf'(x)+f(x)\}dx=xf(x)+c\right]$

Hence, the correct answer is option (D).

6. $\int\frac{\sin^8x-\cos^8x}{(1-2\sin^2x\cos^2x)}dx$ is equal to

(A) $\frac{1}{2}\sin 2x+c$ **(B)** $-\frac{1}{2}\sin 2x+c$

(C) $-\frac{1}{2}\sin x+c$ **(D)** $-\sin^2x+c$

[JEE MAIN 2014 (ONLINE SET-1)]

Solution:

$$\int\frac{\sin^8x-\cos^8x}{1-2\sin^2\cos^2x}dx=\int\frac{(\sin^4x+\cos^4x)(\sin^4x-\cos^4x)dx}{(1-2\sin^2x\cos^2x)}$$
$$=\int\frac{\{(\sin^2x+\cos^2x)^2-2\sin^2x\cos^2x\}\{\sin^2x-\cos^2x\}dx}{(1-2\sin^2x\cos^2x)}$$
$$\Rightarrow\int\frac{\cancel{(1-2\sin^2x\cos^2x)}(-\cos 2x)dx}{\cancel{(1-2\sin^2x\cos^2x)}}\Rightarrow-\frac{\sin 2x}{2}+c$$

Hence, the correct answer is option (B).

7. The integral $\int x\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx\,(x>0)$ is equal to

(A) $-x+(1+x^2)\tan^{-1}x+c$ **(B)** $x-(1+x^2)\cot^{-1}x+c$
(C) $-x+(1+x^2)\cot^{-1}x+c$ **(D)** $x-(1+x^2)\tan^{-1}x+c$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$I=\int x\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx,\,x>0$$

Put $x=\tan\theta$. Then $dx=\sec^2\theta\,d\theta$.

$$I=\int\tan\theta\cos^{-1}(\cos 2\theta)\sec^2\theta\,d\theta=2\int\theta\tan\theta\sec^2\theta\,d\theta$$

Again put $\tan\theta=t$. Then $\sec^2\theta\,d\theta=dt$.
Therefore

$$\mathrm{I}=2\int(\tan^{-1}t)t\,dt$$
$$=2\left[(\tan^{-1}t)\frac{t^2}{2}-\int\frac{1}{1+t^2}\frac{t^2}{2}dt\right]$$
$$=t^2\tan^{-1}t-\left[\int\frac{(t^2+1)}{1+t^2}dt-\int\frac{1dt}{1+t^2}\right]$$

$=t^2\tan^{-1}t-[t-\tan^{-1}t]=t^2\tan^{-1}t-t+\tan^{-1}t=(t^2+1)\tan^{-1}t-t=\{\tan^2\theta+1\}\tan^{-1}(\tan\theta)-\tan\theta=\theta\{\tan^2\theta+1\}-\tan\theta$

Therefore, $I=(1+x^2)\tan^{-1}x-x+c$.

Hence, the correct answer is option (A).

8. The integral $\int\frac{\sin^2x\cos^2x}{(\sin^3x+\cos^3x)^2}dx$ equal to

(A) $\frac{1}{(1+\cot^3x)}+c$ **(B)** $-\frac{1}{3(1+\tan^3x)}+c$
(C) $\frac{\sin^3x}{(1+\cos^3x)}+c$ **(D)** $-\frac{\cos^3x}{3(1+\sin^3x)}+c$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$I=\int\frac{\sin^2x\cos^2x}{(\sin^3x+\cos^3x)^2}dx=\int\frac{\sin^2x\cos^2x}{\cos^6x(1+\tan^3x)^2}dx$$
$$=\int\frac{\sin^2x}{\cos^4x(1+\tan^3x)^2}dx=\int\frac{\tan^2x\sec^2x}{(1+\tan^3x)^2}dx$$

Now put $\tan x=t$. Then $\sec^2x\,dx=dt$.

Therefore,

$$I=-\int\frac{t^2dt}{(1+t^3)^2}$$

Again substituting $1+t^3=\mu$, we get

$$3t^2\,dt=d\mu$$

$$=\frac{1}{3}\int\frac{d\mu}{\mu^2}=-\frac{1}{3}\times\frac{1}{\mu}=-\frac{1}{3}\times\frac{1}{1+t^3}=-\frac{1}{3(1+\tan^3x)}+c$$

Hence, the correct answer is option (B).

9. If m is a non-zero number and $\int\frac{x^{5m-1}+2x^{4m-1}}{(x^{2m}+x^m+1)^3}dx=f(x)+c$, then $f(x)$ is

(A) $\frac{x^{5m}}{2m(x^{2m}+x^m+1)^2}$ **(B)** $\frac{x^{4m}}{2m(x^{2m}+x^m+1)^2}$
(C) $\frac{2m(x^{5m}+x^{4m})}{(x^{2m}+x^m+1)^2}$ **(D)** $\frac{(x^{5m}-x^{4m})}{2m(x^{2m}+x^m+1)^2}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$I=\int\frac{x^{5m-1}+2x^{4m-1}}{(x^{2m}+x^m+1)^3}dx\Rightarrow\int\frac{x^{5m-1}+2x^{4m-1}}{\{x^{2m}(1+x^{-m}+x^{-2m})\}^3}dx$$
$$\Rightarrow\int\frac{x^{5m-1}+2x^{4m-1}}{x^{6m}(1+x^{-m}+x^{-2m})^3}dx\Rightarrow\int\frac{x^{-m-1}+2x^{-2m-1}}{(1+x^{-m}+x^{-2m})^3}dx$$

Now put $1+x^{-m}+x^{-2m}=t$. Then

$$(-mx^{-m-1}-2mx^{-2m-1})dx=dt\Rightarrow-m(x^{-m-1}+2x^{-2m-1})dx=dt$$

Therefore,

$$I=-\frac{1}{m}\int\frac{dt}{t^3}=-\frac{1}{m}\frac{t^{-3+1}}{-3+1}+c=-\frac{1}{m}\frac{t^{-2}}{-2}+c$$
$$=\frac{1}{2m}\frac{1}{t^2}+c=\frac{1}{2m(1+x^{-m}+x^{-2m})^2}+c$$
$$=\frac{1}{2m\left(1+\frac{1}{x^m}+\frac{1}{x^{2m}}\right)^2}+c=\frac{x^{4m}}{2m(x^{2m}+x^m+1)^2}+c$$

Hence, the correct answer is option (B).

10. The integral $\int\frac{dx}{x^2(x^4+1)^{3/4}}$ equals

(A) $(x^4+1)^{\frac{1}{4}}+c$ **(B)** $-(x^4+1)^{\frac{1}{4}}+c$
(C) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$ **(D)** $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$

[JEE MAIN 2015 (OFFLINE)]

Solution:

$$I=\int\frac{dx}{x^2(x^4+1)^{3/4}}=\int\frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

Put $x^{-4}=t$. Then

$$\frac{-4}{x^5}dx=dt$$

$$\Rightarrow I=\int\frac{-dt}{4(1+t)^{3/4}}=-\frac{1}{4}\frac{(1+t)^{1/4}}{1/4}+c=-(1+x^{-4})^{1/4}+c$$

Hence, the correct answer is option (C).

11. The integral $\int\frac{dx}{(x+1)^{3/4}(x-2)^{5/4}}$ is equal to

(A) $4\left(\frac{x+1}{x-2}\right)^{1/4}+C$ **(B)** $4\left(\frac{x-2}{x+1}\right)^{1/4}+C$

(C) $-\frac{4}{3}\left(\frac{x+1}{x-2}\right)^{1/4}+C$ **(D)** $-\frac{4}{3}\left(\frac{x-2}{x+1}\right)^{1/4}+C$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:

$$I=\int\frac{dx}{(x+1)^{3/4}(x-2)^{5/4}}=\int\frac{dx}{(x+1)^{3/4}(x-2)^2(x-2)^{5/4-2}}$$

$$=\int\frac{dx}{(x+1)^{3/4}(x-2)^2(x-2)^{-3/4}}=\int\frac{(x+1)^{-3/4}}{(x-2)^{-3/4}}\cdot\frac{1}{(x-2)^2}dx$$

$$=\frac{1}{-3}\int\left[\frac{x+1}{x-2}\right]^{-3/4}\cdot\left(\frac{-3}{(x-2)^2}\right)dx$$

Put $t=\frac{x+1}{x-2}\Rightarrow dt=\frac{-3}{(x-2)^2}dx$

$$\Rightarrow I=\frac{-1}{3}\int(t)^{-3/4}\,dt=\frac{-4}{3}(t)^{1/4}+C$$

$$\Rightarrow-\frac{4}{3}\left[\frac{x+1}{x-2}\right]^{1/4}+C$$

Hence, the correct answer is option (C).

12. If $\int\frac{\log(t+\sqrt{1+t^2})}{(\sqrt{1+t^2})}dt=\frac{1}{2}(g(t))^2+C$, where C is a constant, then $g(2)$ is equal to

(A) $2\log(2+\sqrt{5})$ **(B)** $\log(2+\sqrt{5})$

(C) $\frac{1}{\sqrt{5}}\log(2+\sqrt{5})$ **(D)** $\frac{1}{2}\log(2+\sqrt{5})$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution:

$$I=\int\frac{\log\left(t+\sqrt{1+t^2}\right)}{\sqrt{1+t^2}}dt$$

Since, $\frac{d}{dt}\left(\log\left(t+\sqrt{1+t^2}\right)\right)=\frac{1}{\sqrt{1+t^2}}$, we get

$$I=\frac{1}{2}\left[\log(t+\sqrt{1+t^2})\right]^2+C$$

$$\Rightarrow g(t)=\log(t+\sqrt{1+t^2})\Rightarrow g(2)=\log(2+\sqrt{5})$$

Hence, the correct answer is option (B).

13. The integral $\int\frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to

(A) $\frac{-x^{10}}{2(x^5+x^3+1)^2}+C$ **(B)** $\frac{-x^5}{(x^5+x^3+1)^2}+C$

(C) $\frac{x^{10}}{2(x^5+x^3+1)^2}+C$ **(D)** $\frac{x^5}{2(x^5+x^3+1)^2}+C$

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$\int\frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx\Rightarrow\int\frac{2x^{12}+5x^9}{x^{15}\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}dx$$

$$\Rightarrow\int\frac{\left(\frac{2}{x^3}+\frac{5}{x^6}\right)}{\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^3}dx$$

Put $1+x^{-2}+x^{-5}=t$. Then

$$\left(-\frac{2}{x^3}-\frac{5}{x^6}\right)dx=dt$$

$$\Rightarrow-\int\frac{1}{t^3}dt=\frac{1}{2t^2}+C$$

$$\Rightarrow\frac{1}{2\left(1+\frac{1}{x^2}+\frac{1}{x^5}\right)^2}+C=\frac{x^{10}}{2(x^5+x^3+1)}+C$$

Hence, the correct answer is option (C).

14. If $\int\frac{dx}{\cos^3x\sqrt{2\sin 2x}}=(\tan x)^A+C(\tan x)^B+k$, where k is a constant of integration, then $A+B+C$ equals

(A) $\frac{16}{5}$ **(B)** $\frac{27}{10}$

(C) $\frac{7}{10}$ **(D)** $\frac{21}{15}$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We have

$$I=\int\frac{dx}{\cos^3x\sqrt{4\tan x\cos^2x}}$$

$$I=\frac{1}{2}\int\frac{dx}{\cos^4x\sqrt{\tan x}}=\frac{1}{2}\int\frac{\sec^4x\,dx}{\sqrt{\tan x}}$$

Put $\tan x=t$. Then $\sec^2x\,dx=dt$. Therefore,

$$I=\frac{1}{2}\int\frac{(1+t^2)dt}{t^{1/2}}=\frac{1}{2}\int(t^{-1/2}+t^{3/2})dt$$

$$=\frac{1}{2}\left(2t^{1/2}+\frac{2}{5}t^{5/2}\right)+k$$

$$=(\tan x)^{1/2}+\frac{1}{5}(\tan x)^{5/2}+k$$

Comparing this with the given equation, we get $A=1/2$, $B=5/2$ and $C=1/5$. Therefore,

$$A+B+C=\frac{1}{2}+\frac{5}{2}+\frac{1}{5}=3+\frac{1}{5}=\frac{16}{5}$$

Hence, the correct answer is option (A).

15. The integral $\int\frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$ is equal to (where C is a constant of integration)

(A) $-2\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}+C$ **(B)** $-\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}+C$

(C) $-2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}+C$ **(D)** $2\sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}+C$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have

$$I=\int\frac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}=\int\frac{dx}{(1+\sqrt{x})^2\sqrt{x\frac{(1-\sqrt{x})}{(1+\sqrt{x})}}}$$

Put $y=\frac{1-\sqrt{x}}{1+\sqrt{x}}$. Then

$$dy=-\frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$$

Therefore,

$$I=-\int\frac{dy}{\sqrt{y}}=-\frac{y^{(-1/2)+1}}{(-1/2)+1}+C=-2\sqrt{y}+C$$

$$I=-2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}+C$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Let $I=\int\frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J=\int\frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. Then, for an arbitrary constant C, the value of $J-I$ equals

(A) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+C$

(B) $\frac{1}{2}\log\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+C$

(C) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+C$

(D) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+C$

[IIT-JEE 2008]

Solution: We have

$$I=\int\frac{e^x dx}{e^{4x}+e^{2x}+1}$$

$$J=\int\frac{e^{-x}dx}{e^{-4x}+e^{-2x}+1}=\int\frac{e^{3x}dx}{e^{4x}+e^{2x}+1}$$

Now,

$$J-I=\int\frac{(e^{2x}-1)e^x dx}{e^{4x}+e^{2x}+1}$$

Put $e^x=t$. Then

$$e^x dx=dt$$

$$J-I=\int\frac{(t^2-1)dt}{t^4+t^2+1}$$

$$=\int\frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)^2-1}$$

Put $t+\frac{1}{t}=\mu\Rightarrow\left(1-\frac{1}{t^2}\right)dt=d\mu$

$$\Rightarrow I-I=\int\frac{d\mu}{\mu^2-1}$$

$$=\frac{1}{2}\log\left|\frac{\mu-1}{\mu+1}\right|$$

$$=\frac{1}{2}\log\left(\frac{e^x+e^{-x}-1}{e^x+e^{-x}+1}\right)+c$$

$$=\frac{1}{2}\log\left(\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right)+c$$

Hence, the correct answer is option (C).

2. The integral $\int\frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$ equals (for some arbitrary constant K)

(A) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(B) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}-\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(C) $-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

(D) $\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+K$

[IIT-JEE 2012]

Solution:

$$I=\int\frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$$

Let $\sec x+\tan x=t$. Then

$$\sec x-\tan x=1/t$$

Now,

$$(\sec x\tan x+\sec^2 x)dx=dt\Rightarrow\sec x(\sec x+\tan x)dx=dt$$

$$\sec x\,dx=\frac{dt}{t},\ \frac{1}{2}\left(t+\frac{1}{t}\right)=\sec x$$

$$I=\frac{1}{2}\int\frac{\left(t+\frac{1}{t}\right)}{t^{9/2}}\frac{dt}{t}=\frac{1}{2}\int(t^{-9/2}+t^{-13/2})dt$$

$$=\frac{1}{2}\left[\frac{t^{-\frac{9}{2}+1}}{-\frac{9}{2}+1}+\frac{t^{-\frac{13}{2}+1}}{-\frac{13}{2}+1}\right]=\frac{1}{2}\left[\frac{t^{-7/2}}{-\frac{7}{2}}+\frac{t^{-11/2}}{-\frac{11}{2}}\right]$$

$$=-\frac{1}{7}t^{-7/2}-\frac{1}{11}t^{-11/2}=-\frac{1}{7}\frac{1}{t^{7/2}}-\frac{1}{11}\frac{1}{t^{11/2}}$$

$$=-\frac{1}{t^{11/2}}\left(\frac{1}{11}+\frac{t^2}{7}\right)$$

$$=-\frac{1}{(\sec x+\tan x)^{11/2}}\left\{\frac{1}{11}+\frac{1}{7}(\sec x+\tan x)^2\right\}+k$$

Hence, the correct answer is option (C).

Practice Exercise 1

1. $\int \frac{dx}{\sqrt{1+x}+\sqrt{x}}$ is equal to

(A) $\frac{2}{3}(1+x)^{2/3}-\frac{2}{3}x^{2/3}+c$ (B) $\frac{3}{2}(1+x)^{2/3}+\frac{3}{2}x^{3/2}+c$

(C) $\frac{3}{2}(1+x)^{3/2}+\frac{3}{2}x^{3/2}+c$ (D) $\frac{2}{3}(1+x)^{3/2}-\frac{2}{3}x^{3/2}+c$

2. The value of $\int \frac{dx}{(e^x+1)(2e^x+3)}$ is equal to

(A) $x+\ln(e^x+1)-\frac{2}{3}\ln(2e^x+1)+c$

(B) $\frac{1}{3}x-\ln\left(e^x+1\right)+\frac{2}{3}\ln\left(2e^x+3\right)+c$

(C) $x-\frac{2}{3}\ln(e^x+1)+\ln(2e^x+3)+c$

(D) None of these

3. The value of $\int \frac{\cos^3 x\,dx}{\sin^2 x+\sin x}$ is equal to

(A) $\log \sin x - \sin x + c$ (B) $\log|\sin x| - \sin x + c$

(C) $\log|\sin x| + c$ (D) None of these

4. The value of $\int \frac{dx}{x(x^n+1)}$ is equal to

(A) $\log\left|\frac{x^n}{1+x^n}\right|+c$ (B) $\frac{1}{n}\log\left|\frac{x^n+1}{x^n}\right|+c$

(C) $\log\left|\frac{x^n+1}{x^n}\right|+c$ (D) $\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+c$

5. The value of $\int \frac{1+x+x^2}{1+x^2}e^{\tan^{-1}x}dx$ is equal to

(A) $x^2e^{\tan^{-1}x}$ (B) $e^{\tan^{-1}x}+c$

(C) $xe^{\tan^{-1}x}+c$ (D) None of these

6. $\int \frac{\tan x}{\sqrt{\cos x}}dx$ is equal to

(A) $\frac{2}{\sqrt{\sin x}}+c$ (B) $\frac{2}{\sqrt{\cos x}}+c$

(C) $\frac{2}{\sqrt{\tan x}}+c$ (D) $\frac{2}{(\sin x)^{3/2}}+c$

7. $\int \frac{dx}{x\ln x\ln(\ln x)}$ is equal to

(A) $\ln|(\ln(\ln x))|+c$ (B) $|\ln x|+c$

(C) $\ln\left|\left(\ln\left(\frac{1}{x}\right)\right)\right|+c$ (D) $\ln|\ln x|+c$

8. Value of the integral $\int e^x\left[\frac{1+\sqrt{1-x^2}\sin^{-1}x}{\sqrt{1-x^2}}\right]dx$ is equal to

(A) $e^x\sin^{-1}x+c$ (B) $\frac{e^x}{\sqrt{1-x^2}}+c$

(C) $e^x\sqrt{1-x^2}+c$ (D) $\sqrt{1-x^2}\sin^{-1}x+c$

9. The value of $\int \frac{x^3}{1+x^8}dx$ is equal to

(A) $\frac{1}{4}\tan^{-1}x^4+c$ (B) $\frac{1}{2}\tan^{-1}x^4+c$

(C) $\frac{1}{4}\cot^{-1}x^2+c$ (D) None of these

10. The value of $\int xe^x dx$ is equal to

(A) xe^x+e^x+c (B) xe^x-e^x+c

(C) $-xe^x+e^x+c$ (D) None of these

11. $\int \frac{dx}{\sqrt{2x-x^2}}$ is equal to

(A) $\sin^{-1}(1-x)+c$ (B) $-\cos^{-1}(1-x)+P$

(C) $\sin^{-1}(x-1)+c$ (D) $\cos^{-1}(x-1)+P$

12. If $I=\int \frac{1}{2p}\sqrt{\frac{p-1}{p+1}}dp = f(p)+c$, then $f(p)$ is equal to

(A) $\frac{1}{2}\ln(p-\sqrt{p^2-1})$ (B) $\left(\frac{1}{2}\cos^{-1}p+\frac{1}{2}\sec^{-1}p\right)$

(C) $\ln\sqrt{p+\sqrt{p^2-1}}-\frac{1}{2}\sec^{-1}p$ (D) None of these

13. $\int \frac{\sin\theta+\cos\theta}{\sqrt{(\sin 2\theta)}}d\theta$ is equal to

(A) $\sin^{-1}(\sin\theta+\cos\theta)$ (B) $\sin^{-1}\theta(\sin\theta-\cos\theta)$

(C) $\sin^{-1}(\cos\theta-\sin\theta)$ (D) None of these

14. If $\int x^6\sin(5x^7)dx=\frac{k}{5}\cos(5x^7)$, $x\neq 0$, then

(A) $k=7$ (B) $k=-7$

(C) $k=\frac{1}{7}$ (D) $k=-\frac{1}{7}$

15. $\int e^x(4x^2+8x+3)dx$ is equal to

(A) $(2x+1)^2e^x+k$ (B) $(x+1)^2e^x+k$

(C) $(4x^2+3)e^x+k$ (D) None of these

16. The anti-derivative of $\frac{2^x}{\sqrt{1-4^x}}$ w.r.t. x is

(A) $\log_2 e\cdot\sin^{-1}(2^x)+k$ (B) $\sin^{-1}(2^x)+k$

(C) $\cos^{-1}(2^x)\log_2 e+k$ (D) None of these

17. $\int \frac{dx}{x\sqrt{x^4-1}}$ is equal to

(A) $\sec^{-1}x^2+c$ (B) $\frac{1}{2}\sec^{-1}x^2+c$

(C) $\tan^{-1}x^2+c$ (D) $\mathrm{cosec}^{-1}x^2+c$

18. $\int \frac{dx}{\sec^2 x+\tan^2 x}$ is equal to

(A) $\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)+x+c$ **(B)** $\sqrt{2}\tan^{-1}(\sqrt{2}\tan x)-x+c$

(C) $\sqrt{2}\tan^{-1}(2\tan x)+c$ **(D)** None of these

19. $\int \frac{x^2+\sin^2 x}{1+x^2}\sec^2 x\,dx$ is equal to

(A) $\tan x+c$ **(B)** $\tan x-\tan^{-1}x+c$

(C) $\tan x+\tan^{-1}x+c$ **(D)** None of these

20. $\int \sin 2x\cdot\log\cos x\,dx$ is equal to

(A) $\cos^2 x\left(\frac{1}{2}+\log\cos x\right)+k$ **(B)** $\cos^2 x\log\cos x+k$

(C) $\cos^2 x\left(\frac{1}{2}-\log\cos x\right)+k$ **(D)** None of these

21. $\int \frac{3+2\cos x}{(2+3\cos x)^2}dx$ is equal to

(A) $\frac{\sin x}{(3\cos x+2)}+c$ **(B)** $\frac{2\cos x}{(3\sin x+2)}+c$

(C) $\left(\frac{2\cos x}{3\cos x+2}\right)+c$ **(D)** $\left(\frac{2\sin x}{3\sin x+2}\right)+c$

22. If $\int f(x)\cos x\,dx=\frac{1}{2}f^2(x)+c$, then $f(x)$ can be

(A) x **(B)** 1

(C) $\cos x$ **(D)** $\sin x$

23. If $\int g(x)dx=g(x)$, then $\int g(x)(f(x)-f''(x))dx$ is equal to

(A) $g(x)(f(x)+f'(x))+c$ **(B)** $g(x)(f(x)-f'(x))+c$

(C) $g(x)f(x)f'(x)+c$ **(D)** None of these

24. $\int e^{\ln(\sin x)}dx$ is equal to

(A) $\sin x+c$ **(B)** $-\cos x+c$

(C) $e^{\log\cos x}+c$ **(D)** None of these

25. $\int\left(\frac{x^2-x-1}{x}+1\right)^{K+7}\left(\frac{x^2+1}{x^2}\right)dx$ is equal to

(A) $\frac{\left(\frac{x^2-1}{x}\right)^{K+7}}{k+7}+c$ **(B)** $\frac{\left(x-\frac{1}{x}\right)^{K+8}}{k+8}+c$

(C) $\left(x-\frac{1}{x}\right)^{K+8}(K+8)+c$ **(D)** None of these

26. $\int \sec^3 2\theta\,d\theta$ is equal to

(A) $\frac{1}{2}\sec\theta\tan\theta+\ln\sqrt{\sec\theta+\tan\theta}+c$

(B) $\frac{1}{4}\sec 2\theta\tan 2\theta+\ln\sqrt{\sec 2\theta+\tan 2\theta}+c$

(C) $\frac{1}{4}\sec 2\theta\tan 2\theta+\ln\sqrt[4]{\sec 2\theta+\tan 2\theta}+c$

(D) None of these

27. $\int \frac{\cos 2x+x+1}{x^2+\sin 2x+2x}dx=$

(A) $\log(x^2+\sin 2x+2x)+c$ **(B)** $-\log(x^2+\sin 2x+2x)+c$

(C) $\frac{1}{2}\log(x^2+\sin 2x+2x)+c$ **(D)** None of these

28. $\int \frac{1+\tan x}{x+\log\sec x}dx$ is equal to

(A) $\log(x+\log\sec x)+c$ **(B)** $-\log(x+\log\sec x)+c$

(C) $\log(x-\log\sec x)+c$ **(D)** None of these

29. $\int \frac{(2x+1)e^x}{(2x+3)^2}dx$ is equal to

(A) $\frac{e^x}{2x+1}+c$ **(B)** $e^x(2x+1)+c$

(C) $e^x(2x+3)+c$ **(D)** $\frac{e^x}{2x+3}+c$

30. $\int\left(\frac{\ln x-1}{(\ln x)^2+1}\right)^2 dx$ is equal to

(A) $\frac{x}{x^2+1}+c$ **(B)** $\frac{\ln x}{(\ln x)^2+1}$

(C) $\frac{x}{(\ln x)^2+1}+c$ **(D)** $e^x\left(\frac{x}{x^2+1}\right)+c$

31. The value of $\int e^x\frac{(x^3+x+1)}{(1+x^2)^{3/2}}dx$ is equal to

(A) $\frac{xe^x}{(1+x^2)^{1/2}}+C$ **(B)** $\frac{x^2e^x}{(1+x^2)^{1/2}}+C$

(C) $\frac{e^x}{(1+x^2)^{1/2}}+C$ **(D)** None of these

32. $\int \frac{\cos 2x}{\cos x}dx$ is equal to

(A) $2\sin x+\log|(\sec x-\tan x)|+c$

(B) $2\sin x-\log|(\sec x-\tan x)|+c$

(C) $2\sin x+\log|(\sec x+\tan x)|+c$

(D) $2\sin x-\log|(\sec x+\tan x)|+c$

33. The value of $\int \frac{\sin x}{\cos^{3/2}x}dx$ is equal to

(A) $2\sqrt{\sin x}+c$ **(B)** $2\sqrt{\cos x}+c$

(C) $2\sqrt{\sec x}+c$ **(D)** $2\sqrt{\operatorname{cosec} x}+c$

34. The value of $\int e^x\left[\frac{x+2}{x+4}\right]^2 dx$ is equal to

(A) $\frac{e^x x}{x+4}+c$ **(B)** $\frac{e^x}{x+4}+c$

(C) $\frac{e^x}{(x+4)^2}+c$ **(D)** $\frac{e^x x^2}{x+4}+c$

35. The value of $\int \sqrt{\frac{x-a}{b-x}}\,dx$ is equal to

(A) $(a+b)\left[\frac{\sin 2\theta}{2}+\theta\right]+c$, where $a\sin^2\theta+b\cos^2\theta=x$

(B) $(a-b)\left[\frac{\sin 2\theta}{2}+\theta\right]+c$, where $a\sin^2\theta+b\cos^2\theta=x$

(C) $(a-b)\left[\frac{\sin 2\theta}{2}+\theta\right]+c$, where $a\sin^2\theta-b\cos^2\theta=x$

(D) $(a+b)\left[\frac{\sin 2\theta}{2}+\theta\right]+c$, where $a\sin^2\theta-b\cos^2\theta=x$

36. If $\int \frac{bx\cos 4x-a\sin 4x}{x^2}dx=\frac{a\sin 4x}{x}+c$, then a and b may be

(A) $a=2, b=2$ **(B)** $a=1, b=4$

(C) $a=-1, b=4$ **(D)** $a=\frac{1}{4}, b=2$

37. If $\int \frac{dx}{x\sqrt{1-x^3}}=a\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+c$, then

(A) $a=\frac{1}{2}$ **(B)** $a=\frac{2}{3}$

(C) $a=\frac{1}{3}$ **(D)** $a=-\frac{2}{3}$

38. If $\int x\log_e(1+1/x)dx=P(x)\ln\left(1+\frac{1}{x}\right)+\frac{1}{2}x-\frac{1}{2}\ln(1+x)+c$, then

(A) $p(x)=\frac{x^2}{2}$ **(B)** $p(x)=-1$

(C) $p(x)=1$ **(D)** None of these

39. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ is equal to

(A) $2\cos\sqrt{x}+c$ **(B)** $\sqrt{\frac{\cos x}{x}}+c$

(C) $\sin\sqrt{x}+c$ **(D)** $2\sin\sqrt{x}+c$

40. $\int \frac{1+\tan x}{x+\log\sec x}dx=$

(A) $\log(x+\log\sec x)+c$ **(B)** $-\log(x+\log\sec x)+c$

(C) $\log(x-\log\sec x)+c$ **(D)** None of these

41. If $\int \frac{(x^2-1)}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)}dx=k\log\left|\tan^{-1}\frac{x^2+1}{x}\right|+c$, then k is equal to

(A) 1 **(B)** 2

(C) 3 **(D)** 5

42. The value of $\int \frac{\ln x}{(1+\ln x)^2}dx$ is equal to

(A) $\frac{x}{1-\ln x}+c$ **(B)** $\frac{x\ln x}{1+\ln x}+c$

(C) $\frac{x}{1+\ln x}+c$ **(D)** $\frac{\ln x}{x+x\ln x}+c$

43. The value of $\int \frac{\log(x/e)}{(\log x)^2}dx$ is equal to

(A) $\frac{x+1}{(\log x)^2}+c$ **(B)** $\frac{x-1}{(\log x)^2}+c$

(C) $\frac{x}{\log x}+c$ **(D)** $\frac{\log x}{x}+c$

44. $\int \frac{dx}{\cos(x-a)\cos(x-b)}=$

(A) $\operatorname{cosec}(a-b)\log\frac{\sin(x-a)}{\sin(x-b)}+c$

(B) $\operatorname{cosec}(a-b)\log\frac{\cos(x-a)}{\cos(x-b)}+c$

(C) $\operatorname{cosec}(a-b)\log\frac{\sin(x-b)}{\sin(x-a)}+c$

(D) $\operatorname{cosec}(a-b)\log\frac{\cos(x-b)}{\cos(x-a)}+c$

45. $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}=$

(A) $\frac{2}{3(b-a)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+c$

(B) $\frac{2}{3(a-b)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+c$

(C) $\frac{2}{3(a-b)}\left[(x+a)^{3/2}+(x+b)^{3/2}\right]+c$

(D) None of these

46. $\int \frac{3\cos x+3\sin x}{4\sin x+5\cos x}dx=$

(A) $\frac{27}{41}x-\frac{3}{41}\log(4\sin x+5\cos x)+c$

(B) $\frac{27}{41}x+\frac{3}{41}\log(4\sin x+5\cos x)+c$

(C) $\frac{27}{41}x-\frac{3}{41}\log(4\sin x-5\cos x)+c$

(D) None of these

47. If $\int(\sin 2x+\cos 2x)dx=\frac{1}{\sqrt{2}}\sin(2x-c)+a$, then the value of a and c is

(A) $c=\pi/4$ and $a=k$ (an arbitrary constant)

(B) $c=-\pi/4$ and $c=\pi/2$ $a=\pi/2$

(C) $c=\pi/2$ and a is an arbitrary constant

(D) None of these

48. $\int \frac{x^3-x-2}{(1-x^2)}dx=$

(A) $\log\left(\frac{x+1}{x-1}\right)-\frac{x^2}{2}+c$ **(B)** $\log\left(\frac{x-1}{x+1}\right)+\frac{x^2}{2}+c$

(C) $\log\left(\frac{x+1}{x-1}\right)+\frac{x^2}{2}+c$ **(D)** $\log\left(\frac{x-1}{x+1}\right)-\frac{x^2}{2}+c$

49. $\int \frac{x^2dx}{(a+bx)^2} =$

(A) $\frac{1}{b^2}\left[x+\frac{2a}{b}\log(a+bx)-\frac{a^2}{b}\frac{1}{a+bx}\right]+c$

(B) $\frac{1}{b^2}\left[x-\frac{2a}{b}\log(a+bx)+\frac{a^2}{b}\frac{1}{a+bx}\right]+c$

(C) $\frac{1}{b^2}\left[x+\frac{2a}{b}\log(a+bx)+\frac{a^2}{b}\frac{1}{a+bx}\right]+c$

(D) $\frac{1}{b^2}\left[x+\frac{a}{b}-\frac{2a}{b}\log(a+bx)-\frac{a^2}{b}\frac{1}{a+bx}\right]+c$

50. $\int \frac{dx}{(1+x^2)\sqrt{p^2+q^2(\tan^{-1}x)^2}} =$

(A) $\frac{1}{q}\log\left[q\tan^{-1}x+\sqrt{p^2+q^2(\tan^{-1}x)^2}\right]+c$

(B) $\log\left[q\tan^{-1}x+\sqrt{p^2+q^2(\tan^{-1}x)^2}\right]+c$

(C) $\frac{2}{3q}(p^2+q^2\tan^{-1}x)^{3/2}+c$

(D) None of these

51. $\int \frac{x^5}{\sqrt{1+x^3}}dx =$

(A) $\frac{2}{9}(1+x^3)^{3/2}+c$

(B) $\frac{2}{9}(1+x^3)^{3/2}+\frac{2}{3}(1+x^3)^{1/2}+c$

(C) $\frac{2}{9}(1+x^3)^{3/2}-\frac{2}{3}(1+x^3)^{1/2}+c$

(D) None of these

52. $\int \frac{dx}{\sin x-\cos x+\sqrt{2}}$ is equal to

(A) $-\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$ **(B)** $\frac{1}{\sqrt{2}}\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$

(C) $\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$ **(D)** $-\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$

53. $\int \frac{a\,dx}{b+ce^x} =$

(A) $\frac{a}{b}\log\left(\frac{e^x}{b+ce^x}\right)+c$ **(B)** $\frac{a}{b}\log\left(\frac{b+ce^x}{e^x}\right)+c$

(C) $\frac{b}{a}\log\left(\frac{e^x}{b+ce^x}\right)+c$ **(D)** $\frac{b}{a}\log\left(\frac{b+ce^x}{e^x}\right)+c$

54. $\int \sin\sqrt{x}\,dx =$

(A) $2\left[\sin\sqrt{x}-\cos\sqrt{x}\right]+c$ **(B)** $2\left[\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x}\right]+c$

(C) $2\left[\sin\sqrt{x}+\cos\sqrt{x}\right]+c$ **(D)** $2\left[\sin\sqrt{x}+\sqrt{x}\cos\sqrt{x}\right]+c$

55. $\int \frac{x^2}{(9-x^2)^{3/2}}dx =$

(A) $\frac{x}{\sqrt{9-x^2}}-\sin^{-1}\frac{x}{3}+c$ **(B)** $\frac{x}{\sqrt{9-x^2}}+\sin^{-1}\frac{x}{3}+c$

(C) $\sin^{-1}\frac{x}{3}-\frac{x}{\sqrt{9-x^2}}+c$ **(D)** None of these

56. $\int x\sqrt{\frac{1-x^2}{1+x^2}}\,dx =$

(A) $\frac{1}{2}\left[\sin^{-1}x^2+\sqrt{1-x^4}\right]+c$

(B) $\frac{1}{2}\left[\sin^{-1}x^2-\sqrt{1-x^4}\right]+c$

(C) $\sin^{-1}x^2+\sqrt{1-x^4}+c$

(D) $\sin^{-1}x^2+\sqrt{1-x^2}+c$

57. If $\int f(x)\sin x\cos x\,dx = \frac{1}{2(b^2-a^2)}\log(f(x))+c$, then $f(x) =$

(A) $\frac{1}{a^2\sin^2 x+b^2\cos^2 x}$ **(B)** $\frac{1}{a^2\sin^2 x-b^2\cos^2 x}$

(C) $\frac{1}{a^2\cos^2 x+b^2\sin^2 x}$ **(D)** $\frac{1}{a^2\cos^2 x-b^2\sin^2 x}$

58. $\int \frac{dx}{\sqrt{(x-a)(b-x)}} =$

(A) $2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)}+c$ **(B)** $\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)}+c$

(C) $2\sin^{-1}\sqrt{\left(\frac{x+a}{b-a}\right)}+c$ **(D)** None of these

59. If $\int xe^x\cos x\,dx = (x)+c$, then $f(x)$ is equal to

(A) $\frac{e^x}{2}\{(1-x)\sin x - x\cos x\}$

(B) $\frac{e^x}{2}\{(1-x)\sin x + x\cos x\}$

(C) $\frac{e^x}{2}\{(1+x)\sin x - x\cos x\}$

(D) None of these

60. If $I = \int \frac{dx}{e^x+4e^{-x}} = f(x)+c$, then $f(x)$ is equal to

(A) $2\tan^{-1}(2e^x)$ **(B)** $\frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right)$

(C) $2\tan^{-1}\frac{e^x}{2}$ **(D)** $\frac{1}{2}\tan^{-1}(2e^{2x})$

61. $\int \frac{dx}{(2x+1)\left(1+\sqrt{(2x+1)}\right)}$ is equal to

(A) $\tan^{-1}\dfrac{\sqrt{2x+1}}{1+\sqrt{2x+1}}+c$ (B) $\log_e \dfrac{\sqrt{2x+1}}{1+\sqrt{2x+1}}+c$

(C) $\log_e\left(\dfrac{1+\sqrt{2x+1}}{\sqrt{2x+1}}\right)+c$ (D) $\tan^{-1}\dfrac{1+\sqrt{2x+1}}{\sqrt{2x+1}}+c$

62. $\int \dfrac{\sin x}{\sin x-\cos x}dx$ is equal to

(A) $\dfrac{x}{2}-\dfrac{1}{2}\log(\sin x-\cos x)+c$ (B) $\dfrac{x}{2}+\dfrac{1}{2}\log(\sin x-\cos x)+c$

(C) $\dfrac{x}{2}+\dfrac{1}{2}\log(\sin x+\cos x)+c$ (D) None of these

63. $I=\int \dfrac{(1+x)}{x(1+xe^x)^2}dx$ is equal to

(A) $\ln\left(\dfrac{xe^x}{1+xe^x}\right)-\left(\dfrac{1}{1+xe^x}\right)+c$

(B) $\ln\left(\dfrac{xe^x}{1+xe^x}\right)+\left(\dfrac{1}{1-xe^x}\right)+c$

(C) $\ln\left(\dfrac{xe^x}{1-xe^x}\right)+\left(\dfrac{1}{1+xe^x}\right)+c$

(D) $\ln\left(\dfrac{xe^x}{1+xe^x}\right)+\left(\dfrac{1}{1+xe^x}\right)+c$

64. $\int \dfrac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$ is equal to

(A) $\dfrac{\sin 2x}{2}-\sin x+c$ (B) $-\dfrac{\sin 2x}{2}+\sin x+c$

(C) $-\dfrac{\sin 2x}{2}-\sin x+c$ (D) $\dfrac{\sin 2x}{2}+\sin x+c$

65. $\int \dfrac{e^x(1+x)}{\cos^2(xe^x)}dx=$

(A) $-\cot(xe^x)$ (B) $\tan(xe^x)$

(C) $\tan(e^x)$ (D) None of these

66. $\int \tan^3 2x\sec 2x dx=$

(A) $\sec^3 2x+3\sec 2x$ (B) $\dfrac{1}{6}\left[\sec^3 2x-3\sec 2x\right]$

(C) $\left[\sec^3 2x-3\sec 2x\right]$ (D) None of these

67. $\int \dfrac{\sec x\,\mathrm{cosec}\,x}{\log\tan x}dx$

(A) $\log(\tan x)+c$ (B) $\cot(\log x)+c$

(C) $\log(\log\tan x)+c$ (D) $\tan(\log x)+c$

68. $\int \cos^3 x\, e^{\log(\sin x)}dx$ is equal to

(A) $-\dfrac{\sin^4 x}{4}+c$ (B) $-\dfrac{\cos^4 x}{4}+c$

(C) $\dfrac{e^{\sin x}}{4}+c$ (D) None of these

69. $\int \dfrac{(x^4-x)^{\frac{1}{4}}}{x^5}dx$ is equal to

(A) $\dfrac{4}{15}\left(1-\dfrac{1}{x^3}\right)^{\frac{5}{4}}+c$ (B) $\dfrac{4}{5}\left(1-\dfrac{1}{x^3}\right)^{\frac{5}{4}}+c$

(C) $\dfrac{4}{15}\left(1+\dfrac{1}{x^3}\right)^{\frac{5}{4}}+c$ (D) None of these

70. $\int \dfrac{\log(x+1)-\log x}{x(x+1)}dx$ is equal to

(A) $-\log\left(\dfrac{x+1}{x}\right)+c$ (B) $-\log\left[\log\left(\dfrac{x+1}{x}\right)\right]+c$

(C) $-\dfrac{1}{2}\left[\log\left(\dfrac{x+1}{x}\right)\right]^2+c$ (D) $c-\dfrac{1}{2}\left[\log(x+1)^2-(\log x)^2\right]$

71. $\int \dfrac{\sin x-\cos x}{\sqrt{1-\sin 2x}}e^{\sin x}\cos x dx$ is equal to

(A) $e^{\sin x}+c$ (B) $e^{\sin x-\cos x}+c$

(C) $e^{\sin x+\cos x}+c$ (D) $e^{\cos x-\sin x}+c$

72. If $\int \dfrac{1}{x\sqrt{1-x^3}}dx=a\log\left|\dfrac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right|+b$ then a is equal to

(A) $\dfrac{1}{3}$ (B) $\dfrac{2}{3}$

(C) $-\dfrac{1}{3}$ (D) $-\dfrac{2}{3}$

73. $\int a^{a^{a^x}}\cdot a^{a^x}\cdot a^x\,dx$ is equal to

(A) $\dfrac{a^{a^x}}{(\log a)^3}+c$ (B) $a^{a^{a^x}}(\log a)^3+c$

(C) $\dfrac{a^{a^{a^x}}}{(\log a)^3}+c$ (D) None of these

74. $\int e^{\log_5 x}dx=$

(A) $\dfrac{x^{\log_5 e}}{\log_5 e}$ (B) $\dfrac{x^{\log_5 5e}}{\log_5 5e}$

(C) $\dfrac{x^{\log_e 5e}+1}{\log_e 5e+1}$ (D) None of these

75. $\int x^2\sin x dx=$

(A) $x^2\sin x-2x\cos x+c$

(B) $x^2\sin x+c$

(C) $-x^2\cos x+2x\sin x+2\cos x+c$

(D) $-x^2\sin x-2x\cos x+\sin x+c$

76. $\int\left\{\dfrac{(1-\cos x)}{[\cos x(1+\cos x)]}\right\}dx=$

(A) $\log(\sec x+\tan x)-2\tan\left(\frac{x}{2}\right)$

(B) $\log(\sec x+\tan x)+2\tan\left(\frac{x}{2}\right)$

(C) $\log(\sec x-\tan x)-2\tan\left(\frac{x}{2}\right)$

(D) None of these

77. $\int(1-x)^{23}x\,dx=$

(A) $\frac{x^{23}}{23}+\frac{x^{24}}{24}+c$ **(B)** $\frac{(x-1)^{23}}{23}+\frac{(x-1)^{24}}{24}+c$

(C) $\frac{(1-x)^{25}}{25}-\frac{(1-x)^{24}}{24}+c$ **(D)** None of these

78. $\int\{\sin(\log x)+\cos(\log x)\}dx=$

(A) $x\sin(\log x)+c$ **(B)** $x\cos(\log x)+c$

(C) $x\log(\sin x)+c$ **(D)** $x\log(\cos x)+c$

79. $\int\frac{dx}{\cos^6 x+\sin^6 x}$ is equal to

(A) $\log_e(\tan x-\cot x)+c$ **(B)** $\log_e(\cot x-\tan x)+c$

(C) $\tan^{-1}(\tan x-\cot x)+c$ **(D)** $\tan^{-1}(2\cot 2x)+c$

80. $\int\frac{3+2\cos x}{(2+3\cos x)^2}dx$ is equal to

(A) $\left(\frac{\sin x}{3\cos x+2}\right)+c$ **(B)** $\left(\frac{2\cos x}{3\sin x+2}\right)+c$

(C) $\left(\frac{2\cos x}{3\cos x+2}\right)+c$ **(D)** $\left(\frac{2\sin x}{3\sin x+2}\right)+c$

81. $I=\int\frac{(x+x^{\frac{2}{3}}+x^{\frac{1}{6}})}{x(1+x^{\frac{1}{3}})}dx$ is equal to

(A) $\frac{3}{2}x^{\frac{2}{3}}+6\tan^{-1}\left(x^{\frac{1}{6}}\right)+c$ **(B)** $\frac{3}{2}x^{\frac{2}{3}}-6\tan^{-1}\left(x^{\frac{1}{6}}\right)+c$

(C) $\frac{3}{2}x^{\frac{2}{3}}+\tan^{-1}\left(x^{\frac{1}{6}}\right)+c$ **(D)** None of these

82. $I=\int\frac{1}{\sqrt{1-e^{2x}}}dx$ is equal to

(A) $\ln\left|\frac{\sin^{-1}e^x}{2}\right|+c$ **(B)** $\tan\ln\left|\frac{\sin^{-1}e^x}{2}\right|+c$

(C) $\ln\left|\tan\left(\frac{\sin^{-1}e^x}{2}\right)\right|+c$ **(D)** None of these

83. $\int(e^{\log x}+\sin x)\cos x\,dx$ is equal to

(A) $x\sin x+\cos x-\sin^2 x+c$

(B) $x\cos x-\sin^2 x+c$

(C) $x\sin x+\cos x-(\cos^2 x)/2+c$

(D) $x^2\sin x+\cos x-\sin^3 x+c$

84. $\int\frac{x+\sin x}{1+\cos x}dx$ is equal to

(A) $x\tan\frac{x}{2}+c$ **(B)** $\tan\frac{x}{2}+c$

(C) $x\tan x+c$ **(D)** $\frac{x}{2}\tan\frac{x}{2}+c$

85. $\int\frac{1}{4\cos^3 x-3\cos x}dx$ is equal to

(A) $\frac{1}{3}\ln|\sec 3x-\tan 3x|+c$ **(B)** $\frac{1}{3}\ln|\sec 3x+\tan 3x|+c$

(C) $\frac{1}{4}\ln|\sec 3x+\tan 3x|+c$ **(D)** $\frac{1}{4}\ln|\sec 3x-\tan 3x|+c$

86. $\int\sin(\log x)dx$ is equal to

(A) $\frac{x}{2}[\sin(\ln x)+\cos(\ln x)]+c$ **(B)** $\frac{x}{2}[\cos(\ln x)-\sin(\ln x)]+c$

(C) $\frac{x}{2}[\sin(\ln x)-\cos(\ln x)]+c$ **(D)** $x[\sin(\ln x)-\cos(\ln x)]+c$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If $I_{m,n}=\int\cos^m x\cos nx\,dx$, then the value of $(m+n)I_{m,n}-m\,I_{m-1,n-1}$ $(m,n\in N)$ is equal to

(A) $\frac{\cos^m x\sin nx}{n}+c$ **(B)** $\cos^m x\sin nx+c$

(C) $\frac{\cos^m x\cos nx}{n}+c$ **(D)** $-\cos^m x\cos nx+c$

2. If the anti-derivative of $\int\frac{\sin^4 x}{x}dx$ is $f(x)$, then $\int\frac{\sin^4((p+q)x)}{x}dx$ in terms of $f(x)$ is

(A) $f((p+q)x)$ **(B)** $\frac{f((p+q)x)}{p+q}$

(C) $f((p+q)x)(p+q)$ **(D)** None of these

3. If $I=\int\frac{x^n dx}{1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}$, then

(A) $I=\frac{1}{n!}\left\{x+\ln\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)\right\}+c$

(B) $I=\frac{x^2-e^x\ln x}{n!(e^x-1)}+c$

(C) $I=n!\ln\left(\frac{e^x}{1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}\right)+c$

(D) None of these

4. Let $f(x)=\int \frac{x^2\,dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $g(x)=\int \frac{x\,dx}{\sqrt{1+x^4}}$, if $f(0)=g(0)=0$. Then the value of $f(1)-2g(1)+\frac{\pi}{4}$ is

(A) 0 **(B)** 1
(C) 2 **(D)** None of these

5. $\int \frac{\sqrt[4]{x^{10}+x^8+1}}{x^6}\left(3x^{10}+2x^8-2\right)dx=$

(A) $\frac{2(x^{10}+x^8+1)^{5/4}}{5x^5}+c$ **(B)** $\frac{4(x^{10}+x^8+1)^{5/4}}{5}+c$

(C) $\frac{2(x^6+x^4+x^{-4})^{5/4}}{5x^5}+c$ **(D)** None of these

6. $\int (x^6+x^4)\sqrt{(2x^4+3x^2)}dx, x>0=$

(A) $\frac{1}{6}(2x^6+3x^4)^{3/2}dx+c$ **(B)** $\frac{1}{18}x^6(2x^2+3)^{\frac{3}{2}}+c$

(C) $\frac{1}{12}(2x^6+3x^4)^{3/2}dx+c$ **(D)** $\frac{1}{18}x^2(2x^2+3)^{\frac{3}{2}}+c$

7. $\int \frac{x^4-1}{(x^5+1)(x+1)}dx=$

(A) $\frac{1}{5}\ln\left(\frac{(x^5+1)}{(x+1)}\right)+c$ **(B)** $\frac{1}{5}\ln\left(\frac{(x^5+1)}{(x+1)}\right)^4+c$

(C) $\ln\left(\frac{(x^5+1)^{1/5}}{(x+1)}\right)+c$ **(D)** None of these

8. $\int \frac{e^{2x}-e^x+1}{(e^x\sin x+\cos x)(e^x\cos x-\sin x)}\,dx=$

(A) $\ln\left(\frac{e^x\cos x-\sin x}{e^x\sin x+\cos x}\right)+c$ **(B)** $\ln\left(\frac{e^x\cos x+\sin x}{e^x\sin x+\cos x}\right)+c$

(C) $\ln\left(\frac{e^x\cos x-\sin x}{e^x\sin x-\cos x}\right)+c$ **(D)** None of these

9. If $\int(\sin 3\theta+\sin\theta)\cos\theta\, e^{\sin\theta}d\theta=(A\sin^3\theta+B\cos^2\theta+C\sin\theta$ $+D\cos\theta+E)e^{\sin\theta}+F$, then

(A) $A=4$ **(B)** $B=-12$
(C) $C=20$ **(D)** $D=0$

10. Let $f(x)=\frac{1}{4-3\cos^2x+5\sin^2x}$ and its anti-derivative $F(x)=\frac{1}{3}\tan^{-1}(g(x))+c$, then

(A) $g(x)$ is equal to 3 tan$x$ **(B)** $g\left(\frac{\pi}{4}\right)$ is equal to 3

(C) $g'\left(\frac{\pi}{3}\right)$ is equal to 6 **(D)** $g'\left(\frac{\pi}{3}\right)$ is equal to 12

11. Let $f(x)=[b^2+(a-1)b+2]x-\int(\sin^2x+\cos^4x)dx$ be an increasing function of $x\in R$ and $b\in R$. Then a can take value(s)

(A) 0 **(B)** 1
(C) 2 **(D)** 4

Comprehension Type Questions

Paragraph for Questions 12 and 13: Let n be a positive integer such that $I_n=\int x^n\sqrt{a^2-x^2}dx$. Then answer the following questions:

12. The value of I_1 is

(A) $\frac{2}{3}(a^2-x^2)^{1/2}+C$ **(B)** $\frac{1}{3}(a^2-x^2)^{3/2}+C$

(C) $-\frac{2}{3}(a^2-x^2)^{3/2}+C$ **(D)** $-\frac{1}{3}(a^2-x^2)^{3/2}+C$

13. If $I_n=\frac{-x^{n-1}(a^2-x^2)^{3/2}}{n+2}+kI_{n-2}$, then the value of k is

(A) $\frac{n-1}{n+2}$ **(B)** $\frac{n+2}{n-1}$

(C) $\left(\frac{n-1}{n+2}\right)a^2$ **(D)** $\left(\frac{n+2}{n-1}\right)a^2$

Matrix Match Type Questions

14. Match the following:

Column I	Column II
(A) If $f_r(x)=\underbrace{\log\log\log\cdots\log x}_{r\text{ times}}$. Then $\int\{xf_1(x)f_2(x)\cdots f_{100}(x)\}^{-1}dx$ $=f_k(x)+c$, where $k=$	**(p)** π
(B) $f(x)=\int\frac{x+\sin x}{1+\cos x}dx$ and $f(0)=0$, then $f\left(\frac{\pi}{2}\right)$ is	**(q)** $\frac{\pi}{4}$
(C) Let $f(x)=\int e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx$ and $f(0)=1$ if $f\left(\frac{1}{2}\right)=\frac{k\sqrt{3}\,e^{\pi/6}}{\pi}$, then k is	**(r)** 101
(D) Let $f(x)=\int\left(\frac{\sqrt{\tan x}}{\sin x.\cos x}\right)dx$ and $f(0)=0$ if $f\left(\frac{\pi}{4}\right)=\frac{2k}{\pi}$, then k is	**(s)** $\frac{\pi}{2}$

Answer Key

Practice Exercise 1

1. (D)	**2.** (B)	**3.** (B)	**4.** (D)	**5.** (C)	**6.** (B)
7. (A)	**8.** (A)	**9.** (A)	**10.** (B)	**11.** (C)	**12.** (C)
13. (B)	**14.** (D)	**15.** (C)	**16.** (A)	**17.** (B)	**18.** (B)
19. (B)	**20.** (C)	**21.** (A)	**22.** (D)	**23.** (B)	**24.** (B)
25. (B)	**26.** (B)	**27.** (C)	**28.** (A)	**29.** (D)	**30.** (C)
31. (D)	**32.** (D)	**33.** (C)	**34.** (A)	**35.** (B)	**36.** (B)
37. (C)	**38.** (A)	**39.** (D)	**40.** (A)	**41.** (A)	**42.** (C)
43. (C)	**44.** (B)	**45.** (B)	**46.** (A)	**47.** (A)	**48.** (D)
49. (D)	**50.** (A)	**51.** (C)	**52.** (D)	**53.** (A)	**54.** (B)
55. (A)	**56.** (A)	**57.** (A)	**58.** (A)	**59.** (A)	**60.** (B)
61. (B)	**62.** (B)	**63.** (D)	**64.** (C)	**65.** (B)	**66.** (B)
67. (C)	**68.** (B)	**69.** (A)	**70.** (C)	**71.** (A)	**72.** (A)
73. (C)	**74.** (D)	**75.** (B)	**76.** (A)	**77.** (C)	**78.** (A)
79. (C)	**80.** (A)	**81.** (A)	**82.** (C)	**83.** (C)	**84.** (A)
85. (B)	**86.** (C)				

Practice Exercise 2

1. (B)	**2.** (A)	**3.** (C)	**4.** (A)	**5.** (A)	**6.** (B)
7. (C)	**8.** (A)	**9.** (A), (B)	**10.** (B), (D)	**11.** (A), (B), (C)	**12.** (D)
13. (C)	**14.** (A) → (r), (B) → (s), (C) → (s), (D) → (p)				

Solutions

Practice Exercise 1

1. $I=\int \frac{\sqrt{1+x}-\sqrt{x}}{(1+x)-(x)}\,dx$

$=\frac{2}{3}(x+1)^{3/2}-\frac{2}{3}x^{3/2}+c$

2. $\int \frac{e^x dx}{e^x(e^x+1)(2e^x+3)}$(Let $e^x=t$)

$I=\frac{1}{2}\int \frac{dt}{t(t+1)\left(t+\frac{3}{2}\right)}$

$I=\frac{1}{3}\int\frac{dt}{t}-\int\frac{dt}{t+1}+\frac{2}{3}\int\frac{dt}{\left(t+\frac{3}{2}\right)}$

$=\frac{x}{3}-\ln(e^x+1)+\frac{2}{3}\ln(2e^x+3)+c$

3. $I=\int\frac{(1-\sin^2 x)\cos x\,dx}{(\sin x+\sin^2 x)}$

$=\int\left(\frac{1-t^2}{t+t^2}\right)dt$ [Put $(\sin x=t)$]

$=\int\left(\frac{1-t}{t}\right)dt$

$=\log|\sin x|-\sin x+\mathrm{c}$

4. $\frac{1}{n}\int\frac{n\cdot x^{n-1}dx}{x^n(x^n+1)}$ (put $x^n=t$)

$=\frac{1}{n}\int\frac{dt}{t(t+1)}=\frac{1}{n}\int\frac{dt}{t}-\frac{1}{n}\int\frac{dt}{(t+1)}$

$\frac{1}{n}\log\left(\frac{t}{t+1}\right)+c=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+c$

5. $I=\int\frac{1+x+x^2}{1+x^2}\,e^{\tan^{-1}x}dx$

Put $\tan^{-1}x=t$. Then $\frac{dx}{1+x^2}=dt$. So,

$I=\int(1+\tan t+\tan^2 t)e^t dt=\int(\sec^2 t+\tan t)e^t dt$

$I=e^t\tan t+c=xe^{\tan^{-1}x}+c$

6. $\int\frac{\tan x}{\sqrt{\cos x}}dx=\int\frac{\sin x}{(\cos x)^{3/2}}dx$ (Let $\cos x=t$)

$=-\int t^{-3/2}dt$

$=-\frac{t^{-1/2}}{-\frac{1}{2}}+c$

$=\frac{2}{\sqrt{\cos x}}+c$

7. $\int\frac{dx}{x\ln x\ln(\ln x)}$ (Put $\ln x=t$)

$$\int \frac{dt}{t \ln t}$$

Let $\ln t = z$. Then, $\frac{1}{t}dt = dz$. So,

$$\int \frac{dz}{z} = \log z = \log(\ln t)$$
$$= \ln|\ln(\ln x)| + c$$

8. $\int e^x \left[\frac{1+\sqrt{1-x^2}\sin^{-1}x}{\sqrt{1-x^2}}\right]dx$

$$= \int \frac{e^x}{\sqrt{1-x^2}}dx + \int e^x \cdot \sin^{-1}x\,dx$$
$$= \int \frac{e^x}{\sqrt{1-x^2}}dx + e^x \sin^{-1}x - \int \frac{e^x}{\sqrt{1-x^2}}dx$$
$$= e^x \sin^{-1}x + c$$

9. $\int \frac{x^3}{1+x^8}dx$

Let $x^4 = t$. Then $4x^3 dx = dt$.

$$I = \frac{1}{4}\int \frac{dt}{1+t^2} = \frac{1}{4}\tan^{-1}t + c$$
$$I = \frac{1}{4}\tan^{-1}x^4 + c$$

10. $\int xe^x dx = x\int e^x dx - \int e^x dx + c = xe^x - e^x + c$

11. $\int \frac{dx}{\sqrt{1-(x^2-2x+1)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}} = \sin^{-1}(x-1) + c$

12. $I = \frac{1}{2}\int \frac{p-1}{p\sqrt{p^2-1}}dp = \frac{1}{2}\int \frac{p}{p\sqrt{p^2-1}}dp - \frac{1}{2}\int \frac{1}{p\sqrt{p^2-1}}dp$

$f(p) = \ln\sqrt{p+\sqrt{p^2-1}} - \frac{1}{2}\sec^{-1}p$

13. From given equation, we have

$$\int \frac{(\sin\theta+\cos\theta)d\theta}{\sqrt{1-(\sin\theta-\cos\theta)^2}}$$

Put $\sin\theta - \cos\theta = t$. Then

$$(\cos\theta + \sin\theta)\,d\theta = dt$$

Therefore,

$$\int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(\sin\theta - \cos\theta) + c$$

14. Put $5x^7 = t$. Then

$$35x^6 dx = dt$$

Therefore, from given equation, we have

$$\frac{1}{35}\int \sin t \cdot dt = \frac{k}{5}\cos(5x^7)$$
$$\Rightarrow \frac{-\cos(5x^7)}{35} = \frac{k}{5}\cos(5x^7)$$
$$\Rightarrow k = -\frac{1}{7}$$

15. $\int e^x(4x^2+8x+3)dx = \int e^x[f(x)+f'(x)] + 3\int e^x dx$

$$= 4x^2e^x + 3e^x + k$$

16. Put $2^x = t$. Then

$2^x \ln 2\, dx = dt$

$$\Rightarrow \frac{1}{\ln 2}\int \frac{dt}{\sqrt{1-t^2}} = \log_2 e \cdot \sin^{-1}(t) + c$$
$$\Rightarrow \int \frac{2^x}{\sqrt{1-4^x}}dx = \log_2 e \cdot \sin^{-1}(2^x) + k$$

17. Put $x^2 = t$. Then

$2x\,dx = dt$

$$\Rightarrow \frac{1}{2}\int \frac{dt}{t\sqrt{t^2-1}} = \frac{1}{2}\sec^{-1}(t) + c$$
$$\Rightarrow \frac{1}{2}\int \frac{2x\,dx}{x^2\sqrt{(x^2)^2-1}} = \frac{1}{2}\sec^{-1}(x^2) + c$$

18. $\int \frac{dx}{\sec^2 x + \tan^2 x} = \int \frac{(1-\sin^2 x)}{(1+\sin^2 x)}dx = \int \frac{2-(1+\sin^2 x)}{(1+\sin^2 x)}dx$

$$= 2\int \frac{\sec^2 x}{2\tan^2 x + 1}dx - \int 1\,dx \left(\text{Let } \tan x = t\right)$$
$$= \frac{2}{2}\int \frac{dt}{t^2 + \left(\frac{1}{\sqrt{2}}\right)^2} - x = \sqrt{2}\tan^{-1}(\sqrt{2}\tan x) - x + c$$

19. $I = \int \frac{x^2}{1+x^2}\sec^2 x + \int \frac{\tan^2 x\,dx}{1+x^2}$

$$= \int \sec^2 x - \int \frac{\sec^2 x}{1+x^2}dx + \int \frac{\tan^2 x\,dx}{1+x^2}$$
$$= \tan x - \tan^{-1}x + c$$

20. $I = -\frac{1}{2}\log\cos x \cdot \cos 2x - \frac{1}{2}\int \frac{\sin x}{\cos x}\cos 2x\,dx$

$$= -\frac{1}{2}\cos 2x \log\cos x - \frac{1}{2}\int \left(\frac{2\cos^2 x - 1}{\cos x}\right)\sin x\,dx$$

Let $\cos x = t$. Then $-\sin x\,dx = dt$.

$$= -\frac{1}{2}\cos 2x \log\cos x + \frac{1}{2}\int \left(\frac{2t^2-1}{t}\right)dt$$
$$= -\frac{1}{2}\cos 2x \log\cos x + \frac{t^2}{2} - \frac{1}{2}\log t$$
$$= -\frac{1}{2}\cos 2x \cdot \log\cos x + \frac{(\cos x)^2}{2} - \frac{1}{2}\log\cos x + k$$
$$= \cos^2 x\left(\frac{1}{2} - \log\cos x\right) + k$$

21. $I=\int \frac{3+2\cos x}{(2+3\cos x)^2}dx$

Let $\frac{\sin x}{3\cos x+2}=t$. Then

$$\frac{(3\cos x+2)\cos x-\sin x(-3\sin x)}{(3\cos x+2)^2}dx=dt$$

$$\Rightarrow \frac{3+2\cos x}{(3\cos x+2)^2}dx=dt$$

$$\Rightarrow I=\int 1\,dt=t+c=\frac{\sin x}{3\cos x+2}+c$$

22. $\int f(x)\cos x\,dx=\frac{1}{2}f^2(x)+c$

By putting $f(x)=\sin x$, we get

$$I=\frac{1}{2}\int \sin 2x\;dx=\frac{1}{4}\cos 2x+c$$

$$=-\frac{1}{4}(1-2\sin^2 x)+c$$

$$=\frac{1}{2}\sin^2 x+k$$

$$=\frac{1}{2}f^2(x)+k$$

Hence, $f(x)=\sin x$.

23. $I=\int g(x)f(x)dx-\int g(x)f''(x)\,dx$

$$=f(x)\cdot\int g(x)dx-\int\left(f'(x)\cdot\int g(x)dx\right)-\int g(x)f''(x)dx$$

$$=f(x)g(x)-[g(x)\cdot f'(x)-\int f''(x)\cdot g(x)dx]-\int g(x)f''(x)dx$$

$$=g(x)\,(f(x))-f'(x)+c$$

24. $\int e^{\ln(\sin x)}dx=\int \sin x\,dx=-\cos x+c$

25. $I=\int\left(\frac{x^2-x-1}{x}+1\right)^{k+7}\left(\frac{x^2+1}{x^2}\right)dx$

$$=\int\left(\frac{x^2-1}{x}\right)^{k+7}\left(\frac{x^2+1}{x^2}\right)dx$$

Now,

$$\left(x-\frac{1}{x}\right)=t\;\Rightarrow\;\left(1+\frac{1}{x^2}\right)dx=dt$$

$$I=\int t^{k+7}dt=\frac{t^{k+8}}{k+8}+c=\frac{\left(x-\frac{1}{x}\right)^{k+8}}{k+8}+c$$

26. $I=\int \sec^3(2\theta)\cdot d\theta=\int \sec(2\theta)\cdot\left[1+\tan^2 2\theta\right]d\theta$

$$=\int \sec 2\theta\, d\theta+\int \sec 2\theta\cdot\tan^2 2\theta\, d\theta$$

$$I_1=\int \sec 2\theta\, d\theta=\frac{1}{2}\ln\left(\sec 2\theta+\tan 2\theta\right)+c_1$$

$$I_2=\int \sec 2\theta.\,\tan^2 2\theta\, d\theta=\frac{1}{2}\int\sqrt{\sec^2 2\theta-1}\cdot 2\sec 2\theta\tan 2\theta\, d\theta$$

$$\Rightarrow I_2=\frac{1}{2}\int\sqrt{t^2-1}\,dt\quad(\text{put }\sec 2\theta=t)$$

$$\Rightarrow I_2=\frac{1}{2}\left[\frac{t}{2}\sqrt{t^2-1}-\frac{1}{2}\ln\left(t+\sqrt{t^2-1}\right)\right]+c_2$$

$$\Rightarrow I_2=\frac{1}{4}\left[\sec 2\theta\,\sqrt{\sec^2 2\theta-1}\right]+\frac{1}{2}\ln\left[\sec 2\theta+\sqrt{\sec^2 2\theta-1}\right]+c$$

$$I=I_1+I_2=\frac{1}{4}\sec 2\theta.\tan 2\theta+\frac{1}{2}\ln\left(\sec 2\theta+\tan 2\theta\right)+c$$

27. Let $x^2+\sin 2x+2x=t$. Then

$$\Rightarrow (2x+2\cos 2x+2)dx=dt$$

$$\Rightarrow (x+\cos 2x+1)dx=\frac{dt}{2}$$

$$\Rightarrow I=\frac{1}{2}\int\frac{dt}{t}=\frac{1}{2}\log t=\frac{1}{2}\log(x^2+\sin 2x+2x)+k$$

28. As $\frac{d}{dx}(x+\log\sec x)=1+\tan x$, so

$$I=\log(x+\log\sec x)+c$$

29.

$$\int\frac{e^x}{(2x+3)}dx-2\int\frac{e^x}{(2x+3)^2}dx$$

$$=\int e^x[f(x)+f'(x)]dx=\frac{e^x}{2x+3}+c$$

30. $\int\frac{(\ln x)^2-2(\ln x)+1}{((\ln x)^2+1)^2}dx=\int\left[\frac{(\ln x)^2+1}{((\ln x)^2+1)^2}-2\frac{\ln x}{((\ln x)^2+1)^2}\right]dx$

$$=\int\frac{1}{(\ln x)^2+1}dx-2\int\frac{\ln x}{(\ln x)^2+1}dx$$

$$=\frac{x}{(\ln x)^2+1}+\int\frac{x\,2\ln x}{x((\ln x)^2+1)^2}dx-2\int\frac{\ln x dx}{(1+(\ln x)^2)^2}+c$$

$$=\frac{x}{(\ln x)^2+1}+c$$

31. $\int\frac{e^x\left[x(x^2+1)+1\right]}{(x^2+1)^{3/2}}dx$

$$=\int e^x\left[\frac{x}{(x^2+1)^{1/2}}+\frac{1}{(x^2+1)^{3/2}}\right]dx$$

(Since, $e^x[f'(x)+f(x)]=e^xf(x)+c$)

$$=\frac{xe^x}{(x^2+1)^{1/2}}+c$$

32. $\int\frac{2\cos^2 x-1}{\cos x}dx$

$$=2\int\cos x-\int\sec x\,dx$$

$$=2\sin x-\log|(\sec x+\tan x)|+c$$

33. $\cos x=t\Rightarrow-\sin x\,dx=dt$

$$I=-\int t^{-3/2}dt=-\frac{t^{-3/2+1}}{-\frac{3}{2}+1}+c=\frac{2}{\sqrt{t}}+c=2\sqrt{\sec x}+c$$

34. $\int e^x\left(\frac{x+2}{x+4}\right)^2 dx = \int e^x\left(1-\frac{2}{x+4}\right)^2 dx$

$$= \int e^x\left[1-\frac{4}{x+4}+\frac{4}{(x+4)^2}\right]dx$$

$$= e^x + \int e^x\left[-\frac{4}{x+4}+\frac{4}{(x+4)^2}\right]dx$$

$$= e^x - \frac{4e^x}{x+4} = \frac{xe^x}{x+4}+c$$

35. $\int\sqrt{\frac{x-a}{b-x}}\,dx$

Let $x = a\sin^2\theta + b\cos^2\theta$. Then

$dx = (2a\sin\theta\cos\theta - 2b\cos\theta\sin\theta)d\theta = (a-b)\sin 2\theta\cdot d\theta$

$$= \int\sqrt{\frac{\cos^2\theta(b-a)}{(b-a)\sin^2\theta}}\times(a-b)\sin 2\theta\, d\theta = 2\int(a-b)(\cos^2\theta)d\theta$$

$$= (a-b)\left(\theta+\frac{1}{2}\sin 2\theta\right)+c$$

36. $\left[\frac{\sin 4x}{4x}-\int\frac{-\sin 4x}{4x^2}dx\right]-\int\frac{a}{x^2}\sin 4x\,dx$

$$= \frac{b}{4}\frac{\sin 4x}{x}+\left(\frac{b-a}{4}\right)\frac{\sin 4x}{x^2}dx$$

Hence, $a = 1, b = 4$.

37. $\int\frac{dx}{x\sqrt{1-x^3}} = \int\frac{x^2dx}{x^3\sqrt{1-x^3}}$

Let $1-x^3 = t^2$. Then $-3x^2dx = 2tdt$.

$$I = -\frac{2}{3}\int\frac{tdt}{(1-t^2)t} = -\frac{2}{3}\int\frac{dt}{1-t^2} = \frac{2}{3}\int\frac{dt}{t^2-1}$$

$$= \frac{1}{3}\log\frac{t-1}{t+1} = \frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1+x^3}+1}\right|$$

$$\Rightarrow a = \frac{1}{3}$$

38. $\int x\log\left(1+\frac{1}{x}\right)dx = P(x)\ln\left(1+\frac{1}{x}\right)+\frac{x}{2}-\frac{1}{2}\ln(1+x)+c$

LHS $= \log\left(1+\frac{1}{x}\right)\cdot\frac{x^2}{2}-\int\frac{-\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)}\cdot\frac{x^2}{2}dx$

$$= \log\left(1+\frac{1}{x}\right)\cdot\frac{x^2}{2}+\frac{1}{2}\int\frac{x}{x+1}dx = \frac{x^2}{2}\ln\left(1+\frac{1}{x}\right)+\frac{x}{2}-\frac{1}{2}\ln(1+x)+c$$

Hence, $p(x) = \frac{x^2}{2}$.

39. Let $\sqrt{x} = t$. Then $\frac{1}{2\sqrt{x}}dx = dt$.

$$I = 2\int\cos dt = 2\sin t + c$$

$$= 2\sin\sqrt{x}+c$$

40. $x+\log\sec x = t \Rightarrow \left(1+\frac{\sec x\tan x}{\sec x}\right)dx = dt$

Hence, $I = \log(x+\log\sec x)+c$.

41. Let $\tan^{-1}\left(\frac{x^2+1}{x}\right) = \theta$. Then

$$\frac{1}{1+\left(\frac{x^2+1}{x}\right)^2}\times\frac{x(2x)-(x^2+1)}{x^2}dx = d\theta$$

$$\Rightarrow \frac{(x^2-1)}{(x^4+3x^2+1)}dx = d\theta$$

$$\Rightarrow I = \log\left|\tan^{-1}\left(\frac{x^2+1}{x}\right)\right|+c$$

Hence, the correct answer is (1).

42. $\int\frac{\ln x}{(1+\ln x)^2}dx = \int\frac{1}{1+\ln x}dx - \int\frac{1}{(1+\ln x)^2}dx$

$$= \frac{x}{1+\ln x}+\int\frac{x\cdot\frac{1}{x}dx}{(1+\ln x)^2}-\int\frac{1}{(1+\ln x)^2}dx$$

$$= \frac{x}{1+\ln x}+c$$

43. $\int\frac{\log(x/e)}{(\log x)^2}dx = \int\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right]dx$

$$= \frac{x}{\log x}+\int\frac{x}{x(\log x)^2}dx-\int\frac{1}{(\log x)^2}dx$$

$$= \frac{x}{\log x}+c$$

44. $\int\frac{dx}{\cos(x-a)\cos(x-b)}$

$$= \frac{1}{\sin(a-b)}\int\frac{\sin((x-b)-(x-a))}{\cos(x-a)\cos(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int\left(\frac{\sin(x-b)}{\cos(x-b)}-\frac{\sin(x-a)}{\cos(x-a)}\right)dx$$

$$= \frac{1}{\sin(a-b)}\ln\left|\frac{\cos(x-a)}{\cos(x-b)}\right|+c$$

45. $\int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}} = \int\frac{(\sqrt{x+a}-\sqrt{x+b})dx}{(x+a-x-b)}$

$$= \frac{1}{(a-b)}\int(\sqrt{x+a}-\sqrt{x+b})dx = \frac{2}{3(a-b)}((x+a)^{3/2}-(x+b)^{3/2})+c$$

46. $3\cos x+3\sin x = a(4\sin x+5\cos x)+b\frac{d}{dx}(4\sin x+5\cos x)$

$$3\cos x+3\sin x = \cos x(5a+4b)+\sin x(4a-5b)$$

Compare the coefficients of $\sin x$ and $\cos x$ on the both sides

$$(5a+4b) = 3,\ (4a-5b) = 3$$

$$a = \frac{27}{41},\ b = -\frac{3}{41}$$

$$\int \frac{3\cos x+3\sin x}{4\sin x+5\cos x}dx=\int \frac{27}{41}dx-\frac{3}{41}\int \frac{4\cos x-5\sin x}{4\sin x+5\cos x}dx$$

$$\int \frac{3\cos x+3\sin x}{4\sin x+5\cos x}dx=\frac{27}{41}x-\frac{3}{41}\ln|4\sin x+5\cos x|+c$$

47. $\int (\sin 2x+\cos 2x)dx=-\frac{\cos 2x}{2}+\frac{\sin 2x}{2}+k$

$$=\frac{1}{\sqrt{2}}\left(-\frac{\cos 2x}{\sqrt{2}}+\frac{\sin 2x}{\sqrt{2}}\right)+k=\frac{1}{\sqrt{2}}\sin\left(2x-\frac{\pi}{4}\right)+k$$

So, $c=\frac{\pi}{4}$ and $a=k$, an arbitrary constant.

48. $\int \frac{x^3-x-2}{1-x^2}dx=\int \frac{x(x^2-1)}{1-x^2}dx-\int \frac{2}{1-x^2}dx$

$$=-\int x\,dx+\int \frac{2}{x^2-1}dx=-\frac{x^2}{2}+\ln\left|\frac{x-1}{x+1}\right|+c$$

49. Put $a+bx=t$. Then

$$x=\frac{t-a}{b}\Rightarrow dx=\frac{dt}{b}$$

$$I=\int\left(\frac{t-a}{b}\right)^2\cdot\frac{dt}{t^2 b}$$

$$I=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+a^2t^{-2}\right)dt=\frac{1}{b^3}(t-2a\ln t-a^2t^{-1})+c$$

$$I=\frac{1}{b^3}\left[(a+bx)-2a\ln(a+bx)-a^2(a+bx)^{-1}\right]+c$$

50. Putting $q\tan^{-1}x=t$. Then $\frac{dx}{1+x^2}=\frac{dt}{q}$.

$$\int \frac{dx}{(1+x^2)\sqrt{p^2+(q\tan^{-1}x)^2}}=\frac{1}{q}\int \frac{dx}{\sqrt{p^2+t^2}}$$

$$=\frac{1}{q}\ln\left|q\tan^{-1}x+\sqrt{p^2+(q\tan^{-1}x)^2}\right|+c$$

51. Put $1+x^3=t^2$. Then $3x^2dx=2tdt$ and $x^3=t^2-1$.

So,

$$\int \frac{x^5}{\sqrt{1+x^3}}dx=\int \frac{x^2\cdot x^3}{\sqrt{1+x^3}}dx$$

$$=\frac{2}{3}\int \frac{(t^2-1)\cdot t}{t}dt=\frac{2}{3}\int (t^2-1)dt=\frac{2}{3}\left(\frac{t^3}{3}-t\right)+c$$

$$=\frac{2}{3}\left(\frac{(1+x^3)^{3/2}}{3}-(1+x^3)^{1/2}\right)+c$$

52. $I=\int \frac{dx}{\sin x-\cos x+\sqrt{2}}=\int \frac{dx}{\sqrt{2}\left(\sin x\sin\frac{\pi}{4}-\cos x\cos\frac{\pi}{4}+1\right)}$

$$=\int \frac{dx}{\sqrt{2}\left(1-\cos\left(x+\frac{\pi}{4}\right)\right)}=\int \frac{dx}{2\sqrt{2}\sin^2\left(\frac{x}{2}+\frac{\pi}{8}\right)}=\int \frac{\operatorname{cosec}^2\left(\frac{x}{2}+\frac{\pi}{8}\right)dx}{2\sqrt{2}}$$

$$=-\frac{2}{2\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+c=-\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$$

53. $\int \frac{a\,dx}{b+ce^x}=\int \frac{ae^x\,dx}{be^x+ce^{2x}}$

Now, put $e^x=t$, then it reduces to

$$a\int \frac{dt}{t(b+ct)}=-\frac{a}{b}\int\left(\frac{c}{(b+ct)}-\frac{1}{t}\right)dt \text{ \{By partial fraction\}}$$

$$=\frac{a}{b}\ln\left|\frac{e^x}{(b+ce^x)}\right|+c$$

54. Put $\sqrt{x}=t$, then $\frac{1}{2\sqrt{x}}dx=dt$.

$$\int \sin\sqrt{x}\,dx=2\int t\sin t\,dt=2(-t\cos t+\sin t)+c$$

$$=2(-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x})+c$$

55. Put $x=3\sin\theta$, then $dx=3\cos\theta\,d\theta$. Therefore,

$$\int \frac{x^2}{(9-x^2)^{3/2}}dx=\int \frac{9\sin^2\theta\cdot 3\cos\theta}{(9-9\sin^2\theta)^{3/2}}d\theta$$

$$=\int \frac{27\sin^2\theta\cdot\cos\theta}{27\cos^3\theta}d\theta=\int \tan^2\theta\,d\theta=\int(\sec^2\theta-1)d\theta=\tan\theta-\theta+c$$

$$=\tan\left(\sin^{-1}\frac{x}{3}\right)-\sin^{-1}\frac{x}{3}+c=\frac{x}{\sqrt{9-x^2}}-\sin^{-1}\frac{x}{3}+c$$

56. $\int x\sqrt{\frac{1-x^2}{1+x^2}}dx=\int \frac{x\cdot(1-x^2)}{\sqrt{1-x^4}}dx=\int \frac{x}{\sqrt{1-x^4}}dx-\int \frac{x^3}{\sqrt{1-x^4}}dx$

$$=\frac{1}{2}(\sin^{-1}x^2+\sqrt{1-x^4})+c$$

57. Since $\int f(x)\sin x\cos x\,dx=\frac{1}{2(b^2-a^2)}\ln|f(x)|+c$

Therefore,

$$f(x)\sin x\cdot\cos x=\frac{1}{2(b^2-a^2)}\frac{f'(x)}{f(x)}$$

$$\Rightarrow 2(b^2-a^2)\sin x\cdot\cos x=\frac{f'(x)}{f^2(x)}$$

$$\Rightarrow 2(b^2\int \sin x\cdot\cos x\,dx-a^2\int \sin x\cdot\cos x\,dx)=\int \frac{f'(x)}{f^2(x)}dx$$

$$\Rightarrow(-b^2\cos^2 x-a^2\sin^2 x)=-\frac{1}{f(x)}$$

$$\Rightarrow -f(x)=\frac{1}{(-b^2\cos^2 x-a^2\sin^2 x)}$$

58. Put $x=a\cos^2\theta+b\sin^2\theta$, the given integral becomes

$$I=\int \frac{2(b-a)\sin\theta\cos\theta\,d\theta}{\left\{(a\cos^2\theta+b\sin^2\theta-a)\ \ (a\cos^2\theta+b\sin^2\theta-b\right\}^{\frac{1}{2}}}$$

$$=\int \frac{2(b-a)\sin\theta\cos\theta\,d\theta}{(b-a)\sin\theta\cos\theta}\quad=\left(\frac{b-a}{b-a}\right)\int 2\,d\theta=2\theta+c$$

$$=2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)}+c$$

59. $I =$ real part of $\int x e^{(1+i)x} dx$

$$= \frac{xe^{(1+i)x}}{1+i} - \int \frac{e^{(1+i)x}}{1+i} dx = \frac{xe^{(1+i)x}}{1+i} - \frac{e^{(1+i)x}}{(1+i)^2}$$

$$= e^{(1+i)x}\left[\frac{x(1+i)-1}{(1+i)^2}\right]$$

$$= e^x(\cos x + i\sin x)\left[\frac{(x-1)+ix}{1+2i-1}\right]$$

$$= \frac{e^x}{-2}[i\cos x - \sin x][(x-1)+ix]$$

$$I = \frac{e^x}{2}[(1-x)\sin x - x\cos x] + c$$

60. $I = \int \frac{dx}{e^x + 4e^{-x}} = f(x) + c$

$$\Rightarrow I = \int \frac{e^x dx}{e^{2x}+4}$$

Let $e^x = t$. Then $e^x\, dx = dt$.

$$I = \int \frac{dt}{t^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right) + c$$

61. $\int \frac{dx}{(2x+1)(1+\sqrt{(2x+1)})} = \int \frac{dx}{(\sqrt{(2x+1)})^2(1+\sqrt{(2x+1)})}$

Put $\sqrt{2x+1} = t$. Then

$$\frac{dx}{\sqrt{2x+1}} = dt \Rightarrow dx = t\cdot dt$$

Therefore,

$$\int \frac{t\cdot dt}{t^2(1+t)} = \int \frac{dt}{t(1+t)} = \int\left(\frac{1}{t} - \frac{1}{(1+t)}\right)dt$$

$$= \ln t - \ln(1+t) + c = \ln\frac{t}{t+1} + c$$

$$= \ln\frac{\sqrt{2x+1}}{\sqrt{2x+1}+1} + c$$

62. Let $\sin x = A(\sin x - \cos x) + \frac{d}{dx}(\sin x - \cos x)$. Then

$$\sin x = A(\sin x - \cos x) + B(\cos x + \sin x)$$
$$\Rightarrow \sin x = (A+B)\sin x + (B-A)\cos x$$

Equating the coefficients of $\sin x$ and $\cos x$, we get

$$A + B = 1 \text{ and } B - A = 0$$
$$A = 1/2, B = 1/2$$

$$I = \int \frac{\frac{1}{2}(\sin x - \cos x) + \frac{1}{2}(\cos x + \sin x)}{\sin x - \cos x}$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x - \cos x} dx = \frac{x}{2} + \frac{1}{2}\log(\sin x - \cos x) + c$$

63. Let $I = \int \frac{(1+x)}{x(1+xe^x)^2} dx = \int \frac{(1+x)e^x}{(xe^x)(1+xe^x)^2} dx$. Then

$$(1+xe^x = p \Rightarrow e^x(1+x)\,dx = dp)$$

$$I = \int \frac{dp}{(p-1)p^2}$$

$$\frac{1}{(p-1)p^2} = \frac{A}{(p-1)} + \frac{B}{p} + \frac{C}{p^2}$$

$$1 = Ap^2 + B(p)(p-1) + C(p-1)$$

For $p = 1$, $p = 0$ and $p = -1$, $A = 1$, $C = -1$ and $B = -1$

$$I = \int \frac{1}{(p-1)}dp - \int \frac{dp}{p} - \int \frac{dp}{p^2} = \ln\frac{(p-1)}{p} + \frac{1}{p} + c =$$

$$\ln\left(\frac{xe^x}{1+xe^x}\right) + \left(\frac{1}{1+xe^x}\right) + c$$

64. $\int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}\cos\frac{3x}{2}}{\left[1-2\left(2\cos^2\frac{3x}{2}-1\right)\right]\cos\frac{3x}{2}} dx = \int \frac{2\cos\frac{9x}{2}\cos\frac{3x}{2}\cos\frac{x}{2}}{3\cos\frac{3x}{2} - 4\cos^3\frac{3x}{2}} dx$

$$= \int \frac{2\cos\frac{9x}{2}\cos\frac{3x}{2}\cos\frac{x}{2}}{-\cos\frac{9x}{2}} dx = -\int(\cos 2x + \cos x)dx$$

$$= -\frac{\sin 2x}{2} - \sin x + c$$

65. Put $x e^x = t$, then $(xe^x + e^x)\,dx = dt$.

Therefore,

$$I = \int \sec^2 t\,dt = \tan t + c = \tan(xe^x) + c$$

66. $I = \int \tan^2 2x \tan 2x \sec 2x dx = \int (\sec^2 2x - 1)\sec 2x \tan 2x dx$

Put $\sec 2x = t$. Then $2\sec 2x \tan 2x\, dx = dt$.

$$I = \frac{1}{2}\int (t^2-1)dt = \frac{1}{2}\left(\frac{t^3}{3} - t\right) = \frac{1}{6}(\sec^3 2x - 3\sec 2x)$$

67. Put $\log\tan x = t$, then

$$\Rightarrow \frac{1}{\tan x}\sec^2 x dx = dt \Rightarrow \sec x \operatorname{cosec} x\, dx = dt$$

$$\int \frac{dt}{t} = \log t + c = \log(\log\tan x) + c$$

68. $$e^{\log\sin x} = \sin x$$

Therefore,

$$\int \cos^3 x \sin x\, dx$$

Put $\cos x = t$, we get

$$-\int t^3 dt = -\frac{t^4}{4} + c = -\frac{\cos^4 x}{4} + c$$

69. $$I = \int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx$$

Put $1 - \frac{1}{x^3} = t$, then

$$\frac{3}{x^4} dx = dt$$

Therefore,

$$I = \frac{1}{3}\int t^{\frac{1}{4}} dt$$

$$=\frac{1}{3}\cdot\frac{t^{5/4}}{5/4}+c=\frac{4}{15}\left(1-\frac{1}{x^3}\right)^{5/4}+c$$

70. $$I=\int\frac{\log\left(\frac{x+1}{x}\right)}{\frac{(x+1)}{x}}\cdot\frac{1}{x^2}dx$$

Now put $t=\frac{x+1}{x}=1+\frac{1}{x}$, then

$$dt=-\frac{1}{x^2}dx$$

Therefore,

$$I=-\int\log t\cdot\frac{1}{t}dt=-\frac{1}{2}(\log t)^2+c=-\frac{1}{2}\left[\log\left(\frac{x+1}{x}\right)\right]^2+c$$

71. $$I=\int e^{\sin x}\cos x\ dx$$

As,

$$1-\sin 2x=(\sin x-\cos x)^2$$

Hence,

$$\int e^{\sin x}\cos x dx$$

Put $\sin x=t$, then $\cos x\,dx=dt$.

$$\int e^t dt=e^t+c=e^{\sin x}+c$$

72. Multiplying above and below by x^2 and put

$$1-x^3=t^2\Rightarrow-3x^2dx=2t\,dt$$

Therefore,

$$I=\frac{2}{3}\int\frac{dt}{t^2-1}=\frac{2}{3}.\frac{1}{2}\log\frac{t-1}{t+1}+c$$

$$I=\frac{1}{3}\log\frac{t-1}{t+1}+c$$

Therefore,

$$a=\frac{1}{3}$$

73. Put $a^x=t$. Then

$$I=\int\frac{a^{a^t}\cdot a^t\ dt}{\log a}$$

Again put $a^t=z$, then

$$I=\int\frac{a^z\ dz}{(\log a)^2}=\frac{a^z}{(\log a)^3}+c=\frac{a^{a^{a^x}}}{(\log a)^3}+c$$

74. $e^{\log_5 x}=x^{\log_5 e}$ (By the property of exponential function)

$$\int e^{\log_5 x}dx=\int x^{\log_5 e}dx=\frac{x^{\log_5 e}+1}{\log_5 e+1}=\frac{x^{\log_5 5e}}{\log_5 5e}$$

75. $\int x^2\sin x dx=-x^2\cos x+\int 2x.\cos dx$

$$=-x^2\cos x+2(x\sin x-\int\sin x\,dx)$$

$$=-x^2\cos x+2x\sin x+2\cos x$$

76. Here, we have $\cos x$ but its differential coefficient, that is $-\sin x$, is not present in the numerator and as such we cannot make the substitution of $\cos x=t$, but we simply put $\cos x=t$ to split the integrand into the partial fractions.

$$\frac{1-\cos x}{\cos x(1+\cos x)}=\frac{1-t}{t(1+t)}$$

$$=\left(\frac{1}{t}-\frac{2}{1+t}\right)=\left(\frac{1}{\cos x}-\frac{2}{1+\cos x}\right)$$

Therefore,

$$I=\int\left(\frac{1}{\cos x}-\frac{2}{1+\cos x}\right)dx=\int\left(\sec x-\sec^2\frac{x}{2}\right)dx$$

$$=\log(\sec x+\tan x)-2\tan\left(\frac{x}{2}\right)$$

77. Put $(1-x)=t$.

Differentiating above equation, we have

$$-dx=dt$$

Now,

$$\int -dt\,t^{23}(1-t)=\int(t^{24}-t^{23})dt$$

$$=\frac{t^{25}}{25}-\frac{t^{24}}{24}+c=\frac{(1-x)^{25}}{25}-\frac{(1-x)^{24}}{24}+c$$

78. Put $\log x=t$. Then

$$dx=e^t\,dt\Rightarrow x=e^t$$

Now,

$$I=\int e^t(\sin t+\cos t)dt=e^t\sin t=x\sin(\log x)+c$$

79. $I=\int\frac{dx}{\cos^6 x+\sin^6 x}=\int\frac{\sec^6 x}{1+\tan^6 x}dx=\int\frac{(1+\tan^2 x)^2\sec^2 x\,dx}{1+\tan^6 x}$

If $\tan x=p$, then $\sec^2 x\,dx=dp$.

$$I=\frac{(1+p^2)^2dp}{1+p^6}=\int\frac{(1+p^2)}{p^4-p^2+1}dp=\int\frac{p^2\left(1+\frac{1}{p^2}\right)}{p^2\left(p^2+\frac{1}{p^2}-1\right)}dp$$

If $p-\frac{1}{p}=k$, then $\left(1+\frac{1}{p^2}\right)dp=dk$.

Therefore,

$$I=\int\frac{dk}{k^2+1}=\tan^{-1}(k)+c$$

$$=\tan^{-1}\left(p-\frac{1}{p}\right)+c$$

$$=\tan^{-1}(\tan x-\cot x)+c$$

80. $I=\int\frac{3+2\cos x}{(2+3\cos x)^2}dx$

Multiplying numerator and denominator by $\operatorname{cosec}^2 x$, we get

$$I=\int\frac{(3\operatorname{cosec}^2 x+2\cot x\operatorname{cosec} x)}{(2\operatorname{cosec} x+3\cot x)^2}dx$$

$$=-\int\frac{-3\operatorname{cosec}^2 x-2\cot x\operatorname{cosec} x}{(2\operatorname{cosec} x+3\cot x)^2}dx$$

$$=\frac{1}{2\operatorname{cosec} x+3\cot x}=\left(\frac{\sin x}{2+3\cos x}\right)+c$$

81. Substituting $x = p^6$, $dx = 6p^5\,dp$, we get

$$I = \int \frac{6p^5(p^6+p^4+p)}{p^6(1+p^2)}dp = \int \frac{6(p^5+p^3+1)}{(p^2+1)}dp$$

$$= \int 6p^3\,dp + \int\left(\frac{6}{p^2+1}\right)dp$$

$$= \frac{6p^4}{4} + 6\tan^{-1}p = \frac{3}{2}x^{\frac{2}{3}} + 6\tan^{-1}\left(x^{\frac{1}{6}}\right) + c$$

82.

$$e^x = t \Rightarrow e^x\,dx = dt$$

Now,

$$\int \frac{dt}{t\sqrt{1-t^2}}$$

Put $\sin^{-1}t = z$, we get

$$\frac{1}{\sqrt{1-t^2}}dt = dz = \int \frac{dz}{\sin z} = \int \operatorname{cosec} z\,dz = \ln\left|\tan\frac{z}{2}\right| + k$$

$$= \ln\left|\tan\left(\frac{\sin^{-1}e^x}{2}\right)\right| + k$$

83.

$$e^{\log x} = x$$

$$I = \int x\cos x\,dx + \int \sin x\cos x\,dx = x\sin x + \cos x - \frac{\cos^2 x}{2} + c$$

84.

$$I = \int \frac{x+\sin x}{1+\cos x}dx = \frac{1}{2}\int \sec^2\frac{x}{2}(x+\sin x)dx$$

$$I = \frac{1}{2}\int x\sec^2\frac{x}{2} + \frac{2}{2}\int \tan\frac{x}{2}dx$$

$$= \frac{1}{2}x\frac{\tan x/2}{1/2} - \frac{1}{2}\int\frac{\tan x/2}{1/2} + \int \tan x/2\,dx = x\tan x/2 + c$$

85.

$$I = \int \frac{1}{4\cos^3 x - 3\cos x}dx = \int \frac{1}{\cos 3x}dx$$

$$= \int \sec 3x\,dx = \frac{1}{3}\ln|\sec 3x + \tan 3x| + c$$

86.

$$I = \int \sin(\ln x)dx$$

Let $\ln x = t$. Then

$$x = e^t \Rightarrow dx = e^t\,dt$$

Therefore,

$$I = \int e^t\cdot\sin t\,dt$$

$$= \sin t\cdot e^t - \int \cos t\cdot e^t\,dt$$

$$= \sin t\cdot e^t - \cos t\cdot e^t - \int \sin t\cdot e^t\,dt$$

$$2I = e^t(\sin t - \cos t)$$

Therefore,

$$I = \frac{1}{2}e^t(\sin t - \cos t)$$

$$= \frac{1}{2}e^{\ln x}[\sin(\ln x) - \cos(\ln x)] + c$$

$$= \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + c$$

Practice Exercise 2

1. $I_{m,n} = \int \cos^m x\cos nx\,dx$

$$= \frac{\cos^m x\sin nx}{n} + \frac{m}{n}\int \cos^{m-1}x\sin x\sin nx\,dx$$

$$= \frac{\cos^m x\sin nx}{n} + \frac{m}{n}\int \cos^{m-1}x(\cos(n-1)x - \cos nx\cos x)dx$$

$$I_{m,n} = \frac{\cos^m x\sin nx}{n} + \frac{m}{n}\int \cos^{m-1}x\cos(n-1)x\,dx - \frac{m}{n}I_{m,n}$$

$$\Rightarrow \frac{m+n}{n}I_{m,n} = \frac{\cos^m x\sin nx}{n} + \frac{m}{n}I_{m-1,n-1} + c_1$$

or

$$(m+n)I_{m,n} = mI_{m-1,n-1} + \cos^m x\sin nx + c$$

2.

$$\int \frac{\sin^4 x}{x}dx = f(x)$$

$$\int \frac{\sin^4(p+q)x}{(p+q)x} = \frac{f((p+q)x)}{p+q}$$

$$\int \frac{\sin^4(p+q)dx}{x} = f((p+q)x)$$

3. $$I = n!\int dx - n!\int \frac{1+\frac{x}{1!}+\cdots+\frac{x^{n-1}}{(n-1)!}}{1+\frac{x}{1!}+\cdots+\frac{x^n}{n!}}dx$$

$$\Rightarrow I = n!\ln\left(\frac{e^x}{1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}\right) + c$$

4. $$f(x) = \int \frac{x^2\,dx}{(1+x^2)(1+\sqrt{1+x^2})} = \int \frac{x^2(\sqrt{1+x^2}-1)dx}{(1+x^2)x^2}$$

$$f(x) = \int \frac{dx}{\sqrt{1+x^2}} - \int \frac{dx}{1+x^2} = \ln\left|x+\sqrt{1+x^2}\right| - \tan^{-1}x + c$$

$$f(0) = 0 = c$$

$$f(1) = \ln\left|1+\sqrt{2}\right| - \frac{\pi}{4}$$

$$g(x) = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{1+x^4}} = \frac{1}{2}\ln\left|x^2+\sqrt{1+x^4}\right| + c$$

$$g(0) = 0 = c$$

$$g(1) = \frac{1}{2}\ln\left|1+\sqrt{2}\right|$$

5.

$$I = \int x.\sqrt[4]{x^6+x^4+x^{-4}}\frac{(3x^{10}+2x^8-2)}{x^6}dx$$

$$= \int \sqrt[4]{x^6+x^4+x^{-4}}(3x^5+2x^3-2x^{-5})dx$$

Put $x^6+x^4+x^{-4} = t$. Then

$$2(3x^5+2x^3-2x^{-5})dx = dt$$

$$I = \frac{1}{2}\int \sqrt[4]{t}\,dt = \frac{2(t)^{5/4}}{5} + c = \frac{2(x^{10}+x^8+1)^{5/4}}{5x^5} + c$$

6. $\int (x^6+x^4)\sqrt{2x^4+3x^2}\,dx = \int (x^5+x^3)\sqrt{2x^6+3x^4}\,dx$

$$=\frac{1}{18}x^6(2x^2+3)^{\frac{3}{2}}+c$$

7. $I=\int \frac{x^4-1}{(x^5+1)(x+1)}dx=\int\frac{(x+1)x^4-(x^5+1)}{(x^5+1)(x+1)}dx$

$$I=\frac{1}{5}\int\frac{5x^4}{(x^5+1)}dx-\int\frac{1}{(x+1)}dx=\ln\left(\frac{(x^5+1)^{1/5}}{(x+1)}\right)+c$$

8. $f(x)=e^x\sin x+\cos x$

$$f'(x)=e^x\cos x+\sin x\,e^x-\sin x$$
$$g(x)=e^x\cos x-\sin x$$
$$g'(x)=\cos x\cdot e^x-e^x\sin x-\cos x$$

Now

$f(x)\cdot g'(x)=(e^x\sin x+\cos x)(\cos x\cdot e^x-e^x\sin x-\cos x)$
$=e^{2x}\sin x\cos x-e^{2x}\sin^2 x-e^x\sin x\cos x+e^x\cos^2 x-e^x\sin x\cos x-\cos^2 x$

and

$g(x)\cdot f'(x)=(e^x\cos x-\sin x)(e^x\cos x+\sin x\,e^x-\sin x)$
$=e^{2x}\cos^2 x+e^{2x}\sin x\cos x-e^x\sin x\cos x-e^x\sin x\cos x-e^x\sin^2 x+\sin^2 x$

$$g(x)\cdot f'(x)-f(x)\cdot g'(x)=e^{2x}-e^x+1$$

$$I=\int\frac{f(x)g'(x)-g(x)f'(x)}{f(x)g(x)}\,dx=\ln\left|\frac{f(x)}{g(x)}\right|+c$$
$$=\ln\left(\frac{e^x\cos x-\sin x}{e^x\sin x+\cos x}\right)+c$$

9. Putting $\sin\theta=t$, we get

$$\int(4t-4t^3)e^t=\left(At^3+B(1-t^2)+Ct+D\sqrt{1-t^2}+E\right)e^t+F$$

It follows immediately that $D=0$.

Differentiating both sides w.r.t. t, we get

$$(4t-4t^3)e^t=[At^3+(3A-B)t^2+(C-2B)t+C+B+E]e^t$$

And hence, $A=-4$, $B=-12$, $C=-20$.

10. $F(x)=\int\frac{1}{4-3\cos^2 x+5\sin^2 x}dx=\int\frac{1}{9-8\cos^2 x}dx$

$$=\int\frac{\sec^2 x}{9\sec^2 x-8}dx=\int\frac{\sec^2 x}{1+9\tan^2 x}dx=\frac{1}{3}\tan^{-1}(3\tan x)+c$$
$$\Rightarrow g(x)=3\tan x$$

Therefore,

$$g\left(\frac{\pi}{4}\right)=3$$

and $$g'\left(\frac{\pi}{3}\right)=12$$

11. $f'(x)=b^2+(a-1)b+2-\sin^2 x-\cos^4 x$

$$f'(x)=b^2+(a-1)b+2-1$$

For increasing function, $f'(x)>0$, so

$$D<0$$
$$\Rightarrow (a-1)^2-4<0$$
$$\Rightarrow a\in(-1,3)$$

12. $$I_1=\int x\sqrt{a^2-x^2}\,dx$$

Put $a^2-x^2=t^2\Rightarrow x\,dx=-t\,dt$

Therefore,

$$I_1=-\int t\cdot t\cdot dt=-\frac{t^3}{3}+c=-\frac{(a^2-x^2)^{3/2}}{3}+c$$

13. $I_n=\int x^n\sqrt{a^2-x^2}\,dx=\int x^{n-1}(x\sqrt{a^2-x^2})\,dx$

$$=x^{n-1}\left[-\frac{1}{3}(a^2-x^2)^{3/2}\right]+\frac{n-1}{3}\int x^{n-2}(a^2-x^2)\sqrt{a^2-x^2}\,dx$$
$$=-\frac{1}{3}x^{n-1}(a^2-x^2)^{3/2}+\frac{n-1}{3}a^2 I_{n-2}-\frac{n-1}{3}I_n$$
$$\Rightarrow\left(1+\frac{n-1}{3}\right)I_n=-\frac{1}{3}x^{n-1}(a^2-x^2)^{3/2}+\frac{a^2(n-1)}{3}I_{n-2}$$
$$\Rightarrow I_n=\frac{-x^{n-1}(a^2-x^2)^{3/2}}{n+2}+\frac{a^2(n-1)}{n+2}I_{n-2}$$

14. **(A)** Put $f_{101}(x)=t$. Then $\frac{1}{xf_1(x)f_2(x)\cdots f_{100}(x)}dx=dt$.

Therefore, given integral $=f_{101}(x)+c$.

(B) $f(x)=\int\frac{x+\sin x}{1+\cos x}dx=\int\left(\frac{x}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx=x\tan\frac{x}{2}+c$

(C) $f(x)=\int e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx$

$$=\int e^{\sin^{-1}x}\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}\right)dx$$
$$f(x)=e^{\sin^{-1}x}\sqrt{1-x^2}+c\Rightarrow f(0)=1+c\Rightarrow c=0$$
$$f\left(\frac{1}{2}\right)=\frac{\sqrt{3}\,e^{\pi/6}}{2}$$

(D) $f(x)=\int\left(\frac{\sqrt{\tan x}}{\sin x\cdot\cos x}\right)dx$

$$=\int(\tan x)^{-1/2}\sec^2 x\,dx=2\sqrt{\tan x}+c$$
$$f(0)=c=0\Rightarrow c=0$$
$$f\left(\frac{\pi}{4}\right)=2=\frac{2k}{\pi}\Rightarrow k=\pi$$

Solved JEE 2017 Questions

JEE Main 2017

1. Let $I_n = \int \tan^n x\, dx, (n > 1)$. If $I_4 + I_6 = a \tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to:

(A) $\left(\frac{1}{5}, 0\right)$ **(B)** $\left(\frac{1}{5}, -1\right)$

(C) $\left(-\frac{1}{5}, 0\right)$ **(D)** $\left(-\frac{1}{5}, 1\right)$

(OFFLINE)

Solution: We have

$$I_n = \int \tan^n x\, dx$$

$$I_4 + I_6 = \int (\tan^4 x + \tan^6 x) dx$$

$$= \int \tan^4 x(1 + \tan^2 x) dx$$

$$= \int \tan^4 x \cdot \sec^2 x dx$$

Substituting $t = \tan x$, we get

$$I_4 + I_6 = \int t^4 \cdot dt = \frac{t^5}{5} + c$$

That is,

$$I_4 + I_6 = \frac{\tan^5 x}{5} + c$$

On comparison, we get $a = \frac{1}{5}, b = 0$.

Hence, the correct answer is option (A).

23 Definite Integration

23.1 Definition

If $f(x)$ is a continuous function on $[a, b]$ and $F(x)$ is any anti-derivative of $f(x)$ on $[a, b]$, that is,

$$\frac{d}{dx}(F(x)) = f(x)\ \forall\ x \in (a,b),$$

then

$$\int_a^b f(x)dx = F(b) - F(a)$$

is called definite integration with limits a and b, where a is called the lower limit and b is called the upper limit of the integral. This formula is known as Newton-Leibnitz formula.

23.2 Geometrical Meaning of Definite Integration

If $f(x) > 0$ for all $x \in [a, b]$; then $\int_a^b f(x)$ is numerically equal to the area bounded by the curve $y = f(x)$, x-axis and the straight lines $x = a$ and $x = b$.

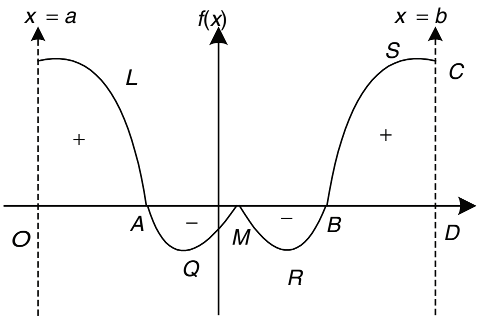


Figure 23.1

In general, $\int_a^b f(x)\,dx$ represents algebraic sum of the figures bounded by the curve (Fig. 23.1) $y = f(x)$, the x-axis and the straight line $x = a$ and $x = b$. The areas above x-axis are taken with plus sign and the areas below the x-axis are taken with minus sign.

That is,

$$\int_a^b f(x)\,dx = \text{area } OLA - \text{area } AQM - \text{area } MRB + \text{area } BSCD$$

Illustration 23.1 Evaluate $\int_0^{\pi} \sin^2 x\, dx$.

Solution:

$$I = \frac{1}{2}\int_0^{\pi} 2\sin^2 x\, dx = \frac{1}{2}\int_0^{\pi} [1 - \cos 2x]\,dx$$

$$\Rightarrow I = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\Rightarrow I = \frac{1}{2}[\pi] = \frac{\pi}{2}$$

Illustration 23.2 Evaluate $\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$.

Solution: Put $\log_e x = t$, then $e^t = x$.

Therefore, $dx = e^t dt$ and limits are adjusted as –1 to 2.

$$I = \int_{-1}^{2} \left|\frac{t}{e^t}\right| e^t dt = \int_{-1}^{2} |\,t\,|\, dt$$

$$\Rightarrow\ I = \int_{-1}^{0} -t\, dt + \int_0^2 t\, dt$$

$$\Rightarrow\ I = \left[\frac{-t^2}{2}\right]_{-1}^{0} + \left[\frac{t^2}{2}\right]_0^2 \Rightarrow I = \frac{5}{2}$$

Illustration 23.3 Evaluate $\int_0^{\pi/2} \frac{dx}{1 + \sin x}$.

Solution:

$$I = \int_0^{\pi/2} \frac{dx}{\sin^2 x/2 + \cos^2 x/2 + 2\sin x/2 \cos x/2}$$

$$I = \int_0^{\pi/2} \frac{dx}{(\sin x/2 + \cos x/2)^2} = \int_0^{\pi/2} \frac{\sec^2 x/2}{(1 + \tan x/2)^2} dx$$

Put $(1 + \tan x/2) = t$. Then

$$\frac{1}{2}\sec^2 x/2\, dx = dt$$

$$I = 2\int_1^2 \frac{dt}{t^2} = -2\left[\frac{1}{t}\right]_1^2 = -2\left[\frac{1}{2} - \frac{1}{1}\right] = 1$$

Illustration 23.4 Let $\alpha = \int_0^{\infty} \frac{dx}{x^4 + 7x^2 + 1}$ and $\beta = \int_0^{\infty} \frac{x^2 dx}{x^4 + 7x^2 + 1}$.

Then show that $\alpha = \beta$.

Solution:

$$\alpha = \int_0^{\infty} \frac{dx}{x^4 + 7x^2 + 1}$$

Put $x = 1/t$, then

$$dx = -1/t^2$$

$$\alpha = \int_{\infty}^{0} \frac{-\frac{1}{t^2}dt}{\frac{1}{t^4}+\frac{7}{t^2}+1} = \int_{0}^{\infty} \frac{dt\, t^2}{t^4+7t^2+1} = \int_{0}^{\infty} \frac{t^2 dt}{t^4+7t^2+1} = \beta$$

23.3 Definite Integration as the Limit of Sum

Let $f(x)$ be a single valued-continuous function defined in the interval $a \le x \le b$, where a and b are both finite. Let this interval be divided into n equal sub-intervals, each of width h by inserting $(n - 1)$ points $a+h, a+2h, a+3h, \cdots, a+(n-1)h$ between a and b. Then

$$nh = b - a$$

Now, we form the sum

$$hf(a)+hf(a+h)+hf(a+2h)+\cdots+hf(a+rh)+\cdots+hf[a+(n-1)h]$$

$$= h[f(a)+f(a+h)+f(a+2h)+\cdots\cdots+f(a+rh)+\cdots\cdots+f\{a+(n-1)h\}]$$

$$= h\sum_{r=0}^{n-1} f(a+rh),$$

where

$$a+nh = b \Rightarrow nh = b-a$$

The $\lim\limits_{h\to 0} h\sum\limits_{r=0}^{n-1} f(a+rh)$, if it exists, is called the **definite integral** of $f(x)$ with respect to x between the limits a and b, and we denote it by the symbol $\int_a^b f(x)\,dx$. Thus,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[\,f(a)+f(a+h)+f(a+2h)+\cdots+f\{a+(n-1)h\}]$$

$$\Rightarrow \int_a^b f(x)\,dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh),$$

where $nh = b-a$, a and b being the limits of integration.

The process of evaluating a definite integral by using the above definition is called integration from the first principle or integration as the limit of a sum.

Illustration 23.5 Evaluate $\int_a^b x^3\,dx$ by first principle.

Solution:

$$I = \int_a^b x^3\,dx$$

$$I = \lim_{h\to 0} h\sum_{r=0}^{n-1} (a+rh)^3$$

$$I = \lim_{h\to 0} h\{(a)^3+(a+h)^3+(a+2h)^3+\cdots+(a+(n-1)h)^3\}$$

$$= \lim_{h\to 0} h\left\{\begin{matrix} na^3+(1^3+2^3+\cdots+(n-1)^3)h^3+3ah((a+h) \\ +2(a+2h)+3(a+3h)+\cdots+(n-1)(a+(n-1)h)) \end{matrix}\right\}$$

$$= \lim_{h\to 0} h\left[\begin{matrix} na^3+(1^3+2^3+\cdots+(n-1)^3)h^3+3ah\{a(1+2+ \\ 3+\cdots+(n-1))+h(1^2+2^2+3^2+\cdots+(n-1)^2)\} \end{matrix}\right]$$

$$= \lim_{h\to 0}\left[nha^3+\left(\frac{n(n-1)}{2}\right)^2 h^4+3a\left\{a\frac{n(n-1)h^2}{2}+\frac{n(n-1)(2n-1)h^3}{6}\right\}\right]$$

$$= \lim_{h\to 0}\left[\begin{matrix} (nh)a^3+\left(\dfrac{nh(nh-h)}{2}\right)^2 \\ 3a\left\{a\dfrac{nh(nh-h)}{2}+\dfrac{nh(nh-h)(2nh-h)}{6}\right\} \end{matrix}\right]$$

$$= (b-a)a^3+\left(\frac{(b-a)^2}{2}\right)^2+3a\left\{a\frac{(b-a)^2}{2}+\frac{2(b-a)^3}{6}\right\}$$

$$= \frac{b-a}{4}(a^3+b^3+ab^2+a^2b)$$

$$= \frac{b^4-a^4}{4}$$

Illustration 23.6 Evaluate $\int_a^b \sin x\,dx$ by first principle.

Solution:

$$I = \int_a^b \sin x\,dx$$

$$I = \lim_{h\to 0} h\sum_{r=0}^{n-1} \sin(a+rh)$$

$$= \lim_{h\to 0} h\{\sin a+\sin(a+h)+\sin(a+2h)+\cdots+\sin(a+(n-1)h)\}$$

$$= \lim_{h\to 0} \frac{2\sin\frac{nh}{2}\sin\left(a+\frac{(n-1)h}{2}\right)\left(\frac{h}{2}\right)}{\sin\frac{h}{2}}$$

$$= 2\sin\left(\frac{b+a}{2}\right)\sin\left(\frac{b-a}{2}\right)$$

$$= \cos a - \cos b$$

23.4 Properties of Definite Integration

1. Change of variable of integration is immaterial so long as limits of integration remain the same, that is,

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

Illustration 23.7 Evaluate $\int_0^1 \ln(x+1)dx$.

Solution:

$$I = \int_0^1 \ln(x+1)dx = (x+1)\ln(x+1)-(x+1)\big|_0^1$$

$$\Rightarrow I = 2\ln 2 - 1$$

Illustration 23.8 Evaluate $\int_0^1 \frac{1}{x+1}dx$.

Solution:

$$I = \int_0^1 \frac{1}{x+1}dx$$

$$I = \ln(x+1)\big|_0^1 = \ln 2$$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

That is, by the interchange in the limits of definite integral, the sign of the integral is changed.

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$

where ($a<c<b$)

Generally, we break the limit first at the points where $f(x)$ is discontinuous and second at the points where definition of $f(x)$ changes.

Or

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_n}^{b} f(x)dx \text{ where}$$

$$(a<c_1<c_2<\cdots c_n<b)$$

Generally, this property is used when the integrand has two or more rules in the integration interval.

Proof: Let $\int f(x)dx = A(x)+k$

$$\int_a^c f(x) + \int_c^b f(x) = (A(x)+k)|_a^c + (A(x)+k)|_c^b$$

$$= A(c) - A(a) + A(b) - A(c)$$

$$= A(b) - A(a)$$

Illustration 23.9 Evaluate $\int_0^{\frac{5\pi}{12}} [\tan x]dx$, where [.] is the greatest integer function.

Solution: Let

$$I = \int_0^{\frac{5\pi}{12}} [\tan x]dx$$

Value of $\tan x$ at $x = \frac{5\pi}{12}$ is $2+\sqrt{3}$.

Value of $\tan x$ at $x=0$ is 0.

Integers between 0 and $2+\sqrt{3}$ are 1, 2 and 3. Therefore,

$$\tan x = 1,$$

$$\tan x = 2,$$

and

$$\tan x = 3$$

$$\Rightarrow x = \tan^{-1} 1,$$

$$x = \tan^{-1} 2,$$

and

$$x = \tan^{-1} 3$$

Therefore,

$$I = \int_0^{\tan^{-1}1} [\tan x]dx + \int_{\tan^{-1}1}^{\tan^{-1}2} [\tan x]dx + \int_{\tan^{-1}2}^{\tan^{-1}3} [\tan x]dx + \int_{\tan^{-1}3}^{\frac{5\pi}{12}} [\tan x]dx$$

$$= \int_0^{\tan^{-1}1} 0\,dx + \int_{\tan^{-1}1}^{\tan^{-1}2} 1dx + \int_{\tan^{-1}2}^{\tan^{-1}3} 2dx + \int_{\tan^{-1}3}^{\frac{5\pi}{12}} 3dx$$

$$= 0 + (\tan^{-1}2 - \tan^{-1}1) + 2(\tan^{-1}3 - \tan^{-1}2) + 3\left(\frac{5\pi}{12} - \tan^{-1}3\right)$$

$$= \frac{5\pi}{4} - \frac{\pi}{4} - \tan^{-1}3 - \tan^{-1}2$$

$$= \pi - \left[\tan^{-1}\left(\frac{3+2}{1-6}\right) + \pi\right] = -\tan^{-1}(-1)$$

$$= \frac{\pi}{4}$$

Illustration 23.10 Evaluate $\int_{-2}^{2} |1-x^2|\,dx$

Solution:

$$I = \int_{-2}^{2} |1-x^2|\,dx = \int_{-2}^{-1} |1-x^2|\,dx + \int_{-1}^{1} |1-x^2|\,dx + \int_{1}^{2} |1-x^2|\,dx$$

$$\Rightarrow I = -\int_{-2}^{-1} (1-x^2)dx + \int_{-1}^{1} (1-x^2)dx - \int_{1}^{2} (1-x^2)dx$$

$$\Rightarrow I = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$$

Illustration 23.11 Evaluate $\int_0^{1.5} [x^2]dx$, where [.] denotes the greatest integer function.

Solution:

$$I = \int_0^{1.5} [x^2]dx = \int_0^{1} [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{1.5} [x^2]dx$$

$$\Rightarrow I = 0 + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{1.5} 2\,dx = \sqrt{2} - 1 + 3 - 2\sqrt{2}$$

$$\Rightarrow I = 2 - \sqrt{2}$$

Illustration 23.12 Evaluate $\int_0^{2\pi} \frac{|\cos x|}{\cos x} dx$

Solution: Let

$$I = \int_0^{2\pi} \frac{|\cos x|}{\cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x} dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos x}{\cos x} dx + \int_{\frac{3\pi}{2}}^{2\pi} \frac{\cos x}{\cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} 1dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 1dx + \int_{\frac{3\pi}{2}}^{2\pi} 1dx$$

$$= \frac{\pi}{2} - \pi + \frac{\pi}{2} = 0$$

Illustration 23.13 Evaluate $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x\,dx}$

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x\,dx} = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x(1-\cos^2 x)dx}$$

$$I = \int_{-\pi/2}^{\pi/2} |\sin| \sqrt{\cos x}\,dx = -\int_{-\pi/2}^{0} \sin x\sqrt{\cos x}\,dx + \int_0^{\pi/2} \sin x\sqrt{\cos x}\,dx$$

Put $\cos x = z$, then $\sin x\,dx = -dz$

$$I = \int_0^1 \sqrt{z}\,dz - \int_1^0 \sqrt{z}\,dz = 2\int_0^1 \sqrt{z}\,dz \qquad \left(\text{As} \int_a^b f(x) = -\int_b^a f(x)\right)$$

$$I = 2\cdot\frac{2}{3} z^{3/2}\Big|_0^1 = \frac{4}{3}$$

4. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$. In particular $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof: Let

$$I=\int_a^b f(a+b-x)dx$$

$$a+b-x=z \Rightarrow dx=-dz$$

$$x=a \Rightarrow z=b, x=b \Rightarrow z=a$$

$$I=-\int_b^a f(z)dz \Rightarrow I=\int_a^b f(z)dz$$

$$I=\int_a^b f(x)dx$$

(A) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx=\int_0^{\pi/2} f(\sin 2x)\cos x\,dx$

(B) $\int_0^{\pi/2} f(\sin x)dx=\int_0^{\pi/2} f(\cos x)dx$

(C) $\int_0^{\pi/2} f(\tan x)dx=\int_0^{\pi/2} f(\cot x)dx$

(D) $\int_0^{1} f(\ln x)\,dx=\int_0^{1} f[\ln(1-x)]dx$

Illustration 23.14 Evaluate $\int_0^{\pi/2}\frac{\sin^n x}{\sin^n x+\cos^n x}dx.$

Solution:

$$I=\int_0^{\pi/2}\frac{\sin^n x}{\sin^n x+\cos^n x}dx$$

$$I=\int_0^{\pi/2}\frac{\cos^n x}{\sin^n x+\cos^n x}dx$$

$$2I=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

Similarly we can solve these examples:

(A) $\int_0^{\pi/2}\frac{\tan^n x}{1+\tan^n x}dx=\int_0^{\pi/2}\frac{\cot^n x}{1+\cot^n x}dx=\frac{\pi}{4}$

(B) $\int_0^{\pi/2}\frac{1}{1+\tan^n x}dx=\int_0^{\pi/2}\frac{1}{1+\cot^n x}dx=\frac{\pi}{4}$

(C) $\int_0^{\pi/2}\frac{\sec^n x}{\sec^n x+\mathrm{cosec}^n x}dx=\int_0^{\pi/2}\frac{\mathrm{cosec}^n x}{\mathrm{cosec}^n x+\sec^n x}dx=\frac{\pi}{4}$

Illustration 23.15 Evaluate $\int_0^{\pi/2}\ln(\tan x)dx.$

Solution:

$$I=\int_0^{\pi/2}\ln(\tan x)dx$$

$$I=\int_0^{\pi/2}\ln(\cot x)dx$$

$$2I=\int_0^{\pi/2}(\ln(\tan x)+\ln(\cot x))dx=\int_0^{\pi/2}\ln(\tan x\cdot \cot x)\,dx$$

$$2I=0\Rightarrow I=0$$

Illustration 23.16 Evaluate $\int_0^{\pi/4}\ln(1+\tan x)dx.$

Solution:

$$I=\int_0^{\pi/4}\ln(1+\tan x)dx$$

$$\Rightarrow I=\int_0^{\pi/4}\ln\left(\frac{2}{1+\tan x}\right)dx$$

$$\Rightarrow I=\int_0^{\pi/4}\ln 2\,dx-\int_0^{\pi/4}\ln(1+\tan x)dx$$

$$\Rightarrow 2I=\frac{\pi}{4}\ln 2\Rightarrow I=\frac{\pi}{8}\ln 2$$

Illustration 23.17 Evaluate $\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx.$

Solution:

$$I=\int_0^{\pi/2}\frac{a\sin x+b\cos x}{\sin x+\cos x}dx$$

$$I=\int_0^{\pi/2}\frac{a\sin\left(\frac{\pi}{2}-x\right)+b\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx\Rightarrow I=\int_0^{\pi/2}\frac{a\cos x+b\sin x}{\sin x+\cos x}dx$$

$$2I=\int_0^{\pi/2}(a+b)\frac{\cos x+\sin x}{\cos x+\sin x}dx\Rightarrow 2I=\frac{\pi}{2}(a+b)$$

$$I=\frac{\pi}{4}(a+b)$$

Illustration 23.18 Evaluate $\int_0^1 (x-1)(1-x)^{99}\,dx.$

Solution:

$$I=\int_0^1 (x-1)(1-x)^{99}dx$$

$$I=\int_0^1(-x)(x)^{99}dx\Rightarrow I=-\int_0^1 x^{100}dx$$

$$I=-\left.\frac{x^{101}}{101}\right|_0^1=-\frac{1}{101}$$

Illustration 23.19 Evaluate $\int_0^{\pi} e^{\sin^2 x}\cos^3 x\,dx.$

Solution:

$$I=\int_0^{\pi} e^{\sin^2 x}\cos^3 x\,dx$$

$$I=\int_0^{\pi} e^{\sin^2(\pi-x)}\cos^3(\pi-x)\,dx$$

$$I=-\int_0^{\pi} e^{\sin^2 x}\cos^3 x\,dx\Rightarrow I=-I\Rightarrow I=0$$

Illustration 23.20 Evaluate $\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx.$

Solution:

$$I=\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx=\int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f(x)}dx$$

$$2I=\int_0^{2a}\frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}dx=\int_0^{2a}dx=[x]_0^{2a}=2a$$

$$\Rightarrow I=a$$

Illustration 23.21 Evaluate $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx.$

Solution:

$$I=\int_0^{\pi/2} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx \Rightarrow I=\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$$

$$I=\int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}} dx \Rightarrow I=\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$$

$$2I=\int_0^{\pi/2} 1dx \Rightarrow I=\frac{\pi}{4}$$

Illustration 23.22 Evaluate $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}.$

Solution:

$$I=\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$$

$$I=\int_{\pi/4}^{3\pi/4} \frac{1}{1-\cos x} dx \quad \left[\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}-x\right)=-\cos x\right]$$

$$2I=\int_{\pi/4}^{3\pi/4} \frac{2}{1-\cos^2 x} dx = \int_{\pi/4}^{3\pi/4} \frac{2}{\sin^2 x} dx$$

$$\Rightarrow 2I=2\int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x\, dx$$

$$\Rightarrow 2I=-2[\cot x]_{\pi/4}^{3\pi/4}=4 \Rightarrow I=2$$

Illustration 23.23 Evaluate $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx.$

Solution:

$$I=\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$$

$$I=\int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}} dx$$

$$2I=\int_2^3 1dx \Rightarrow 2I=[x]_2^3=1$$

$$\Rightarrow I=1/2$$

Illustration 23.24 Evaluate $\int_0^1 \ln\left(\frac{1}{x}-1\right) dx.$

Solution:

$$I=\int_0^1 \ln\left(\frac{1-x}{x}\right) dx=\int_0^1 \ln\frac{x}{1-x} dx=-\int_0^1 \ln\frac{1-x}{x} dx$$

$$\Rightarrow I=-I \Rightarrow I=0$$

5. $\int_0^{2a} f(x)dx=\int_0^a (f(x)+f(2a-x))dx$

Special case: $\int_0^{2a} f(x)dx=\begin{cases}0, & \text{if } f(2a-x)=-f(x)\\ 2\int_0^a f(x)dx, & \text{if } f(2a-x)=f(x)\end{cases}$

Proof: Let

$$I=\int_0^{2a} f(x)dx=\int_0^a f(x)dx+\int_a^{2a} f(x)dx$$

$$I=I_1+I_2$$

$$I_2=\int_a^{2a} f(x)dx$$

Put

$$x=2a-t \Rightarrow dx=-dt$$

$$x=a \Rightarrow t=a, x=2a \Rightarrow t=0$$

$$I_2=-\int_a^0 f(2a-t)dt=\int_0^a f(2a-t)dt$$

$$I_2=\int_0^a f(2a-x)dx$$

$$I=\int_0^a (f(x)+f(2a-x))dx$$

Illustration 23.25 Evaluate $\int_0^{2\pi} \frac{x\cdot \sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx\ x>0.$

Solution:

$$I=\int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}(2\pi-x)}{\sin^{2n}(2\pi-x)+\cos^{2n}(2\pi-x)} dx=\int_0^{2\pi} \frac{(2\pi-x)\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x}$$

$$\Rightarrow 2I=\int_0^{2\pi} \frac{2\pi\cdot \sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx$$

$$\Rightarrow 2I=4\pi\int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx$$

$$\Rightarrow I=4\pi\int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x}$$

$$\Rightarrow I=4\pi\int_0^{\pi/2} \frac{\cos^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx \left(\text{As} \int_0^a f(x)_2 \int_0^a f(a-x)\right)$$

$$\Rightarrow 2I=4\int_0^{\pi/2} \pi dx=2\pi^2 \Rightarrow I=\pi^2$$

Illustration 23.26 Evaluate $\int_0^{\pi} x\ \ln\sin x\, dx.$

Solution:

$$I=\int_0^{\pi} x\ln\sin x\, dx$$

$$I=\int_0^{\pi} (\pi-x)\ln\sin(\pi-x)dx=\int_0^{\pi} (\pi-x)\ln\sin x\, dx$$

$$2I=\pi\int_0^{\pi} \ln\sin x\, dx=2\pi\int_0^{\pi/2} \ln\sin x\, dx$$

$$I=\pi\int_0^{\pi/2} \ln\sin x\, dx=\pi\int_0^{\pi/2} \ln\cos x\, dx$$

$$2I=\pi\int_0^{\pi/2} \ln(\sin\cdot\cos x)dx=\pi\int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right)dx$$

$$2I = \pi\int_0^{\pi/2} \ln\sin 2x\,dx - \pi\int_0^{\pi/2} \ln 2\,dx$$

Put $2x = t$, then $2dx = dt$.

$$2I = \frac{\pi}{2}\int_0^{\pi} \ln\sin t\,dt - \frac{\pi^2}{2}\ln 2$$

$$2I = \pi\int_0^{\pi/2} \ln\sin t\,dt - \frac{\pi^2}{2}\ln 2$$

$$2I = I - \frac{\pi^2}{2}\ln 2 \Rightarrow I = -\frac{\pi^2}{2}\ln 2$$

Illustration 23.27 Evaluate $\int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx, (n\in I)$.

Solution:

$$I = \int_0^{\pi} e^{\cos^2(\pi-x)}\cdot\cos^3(2n+1)(\pi-x)dx$$

$$I = -\int_0^{\pi} e^{\cos^2 x}\cdot\cos^3(2n+1)x\,dx \quad \Rightarrow I = -I$$

$$\Rightarrow 2I = 0 \quad \Rightarrow I = 0$$

6. $\int_{-a}^{a} f(x)dx = \int_0^{a}((f(x)+f(-x))dx$

Special case:

$$\int_0^{a} f(x)dx = \begin{cases} 2\int_0^{a} f(x)dx, & \text{if } f(x) \text{ is even funciton } f(-x)=f(x). \\ 0, & \text{if } f(x) \text{ is odd function } f(-x)=-f(x). \end{cases}$$

Proof: Let

$$I = \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^{a} f(x)dx$$

$$I = I_1 + I_2$$

$$I_1 = \int_{-a}^{0} f(x)dx$$

Put

$$x = -t \Rightarrow dx = -dt$$

$$x = -a \Rightarrow t = a,\ x = 0 \Rightarrow t = 0$$

$$I_1 = -\int_a^{0} f(-t)dt = \int_0^{a} f(-t)dt$$

$$I_1 = \int_0^{a} f(-x)dx$$

$$I = \int_0^{a}(f(x)+f(-x))\,dx$$

Illustration 23.28 Evaluate $\int_{-4}^{4}\frac{x^2}{(x^2+16)(1+e^{x^5})}dx$

Solution:

$$I = \int_{-4}^{4}\frac{x^2}{(x^2+16)\left(1+e^{x^5}\right)}dx$$

$$I = \int_{-4}^{4}\left\{\left(\frac{x^2}{(x^2+16)(1+e^{x^5})} + \frac{x^2e^{x^5}}{(x^2+16)(1+e^{x^5})}\right)\right\}dx$$

$$I = \int_{-4}^{4}\frac{x^2}{x^2+16}dx$$

$$I = 2\int_0^{4}\frac{x^2}{x^2+16}dx \Rightarrow I = 2\int_0^{4}\frac{x^2+16-16}{x^2+16}dx$$

$$I = 2\left(x - \frac{16}{4}\tan^{-1}\frac{x}{4}\right)\Bigg|_0^4 = 2\left(4 - 4\cdot\frac{\pi}{4}\right) = 8 - 2\pi$$

Illustration 23.29 Evaluate $\int_0^{\pi}\frac{dx}{1+5^{\cos x}} + \int_{-2}^{2}\ln\left(\frac{5-x}{5+x}\right)dx.$

Solution: Let

$$I_1 = \int_0^{\pi}\frac{dx}{1+5^{\cos x}} \tag{1}$$

Now

$$I_1 = \int_0^{\pi}\frac{dx}{1+5^{\cos(\pi-x)}} = \int_0^{\pi}\frac{dx}{1+5^{-\cos x}} = \int_0^{\pi}\frac{5^{\cos x}dx}{5^{\cos x}+1} \tag{2}$$

Adding Eqs. (1) and (2), we get

$$2I_1 = \int_0^{\pi}\frac{dx}{1+5^{\cos x}} + \int_0^{\pi}\frac{5^{\cos x}dx}{5^{\cos x}+1} = \int_0^{\pi}1\cdot dx = \pi$$

$$I_1 = \pi/2$$

Consider

$$I_2 = \int_{-2}^{2}\log\left(\frac{5-x}{5+x}\right)dx$$

Let

$$g(x) = \log\left(\frac{5-x}{5+x}\right)$$

Now

$$g(-x) = \log\left(\frac{5-(-x)}{5+(-x)}\right) = -\log\frac{5-x}{5+x} = -g(x)$$

Therefore, $g(x)$ is an odd function.
Therefore,

$$\int_{-2}^{2} g(x)dx = 0 \Rightarrow I_2 = 0$$

$$I = I_1 + I_2 = \pi/2 + 0 = \pi/2$$

Illustration 23.30 Evaluate $\int_{\log 1/2}^{\log 2}\sin\left(\frac{e^x-1}{e^x+1}\right)dx.$

Solution:

$$f(-x) = \sin\left(\frac{e^{-x}-1}{e^{-x}+1}\right) = -\sin\frac{e^x-1}{e^x+1} = -f(x)$$

$$\int_{-\log 2}^{\log 2}\sin\left(\frac{e^x-1}{e^x+1}\right)dx = 0$$

Illustration 23.31 Evaluate $\int_{-1}^{1}\frac{x^3+|x|+1}{x^2+2|x|+1}dx.$

Solution:

$$I = \int_{-1}^{1}\frac{x^3}{x^2+2|x|+1}dx + \int_{-1}^{1}\frac{|x|+1}{x^2+2|x|+1}dx$$

$$= 0 + 2\int_0^1 \frac{x+1}{x^2+2x+1}dx = \ln(x^2+2x+1)\Big|_0^1$$

$$= \ln 4$$

Illustration 23.32 If $I_1 = \int_0^{\pi/2} \ln(\sin x)dx$ and $I_2 = \int_{-\pi/4}^{\pi/4} \ln(\sin x + \cos x)dx$, then find $\frac{I_1}{I_2}$.

Solution:

$$I_2 = \int_{-\pi/4}^{\pi/4} \ln(\cos x + \sin x)dx$$

Using

$$\int_{-a}^{a} f(x)dx = \int_0^a (f(x)dx + f(-x)dx)$$

$$\Rightarrow I_2 = \int_0^{\pi/4} \ln(\cos^2 x - \sin^2 x)dx = \int_0^{\pi/4} \ln\cos 2x\, dx = \int_0^{\pi/4} \ln\sin 2x\, dx$$

Put

$$2x = y \Rightarrow 2dx = dy$$

$$I_2 = \frac{1}{2}\int_0^{\pi/2} \ln\sin y\, dy$$

$$\Rightarrow 2I_2 = \int_0^{\pi/2} \ln\sin x\, dx$$

$$\Rightarrow 2I_2 = I_1 \Rightarrow \frac{I_1}{I_2} = \frac{1}{2}$$

Illustration 23.33 Evaluate $\int_{-1/2}^{1/2}\left([x] + \ln\left(\frac{1+x}{1-x}\right)\right)dx$.

Solution: $\ln\left(\frac{1+x}{1-x}\right)$ is an odd function of x as $f(-x) = -f(x)$, therefore

$$I = \int_{-1/2}^{1/2} [x]dx + 0$$

$$I = \int_{-1/2}^{0} [x]dx + \int_0^{1/2} [x]dx$$

$$I = \int_{-1/2}^{0} -1dx + 0 \quad = -[x]\Big|_{-1/2}^{0} = -\frac{1}{2}$$

Illustration 23.34 Evaluate $\int_{-\pi}^{\pi}(1-x^2)\sin x \cos^2 x\, dx$.

Solution: Let

$$I = \int_{-\pi}^{\pi}(1-x^2)\sin x \cos^2 x\, dx$$

$f(x) = (1-x^2)\cdot \sin x \cdot \cos^2 x$ is an odd function as

$$f(-x) = -(1-x^2)\sin x \cos^2 x = -f(x)$$

$$I = \int_0^{\pi}(f(x) + f(-x))dx = 0$$

Illustration 23.35 Evaluate $\int_{-1/2}^{1/2} \cos x \cdot \ln\left(\frac{1-x}{1+x}\right)dx$.

Solution: Let

$$I = \int_{-1/2}^{1/2} \cos x \cdot \ln\left(\frac{1-x}{1+x}\right)dx$$

$$f(x) = \cos x \cdot \ln\left(\frac{1-x}{1+x}\right)$$

$$f(-x) = \cos x \cdot \ln\left(\frac{1+x}{1-x}\right) = -f(x)$$

$$\Rightarrow f(x) + f(-x) = 0$$

$$\Rightarrow I = \int_0^{1/2}(f(x) + f(-x))dx = 0$$

7. $\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)x + a)dx$

Proof: Let

$$I = (b-a)\int_0^1 f((b-a)x + a)dx$$

Put

$$t = (b-a)x + a \Rightarrow dx = \frac{dt}{(b-a)}$$

$$x = 0 \Rightarrow t = a, x = 1 \Rightarrow t = b$$

$$I = (b-a)\int_a^b \frac{f(t)}{(b-a)}dt = \int_a^b f(t)dt$$

$$I = \int_a^b f(x)dx$$

Illustration 23.36 Evaluate $\int_{-4}^{-5} e^{(x+5)^2}dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx$.

Solution:

$$I = I_1 + I_2$$

$$I_1 = \int_{-4}^{-5} e^{(x+5)^2}dx$$

$$= (-5+4)\int_0^1 e^{((-5+4)x-4+5)^2}dx$$

$$I_1 = -\int_0^1 e^{(x-1)^2}dx \qquad (1)$$

Again, let

$$I_2 = -3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}dx$$

$$I_2 = 3\left(\frac{2-1}{3}\right)\int_0^1 e^{\left(\left(\left(\frac{2-1}{3}\right)\right)x-\left(\frac{2-1}{3}\right)\right)^2}dx$$

$$= \int_0^1 e^{(x-1)^2}dx$$

$$I_2 = -I_1$$

$$\Rightarrow I = I_1 - I_1 = 0$$

23.5 Properties Based on Periodic Function

If $f(x)$ is a periodic function with period T $(f(x+T)=f(x))$ and $m,n\in I$, $a\in R^+$, then

8. $\int_{nT}^{a+nT} f(x)dx=\int_0^a f(x)dx$

Proof: Let

$$I=\int_{nT}^{a+nT} f(x)dx$$

Put

$$x=z+nT \Rightarrow dx=dz$$
$$x=nT \Rightarrow z=0, x=a+nT \Rightarrow z=a$$
$$f(z+nT)=f(z)$$
$$I=\int_0^a f(z+nT)dz=\int_0^a f(z)dz$$
$$I=\int_0^a f(x)dx$$

9. $\int_0^{nT} f(x)dx=n\int_0^T f(x)dx$

Proof: Let

$$I=\int_0^{nT} f(x)dx=\int_0^T f(x)dx+\int_T^{2T} f(x)dx+\cdots+\int_{(n-1)T}^{nT} f(x)dx$$
$$I=\int_0^T f(x)dx+\int_0^T f(x+T)dx+\int_0^T f(x+2T)dx+\cdots+\int_0^T f(x+(n-1)T)dx$$
$$=\int_0^T f(x)dx+\int_0^T f(x)dx+\int_0^T f(x)dx+\cdots+\int_0^T f(x)dx \quad \text{(upto } n \text{ times)}$$
$$f(x+T)=f(x), f(x+2T)=f(x),\ldots,f(x+(n-1)T)=f(x)$$
$$I=n\int_0^T f(x)dx$$

10. $\int_a^{a+nT} f(x)dx=n\int_0^T f(x)dx$

Proof: Let

$$I=\int_a^{a+nT} f(x)dx=\int_a^0 f(x)dx+\int_0^{nT} f(x)dx+\int_{nT}^{a+nT} f(x)dx$$
$$I=I_1+I_2+I_3$$
$$I_3=\int_{nT}^{a+nT} f(x)dx$$

Put

$$x=z+nT \Rightarrow dx=dz$$
$$x=nT \Rightarrow z=0, x=a+nT \Rightarrow z=a$$
$$f(z+nT)=f(z)$$
$$I_3=\int_0^a f(z+nT)dz=\int_0^a f(z)dz$$
$$I_3=\int_0^a f(x)dx=-I_1$$
$$I_3+I_1=0$$
$$I=\int_0^{nT} f(x)dx=n\int_0^T f(x)dx$$

11. $\int_{mT}^{nT} f(x)dx=(n-m)\int_0^T f(x)dx$

Proof: Let

$$I=\int_{mT}^{nT} f(x)dx$$

Put

$$x=z+mT \Rightarrow dx=dz$$
$$x=nT \Rightarrow z=(n-m)T, x=mT \Rightarrow z=0$$
$$I=\int_0^{(n-m)T} f(z+mT)dz=\int_0^{(n-m)T} f(z)dz$$
$$I=(n-m)\int_0^T f(x)dx$$

12. $\int_{a+nT}^{b+nT} f(x)dx=\int_a^b f(x)dx$

Proof: Let

$$I=\int_{a+nT}^{b+nT} f(x)dx$$

Put

$$x=z+nT \Rightarrow dx=dz$$
$$x=a+nT \Rightarrow z=a,\ x=b+nT \Rightarrow z=b$$
$$I=\int_a^b f(z+nT)dz=\int_a^b f(z)dz$$
$$I=\int_a^b f(x)dx$$

13. If $f(x)$ is a periodic function with period T, then $\int_a^{a+T} f(x)$ is independent of a.

Proof: Let

$$I=\int_a^{a+T} f(x)dx=\int_a^0 f(x)dx+\int_0^T f(x)dx+\int_T^{a+T} f(x)dx$$
$$I=I_1+I_2+I_3$$
$$I_3=\int_T^{a+T} f(x)dx$$

Put

$$x=z+T \Rightarrow dx=dz$$
$$x=T \Rightarrow z=0,\ x=a+T \Rightarrow z=a$$
$$f(z+T)=f(z)$$
$$I_3=\int_0^a f(z+T)dz=\int_0^a f(z)dz$$
$$I_3=\int_0^a f(x)dx=-I_1$$

$$I_3 + I_1 = 0$$

$$I = \int_0^T f(x)dx = \int_0^T f(x)dx$$

Illustration 23.37 Evaluate $\int_0^{10\pi} |\sin x|\, dx$.

Solution: Let

$$I = \int_0^{10\pi} |\sin x|\, dx$$

We know that $|\sin x|$ is a periodic function with period π

$$I = 10\int_0^{\pi} |\sin x|\, dx \Rightarrow I = 20\int_0^{\pi/2} \sin x\, dx$$

$$I = -20\cos x\Big|_0^{\pi/2} \Rightarrow I = 20$$

Illustration 23.38 If $f(x)$ is a continuous periodic function with period T, then prove that the integral of $I = \int_a^{a+T} f(x)dx$ is dependent of a.

Solution: Consider the function

$$g(a) = \int_a^{a+T} f(x)dx = \int_a^0 f(x)dx + \int_0^T f(x)dx + \int_T^{a+T} f(x)dx$$

Putting $x - T = y$ in last integral, we get

$$\int_T^{a+T} f(x)dx = \int_0^a f(y+T)dy = \int_0^a f(y)dy$$

$$\Rightarrow g(a) = \int_a^0 f(x)dx + \int_0^T f(x)dx + \int_0^a f(x)dx = \int_0^T f(x)dx$$

Hence, $g(a)$ is independent of a.

Illustration 23.39 If $f(x)$ is a function satisfying $f(x + a) + f(x) = 0$ for all $x \Rightarrow R$ and constant a, such that $\int_b^{c+b} f(x)dx$ is independent of b, then find the least positive value of c.

Solution: We have

$f(x + a) + f(x) = 0$ for all $x \in R$ (1)

$\Rightarrow f(x + a + a) + f(x + a) = 0$ [Replacing x by $x + a$] (2)

$\Rightarrow f(x + 2a) + f(x + a) = 0$ (3)

Subtracting Eqs. (1) from (2), we get

$f(x + 2a) - f(x) = 0$ for all $x \in R$

$\Rightarrow f(x + 2a) = f(x)$ for all $x \in R$

So, $f(x)$ is periodic with period $2a$.

It is given that $\int_b^{c+b} f(x)dx$ is independent of b.

The minimum value of 'c' is equal to the period of $f(x)$, that is, $2a$.

Illustration 23.40 Evaluate $\int_0^{200} \sqrt{\dfrac{1-\cos \pi x}{2}}\, dx$.

Solution:

$$I = \int_0^{200} \left|\sin \frac{\pi x}{2}\right| dx$$

$$= 100\int_0^2 \sin\frac{\pi x}{2} dx = \frac{2\times 50}{\pi}\left(-\cos\frac{\pi x}{2}\right)_0^2$$

$$= \frac{100}{\pi}(1+1) = \frac{200}{\pi}$$

Your Turn 1

1. The value of $\int_2^3 \dfrac{x+1}{x^2(x-1)} dx$ is

   (A) $2\log 2 - \dfrac{1}{6}$ **(B)** $\log\dfrac{16}{9} - \dfrac{1}{6}$

   (C) $\log\dfrac{4}{3} - \dfrac{1}{6}$ **(D)** $\log\dfrac{16}{9} + \dfrac{1}{6}$

   Ans. (B)

2. The value of $\int_1^e \log x\, dx$ is

   (A) 0 **(B)** 1 **(C)** $e-1$ **(D)** $e+1$

   Ans. (B)

3. The value of $I = \int_0^{\pi/2} \dfrac{(\sin x + \cos x)^2}{\sqrt{1+\sin 2x}} dx$ is

   (A) 3 **(B)** 1 **(C)** 2 **(D)** 0

   Ans. (C)

4. $\int_0^{\pi/8} \cos^3 4\theta\, d\theta =$

   (A) $\dfrac{2}{3}$ **(B)** $\dfrac{1}{4}$ **(C)** $\dfrac{1}{3}$ **(D)** $\dfrac{1}{6}$

   Ans. (D)

5. $\int_3^8 \dfrac{2-3x}{x\sqrt{(1+x)}} dx$ is equal to

   (A) $2\log(3/2e^3)$ **(B)** $\log(3/e^3)$

   (C) $4\log(3/e^3)$ **(D)** None of these

   Ans. (A)

6. The value of $\int_0^1 x^2 e^x dx$ is equal to

   (A) $e-2$ **(B)** $e+2$ **(C)** e^2-2 **(D)** e^2

   Ans. (A)

7. Let $I_1 = \int_1^2 \dfrac{dx}{\sqrt{1+x^2}}$ and $I_2 = \int_1^2 \dfrac{dx}{x}$. Then

   (A) $I_1 > I_2$ **(B)** $I_2 > I_1$

   (C) $I_1 = I_2$ **(D)** $I_1 > 2I_2$ **Ans.** (B)

8. The value of $\int_{1/e}^{\tan x} \dfrac{t\,dt}{1+t^2} + \int_{1/e}^{\cot x} \dfrac{dt}{t(1+t^2)} =$

   (A) -1 **(B)** 1 **(C)** 0 **(D)** None of these

   Ans. (B)

9. $\int_{\pi/4}^{3\pi/4} \dfrac{dx}{1+\cos x}$ is equal to

   (A) 2 **(B)** -2 **(C)** $\dfrac{1}{2}$ **(D)** $-\dfrac{1}{2}$

   Ans. (A)

10. The value of $\int_1^{e^2} \frac{dx}{x(1+\ln x)^2}$ is

(A) 2/3 **(B)** 1/3

(C) 3/2 **(D)** ln2

Ans. (A)

23.6 Properties Based on Inequality

14. If functions $A(x)$, $B(x)$ and $C(x)$ are continuous in $x \in [a,b]$ and satisfying the condition

$$A(x) \le B(x) \le C(x)$$

then

$$\int_a^b A(x)dx \le \int_a^b B(x)dx \le \int_a^b C(x)dx$$

Illustration 23.41 For $n \ge 1$, prove that $\frac{1}{2} \le \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} \le \frac{\pi}{6}$.

Solution: For $n \ge 1$ and $-1 \le x \le 1$,

$$\sqrt{1-x^2} \le \sqrt{1-x^{2n}} \le 1$$

and

$$1 \le \frac{1}{\sqrt{1-x^{2n}}} \le \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \int_0^{1/2} 1dx \le \int_0^{1/2} \frac{1}{\sqrt{1-x^{2n}}}dx \le \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow \frac{1}{2} \le \int_0^{1/2} \frac{1}{\sqrt{1-x^{2n}}}dx \le \sin^{-1} x\Big|_0^{1/2}$$

$$\Rightarrow \frac{1}{2} \le \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}} \le \frac{\pi}{6}$$

15. If m and M are the smallest and greatest values of a function $A(x)$ on an interval $[a, b]$, then

$$m(b-a) \le \int_a^b A(x)dx \le M(b-a)$$

Illustration 23.42 Prove that $2 \le \int_{-1}^{1} \sqrt{2+x^5}\, dx \le 2\sqrt{3}$.

Solution: Let

$$f(x) = \sqrt{2+x^5}$$

$f(x)$ is increasing function for all defined real value and the minimum and maximum value of $f(x)$ are respectively 1 and $\sqrt{3}$ in $x \in [-1, 1]$. Therefore,

$$\int_{-1}^{1} 1\,dx \le \int_{-1}^{1} f(x)dx \le \int_{-1}^{1} \sqrt{3}\,dx$$

$$\Rightarrow 1(1+1)dx \le \int_{-1}^{1} f(x)dx \le \sqrt{3}(1+1)$$

$$\Rightarrow 2 \le \int_{-1}^{1} \sqrt{2+x^5}\,dx \le 2\sqrt{3}$$

Illustration 23.43 Prove that $1 \le \int_0^2 \frac{5-x}{9-x^2}\,dx \le \frac{6}{5}$.

Solution: Let

$$f(x) = \frac{5-x}{9-x^2}$$

$$f'(x) = \frac{(x^2-9)+2x(5-x)}{(9-x^2)^2} = \frac{(-x^2+10x-9)}{(9-x^2)^2}$$

$$\Rightarrow f'(x) = 0 \Rightarrow x = 1, 9$$

$$\Rightarrow f(0) = \frac{5}{9},\ f(1) = \frac{1}{2},\ f(2) = \frac{3}{5}$$

The maximum and minimum value of $f(x)$ in $x \in [0,2]$ is $f(1)$ and $f(2)$.

$$\int_0^2 f(1)dx \le \int_0^2 \frac{5-x}{9-x^2}\,dx \le \int_0^2 f(2)dx$$

$$\Rightarrow 2f(1) \le \int_0^2 \frac{5-x}{9-x^2}\,dx \le 2f(2)$$

$$\Rightarrow 1 \le \int_0^2 \frac{5-x}{9-x^2}\,dx \le \frac{6}{5}$$

16. $\left|\int_a^b A(x)dx\right| \le \int_a^b |A(x)|\,dx$

Illustration 23.44 Suppose α is real number then prove that

$$|e2^{\alpha\pi i} - 1| \le 2\pi|\alpha|$$

Solution: Let $A(x) = e^{\alpha x i}$, α and x are real.

$$\left|\int_0^{2\pi} e^{\alpha x i}dx\right| \le \int_0^{2\pi} |e^{\alpha x i}|dx$$

$$\left|\int_0^{2\pi} e^{\alpha x i}dx\right| \le \int_0^{2\pi} 1dx \Rightarrow \left|\int_0^{2\pi} e^{\alpha x i}dx\right| \le 2\pi$$

$$\left|\frac{e^{\alpha x i}}{\alpha i}\Big|_0^{2\pi}\right| \le 2\pi \Rightarrow \frac{|e^{2\alpha\pi i}-1|}{|\alpha|} \le 2\pi \Rightarrow |e^{2\alpha\pi i}-1| \le 2\pi|\alpha|$$

17. If $A^2(x)$ and $B^2(x)$ are integral on $[a, b]$, then

$$\left|\int_a^b A(x)B(x)dx\right| \le \left(\int_a^b A^2(x)dx\right)^{1/2}\left(\int_a^b B^2(x)dx\right)^{1/2}$$

23.7 Newton–Leibnitz Rule

1. If $f(x)$ is continuous and $p(x)$, $q(x)$ are differentiable functions in the interval $[a, b]$, then

$$\frac{d}{dx}\int_{p(x)}^{q(x)} f(t)dt = f\{q(x)\}\frac{d}{dx}\{q(x)\} - f(p(x))\frac{d}{dx}\{p(x)\}$$

2. If $f(x)$ is continuous and $p(x)$, $q(x)$ are differentiable functions at a point $x \in (a,b)$ and $f(x,t)$ is continuous, then

$$\frac{d}{dx}\left[\int_{p(x)}^{q(x)} f(x,t)dt\right] = \int_{p(x)}^{q(x)} \frac{d}{dx}f(x,t)dt + \left\{\frac{d\,q(x)}{dx}\right\}f(x,q(x)) - \left\{\frac{d\,p(x)}{dx}\right\}f(x,p(x))$$

Illustration 23.45 Let $f(x) = \int_1^x \sqrt{2-t^2}\,dt$. Then find the real roots of the equation $x^2 - f'(x) = 0$.

Solution:

$$f(x)=\int_1^x \sqrt{2-t^2}\,dt$$
$$f'(x)=\sqrt{2-x^2}\cdot 1-\sqrt{2-1}\cdot 0=\sqrt{2-x^2}$$
$$x^2=f'(x)\Rightarrow x^2=\sqrt{2-x^2}$$
$$x^4+x^2-2=0\Rightarrow (x^2+2)(x^2-1)=0$$
$$x=\pm 1$$

Illustration 23.46 If $f(x)=\cos x-\int\limits_0^x (x-t)f(t)dt$, then calculate $f''(x)+f(x)$.

Solution:

$$f'(x)=-\sin x-\left(\int\limits_0^x f(t)dt+xf(x)\right)+xf(x)$$
$$f'(x)=-\sin x-\int\limits_0^x f(t)dt$$
$$f''(x)=-\cos x-f(x)$$
$$f''(x)+f(x)=-\cos x$$

Illustration 23.47 Let $f:(0,\infty)\to R$ and $f(x)=\int\limits_0^x f(t)dt$. If $f(x^2)=x^2(1+x)$, then find $f(4)$?

Solution: By definition of $f(x)$ we have

$$f(x^2)=\int_0^{x^2} f(t)dt=x^2+x^3$$

Differentiate both sides,

$$f(x^2)\cdot 2x+0=2x+3x^2$$

Put

$$x=2\Rightarrow 4f(4)=16\Rightarrow f(4)=4$$

Illustration 23.48 If a function $f(x)$ is defined $\forall\ x\in R$ such that $\int\limits_0^a f(x)dx, a\in R^+$ exist. If $g(x)=\int\limits_x^a \frac{f(t)}{t}dt$, then prove that $\int\limits_0^a g(x)dx=\int\limits_0^a f(x)dx$.

Solution:

$$g(x)=\int\limits_x^a \frac{f(t)}{t}dt$$
$$g(a)=0$$

Differentiate w.r.t. x

$$g'(x)=-\frac{f(x)}{x}\Rightarrow f(x)=-xg'(x)$$
$$\int\limits_0^a f(x)dx=-\int\limits_0^a xg'(x)dx$$
$$\int\limits_0^a f(x)dx=-xg(x)\Big|_0^a+\int\limits_0^a g(x)dx$$
$$\int\limits_0^a f(x)dx=-ag(a)+\int\limits_0^a g(x)dx$$
$$\int\limits_0^a f(x)dx=\int\limits_0^a g(x)dx$$

23.8 Summation of Series by Integration

An alternative way of describing $\int\limits_a^b f(x)dx$ is that the definite integral $\int\limits_a^b f(x)dx$ is a limiting case of the summation of an infinite series, provided $f(x)$ is continuous on $[a, b]$, that is,

$$\int\limits_a^b f(x)dx=\lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh),$$

where

$$h=\frac{b-a}{n}$$

The converse is also true, that is, if we have an infinite series of the above form, it can be expressed as a definite integral.

23.8.1 Method to Express the Infinite Series as Definite Integral

1. Express the given series in the form $\sum \frac{1}{n}f\left(\frac{r}{n}\right)$.
2. Then the limit is its sum when $n\to\infty$, that is, $\lim\limits_{n\to\infty}\sum \frac{1}{n}f\left(\frac{r}{n}\right)$.
3. Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and $\lim\limits_{n\to\infty}\sum$ by the sign of $\int$.
4. The lower and the upper limit of integration are the limiting values of $\frac{r}{n}$ for the first and the last term of r, respectively.

Some particular cases of the above are

(A) $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$ or $\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n-1}\frac{1}{n}f\left(\frac{r}{n}\right)=\int\limits_0^1 f(x)dx$

(B) $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{pn}\frac{1}{n}f\left(\frac{r}{n}\right)=\int\limits_\alpha^\beta f(x)dx$

where

$$\alpha=\lim_{n\to\infty}\frac{r}{n}=0 \quad (\text{as } r=1)$$

and

$$\beta=\lim_{n\to\infty}\frac{r}{n}=p \quad (\text{as } r=pn)$$

Illustration 23.49 If $S_n=\frac{1}{1+\sqrt{n}}+\frac{1}{2+\sqrt{2n}}+\cdots+\frac{1}{n+\sqrt{n^2}}$, then $\lim\limits_{n\to\infty}S_n$ is equal to.

Solution:

$$\lim_{n\to\infty}S_n=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r+\sqrt{rn}}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\left[\frac{r}{n}+\sqrt{\frac{r}{n}}\right]}$$

Now

$$\alpha=\lim_{n\to\infty}\frac{1}{n}=0$$

and

$$\beta=\lim_{n\to\infty}\frac{r}{n}=1$$

$$\lim_{n\to\infty}S_n=\int_0^1\frac{1}{\sqrt{x}(1+\sqrt{x})}dx$$
$$=2[\ln(1+\sqrt{x})]_0^1=2\ln 2$$

Illustration 23.50 Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right)$$

Solution: Let

$$I=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+\frac{r}{n}}$$

Now

$$\alpha=\lim_{n\to\infty}\frac{1}{n}=0$$

and

$$\beta=\lim_{n\to\infty}\frac{r}{n}=1$$

$$I=\int_0^1\frac{1}{1+x}dx=[\ln(1+x)]_0^1$$

$$I=\ln 2$$

Illustration 23.51 Evaluate $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$.

Solution: Let

$$A=\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}\Rightarrow A=\lim_{n\to\infty}\left(\frac{1\cdot2\cdot3\ldots n}{n^n}\right)^{1/n}$$

$$\Rightarrow \ln A=\lim_{n\to\infty}\ln\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\right)^{1/n}$$

$$\Rightarrow \ln A=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left[\ln\left(\frac{r}{n}\right)\right]$$

$$\Rightarrow \ln A=\int_0^1\ln x\,dx=[x\ln x-x]_0^1$$

$$\Rightarrow \ln A=-1\Rightarrow A=\frac{1}{e}$$

Illustration 23.52 Evaluate $\lim_{n\to\infty}\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}(p>0)$.

Solution:

$$\lim_{n\to\infty}\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^p}{n\cdot n^p}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\left(\frac{r}{n}\right)^p$$

Now

$$\alpha=\lim_{n\to\infty}\frac{1}{n}=0$$

and

$$\beta=\lim_{n\to\infty}\frac{r}{n}=1$$

$$\Rightarrow\int_0^1 x^p\,dx=\left.\frac{x^{p+1}}{p+1}\right|_0^1=\frac{1}{p+1}$$

23.9 Reduction Formulae for Definite Integration

1. If $I_n=\int_0^{\pi/4}\tan^n x\,dx$, then
$$I_n+I_{n-2}=\frac{1}{n-1}$$

2. If $I_n=\int_0^{\pi/4}\cot^n x\,dx$, then
$$I_n+I_{n-2}=\frac{1}{1-n}$$

3. If $I_n=\int_0^{\pi/4}\sec^n x\,dx$, then
$$(n-1)I_n-(n-2)I_{n-2}=(\sqrt{2})^{n-2}$$

4. If $I_n=\int_0^{\pi/4}\operatorname{cosec}^n x\,dx$, then
$$(n-1)I_n-(n-2)I_{n-2}=-(\sqrt{2})^{n-2}$$

5. If $I_n=\int_0^{\pi/2}\sin^n x\,dx$, then
$$I_n=\frac{n-1}{n}I_{n-2}$$

6. If $I_n=\int_0^{\pi/2}\cos^n x\,dx$, then
$$I_n=\frac{n-1}{n}I_{n-2}$$

7. If $I_n=\int_0^{\pi/2}x^n\sin x\,dx$, then
$$I_n+n(n-1)I_{n-2}=n(\pi/2)^{n-1}$$

8. If $I_n=\int_0^{\pi/2}x^n\cos x\,dx$, then
$$I_n+n(n-1)I_{n-2}=(\pi/2)^n$$

9. $\int_0^{\infty}e^{-ax}\sin bx\,dx=-\frac{b}{a^2+b^2}$

10. $\int_0^{\infty}e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$

11. $\int_0^{\infty}e^{-ax}x^n\,dx=\frac{n!}{a^2+1}$

Illustration 23.53 Determine a positive integer $n\le 5$, such that

$$\int_0^1 e^x(x-1)^n dx=16-6e$$

Solution: Let

$$I_n=\int_0^1 e^x(x-1)^n dx$$

Integrating by parts,

$$I_n=e^x(x-1)^n\Big|_0^1-nI_{n-1}=-(-1)^n-nI_{n-1}$$

$$I_n = (-1)^{n+1} - nI_{n-1}$$
$$I_0 = \int_0^1 e^x (x-1)^0 dx = e^x\Big|_0^1 = e-1$$
$$I_1 = (-1)^{1+1} - I_0$$
$$I_1 = (-1)^{1+1} - I_0 = 2-e,$$
$$I_2 = (-1)^{1+2} - 2I_1 = -1-2(2-e) = -5+2e,$$
$$I_3 = (-1)^{1+3} - 3I_2 = 1-3(-5+2e) = 16-6e$$

Therefore,

$$n = 3$$

23.9.1 Gamma Function

$\int_0^\infty x^{n-1}e^{-x}dx,\ \ n>0$ is called Gamma function and denoted by Γn.

If m and n are non-negative integers, then

$$\int_0^{\pi/2} \sin^m x \cos^n x\ dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

where $\Gamma(n)$ is called gamma function which satisfy the following properties:

$$\Gamma(n+1) = n\Gamma(n) = n!$$

That is,

$$\Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi}$$

In place of gamma function, we can also use the following formula

$$\int_0^{\pi/2} \sin^m x \cos^n x\ dx = \frac{(m-1)(m-3)\cdots(2 \text{ or } 1)(n-1)(n-3)\cdots(2 \text{ or } 1)}{(m+n)(m+n-2)\cdots(2 \text{ or } 1)}$$

It is important to note that we multiply by ($\pi/2$); when both m and n are even.

23.10 Wallis Formulae

1. $$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & \text{when } n \text{ is even} \end{cases}$$

2. $$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(m-1)(m-3)\cdots(n-1)(n-3)}{(m+n)(m+n-2)\cdots(2 \text{ or } 1)}\cdot\frac{\pi}{2},$$

[If m, n are both positive integers]

Your Turn 2

1. The value of $\int_0^\pi e^{\cos^2 x}\cos^5 3x\, dx$ is

(A) 1 (B) −1 (C) 0 (D) None of these

Ans. (C)

2. The value of $\int_{-1}^1 \frac{\sin x - x^2}{3-|x|}dx$ is

(A) 0 (B) $2\int_0^1 \frac{\sin x}{3-|x|}dx$

(C) $2\int_0^1 \frac{-x^2}{3-|x|}dx$ (D) $2\int_0^1 \frac{\sin x - x^2}{3-|x|}dx$

Ans. (C)

3. If $(n-m)$ is odd and $|m| \neq |n|$, then $\int_0^\pi \cos mx \sin nx\, dx$ is

(A) $\frac{2n}{n^2-m^2}$ (B) 0

(C) $\frac{2n}{m^2-n^2}$ (D) $\frac{2m}{n^2-m^2}$ **Ans.** (A)

4. To find the numerical value of $\int_{-2}^2 (px^2+qx+s)dx$, it is necessary to know the values of constants

[IIT 1992]

(A) p (B) q (C) s (D) p and s

Ans. (D)

5. If $I = \int_0^{100\pi} \sqrt{(1-\cos 2x)}\, dx$, then the value of I is

(A) $100\sqrt{2}$ (B) $200\sqrt{2}$

(C) $50\sqrt{2}$ (D) None of these **Ans.** (B)

6. $\int_{-1}^1 \log\left(\frac{1+x}{1-x}\right) dx =$

(A) 2 (B) 1 (C) 0 (D) π

Ans. (C)

7. If $\int_{-1}^4 f(x)dx = 4$ and $\int_2^4 (3-f(x))dx = 7$, then the value of $\int_2^{-1} f(x)dx$ is

(A) 2 (B) −3 (C) −5 (D) None of these

Ans. (C)

8. $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}dx =$

(A) 1 (B) 0 (C) −1 (D) None of these

Ans. (A)

9. The least value of the function $F(x) = \int_{5\pi/4}^x (3\sin u + 4\cos u)du$ on the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$ is

(A) $\sqrt{3}+\frac{3}{2}$ (B) $-2\sqrt{3}+\frac{3}{2}+\frac{1}{\sqrt{2}}$

(C) $\frac{3}{2}+\frac{1}{\sqrt{2}}$ (D) None of these

Ans. (B)

10. $\int_0^\infty e^{-2x}(\sin 2x + \cos 2x)dx =$

(A) 1 (B) 0

(C) $\frac{1}{2}$ (D) ∞

Ans. (C)

Additional Solved Examples

1. The value of $\int_0^{2\pi} [2\sin x]\ dx$, where [ ] represents greatest integer function is

(A) $\frac{-5\pi}{3}$ **(B)** $-\pi$ **(C)** $\frac{5\pi}{3}$ **(D)** $\frac{-2}{7}$

Solution: $\int_0^{2\pi} [2\sin x]\ dx =$

$$\int_0^{\frac{\pi}{6}} 0\ dx + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} dx + \int_{\frac{5\pi}{6}}^{\pi} 0\ dx + \int_{\pi}^{\frac{7\pi}{6}} (-1)\ dx + \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (-2)\ dx + \int_{\frac{11\pi}{6}}^{2\pi} (-1)\ dx$$

$$= -\pi$$

Hence, the correct answer is option (B).

2. If $f(x) = a\ \sin\left(\frac{\pi x}{2}\right) + b, f^1\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)dx = \frac{2a}{\pi}$, then

(A) $a = \frac{\pi}{2}, b = \frac{\pi}{2}$ **(B)** $a = \frac{2}{\pi}, b = \frac{3}{\pi}$

(C) $a = 0, b = \frac{-4}{\pi}$ **(D)** $a = \frac{4}{\pi}, b = 0$

Solution:

$$f'(x) = \frac{a\pi}{2}\cos\left(\frac{\pi}{2x}\right)$$

$$\Rightarrow f'\left(\frac{1}{2}\right) = a\frac{\pi}{2} \times \frac{1}{\sqrt{2}} = \sqrt{2} \Rightarrow a = \frac{4}{\pi};$$

$$\int_0^1 f(x)dx = \left[\frac{-2a}{\pi}\cos\left(\frac{\pi x}{2}\right) + bx\right]_0^1 = b + \frac{2a}{\pi} = \frac{2a}{\pi} \text{ (given)}$$

Therefore,

$$b = 0$$

The only choice with $a = 4/\pi$.

Hence, the correct answer is option (D).

3. $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x} =$

(A) 0 **(B)** 1

(C) $\frac{\pi}{2}$ **(D)** $\frac{\pi}{4}$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^3 x} = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot^3 x}, \text{ using } \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\tan^3 x}{1+\tan^3 x}dx \Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

Hence, the correct answer is option (D).

4. Let $f: R \to R$ and $g: R \to R$ be two continuous functions. Then the value of integral $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} [f(x) + f(-x)][g(x) - g(-x)]\ dx$ is

(A) π **(B)** 1

(C) -1 **(D)** 0

Solution: If the given function be $F(x)$, then clearly

$$F(-x) = -F(x)$$

and hence

$$I = 0$$

Hence, the correct answer is option (D).

5. Let $f: R \to R$ be a differentiable function and $f(1) = 4$. Then the value of $\lim_{x\to 1} \int_4^{f(x)} \frac{2t}{x-1} dt$ is

(A) $8f'(1)$ **(B)** $4f'(1)$

(C) $2f'(1)$ **(D)** $f'(1)$

Solution:

$$\lim_{x\to 1} \int_4^{f(x)} \frac{2t}{x-1}dt = \lim_{x\to 1}\left[\frac{t^2}{x-1}\right]_4^{f(x)}$$

$$= \lim_{x\to 1} \frac{[f(x)]^2 - 4^2}{x-1} \qquad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 1} \frac{2f(x)\cdot f'(x)}{1} \qquad \text{[By applying L' Hospital rule]}$$

$$= 2f(1)\,f'(1) = 8f'(1)$$

Hence, the correct answer is option (A).

6. The value of $\int_0^{1.5} [x^2]\ dx$ is.

Solution: For

$$0 \le x < 1, 0 \le x^2 < 1$$

and

$$[x^2] = 0$$

$$1 \le x < \sqrt{2},\ 1 \le x^2 < 2 \text{ and } [x^2] = 1$$

$$\sqrt{2} \le x < 1{\cdot}5,\ 2 \le x^2 < 2{\cdot}25 \text{ and } [x^2] = 2$$

Therefore,

$$\int_0^{1.5} [x^2]dx = \int_0^1 0\ dx + \int_1^{\sqrt{2}} 1dx + \int_{\sqrt{2}}^{1.5} 2dx$$

$$= \sqrt{2} - 1 + 2(1{\cdot}5 - \sqrt{2}) = 2 - \sqrt{2}$$

7. The value of $\int_{-2}^{2}\left|1-x^2\right| dx$ is.

Solution:

$$\int_{-2}^{2}\left|1-x^2\right| dx = 2\int_{0}^{2}\left|1-x^2\right| dx$$

$$= 2\left[\int_{0}^{1}\left|1-x^2\right| dx + \int_{1}^{2}\left|1-x^2\right| dx\right] = 2\left[\int_{0}^{1}(1-x^2)dx + \int_{1}^{2}(x^2-1)dx\right]$$

$$= 2\left\{\left[x-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}-x\right]_1^2\right\} = 2\left\{1-\frac{1}{3}+\frac{8}{3}-2-\frac{1}{3}+1\right\} = 2[2] = 4$$

8. $\lim_{n\to\infty}\left\{\frac{1}{1-n^2}+\frac{2}{1-n^2}+\dots+\frac{2}{1-n^2}\right\}$ is equal to

(A) 0 **(B)** $-\frac{1}{2}$

(C) $\frac{1}{2}$ **(D)** None of these

Solution:

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{1-n^2} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2}\frac{\frac{r}{n}}{\left(\frac{1}{n^2}-1\right)} = \lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{r=1}^{n}\left(\frac{r}{n}\right)\lim_{n\to\infty}\frac{1}{\left\{\left(\frac{1}{n^2}\right)-1\right\}}$$

$$= \int_{0}^{1} x\, dx(-1) = \frac{1}{2}(-1) = \frac{-1}{2}$$

Hence, the correct answer is option (B).

9. The value of $\lim_{x\to\infty}\left\{\frac{1}{(x+1)}+\frac{1}{(x+2)}+\dots+\frac{1}{(x+5x)}\right\}$ is equal to

(A) ln 2 **(B)** ln 3

(C) ln 6 **(D)** None of these

Solution:

$$\lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+\left(\frac{r}{n}\right)}$$

$$\int_{0}^{5}\frac{dx}{1+x} = [\ln(1+x)]_0^5 = \ln 6$$

Hence, the correct answer is option (C).

10. The value of $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx, a>0$ is

(A) π **(B)** $a\pi$

(C) $\pi/2$ **(D)** 2π

Solution:

$$I = \int_{-\pi}^{\pi}\frac{\cos^2 x dx}{1+a^x} = \int_{0}^{\pi}\left(\frac{\cos^2 x}{1+a^x}+\frac{\cos^2 x}{1+a^{-x}}\right)dx = \int_{0}^{\pi}\cos^2 x dx = \frac{\pi}{2}$$

Hence, the correct answer is option (C).

11. Let f: $(0, \infty) \to R$ and $F(x) = \int_{0}^{x} f(t)\, dt$. If $F(x^2) = x^2(1 + x)$, then $f(4)$ equals

(A) 5/4 **(B)** 7

(C) 4 **(D)** 2

Solution: We have

$$F(x^2) = \int_{0}^{x^2} f(t)\; dt = x^2 + x^3$$

Differentiating both sides, we get

$$f(x^2) \cdot 2x = 2x + 3x^2$$

$$\Rightarrow f(x^2) = 1 + (3/2)x$$

$$\Rightarrow f(4) = 1 \pm (3/2)(2) = 4, -2$$

Hence, the correct answer is option (C).

12. Let $g(x) = \int_{0}^{x} f(t)dt$, where f is such that $\frac{1}{2} \le f(t) \le 1$ for $t \in [0, 1]$ and $0 \le f(t) \le \frac{1}{2}$ for $t \in (1, 2]$. Then $g(2)$ satisfies the inequality

(A) $-\frac{3}{2} \le g(2) \le \frac{1}{2}$ **(B)** $0 \le g(2) < 2$

(C) $\frac{3}{2} < g(2) \le \frac{5}{2}$ **(D)** $2 < g(2) < 4$

Solution:

$$g(2) = \int_{0}^{2} f(t)\, dt = \int_{0}^{1} f(t)\, dt + \int_{1}^{2} f(t) dt$$

Using $m(b-a) \le \int_{a}^{b} f(t)\, dt \le M(b-a)$, we get

$$\frac{1}{2} \times 1 + 0 \le g(2) \le 1 \times 1 + \frac{1}{2}$$

$$\frac{1}{2} \le g(2) \le \frac{3}{2} \Rightarrow 0 \le g(2) < 2$$

Hence, the correct answer is option (B).

13. If $f(x) = \begin{cases} e^{\cos x}\sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)dx =$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

Solution:

$$\int_{-2}^{3} f(x)dx = \int_{-2}^{2} e^{\cos x}\; \sin x\; dx + \int_{2}^{3} 2\; dx = 0 + 2 \quad (\text{since } e^{\cos x}\sin x \text{ is odd})$$

Hence, the correct answer is option (C).

14. The values of the integral $\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right|dx$ is

(A) $\frac{3}{2}$ **(B)** $\frac{5}{2}$

(C) 3 **(D)** 5

Solution: Let

$$I=\int_{e^{-1}}^{e^2}\left|\frac{\log_e x}{x}\right| dx$$

Note $x>0$ for $x\in\left(\frac{1}{e}, e^2\right)$

and

$$\log_e x \text{ is } <0 \text{ for } \frac{1}{e}<x<1$$

and

$$\text{for } 1<x<e^2, \log e^x>0$$

Therefore,

$$I=-\int_{1/e}^{1}\frac{\log_e x}{x}dx+\int_{1}^{e^2}\frac{\log_e x}{x}dx=\frac{-1}{2}+2=\frac{3}{2}$$

Hence, the correct answer is option (A).

Previous Years' Solved JEE Main/AIEEE Questions

1. Let $F(x)=f(x)+f\left(\frac{1}{x}\right)$, where $f(x)=\int_1^x\frac{\log t}{1+t}dt$. Then $F(e)$ equals

(A) $\frac{1}{2}$ **(B)** 0

(C) 1 **(D)** 2

[AIEEE 2007]

Solution: We have

$$f(x)=\int_1^x\frac{\log t}{1+t}dt$$

That is,

$$F(e)=f(e)+f\left(\frac{1}{e}\right)\Rightarrow F(e)=\int_1^e\frac{\log t}{1+t}dt+\int_1^{1/e}\frac{\log t}{1+t}dt$$

$$=\int_1^e\frac{\log t}{1+t}+\int_1^e\frac{\log t}{t(1+t)}dt=\int_1^e\frac{\log t}{t}dt=\frac{1}{2}$$

Hence, the correct answer is option (A).

2. The solution for x of the equation $\int_{\sqrt{2}}^{x}\frac{dt}{t\sqrt{t^2-1}}=\frac{\pi}{2}$ is

(A) 2 **(B)** π

(C) $\frac{\sqrt{3}}{2}$ **(D)** $2\sqrt{2}$

[AIEEE 2007]

Solution:

$$\int_{\sqrt{2}}^{x}\frac{dt}{t\sqrt{t^2-1}}=\frac{\pi}{2}$$

$$\Rightarrow[\sec^{-1}t]_{\sqrt{2}}^{x}=\frac{\pi}{2}\Rightarrow\sec^{-1}x-\frac{\pi}{4}=\frac{\pi}{2}\Rightarrow\sec^{-1}x=\frac{3\pi}{4}$$

$$\Rightarrow x=\sec\left(\pi-\frac{\pi}{4}\right)=-\sec\frac{\pi}{4}\Rightarrow x=-\sqrt{2}$$

There is no correct option for this question.

3. Let $I=\int_0^1\frac{\sin x}{\sqrt{x}}dx$ and $J=\int_0^1\frac{\cos x}{\sqrt{x}}dx$. Then which one of the following is true?

(A) $I>\frac{2}{3}$ and $J>2$ **(B)** $I<\frac{2}{3}$ and $J<2$

(C) $I<\frac{2}{3}$ and $J>2$ **(D)** $I>\frac{2}{3}$ and $J<2$

[AIEEE 2008]

Solution:

$$I=\int_0^1\frac{\sin x}{\sqrt{x}}dx<\int_0^1\frac{x}{\sqrt{x}}dx=\int_0^1\sqrt{x}dx=\frac{2}{3}x^{3/2}\Big|_0^1=\frac{2}{3}\Rightarrow I<\frac{2}{3}$$

$$J=\int_0^1\frac{\cos x}{\sqrt{x}}dx<\int_0^1\frac{1}{\sqrt{x}}dx=2\sqrt{x}\Big|_0^1=2$$

Therefore, $J<2$.

Hence, the correct answer is option (B).

4. Let $p(x)$ be a function defined on R such that $p'(x)=p'(1-x)$, for all $x\in[0,1]$, $p(0)=1$ and $p(1)=41$. Then $\int_0^1 p(x)dx$ equals

(A) 21 **(B)** 41

(C) 42 **(D)** $\sqrt{41}$

[AIEEE 2010]

Solution: We have

$$p'(x)=p'(1-x)\Rightarrow p(x)=-p(1-x)+c$$

$$\text{At } x=0,\ p(0)=-p(1)+c\ \Rightarrow 42=c$$

Now,

$$p(x)=-p(1-x)+42$$

which implies that

$$p(x)+p(1-x)=42$$

$$I=\int_0^1 p(x)dx=\int_0^1 p(1-x)dx\Rightarrow 2I=\int_0^1(42)dx\Rightarrow I=21$$

Hence, the correct answer is option (A).

5. The value of $\int_0^1\frac{8\log(1+x)}{1+x^2}dx$ is

(A) $\frac{\pi}{8}\log 2$ **(B)** $\frac{\pi}{2}\log 2$

(C) $\log 2$ **(D)** $\pi\log 2$

[AIEEE 2011]

Solution: We have

$$I=8\int_0^1\frac{\log(1+x)}{1+x^2}dx$$

Put $x=\tan\theta$, then

$$dx=\sec^2\theta\, d\theta$$

That is,

$$I=8\int_0^1\frac{\log(1+x)}{1+x^2}dx=8\int_0^{\frac{\pi}{4}}\frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

$$=8\int_0^{\frac{\pi}{4}}\frac{\log(1+\tan\theta)}{\cancel{\sec^2}}\cancel{\sec^2}\theta\, d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta = 8\int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \log 2\, d\theta - 8\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta$$

$$= 8\log 2\frac{\pi}{4} - I \Rightarrow 2I = 2\pi\log 2 \Rightarrow I = \pi\log 2$$

Hence, the correct answer is option (D).

6. If $g(x)=\int_0^x \cos 4t\, dt$, then $g(x+\pi)$ equals

(A) $\dfrac{g(x)}{g(\pi)}$ **(B)** $g(x)+g(\pi)$

(C) $g(x)-g(\pi)$ **(D)** $g(x)\cdot g(\pi)$

[AIEEE 2012]

Solution:

$$g(x)=\int_0^x \cos 4t\, dt$$

$$\Rightarrow g'(x)=\cos 4x \Rightarrow g(x)=\frac{\sin 4x}{4}+k \Rightarrow g(x)=\frac{\sin 4x}{4} \quad [\text{since, } g(0)=0]$$

Therefore,

$$g(x+\pi)=g(x)+g(\pi)=g(x)-g(\pi) \quad (\text{since, } g(\pi)=0)$$

Hence, the correct answers are options (B) and (C).

7. Statement-I: The value of the integral $\displaystyle\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\dfrac{\pi}{6}$.

Statement-II: $\displaystyle\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$.

(A) Statement-I is True; Statement-II is true; Statement-II is **not** a correct explanation for Statement-I.
(B) Statement-I is True; Statement-II is False.
(C) Statement-I is False; Statement-II is True.
(D) Statement-I is True; Statement-II is True; Statement-II is a correct explanation for Statement-I.

[JEE MAIN 2013]

Solution: We have

$$I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$$

$$=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$$

$$I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\frac{1}{\tan x}}}\Rightarrow\int_{\pi/6}^{\pi/3}\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}dx\Rightarrow 2I=\int_{\pi/6}^{\pi/3}\frac{\cancel{(1+\sqrt{\tan x})}}{\cancel{(1+\sqrt{\tan x})}}dx$$

$$\Rightarrow I=\frac{1}{2}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]\Rightarrow I=\frac{\pi}{12}$$

Hence, the correct answer is option (C).

8. The integral $\displaystyle\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals

(A) $4\sqrt{3}-4$ **(B)** $4\sqrt{3}-4-\dfrac{\pi}{3}$

(C) $\pi-4$ **(D)** $\dfrac{2\pi}{3}-4-4\sqrt{3}$

[JEE MAIN 2014 (OFFLINE)]

Solution: See Fig. 23.2.

$$\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx=\int_0^{\pi}\sqrt{\left(1-2\sin\frac{x}{2}\right)^2}$$

$$=\int_0^{\pi}\left|1-2\sin\frac{x}{2}\right|dx=\int_0^{\frac{\pi}{3}}\left(1-2\sin\frac{x}{2}\right)dx+\int_{\frac{\pi}{3}}^{\pi}-\left(1-2\sin\frac{x}{2}\right)dx$$

$$=\left[x+\frac{2\cos\frac{x}{2}}{(1/2)}\right]_0^{\frac{\pi}{3}}-\left[x+\frac{2\cos\frac{x}{2}}{(1/2)}\right]_{\frac{\pi}{3}}^{\pi}$$

$$=\left(\frac{\pi}{3}+2\sqrt{3}\right)-(4)-\left[(\pi+4(0))-\left(\frac{\pi}{3}+2\sqrt{3}\right)\right]$$

$$=\frac{\pi}{3}+2\sqrt{3}-4-\pi+\frac{\pi}{3}+2\sqrt{3}$$

$$=\frac{2\pi}{3}-\pi+4\sqrt{3}-4=4\sqrt{3}-\frac{\pi}{3}-4$$

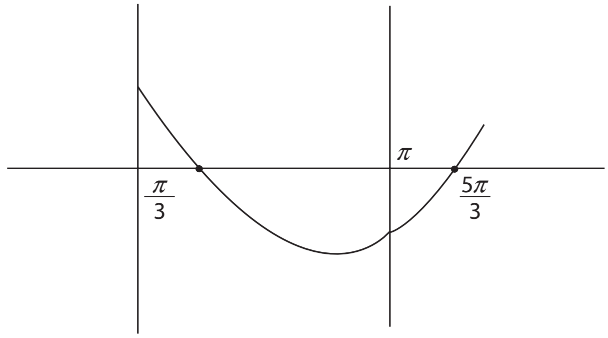


Figure 23.2

Hence, the correct answer is option (B).

9. The integral $\displaystyle\int_0^{\frac{1}{2}}\frac{\ln(1+2x)}{1+4x^2}dx$, equals

(A) $\dfrac{\pi}{4}\ln 2$ **(B)** $\dfrac{\pi}{8}\ln 2$

(C) $\dfrac{\pi}{16}\ln 2$ **(D)** $\dfrac{\pi}{32}\ln 2$

[JEE MAIN 2014 (ONLINE SET-1)]

Solution:

$$\int_0^{\frac{1}{2}}\frac{\ln(1+2x)}{1+4x^2}dx$$

Put $2x=y$. Therefore,

$$2dx=dy$$

Therefore, integration changes to

$$I=\frac{1}{2}\int_0^1\frac{\ln(1+y)}{1+y^2}dy$$

Now, put

$$y=\tan\theta \Rightarrow dy=\sec^2\theta\, d\theta$$

$$I=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\ln(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$I=\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right)d\theta=\int_0^{\pi/4}\ln\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln 2d\theta-\frac{1}{2}\int_0^{\frac{\pi}{4}}\ln(1+\tan\theta)d\theta$$

$$=\frac{\ln 2}{2}\int_0^{\frac{\pi}{4}}d\theta-I$$

Therefore,

$$2I=\frac{1}{2}(\ln 2)\left(\frac{\pi}{4}-0\right)\Rightarrow 2I=\frac{\pi}{8}\ln 2\Rightarrow I=\frac{\pi}{16}\ln 2$$

Hence, the correct answer is option (C).

10. If for $n \geq 1$, $P_n=\int_1^e (\log x)^n\, dx$, then $P_{10}-90P_8$ is equal to

(A) -9 **(B)** $10e$

(C) $-9e$ **(D)** 10

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$P_n=\int_1^e(\log x)^n dx$$

Put

$$\log x=t\Rightarrow e^t=x$$

Therefore,

$$dx=e^t\,dt$$

$$P_n=\int_0^1 t^n e^t dt=[t^n e^t]_0^1-\int_0^1 nt^{n-1}e^t dt=1-n\,P_{n-1}$$

$$=1-n\,[1-(n-1)\,P_{n-2}]=1-n+n\,(n-1)\,P_{n-2}$$

Therefore,

$$P_{10}=1-10+10\,(10-1)\,P_8\Rightarrow P_{10}-90\,P_8=-9$$

Hence, the correct answer is option (A).

11. If [ ] denotes the greatest integer function, then the integral $\int_0^{\pi}[\cos x]dx$ is equal to

(A) $\frac{\pi}{2}$ **(B)** 0

(C) -1 **(D)** $-\frac{\pi}{2}$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: See Fig. 23.3.

$$\int_0^{\pi}[\cos x]dx=\int_0^{\pi/2}0dx+\int_{\frac{\pi}{2}}^{\pi}(-1)dx=-[x]_{\frac{\pi}{2}}^{\pi}=-\left[\pi-\frac{\pi}{2}\right]=-\frac{\pi}{2}$$

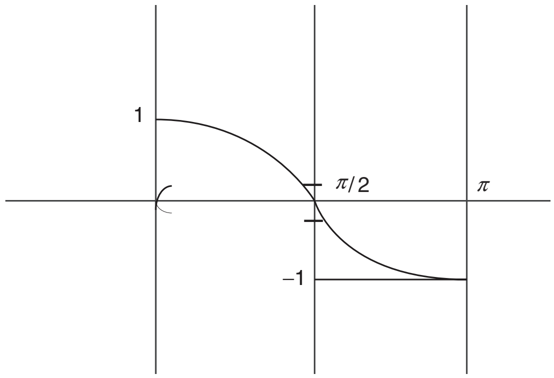


Figure 23.3

Hence, the correct answer is option (D).

12. If for a continuous function $f(x)$, $\int_{-\pi}^{t}(f(x)+x)dx=\pi^2-t^2$, for all $t \geq -\pi$, then $f\left(-\frac{\pi}{3}\right)$ is equal to

(A) π **(B)** $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ **(D)** $\frac{\pi}{6}$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$\int_{-\pi}^{t}(f(x)+x)dx=\pi^2-t^2$$

Therefore,

$$\frac{d}{dt}\left(\int_{-\pi}^{t}(f(x)+x)dx\right)=\frac{d}{dt}(\pi^2-t^2)$$

$$\Rightarrow (f(t)+t)\frac{d}{dt}t=-2t\Rightarrow f(t)=-3t$$

Therefore,

$$f\left(\frac{-\pi}{3}\right)=-3\times\left(\frac{-\pi}{3}\right)=\pi$$

Hence, the correct answer is option (A).

13. Let function F be defined as $F(x)=\int_1^x\frac{e^t}{t}dt, x>0$. Then the value of the integral $\int_1^x\frac{e^t}{t+a}dt$, where $a>0$, is

(A) $e^a[F(x)-F(1+a)]$ **(B)** $e^{-a}[F(x+a)-F(a)]$

(C) $e^a[F(x+a)-F(1+a)]$ **(D)** $e^{-a}[F(x+a)-F(1+a)]$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$f(x)=\int_1^x\frac{e^t}{t}dt, x>0$$

Now

$$I=\int_1^x\frac{e^t}{t+a}dt$$

Put

$$t+a=\mu \Rightarrow t=\mu-a \Rightarrow dt=d\mu$$

Therefore,

$$I=\int_{1+a}^{x+a}\frac{e^{\mu-a}}{\mu}d\mu = e^{-a}\int_{1+a}^{x+a}\frac{e^{\mu}}{\mu}du = e^{-a}\int_{1+a}^{x+a}\frac{e^{t}}{t}dt$$

$$=e^{-a}\left[\int_{1+a}^{1}\frac{e^{t}}{t}dt+\int_{1}^{x+a}\frac{e^{t}}{t}dt\right]$$

$$=e^{-a}\left[\int_{1}^{x+a}\frac{e^{t}}{t}dt-\int_{1}^{1+a}\frac{e^{t}}{t}dt\right]=e^{-a}[F(x+a)-F(1+a)]$$

Hence, the correct answer is option (D).

14. The integral $\int_{2}^{4}\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$ is equal to

(A) 4 **(B)** 1

(C) 6 **(D)** 2

[JEE MAIN 2015 (OFFLINE)]

Solution:

$$I=\int_{2}^{4}\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx$$

$$=\int_{2}^{4}\frac{\log x^2}{\log x^2+\log(6-x)^2}dx$$

$$=\int_{2}^{4}\frac{\log x}{\log x+\log(6-x)}dx \quad (1)$$

$$\Rightarrow I=\int_{2}^{4}\frac{\log(2+4-x)}{\log(2+4-x)+\log(x)}dx$$

By using property $\int_{a}^{b}f(x)dx=\int_{a}^{b}f(a+b-x)dx$, we get

$$I=\int_{2}^{4}\frac{\log(6-x)}{\log(6-x)+\log x} \quad (2)$$

Eqs. (1) + (2) gives,

$$2I=\int_{2}^{4}1dx=2 \Rightarrow I=1$$

Hence, the correct answer is option (B).

15. For $x>0$, let $f(x)=\int_{1}^{x}\frac{\log t}{1+t}dt$. Then $f(x)+f\left(\frac{1}{x}\right)$ is equal to

(A) $\frac{1}{4}(\log x)^2$ **(B)** $\frac{1}{2}(\log x)^2$

(C) $\log x$ **(D)** $\frac{1}{4}\log x^2$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: $x>0$;

$$f(x)+f\left(\frac{1}{x}\right)=\int_{1}^{x}\frac{\log t}{1+t}dt+\int_{1}^{1/x}\frac{\log t}{1+t}dt$$

In the second integral, put

$$t=\frac{1}{\mu} \Rightarrow dt=\frac{-1}{u^2}du$$

When $t=1, u=1$; When $t=\frac{1}{x}, u=x$

Therefore,

$$f(x)+f\left(\frac{1}{x}\right)=\int_{1}^{x}\frac{\log t}{1+t}dt-\int_{1}^{x}\frac{-\log u}{1+\frac{1}{u}}\cdot\frac{1}{u^2}du$$

$$=\int_{1}^{x}\frac{\log t}{1+t}dt+\int_{1}^{x}\frac{\log u}{u(1+u)}du=\int_{1}^{x}\frac{\log t}{1+t}dt+\int_{1}^{x}\frac{\log t}{t(1+t)}dt$$

$$=\int_{1}^{x}\left(\frac{\log t}{1+t}\right)\cdot\left(1+\frac{1}{t}\right)dt=\int_{1}^{x}\frac{1}{t}\log t\,dt=\frac{(\log x)^2}{2}$$

Hence, the correct answer is option (B).

16. If $2\int_{0}^{1}\tan^{-1}x dx=\int_{0}^{1}\cot^{-1}(1-x+x^2)dx$, then $\int_{0}^{1}\tan^{-1}(1-x+x^2)dx$ is equal to

(A) $\frac{\pi}{2}+\log 2$ **(B)** $\log 2$

(C) $\frac{\pi}{2}-\log 4$ **(D)** $\log 4$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We have

$$2\int_{0}^{1}\tan^{-1}x\,dx=\int_{0}^{1}\cot^{-1}(1-x+x^2)dx$$

$$=\int_{0}^{1}\left(\frac{\pi}{2}-\tan^{-1}(1-x+x^2)\right)dx$$

$$=\frac{\pi}{2}-\int_{0}^{1}\tan^{-1}(1-x+x^2)dx$$

Therefore,

$$\int_{0}^{1}\tan^{-1}(1-x+x^2)dx=\frac{\pi}{2}-2\int_{0}^{1}\tan^{-1}dx=\frac{\pi}{2}-2I$$

$$\Rightarrow I=\int_{0}^{1}\tan^{-1}x dx=(x)\tan^{-1}x\Big|_{0}^{1}-\frac{1}{2}\int_{0}^{1}\frac{2x}{1+x^2}dx$$

$$=x\tan^{-1}x\Big|_{0}^{1}-\frac{1}{2}\log(1+x^2)\Big|_{0}^{1}$$

$$=\frac{\pi}{4}-\frac{1}{2}(\log 2-0)=\frac{\pi}{4}-\frac{1}{2}\log 2$$

Hence,

$$\int_{0}^{1}\tan^{-1}(1-x+x^2)dx=\frac{\pi}{2}-\frac{\pi}{2}+\log 2=\log 2$$

Hence, the correct answer is option (B).

17. The value of the integral

$$\int_4^{10} \frac{[x^2]dx}{[x^2-28x+196]+[x^2]}$$

where $[x]$ denotes the greatest integer less than or equal to x, is

(A) $\frac{1}{3}$ **(B)** 6

(C) 7 **(D)** 3

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: We have

$$I = \int_4^{10} \frac{[x^2]dx}{[x^2-28x+196]+[x^2]}$$

$$= \int_4^{10} \frac{[x^2]dx}{[(x-14)^2]+[x^2]}$$

Let us use the following property:

$$I = \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Therefore,

$$I = \int_4^{10} \frac{[(x-14)^2]dx}{[x^2]+[(x-14)^2]}$$

$$2I = \int_4^{10} 1dx = 6 \Rightarrow I = 3$$

Hence, the correct answer is option (D).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2-\frac{\pi^2}{16}}$ equals

(A) $\frac{8}{\pi}f(2)$ **(B)** $\frac{2}{\pi}f(2)$

(C) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ **(D)** $4f(2)$

[IIT-JEE 2007]

Solution: We have

$$\lim_{x\to\frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)dt}{x^2-(\pi^2/16)} \quad \left(\frac{0}{0}\text{form}\right)$$

Using L' Hospital's rule, we get

$$\lim_{x\to\frac{\pi}{4}} \frac{f(\sec^2 x)\cdot(2\sec x)\cdot(\sec x)\cdot(\tan x)}{2x} = \frac{f(2)\cdot 2\cdot\sqrt{2}\cdot\sqrt{2}\cdot 1}{2\cdot(\pi/4)} = \frac{8f(2)}{\pi}$$

Hence, the correct answer is option (A).

2. Match the Column I to Column II.

Column I	Column II
(A) $\int_{-1}^{1} \frac{dx}{1+x^2}$	**(P)** $\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	**(Q)** $2\log\left(\frac{2}{3}\right)$
(C) $\int_2^3 \frac{dx}{1-x^2}$	**(R)** $\frac{\pi}{3}$
(D) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	**(S)** $\frac{\pi}{2}$

[IIT-JEE 2007]

Solution:

(A) → (S)

$$I = \int_{-1}^{1} \frac{dx}{1+x^2}$$
$$= [\tan^{-1} x]_{-1}^{1}$$
$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

(B) → (S)

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$$
$$= [\sin^{-1} x]_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(C) → (P)

$$I = \int_2^3 \frac{dx}{1-x^2}$$
$$= \frac{1}{2}\left[\log\left|\frac{1+x}{1-x}\right|\right]_2^3$$
$$= \frac{1}{2}\left[\log\frac{4}{2} - \log\frac{3}{1}\right]$$
$$= \frac{1}{2}\log\left(\frac{2}{3}\right)$$

(D) → (R)

$$I = \int_1^2 \frac{dx}{x\sqrt{x^2-1}}$$
$$= [\sec^{-1} x]_1^2$$
$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

Hence, the correct matches are (A)→(S); (B)→(S); (C)→(P); (D)→(R).

3. Let f be a non-negative function defined on the interval [0, 1]. If $\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)dt, 0 \le x \le 1$, and $f(0) = 0$, then

(A) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(B) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)>\frac{1}{3}$

(C) $f\left(\frac{1}{2}\right)<\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

(D) $f\left(\frac{1}{2}\right)>\frac{1}{2}$ and $f\left(\frac{1}{3}\right)<\frac{1}{3}$

[IIT-JEE 2009]

Solution:

$$\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_o^x f'(t)dt$$
$$\Rightarrow \sqrt{1-(f'(x))^2} = f(x)$$
$$\Rightarrow f'(x) = \pm\sqrt{1-(f(x))^2}$$
$$\Rightarrow f(x) = \sin x \text{ or } f(x) = -\sin x \text{ (not possible)}$$
$$\Rightarrow f(x) = \sin x$$

Also,

$$x > \sin x \forall x > 0$$

Hence, the correct answer is option (C).

4. If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx, n = 0, 1, 2, \cdots,$ then

(A) $I_n = I_{n+2}$ **(B)** $\sum_{m=1}^{10} I_{2m+1} = 10\pi$

(C) $\sum_{m=1}^{10} I_{2m} = 0$ **(D)** $I_n = I_{n+1}$

[IIT-JEE 2009]

Solution:

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+\pi^x)\sin x}dx$$
$$= \int_0^{\pi}\left(\frac{\sin nx}{(1+\pi^x)\sin x} + \frac{\pi^x \sin nx}{(1+\pi^x)\sin x}\right)dx = \int_0^{\pi} \frac{\sin nx}{\sin x}$$

Now,

$$I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x}dx$$
$$= \int_0^{\pi} \frac{2\cos(n+1)x \cdot \sin x}{\sin x}dx = 0$$
$$\Rightarrow I_1 = \pi, I_2 = \int_0^{\pi} 2\cos x dx = 0$$

Hence, the correct answers are options (A), (B) and (C).

5. Let $f: R \to R$ be a continuous function which satisfies $f(x) = \int_0^x f(t)dt$. Then the value of $f(\ln 5)$ is ______.

[IIT-JEE 2009]

Solution:

$$f(x) = \int_0^x f(t)dt \Rightarrow f(0) = 0$$

also,

$$f'(x) = f(x), x > 0$$
$$\Rightarrow f(x) = k, x > 0$$

Since, $f(0) = 0$ and $f(x)$ is continuous, so

$$f(x) = 0 \;\forall\; x > 0$$

Therefore, $f(\ln 5) = 0$.

Hence, the correct answer is (0).

6. The value of $\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t\ln(1+t)}{t^4+4}dt$ is

(A) 0 **(B)** $\frac{1}{12}$

(C) $\frac{1}{24}$ **(D)** $\frac{1}{64}$

[IIT-JEE 2010]

Solution:

$$\lim_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ln(1+t)dt}{t^4+4} = \left(\frac{0}{0}\text{form}\right)$$

Using L' Hospitals rule, we get

$$\lim_{x\to 0}\frac{x\ln(1+x)}{(x^4+4)3x^2} = \lim_{x\to 0}\frac{1}{3}\frac{\ln(1+x)}{x(x^4+4)} = \frac{1}{3(1+x)[x^2+4+4x^4]} = \frac{1}{12}$$

Hence, the correct answer is option (B).

7. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are)

(A) $\frac{22}{7}-\pi$ **(B)** $\frac{2}{105}$

(C) 0 **(D)** $\frac{71}{15}-\frac{3\pi}{2}$

[IIT-JEE 2010]

Solution:

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$$
$$= \int_0^1\left(x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2}\right)dx$$
$$= \left[\frac{x^7}{7}-\frac{2x^6}{3}+x^5-\frac{4x^3}{3}+4x\right]_0^1 - \pi$$
$$= \frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-\pi = \frac{22}{7}-\pi$$

Hence, the correct answer is option (A).

8. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x\sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)}dx$ is

(A) $\frac{1}{4}\ln\frac{3}{2}$ **(B)** $\frac{1}{2}\ln\frac{3}{2}$

(C) $\ln\frac{3}{2}$ **(D)** $\frac{1}{6}\ln\frac{3}{2}$

[IIT-JEE 2011]

Solution: Put

$$x^2 = t \Rightarrow 2x\,dx = dt$$

$$I = \frac{1}{2}\int_{\ln 2}^{\ln 3}\frac{\sin t}{\sin t + \sin(\ln 6 - t)}\,dt \;\Rightarrow\; I = \frac{1}{2}\int_{\ln 2}^{\ln 3}\frac{\sin(\ln 6 - t)}{\sin(\ln 6 - t) + \sin t}\,dt$$

$$2I = \frac{1}{2}\int_{\ln 2}^{\ln 3} 1\,dt \Rightarrow I = \frac{1}{4}\ln\frac{3}{2}$$

Hence, the correct answer is option (A).

9. Let $f:[1,\infty)\to[2,\infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)\,dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is ______.

[IIT-JEE 2011]

Solution:

$$6\int_1^x f(t)dt = 3x\,f(x) - x^3 \Rightarrow 6f(x) = 3f(x) + 3xf'(x) - 3x^2$$

$$\Rightarrow 3f(x) = 3xf'(x) - 3x^2 \Rightarrow xf'(x) - f(x) = x^2$$

$$\Rightarrow x\frac{dy}{dx} - y = x^2 \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = x \qquad (1)$$

$$\text{I.F} = e^{\int -\frac{1}{x}dx} = e^{-\log_e x}$$

Multiplying both sides of Eq. (1) by $\frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 1 \Rightarrow \frac{d}{dx}\left(y\cdot\frac{1}{x}\right) = 1$$

Integrating

$$\frac{y}{x} = x + c$$

Put $x = 1, y = 2$

$$\Rightarrow 2 = 1 + c \Rightarrow c = 1 \Rightarrow y = x^2 + x$$
$$\Rightarrow f(x) = x^2 + x \Rightarrow f(2) = 6$$

Note: If we put $x = 1$ in the given equation we get $f(1) = 1/3$.

Hence, the correct answer is (6).**

***Question is ambiguous as 8/3 can also be the answer.*

10. Match the statements given in **Column I** with the values given in **Column II.**

Column I	Column II
(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}, \vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	**(P)** $\frac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	**(Q)** $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6}\sec(\pi x)dx$ is	**(R)** $\frac{\pi}{3}$
(D) The maximum value of $\left\lvert \text{Arg}\left(\frac{1}{1-z}\right)\right\rvert$ for $\lvert z\rvert = 1, z \neq 1$ is given by	**(S)** π
	(T) $\frac{\pi}{2}$

[IIT-JEE 2011]

Solution:

(A)

$$\vec{a}\cdot\vec{b} = -1 + 3 = 2$$
$$\lvert\vec{a}\rvert = 2, \lvert\vec{b}\rvert = 2$$
$$\cos\theta = \frac{2}{2\times 2} = \frac{1}{2}$$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ but $\frac{2\pi}{3}$ is opposite to the side of the maximum length.

(B)

$$\int_a^b (f(x) - 3x)dx = a^2 - b^2$$

$$\int_a^b f(x)dx = \frac{3}{2}(b^2 - a^2) + a^2 - b^2 = \frac{-a^2 + b^2}{2}$$

$$\Rightarrow f(x) = x$$

(C)

$$\frac{\pi^2}{\ln 3}\left(\frac{\ln\lvert(\sec\pi x + \tan\pi x)\rvert_{7/6}^{5/6}}{\pi}\right)$$

$$= \frac{\pi}{\ln 3}\left(\ln\left\lvert\sec\frac{5\pi}{6} + \tan\frac{5\pi}{6}\right\rvert - \ln\left\lvert\sec\frac{7\pi}{6} + \tan\frac{7\pi}{6}\right\rvert\right) = \pi$$

(D) Let

$$u = \frac{1}{1-z} \Rightarrow z = 1 - \frac{1}{u}$$

$$\lvert z\rvert = 1 \Rightarrow \left\lvert 1 - \frac{1}{u}\right\rvert = 1$$

$$\Rightarrow \lvert u - 1\rvert = \lvert u\rvert$$

Therefore, locus of u is perpendicular bisector of line segment joining 0 and 1, so maximum arg u approaches $\frac{\pi}{2}$ but will not attain.

Hence, the correct matches are (A)→(Q); (B)→(P); (C)→(S); (D)→(T).

11. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2 + \ln\frac{\pi + x}{\pi - x}\right)\cos x\,dx$ is

(A) 0

(B) $\frac{\pi^2}{2} - 4$

(C) $\frac{\pi^2}{2} + 4$

(D) $\frac{\pi^2}{2}$

[IIT-JEE 2012]

Solution:

$$\int_{-\pi/2}^{\pi/2}\left\{x^2+\ln\left(\frac{\pi+x}{\pi-x}\right)\right\}\cos x\,dx$$

$$=\int_{-\pi/2}^{\pi/2}x^2\cos x\,dx+\int_{-\pi/2}^{\pi/2}\ln\left(\frac{\pi+x}{\pi-x}\right)\cos x\,dx$$

$$=2\int_0^{\pi/2}x^2\cos x\,dx+0$$

$$=2[x^2\sin x+2x\cos x-2\sin x]_0^{\pi/2}$$

$$=2\left[\frac{\pi^2}{4}-2\right]=\frac{\pi^2}{2}-4$$

Hence, the correct answer is option (B).

12. Let $f:\left[\frac{1}{2},1\right]\to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x)<2f(x)$ and $f\left(\frac{1}{2}\right)=1$. Then the value of $\int_{1/2}^{1}f(x)dx$ lies in the interval

(A) $(2e-1, 2e)$ **(B)** $(e-1, 2e-1)$

(C) $\left(\frac{e-1}{2}, e-1\right)$ **(D)** $\left(0, \frac{e-1}{2}\right)$

[JEE ADVANCED 2013]

Solution: We have

$$\frac{dy}{dx}<2y$$

That is,

$$e^{-2x}\frac{dy}{dx}<2ye^{-2x}$$

$$\frac{d}{dx}(ye^{-2x})<0$$

which implies that ye^{-2x} is a decreasing function.

As $\frac{1}{2}<x<1$, we have

$$e^{-1}>ye^{-2x}>y(1)e^{-2}$$

$$\Rightarrow e^{2x-1}>y>y(1)e^{2x-2}$$

$$\Rightarrow \int_{1/2}^{1}e^{2x-1}dx>\int_{1/2}^{1}ydx>\int_{1/2}^{1}y(1)e^{2x-2}>0$$

Therefore,

$$0<\int_{1/2}^{1}ydx<\frac{e-1}{2}$$

Hence, the correct answer is option (D).

13. The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$ is ______.

[JEE ADVANCED 2014]

Solution:

$$\int_0^1 \underbrace{4x^3}_{\text{I}}\underbrace{\frac{d^2}{dx^2}(1-x^2)^5}_{\text{II}}dx$$

Integrating by parts

$$I=4x^3\left[\frac{d}{dx}(1-x^2)^5\right]_0^1-\int_0^1 12x^2\frac{d}{dx}(1-x^2)^5dx$$

$$=4x^3[5(1-x^2)^4(-2x)]_0^1-12\left[[x^2(1-x^2)^5]_0^1-\int_0^1 2x(1-x^2)^5dx\right]$$

$$=0-0+12\int_0^1 2x(1-x^2)^5dx$$

Now putting $1-x^2=t$. Then

$$-2xdx=dt$$

Where $x=0, t=1$

Where $x=1, t=0$

Therefore,

$$I=-12\int_1^0 t^5dt$$

$$I=12\times\int_0^1 t^5dt=12\times\left[\frac{t^6}{6}\right]_0^1=12\times\frac{1}{6}=2$$

Hence, the correct answer is (2).

14. The following integral $\int_{\pi/4}^{\pi/2}(2\operatorname{cosec}x)^{17}dx$ is equal to

(A) $\int_0^{\log(1+\sqrt{2})}2(e^u+e^{-u})^{16}du$

(B) $\int_0^{\log(1+\sqrt{2})}2(e^u+e^{-u})^{17}du$

(C) $\int_0^{\log(1+\sqrt{2})}2(e^u-e^{-u})^{17}du$

(D) $\int_0^{\log(1+\sqrt{2})}2(e^u-e^{-u})^{16}du$

[JEE ADVANCED 2014]

Solution: $\int_{\pi/4}^{\pi/2}(2\operatorname{cosec}x)^{17}dx$

Let $\operatorname{cosec}x+\cot x=e^t$. Then

$$(-\operatorname{cosec}x\cot x-\operatorname{cosec}^2x)dx=e^tdt$$

$$\Rightarrow -\operatorname{cosec}x(\cot x+\operatorname{cosec}x)dx=e^tdt$$

$$\Rightarrow -(\operatorname{cosec}x)e^tdx=e^tdt$$

Therefore,

$$(\operatorname{cosec}x)dx=-dt$$

Now when $x=\frac{\pi}{4}$,

$$\operatorname{cosec}\frac{\pi}{4}+\cot\frac{\pi}{4}=e^t$$

Therefore,

$$e^t=\sqrt{2}+1$$

$$\Rightarrow t=\log_e(1+\sqrt{2})$$

when $x=\dfrac{\pi}{2}$,

$$\operatorname{cosec}\frac{\pi}{2}+\cot\frac{\pi}{2}=e^t$$

Therefore,

$$1-0=e^t$$
$$\Rightarrow t=0$$

Therefore,

Also $\operatorname{cosec}^2 x-\cot^2 x=1$

$$\Rightarrow \operatorname{cosec}x-\cot x=\frac{1}{e^t}=e^{-t}$$

Therefore,

$$2\operatorname{cosec}x=e^t+e^{-t}$$

Therefore, integral reduces to

$$-\int_{\log_e(1+\sqrt{2})}^{0}(e^t+e^{-t})^{16}2dt=+2\int_0^{\log_e(1+\sqrt{2})}(e^t+e^{-t})dt$$

Note: t can be replaced by μ.

Hence, the correct answer is option (A).

Paragraph for Questions 15 and 16: Given that for each $a\in(0,1)$. $\lim\limits_{h\to0^+}\int_h^{1-h}t^{-a}(1-t)^{a-1}dt$ exists. Let this limit be $g(a)$. In addtion, it is given that function $g(a)$ is differentiable on (0, 1).

[JEE ADVANCED 2014]

15. The value of $g\left(\dfrac{1}{2}\right)$ is

(A) π **(B)** 2π

(C) $\dfrac{\pi}{2}$ **(D)** $\dfrac{\pi}{4}$

Solution: See Fig. 23.4.

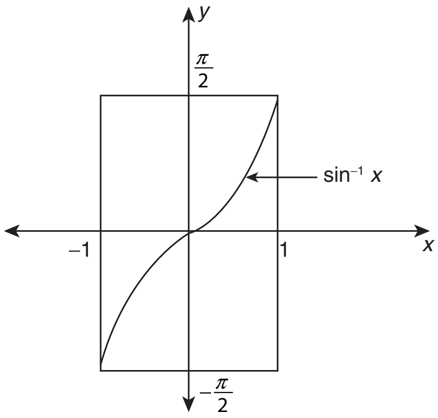


Figure 23.4

Given

$$g(a)=\lim_{h\to0^+}\int_h^{1-h}t^{-a}(1-t)^{a-1}dt$$

$$g\left(\frac{1}{2}\right)=\lim_{h\to0^+}\int_h^{1-h}t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt=\lim_{h\to0^+}\int_h^{1-h}\frac{1}{\sqrt{t(1-t)}}dt$$

$$=\lim_{h\to0^+}\int_h^{1-h}\frac{1}{\sqrt{t-t^2}}dt=\lim_{h\to0^+}\int_h^{1-h}\frac{1}{\sqrt{-\left(t^2-t+\frac{1}{4}-\frac{1}{4}\right)}}dt$$

$$=\lim_{h\to0^+}\int_h^{1-h}\frac{1}{\sqrt{-\left\{t-\frac{1}{2}\right\}^2+\frac{1}{4}}}dt=\lim_{h\to0^+}\int_h^{1-h}\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(t-\frac{1}{2}\right)^2}}dt$$

$$=\lim_{h\to0^+}\left[\sin^{-1}\left(\frac{t-1/2}{1/2}\right)\right]_h^{1-h}=\lim_{h\to0^+}\left[\sin^{-1}(2t-1)\right]_h^{1-h}$$

$$=\lim_{h\to0^+}[\sin^{-1}\{2-2h-1\}-\sin^{-1}(2h-1)]$$

$$=\lim_{h\to0^+}+\sin^{-1}(1-2h)-\lim_{h\to0^+}+\sin^{-1}(2h-1)$$

$$=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)$$

$$=\pi$$

Hence, the correct answer is option (A).

16. The value of $g'\left(\dfrac{1}{2}\right)$ is

(A) $\dfrac{\pi}{2}$ **(B)** π

(C) $-\dfrac{\pi}{2}$ **(D)** 0

Solution:

$$g(a)=\lim_{h\to0^+}\int_h^{1-h}t^{-a}(1-t)^{a-1}dt \quad (1)$$

Now

$$g(1-a)=\lim_{h\to0^+}\int_h^{1-h}t^{-(1-a)}(1-t)^{1-a-1}dt$$

$$=\lim_{h\to0^+}\int_h^{1-h}t^{a-1}(1-t)^{-a}dt$$

$$=\lim_{h\to0^+}\int_h^{1-h}(\cancel{h}+1-\cancel{h}-t)^{a-1}\{1-(\cancel{h}+1-\cancel{h}-t)\}^{-a}dt$$

$$=\lim_{h\to0^+}\int_h^{1-h}(1-t)^{a-1}(\cancel{1}-\cancel{1}+t)^{-a}dt$$

$$=\lim_{h\to0^+}\int_h^{1-h}t^{-a}(1-t)^{a-1}dt \quad (2)$$

From Eqs. (1) and (2)

$$g(a)=g(1-a)$$

Therefore,

$$g'(a)=-g'(1-a)$$

when

$$a=\frac{1}{2},\ g'\left(\frac{1}{2}\right)=-g'\left(\frac{1}{2}\right)$$

$$\Rightarrow g'\left(\frac{1}{2}\right)=0$$

Hence, the correct answer is option (D).

17. Match the List I with List II.

List I	List II
P. The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f(0)=0$ and $\int_0^1 f(x)dx=1$, is	**1.** 8

Q.	The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	**2.** 2
R.	$\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equals	**3.** 4
S.	$\dfrac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}{\left(\int_{0}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right) dx\right)}$ equals	**4.** 0

	P	Q	R	S
(A)	3	2	4	1
(B)	2	3	4	1
(C)	3	2	1	4
(D)	2	3	1	4

[JEE ADVANCED 2014]

Solution:

For (P) in List I: Let

$$f(x) = ax^2 + bx \qquad [\text{As } f(0) = 0]$$

Also

$$\int_0^1 f(x)dx = 1 \Rightarrow \int_0^1 (ax^2 + bx)dx = 1$$

$$\Rightarrow \left[\frac{ax^3}{3} + \frac{bx^2}{2}\right]_0^1 = 1 \Rightarrow \frac{a}{3} + \frac{b}{2} = 1$$

Since, $a \ge 0, b \ge 0$. Therefore,

$$\frac{0}{3} + \frac{2}{2} = 1$$

So, $a = 0$ and $b = 2$. Now

$$\frac{3}{3} + \frac{0}{2} = 1$$

So, $a = 3$ and $b = 2$. Thus, the possible polynomials are

$$f(x) = bx \text{ or } f(x) = 3x^2$$

Hence,

$$\textbf{(P)} \rightarrow \textbf{(2)}$$

For (Q) is List I:

$$f(x) = \sin x^2 + \cos x^2 = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x^2 + \frac{1}{\sqrt{2}}\cos x^2\right]$$

$$= \sqrt{2}\sin\left(x^2 + \frac{\pi}{4}\right)$$

For maximum value of $f(x)$, we have

$$x^2 + \frac{\pi}{4} = \frac{\pi}{2} + 2n\pi \Rightarrow x^2 = 2n\pi + \frac{\pi}{4}$$

$$\Rightarrow x = \pm\sqrt{\frac{\pi}{4}}, \text{ for } n = 0; x = \pm\sqrt{\frac{9\pi}{4}}, \text{ for } n = 1$$

Hence,

$$\textbf{(Q)} \rightarrow \textbf{(3)}$$

For (R) in List I:

$$\int_{-2}^{2} \frac{3x^2}{1+e^x} dx = \int_0^2 \left(\frac{3x^2}{1+e^x} + \frac{e^x 3x^2}{1+e^x}\right) dx$$

$$= \int_0^2 3x^2\, dx = 8$$

Hence,

$$\textbf{(R)} \rightarrow \textbf{(1)}$$

For (S) in List I: Let

$$f(x) = \cos 2x \ln\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow f(-x) = \cos 2x \ln\left(\frac{1-x}{1+x}\right)$$

$$\Rightarrow f(x) + f(-x) = \cos 2x\left[\ln\left(\frac{1+x}{1-x} \times \frac{1-x}{1+x}\right)\right] = 0$$

Therefore, $f(x)$ is an odd function. Thus,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \ln\left(\frac{1+x}{1-x}\right) dx = 0$$

So, the denominator is non-zero. Hence,

$$\textbf{(S)} \rightarrow \textbf{(4)}$$

Hence, the correct answer is option (D).

18. Let $f: R \rightarrow R$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$, where $[x]$ is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2 + f(x+1)} dx$, then the value of $(4I - 1)$ is ______.

[JEE ADVANCED 2015]

Solution: $f: R \rightarrow R$ is

$$f(x) = \begin{cases} [x]; & x \le 2 \\ 0; & x > 2 \end{cases}$$

$$I = \int_{-1}^{2} \frac{xf(x^2)}{2 + f(x+1)} dx;$$

$$I = \int_{-1}^{0} \frac{xf(x^2)dx}{2+f(x+1)} + \int_{0}^{1} \frac{xf(x^2)dx}{2+f(x+1)} + \int_{1}^{\sqrt{2}} \frac{xf(x^2)dx}{2+f(x+1)} + \int_{\sqrt{2}}^{\sqrt{3}} \frac{xf(x^2)dx}{2+f(x+1)} + \int_{\sqrt{3}}^{2} \frac{xf(x^2)dx}{2+f(x+1)}$$

$$= \int_{-1}^{0} \frac{x[x^2]dx}{2+[x+1]} + \int_{0}^{1} \frac{x[x^2]}{2+[x+1]}dx + \int_{1}^{\sqrt{2}} \frac{x[x^2]dx}{2+0} + \int_{\sqrt{2}}^{\sqrt{3}} \frac{x \cdot 0\, dx}{x+0} + \int_{\sqrt{3}}^{2} \frac{x \cdot 0}{2+0}dx$$

$$= 0 + 0 + \int_{1}^{\sqrt{2}} \frac{x(1)}{2}dx = \left[\frac{x^2}{4}\right]_1^{\sqrt{2}} = \frac{1}{4} \Rightarrow 4I - 1 = 0$$

Hence, the correct answer is (0).

19. If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x})\left(\frac{12+9x^2}{1+x^2}\right)dx$, where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is ______.

[JEE ADVANCED 2015]

Solution:

$$\alpha=\int_0^1 (e^{9x+3\tan^{-1}x})\left(\frac{12+9x^2}{1+x^2}\right)dx$$

Here

$$\tan^{-1}x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Put

$$\tan^{-1}x=t\Rightarrow\frac{1}{1+x^2}dx=dt$$

$$\Rightarrow\ \alpha=\int_0^{\pi/4} e^{9\tan t+3t}\cdot(12+9\tan^2 t)\cdot dt$$

Again, put

$$9\tan t+3t=z$$

$$\Rightarrow\ (9(\sec^2 t)+3)dt=dz$$

$$\Rightarrow\ (9(\tan^2 t)+12)dt=dz$$

$$\Rightarrow\ \alpha=\int_0^{9+3\pi/4} e^z dz=e^{\left(9+\frac{3\pi}{4}\right)}-1$$

$$\Rightarrow\ 1+\alpha=e^{9+3\pi/4}$$

$$\Rightarrow\ \ln|1+\alpha|=9+3\pi/4$$

$$\Rightarrow\ \ln|(1+\alpha)|-\frac{3\pi}{4}=9$$

Hence, the correct answer is (9).

20. Let $f:R\to R$ be a continuous odd function, which vanishes exactly at one point and $f(1)=\frac{1}{2}$. Suppose that $F(x)=\int_{-1}^{x} f(t)\,dt$ for all $x\in[-1,2]$ and $G(x)=\int_{-1}^{x} t\,|f(f(t))|\,dt$ for all $x\in[-1,2]$. If $\lim_{x\to1}\frac{F(x)}{G(x)}=\frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is ______.

[JEE ADVANCED 2015]

Solution: $f:\mathrm{R}\to\mathrm{R}$ is a continuous odd function having a single root.

$$f(1)=\frac{1}{2},\quad F(x)=\int_{-1}^{x} f(t)dt \qquad \forall x\in[-1,2]$$

$$G(x)=\int_{-1}^{x} t|f(f(t))|dt \qquad \forall x\in[1,2]$$

$$\lim_{x\to1}\frac{F(x)}{G(x)}=\frac{1}{14}, f\left(\frac{1}{2}\right)=?$$

Clearly $f(t)$ and $t|f(f(t))|$ are odd functions for $t\in(-1,x)$ and $x\in[-1,2]$. Therefore,

$$\lim_{x\to1}\frac{F(x)}{G(x)}=\frac{1}{14}\ \left(\text{limit is of }\frac{0}{0}\text{ form}\right)$$

$$\Rightarrow\ \lim_{x\to1}\frac{f(x)}{x|f(f(x))|}=\frac{1}{14}$$

$$\Rightarrow\ \frac{f(1)}{1|f(f(1))|}=\frac{1}{14}$$

$$\Rightarrow\ \frac{1/2}{\left|f\left(\frac{1}{2}\right)\right|}=\frac{1}{14}$$

$$\Rightarrow\ \left|f\left(\frac{1}{2}\right)\right|=7$$

$$\Rightarrow\ f\left(\frac{1}{2}\right)=\pm7,$$

but being odd function and continuous $f(0)=0$, thus if

$$f\left(\frac{1}{2}\right)=-7,$$

then $f(x)$ must have another root in $\left(\frac{1}{2},1\right)$. So

$$f\left(\frac{1}{2}\right)\neq-7$$

$$\Rightarrow\ f\left(\frac{1}{2}\right)=7$$

Hence, the correct answer is (7).

21. The option(s) with the value of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at+\cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at+\cos^4 at)dt}=L$$

(A) $a=2,\ L=\frac{e^{4\pi}-1}{e^{\pi}-1}$

(B) $a=2,\ L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) $a=4,\ L=\frac{e^{4\pi}-1}{e^{\pi}-1}$

(D) $a=4,\ L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

[JEE ADVANCED 2015]

Solution: For $a=2$,

$$L=\int_0^{4\pi}\frac{e^t(\sin^6 2t+\cos^4 2t)}{\int_0^{\pi} e^t(\sin^6 2t+\cos^4 2t)dt}dt$$

Let

$$L_1=\int_0^{4\pi} e^t(\sin^6 2t+\cos^4 2t)dt$$

$$=\int_0^{\pi} f(t)dt+\int_{\pi}^{2\pi} f(t)dt+\int_{2\pi}^{3\pi} f(t)dt+\int_{2\pi}^{4\pi} f(t)dt=I_1+I_2+I_3+I_4$$

In the 2nd integration, put

$$t=\pi+x\Rightarrow dt=dx$$

and

$$t=\pi\Rightarrow x=0,\ t=2\pi\Rightarrow x=\pi$$

That is,

$$I_2=\int_0^{\pi} f(\pi+x)dx$$

In I_3, put

$$t=2\pi+x$$

$$\Rightarrow dt=dx,\ t=2\pi\Rightarrow x=0,\ t=3\pi, x=\pi$$

$$\Rightarrow\ I_3=\int_0^{\pi} f(2\pi+x)dx$$

In I_4, put

$$t=3\pi+x\Rightarrow dt=dx$$

$$t = 3\pi \Rightarrow x = 0,\ t = 4\pi \Rightarrow x = \pi$$

$$\Rightarrow I_4 = \int_0^{\pi} f(3\pi + x)dx$$

Therefore,

$$L = \frac{L_1}{I_1} = \frac{I_2 + I_2 + I_3 + I_4}{I_1}$$

$$= \frac{I_1 + e^{\pi} I_1 + e^{2\pi} I_1 + e^{3\pi} I_1}{I_1}$$

$$\Rightarrow L = (1 + e^{\pi} + e^{2\pi} + e^{2\pi}) = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

Similarly,

$$L = \frac{e^{4\pi} - 1}{e^{\pi} - 1} \text{ for } a = 4$$

Hence, the correct answers are options (A) and (C).

22. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Then the correct expression(s) is (are)

(A) $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ **(B)** $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} xf(x)dx = \frac{1}{6}$ **(D)** $\int_0^{\pi/4} f(x)dx = 1$

[JEE ADVANCED 2015]

Solution:

$$f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x \ \forall\ x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

$$= 7\tan^6 x \cdot \sec^2 x - 3\tan^2 x \cdot \sec^2 x$$

$$= (7\tan^6 x - 3\tan^2 x) \cdot \sec^2 x$$

$$\Rightarrow \int_0^{\pi/4} f(x)dx = \int_0^{\pi/4} (7\tan^6 x - 3\tan^2 x)\sec^2 dx$$

$$= \int_0^1 (7t^6 - 3t^2)dt = [t^7 - t^3]_0^1 = 0$$

Also,

$$I = \int_0^{\pi/4} xf(x)dx$$

$$= \left| x \cdot \int (7\tan^6 x - 3\tan^2 x)\sec^2 x\, dx \right|_0^{\pi/4}$$

$$- \int_0^{\pi/4} 1 \cdot \int (7\tan^6 x - 3\tan^2 x)\sec^2 x\, dx$$

$$= \left| x \cdot (\tan^7 x - \tan^3 x) \right|_0^{\pi/4} - \int_0^{\pi/4} (\tan^7 x - \tan^3 x)dx$$

$$= 0 - \int_0^{\pi/4} \tan^3 x(\tan^4 x - 1)dx$$

$$= -\int_0^{\pi/4} \tan^3 x(\tan^2 x - 1)(\sec^2 x)dx$$

$$= -\int_0^1 (t^5 - t^3)dt$$

$$= -\left[\frac{t^6}{6} - \frac{t^4}{4}\right]_0^1 = \left[\frac{1}{4} - \frac{1}{6}\right] = \frac{1}{12}$$

Hence, the correct answers are options (A) and (B).

23. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$ is ________.

[JEE ADVANCED 2016]

Solution: See Fig. 23.5. We have $x \in [0,1]$ for which

$$\int_0^x \frac{t^2}{1+t^4}dt = 2x - 1$$

Let

$$F(x) = \int_0^x \frac{t^2}{1+t^4}dt$$

and

$$f(x) = 2x - 1$$

Now, if $F'(x) = \frac{x^2}{1+x^4} > 0$, it means that $F(x)$ is an increasing function.

$F(0) = 0$

$$F(x) = \frac{1}{2}\int_0^x \frac{(t^2+1)+(t^2-1)}{1+t^4}dt$$

$$= \frac{1}{2}\int_0^x \left(\frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} + \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2} \right) dt$$

$$= \frac{1}{2}\int_0^x \frac{\left(1 + \frac{1}{t^2}\right)dt}{\left(t - \frac{1}{t}\right)^2 + 2} + \frac{1}{2}\int_0^x \frac{\left(1 - \frac{1}{t^2}\right)}{\left(t + \frac{1}{t}\right)^2 - 2}dt$$

$$= \frac{1}{2\sqrt{2}} \left\{ \tan^{-1}\left(t - \frac{1}{t}\right) + \ln\left(\frac{t + \frac{1}{t} - \sqrt{2}}{t + \frac{1}{t} + \sqrt{2}} \right) \right\} \Bigg|_0^x$$

$$= \frac{1}{2\sqrt{2}} \left\{ \tan^{-1}\left(\frac{t^2 - 1}{t}\right) + \ln\left(\frac{t^2 + 1 - \sqrt{2}t}{t^2 + 1 + \sqrt{2}t} \right) \right\} \Bigg|_0^x$$

$$= \frac{1}{2\sqrt{2}} \left\{ \tan^{-1}\left(\frac{x^2 - 1}{x}\right) + \ln\left(\frac{x^2 + 1 - \sqrt{2}t}{x^2 + 1 + \sqrt{2}t} \right) - \left(-\frac{\pi}{2} + 0\right) \right\}$$

Therefore,

$$F(x) = \frac{1}{2\sqrt{2}}\frac{\pi}{2} + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2 - 1}{x}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{x^2 + 1 - \sqrt{2}x}{x^2 + 1 + \sqrt{2}x}\right)$$

$$F(1) = \frac{\pi}{4\sqrt{2}} + \frac{1}{2\sqrt{2}}(0) + \frac{1}{2\sqrt{2}}\ln\left(\frac{2 - \sqrt{2}}{2 + \sqrt{2}}\right) < 1$$

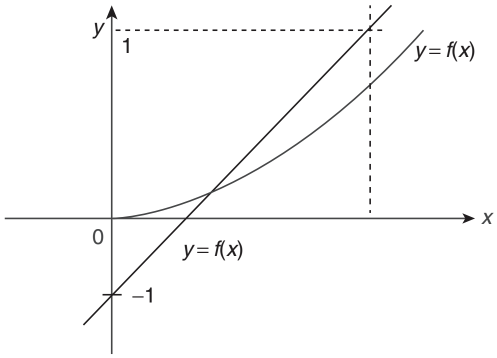


Figure 23.5

Now,

$$f(x) = 2x - 1 \Rightarrow f(0) = -1$$

and

$$f(1) = 2 - 1 = 1$$

Therefore, the total number of distinct values of $x \in [0.1]$ is only one.

Hence, the correct answer is (1).

24. The value of $\int_{-(\pi/2)}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx$ is equal to

(A) $\frac{\pi^2}{4} - 2$ **(B)** $\frac{\pi^2}{4} + 2$

(C) $\pi^2 - e^{-(\pi/2)}$ **(D)** $\pi^2 + e^{(\pi/2)}$

[JEE ADVANCED 2016]

Solution: The given integral is

$$I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx$$

Using the integral property, we get

$$I = \int_0^{\pi/2} \left(\frac{x^2 \cos x}{1+e^x} + \frac{x^2 \cos x}{1+e^{-x}} \right) dx$$

$$I = \int_0^{\pi/2} x^2 \cos x \, dx$$

That is,

$$\int x^2 \cos x \, dx = x^2 \sin x - 2\int x \sin x \, dx$$

$$= x^2 \sin x - 2\left\{ -x \cos x + \int \cos x \, dx \right\}$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x$$

Therefore,

$$I = x^2 \sin x + 2x \cos x - 2 \sin x \Big|_0^{\pi/2} = \left(\frac{\pi^2}{4} + 0 - 2 \right) - (0) = \frac{\pi^2}{4} - 2$$

Hence, the correct answer is option (A).

Practice Exercise 1

1. If $f(x) = x - [x]$: for every real number of x, when $[x]$ is the integral point of x. Then $\int_{-1}^{1} f(x) dx$ is equal to

(A) 1 **(B)** 2

(C) 0 **(D)** $-\frac{1}{2}$

2. Let $f(x) = \int_1^x \sqrt{2 - t^2} dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(A) (0, 1) **(B)** $\pm\frac{1}{\sqrt{2}}$

(C) $\pm\frac{1}{2}$ **(D)** ± 1

3. Let $T > 0$, be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x + T) = f(x)$. If $I = \int_0^T f(x) dx$, then value of $\int_3^{3+3T} f(2x) dx$ is

(A) $-\frac{3}{2}I$ **(B)** $2I$

(C) $3I$ **(D)** $6I$

4. The value of $\int_0^{100} a^{x-[x]} dx$ is

(A) $\frac{100(a-1)}{\log a}$ **(B)** $\frac{100(a+1)}{\log a}$

(C) $100(a - 1)$ **(D)** None of these

5. Value of $\int_0^x [\sin t] dt$, $(2n+1)\pi < x < (2n+2)\pi$, $n \in N$ is equal to

(A) $(n+1)\pi p + x$ **(B)** $n\pi + x$

(C) $n\pi - x$ **(D)** $(n+1)\pi - x$

6. The value of $\int_0^{\pi} \frac{dx}{1+2^{\tan x}}$ is equal to

(A) 0 **(B)** $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$ **(D)** π

7. $\int_{-\pi/4}^{\pi/4} \frac{\sec^2 x dx}{1+e^x}$ equals

(A) 2 **(B)** 0

(C) –1 **(D)** 1

8. If $f(x) = f(a - x)$, then $\int_0^a x f(x) \, dx$ equals

(A) $\frac{a}{2}\int_0^a x f(x) \, dx$ **(B)** $a\int_0^a x f(x) \, dx$

(C) $\int_0^a x f(x) \, dx$ **(D)** None of these

9. If $I_p = \int_1^e (\ln x)^p dx$, then $I_p + pI_{p-1}$ is less than

(A) 1 **(B)** 2

(C) 3 **(D)** None of these

10. If $\int_0^{\frac{\pi}{2}} \ln|\tan x + \cot x|\, dx = \lambda \ln 2$, then λ equals

(A) $\frac{\pi}{2}$ **(B)** π

(C) 2π **(D)** None of these

11. If $F(x) = \int_0^x \ln\left(\frac{1-t}{1+t}\right) dt$, then $F'(-x) + F'(x)$ equals

(A) 0 **(B)** e

(C) $\frac{1}{e}$ **(D)** None of these

12. $\int_0^{\frac{\pi}{4}} [\sin x + \cos x]\, d(x - [x])$, [.] greatest integer function, equals

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{2}$

(C) $\frac{3\pi}{4}$ **(D)** None of these

13. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$ equals

(A) $\sqrt{5}+1$ **(B)** $\sqrt{5}-1$

(C) $\sqrt{2}+1$ **(D)** $\sqrt{2}-1$

14. $\int_0^{[x]} \frac{2^x}{2^{[x]}} dx$ equals (where [.] denotes the greatest integer function)

(A) $\ln 2$ **(B)** $\frac{[x]}{\ln 2}$

(C) $\frac{1}{2}\frac{[x]}{\ln 2}$ **(D)** None of these

15. The value of integral $\int_0^1 \frac{x^b - 1}{\log x} dx$ is

(A) $\log b$ **(B)** $2\log(b+1)$

(C) $3\log b$ **(D)** None of these

16. The value of the integral $\int_{-1}^{1} \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) dx$ is

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{4}$

(C) $-\frac{\pi}{2}$ **(D)** None of these

17. The least value of the function

$$F(x) = \int_{5\pi/4}^{x} (3\sin u + 4\cos u)\, du$$

on the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$ is

(A) $\sqrt{3} + \frac{3}{2}$ **(B)** $-2\sqrt{3} + \frac{3}{2} + \frac{1}{\sqrt{2}}$

(C) $\frac{3}{2} + \frac{1}{\sqrt{2}}$ **(D)** None of these

18. $\int_0^{\infty} e^{-2x}(\sin 2x + \cos 2x)\, dx =$

(A) 1 **(B)** 0

(C) $\frac{1}{2}$ **(D)** ∞

19. $\int_0^{b-c} f''(x+a)\, dx =$

(A) $f'(a) - f'(b)$ **(B)** $f'(b-c+a) - f'(a)$

(C) $f'(b+c-a) + f'(a)$ **(D)** None of these

20. The greatest value of the function $F(x) = \int_1^x |t|\, dt$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is given by

(A) $\frac{3}{8}$ **(B)** $-\frac{1}{2}$

(C) $-\frac{3}{8}$ **(D)** $\frac{2}{5}$

21. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^2 x (\sin x + \cos x)\, dx =$

(A) $\frac{2}{15}$ **(B)** $\frac{4}{15}$

(C) $\frac{6}{15}$ **(D)** $\frac{8}{15}$

22. $\int_0^{\infty} \frac{dx}{\left(x + \sqrt{x^2+1}\right)^3} =$

(A) $\frac{3}{8}$ **(B)** $\frac{1}{8}$

(C) $-\frac{3}{8}$ **(D)** None of these

23. The derivative of $F(x) = \int_{x^2}^{x^3} \frac{1}{\log t} dt$, $(x > 0)$ is

(A) $\frac{1}{3\log x} - \frac{1}{2\log x}$ **(B)** $\frac{1}{3\log x}$

(C) $\frac{3x^2}{3\log x}$ **(D)** $(\log x)^{-1} \cdot x(x-1)$

24. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in

(A) (2, 2) **(B)** No value of x

(C) $(0, \infty)$ **(D)** $(-\infty, 0)$

25. If $f(x) = \int_{x^2}^{x^4} \sin\sqrt{t}\, dt$, then $f'(x)$ equals

(A) $\sin x^2 - \sin x$ **(B)** $4x^3 \sin x^2 - 2x \sin x$

(C) $x^4 \sin x^2 - x \sin x$ **(D)** None of these

26. If $F(x) = \frac{1}{x^2}\int_4^x (4t^2 - 2F'(t))\,dt$, then $F'(4)$ equals

(A) 32 **(B)** $\frac{32}{3}$

(C) $\frac{32}{9}$ **(D)** None of these

27. The value of the integral $\sum_{k=1}^{n}\int_0^1 f(k-1+x)\,dx$ is

(A) $\int_0^1 f(x)\,dx$ **(B)** $\int_0^2 f(x)\,dx$

(C) $\int_0^n f(x)\,dx$ **(D)** $n\int_0^1 f(x)\,dx$

28. The value of $\int_a^{a+(\pi/2)} (\sin^4 x + \cos^4 x)\,dx$ is

(A) Independent of a **(B)** $a\left(\frac{\pi}{2}\right)^2$

(C) $\frac{3\pi}{8}$ **(D)** $\frac{3\pi a^2}{8}$

29. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)dx$ equals

(A) $\frac{16}{15}$ **(B)** $\frac{32}{15}$

(C) $\frac{8}{15}$ **(D)** $\frac{5}{6}$

30. $\lim_{n\to\infty}\left\{\frac{1}{1-n^2} + \frac{2}{1-n^2} + \frac{3}{1-n^2} + \cdots + \frac{n}{1-n^2}\right\} =$

(A) 0 **(B)** $-1/2$

(C) $1/2$ **(D)** ∞

31. $\int_0^{\infty}\left[\frac{2}{e^x}\right]dx$ is [where [ ] represents the greatest integer function]

(A) 0 **(B)** $2/e$

(C) e^2 **(D)** $\log_e 2$

32. Evaluate

$$\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{\left(\sin^3 x + \cos^3 x\right)^2}\,dx$$

33. If $f(x) = \sqrt{x - m + 1}$, $x \in [m-1, m)$, $(m \in I)$, then evaluate $\int_0^n f(x)\,dx$, $(n \in N)$.

34. If $F(x) = \int_{x^2}^{x^3} \log t\,dt$, $(x > 0)$, then $F'(x) =$

(A) $(9x^2 - 4x)\log x$ **(B)** $(4x - 9x^2)\log x$

(C) $(9x^2 + 4x)\log x$ **(D)** None of these

35. $\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^6 x\,dx =$

(A) $\frac{3\pi}{64}$ **(B)** $\frac{3\pi}{572}$

(C) $\frac{3\pi}{256}$ **(D)** $\frac{3\pi}{128}$

36. $\int_0^1 \frac{d}{dx}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right]dx$ is equal to

(A) 0 **(B)** π

(C) $\pi/2$ **(D)** $\pi/4$

37. $\int_0^{\infty} \frac{x\,dx}{(1+x)(1+x^2)} =$

(A) 0 **(B)** $\pi/2$

(C) $\pi/4$ **(D)** 1

38. $\int_0^a x^4\sqrt{a^2 - x^2}\,dx =$

(A) $\frac{\pi}{32}$ **(B)** $\frac{\pi}{32}a^6$

(C) $\frac{\pi}{16}a^6$ **(D)** $\frac{\pi}{8}a^6$

39. $\int_0^a x(2ax - x^2)^{\frac{3}{2}}\,dx =$

(A) $a^5\left[\frac{3\pi}{16} - 1\right]$ **(B)** $a^5\left[\frac{3\pi}{16} + 1\right]$

(C) $a^5\left[\frac{3\pi}{16} - \frac{1}{5}\right]$ **(D)** None of these

40. $\int_0^a x^2(a^2 - x^2)^{3/2}\,dx =$

(A) $\frac{\pi a^6}{32}$ **(B)** $\frac{2a^5}{15}$

(C) $\frac{a^6}{32}$ **(D)** None of these

41. Let $\frac{d}{dx}F(x) = \left(\frac{e^{\sin x}}{x}\right)$; $x > 0$. If $\int_1^4 \frac{3}{x}e^{\sin x^3}\,dx = F(k) - F(1)$, then one of the possible value of k is

(A) 15 **(B)** 16

(C) 63 **(D)** 64

42. If $f(x) = \int_0^x t\sin t\,dt$, then $f'(x) =$

(A) $\cos x + x\sin x$ **(B)** $x\sin x$

(C) $x\cos x$ **(D)** None of these

43. Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\cos^2 x + \sin^4 x}\,dx.$

44. Prove that $\frac{1}{2}\int_0^x e^{zx} e^{-z^2} dz = e^{x^2/4} \int_0^{x/2} e^{-z^2} dz$.

45. Evaluate $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$.

46. Evaluate $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$.

47. Show that $\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x) dx$.

48. Find the value of $\int_{-1}^{3/2} |x \sin \pi x|\, dx$.

49. Evaluate $\int_0^{\pi} \frac{x^2 \sin 2x \sin\left\{\left(\frac{\pi}{2}\right)\cos x\right\} dx}{(2x - \pi)}$.

50. Show that $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{6n}\right) = \ln 6$.

51. Find $\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{\frac{1}{n}}$.

52. Evaluate: $\int_0^{\pi} \frac{x \sin x}{2 - \sin^2 x} dx =$

(A) $\frac{\pi^2}{4}$ **(B)** $\frac{\pi^2}{2}$

(C) $\frac{\pi}{2}$ **(D)** $\frac{\pi}{4}$

53. $\int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}}\, dx =$

(A) 2 **(B)** 0 **(C)** –1 **(D)** 1

54. $\int_{-1}^{3/2} |\sin \pi x|\, dx =$

(A) $\frac{2}{\pi}$ **(B)** $\frac{3}{\pi}$

(C) $\frac{4}{\pi}$ **(D)** $\frac{5}{\pi}$

55. $\int_{\pi/8}^{7\pi/8} \frac{x\, dx}{\cos^2 x - \sin^2 x} =$

(A) $\frac{\pi}{2}\log(3 - 2\sqrt{2})$ **(B)** $\frac{\pi}{4}\log(3 - 2\sqrt{2})$

(C) $\pi \log(\sqrt{2} - 1)$ **(D)** $\pi \log(\sqrt{2} + 1)$

56. If for a real number y, $[y]$ is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x]\, dx$ is

(A) $-\pi$ **(B)** 0

(C) $-\frac{\pi}{2}$ **(D)** $\frac{\pi}{2}$

57. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ is equal to

(A) 2 **(B)** –2 **(C)** $\frac{1}{2}$ **(D)** $-\frac{1}{2}$

58. $\int_0^x f(t)dt = x + \int_x^1 t\ f(t)dt$, then the value of $f(1)$ is

(A) $\frac{1}{2}$ **(B)** 0 **(C)** 1 **(D)** $-\frac{1}{2}$

59. If $g(x) = \int_0^x \cos^4 t\, dt$, then $g(x + \pi)$ equals

(A) $g(x) + g(\pi)$ **(B)** $g(x) - g(\pi)$

(C) $g(x)\, g(\pi)$ **(D)** $\frac{g(x)}{g(\pi)}$

60. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is ______.

61. Let $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}$, $x > 0$ if $\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(x) - F(1)$ then one of the possible values of k is ______.

62. If for non-zero x, a $f(x)$ + b $f\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \neq b$, then $\int_1^2 f(x) dx =$ ______.

63. Let $T > 0$ be a fixed real number. Suppose f is a continuous function such that for all $x \in R$, $f(x + T) = f(x)$. If $I = \int_0^T f(x)\ dx$ then the value of $\int_3^{3+3T} f(2x)\, dx$ is

(A) $\frac{3}{2} I$ **(B)** $2I$ **(C)** $3I$ **(D)** $6I$

64. Evaluate $\int_0^{\pi} \frac{x\, dx}{1 + \cos\alpha \sin x}$, $0 < a < \pi$.

65. Evaluate $\int_0^{\pi/2} \frac{x \sin x \cos x dx}{\cos^4 x + \sin^4 x}$.

66. A function $f(x)$ is such that it is integrable has interval over every interval on the real line

$$\int_0^{\pi/2} \sin 2kx$$

$$\cot x\ dx = \int_0^{\pi/2} \frac{(\sin 2kx)\cos x}{\sin x} dx$$

for every x and real t show that

$$\int_0^{\pi/2} 2\cos x[\cos x + \cos 3x + \cdots + \cos(2k-1)x] dx$$

is independent of a.

67. Evaluate $\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$.

68. If $f(x) = \begin{cases} e^{\cos x}.\sin x, & |x| \leq 2 \\ 2, & \text{otherwise} \end{cases}$, then $\int_{-2}^{3} f(x)dx =$ ________.

69. If $\int_{n}^{n+1} f(x)dx = \frac{n^2}{2}$, where $n \in I$, then evaluate $\int_{-3}^{5} f(|x|)dx$.

70. Evaluate $\int_0^{\pi/2} \ln \sin x \, dx$.

71. If f, g, h be continuous function on $[0, a]$ such that $f(a-x) = f(x)$, $g(a-x) = -g(x)$ and $3h(x) - 4h(a-x) = 5$, then calculate $\int_0^a f(x)\cdot g(x)\cdot h(x)\,dx$.

72. Evaluate $\int_0^{\pi} x\cdot \sin 2x \cdot \sin\left(\frac{\pi}{2}\cos x\right)dx$.

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If $\int_2^3 \frac{x^2dx}{\sqrt{x^4-x^2+1}} = I$, then the value of

$$\int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} - \int_{1/2}^{1/3} \frac{dx}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}$$ is equal to

(A) $-I$
(B) 0
(C) I
(D) $2I$

2. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and g be the function satisfying $f(x) + g(x) = x$. Then the value of $\int_0^1 f(x)g(x)dx$ is

(A) $\frac{3-e^2}{2}$
(B) $\frac{e^2-3}{2}$
(C) $\frac{e^2}{2}$
(D) $\frac{e-2}{4}$

3. The value of $\int_{-1/2}^{1/2} \cos^{-1}(4x^3-3x)dx$ is equal to

(A) 2π
(B) π
(C) $\frac{\pi}{2}$
(D) $\frac{\pi}{4}$

4. The value of $\int_0^{\pi/4} \ln\left(\frac{1-\tan x}{1+\tan^2 x}\right)dx$ is

(A) $\frac{\pi}{4}\ln\left(\frac{1}{2}\right)$
(B) $\frac{3\pi}{8}\ln 2$
(C) $\frac{\pi}{8}\ln 2$
(D) $-\frac{3\pi}{8}\ln 2$

5. Let $\lim_{n\to\infty}\left(\frac{1}{n^2}\sin\left(\frac{1+n^2}{n^2}\right) + \frac{2}{n^2}\sin\left(\frac{4+n^2}{n^2}\right) + \frac{3}{n^2}\sin\left(\frac{9+n^2}{n^2}\right) + \cdots + \frac{2}{n}\cdot\sin(5)\right) = y$, then the value of y is

(A) $\cos 2\cdot\sin 3$
(B) $\sin 2\cdot\cos 3$
(C) $\sin 2\cdot\sin 3$
(D) $\cos 2\cdot\cos 3$

6. Let $f(x)$ and $g(x)$ be two differentiable functions satisfying $xf'(x) + f(x) = g(x)$ where $f'(x) < 0\ \forall\ x \in R$. Then

(A) $2g(2) > \int_0^2 g(x)dx$
(B) $3g(3) > \int_0^3 g(x)dx$
(C) $2g(2) < \int_0^2 g(x)dx$
(D) $3g(3) < \int_0^3 g(x)dx$

7. If $f(x) = \left[\tan^{100} x\right]$ (where $[\cdot]$ denotes greatest integer function), then

(A) $f(x)$ is discontinuous at $x = \tan^{-1}\sqrt[100]{3}, \tan^{-1}\sqrt[100]{100}$
(B) $f(x)$ is discontinuous at $x = \tan^{-1}\sqrt[3]{100}, \tan^{-1}\sqrt[100]{99}$
(C) $\int_0^{\tan^{-1}\sqrt[100]{4}} f(x)dx = 3\tan^{-1}\sqrt[100]{4} - \tan^{-1}\sqrt[100]{3} - \tan^{-1}\sqrt[100]{2} - 1$
(D) $\int_0^{\tan^{-1}\sqrt[100]{4}} f(x)dx = 3\tan^{-1}\sqrt[100]{4} - 2\tan^{-1}\sqrt[100]{3}$

8. If $I_n = \int_0^{\infty} e^{-x}(\sin x)^n dx (n \in N, > 1)$, then

(A) $I_n = \frac{n(n-1)}{n^2+1}I_{n-2}$
(B) $I_n = \frac{n(n-1)}{(n-1)^2+1}I_{n-1}$
(C) $101\frac{I_{10}}{I_8} = 90$
(D) $82\frac{I_{10}}{I_9} = 90$

9. If $\int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, then

(A) $\int_0^{\infty} e^{-2x^2}dx = \frac{\sqrt{\pi}}{\sqrt{2}}$
(B) $\int_0^{\infty} xe^{-x^2}dx = \frac{1}{2}$
(C) $\int_0^{\infty} x^2e^{-x^2}dx = \frac{\pi}{4}$
(D) $\int_0^{\infty} x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$

10. Let

$$f(x) = \begin{cases} \int_0^x (5+|1-t|)dt, & \text{if } x > 2 \\ 5x+1, & \text{if } x \le 2 \end{cases}$$

Then the function is

(A) Continuous at $x = 2$
(B) Differentiable at $x = 2$
(C) Discontinuous at $x = 2$
(D) Not differentiable at $x = 2$

Comprehension Type Questions

Paragraph for Questions 11–13: Let $f: R \to R$ be a continuous and bijective function defined such that $f(\alpha) = 0$ $(\alpha \neq 0)$. The area bounded by $y = f(x)$, $x = \alpha$, $x = \alpha - t$ is equal to the area bounded by $y = f(x)$, $x = \alpha$, $x = \alpha + t\ \forall\ t \in R$ then

11. Graph of $y = f(x)$ is symmetrical about point

(A) $(0, 0)$ **(B)** $(0, \alpha)$ **(C)** $(\alpha, 0)$ **(D)** (α, α)

12. The value of $f(2\alpha)$ is equal to

(A) $f(\alpha)$ **(B)** $-f(\alpha)$
(C) $f(0)$ **(D)** $-f(0)$

13. The value of $\int_{-\beta}^{\beta} f^{-1}(t)dt$ is equal to

(A) 0 **(B)** $2\alpha\beta$
(C) $\alpha\beta$ **(D)** None of these

Paragraph for Questions 14–16: Consider a polynomial $f(x)$, which satisfies the following conditions:

(i) $f(x) = \{f'(x)\}^2, \forall x$

(ii) $\int_0^1 f(x)dx = \frac{19}{12}$

(iii) $f'(0) > 0$

14. The function $f(x)$ can be

(A) A linear function **(B)** A quadratic function
(C) A cubic function **(D)** Any polynomial of even degree

15. The value of $f'(0)$ is

(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1

16. The function $f(x)$ is

(A) Even **(B)** Odd
(C) Neither even nor odd **(D)** May be either even or odd

Paragraph for Questions 17–19: If $m > 0$, $n > 0$, the definite integral $I = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ depends upon the values of m and n and is denoted by $\beta(m, n)$, called the beta function. That is, $\int_0^1 x^4(1-x)^5 dx = \int_0^1 x^{5-1}(1-x)^{6-1}dx = \beta(5, 6)$ and $\int_0^1 x^{5/2}(1-x)^{-1/2}dx = \int_0^1 x^{7/2-1}(1-x)^{1/2-1}dx = \beta\left(\frac{7}{2}, \frac{1}{2}\right)$. Obviously, $\beta(n, m) = \beta(m, n)$.

17. The integral $\int_0^{\pi/2} \cos^{2m}\theta \sin^{2n}\theta \, d\theta$ is equal to

(A) $\frac{1}{2}\beta\left(m+\frac{1}{2}, n+\frac{1}{2}\right)$ **(B)** $2\beta(2m, 2n)$
(C) $\beta(2m+1, 2n+1)$ **(D)** None of these

18. If $\int_0^n \left(1-\frac{x}{n}\right)^n x^{k-1}dx = R\beta(k, n+1)$, then R is equal to

(A) n **(B)** n^{kn}
(C) n^k **(D)** None of these

19. If $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = k\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$, then k is equal to

(A) $\frac{m}{n}$ **(B)** 1
(C) $\frac{n}{m}$ **(D)** None of these

Paragraph for Questions 20–22: Let n be a positive integer such that $I_n = \int x^n\sqrt{a^2 - x^2}dx$, then answer the following questions:

20. The value of I_1 is

(A) $\frac{2}{3}(a^2 - x^2)^{1/2}$ **(B)** $\frac{1}{3}(a^2 - x^2)^{3/2}$
(C) $-\frac{2}{3}(a^2 - x^2)^{3/2}$ **(D)** $-\frac{1}{3}(a^2 - x^2)^{3/2}$

21. The value of the expression $\frac{\int_0^a x^4\sqrt{a^2-x^2}dx}{\int_0^a x^2\sqrt{a^2-x^2}dx}$ is equal to

(A) $\frac{a^2}{6}$ **(B)** $\frac{3a^2}{2}$
(C) $\frac{3a^2}{4}$ **(D)** $\frac{a^2}{2}$

22. If $I_n = \frac{-x^{n-1}(a^2 - x^2)^{3/2}}{n+2} + kI_{n-2}$, then the value of k is

(A) $\frac{n-1}{n+2}$ **(B)** $\frac{n+2}{n-1}$
(C) $\left(\frac{n-1}{n+2}\right)a^2$ **(D)** $\left(\frac{n+2}{n-1}\right)a^2$

Matrix Match Type Questions

23. Match the following:

Column I	Column II
(A) If $y = \cos^{-1}\left(\frac{2\cos x - 3\sin x}{\sqrt{13}}\right)$, then $\frac{dy}{dx}$ is	**(p)** 2π
(B) The value of $\int_{-1}^{1} \sin^{-1}\left[x^2 + \frac{1}{2}\right]dx + \int_{-1}^{1} \cos^{-1}\left[x^2 - \frac{1}{2}\right]dx$ is equal to, where [·] denotes the greatest integer function	**(q)** 0
(C) The value of $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+x^2)(1+e^{\sin x})}$ is equal to	**(r)** $\pi/3$
(D) The value of $\int_0^{6\pi} \sin^{-1}(\sin x)dx$ is equal to	**(s)** 1

24. Match the following:

Column I	Column II
(A) $\int_0^{\pi/2} \frac{dx}{1+\tan x}$	**(p)** $\frac{1}{117}$

Column I	Column II
(B) If $\int_0^{x^2(1+x^5+7x^{12})} f(t)dt = x$, then $f(9)$ is equal to	**(q)** $\frac{\pi}{2}-\log 2$
(C) $\int_0^{\infty} e^{-2x}(\sin 2x+\cos 2x)dx$ is equal to	**(r)** $\frac{1}{2\sqrt{2}}$
(D) $\int_0^1 \cot^{-1}(1+x^2-x)dx$ is equal to	**(s)** $\frac{\pi}{4}$

25. Match the following:

Column I	Column II
(A) For any integer n, the integral $\int_0^{\pi} e^{\cos^2 t}\cos^3(2n+1)tdt$ is equal to	**(p)** $\frac{\pi}{2}$
(B) If $I_n = \int_0^{\pi/4}\tan^n\theta\, d\theta$, then for any positive integer n, $n(I_{n-1}+I_{n+1}) =$	**(q)** 0
(C) The value of $\int_0^{\infty}\frac{x\log x}{(1+x^2)^2}dx$ is	**(r)** 1
(D) The value of $\int_{-1}^{1}\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx$ is	**(s)** $-\frac{\pi}{2}$

26. Match the following:

Column I	Column II
(A) $\int_0^{\pi} xf(\sin x)dx : \pi\int_0^{\pi} f(\sin x)dx$	**(p)** is equal to the ratio 1:1
(B) $\int_0^{\pi} xf(\sin x)dx : \pi\int_0^{\pi/2} f(\cos x)dx$	**(q)** is equal to the ratio 1:2
(C) $\int_{1-k}^{k} xf(x(1-x))dx : \int_{1-k}^{k} f(x(1-x))dx$	**(r)** is equal to the ratio 2:1
(D) $\int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))dx : \int_0^{\cos^2 t} f(1-x^2)dx$	**(s)** is equal to the ratio k:1

Integer Type Questions

27. If $I = \frac{1}{\pi}\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)d\theta$, then find the value of I.

28. Let $f: R \to R$ defined by $f(x) = \cos x + x$ if $\int_0^{\pi} f^{-1}(x)dx = \frac{\pi^2}{2} - k$, then the value of k is _________.

29. If $\int_{\cos x}^{1} t^2 f(t)dt = 1-\cos x \quad \forall\, x \in \left(0, \frac{\pi}{2}\right)$, then the value of $\left[f\left(\frac{\sqrt{3}}{4}\right)\right]$ is ________ (where [·] denotes the greatest integer function).

Answer Key

Practice Exercise 1

1. (A)	**2.** (D)	**3.** (C)	**4.** (A)	**5.** (D)	**6.** (C)
7. (D)	**8.** (D)	**9.** (C)	**10.** (B)	**11.** (A)	**12.** (A)
13. (B)	**14.** (B)	**15.** (D)	**16.** (C)	**17.** (B)	**18.** (C)
19. (B)	**20.** (C)	**21.** (B)	**22.** (A)	**23.** (D)	**24.** (D)
25. (B)	**26.** (C)	**27.** (C)	**28.** (C)	**29.** (A)	**30.** (B)
31. (D)	**32.** $\frac{1}{6}$	**33.** $\frac{2n}{3}$	**34.** (A)	**35.** (C)	**36.** (C)
37. (C)	**38.** (B)	**39.** (C)	**40.** (A)	**41.** (D)	**42.** (B)
43. $\frac{\pi}{4}-\frac{1}{\sqrt{3}}\log\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)$		**45.** $\frac{4}{3}$	**46.** $\frac{1}{20}\ln 3$	**48.** $\frac{3}{\pi}+\frac{1}{\pi^2}$	
49. $\frac{16}{\pi}$	**50.** ln6	**51.** $2e^{\frac{\pi-4}{2}}$	**52.** (A)	**53.** (A)	**54.** (D)
55. (B)	**56.** (C)	**57.** (A)	**58.** (A)	**59.** (A)	**60.** 2
61. $k=16$	**62.** $\frac{1}{a^2-b^2}\left[a\ln 2+\frac{7b}{2}-5a\right]$		**63.** (C)	**64.** $\pi\alpha\operatorname{cosec}\alpha$	**65.** $\frac{\pi^2}{16}$
67. $\frac{6-\pi\sqrt{3}}{12}$	**68.** 2	**69.** $\frac{35}{2}$	**70.** $\frac{-\pi}{2}\ln 2$	**71.** 0	**72.** $\frac{8}{\pi}$

Practice Exercise 2

1. (D) **2.** (A) **3.** (C) **4.** (D) **5.** (C) **6.** (C), (D)
7. (A), (C) **8.** (A), (C) **9.** (B), (D) **10.** (A), (D) **11.** (C) **12.** (D)
13. (B) **14.** (B) **15.** (D) **16.** (C) **17.** (A) **18.** (C)
19. (B) **20.** (D) **21.** (D) **22.** (C) **23.** (A) → (s), (B) → (p), (C) → (r), (D) → (q)
24. (A) → (s), (B) → (p), (C) → (r), (D) → (q) **25.** (A) → (q), (B) → (r), (C) → (q), (D) → (s) **26.** (A) → (q), (B) → (p), (C) → (q), (D) → (r)
27. 2 **28.** 0 **29.** 5

Solutions

Practice Exercise 1

1. $$\int_{-1}^{1}(x-[x])dx=\int_{-1}^{1}x\,dx-\int_{-1}^{1}[x]dx=\int_{-1}^{1}x\,dx-\int_{-1}^{0}[x]dx-\int_{0}^{1}[x]dx$$

$$=\left(\frac{x^2}{2}\right)_{-1}^{1}-\int_{-1}^{0}(-1)dx-\int_{0}^{1}(0)dx$$

$$=0+(x)_{-1}^{0}-(0)_0^1=1-0=1$$

2. $$x^2-f'(x)=0$$

$$\Rightarrow x^2-\sqrt{2-x^2}-0=0$$

$$\Rightarrow x^2=\sqrt{2-x^2}$$

$$\Rightarrow x^4=2-x^2$$

$$\Rightarrow x^4+x^2-2=0$$

$$\Rightarrow x^4+2x^2-x^2-2=0$$

$$\Rightarrow (x^2+2)(x^2-1)=0$$

$$\Rightarrow x^2=1,-2$$

$$\Rightarrow x=\pm 1$$

3. $$\int_{3}^{3+3T}f(2x)dx=\int_{0}^{3T}f(2x)dx=3\int_{0}^{T}f(2x)dx$$

Let $2x=t$. Then

$$\frac{3}{2}\int_{0}^{2T}f(t)dt=3\int_{0}^{T}f(t)dt=3I$$

4. $$\int_{0}^{100}a^{x-[x]}dx$$

$$=\int_{0}^{100}\frac{a^x}{a^{[x]}}dx$$

$$=\int_{0}^{1}\frac{a^x}{a^0}dx+\int_{1}^{2}\frac{a^x}{a}dx+\int_{2}^{3}\frac{a^x}{a^2}dx+\cdots+\int_{99}^{100}\frac{a^x}{a^{99}}dx$$

$$=\left(\frac{a^x}{\log a}\right)_0^1+\left(\frac{a^x}{a\cdot\log a}\right)_1^2+\left(\frac{a^x}{a^2\log a}\right)_2^3+\cdots+\left(\frac{a^x}{a^{99}\log a}\right)_{99}^{100}$$

$$=\frac{a-1}{\log a}+\frac{a^2-a^1}{a\log a}+\frac{a^3-a^2}{a^2\log a}+\cdots+\frac{a^{100}-a^{99}}{a^{99}\log a}$$

$$=\frac{100(a-1)}{\log a}$$

5. $$\int_{0}^{x}[\sin t]dt \quad ((2n+1)\pi<x<(2n+2)\pi)$$

$$\int_{0}^{\pi/2}0\cdot dt+\int_{\pi/2}^{\pi}0\cdot dt+\int_{\pi}^{3\pi/2}-1\cdot dt+\int_{\frac{3\pi}{2}}^{2\pi}(-1)dt+\cdots$$

$$=\int_{0}^{\pi}0\cdot dt-\int_{\pi}^{2\pi}1\cdot dt+\int_{2\pi}^{3\pi}0\cdot dt+\int_{3\pi}^{4\pi}-1dt+\cdots+\int_{(2n+1)\pi}^{x}-1dt$$

$$=-\pi-\pi-\cdots n\text{ times}-(n)_{(2n+1)\pi}^{x}$$

$$=-n\pi-(x-(2n+1)\pi)=-x+(n+1)\pi$$

6. $$I=\int_{0}^{\pi}\frac{dx}{1+2^{\tan x}}=\int_{0}^{\pi}\frac{dx}{1+2^{-\tan x}}$$

$$\Rightarrow 2I=\int_{0}^{\pi}dx=\pi\Rightarrow I=\frac{\pi}{2}$$

7. $$\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{1+e^x}dx=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\frac{(1+e^x)+(1-e^x)}{(1+e^x)}\cdot\sec^2 x\,dx$$

$$=\frac{1}{2}\int_{-\pi/4}^{\pi/4}\sec^2 x\,dx+\frac{1}{2}\int_{-\pi/4}^{\pi/4}\frac{1-e^x}{1+e^x}\sec^2 x\,dx$$

$$=\frac{1}{2}(\tan x)_{-\pi/4}^{\pi/4}+0$$

$$\left(\text{since, } f(-x)=\frac{1-e^{-x}}{1+e^{-x}}\sec^2(-x)=-\frac{1-e^x}{1+e^x}\sec^2 x=-f(x)\right)$$

$$=\frac{1}{2}\cdot 2=1$$

8. $$I=\int_{0}^{a}(a-x)f(a-x)dx$$

$$=a\int_{0}^{a}f(x)dx-\int_{0}^{a}x\,f(x)dx$$

$$\Rightarrow 2I=a\int_{0}^{a}f(x)dx$$

$$\Rightarrow I=\frac{a}{2}\int_{0}^{a}f(x)dx$$

9. $I_p = \int_1^e (\ln x)^p dx$

$$I_p + pI_{p-1} = \int_1^e (\ln x)^p dx + p\int_1^e (\ln x)^{p-1} dx$$

$$= \left[x\cdot(\ln x)^p\right]_1^e - \int_1^e \frac{p}{x}(\ln x)^{p-1}\cdot x dx + p\int_1^e (\ln x)^{p-1} dx = e$$

which is less than 3.

10. $I = \int_0^{\frac{\pi}{2}} \ln|\tan x + \cot x|\, dx \;= -\int_0^{\pi/2} \ln\left|\frac{\sin 2x}{2}\right| dx \;\; I = -2\int_0^{\pi/2} \ln \sin x\, dx$

$$= -\int_0^{\pi/2} [\ln|\sin 2x| - \ln 2] dx$$

$$I = -2I_1 = -I_1 + \int_0^{\pi/2} \ln 2\, dx = -I_1 + \frac{\pi}{2}\log_e 2 \;\; \Rightarrow \;\; I_1 = -\frac{\pi}{2}\ln 2$$

Hence,

$$-2I_1 = -2\left(-\frac{\pi}{2}\ln 2\right)$$

$$\Rightarrow I = \pi \ln 2$$

11.

$$F(x) = \int_0^x \ln\left(\frac{1-t}{1+t}\right) dt$$

$$\Rightarrow \; F'(x) + F'(-x) = \ln\left(\frac{1-x}{1+x}\right) + \ln\left(\frac{1+x}{1-x}\right) = 0$$

12. $\int_0^{\pi/4} 1\cdot d\left(x - [x]\right) = \int_0^{\pi/4} d\{x\} = \left[\{x\}\right]_0^{\pi/4} \;= \frac{\pi}{4}$

13.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$$

$$\lim_{n\to\infty} = \frac{1}{n}\sum_{r=1}^{2n} \frac{r/n}{\sqrt{1 + (r/n)^2}}$$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}} dx$$

Let $1 + x^2 = t^2$. Then $xdx = tdt$

$$\int_1^{\sqrt{5}} \frac{t}{t} dt = (t)_1^{\sqrt{5}} = \sqrt{5} - 1$$

14. $\int_0^{[x]} \frac{2^x}{2^{[x]}} dx$

$$= \int_0^1 2^x dx + \int_1^2 \frac{2^x}{2} dx + \int_2^3 \frac{2^x}{2^2} dx + \cdots$$

$$= \left(\frac{2^x}{\log 2}\right)_0^1 + \frac{1}{2}\left(\frac{2^x}{\log 2}\right)_1^2 + \cdots$$

$$= \frac{1}{\ln 2}(2^1 - 2^0) + \frac{1}{2\ln 2}(2^2 - 2^1) + \frac{1}{2^2 \log 2}(2^3 - 2^2) + \cdots$$

$$= \frac{1}{\ln 2} + \frac{1}{\ln 2} + \frac{1}{\ln 2} + \cdots + [x] \text{ times}$$

$$= \frac{[x]}{\ln 2}$$

15. Let

$$I(b) = \int_0^1 \frac{x^b - 1}{\log x} dx \Rightarrow I'(b) = \int_0^1 \frac{x^b \log x}{\log x} dx$$

(If $I(\alpha) = \int_0^b f(x,\alpha)dx$, then $I'(\alpha) = \int_0^b f'(x,\alpha)dx$, where $f'(x,\alpha)$ is derivative of $f(x,\alpha)$ w.r.t. α keeping x constant)

$$I'(b) = \int_0^1 x^b dx = \frac{1}{b+1}$$

$$\Rightarrow I(b) = \int \frac{db}{b+1} + c = \log(b+1) + c$$

If $b = 0$, then $I(b) = 0$. So,

$$c = 0 \;\Rightarrow I(b) = \log(b+1)$$

16. $\int_{-1}^1 \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) dx = -2[\tan^{-1}(x)]_0^1 = -\frac{\pi}{2}$

17. We have

$$F'(x) = 3\sin x + 4\cos x$$

Since in $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$, $F'(x) < 0$, so assume the least value at the point $x = \frac{4\pi}{3}$. Thus,

$$f\left(\frac{4\pi}{3}\right) = \int_{5\pi/4}^{4\pi/3} (3\sin u + 4\cos u) du$$

$$= \frac{3}{2} - 2\sqrt{3} + \frac{1}{\sqrt{2}}$$

18. $\int_0^\infty e^{-2x}(\sin 2x + \cos 2x) dx$

$$= \left[-e^{-2x}\frac{\cos 2x}{2}\right]_0^\infty - \int_0^\infty (-2e^{-2x})\left(\frac{-\sin 2x}{2}\right) dx + \int_0^\infty e^{-2x}\sin 2x\, dx$$

$$= \frac{1}{2}$$

19. $\int_0^{b-c} f''(x+a)dx \; = [f'(x+a)]_0^{b-c} = f'(b-c+a) - f'(a)$

20. $F'(x) = |x| > 0 \,\forall x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Hence, the function is increasing on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and therefore $F(x)$ has maxima at the right end point of $\left[-\frac{1}{2}, \frac{1}{2}\right]$. So,

$$\text{Max } F(x) = F\left(\frac{1}{2}\right) = \int_1^{1/2} |t|\, dt = -\frac{3}{8}$$

21. $\int_{-\pi/2}^{\pi/2}\sin^2 x\cos^2 x(\sin x+\cos x)dx$

$$=\int_{-\pi/2}^{\pi/2}\sin^3 x\cos^2 x dx+\int_{-\pi/2}^{\pi/2}\sin^2 x\cos^3 x\, dx$$

$$=0+2\int_0^{\pi/2}\sin^2 x\cos^3 x dx \;=0+2\times\frac{2}{15}=\frac{4}{15}$$

22. Putting $x=\tan\theta$, we get

$$\int_0^{\infty}\frac{dx}{(x+\sqrt{x^2+1})^3}=\int_0^{\pi/2}\frac{\sec^2\theta\, d\theta}{(\tan\theta+\sec\theta)^3}=\int_0^{\pi/2}\frac{\cos\theta}{(1+\sin\theta)^3}d\theta$$

$$=\left[-\frac{1}{2(1+\sin\theta)^2}\right]_0^{\pi/2}=-\frac{1}{8}+\frac{1}{2}=\frac{3}{8}$$

23. We know that

$$\frac{d}{dx}\left(\int_a^b f(t)dt\right)=\frac{db}{dx}f(b)-\frac{da}{dx}f(a)$$

a and b are functions of x.

Therefore,

$$F(x)=\int_{x^2}^{x^3}\frac{1}{\log t}dt$$

$$\Rightarrow F'(x)=\frac{d}{dx}(x^3)\frac{1}{\log x^3}-\frac{d}{dx}(x^2)\frac{1}{\log x^2}$$

$$=\frac{3x^2}{3\log x}-\frac{2x}{2\log x}=x(x-1)(\log x)^{-1}$$

24. $f'(x)=e^{-(x^2+1)^2}\cdot 2x-e^{-(x^2)^2}\cdot 2x=2xe^{-(x^4+1+2x^2)}(1-e^{2x^2+1})$

$$\Rightarrow f'(x)>0,\forall x\in(-\infty,0)$$

25. We have

$$f(x)=\int_{x^2}^{x^4}\sin\sqrt{t}\,dt$$

Therefore,

$$f'(x)=\frac{d}{dx}(x^4)(\sin\sqrt{x^4})-\frac{d}{dx}(x^2)(\sin\sqrt{x^2})$$

$$=4x^3\sin x^2-2x\sin x$$

26. We have

$$F(x)=\frac{1}{x^2}\int_4^x(4t^2-2F'(t))dt$$

Therefore,

$$F'(x)=\frac{1}{x^2}(4x^2-2F'(x))-\frac{2}{x^3}\int_4^x(4t^2-2F'(t))dt$$

$$\Rightarrow F'(4)=\frac{1}{16}[64-2F'(4)]-0\Rightarrow F'(4)=\frac{32}{9}$$

27. Let

$$I=\int_0^1 f(k-1+x)dx$$

$$\Rightarrow I=\int_{k-1}^k f(t)dt,\text{ where } t=k-1+x$$

Therefore,

$$\sum_{k=1}^{n}\int_{k-1}^{k}f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx+\cdots+\int_{n-1}^{n}f(x)dx$$

$$=\int_0^n f(x)dx$$

28. Since $\sin^4 x+\cos^4 x$ is a periodic function with period $\frac{\pi}{2}$, therefore

$$\int_a^{a+(\pi/2)}(\sin^4 x+\cos^4 x)\,dx$$

$$=\int_0^{\pi/2}(\sin^4 x+\cos^4 x)dx$$

$$=2\int_0^{\pi/2}\sin^4 x\,dx=\frac{3\Gamma(5/2)\Gamma(1/2)}{2\Gamma\left(\frac{4+0+2}{2}\right)}=\frac{3\pi}{8}$$

29. $\int_0^{\pi}\sin^5\frac{x}{2}dx=2\int_0^{\pi/2}\sin^5 t dt=2\cdot\frac{\Gamma\frac{6}{2}.\Gamma\frac{1}{2}}{2\Gamma\frac{7}{2}}=\frac{16}{15}$

30. $\lim_{n\to\infty}\left\{\frac{1}{1-n^2}+\frac{2}{1-n^2}+\frac{3}{1-n^2}+\cdots+\frac{n}{1-n^2}\right\}$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{1-n^2}=\lim_{n\to\infty}\frac{1}{\left(\frac{1}{n^2}-1\right)}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{r}{n}\right)$$

$$=\lim_{n\to\infty}\frac{1}{\left(\frac{1}{n^2}-1\right)}\int_0^1 x\,dx=\lim_{n\to\infty}\frac{1}{2}\times\frac{1}{\left(\frac{1}{n^2}-1\right)}=\frac{-1}{2}$$

31. $\int_0^{\infty}\left[\frac{2}{e^x}\right]dx$

$x\in(0,\log_e 2)$, $\frac{2}{e^x}$ varies from 2 to 1 and hence $\left[\frac{2}{e^x}\right]=1$

$x\in(\log_e 2,\infty)$, $\frac{2}{e^x}$ varies from 1 to 0 and hence $\left[\frac{2}{e^x}\right]=0$

Therefore,

$$\int_0^{\infty}\left[\frac{2}{e^x}\right]dx=\int_0^{\log_e 2}1\,dx+\int_{\log_e 2}^{\infty}0\,dx$$

$$=[x]_0^{\log_e 2}+0=\log_e 2$$

32. $\int_0^{\pi/4}\frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx=\int_0^{\pi/4}\frac{\tan^2 x\sec^2 x}{(1+\tan^3 x)^2}dx,$

(Dividing numerator and denominator by $\cos^6 x$.)

Put $z=1+\tan^3 x$. Therefore, $dz=3\tan^2 x\sec^2 x\,dx$

When $x=0, z=1$ and when $x=\frac{\pi}{4}, z=2$

Therefore,

$$\int_0^{\pi/4}\frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx=\int_1^2\frac{1}{3}\frac{dz}{z^2}=\left[\frac{1}{3}\left(\frac{-1}{z}\right)\right]_1^2$$

$$=\frac{1}{3}-\frac{1}{6}=\frac{1}{6}$$

33. $\int_0^n f(x)\,dx = \sum_{m=1}^{n}\int_{m-1}^{m} f(x)\,dx = \sum_{m=1}^{n}\int_{m-1}^{m}(\sqrt{x-m+1})\,dx$

$$= \sum_{m=1}^{n}\frac{2}{3}(x-m+1)^{3/2}\Big|_{m-1}^{m} = \sum_{m=1}^{n}\frac{2}{3} = \frac{2n}{3}$$

34. $F(x) = \int_{x^2}^{x^3}\log t\,dt$

Applying Leibnitz's theorem,

$$F'(x) = \log x^3\cdot\frac{d}{dx}x^3 - \log x^2\cdot\frac{d}{dx}x^2$$
$$= 3\log x\cdot 3x^2 - 2\log x\cdot 2x = (9x^2-4x)\log x$$

35. $I = \int_{-\pi/2}^{\pi/2}\sin^4 x\cos^6 x\,dx = 2\int_0^{\pi/2}\sin^4 x\cos^6 x\cdot dx$

since, $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx,\quad \text{if } f(-x) = f(x)$

$= 0,\quad \text{if } f(-x) = -f(x)$

Applying Gamma function, we get

$$I = \frac{2\Gamma 5/2\cdot\Gamma 7/2}{2\cdot\Gamma 6}$$
$$= \frac{3/2\cdot 1/2\cdot\sqrt{\pi}\cdot 5/2\cdot 3/2\cdot 1/2\cdot\sqrt{\pi}}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{3\pi}{2^8} = \frac{3\pi}{256}$$

36. $I = \left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right]_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2}$

37. $\int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)} = \int_0^\infty \frac{-\frac{1}{2}dx}{(1+x)} + \int_0^\infty \frac{\left(\frac{1}{2}x+\frac{1}{2}\right)}{1+x^2}dx$

$$= \left[\frac{-1}{2}\log(1+x)\right]_0^\infty + \frac{1}{2}\times\frac{1}{2}[\log(1+x^2)]_0^\infty + \frac{1}{2}[\tan^{-1}x]_0^\infty$$

$$= 0+0+\frac{1}{2}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{4}$$

38. Put

$$x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$$

Now

$$\int_0^a x^4\sqrt{a^2-x^2}\,dx = a^6\int_0^{\pi/2}\sin^4\theta\cos\theta\cos\theta\,d\theta$$

$$= a^6\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta = a^6\frac{\Gamma\left(\frac{5}{2}\right).\Gamma\left(\frac{3}{2}\right)}{2\Gamma 4} = \frac{\pi}{32}a^6$$

(Using gamma function)

39. Put

$$x = a(1-\cos 2\theta) \Rightarrow dx = 2a\sin 2\theta\,d\theta$$

Therefore,

$$\int_0^a x(2ax-x^2)^{3/2}dx$$
$$= \int_0^{\pi/4} 2a^5(1-\cos 2\theta)\sin^4 2\theta\,d\theta$$

Now again, put $2\theta = \varphi$. Therefore,

$$a^5\left[\int_0^{\pi/2}\sin^4\varphi\,d\varphi - \int_0^{\pi/2}\sin^4\varphi\cos\varphi\,d\varphi\right] = a^5\left[\frac{3\pi}{16}-\frac{1}{5}\right]$$

40. $I = \int_0^a x^2(a^2-x^2)^{3/2}dx$

Put

$$x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$$
$$I = \int_0^{\pi/2} a^2\sin^2\theta\cdot a^3\cos^3\theta\cdot a\cos\theta\,d\theta$$
$$= a^6\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta = a^6\frac{\Gamma\frac{3}{2}\cdot\Gamma\frac{5}{2}}{2\cdot\Gamma\frac{8}{2}}$$
$$= a^6\frac{\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot 3\cdot 2\cdot 1} = \frac{\pi a^6}{32}$$

41. $\int_1^4 \frac{3}{x}e^{\sin x^3}dx = \int_1^4 \frac{3x^2}{x^3}e^{\sin x^3}dx$

Put

$$x^3 = t \Rightarrow 3x^2dx = dt$$
$$F(t) = \int_1^{64}\frac{e^{\sin t}}{t}dt = \int_1^{64}\frac{d}{dt}F(t)dt = F(64)-F(1),$$

On comparing, we get

$$k = 64$$

42. Since, $f(x) = \int_0^x t\sin t\,dt$. Now, according to Leibnitz's rule,

$$f'(x) = x\sin x\cdot(1) - 0 = x\sin x$$

43. Observe that $\sin x + \cos x = \frac{d}{dx}(\sin x - \cos x)$

Therefore, express the denominator as a function of $\sin x - \cos x$.

$$\cos^2 x + \sin^4 x = \frac{1+\cos 2x}{2} + \left(\frac{1-\cos 2x}{2}\right)^2$$
$$= \frac{3}{4} + \frac{\cos^2 2x}{4} = \frac{4-\sin^2 2x}{4} = 1-\sin^2 x\cos^2 x$$
$$= 1-\left\{\frac{1-(\sin x-\cos x)^2}{2}\right\}^2$$

Let $z = \sin x - \cos x;\ dz = (\cos x + \sin x)\,dx$

When $x = 0, z = -1$, when $x = \pi/4,\ z = 0$

$$I = \int_{-1}^{0}\frac{dz}{1-\left(\frac{1-z^2}{2}\right)^2} = \int_{-1}^{0}\frac{4\,dz}{2^2-(1-z^2)^2}$$
$$= \int_{-1}^{0}\frac{4\,dz}{(3-z^2)\ (1+z^2)}$$
$$= \int_{-1}^{0}\left\{\frac{1}{1+z^2}+\frac{1}{3-z^2}\right\}dz$$

$$=\left[\tan^{-1}z+\frac{1}{2\sqrt{3}}\log\frac{\sqrt{3}+z}{\sqrt{3}-z}\right]_{-1}^{0}$$

$$=0-\left\{\tan^{-1}(-1)+\frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\right\}$$

$$=\frac{\pi}{4}-\frac{1}{2\sqrt{3}}\log\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)=\frac{\pi}{4}-\frac{1}{2\sqrt{3}}\log\frac{(\sqrt{3}-1)^2}{2}$$

$$=\frac{\pi}{4}-\frac{1}{\sqrt{3}}\log\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)$$

44. Put $z=\frac{x+2u}{2}$. Therefore, $dz=du$.

$z=0\Rightarrow u=-\frac{x}{2}$ and $z=x\Rightarrow u=\frac{x}{2}$

Therefore,

$$\text{LHS}=I=\frac{1}{2}\int_{u=-\frac{x}{2}}^{x/2}e^{\left(\frac{x+2u}{2}\right)\left(x-\frac{x+2u}{2}\right)}.du$$

$$=\frac{1}{2}\int_{-x/2}^{x/2}e^{\frac{x^2-4u^2}{4}}du=\frac{1}{2}e^{\frac{x^2}{4}}\int_{-\frac{x}{2}}^{\frac{x}{2}}e^{-u^2}du$$

$$=\frac{1}{2}\cdot 2e^{\frac{x^2}{4}}\int_{0}^{\frac{x}{2}}e^{-u^2}du=e^{\frac{x^2}{4}}\int_{0}^{\frac{x}{2}}e^{-u^2}du=e^{x^2/4}\int_{0}^{x/2}e^{-z^2}dz=\text{RHS}$$

45. $I=\int_{-\frac{\pi}{2}}^{\pi/2}\sqrt{\cos x-\cos^3 x}\,dx$

$$=2\int_{0}^{\pi/2}\sqrt{\cos x-\cos^3 x}\,dx \text{ (as }\cos x\text{ is an even function)}$$

$$=2\int_{0}^{\pi/2}\sqrt{\cos x}\sqrt{\sin^2 x}\,dx$$

$$=2\int_{0}^{\pi/2}\sqrt{\cos x}\,\sin x\,dx\ (\text{since }\sin x\text{ is positive in }(0,\pi/2))$$

$$=\left[-2(\cos x)^{3/2}\cdot\frac{2}{3}\right]_{0}^{\pi/2}=\frac{4}{3}$$

46. $I=\int_{0}^{\pi/4}\frac{(\sin x+\cos x)}{9+16\sin 2x}dx$

Put $y=\sin x-\cos x$

$dy=(\cos x+\sin x)\,dx$

$y^2=\sin^2 x+\cos^2 x-2\sin x\cos x=1-\sin 2x$

Therefore, $\sin 2x=1-y^2$

$$x=\frac{\pi}{4},\ y=0$$

Therefore,

$$\int_{-1}^{0}\frac{dy}{9+16(1-y^2)}dx$$

$$=-\frac{1}{16}\int_{0}^{1}\frac{dy}{y^2-\left(\frac{5}{4}\right)^2}=-\frac{1}{16}\times\frac{2}{5}\int_{0}^{1}\left[\frac{1}{\left(y-\frac{5}{4}\right)}-\frac{1}{\left(y+\frac{5}{4}\right)}\right]dy$$

$$=-\frac{1}{40}\left[\ln\left(y-\frac{5}{4}\right)-\ln\left(y+\frac{5}{4}\right)\right]_{0}^{1}$$

$$=\frac{1}{40}\ln\left[\frac{y+\frac{5}{4}}{y-\frac{5}{4}}\right]_{0}^{1}=\frac{1}{40}\ln 9=\frac{1}{40}\ln 3^2=\frac{1}{20}\ln 3$$

Therefore,

$$\int_{0}^{\pi/4}\left(\frac{\sin x+\cos x}{9+16\sin 2x}\right)dx=\frac{1}{20}\ln 3$$

47. Let

$$I=\int_{0}^{\pi}xf(\sin x)dx=\int_{0}^{\pi}(\pi-x)f\{\sin(\pi-x)\}dx$$

$$=\pi\int_{0}^{\pi}f(\sin x)dx-\int_{0}^{\pi}xf(\sin x)\,dx=\pi\int_{0}^{\pi}f(\sin x)\,dx-I$$

$$\Rightarrow\ 2I=\pi\int_{0}^{\pi}f(\sin x)\,dx\ \text{ or }\ I=\frac{\pi}{2}\int_{0}^{\pi}f(\sin x)\,dx$$

48. $\sin x$ is positive for $0\le x\le\pi$

$\sin \pi x$ is positive for $0\le\pi x\le\pi$, that is, $0\le x\le 1$

For $-1\le x<0$, $-\pi\le\pi x<0$, for such x, $\sin\pi x<0$.

For $1<x\le\left(\frac{3}{2}\right)$, $\pi\le\pi x\le\left(\frac{3\pi}{2}\right)\Rightarrow\sin\pi x<0$

Therefore,

$-1\le x<0,$	$x\sin\pi x>0$
$0\le x\le 1,$	$x\sin\pi x>0$
$1<x\le\frac{3}{2}$	$x\sin\pi x<0$

Therefore,

$$\int_{-1}^{\frac{3}{2}}|x\sin\pi x|dx=\int_{-1}^{1}x\sin\pi x\,dx+\int_{1}^{\frac{3}{2}}(-x\sin\pi x)$$

$$=2\int_{0}^{1}x\sin\pi x\,dx-\int_{1}^{\frac{3}{2}}x\sin\pi x\,dx$$

$$=2\left[x\left(\frac{-\cos\pi x}{\pi}\right)_{0}^{1}+\left(\frac{\sin\pi x}{\pi^2}\right)_{0}^{1}\right]-\left[x\left(\frac{-\cos\pi}{\pi}\right)\right]_{1}^{\frac{3}{2}}+\left(\frac{\sin\pi x}{\pi^2}\right)_{1}^{\frac{3}{2}}$$

$$=2\left(\frac{1}{\pi}\right)-\left\{-\left(\frac{1}{\pi}\right)-\left(\frac{1}{\pi^2}\right)\right\}=\frac{3}{\pi}+\frac{1}{\pi^2}$$

(Since, $\sin\frac{3\pi}{2}=-1$ and $\cos\pi=-1$)

49. Put

$$x=\left(\frac{\pi}{2}\right)-y\ \Rightarrow dx=-dy$$

and when $x=\pi, y=\frac{-\pi}{2}$ or when $x=0, y=\frac{\pi}{2}$.

$$I=\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}}\frac{\left(\frac{\pi}{2}-y\right)^2\sin 2y\,\sin\left(\left(\frac{\pi}{2}\right)\sin y\right)}{(-y)}(-dy)$$

$$=\frac{-1}{2}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\frac{\left(\frac{\pi^2}{4}+y^2\right)\sin 2y\,\sin\left\{\frac{\pi}{2}\sin y\right\}}{y}dy$$

$$+\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\pi y\frac{\sin 2y\,\sin\left\{\frac{\pi}{2}\sin y\right\}}{y}dy$$

$$=-I_1+I_2$$

$$I_1=0 \qquad \text{(Since integrand is odd)}$$

Therefore,

$$I=2\pi\int_0^{\frac{\pi}{2}}2\sin y\,\cos y\,\sin\left\{\left(\frac{\pi}{2}\right)\sin y\right\}dy$$

Put

$$\left(\frac{\pi}{2}\right)\sin y=t\Rightarrow\frac{\pi}{2}\cos y\,dy=dt$$

$$\Rightarrow I=2\pi\times 2\int_0^{\frac{\pi}{2}}\frac{2t}{\pi}\cdot\frac{2}{\pi}\sin t\,dt=\frac{16}{\pi}\left[\{t(-\cos t)\}_0^{\pi/2}-\int_0^{\frac{\pi}{2}}(-\cos t)dt\right]$$

$$=\frac{16}{\pi}$$

50. $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{6n}\right)$

$$=\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+5n}\right)=\lim_{n\to\infty}\sum_{r=1}^{5n}\frac{1}{n+r}$$

$$=\int_0^5\frac{dx}{1+x}=\ln(1+x)\Big|_0^5=\ln 6$$

51. Let

$$A=\lim_{n\to\infty}\left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right\}^{\frac{1}{n}}$$

$$\Rightarrow \log_e A=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left\{1+\left(\frac{r}{n}\right)^2\right\}=\int_0^1\log(1+x^2)\,dx$$

$$=[x\log(1+x^2)]_0^1-\int_0^1\frac{x\cdot 2x\,dx}{1+x^2}$$

$$=1\log 2-\int_0^1\frac{2(1+x^2)-2}{1+x^2}dx$$

$$=\log 2-\left[2x-2\tan^{-1}x\right]_0^1$$

$$=\log 2-\left[2-\frac{2\pi}{4}\right]=\log 2+\frac{\pi-4}{2}$$

$$\Rightarrow A=e^{\log 2+\frac{\pi-4}{2}}=2e^{\frac{\pi-4}{2}}$$

52. $$I=\int_0^{\pi}\frac{x\sin x}{2-\sin^2 x}dx=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{2-\sin^2(\pi-x)}dx,$$

$$\left(\text{using }\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x}{2-\sin^2 x}dx$$

Therefore,

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{-d(\cos x)}{1+\cos^2 x}=\left[-\frac{\pi}{2}\tan^{-1}(\cos x)\right]_0^{\pi}$$

$$=\left(\frac{-\pi}{2}\right)\left(\frac{-\pi}{4}\right)+\frac{\pi}{2}\left(\frac{\pi}{4}\right)=\frac{\pi^2}{4}$$

53. $$\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx=\int_0^{\pi}\sqrt{\cos^2 x}\,dx=\int_0^{\pi}|\cos x|\,dx$$

$$=\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}-\cos x\,dx$$

$$=[\sin x]_0^{\pi/2}-[\sin x]_{\pi/2}^{\pi}=1-(-1)=2$$

54. See Fig. 23.6.

$$\int_{-1}^{3/2}|\sin\pi x|\,dx=\int_{-1}^{0}-\sin(\pi x)dx+\int_0^1\sin(\pi x)dx+\int_1^{3/2}-\sin(\pi x)dx$$

$$=\left[\frac{\cos\pi x}{\pi}\right]_{-1}^{0}+\left[\frac{-\cos\pi x}{\pi}\right]_0^1+\left[\frac{\cos\pi x}{\pi}\right]_1^{3/2}$$

$$=\frac{1}{\pi}\{1-(-1)+1-(-1)+0-(-1)\}=\frac{5}{\pi}$$

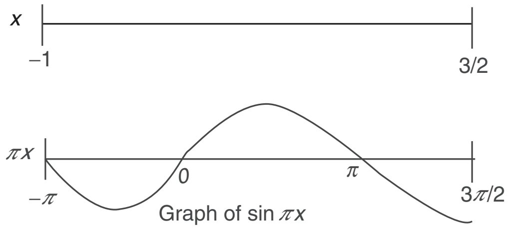


Figure 23.6

55. $$I=\int_{\pi/8}^{7\pi/8}\frac{x\,dx}{\cos^2 x-\sin^2 x}=\int_{\pi/8}^{7\pi/8}\frac{\pi-x}{\cos 2(\pi-x)}dx$$

$$=\int_{\pi/8}^{7\pi/8}\frac{(\pi-x)\,dx}{\cos 2x}=\pi\int_{\pi/8}^{7\pi/8}\frac{dx}{\cos 2x}-I$$

Therefore,

$$2I = \frac{\pi}{2}[\log(\sec 2x + \tan 2x)]_{\pi/8}^{7\pi/8}$$

$$I = \frac{\pi}{4}\left[\log\left(\sec\frac{7\pi}{4} + \tan\frac{7\pi}{4}\right) - \log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{4}[\log(\sqrt{2}-1) - \log(\sqrt{2}+1)] = \frac{\pi}{4}\log(3-2\sqrt{2})$$

56. $\int_{\pi/2}^{3\pi/2}[2\sin x]dx$, taking points where for $2\sin x = \pm 2, 0, \pm 1$

$$I = \int_{\pi/2}^{5\pi/6} dx + \int_{5\pi/6}^{\pi} 0\,dx + \int_{\pi}^{7\pi/6}(-1)\,dx + \int_{7\pi/2}^{3\pi/2}(-2)\,dx$$

$$= \frac{5\pi}{6} - \frac{\pi}{2} + \pi - \frac{7\pi}{6} + \frac{7\pi}{3} - 3\pi$$

$$= \frac{5\pi - 3\pi + 6\pi - 7\pi + 14\pi - 18\pi}{6} = -\frac{3\pi}{6} = -\frac{\pi}{2}$$

57. $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x} = \frac{1}{2}\int_{\pi/4}^{3\pi/4}\sec^2\frac{x}{2}dx = \left[\tan\frac{x}{2}\right]_{\pi/4}^{3\pi/4} = \tan\frac{3\pi}{8} - \tan\frac{\pi}{8}$

$$= \frac{1}{-1+\sqrt{2}} - \left(-1+\sqrt{2}\right) \qquad \left[\text{since} \quad \tan\frac{\pi}{8} = -1+\sqrt{2}\right]$$

$$= \frac{1-\left(-1+\sqrt{2}\right)^2}{-1+\sqrt{2}}$$

$$= \frac{-2+2\sqrt{2}}{-1+\sqrt{2}} = 2$$

58. Differentiate both side w.r.t. x

$$f(x)\cdot 1 - f(0)\cdot 0 = 1 + 1\cdot f(1)\cdot 0 - x\cdot f(x)\cdot 1$$

$$\Rightarrow (x+1)f(x) = 1 \Rightarrow f(x) = \frac{1}{x+1}$$

At $x = 1$, we have

$$f(1) = \frac{1}{1+1} = \frac{1}{2}$$

59. $g(x+\pi) = \int_0^{x+\pi}\cos^4 t\,dt = \int_0^x \cos^4 t\,dt + \int_x^{x+\pi}\cos^4 t\,dt$

$$= g(x) + \int_x^{\pi}\cos^4 t\,dt + \int_{\pi}^{\pi+x}\cos^4 t\,dt$$

$$= g(x) + \int_0^{\pi}\cos^4 t\,dt - \int_0^x \cos^4 t\,dt + \int_0^x \cos^4 t\,dt$$

By taking $t = x + \pi$ in $\int_{\pi}^{\pi+x}\cos^4 t dt$

$$g(x+\pi) = g(x) + g(\pi)$$

60. $$I = \int_1^{e^{37}}\frac{\pi\sin(\pi\ln x)}{x}dx$$

Let

$$\pi\ln x = t \Rightarrow \frac{\pi}{x}dx = dt$$

upper limit $= 37\pi$
lower limit $= 0$

$$I = \int_0^{37\pi}\sin t\,dt = -\cos t\Big|_0^{37\pi}$$

$$= -\{\cos 37\pi - \cos 0\} = 1 + 1 = 2$$

$$I = 2$$

61. $$I = \int_1^4 \frac{2e^{\sin x^2}}{x}dx \quad \text{Let } x^2 = t \Rightarrow x\,dx = \frac{dt}{2}$$

$$= \int_1^{16}\frac{2e^{\sin t}}{2t}dt = \int_1^{16}\left(\frac{d}{dt}F(t)\right)dt$$

$$= F(t)\Big|_1^{16} = F(16) - F(1)$$

Hence, possible value of $k = 16$.

62. $$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \qquad (1)$$

Replacing x by $\left(\frac{1}{x}\right)$,

$$af\left(\frac{1}{x}\right) + bf(x) = x - 5 \qquad (2)$$

By Eqs. (1) and (2),

$$f(x) = \frac{1}{a^2-b^2}\left\{\frac{a}{x} - bx + 5(b-a)\right\}$$

Then

$$\int_1^2 f(x)dx = \frac{1}{a^2-b^2}\int_1^2\left\{\frac{a}{x} - bx + 5(b-a)\right\}dx$$

$$= \frac{1}{a^2-b^2}\left[a\ln x - \frac{bx^2}{2} + 5(b-a)x\right]_1^2$$

$$= \frac{1}{a^2-b^2}\left[a\ln 2 - 2b + 10(b-a) + \frac{b}{2} - 5(b-a)\right]$$

$$= \frac{1}{a^2-b^2}\left[a\ln 2 + 5b - 5a - \frac{3}{2}b\right] = \frac{1}{a^2-b^2}\left[a\ln 2 + \frac{7b}{2} - 5a\right]$$

63. $\int_3^{3+3T} f(2x)\,dx = \int_6^{6+6T} f(t)\frac{dt}{2} = \frac{1}{2}\int_0^{6T} f(t)dt = 3\int_0^T f(t) = 3I$

64. $$I = \int_0^{\pi}\frac{(\pi-x)}{1+\cos\alpha\sin(\pi-x)}dx$$

$$= \int_0^{\pi}\frac{\pi}{1+\cos\alpha\sin x}dx - \int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}dx$$

$$\Rightarrow 2I = \int_0^{\pi}\frac{\pi}{1+\cos\alpha\sin x}dx$$

$$I = \frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\cos\alpha\sin x}dx = \frac{\pi}{2}\int_0^{\pi}\frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\cos\alpha}dx$$

Let

$$\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

$$I=\frac{\pi}{2}\int_0^{\infty}\frac{2dt}{1+t^2+2t\cos\alpha}dx=\frac{\pi}{2}\times 2\int_0^{\infty}\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}dx$$

$$=\frac{\pi}{\sin\alpha}\left(\tan^{-1}\frac{t+\cos\alpha}{\sin\alpha}\bigg|_0^{\infty}\right)=\frac{\pi}{\sin\alpha}\left(\frac{\pi}{2}-\tan^{-1}\cot\alpha\right)$$

$$=\frac{\pi}{\sin\alpha}\left(\frac{\pi}{2}-\tan^{-1}\tan\left(\frac{\pi}{2}-\alpha\right)\right)$$

$$I=\frac{\pi}{\sin\alpha}\left(\frac{\pi}{2}-\frac{\pi}{2}+\alpha\right)=\frac{\pi\alpha}{\sin\alpha}$$

Hence, $I=\pi\alpha\operatorname{cosec}\alpha$.

65. $$I=\int_0^{\frac{\pi}{2}}\frac{\left(\frac{\pi}{2}-x\right)\cos x\ \sin x\ dx}{\sin^4 x+\cos^4 x}\quad\left(\text{Using }\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\cos x\ \sin x\ dx}{\sin^4 x+\cos^4 x}-I\Rightarrow I=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{\cos x\ \sin x\ dx}{\sin^4 x+\cos^4 x}$$

$$I=\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\frac{\tan x\ \sec^2 x}{1+\tan^4 x}dx=\frac{\pi}{4}\int_0^{\infty}\frac{tdt}{1+(t^2)^2},\text{ where } t=\tan x$$

$$=\frac{\pi}{4}\cdot\frac{1}{2}\left[\tan^{-1}t^2\right]_0^{\infty}=\frac{\pi^2}{16}$$

66. $$\int_a^{a+t}f(x)dx=\int_a^0 f(x)dx+\int_0^t f(x)dx+\int_t^{a+t}f(x)dx$$

$$I_3=\int_t^{a+t}f(x)dx$$

Put $x=t+y$, then

$$I_3=\int_0^a f(t+y)dy=\int_0^a f(y)dy=-\int_a^0 f(x)dx\quad(\text{since, } f(t+y)=f(y))$$

Therefore,

$$I=I_1+I_2+I_3=\int_0^t f(x)dx\quad(\text{Independent of } a).$$

67. Let

$$I=\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

Put $x=\sin\theta$ when $x=0$, $\theta=0$

$dx=\cos\theta\, d\theta$ when $x=\frac{1}{2}$, $\theta=\frac{\pi}{6}$.

Thus,

$$I=\int_0^{\pi/6}\frac{\sin\theta\sin^{-1}(\sin\theta)}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$$

$$=\int_0^{\pi/6}\theta\sin\theta\, d\theta$$

$$=\left[\theta(-\cos\theta)\right]_0^{\pi/6}+\int_0^{\pi/6}1\cdot\cos\theta\, d\theta$$

$$=\left[-\theta\cos\theta+\sin\theta\right]_0^{\pi/6}$$

$$=-\frac{\pi}{6}\times\frac{\sqrt{3}}{2}+\frac{1}{2}$$

Hence,

$$I=\frac{6-\pi\sqrt{3}}{12}$$

68. $f(x)=e^{\cos x}\cdot\sin x$ is an odd function and

$$|x|\le 2\Rightarrow -2\le x\le 2$$

Therefore,

$$I=\int_{-2}^{3}f(x)dx=\int_{-2}^{2}f(x)dx+\int_2^3 f(x)dx$$

$$\Rightarrow I=0+\int_2^3 2dx=[2x]_2^3=2$$

69. $$I=2\int_0^3 f(x)dx+\int_3^5 f(x)dx$$

$$=2\left(\int_0^1 f(x)dx+\int_1^2 f(x)dx+\int_2^3 f(x)dx\right)+\int_3^4 f(x)dx+\int_4^5 f(x)dx$$

$$=2\left(\frac{0^2}{2}+\frac{1^2}{2}+\frac{2^2}{2}\right)+\frac{3^2}{2}+\frac{4^2}{2}$$

$$=1+4+\frac{25}{2}=\frac{35}{2}$$

70. $I=\int_0^{\pi/2}\ln\sin x\,dx$

$$I=\int_0^{\pi/2}\ln\sin\left(\frac{\pi}{2}-x\right)dx\Rightarrow I=\int_0^{\pi/2}\ln\cos x\,dx$$

$$2I=\int_0^{\pi/2}\ln(\sin x\cdot\cos x)dx\Rightarrow 2I=\int_0^{\pi/2}\ln\left(\frac{\sin 2x}{2}\right)dx$$

$$2I=\int_0^{\pi/2}\ln(\sin 2x)dx-\int_0^{\pi/2}\ln 2dx\Rightarrow 2I=\int_0^{\pi/2}\ln(\sin 2x)dx-\frac{\pi}{2}\ln 2$$

Put $2x=t$, then

$$dx=\frac{1}{2}dt$$

$$\Rightarrow 2I=\frac{1}{2}\int_0^{\pi}\ln(\sin t)dt-\frac{\pi}{2}\ln 2\ \Rightarrow 2I=\int_0^{\pi/2}\ln(\sin t)dt-\frac{\pi}{2}\ln 2$$

$$\Rightarrow 2I=\int_0^{\pi/2}\ln(\sin x)dx-\frac{\pi}{2}\ln 2\Rightarrow 2I=I-\frac{\pi}{2}\ln 2$$

$$\Rightarrow I=-\frac{\pi}{2}\ln 2$$

71. $$I=\int_0^a f(x)g(x)h(x)dx = \int_0^a f(a-x)g(a-x)h(a-x)\,dx$$

$$=-\int_0^a f(x)g(x)h(a-x)dx$$

$$7I = 3I + 4I$$

$$\int_0^a f(x)g(x)\{3h(x)-4h(a-x)\}dx = 5\int_0^a f(x)g(x)dx = 0$$

$$\Rightarrow I = 0$$

72. Let

$$I=\int_0^\pi x\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx \qquad (1)$$

$$=\int_0^\pi (\pi-x)\sin 2(\pi-x)\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)dx$$

$$=\int_0^\pi (\pi-x)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx \qquad (2)$$

Adding Eqs. (1) and (2), we get

$$2I=\pi\int_0^\pi \sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$\Rightarrow I=\pi\int_0^{\frac{\pi}{2}} 2\sin x\cos x\sin\left(\frac{\pi}{2}\cos x\right)dx$$

Put

$$\frac{\pi}{2}\cos x = z \Rightarrow -\frac{\pi}{2}\sin x dx = dz$$

$$\pi\int_{\frac{\pi}{2}}^0 2\cdot\frac{2z}{\pi}\left(-\frac{2}{\pi}\right)\sin z\,dz = -\frac{8}{\pi}\int_{\frac{\pi}{2}}^0 z\sin z\; dz = \frac{8}{\pi}$$

Practice Exercise 2

1. $$\int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} - \int_{1/2}^{1/3}\frac{dx}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}$$

$$=\int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} - \int_{1/2}^{1/3}\frac{dx}{x^3\sqrt{1+\left(x-\frac{1}{x}\right)^2}}$$

$$=\int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} + \int_2^3\frac{t^3dt}{t^2\sqrt{1+\left(\frac{1}{t}-t\right)^2}}$$

$$=\int_2^3 \frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}} + \int_2^3\frac{tdt}{\sqrt{1+\left(t-\frac{1}{t}\right)^2}} = I + I = 2I$$

Here, $I=\int_2^3\frac{x^2dx}{\sqrt{x^4-x^2+1}}$

$$=\int_2^3\frac{xdx}{\sqrt{x^2-1+\frac{1}{x^2}}}=\int_2^3\frac{xdx}{\sqrt{1+\left(x-\frac{1}{x}\right)^2}}=\int_2^3\frac{tdt}{\sqrt{1+\left(t-\frac{1}{t}\right)^2}}$$

2. $$f'(x)=f(x)$$

Integrating, we get

$$\log f(x) = x + k$$

$$\Rightarrow f(x) = e^{x+k}$$

Now, $f(0) = 1$

Therefore,

$$1 = e^{0}\cdot e^{k}$$

$$\Rightarrow k = 0$$

Therefore, $f(x) = e^x$

and

$$g(x) = x - f(x) = x - e^x$$

Therefore,

$$I=\int_0^1 e^x(x-e^x)dx = \int_0^1 e^x x\,dx - \int_0^1 e^{2x}dx$$

$$=e-(e-1)-\frac{e^2}{2}+\frac{1}{2} = \frac{3}{2}-\frac{e^2}{2}=\frac{3-e^2}{2}$$

3. $I=\int_{-1/2}^{1/2}\cos^{-1}(4x^3-3x)dx$

and

$$x=\cos\theta,\quad \theta\in\left[\frac{\pi}{3},\frac{2\pi}{3}\right]$$

Now,

$$\cos^{-1}(4x^3-3x)=\cos^{-1}(\cos 3\theta)=2\pi-3\theta$$

Therefore,

$$I=-\int_{2\pi/3}^{\pi/3}(2\pi-3\theta)\sin\theta\,d\theta$$

$$=2\pi\int_{\pi/3}^{2\pi/3}\sin\theta\,d\theta-3\int_{\pi/3}^{2\pi/3}\theta\sin\theta\,d\theta=\frac{\pi}{2}$$

4. $I=\int_0^{\pi/4}\ln\cos 2x dx-\int_0^{\pi/4}\ln(1+\tan x)dx$

Now,

$$I_1=\frac{1}{2}\int_0^{\pi/2}\ln\cos x dx=\frac{1}{2}\left(\frac{\pi}{2}\ln\frac{1}{2}\right)=-\frac{\pi}{4}\ln 2$$

and

$$I_2=\frac{\pi}{8}\ln 2$$

Therefore,

$$I=I_1-I_2=-\frac{3\pi}{8}\ln 2$$

5. $y=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\left(\frac{r}{n}\right)\cdot\sin\left(\frac{r^2}{n^2}+1\right)$

$$=\int_0^2 x\sin(x^2+1)dx=\sin 2\cdot\sin 3$$

6. $g(x)<f(x)\ \forall\ x\in(0,\infty)$

Also,

$$d(x(f(x))=g(x)dx$$

Therefore,

$$xf(x)=\int_0^x g(x)dx\ \forall\ x\in(0,\infty)$$

and

$$xg(x)<\int_0^x g(x)dx\ \forall\ x\in(0,\infty)$$

7. $f(x)=[\tan^{100}x]$

$$\Rightarrow f(3^{1/100}-)\neq f(3^{1/100})=f(3^{1/100}+)\text{ and}$$

$$f(100^{1/100}-)\neq f(100^{1/100})=f(100^{1/100}+)$$

Hence, $f(x)$ is discontinuous at these points.

Now,

$$\int_0^{\tan^{-1}4^{1/100}}[\tan^{100}x]\,dx=\int_0^{\tan^{-1}1^{1/100}}[\tan^{100}x]dx$$

$$+\int_{\tan^{-1}1^{1/100}}^{\tan^{-1}2^{1/100}}[\tan^{100}x]dx+\int_{\tan^{-1}2^{1/100}}^{\tan^{-1}3^{1/100}}[\tan^{100}x]dx+\int_{\tan^{-1}3^{1/100}}^{\tan^{-1}4^{1/100}}[\tan^{100}x]dx$$

Therefore,

$$\int_0^{\tan^{-1}\sqrt[100]{4}}f(x)dx=3\tan^{-1}\sqrt[100]{4}-\tan^{-1}\sqrt[100]{3}-\tan^{-1}\sqrt[100]{2}-1$$

8. $I_n=\left(\frac{e^{-x}(\sin x)^n}{-1}\right)\Bigg|_0^{\infty}+n\int_0^{\infty}(\sin x)^{n-1}\cos xe^{-x}dx$

$$=\int_0^{\infty}(-(\sin x)^n+(n-1)(1-\sin^2x)(\sin x)^{n-2})e^{-x}dx$$

$$=\frac{n(n-1)}{n^2+1}I_{n-2}$$

Hence,

$$\frac{I_{10}}{I_8}=\frac{90}{101}$$

9. In option (B), we have

$$\int_0^{\infty}xe^{-x^2}dx=-\frac{1}{2}e^{-x^2}\Bigg|_0^{\infty}=\frac{1}{2}$$

Therefore,

$$\int_0^{\infty}x^2e^{-x^2}dx=x\left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_0^{\infty}+\frac{1}{2}\int_0^{\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{4}$$

10. For $x>2$, we have

$$\int_0^x(5+|1-t|)dt=\int_0^1(6-t)dt+\int_1^x(4+t)dt\ =1+4x+\frac{x^2}{2}$$

$$\Rightarrow f(x)=\begin{cases}1+4x+\dfrac{x^2}{2}, & x>2\\ 5x+1, & x\le 2\end{cases}$$

Now,

$$f'(x)=\begin{cases}4+x, & x>2\\ 5, & x\le 2\end{cases}$$

Therefore,

$$f(2^+)=f(2^-)=f(2)=11\text{ is continuous at }x=2$$

and

$$f'(2^+)\neq f'(2^-)\Rightarrow\text{not differentiable at }x=2$$

11. $\int_{\alpha-t}^{\alpha}f(t)\,dt=-\int_{\alpha}^{\alpha+t}f(t)\,dt\ \Rightarrow f(\alpha-t)=-f(\alpha+t)\ \ \forall\ t\in R$

Hence, graph of $f(x)$ is symmetrical about $(\alpha, 0)$.

12. $f(\alpha-t)=-f(\alpha+t)\ \ \forall\ t\in R$

Put $x=\alpha$, then

$$f(2\alpha)=-f(0)$$

13. $\int_{\alpha-t}^{\alpha}f(t)\,dt=-\int_{\alpha}^{\alpha+t}f(t)\,dt$

$$\Rightarrow f(\alpha-t)=-f(\alpha+t)\ \forall\ t\in R$$

$$\Rightarrow f(\alpha-t)=-f(\alpha+t)=x$$

$$\Rightarrow t=\alpha-f^{-1}(x)=f^{-1}(-x)-\alpha$$

$$\Rightarrow f^{-1}(x)+f^{-1}(-x)=2\alpha$$

$$\Rightarrow\int_{-\beta}^{\beta}f^{-1}(x)dx\ =2\alpha\beta$$

14. On differentiating a polynomial of n^{th} degree, we get another polynomial of $(n-1)$ degrees. So,

$$f(x)=\{f'(x)\}^2\ \ \Rightarrow n=2(n-1)\ \Rightarrow n=2$$

15. Let

$$f(x)=ax^2+bx+c\Rightarrow f'(0)=b>0$$

Also,

$$f(x)=\{f'(x)\}^2\Rightarrow ax^2+bx+c=4a^2x^2+4abx+b^2\ \forall\ x$$

Thus, $a=4a^2$, $b=4ab$ and $c=b^2$

From which, we get $a=\frac{1}{4}$, since $(b\neq 0)$

Again,

$$\int_0^1 f(x)dx=\frac{19}{12}\ \Rightarrow\ \frac{a}{3}+\frac{b}{2}+c=\frac{19}{12}$$

Therefore,

$$\frac{b}{2}+b^2=\frac{3}{2}\Rightarrow b=1,$$

(since, $(b>0)$ and so $c=1$)

Therefore,

$$f'(0)=b=1$$

16. By putting the value of a, b and c in $f(x)$, we have

$$f(x)=\frac{x^2}{4}+x+1$$

17. Writing $\sin^2\theta = x$, we get $2\sin\theta\cos\theta\, d\theta = dx$, and hence the given integral is

$$\frac{1}{2}\int_0^{\pi/2}\cos^{2m-1}\sin^{2n-1}\theta\,(2\sin\theta\cos\theta)\,d\theta$$

$$=\frac{1}{2}\int_0^1(\cos^2\theta)^{\frac{2m-1}{2}}(\sin^2\theta)^{\frac{2n-1}{2}}dx$$

$$=\frac{1}{2}\int_0^1(1-x)^{m-1/2}x^{n-1/2}dx$$

$$=\frac{1}{2}\beta\left(m+\frac{1}{2},\ n+\frac{1}{2}\right)$$

18. Writing $\frac{x}{n}=z$, we get

$$\int_0^n\left(1-\frac{x}{n}\right)^n x^{k-1}dx=\int_0^1(1-z)^n(nz)^{k-1}n\,dz$$

$$=n^k\int_0^1(1-z)^n z^{k-1}dz=n^k\,\beta(k,\ n+1)$$

19. Writing $\frac{x}{1+x}=z$, we get $x=\frac{z}{1-z}, 1+x=\frac{1}{1-z}$ and $dx=\frac{dz}{(1-z)^2}$

$$\text{LHS}=\int_0^1\frac{z^{m-1}}{(1-z)^{m-1}}(1-z)^{m+n}\frac{dz}{(1-z)^2}=\int_0^1 z^{m-1}(1-z)^{n-1}dz=\beta(m,n)$$

$$=\int_0^\infty\frac{x^{1-n}}{(1+x)^{m+n}}dx$$

Common Explanation for Questions 20–22:

$$\int_0^a x^4\sqrt{a^2-x^2}\,dx=\left[\frac{-x^3(a^2-x^2)^{3/2}}{3}\right]_0^a+a^2\cdot\frac{3}{6}\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

$$=\frac{a^2}{2}\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

$$I_n=\int x^n\sqrt{a^2-x^2}\,dx=\int x^{n-1}\left(x\sqrt{a^2-x^2}\right)dx$$

$$=x^{n-1}\left[-\frac{1}{3}(a^2-x^2)^{3/2}\right]+\frac{n-1}{3}\int x^{n-2}(a^2-x^2)\sqrt{a^2-x^2}\,dx$$

$$=-\frac{1}{3}x^{n-1}(a^2-x^2)^{3/2}+\frac{n-1}{3}a^2I_{n-2}-\frac{n-1}{3}I_n$$

$$\Rightarrow\left(1+\frac{n-1}{3}\right)I_n=-\frac{1}{3}x^{n-1}(a^2-x^2)^{3/2}+\frac{a^2(n-1)}{3}I_{n-2}$$

$$\Rightarrow I_n=-\frac{x^{n-1}(a^2-x^2)^{3/2}}{(n+2)}+\frac{a^2(n-1)}{(n+2)}I_{n-2} \quad (1)$$

20. $I_1=\int x\sqrt{a^2-x^2}\,dx=\frac{1}{2}\int 2x\sqrt{a^2-x^2}\,dx$

$$=-\frac{2(a^2-x^2)^{3/2}}{2.3}+c=-\frac{(a^2-x^2)^{3/2}}{3}+c$$

21. $I_4=-\left.\frac{x^3(a^2-x^2)^{3/2}}{6}\right|_0^a+\frac{3a^2}{6}I_2=\frac{a^2}{2}I_2\Rightarrow\frac{I_4}{I_2}=\frac{a^2}{2}$

22. From equation (1), we have

$$k=\frac{a^2(n-1)}{(n+2)}$$

23. (A) $y=\cos^{-1}\left(\frac{2\cos x-3\sin x}{\sqrt{13}}\right)$

$y=\cos^{-1}\cos(\alpha+x)\ \forall\ \alpha=\tan^{-1}\frac{3}{2}$

Therefore,

$$\frac{dy}{dx}=1$$

(B) $I=\int_{-1}^1\sin^{-1}\left[x^2+\frac{1}{2}\right]dx+\int_{-1}^1\cos^{-1}\left[x^2-\frac{1}{2}\right]dx$

$$=2\int_0^1\left(\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2-\frac{1}{2}\right]\right)dx$$

Let

$$f(x)=\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2-\frac{1}{2}\right]$$

$$=\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left(\left[x^2+\frac{1}{2}\right]-1\right)$$

which is defined if $\left[x^2+\frac{1}{2}\right]=0$ or 1

if $\left[x^2+\frac{1}{2}\right]=0$ then $f(x)=\sin^{-1}0+\cos^{-1}(-1)=0+\pi=\pi$

if $\left[x^2+\frac{1}{2}\right]=1$ then $f(x)=\sin^{-1}1+\cos^{-1}0=\pi$

$I=2\int_0^1\pi dx=2\pi$

(C) $\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{(1+x^2)(1+e^{\sin x})}=\int_0^{\sqrt{3}}\frac{dx}{1+x^2}=\frac{\pi}{3}$

(D) $\int_0^{6\pi}\sin^{-1}(\sin x)dx=3\int_0^{2\pi}\sin^{-1}(\sin x)dx=0$

24. (A) $I=\int_0^{\pi/2}\frac{1}{1+\tan\left(\frac{\pi}{2}-x\right)}dx=\int_0^{\pi/2}\frac{1}{1+\cot x}dx$

$$=\int_0^{\pi/2}\frac{\tan x}{1+\tan x}dx=\int_0^{\pi/2}\left[1-\frac{1}{1+\tan x}\right]dx$$

Therefore,

$$2I=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

(B) Differentiating $[f(x^2(1+x^5+7x^{12}))](2x+7x^6+98x^{13})=1$

Put $x=1$, we get

$$f(9)=\frac{1}{117}$$

(C) $\int_0^\infty e^{-2x}(\sin 2x+\cos 2x)dx$

$$=\sqrt{2}\int_0^\infty e^{-2x}\sin\left(2x+\frac{\pi}{4}\right)dx$$

Therefore,

$$I = \left[-e^{-2x}\frac{\sin\left(2x+\frac{\pi}{4}\right)}{2}\right]_0^{\infty} + \int_0^{\infty} e^{-2x}\cos\left(2x+\frac{\pi}{4}\right)dx$$

$$= \frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}} - I \Rightarrow 2I = \frac{1}{\sqrt{2}} \Rightarrow \text{integral value} = \frac{1}{2\sqrt{2}}$$

(D) $\cot^{-1}(1+x^2-x) = \tan^{-1}\left(\dfrac{x+1-x}{1-x(1-x)}\right)$

Therefore,

$$I = \int_0^1 \cot^{-1}(1+x^2-x)\,dx$$

$$= \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

$$= 2\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{2} - \log 2$$

25. (A) $\int_0^{\pi} f(x)dx = 0$ if $f(a-x) = -f(x)$

(B) $I_{n+1} = \int_0^{\pi/4} \tan^{n-1}\theta(\sec^2\theta - 1)\,d\theta$

$$= \int_0^{\pi/4} \tan^{n-1}\theta\sec^2\theta\,d\theta - I_{n-1}$$

Therefore,

$$I_{n+1} + I_{n-1} = \left[\frac{\tan^n\theta}{n}\right]_0^{\pi/4} = \frac{1}{n}$$

(C) $\displaystyle\int_0^{\infty}\frac{x\log x}{(1+x^2)^2}dx = \int_0^1 \frac{x\log x}{(1+x^2)^2}dx + \int_1^{\infty}\frac{x\log x}{(1+x^2)^2}dx$

Put $x = \dfrac{1}{y}$ in the last integral

Therefore,

$$dx = -\frac{1}{y^2}dy$$

$$\int_0^{\infty}\frac{x\log x}{(1+x^2)^2}dx = \int_1^0 \frac{y^4}{y^3(1+y^2)^2}dy$$

$$= -\int_0^1 \frac{y\log y}{(1+y^2)^2}dy$$

Therefore, given integral equals 0.

(D) $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{1}{x}\right) = \dfrac{1}{1+\dfrac{1}{x^2}}\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{1+x^2}$

$$I = \int_{-1}^{1}\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx = -\int_{-1}^{1}\frac{dx}{1+x^2} = -\frac{\pi}{2}$$

26. (A) $\int_0^{\pi} xf(\sin x)dx = \int_0^{\pi}(\pi - x)f(\sin x)dx$

$$\Rightarrow 2\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\pi} f(\sin x)dx$$

(B) $2\int_0^{\pi} xf(\sin x)dx = \pi\cdot 2\int_0^{\pi/2} f(\sin x)dx$

$$= 2\pi\int_0^{\pi/2} f\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx$$

$$= 2\pi\int_0^{\pi/2} f(\cos x)dx$$

(C) $\int_{1-k}^{k} xf(x(1-x))dx = \int_{1-k}^{k}(k+(1-k)-x)$

$$f(k+1-k-x)(1-(k+1-k-x))dx$$

$$= \int_{1-k}^{k}(1-x)f((1-x)x)dx$$

$$\Rightarrow 2\int_{1-k}^{k} xf(x(1-x))dx = \int_{1-k}^{k} f(x(1-x))dx$$

(D) $\int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))dx = \int_{\sin^2 t}^{1+\cos^2 t}(2-x)f(x(2-x))dx$

$$\Rightarrow \int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))dx = \int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))dx$$

$$= \int_{-\cos^2 t}^{\cos^2 t} f(1-x^2)dx \quad (\text{write } x = 1-y)$$

27. $I = \text{Real}\left\{\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} 1 + (\cos\theta + i\sin\theta) + \dfrac{(\cos\theta+i\sin\theta)^2}{2!} + \cdots\right\}d\theta$

$$= \frac{1}{\pi}\left[\theta + \sin\theta + \frac{\sin 2\theta}{2(2!)} + \cdots\right]_0^{2\pi} = 2$$

28. $\displaystyle\int_0^{\pi} f^{-1}(x)dx = \int_{f^{-1}(0)}^{f^{-1}(\pi)} tf'(t)dt = [tf(t) - f(t)]_{f^{-1}(0)}^{f^{-1}(t)}$

$$= \int_0^{\pi} f^{-1}(x)dx = \pi^2 - \int_0^{\pi}(t+\cos t)dt$$

$$= \pi^2 - \left(\frac{t^2}{2} + \sin t\right)_0^{\pi} = \pi^2 - \frac{\pi^2}{2} = \frac{\pi^2}{2}$$

Therefore, $k = 0$.

29. Differentiating both sides of the given equation, we have

$$\frac{d}{dx}\int_{\cos x}^{1} t^2 f(t)dt = \frac{d}{dx}(1-\cos x)$$

$\Rightarrow 1\cdot f(1)\cdot 0 - \cos^2 x\, f(\cos x)\,(-\sin x) = \sin x$

$\Rightarrow \cos^2 x f(\cos x)\sin x = \sin x$

$$\Rightarrow f(\cos x) = \frac{1}{\cos^2 x}$$

Therefore, $f\left(\dfrac{\sqrt{3}}{4}\right)$ is attained when $\cos x = \dfrac{\sqrt{3}}{4}$. So,

$$f\left(\frac{\sqrt{3}}{4}\right) = \frac{16}{3} = 5.33$$

$$\Rightarrow \left[f\left(\frac{\sqrt{3}}{4}\right)\right] = 5$$

Solved JEE 2017 Questions

JEE Main 2017

1. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to

(A) 2 **(B)** 4
(C) −1 **(D)** −2

(OFFLINE)

Solution: The given integral is

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$$

That is,

$$I = \int \frac{dx}{1+\cos(\pi - x)}$$

By using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\right)dx$$

$$= \int_{\pi/4}^{3\pi/4} \left(\frac{2}{1-\cos^2 x}\right)dx$$

$$2I = 2\int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x dx$$

Therefore,

$$I = 2\int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x dx$$

That is,

$$I = (-\cot x)\Big|_{\pi/4}^{3\pi/4} = -(-1-1) = 2$$

Hence, the correct answer is option (A).

2. The integral $\int_{\pi/2}^{\pi/4} \frac{8\cos 2x}{(\tan x + \cot x)^3} dx$ equals

(A) $\frac{15}{128}$ **(B)** $\frac{13}{32}$
(C) $\frac{13}{256}$ **(D)** $\frac{15}{64}$

(ONLINE)

Solution: The given integral is

$$I = \int_{\pi/12}^{\pi/4} \frac{8\cos 2x}{(\tan x + \cos x)^3} dx$$

That is,

$$I = \int_{\pi/12}^{\pi/4} \frac{\cos 2x}{\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)^3} dx \quad \left(\text{since } \tan x = \frac{\sin x}{\cos x};\ \cot x = \frac{\cos x}{\sin x}\right)$$

$$I = \int_{\pi/12}^{\pi/4} \frac{8\cos 2x}{\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right)^3} dx \Rightarrow I = \int_{\pi/12}^{\pi/4} \frac{8\cos 2x}{\left(\frac{1}{\cos x \sin x}\right)^3} dx$$

Multiplying numerator and denominator of denominator by 2, we get

$$I = \int_{\pi/12}^{\pi/4} \frac{8\cos 2x}{\left(\frac{2}{2\sin x \cos x}\right)^3} dx = \int_{\pi/12}^{\pi/4} \frac{8\cos 2x}{\left(\frac{2}{\sin 2x}\right)^3} dx$$

(since $\sin 2x = 2\sin x + \cos x$)

$$I = \int_{\pi/12}^{\pi/4} 8\cos 2x \times \frac{\sin^3 2x}{8} dx = \int_{\pi/12}^{\pi/4} \cos 2x \sin^3 2x\, dx$$

$$I = \int_{\pi/12}^{\pi/4} (\sin 2x \cos 2x).\sin^2 2x\, dx$$

$$I = \frac{1}{2}\int_{\pi/12}^{\pi/4} (\sin 4x)\sin^2 2x\, dx \qquad (\text{since } \sin 2x = 2 \sin x \cos x)$$

$$I = \frac{1}{2}\int_{\pi/12}^{\pi/4} \sin 4x\left(\frac{1-\cos 4x}{2}\right)dx \qquad (\text{since } \cos 2x = 1 - 2\sin^2 x)$$

$$I = \frac{1}{4}\int_{\pi/12}^{\pi/4} (\sin 4x - \sin 4x \cos 4x)dx$$

$$I = \frac{1}{4}\int_{\pi/12}^{\pi/4} \sin 4x dx - \frac{1}{4}\int_{\pi/12}^{\pi/4} \sin 4x \cos 4x\, dx$$

$$I = \frac{1}{4}\int_{\pi/12}^{\pi/4} \sin 4x dx - \frac{1}{4}\int_{\pi/12}^{\pi/4} \frac{\sin 8x}{2} dx \qquad (\text{since } \sin 2x = 2\sin x\cos x)$$

$$I = \frac{1}{4}\int_{\pi/12}^{\pi/4} \sin 4x\, dx - \frac{1}{8}\int_{\pi/12}^{\pi/4} \sin 8x\, dx$$

$$\Rightarrow I = \frac{1}{4}\left(\frac{-\cos 4x}{4}\Big|_{\pi/12}^{\pi/4}\right) - \frac{1}{8}\left(\frac{-\cos 8x}{8}\Big|_{\pi/12}^{\pi/4}\right)$$

$$I = \frac{1}{4}\left[\frac{-1}{4}\cos\left(4\times\frac{\pi}{4}\right) + \frac{1}{4}\cos\left(4\times\frac{\pi}{12}\right)\right] - \frac{1}{8}\left[\frac{-1}{8}\cos\left(8\times\frac{\pi}{4}\right) + \frac{1}{8}\cos\left(8\times\frac{\pi}{12}\right)\right]$$

$$I = \frac{1}{4}\left[\frac{-1}{4}\cos\pi + \frac{1}{4}\cos\frac{\pi}{3}\right] - \frac{1}{8}\left[\frac{-1}{8}\cos 2\pi + \frac{1}{8}\cos\frac{2\pi}{3}\right]$$

$$I = \frac{1}{4}\left[\frac{-1}{4}(-1) + \frac{1}{4}\left(\frac{1}{2}\right)\right] - \frac{1}{8}\left[\frac{-1}{8}(1) + \frac{1}{8}\left(\frac{-1}{2}\right)\right]$$

$$I = \frac{1}{4}\times\frac{1}{4}\left[1+\frac{1}{2}\right] - \frac{1}{8}\times\frac{1}{8}\left[-1-\frac{1}{2}\right]$$

$$\Rightarrow I = \frac{1}{16}\left(\frac{3}{2}\right) + \frac{1}{64}\left(\frac{3}{2}\right)$$

$$\Rightarrow I = \frac{3}{32} + \frac{3}{128} = \frac{(4\times 3)+3}{128} = \frac{12+3}{128} = \frac{15}{128}$$

Hence, the correct answer is option (A).

3. The integral $\int \sqrt{1+2\cot x(\operatorname{cosec} x+\cot x)}dx$ $\left(0<x<\frac{\pi}{2}\right)$ is equal to (where C is a constant of integration)

(A) $2\log\left(\sin\frac{x}{2}\right)+C$ (B) $2\log\left(\cos\frac{x}{2}\right)+C$

(C) $4\log\left(\cos\frac{x}{2}\right)+C$ (D) $4\log\left(\sin\frac{x}{2}\right)+C$

(ONLINE)

Solution: The given integral is

$$I = \int \sqrt{1+2\cot x(\operatorname{cosec} x+\cot x)}dx$$

$$= \int \sqrt{1+2\cot x + \operatorname{cosec} x + 2\cot^2 x}\ dx$$

$$= \int \sqrt{1+\cot^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x}\ dx$$

Using $1 + \cot^2 x = \operatorname{cosec}^2 x$

$$I = \int \sqrt{\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x}\ dx$$

$(a + b)^2 = a^2 + b^2 + 2ab$

$$\Rightarrow I = \int \sqrt{(\cot x + \operatorname{cosec} x)^2}\ dx$$

$$I = \int (\cot x + \operatorname{cosec} x)dx$$

Substituting $\operatorname{cosec} x = \frac{1}{\sin x}$ and $\cot x = \frac{\cos x}{\sin x}$, we get

$$I = \int\left(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right)dx = \int \frac{\cos x + 1}{\sin x}dx$$

Substituting $\cos x + 1 = 2\cos^2\left(\frac{x}{2}\right)$ and $\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$, we get

$$I = \int \frac{2\cos^2(x/2)}{\sin(x/2)\cos(x/2)}dx = \int \frac{2\cos(x/2)}{\sin(x/2)}dx$$

$$\Rightarrow I = 2\int \cot\left(\frac{x}{2}\right)dx \qquad \left(\text{since } \cot x = \frac{\cos x}{\sin x}\right)$$

Using the standard integral $\int \cot x\ dx = \log|\sin x| + C$, we get

$$I = 2\log\left|\sin\left(\frac{x}{2}\right)\right| + C$$

Hence, the correct answer is option (A).

4. If $\int_1^2 \frac{dx}{(x^2-2x+4)^{3/2}} = \frac{k}{k+5}$, then k is equal to

(A) 1 (B) 3

(C) 4 (D) 2

(ONLINE)

Solution: We have

$$\int_1^2 \frac{dx}{(x^2-2x+4)^{3/2}} \Rightarrow \int_1^2 \frac{dx}{\left[(x-1)^2+3\right]^{3/2}}$$

Substituting $x - 1 = \sqrt{3}\tan\theta$, where $x = 1$, $\theta = 0$.

When $x = 2$, $\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

$$L = \int_0^{\pi/6} \frac{\sqrt{3}\sec^2\theta}{\left[3\tan^2\theta+3\right]^{3/2}}d\theta$$

$$= \int_0^{\pi/6} \frac{\sqrt{3}\sec^2\theta d\theta}{[3\sec^2\theta]^{3/2}} = \int_0^{\pi/6} \frac{\sqrt{3}\sec^2\theta\ d\theta}{3\sqrt{3}\sec^2\theta} = \frac{1}{3}\int_0^{\pi/6} \cos\theta\ d\theta$$

$$= \frac{1}{3}\sin\theta\Big|_0^{\pi/6}$$

$$\Rightarrow \frac{1}{3}\left[\sin\frac{x}{6} - \sin 0\right]$$

$$\Rightarrow \frac{1}{3}\left[\frac{1}{2} - 0\right] = \frac{1}{6}$$

$$\Rightarrow \frac{1}{6} = \frac{k}{k+5} \Rightarrow k+5 = 6k \Rightarrow 5 = 5k \Rightarrow k = 1$$

Hence, the correct answer is option (A).

JEE Advanced 2017

1. If $I = \sum_{k=1}^{98}\int_k^{k+1} \frac{k+1}{x(x+1)}dx$, then

(A) $I > \log_e 99$ (B) $I < \log_e 99$

(C) $I < \frac{49}{50}$ (D) $I > \frac{49}{50}$

Solution: It is given that

$$I = \sum_{k=1}^{98}\int_k^{k+1} \frac{k+1}{x(x+1)}dx$$

$$= \sum_{k=1}^{98}(k+1)\int_k^{k+1} \frac{1}{x(x+1)}dx = \sum_{k=1}^{98}(k+1)\int_k^{k+1}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \sum_{k=1}^{98}(k+1)\left[\ln x - \ln(x+1)\Big|_k^{k+1}\right]$$

$$= \sum_{k=1}^{98}(k+1)\left[(\ln(k+1) - \ln k) - (\ln(k+1+1) - \ln(k+1)\right]$$

$$= \sum_{k=1}^{98} (k+1)[\ln(k+1) - \ln k - \ln(k+2) + \ln(k+1)]$$

$$= \sum_{k=1}^{98} (k+1)\ln(k+1) - (k+1)\ln k - (k+1)\ln(k+2) + (k+1)\ln(k+1)$$

$$= \sum_{k=1}^{98} (k+1)\ln(k+1) - k\ln k - \ln k - (k+1)\ln(k+2) + k\ln(k+1) + \ln(k+1)$$

Rearranging the terms, we get

$$I = \sum_{k=1}^{98} (k+1)\ln(k+1) - k\ln k - \sum_{k=1}^{98} (k+1)\ln(k+2) - k\ln(k+1)$$
$$+ \sum_{k=1}^{98} \ln(k+1) - \ln k$$

That is,

$I = 99 \ln 99 - 99 \ln 100 + \ln 2 + \ln 99 = \ln 99^{99} - \ln 100^{99} + \ln 2 + \ln 99$

$$I = \ln\left(\frac{99^{99}}{100^{99}} \times 2 \times 99\right) = \ln\left(\frac{99^{100} \times 2}{100^{99}}\right) \quad (1)$$

Now, let us consider that 100^{99} is written as

$$100^{99} = (99 + 1)^{99}$$

Using binomial expansion, we get

$$(100)^{99} = {}^{99}C_0 + {}^{99}C_1(99)^1 + {}^{99}C_2(99)^2 + \cdots + {}^{99}C_{98}(99)^{98} + {}^{99}C_{99}(99)^{99}$$

$$= {}^{99}C_0 + {}^{99}C_1(99)^1 + {}^{99}C_2(99)^2 + \cdots + (99)^{99} + (99)^{99}$$

$$\Rightarrow (100)^{99} > 2\cdot(99)^{99} \quad \text{(considering last two terms)}$$

$$\Rightarrow \frac{2\cdot(99)^{99}}{(100)^{99}} < 1$$

Multiplying both sides by 99, we get

$$\frac{2\cdot(99)^{100}}{(100)^{99}} < 99$$

Taking natural logarithm on both sides, we get

$$\ln\frac{2(99)^{100}}{(100)^{99}} < \ln 99 \Rightarrow I < \ln 99 \quad (\ln \equiv \log_e)$$

Now, we know that

$$I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$$

$$\sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx > \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{(x+1)^2} dx$$

$$\Rightarrow I > \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{(x+1)^2} dx = \sum_{k=1}^{98} (k+1) \int_k^{k+1} (x+1)^{-2} dx$$

$$\Rightarrow I > \sum_{k=1}^{98} (k+1)\left(\left.\frac{(x+1)^{-2+1}}{-2+1}\right|_k^{k+1}\right) = \sum_{k=1}^{98} (k+1)\left(\left.\frac{-1}{(x+1)}\right|_k^{k+1}\right)$$

$$\Rightarrow I > \sum_{k=1}^{98} (k+1)\left(\frac{-1}{(k+1+1)} + \frac{1}{(k+1)}\right) = \sum_{k=1}^{98} (k+1)\left(\frac{1}{(k+1)} - \frac{1}{(k+2)}\right)$$

$$\Rightarrow I > \sum_{k=1}^{98} (k+1)\left(\frac{k+2-k-1}{(k+1)(k+2)}\right) = \sum_{k=1}^{98} (k+1)\frac{1}{(k+1)(k+2)}$$

$$\Rightarrow I > \sum_{k=1}^{98} \frac{1}{(k+2)} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{100}$$

These are total 98 terms. Now,

$$\frac{1}{100} + \frac{1}{100} + \cdots + \frac{1}{100} \text{ (98 terms)} = \frac{98}{100} < \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{100}$$

Therefore,

$$I > \frac{98}{100} = \frac{49}{50} \Rightarrow I > \frac{49}{50}$$

Hence, the correct answers are options (B) and (D).

24 Area Under the Curves

24.1 Curve Tracing

To find the approximate shape of a curve, the following procedure is adopted in order:

1. **Symmetry:**
   (a) ***Symmetry about x-axis:*** If all the powers of 'y' in the equation are even, then the curve is symmetrical about the x-axis. For example,
   $$y^2 = 4\,a\,x$$
   (b) ***Symmetry about y-axis:*** If all the powers of 'x' in the equation are even, then the curve is symmetrical about the y-axis. For example,
   $$x^2 = 4\,a\,y$$
   (c) ***Symmetry about both axes:*** If all the powers of 'x' and 'y' in the equation are even, then the curve is symmetrical about the axis of 'x' as well as 'y'. For example,
   $$x^2 + y^2 = a^2$$
   (d) ***Symmetry about the line y = x:*** If the equation of the curve remains unchanged on interchanging 'x' and 'y', then the curve is symmetrical about the line $y = x$. For example,
   $$x^3 + y^3 = 3\,ax\,y$$
   (e) ***Symmetry in opposite quadrants:*** If the equation of the curve remains unaltered when 'x' and 'y' are replaced by $-x$ and $-y$ respectively, then there is symmetry in opposite quadrants. For example,
   $$x\,y = c^2$$
2. If the equation of curve contains no constant terms, then it passes through the origin. Find whether the curve passes through the origin or not.
   $$x^2 + y^2 - 4ax = 0$$
3. Find the points where the curve crosses the x-axis and also the y-axis. For example, the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect the axes at points $(\pm a, 0)$ and $(0, \pm b)$.
4. Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve where you have horizontal tangents. And also find the points at which $\frac{dx}{dy} = 0$ at these points the tangent to the curve is parallel to y-axis.
5. Examine if possible the intervals when $f(x)$ is increasing or decreasing.
6. Write the given equation as $y = f(x)$, and find minimum and maximum values of x which determine the region of the curve. For example, for the curve $xy^2 = a^2(a - x) \Rightarrow y = a\sqrt{\frac{a - x}{x}}$.
   Now y is real, if $0 \le x \le a$, so its region lies between the lines $x = 0$ and $x = a$.
7. Examine what happens to y when $x \to \infty$ or $x \to -\infty$.
8. **Asymptotes:**
   Asymptote(s) is (are) line(s) whose distance from the curve tends to zero as point on curve moves towards infinity along branch of the curve.
   (a) If $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$, then $x = a$ is asymptote of $y = f(x)$.
   (b) If $\lim_{x\to\infty} f(x) = k$ or $\lim_{x\to-\infty} f(x) = k$, then $y = k$ is asymptote of $y = f(x)$.
   (c) If $\lim_{x\to\infty} \frac{f(x)}{x} = m$ or $\lim_{x\to\infty}(f(x) - mx) = c$, then $y = mx + c$ is an asymptote (inclined to right).
   (d) If $\lim_{x\to-\infty} \frac{f(x)}{x} = m$ or $\lim_{x\to-\infty}(f(x) - mx) = c$, then $y = mx + c$ is an asymptote (inclined to left).

24.2 Steps to Draw Curve

For the evaluation of area of bounded regions, it is very essential to know the rough sketch of the curves. The following points are very useful to draw a rough sketch of a curve.

While constructing the graph of $f(x, y) = 0$, it is expedient to follow the procedure given below:

1. Find the set of permissible values of x.
2. Check if the curve is symmetrical about x-axis, y-axis, origin, etc.
3. Find the period of the curve if it is periodic.
4. Find the asymptote(s) of the curve, if any.
5. Find the intervals of increase and decrease of the curve. Hence, determine the greatest and the least values of the curve, if any.

Illustration 24.1 Sketch the region bounded by $3x + 4y \le 12$.

Solution: Converting the inequality into equation we get $3x + 4y = 12$. This line meets x-axis at (4, 0) and y-axis at (0, 3). Joining these two points we obtain the straight line represented by $3x + 4y = 12$. This straight line divides the plane in two parts. One part contains the origin the other does not contain the origin. Clearly, (0, 0) satisfies the inequality $3x + 4y \le 12$. So, the region represented by $3x + 4y \le 12$ is the region containing the origin as shown in Fig. 24.1.

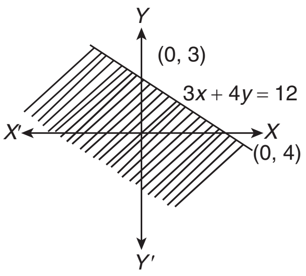


Figure 24.1

Illustration 24.2 Sketch the graph for $y = x^2 - x$.

Solution: We note the following points about the curve:

1. The curve does not have any kind of symmetry.
2. The curve passes through the origin and the tangent at the origin is obtained by equating the lowest degree term to zero.
   The lowest degree term is $x + y$. Equation it to zero, we get $x + y = 0$ as the equation of tangent at the origin.
3. Putting $y = 0$ in the equation of curve, we get
   $$x^2 - x = 0 \Rightarrow x = 0,1$$
   So, the curve crosses x-axis at (0, 0) and (1, 0).
   Putting $x = 0$ in the equation of the curve, we obtain $y = 0$.
   So, the curve meets y-axis at (0, 0) only (Fig. 24.2).
4. $$y = x^2 - x \Rightarrow \frac{dy}{dx} = 2x - 1 \text{ and } \frac{d^2y}{dx^2} = 2$$
   Now,
   $$\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{2}$$
   At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} > 0$, so $x = \frac{1}{2}$ is point of local minima.
5. $$\frac{dy}{dx} > 0 \Rightarrow 2x - 1 > 0 \Rightarrow x > \frac{1}{2}$$
   So, the curve increases for all $x > \frac{1}{2}$ and decreases for all $x < \frac{1}{2}$.

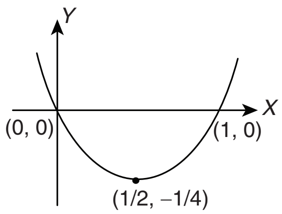


Figure 24.2

Illustration 24.3 Trace the curve $y^2 (2a - x) = x^3$, $a > 0$.

Solution: Note that the curve passes through the origin and is symmetrical about the x-axis.

$$y^2 = \frac{x^3}{2a - x}$$

LHS is positive. If x is negative or if x is greater than $2a$, RHS becomes negative. Hence the curve lies only in the interval 0 to $2a$. When $x \to 2a$, $y \to \infty$. Therefore, the line $x = 2a$ is an asymptote for the curve. A rough figure is shown (Fig. 24.3).

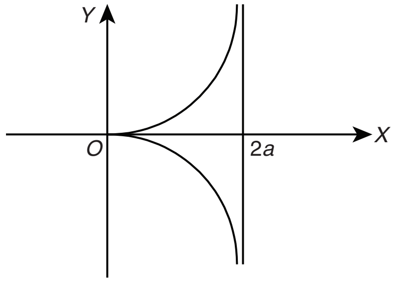


Figure 24.3

Illustration 24.4 Trace the curve $y^2 = \frac{x^2(1+x)}{1-x}$.

Solution: The curve passes through the origin and is symmetrical about the x-axis. It intersects the x-axis at $x = -1$ and $x = 0$. If $x < -1$ or $x > 1$ the curve is non-existent. As $x \to 1$, $y \to \pm\infty$ a rough diagram is shown below (Fig. 24.4).

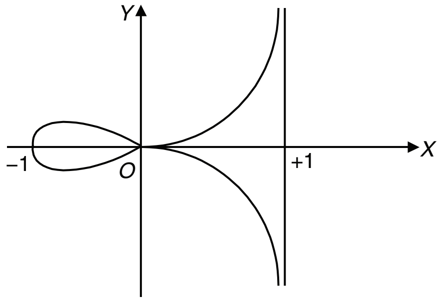


Figure 24.4

The curve has a loop between –1 and 0.

24.3 Area of Bounded Region

Case I: Let $f(x)$ be a continuous function in (a, b) (Fig. 24.5). Then the area bounded by the curve $y = f(x)$, x-axis and lines $x = a$ and $x = b$ is given by the formula

$$A = \left|\int_a^b f(x)dx\right|,$$

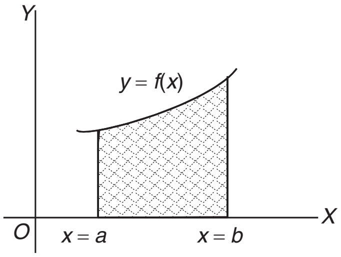


Figure 24.5

Provided $f(x) \geq 0$ (or $f(x) \leq 0$) $\forall\, x \in (a, b)$

Case II: It is sometimes convenient to use formula for area with respect to y, i.e. regarding x as a function of y (Fig. 24.6).

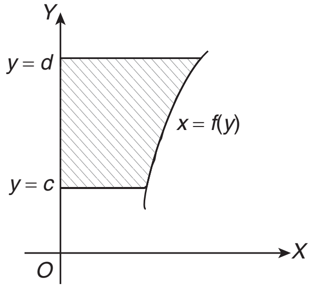


Figure 24.6

The area between $x = f(y)$, y-axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d f(y)dy$$

Note: Sometimes it is better to use the formula $\int_c^d x\,dy$ instead of $\int_a^b y\,dx$ in the computation of area to simplify calculations.

24.4 Area Enclosed Between Two Curves

Case I: Figure 24.7 encloses an area between two curves one of which is represented by PQ with equation $y = f(x)$ and the other by AB with the equation $y = g(x)$.

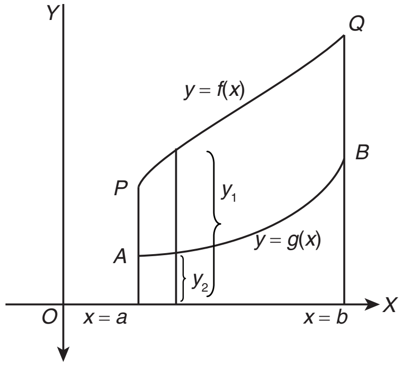


Figure 24.7

$$\text{Area } PABQ = \int_a^b (y_1 - y_2)\,dx$$

where $y_1 = f(x)$ and $y_2 = g(x)$

$$\text{Area } PABQ = \int_a^b \{f(x) - g(x)\}\,dx$$

Case II: Figure 24.8 represents the region bounded by a closed curve $ACQBP$. The area of the region bounded by a closed curve $ACQBP$ is

$$\int_a^b (y_1 - y_2)\,dx,\ y_1 > y_2$$

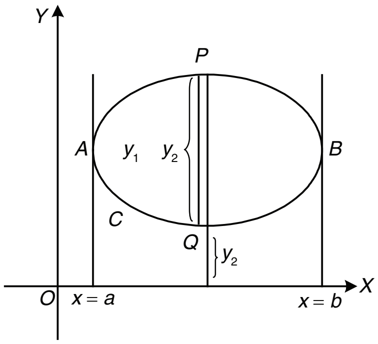


Figure 24.8

The values of y_1 and y_2 are obtained by solving the equation of the curve as a quadratic in y whose larger root y_1 and smaller root y_2 are functions of x.

a and b are the coordinates of the points of contact of tangents drawn parallel to the y-axis.

Case III: When two curves (Fig. 24.9) intersect at a point and the area between them is bounded by x-axis, area bounded by the curves

$$y = f_1(x),\ y_2 = f_2(x) \text{ and } x\text{-axis is } \int_a^{\alpha} f_1(x)dx + \int_{\alpha}^{b} f_2(x)dx,$$

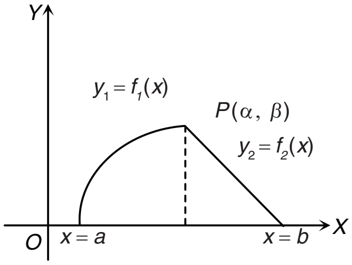


Figure 24.9

Note:

1. If curve lies completely above the x-axis, then the area is positive but when it lies completely below x-axis, then the area is negative, however, we have the convention to consider the magnitude only.
2. If curve lies on both the sides of x-axis, that is, above the x-axis as well as below the x-axis, then calculate both areas separately and add their modulus to get the total area (Fig. 24.10).

   In general if curve $y = f(x)$ crosses the x-axis n times when x varies from a to b, then the area between $y = f(x)$, x-axis and lines $x = a$ and $x = b$ is given by

$$A = |A_1| + |A_2| + \cdots + |A_n|$$

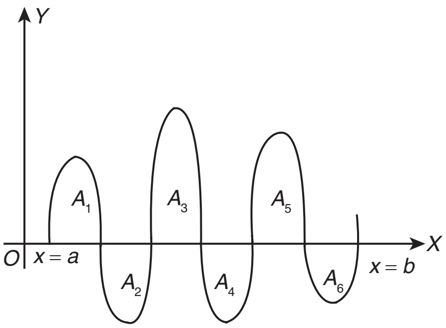


Figure 24.10

3. If the curve is symmetrical about x-axis, or y-axis, or both, then calculate the area of one symmetrical part and multiply it by the number of symmetrical parts to get the whole area.

Illustration 24.5 Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$.

Solution: See Fig. 24.11. The ellipse is symmetrical about both axes and hence the area enclosed is

$$4\times(\text{area under the curve in a quadrant})$$

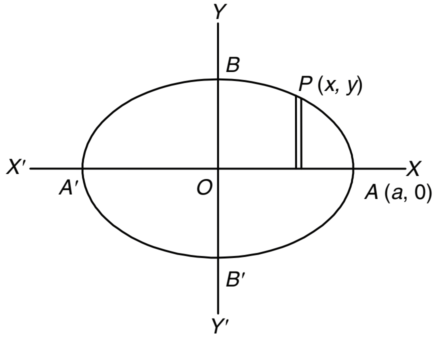


Figure 24.11

$$=4\int_0^a y\,dx=4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

$$=\frac{4b}{a}\int_0^a\sqrt{(a^2-x^2)}\,dx=\frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\frac{4b}{a}\left[\frac{a^2\pi}{4}\right]=\pi ab \text{ sq. units}$$

Illustration 24.6 Find the area included between the parabolas $y^2=4ax$ and $x^2=4ay$.

Solution: See Fig. 24.12. The two parabolas intersect at O (0, 0) and A ($4a$, $4a$).

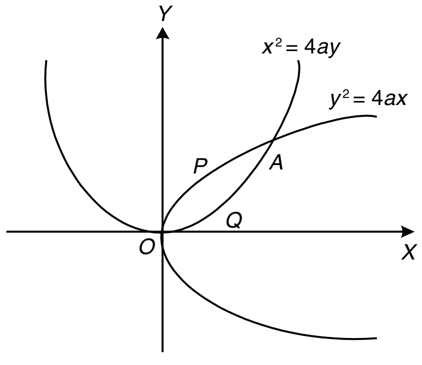


Figure 24.12

The area included between the two curves is area $OQAP$, that is,

$$\int_{x=0}^{x=4a}(y_1-y_2)\,dx=\int_0^{4a}\left(2\sqrt{a}\,\sqrt{x}-\frac{x^2}{4a}\right)dx$$

$$=\left[2\sqrt{a}\,\frac{2}{3}x^{3/2}-\frac{x^3}{12a}\right]_0^{4a}=4\frac{\sqrt{a}}{3}8.a^{3/2}-\frac{64a^3}{12a}=\frac{16a^2}{3} \text{ sq. units}$$

Illustration 24.7 Find the area of the segment cut off from the parabola $y^2=2x$ by the line $y=4x-1$.

Solution: The line $y=4x-1$ intersects the parabola $y^2=2x$ at A and B

$$2x=(4x-1)^2 \Rightarrow 16x^2-10x+1=0$$
$$\Rightarrow (8x-1)(2x-1)=0$$

Therefore,

$$A=\left(\frac{1}{2},1\right) \text{ and } B=\left(\frac{1}{8},-\frac{1}{2}\right)$$

If the formula $\int y\,dx$ is to be used then the area will have to be split up as OBC and CBA (Fig. 24.13). Instead the problem can be done directly by using the formula $\int(x_2-x_1)\,dy$.

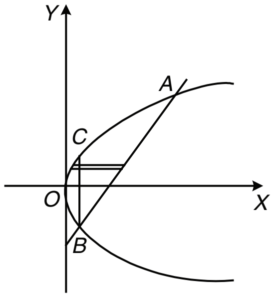


Figure 24.13

$$\text{Required area}=\int_{y=-1/2}^{1}(x_2-x_1)\,dy=\int_{-1/2}^{1}\left(\frac{y+1}{4}-\frac{y^2}{2}\right)dy$$

$$=\left[\frac{y^2}{8}+\frac{y}{4}-\frac{y^3}{6}\right]_{-1/2}^{1}=\left(\frac{1}{8}+\frac{1}{4}-\frac{1}{6}\right)-\left(\frac{1}{32}-\frac{1}{8}+\frac{1}{48}\right)$$

$$=\frac{(3+6-4)}{24}-\frac{(3-12+2)}{96}$$

$$=\frac{5}{24}+\frac{7}{96}=\frac{27}{96}=\frac{9}{32} \text{ sq. units}$$

Illustration 24.8 Find the area between the curves $y=x^2+x-2$ and $y=2x$, for which $|x^2+x-2|+|2x|=|x^2+3x-2|$ is satisfied.

Solution: To find the area between the curve $y=x^2+x-2$ and $y=2x$ such that

$$|x^2+x-2|+|2x|=|x^2+3x-2|$$

So, (x^2+x-2) and $2x$ have same sign (Fig. 24.14).

Thus,

$$\text{Required area}=ar\,(PQR)+ar\,(STN)$$

$$=\int_{-1}^{0}[2x-(x^2+x-2)]\,dx+\int_{1}^{2}[2x-(x^2+x-2)]\,dx$$

$$=\left[\frac{x^2}{2}-\frac{x^3}{3}+2x\right]_{-1}^{0}+\left[\frac{x^2}{2}-\frac{x^3}{3}+2x\right]_{1}^{2}$$

$$=\frac{7}{6}+\left[\frac{10}{3}-\frac{13}{6}\right]=\frac{7}{6}+\frac{7}{6}=\frac{7}{3}\text{ sq. units}$$

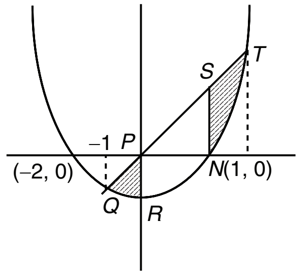


Figure 24.14

Illustration 24.9 Find out the area enclosed by circle $|z| = 2$, parabola $y = x^2 + x + 1$, the curve $y=\left[\sin^2\frac{x}{4}+\cos\frac{x}{4}\right]$ and x-axis (where $[\cdot]$ is the greatest integer function).

Solution: See Fig. 24.15. For $x \in [-2, 2]$

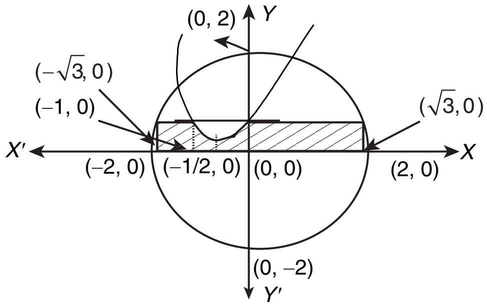


Figure 24.15

$$1<\sin^2\frac{x}{4}+\cos\frac{x}{4}<2$$

Therefore,

$$\left[\sin^2\frac{x}{4}+\cos\frac{x}{4}\right]=1$$

Now, we have to find out the area enclosed by the circle $|z| = 2$, parabola $\left(y-\frac{3}{4}\right)=\left(x+\frac{1}{2}\right)^2$, line $y = 1$ and x-axis.

Therefore, required area is shaded area in Figure 24.15.
Hence, required area is

$$\sqrt{3}\times 1+(\sqrt{3}-1)+\int_{-1}^{0}(x^2+x+1).dx+2\int_{\sqrt{3}}^{2}\sqrt{4-x^2}\,dx$$

$$=\left(\frac{2\pi}{3}+\sqrt{3}-\frac{1}{6}\right)\text{sq. units}$$

Illustration 24.10 Let $f(x) = \text{Max.}\left\{\sin x, \cos x, \frac{1}{2}\right\}$. Then determine the area of the region bounded by the curves $y = f(x)$, x-axis, y-axis and $x = 2\pi$.

Solution: See Fig. 24.16. Since,

$$f(x)=\text{Max}\left\{\sin x, \cos x, \frac{1}{2}\right\}$$

So, interval value of $f(x)$ is

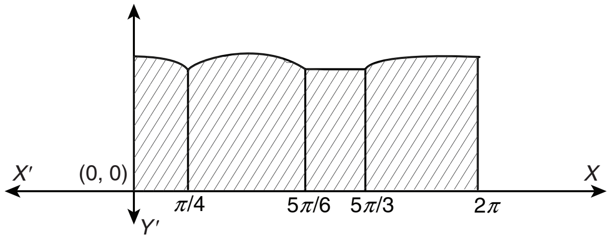


Figure 24.16

for $0\le x<\frac{\pi}{4}$, $f(x)=\cos x$

for $\frac{\pi}{4}\le x<\frac{5\pi}{6}$, $f(x)=\sin x$

for $\frac{5\pi}{6}\le x<\frac{5\pi}{3}$, $f(x)=1/2$

for $\frac{5\pi}{3}\le x\le 2\pi$, $f(x)=\cos x$

Hence, required area is

$$\int_{0}^{\pi/4}\cos x\,dx+\int_{\pi/4}^{5\pi/6}\sin x\,dx+\int_{5\pi/6}^{5\pi/3}\frac{1}{2}dx+\int_{5\pi/3}^{2\pi}\cos x\,dx$$

$$=[\sin x]_0^{\pi/4}+[-\cos x]_{\pi/4}^{5\pi/6}+\frac{1}{2}[x]_{5\pi/6}^{5\pi/3}+[\sin x]_{5\pi/3}^{2\pi}$$

$$=\left(\frac{5\pi}{12}+\sqrt{2}+\sqrt{3}\right)\text{sq. units}$$

Your Turn 1

1. If A is the area of the region bounded by the curve $y=\sqrt{3x+4}$, x-axis and the line $x = -1$ and $x = 4$ and B is that area bounded by curve $y^2 = 3x + 4$, x-axis and the lines $x = -1$ and $x = 4$, then $A : B$ is equal to
   (A) 1 : 1 **(B)** 2 : 1
   (C) 1 : 2 **(D)** None of these **Ans.** (A)
2. The area of the region bounded by the curve $9x^2 + 4y^2 - 36 = 0$ is
   (A) 9π **(B)** 4π
   (C) 36π **(D)** 6π **Ans.** (D)
3. The area bounded by the curve $y = (x + 1)^2$, $y = (x - 1)^2$ and the line $y=\frac{1}{4}$ is
   (A) 1/6 **(B)** 2/3 **(C)** 1/4 **(D)** 1/3 **Ans.** (D)
4. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x=\frac{\pi}{4}$, $x=\beta>\frac{\pi}{4}$ is $\left(\beta\sin\beta+\frac{\pi}{4}\cos\beta+\sqrt{2}\beta\right)$. Then $f\left(\frac{\pi}{2}\right)$ is
   (A) $\left(1-\frac{\pi}{4}-\sqrt{2}\right)$ **(B)** $\left(1-\frac{\pi}{4}+\sqrt{2}\right)$
   (C) $\left(\frac{\pi}{4}+\sqrt{2}-1\right)$ **(D)** $\left(\frac{\pi}{4}-\sqrt{2}+1\right)$ **Ans.** (B)
5. Area bounded by curves $y = x^2$ and $y = 2 - x^2$ is
   (A) 8/3 **(B)** 3/8
   (C) 3/2 **(D)** None of these **Ans.** (A)

6. Let y be the function that passes through (1, 2) having slope $(2x+1)$. The area bounded between the curve and x-axis is

(A) 6 sq. units **(B)** 5/6 sq. units

(C) 1/6 sq. units **(D)** None of these **Ans.** (C)

7. Area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is given by

(A) $\frac{8}{9}$ sq. units **(B)** $\frac{9}{8}$ sq. units

(C) $\frac{4}{3}$ sq. units **(D)** None of these **Ans.** (B)

8. The area of the region bounded by the curve $y = x\,|x|$, x-axis and the ordinates $x = 1$, $x = -1$ is given by

(A) Zero **(B)** $\frac{1}{3}$

(C) $\frac{2}{3}$ **(D)** 1 **Ans.** (C)

9. If the area bounded by $y = ax^2$ and $x = ay^2$, $a > 0$, is 1, then $a =$

(A) 1 **(B)** $\frac{1}{\sqrt{3}}$

(C) $\frac{1}{3}$ **(D)** None of these **Ans.** (B)

10. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and x-axis in the first quadrant is

(A) 9 **(B)** $\frac{27}{4}$ **(C)** 36 **(D)** 18 **Ans.** (A)

11. The area enclosed between the curve $y = \log_e(x + e)$ and the coordinate axes is

(A) 3 **(B)** 4 **(C)** 1 **(D)** 2 **Ans.** (C)

12. The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and the coordinate axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom, then $S_1 : S_2 : S_3$ is

(A) 2 : 1 : 2 **(B)** 1 : 1 : 1 **(C)** 1 : 2 : 1 **(D)** 1 : 2 : 3 **Ans.** (B)

Additional Solved Examples

1. The total area enclosed by the lines $y = |x|$, $|x| = 1$ and $y = 0$ is

(A) 1 **(B)** 2 **(C)** $\frac{1}{2}$ **(D)** None of these

Solution: See Fig. 24.17.

$$y = |x|,\ |x| = 1,\ y = 0$$

$$\text{Total area } = \frac{1}{2}\times 1\times 1 + \frac{1}{2}\times 1\times 1 = \frac{1}{2} + \frac{1}{2} = 1 \text{ sq. units}$$

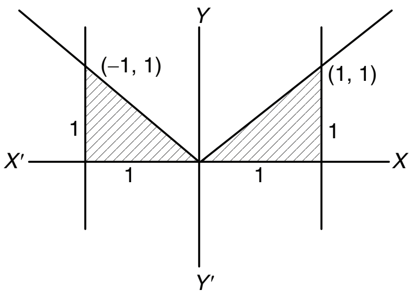


Figure 24.17

Hence, the correct answer is option (A).

2. The area bounded by $y = \frac{|x|}{x}$, $x \neq 0$, and the lines $y(x-1)(x-3) = 0$ is

(A) 3 **(B)** 1 **(C)** 2 **(D)** None of these

Solution: See Fig. 24.18.

$$y = \frac{|x|}{x}, x \neq 0,$$

$$y(x-1)(x-3) = 0$$

$$\text{Area} = 2 \times 1$$

$$= 2 \text{ sq. units}$$

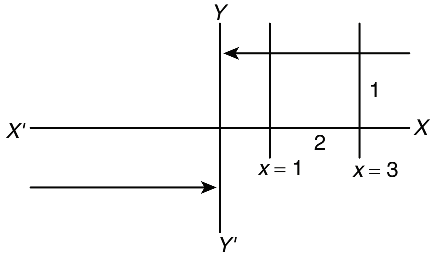


Figure 24.18

Hence, the correct answer is option (C).

3. The area bounded by the curve $|x| = \cos^{-1}y$ and the line $|x| = 1$ and the x-axis is

(A) cos 1 **(B)** sin 1 **(C)** 2 cos 1 **(D)** 2 sin 1

Solution: See Fig. 24.19.

$|x| = \cos^{-1}y$ and line $|x| = 1$ and x-axis

$y = \cos|x|$

$|x| = 1$, x-axis

$$\text{Area} = 2\int_0^1 \cos x\, dx$$

$$= 2\left[\sin x\right]_0^1 = 2\sin 1 \text{ sq. units}$$

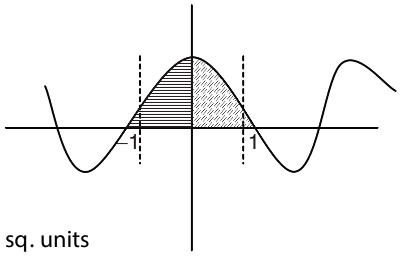


Figure 24.19

Hence, the correct answer is option (D).

4. Area bounded by the curve $f(x) = \begin{cases} \log_e|x|, & |x| \geq \frac{1}{e} \\ |x| - 1 - \frac{1}{e}, & |x| < \frac{1}{e} \end{cases}$ and x-axis is

(A) $\frac{1}{e^2} + 2 - \frac{2}{e}$ **(B)** $\frac{1}{e^2} + 2 + \frac{2}{e}$

(C) $\frac{1}{e^2} + \frac{2}{e}$ **(D)** None of these

Solution: See Fig. 24.20.

$$f(x) = \begin{cases} \log_e|x|, & |x| \geq \frac{1}{e} \\ |x| - 1 - \frac{1}{e}, & |x| < \frac{1}{e} \end{cases}$$

$$\text{Area required} = 2\left[\int_0^{1/e} -\left(x - 1 - \frac{1}{e}\right)dx + \int_{1/e}^{1} -\ln x\, dx\right]$$

$$= 2\left[-\left\{\frac{x^2}{2} - \left(1 + \frac{1}{e}\right)x\right\}_0^{1/e} - (x(\ln x - 1))_{1/e}^{1}\right]$$

$$= -2\left[\frac{1}{2e^2} - \frac{1}{e} - \frac{1}{e^2}\right] - 2\left[(-1) + \left(\frac{2}{e}\right)\right]$$

$$= -\frac{1}{e^2} + \frac{2}{e} + \frac{2}{e^2} + 2 - \frac{4}{e}$$

$$= \frac{1}{e^2} + 2 - \frac{2}{e} \text{ sq. units}$$

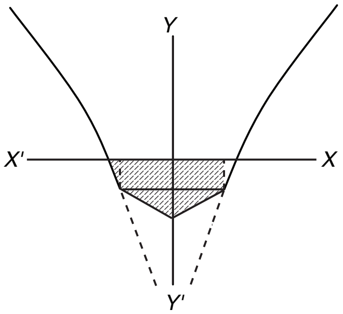


Figure 24.20

Hence, the correct answer is option (A).

5. The whole area of the curves $x = a\cos^3 t$, $y = b\sin^3 t$ is given by

(A) $\frac{3}{8}\pi ab$ **(B)** $\frac{5}{8}\pi ab$ **(C)** $\frac{1}{8}\pi ab$ **(D)** None of these

Solution:

$$\text{Area} = 4\int_0^a y\frac{dx}{dt}\cdot dt$$

$$= 4\int_{\pi/2}^{0} -3ab\sin^4 t\cdot\cos^2 t\cdot dt$$

$$= 4\times 3ab\cdot\frac{3}{6}\cdot\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{3}{8}\pi ab \text{ sq. units}$$

Hence, the correct answer is option (A).

6. Area common to the curves $y^2 = ax$ and $x^2 + y^2 = 4ax$ is equal to

(A) $(9\sqrt{3}+4\pi)\frac{a^2}{3}$ **(B)** $(9\sqrt{3}+4\pi)a^2$

(C) $(9\sqrt{3}-4\pi)\frac{a^2}{3}$ **(D)** None of these

Solution: See Fig. 24.21.

$y^2 = ax, x^2 + y^2 - 4ax = 0$

$y^2 = 4ax - x^2$

$x^2 + ax - 4ax = 0$

$x^2 - 3ax = 0$

$x(x - 3a) = 0$

$x = 0, x = 3a$

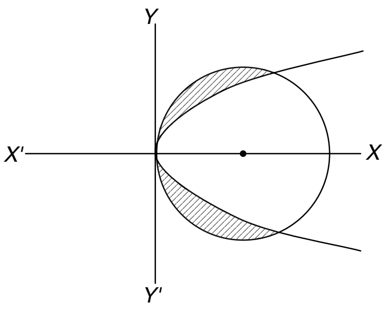


Figure 24.21

$$\text{Required area} = 2\int_0^{3a}(\sqrt{4ax - x^2} - \sqrt{ax})\,dx$$

$$= 2\int_0^{3a}(\sqrt{4a^2 - (x-2a)^2} - \sqrt{ax})\,dx = \frac{(8\pi - 9\sqrt{3})a^2}{3} \text{ sq. units}$$

Hence, the correct answer is option (D).

7. The area $\{(x, y); x^2 \le y \le \sqrt{x}\}$ is equal to

(A) $\frac{1}{3}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{1}{6}$ **(D)** None of these

Solution: See Fig. 24.22.

$$\{(x, y); x^2 \le y \le \sqrt{x}\}$$

$$\text{Area} = \int_0^1(\sqrt{x} - x^2)dx$$

$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units}$$

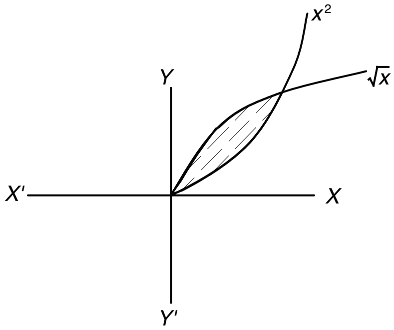


Figure 24.22

Hence, the correct answer is option (A).

8. The area enclosed by the curve $y = x^5$, the x-axis and the ordinates $x = -1$, $x = 1$ is

(A) $\frac{1}{2}$ **(B)** 1 **(C)** $\frac{1}{3}$ **(D)** 0

Solution: See Fig. 24.23.

$$y = x^5, x = \pm 1$$

$$\text{Area} = 2\int_0^1 x^5 dx = 2\left.\frac{x^6}{6}\right|_0^1 = \frac{2}{6} = \frac{1}{3} \text{ sq. units}$$

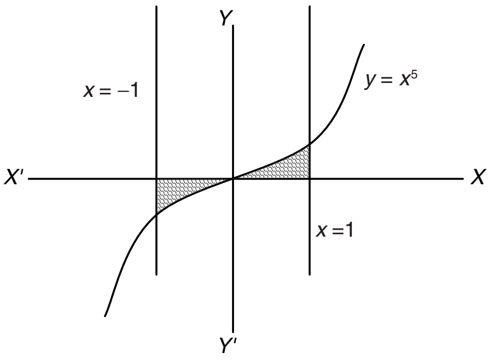


Figure 24.23

Hence, the correct answer is option (C).

9. The area bounded by the curve $y^2 = 9x$ and the lines $x = 1, x = 4$ and $y = 0$ in the first quadrant is

(A) 14 **(B)** 7 **(C)** 28 **(D)** None of these

Solution: See Fig. 24.24.

$$\text{Area } = \int_1^4 3\sqrt{x}\, dx = 2\times 3\frac{x^{3/2}}{3}\bigg|_1^4 = 2\left[x^{3/2}\right]_1^4$$

$$= 2\left[2^3 - 1\right] = 14 \text{ sq. units}$$

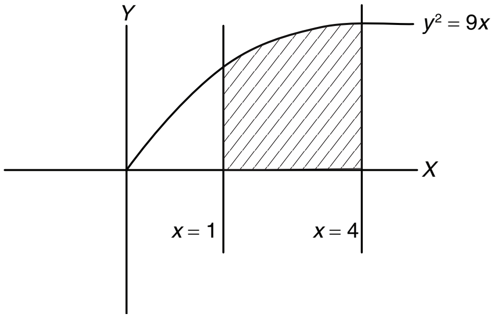


Figure 24.24

Hence, the correct answer is option (A).

10. The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$. If the curve passes through the point (1, 2), then the area of the region bounded by the curve, the x-axis and the line $x = 1$ is

(A) $\frac{5}{6}$ **(B)** $\frac{6}{5}$ **(C)** $\frac{1}{6}$ **(D)** 6

Solution: See Fig. 24.25.

$$f'(x) = 2x + 1$$

$$\Rightarrow f(x) = x^2 + x + c$$

The curve passes through (1, 2), so

$$2 = 1 + 1 + c$$

$$\Rightarrow c = 0$$

$$f(x) = x^2 + x$$

$$\int_0^1 (x^2 + x)dx = \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \text{ sq. units}$$

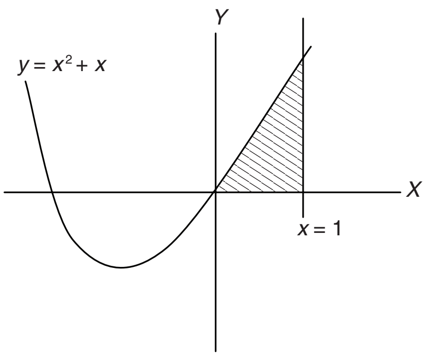


Figure 24.25

Hence, the correct answer is option (A).

11. The area bounded by the axes of reference and normal to $y = \log_e x$ at the point (1, 0) is

(A) 1 sq. units **(B)** 2 sq. units

(C) $\frac{1}{2}$ sq. units **(D)** None of these

Solution: See Fig. 24.26.

$$y = \ln x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

At $x = 1$,

$$\text{slope of normal} = -1$$

$$y = -1(x - 1)$$

$$y + x = 1$$

$$\text{Area } = \frac{1}{2}\times 1\times 1 = \frac{1}{2} \text{ sq. units}$$

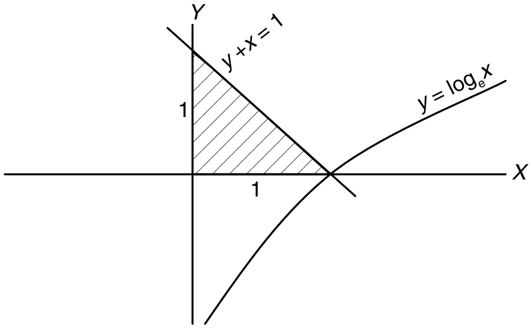


Figure 24.26

Hence, the correct answer is option (C).

12. The area bounded by the line $|x| + |y| = 1$ is

(A) 4 **(B)** 2 **(C)** 1 **(D)** None of these

Solution: See Fig. 24.27.

$$|x| + |y| = 1$$

$$\text{Area } = 4\times\left(\frac{1}{2}\times 1\times 1\right) = 2 \text{ sq. units}$$

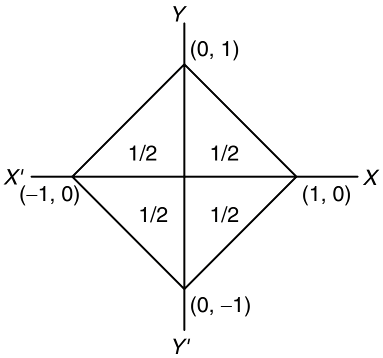


Figure 24.27

Hence, the correct answer is option (B).

13. If area bounded by curve $f(x)$ and x-axis, $x = 1$ to $x = b$ is $(b - 1)\sin(3b + 4)$, then $f(x)$ is

(A) $3x \cos(3x+4) + \sin(3x+4)$
(B) $3(x-1)\cos(3x+4) + \sin(3x+4)$
(C) $x \cos(3x+4) + \sin(3x+4)$
(D) None of these

Solution:

$$\int_1^b f(x)\,dx = (b-1)\sin(3b+4)$$

$$\int_1^x f(x)\,dx = (x-1)\sin(3x+4) \text{ (replacing } b \text{ by } x)$$

$$f(x) = 3(x-1)\cos(3x+4) + \sin(3x+4)$$

Hence, the correct answer is option (B).

14. The area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is

(A) $\frac{15}{4}$ **(B)** $\frac{17}{4}$ **(C)** $\frac{9}{4}$ **(D)** None of these

Solution: See Fig. 24.28.

$$\text{Area} = \left|\int_{-2}^{0} x^3 dx\right| + \int_0^1 x^3 dx = \left|\frac{x^4}{4}\right|_{-2}^{0} + \left.\frac{x^4}{4}\right|_0^1$$

$$= \frac{16}{4} + \frac{1}{4} = \frac{17}{4} \text{ sq. units}$$

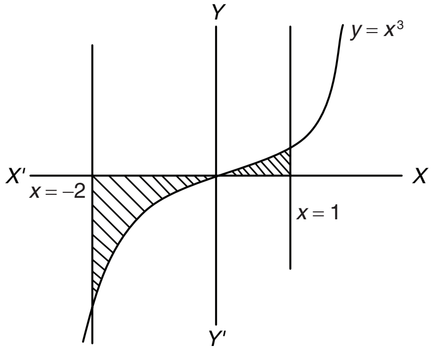


Figure 24.28

Hence, the correct answer is option (B).

15. The area of the region bounded by $y = |x-1|$ and $y = 1$ is

(A) $\frac{1}{2}$ **(B)** 1
(C) 2 **(D)** None of these

Solution:

$$\text{Area} = \frac{1}{2} \times 1 \times 2$$

$$= 1 \text{ sq. unit}$$

Hence, the correct answer is option (B).

Previous Years' Solved JEE Main/AIEEE Questions

1. The area enclosed between the curves $y^2 = x$ and $y = |x|$ is

(A) 2/3 **(B)** 1
(C) 1/6 **(D)** 1/3

[AIEEE 2007]

Solution: Solving $y^2 = x$ and $y = x$, we get, $y = 0, x = 0, y = 1, x = 1$
Therefore,

$$\text{Area} = \int_0^1 (\sqrt{x} - x)dx = \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ sq. units}$$

Hence, the correct answer is option (C).

2. The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to

(A) $\frac{5}{3}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{4}{3}$

[AIEEE 2008]

Solution: Solving the equations, we get the points of intersection as $(-2, 1)$ and $(-2, -1)$. The bounded region is shown as shaded region in Fig. 24.29.

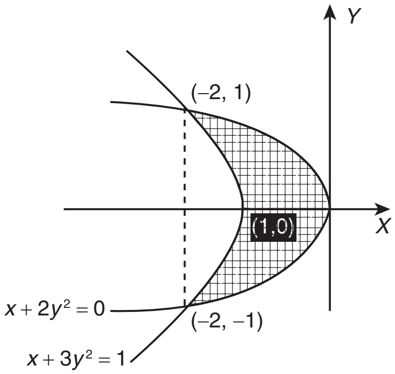


Figure 24.29

The required area is

$$2\int_0^1 \left[(1-3y^2) - (-2y^2)\right] dy = 2\int_0^1 (1-y^2)dy = 2\left[y - \frac{y^3}{3}\right]_0^1$$

$$= 2 \times \frac{2}{3} = \frac{4}{3} \text{ sq. units}$$

Hence, the correct answer is option (D).

3. The area of the region bounded by the parabola $(y-2)^2 = x - 1$, the tangent to the parabola at the point (2, 3) and the x-axis is

(A) 3 **(B)** 6 **(C)** 9 **(D)** 12

[AIEEE 2009]

Solution: Equation of tangent at (2, 3) for the parabola, $(y-2)^2 = x - 1$, is $S_1 = 0$, which implies that $x - 2y + 4 = 0$.

See Fig. 24.30. The required area is

Area of ΔOCB + Area of $OAPD$ − Area of ΔPCD

$$= \frac{1}{2}(4 \times 2) + \int_0^3 (y^2 - 4y + 5)dy - \left[\frac{1}{2}(1 \times 2)\right]$$

$$= 4 + \left[\frac{y^3}{3} - 2y^2 + 5y\right]_0^3 - 1 = 4 + 9 - 18 + 15 - 1$$

$$= 28 - 19 = 9 \text{ sq. units}$$

Alternate solution:

The area is

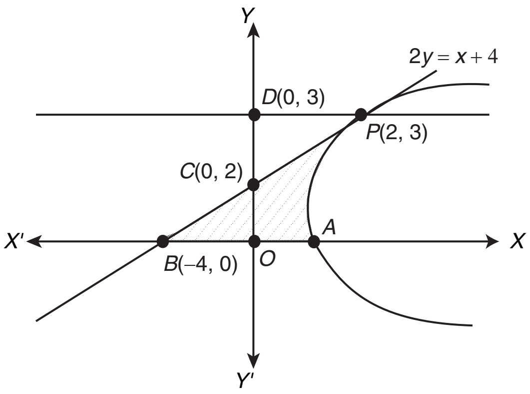


Figure 24.30

$$A\int_0^3 (2y-4-y^2+4y-5)dy = \int_0^3 (-y^2+6y-9)dy$$

$$= -\int_0^3 (3-y)^2 dy = \left[\frac{(y-3)^3}{3}\right]_0^3 = \frac{27}{3} = 9 \text{ sq. units}$$

Hence, the correct answer is option (C).

4. The area bounded by the curves $y=\cos x$ and $y=\sin x$ between the ordinates $x=0$ and $x=\dfrac{3\pi}{2}$ is

(A) $4\sqrt{2}+2$ **(B)** $4\sqrt{2}-1$

(C) $4\sqrt{2}+1$ **(D)** $4\sqrt{2}-2$

[AIEEE 2010]

Solution: The required area is

$$\int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx + \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (\cos x - \sin x)dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} + [\sin x + \cos x]_{\frac{5\pi}{4}}^{\frac{3\pi}{2}}$$

$$= (\sqrt{2}-1) + 2\sqrt{2} + (-1+\sqrt{2}) = 4\sqrt{2}-2 \text{ sq. units}$$

Hence, the correct answer is option (D).

5. The area of the region enclosed by the curves $y=x$, $x=e$, $y=\dfrac{1}{x}$ and the positive x-axis is

(A) 1 sq. units **(B)** $\dfrac{3}{2}$ sq. units

(C) $\dfrac{5}{2}$ sq. units **(D)** $\dfrac{1}{2}$ sq. units

[AIEEE 2011]

Solution: From Figure 24.31, we have,

$$\text{Area} = \int_0^1 x\,dx + \int_1^e \frac{1}{x}dx = \frac{1}{2} + 1 = \frac{3}{2} \text{ sq. units}$$

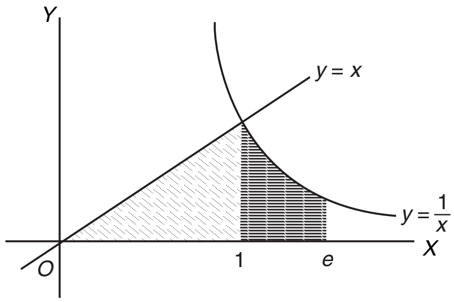


Figure 24.31

Hence, the correct answer is option (B).

6. The area bounded between the parabolas $x^2=\dfrac{y}{4}$ and $x^2=9y$, and the straight line $y=2$ is

(A) $20\sqrt{2}$ **(B)** $\dfrac{10\sqrt{2}}{3}$

(C) $\dfrac{20\sqrt{2}}{3}$ **(D)** $10\sqrt{2}$

[AIEEE 2012]

Solution: From Figure 24.32, the required area is calculated as

$$A = 2\left[\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right)dy\right] = 2\int_0^2 \frac{5\sqrt{y}}{2}dy = 5\left[\frac{y^{3/2}}{3/2}\right]_0^2$$

$$= \frac{10}{3}[2^{3/2}-0] = \frac{20\sqrt{2}}{3} \text{ sq. units}$$

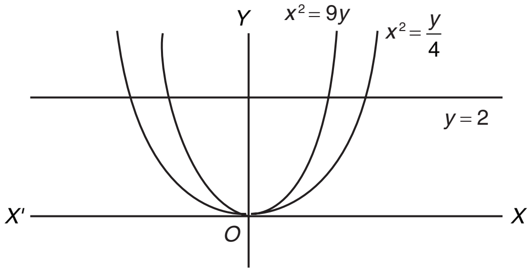


Figure 24.32

Hence, the correct answer is option (C).

7. The area (in square units) bounded by the curves $y=\sqrt{x}$, $2y-x+3=0$, x-axis, and lying in the first quadrant is

(A) 36 **(B)** 18

(C) $\dfrac{27}{4}$ **(D)** 9

[JEE MAIN 2013]

Solution: First solving the equations, we have

$$2\sqrt{x} = x-3 \quad (1)$$

Squaring on both sides of Eq. (1), we get

$$4x = x^2-6x+9 \Rightarrow x^2-10x+9 \Rightarrow x=9,\ x=1$$

Since $x=1$ intersects the parabola below the x-axis, this point is extraneous.

So, for $x = 9$ we have, $y = 3$.

Therefore, the required area under the curve (see Fig. 24.33) is

$$\int_0^3 \left[(2y+3)-y^2\right]dy \Rightarrow \left[y^2+3y-\frac{y^3}{3}\right]_0^3 = 9+9-9 = 9 \text{ sq. units}$$

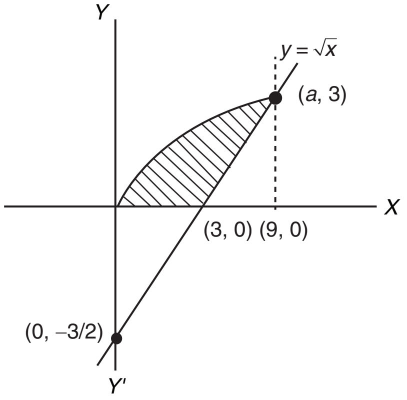


Figure 24.33

Hence, the correct answer is option (D).

8. The area of the region described by $A = \{(x, y): x^2 + y^2 \le 1$ and $y^2 \le 1 - x\}$ is

(A) $\frac{\pi}{2}-\frac{2}{3}$ **(B)** $\frac{\pi}{2}+\frac{2}{3}$

(C) $\frac{\pi}{2}+\frac{4}{3}$ **(D)** $\frac{\pi}{2}-\frac{4}{3}$

[JEE MAIN 2014 (OFFLINE)]

Solution: See Fig. 24.34.

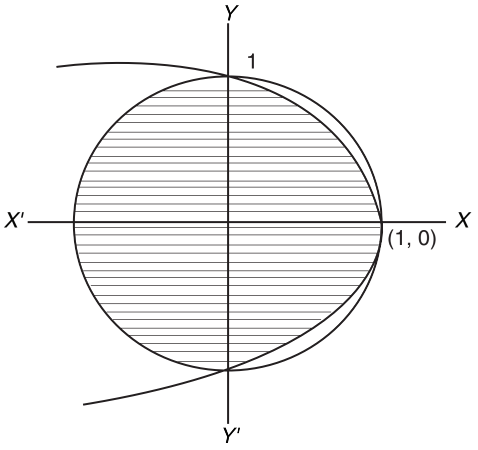


Figure 24.34

$$y^2 = 1 - x \Rightarrow x = 1 - y^2$$

Required area $= \frac{1}{2}(\pi \times 1^2) + 2\int_0^1 (1-y^2)dy$

$$= \frac{\pi}{2} + 2\left[y-\frac{y^3}{3}\right]_0^1 = \frac{\pi}{2}+2\left[\left(1-\frac{1}{3}\right)-0\right] = \frac{\pi}{2}+\frac{4}{3} \text{ sq. units}$$

Hence, the correct answer is option (C).

9. Let $A = \{(x, y): y^2 \le 4x, y - 2x \ge -4\}$. Then the area (in square units) of the region A is

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: See Fig. 24.35. Finding points of intersection,

$$\frac{y^2}{4} = \frac{y+4}{2}$$

Therefore,

$$2y^2 = 4y + 16 \text{ or } y^2 = 2y + 8 \Rightarrow y^2 - 2y - 8 = 0$$

$$y = \frac{2 \pm \sqrt{4+32}}{2} = \frac{2 \pm 6}{2} = 4, -2$$

Therefore,

$$x = 4, 1 \text{ and } P \text{ is } (1, -2) \text{ and } Q \text{ is } (4, 4)$$

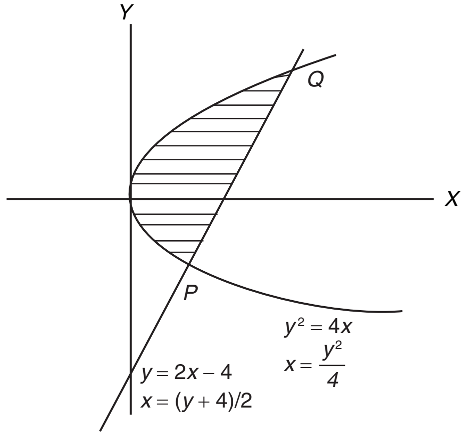


Figure 24.35

$$\text{Required area } = \int_{-2}^{4}\left\{\left(\frac{4+y}{2}\right)-\frac{y^2}{4}\right\}dy$$

$$= \int_{-2}^{4}\left(2+\frac{1}{2}y-\frac{y^2}{4}\right)dy = \left[2y+\frac{y^2}{4}-\frac{y^3}{12}\right]_{-2}^{4}$$

$$= \left(8+4-\frac{64}{12}\right)-\left(2(-2)+\frac{4}{4}+\frac{8}{12}\right)$$

$$= 12+4-1-\left(\frac{72}{12}\right) = 15-6 = 9 \text{ sq. units}$$

Hence, the correct answer is option (B).

10. The area of the region above the x-axis bounded by the curve $y = \tan x, 0 \le x \le \frac{\pi}{2}$ and the tangent to the curve at $x = \frac{\pi}{4}$ is

(A) $\frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$ **(B)** $\frac{1}{2}\left(\log 2 + \frac{1}{2}\right)$

(C) $\frac{1}{2}(1-\log 2)$ **(D)** $\frac{1}{2}(1+\log 2)$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: See Fig. 24.36.

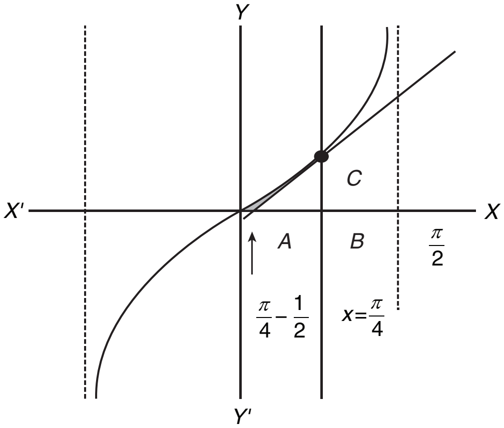


Figure 24.36

$$\text{Required area } = \int_0^{\pi} \tan x - \text{area under tangent at} \left(\frac{\pi}{4}, 1\right) \quad (1)$$

Now slope of tangent is $\dfrac{d}{dx}\tan x$ at $x = \dfrac{\pi}{4} = \sec^2 x\Big|_{\text{at } x = \frac{\pi}{4}} = 2$

Therefore, equation of tangent is $y - 1 = 2\left(x - \dfrac{\pi}{4}\right)$ or $y = 2x + \left(1 - \dfrac{\pi}{2}\right)$

This tangent cuts x-axis when $y = 0$
Therefore,

$$x = \frac{\frac{\pi}{2} - 1}{2} = \frac{\pi}{4} - \frac{1}{2}$$

Thus, required area is

$$[\log \sec x]_0^{\frac{\pi}{4}} - \text{area triangle } ABC$$

$$= \log\sqrt{2} - 0 - \frac{1}{2} \times \left(\frac{\pi}{4} - \frac{\pi}{4} + \frac{1}{2}\right) \times 1 = \frac{1}{2}\left[\log 2 - \frac{1}{2}\right] \text{ sq. units}$$

Hence, the correct answer is option (A).

11. The area (in sq. units) of the region described by $\{(x, y): y^2 \le 2x$ and $y \ge 4x - 1\}$ is

(A) $\dfrac{5}{64}$ **(B)** $\dfrac{15}{64}$ **(C)** $\dfrac{9}{32}$ **(D)** $\dfrac{7}{32}$

[JEE MAIN 2015 (OFFLINE)]

Solution: See Fig. 24.37.

$$R = \{(x, y): y^2 \le 2x \text{ and } y \ge 4x - 1\}$$

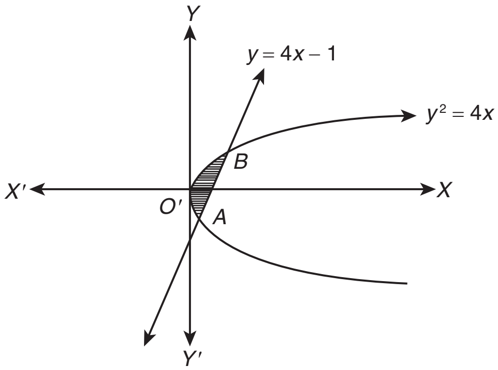


Figure 24.37

Finding points of intersection,

$$y^2 = 2\left(\frac{y+1}{4}\right) \Rightarrow 2y^2 = y + 1 \Rightarrow 2y^2 - y - 1 = 0$$

$$\Rightarrow (y-1)(2y+1) = 0$$

$$\Rightarrow y = 1 \text{ and } y = \frac{-1}{2}$$

$$\Rightarrow x = \frac{1}{2} \text{ and } x = \frac{1}{8}$$

So, point A is (1/8, – 1/2) and B is (1/2, 1).

$$R = \text{shaded area } \int_{y_A}^{y_B} (x_{\text{line}})dy - \int_{y_A}^{y_B} (x_{\text{parabola}})dy$$

$$= \int_{-1/2}^{1} \frac{1}{4}(y+1)dy - \int_{-1/2}^{1} \frac{y^2}{2}dy$$

$$= \frac{1}{4}\left[\frac{y^2}{2} + y\right]_{-1/2}^{1} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-1/2}^{1} = \frac{9}{32} \text{ sq. units}$$

Hence, the correct answer is option (C).

12. The area (in square units) of the region bounded by the curves $y + 2x^2 = 0$ and $y + 3x^2 = 1$ is equal to

(A) $\dfrac{3}{5}$ **(B)** $\dfrac{3}{4}$ **(C)** $\dfrac{1}{3}$ **(D)** $\dfrac{4}{3}$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution: See Fig. 24.38.

$C_1: y + 2x^2 = 0;$

$C_2: y + 3x^2 = 1$

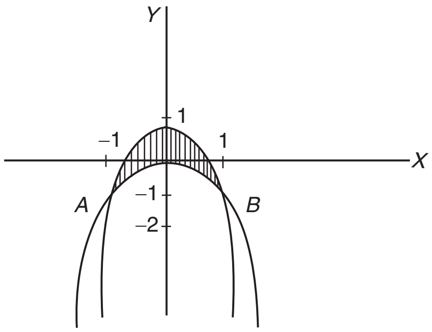


Figure 24.38

At the point of intersection of C_1 and C_2

$$-2x^2 = 1 - 3x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Therefore, A (–1, –2) and B (1, –2) are points of intersection as shown above.
So, required area is

$$2\int_{-1}^{0} [(1 - 3x^2) - (-2x^2)]dx$$

$$= 2\int_{-1}^{0} (1 - x^2)dx = 2\left[x - \frac{x^3}{3}\right]_{-1}^{0} = \frac{4}{3} \text{ sq. units}$$

Hence, the correct answer is option (D).

13. The area (in sq. units) of the region $\{(x, y): y^2 \geq 2x \text{ and } x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is

(A) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ **(B)** $\pi - \frac{4}{3}$

(C) $\pi - \frac{8}{3}$ **(D)** $\pi - \frac{4\sqrt{2}}{3}$

[JEE MAIN 2016 (OFFLINE)]

Solution: We have $y^2 - 2x \geq 0$ and $x^2 + y^2 - 4x \leq 0, x \geq 0, y \geq 0$.

$$(x-2)^2 + y^2 \leq 4$$

Point of intersection of both curves $y^2 = 2x$ and $(x-2)^2 + y^2 = 4$ is (0, 0) and (2, 2) (Fig. 24.39).

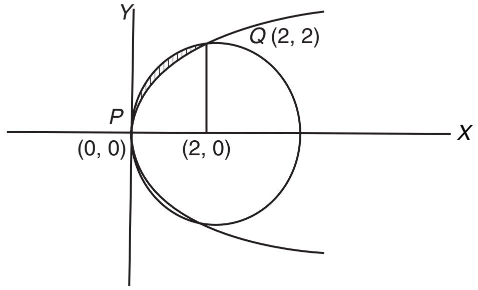


Figure 24.39

The required area is

$$\int_0^2 (y_1 - y_2)dx = \int_0^2 (\sqrt{4x - x^2} - \sqrt{2x})dx$$

$$= \frac{\pi(2)^2}{4} - \sqrt{2}\int_0^2 \sqrt{x}\, dx$$

$$= \left(\pi - \frac{8}{3}\right) \text{sq. units}$$

Hence, the correct answer is option (C).

14. The area (in sq. units) of the region described by $A = \{(x, y) \mid y \geq x^2 - 5x + 4, x + y \geq 1, y \leq 0\}$ is

(A) $\frac{19}{6}$ **(B)** $\frac{17}{6}$

(C) $\frac{7}{2}$ **(D)** $\frac{13}{6}$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: See Fig. 24.40.

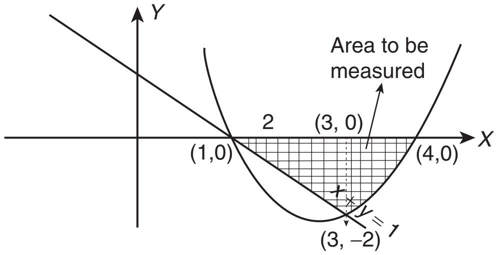


Figure 24.40

We have

$$A = \{(x, y) \mid y \geq x^2 - 5x + 4, x + y \geq 1, y \leq 0\}$$

$$y \geq x^2 - 5x + 4$$
$$y \geq (x-1)(x-4)$$

The area of the region to be measured is

$$\frac{1}{2} \times 2 \times 2 + \left|\int_3^4 (x^2 - 5x + 4)dx\right| = 2 + \left|\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right|_3^4$$

$$= 2 + \left|\frac{64}{3} - \frac{5 \times 16}{2} + 16 - \frac{27}{3} + \frac{5 \times 9}{2} - 12\right|$$

$$= \frac{12+7}{6} = \frac{19}{6} \text{ sq. units}$$

Hence, the correct answer is option (A).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

Paragraph for Questions 1–3: Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

[IIT-JEE 2008]

1. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ **(B)** $-\frac{4\sqrt{2}}{7^3 3^2}$

(C) $\frac{4\sqrt{2}}{7^3 3}$ **(D)** $-\frac{4\sqrt{2}}{7^3 3}$

Solution: We have

$$y^3 - 3y + x = 0$$

Differentiate both sides, we get

$$3y^2 y' - 3y' + 1 = 0 \quad (1)$$

Put $y = 2\sqrt{2}$, $x = -10\sqrt{2}$. Then

$$y'(-10\sqrt{2}) = \frac{-1}{21}$$

Differentiate equation (1), we get

$$3y^2 y'' + 6y(y')^2 - 3y'' = 0$$

Put $y = 2\sqrt{2}, x = -10\sqrt{2}, y' = \frac{-1}{21}$. Then

$$y''(-10\sqrt{2}) = -\frac{4\sqrt{2}}{7^3 \cdot 3^2}$$

Hence, the correct answer is option (B).

2. The area of the region bounded by the curves $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(A) $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$

(B) $-\int_a^b \frac{x}{3((f(x))^2 - 1)} dx + bf(b) - af(a)$

(C) $\int_a^b \frac{x}{3((f(x))^2 - 1)} dx - bf(b) + af(a)$

(D) $-\int_a^b \frac{x}{3((f(x))^2-1)}dx - bf(b) + af(a)$

Solution:

$$\text{Required area} = \int_a^b f(x)dx$$
$$= [xf(x)]_a^b - \int_a^b xf'(x)dx \qquad \text{(By parts)}$$
$$= bf(b) - af(a) + \int_a^b \frac{xdx}{3[f(x)^2-1]}$$

Hence, the correct answer is option (A).

3. $\int_{-1}^{1} g'(x)dx =$

(A) $2g(-1)$ **(B)** 0 **(C)** $-2g(1)$ **(D)** $2g(1)$

Solution:

$$y' = \frac{1}{3[1-(f(x))^2]}$$

Clearly $f(x)$ is an odd function, then $g'(x)$ is an even function, so

$$\int_{-1}^{1} g'(x)dx = 2\int_0^1 g'(x)dx$$
$$= 2[g(x)]_0^1$$
$$= 2[g(1)-g(0)]$$
$$= 2g(1) \qquad (\text{As } g(0)=0)$$

Hence, the correct answer is option (D).

4. The area of the region between the curves $y=\sqrt{\frac{1+\sin x}{\cos x}}$ and $y=\sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x=0$ and $x=\frac{\pi}{4}$ is

(A) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$ **(B)** $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$

(C) $\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$ **(D)** $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}dt$

[IIT-JEE 2008]

Solution: Since, both curves lie above x-axis in $x \in \left(0, \frac{\pi}{4}\right)$.

Therefore, area bounded between the curve is

$$\int_0^{\frac{\pi}{4}} \left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right)dx$$
$$= \int_0^{\frac{\pi}{4}} \left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}} - \sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$
$$= \int_0^{\frac{\pi}{4}} \left(\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\right)dx$$

Put $\tan\frac{x}{2} = t$. Then

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$
$$\Rightarrow dx = \frac{2dt}{1+t^2}$$
$$\Rightarrow \int_0^{\sqrt{2}-1} \left(\frac{4t}{(1+t^2)\sqrt{1-t^2}}\right)dt$$

Hence, the correct answer is option (B).

5. Area of the region bounded by the curve $y=e^x$ and lines $x=0$ and $y=e$ is

(A) $e-1$ **(B)** $\int_1^e \ln(e+1-y)dy$

(C) $e-\int_0^1 e^x dx$ **(D)** $\int_1^e \ln y\,dy$

[IIT-JEE 2009]

Solution: See Fig. 24.41.

$$\text{Required area } = \int_1^e \ln y\,dy$$
$$= (y\ln y - y)_1^e = (e-e) - \{-1\} = 1$$

Also,

$$\int_1^e \ln y\,dy = \int_1^e \ln(e+1-y)dy$$

Further the required area can be written as

$$e\times 1 - \int_0^1 e^x dx$$

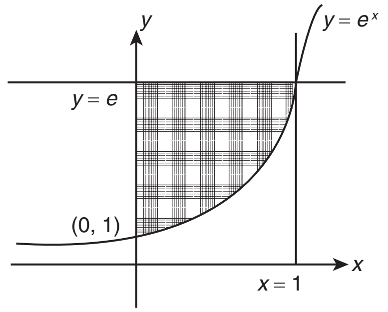


Figure 24.41

Hence, the correct answers are options (B), (C) and (D).

Paragraph for questions 6–8: Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

[IIT-JEE 2010]

6. The real number s lies in the interval

(A) $\left(-\frac{1}{4}, 0\right)$ **(B)** $\left(-11, -\frac{3}{4}\right)$

(C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ **(D)** $\left(0, \frac{1}{4}\right)$

Solution: Since,

$$f\left(-\frac{1}{2}\right)\cdot f\left(-\frac{3}{4}\right)<0$$

$$\Rightarrow S \text{ lie in } \left(-\frac{3}{4}, -\frac{1}{2}\right)$$

Hence, the correct answer is option (C).

7. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

(A) $\left(\frac{3}{4}, 3\right)$ **(B)** $\left(\frac{21}{64}, \frac{11}{16}\right)$

(C) (9, 10) **(D)** $\left(0, \frac{21}{64}\right)$

Solution:

$$-\frac{3}{4}<s<-\frac{1}{2}$$

$$\frac{1}{2}<t<\frac{3}{4}$$

$$\int_0^{1/2}(4x^3+3x^2+2x+1)dx < \text{area} < \int_0^{3/4}(4x^3+3x^2+2x+1)dx$$

$$[x^4+x^3+x^2+x]_0^{1/2} < \text{area} < [x^4+x^3+x^2+x]_0^{3/4}$$

$$\frac{1}{16}+\frac{1}{8}+\frac{1}{4}+\frac{1}{2} < \text{area} < \frac{81}{256}+\frac{27}{64}+\frac{9}{16}+\frac{3}{4}$$

$$\frac{15}{16} < \text{area} < \frac{525}{256}$$

Hence, the correct answer is option (A).

8. The function $f'(x)$ is

(A) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(B) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(C) increasing in $(-t, t)$

(D) decreasing in $(-t, t)$

Solution:

$$f(x) = 4x^3 + 3x^2 + 2x + 1$$
$$f'(x) = 12x^2 + 6x + 2$$
$$f''(x) = 2[12x + 3] = 0 \Rightarrow x = -1/4$$
$$f'''(x) = 24$$

So, the funtion is decreasing in $(-t, t)$.

Hence, the correct answer is option (D).

9. Let the straight line $x = b$ divides the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts $R_1(0 \le x \le b)$ and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

(A) $\frac{3}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{4}$

[IIT-JEE 2011]

Solution: See Fig. 24.42.

Therefore,

$$\int_0^b(1-x)^2dx - \int_b^1(1-x)^2dx = \frac{1}{4}$$

$$\Rightarrow \left[\frac{(x-1)^3}{3}\right]_0^b - \left[\frac{(x-1)^3}{3}\right]_b^1 = \frac{1}{4}$$

$$\Rightarrow \frac{(b-1)^3}{3}+\frac{1}{3}-\left(0-\frac{(b-1)^3}{3}\right)=\frac{1}{4}$$

$$\Rightarrow \frac{2(b-1)^3}{3} = -\frac{1}{12} \Rightarrow (b-1)^3 = -\frac{1}{8} \Rightarrow b = \frac{1}{2}$$

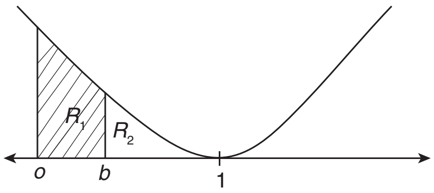


Figure 24.42

Hence, the correct answer is option (B).

10. Let f: $[-1, 2] \to [0, \infty]$ be a continuous function such that $f(x) = f(1 - x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} xf(x)dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$, and the x-axis. Then

(A) $R_1 = 2R_2$ **(B)** $R_1 = 3R_2$

(C) $2R_1 = R_2$ **(D)** $3R_1 = R_2$

[IIT-JEE 2011]

Solution:

$$R_1 = \int_{-1}^{2} xf(x)dx = \int_{-1}^{2}(2-1-x)f(2-1-x)dx$$

$$= \int_{-1}^{2}(1-x)f(1-x)dx = \int_{-1}^{2}(1-x)f(x)dx$$

Hence, $2R_1 = \int_{-1}^{2} f(x)dx = R_2$.

Hence, the correct answer is option (C).

11. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then

(A) $S \ge \frac{1}{e}$ **(B)** $S \ge 1-\frac{1}{e}$

(C) $S \le \frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)$ **(D)** $S \le \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$

[IIT-JEE 2012]

Solution: See Fig. 24.43.

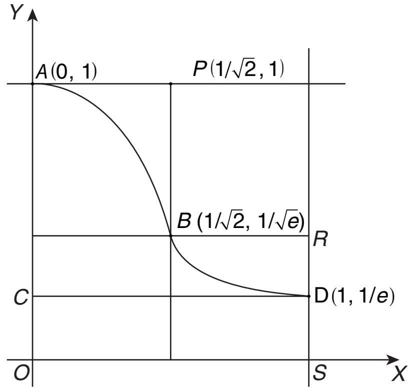


Figure 24.43

$$S > \frac{1}{e} \quad \text{(As area of rectangle } OCDS = 1/e\text{)}$$

Since,

$$e^{-x^2} \geq e^{-x} \ \forall\ x \in [0, 1]$$

$$\Rightarrow S > \int_0^1 e^{-x}dx = \left(1 - \frac{1}{e}\right)$$

Area of rectangle $OAPQ$ + Area of rectangle $QBRS > S$

$$S < \frac{1}{\sqrt{2}}(1) + \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$$

Since,

$$\frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right) < 1 - \frac{1}{e}$$

Hence, the correct answers are options (A), (B) and (D).

12. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

(A) $4(\sqrt{2} - 1)$ **(B)** $2\sqrt{2}(\sqrt{2} - 1)$

(C) $2(\sqrt{2} + 1)$ **(D)** $2\sqrt{2}(\sqrt{2} + 1)$

[JEE ADVANCED 2013]

Solution: Figure 24.44 depicts the area enclosed by the given curves, we have

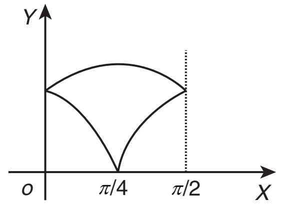


Figure 24.44

$$\int_0^{\pi/2} (\sin x + \cos x)dx - \left[\int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx\right]$$

$$= -|\cos x|_0^{\pi/2} + |\sin x|_0^{\pi/2} - \left[|\sin x|_0^{\pi/4} + |\cos x|_0^{\pi/4} - |\cos x|_{\pi/4}^{\pi/2} - |\sin x|_{\pi/4}^{\pi/2}\right]$$

$$= -(0-1) + (1-0) - \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 - \left(0 - \frac{1}{\sqrt{2}}\right) - \left(1 - \frac{1}{\sqrt{2}}\right)\right]$$

$$= 2 - \left[\sqrt{2} - 1 + \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}\right] = 2 - \left[2\sqrt{2} - 2\right]$$

$$= 4 - 2\sqrt{2} = 2\sqrt{2}(\sqrt{2} - 1)$$

Hence, the correct answer is option (B).

13. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is ______.

[JEE ADVANCED 2014]

Solution:

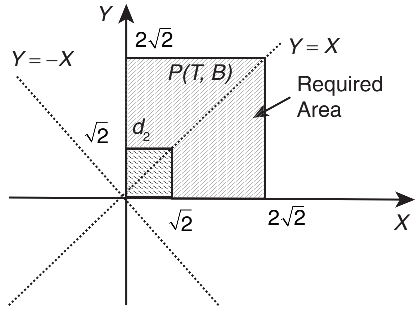


Figure 24.45

$$d_1 = \frac{|x - y|}{\sqrt{2}}$$

$$d_2 = \frac{|x + y|}{\sqrt{2}}$$

Therefore, according to the question (Fig. 24.45)

$$2 \leq \frac{|x - y|}{\sqrt{2}} + \frac{|x + y|}{\sqrt{2}} \leq 4$$

$$\Rightarrow 2\sqrt{2} \leq |x - y| + |x + y| \leq 4\sqrt{2} \qquad (1)$$

Since $x, y \geq 0$ in the first quadrant.

When $x > y$ (or $y - x < 0$),

$$|x - y| = x - y \text{ and } |x + y| = x + y$$

Therefore, Eq. (1) is true given that,

$$2\sqrt{2} \leq x - y + x + y \leq 4\sqrt{2} \Rightarrow \sqrt{2} \leq x \leq 2\sqrt{2}$$

checking with (2, 1) in region $x > y$, i.e. $2 > 1$.

Therefore, we shade area below $y = x$ from $[\sqrt{2}, 2\sqrt{2}]$.

Area of this region $= \frac{1}{2}(2\sqrt{2} \times 2\sqrt{2}) - \frac{1}{2}\sqrt{2} \times \sqrt{2} = 4 - 1 = 3$ sq. units

By symmetry about $y = x$, total area required = 6 sq. units

Hence, the correct answer is (6).

14. Let $F(x)=\int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x\in\mathbb{R}$ and $f:\left[0,\frac{1}{2}\right]\to[0,\infty)$ be a continuous function. For $a\in\left[0,\frac{1}{2}\right]$, if $F'(a)+2$ is the area of the region bounded by $x=0$, $y=0$, $y=f(x)$ and $x=a$, then $f(0)$ is ______.

[JEE ADVANCED 2015]

Solution: We have

$$F'(a)+2=\int_0^a f(x)dx$$

Differentiating both sides, we get

$$F''(a)=f(a)$$

Now,

$$F(x)=\int_x^{x^2+\pi/6} 2\cos^2 t dt$$

$$F'(x)=2x\cdot 2\cos^2\left(x^2+\frac{\pi}{6}\right)-2\cos^2 x$$

$$\Rightarrow F''(x)=-16x^2\cos\left(x^2+\frac{\pi}{6}\right)\sin\left(x^2+\frac{\pi}{6}\right)+4\cos x\sin x$$
$$+4\cos^2\left(x^2+\frac{\pi}{6}\right)$$

$$\Rightarrow F''(a)=-16a^2\cos\left(a^2+\frac{\pi}{6}\right)\sin\left(a^2+\frac{\pi}{6}\right)+4\cos a\sin a$$
$$+4\cos^2\left(a^2+\frac{\pi}{6}\right)$$

$$\Rightarrow f(0)=4\cos^2\left(\frac{\pi}{6}\right)=4\left(\frac{3}{4}\right)=3$$

Hence, the correct answer is (3).

15. Match the Column I to Column II.

	Column I	Column II
(A)	In ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2-b^2)=c^2$ and $\lambda=\dfrac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda)=0$ is (are)	**(P)** 1
(B)	In ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1+\cos 2X-2\cos 2Y=2\sin X\sin Y$, then possible value(s) of $\dfrac{a}{b}$ is (are)	**(Q)** 2
(C)	In $\mathbb{R}^3$, let $\sqrt{3}\hat{i}+\hat{j}$, $\hat{i}+\sqrt{3}\hat{j}$, and $\beta\hat{i}+(1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ with $\overrightarrow{OY}$ is $\dfrac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	**(R)** 3
(D)	Suppose that $F(\alpha)$ denotes the area of the region bounded by $x=0$, $x=2$, $y^2=4x$ and $y=\lvert\alpha x-1\rvert+\lvert\alpha x-2\rvert+\alpha x$, where $\alpha\in\{0,1\}$ Then the value(s) of $F(\alpha)+\dfrac{8}{3}\sqrt{2}$, when $\alpha=0$ and $\alpha=1$, is (are)	**(S)** 5
		(T) 6

[JEE ADVANCED 2015]

Solution: See Fig. 24.46.

$$2(a^2-b^2)=c^2 \quad (1)$$

$$\lambda=\frac{\sin(x-y)}{\sin z} \quad (2)$$

$$\cos(n\pi\lambda)=0 \quad (3)$$

$$\Rightarrow n\lambda=\frac{(2m+1)}{2} \quad (4)$$

From Eq. (2),

$$\lambda=\frac{\sin x\cos y-\cos x\sin y}{\sin z}$$

$$\Rightarrow\lambda=\frac{a\cos y-b\cos x}{c}\ \text{(By Sine formula)}$$

$$\Rightarrow\lambda=\frac{a\left(\dfrac{a^2+c^2-b^2}{2ac}\right)-b\left(\dfrac{b^2+c^2-a^2}{2bc}\right)}{c}$$

$$\Rightarrow\lambda=\frac{2(a^2-b^2)}{2c^2}=\frac{1}{2} \quad (5)$$

Therefore, from Eqs. (4) and (5),

$$\frac{n}{2}=\frac{2m+1}{2}\Rightarrow n=(2m+1)$$

So,

$$(A)\to(P),(R),(S)$$

Checking option (B):

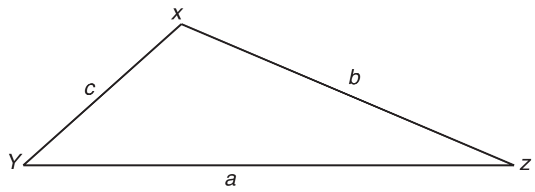


Figure 24.46

$$1+\cos 2x-2\cos 2y=2\sin x\sin y$$
$$\Rightarrow 2\cos^2 x-2(2\cos^2 y-1)=2\sin x\sin y$$
$$\Rightarrow 2\cos^2 x-4\cos^2 y+2=2\sin x\sin y$$
$$\Rightarrow 2\sin^2 y-2\sin x\sin y+\sin x\sin y-\sin^2 x=0$$
$$\Rightarrow 2\sin y(\sin y-\sin x)+\sin x(\sin y-\sin x)=0$$
$$\Rightarrow(\sin y-\sin x)(2\sin y+\sin x)=0$$
$$\Rightarrow b=a \text{ or } 2b=-a \text{ (impossible)}$$
$$\Rightarrow\frac{a}{b}=1$$

So,

$$(B)\to(P)$$

Checking option (C): See Fig. 24.47.

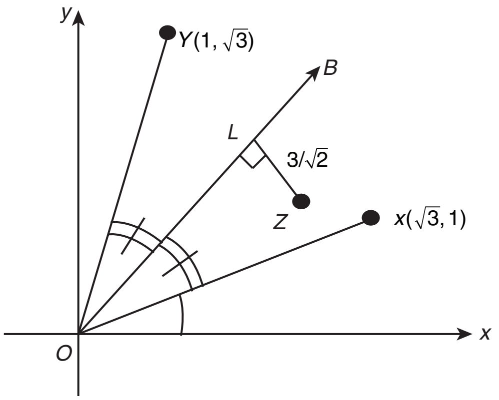


Figure 24.47

Vector along the bisector of acute angle between $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is ,

$$\frac{\sqrt{3}\hat{i}+\hat{j}}{2}+\frac{\hat{i}+\sqrt{3}\hat{j}}{2}=\frac{(\sqrt{3}+1)}{2}(\hat{i}+\hat{j})$$

Slope of $\overrightarrow{OB}=\tan(\pi/4)=1$

$\Rightarrow$ Equation of OB is $y=x$

Since,

$$ZL=3/\sqrt{2},\Rightarrow \frac{|\beta-(1-\beta)|}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$

$\Rightarrow |2\beta-1|=3$
$\Rightarrow (2\beta-1)=\pm 3$
$\Rightarrow \beta=2$ or $\beta=-1$
$\Rightarrow |\beta|=1$ or 2

Therefore, (C)$\rightarrow$ (P), (Q).

Checking option (D): See Fig. 24.48.

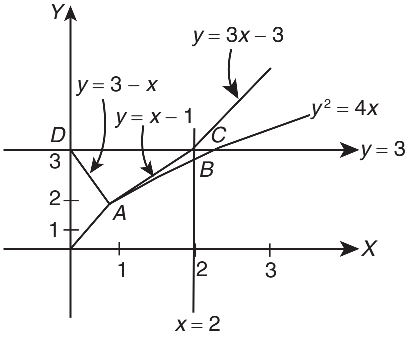


Figure 24.48

$$y=|\alpha x-1|+|\alpha x-2|+\alpha x;\ \alpha\in\{0,1\}$$

Case (I) For $\alpha=0, y=3$

Case (II) For $\alpha=1, y=|x-1|+|x-2|+x$

$$\Rightarrow y=\begin{cases}3-x; & x\le 1\\ x+1; & 1<x<2\\ 3x-3; & x\ge 2\end{cases}$$

Therefore,

$$F(0)=\int_0^2(3-2\sqrt{x})dx=\left[3x-\frac{4}{3}x^{3/2}\right]_0^2$$

$$=\left[6-\frac{4}{3}(2\sqrt{2})\right]=6-\frac{8}{3}\sqrt{2}$$

$$\Rightarrow F(0)+\frac{8}{3}\sqrt{2}=6\Rightarrow (T)$$

and $F(1)=F(0)-$ area of ΔACD

$$=\left(6-\frac{8}{3}\sqrt{2}\right)-\frac{1}{2}(2)(1)=5-\frac{8}{3}\sqrt{2}$$

$$\Rightarrow F(1)+\frac{8}{3}\sqrt{2}=5\Rightarrow\ (S)$$

Therefore,

(D) $\rightarrow$ (T), (S)

Hence, the correct matches are (A) $\rightarrow$ (P), (R), (S); (B) $\rightarrow$ (P); (C) $\rightarrow$ (P), (Q); (D) $\rightarrow$ (S), (T).

16. The area of the region $\{(x,y)\in\mathbb{R}^2 : y\ge\sqrt{|x+3|}, 5y\le x+9\le 15\}$ is equal to

(A) $\frac{1}{6}$ **(B)** $\frac{4}{3}$

(C) $\frac{3}{2}$ **(D)** $\frac{5}{3}$

[JEE ADVANCED 2016]

Solution: It is given that

$$y\ge\sqrt{|x+3|}$$

That is,

$$\sqrt{|x+3|}=\begin{cases}\sqrt{(x+3)}, & x\ge -3\\ \sqrt{-x-3}, & x<-3\end{cases}$$

It is also given that

$$5y\le x+9\le 15$$

That is,

$$x+9\le 15\Rightarrow x\le 6$$
$$5y\le 15\Rightarrow y\le 3$$
$$5y\le x+9$$

From Fig. 24.49, we have

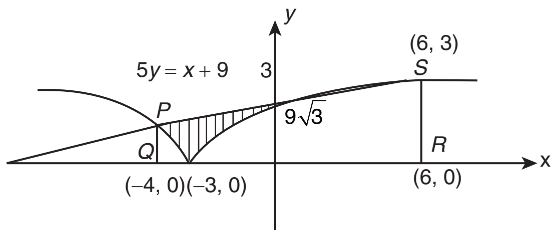


Figure 24.49

$$5y = x + 9 \quad (1)$$

and

$$y = \sqrt{-x-3}$$

That is,

$$\left(\frac{x+9}{5}\right)^2 = -(x+3)$$

$$x^2 + 81 + 18x = -25x - 75$$

$$x^2 + 43x + 156 = 0$$

$$(x+39)(x+4) = 0 \Rightarrow x = -4$$

(Since, $x \neq -39$)

Substituting the value of x in Eq. (1), we get the coordinates of point P as follows:

$$5y = -4 + 9 \Rightarrow y = 1 \Rightarrow P(-4, 1)$$

The area of trapezium $PQRS$ is

$$\frac{1}{2} \times 10 \times 4 = 20$$

Hence, the area of the given region is

$$20 - \int_{-4}^{-3} \sqrt{-x-3}\, dx - \int_{-3}^{6} \sqrt{x+3}\, dx$$

$$= 20 + \frac{2}{3}(-x-3)^{3/2}\Big|_{-4}^{-3} - \frac{2}{3}(x+3)^{3/2}\Big|_{-3}^{6}$$

$$= 20 + \frac{2}{3}(0-1) - \frac{2}{3}9^{3/2}$$

$$= 20 - \frac{2}{3} - \frac{2}{3} \times 27$$

$$= 2 - \frac{2}{3} = \frac{4}{3} \text{ sq. units.}$$

Hence, the correct answer is option (B).

Practice Exercise 1

1. The area of the region bounded by $y = |x-1|$ and $y = 1$ is

(A) 2 **(B)** 1

(C) $\frac{1}{2}$ **(D)** None of these

2. The area between the curve $y^2 = 4ax$, x-axis and the ordinates $x = 0$ and $x = a$ is

(A) $\frac{4}{3}a^2$ **(B)** $\frac{8}{3}a^2$

(C) $\frac{2}{3}a^2$ **(D)** $\frac{5}{3}a^2$

3. The area of the curve $xy^2 = a^2(a-x)$ bounded by y-axis is

(A) πa^2 **(B)** $2\pi a^2$

(C) $3\pi a^2$ **(D)** $4\pi a^2$

4. The area enclosed by the parabolas $y = x^2 - 1$ and $y = 1 - x^2$ is

(A) 1/3 **(B)** 2/3

(C) 4/3 **(D)** 8/3

5. The area of the smaller segment cut off from the circle $x^2 + y^2 = 9$ by $x = 1$ is

(A) $\frac{1}{2}(9\sec^{-1}3 - \sqrt{8})$ **(B)** $9\sec^{-1}(3) - \sqrt{8}$

(C) $\sqrt{8} - 9\sec^{-1}(3)$ **(D)** None of these

6. The area of the region bounded by the curves $y = |x-2|$, $x = 1$, $x = 3$ and the x-axis is

(A) 4 **(B)** 2

(C) 3 **(D)** 1

7. The area enclosed between the parabolas $y^2 = 4x$ and $x^2 = 4y$ is

(A) $\frac{14}{3}$ sq. units **(B)** $\frac{3}{4}$ sq. units

(C) $\frac{3}{16}$ sq. units **(D)** $\frac{16}{3}$ sq. units

8. The area bounded by the curves $y^2 = 8x$ and $y = x$ is

(A) $\frac{128}{3}$ sq. units **(B)** $\frac{32}{3}$ sq. units

(C) $\frac{64}{3}$ sq. units **(D)** 32 sq. units

9. The area bounded by the curves $y = \log_e x$ and $y = (\log_e x)^2$ is

(A) $3 - e$ **(B)** $e - 3$

(C) $\frac{1}{2}(3-e)$ **(D)** $\frac{1}{2}(e-3)$

10. The area between the parabola $y^2 = 4ax$ and $x^2 = 8ay$ is

(A) $\frac{8}{3}a^2$ **(B)** $\frac{4}{3}a^2$

(C) $\frac{32}{3}a^2$ **(D)** $\frac{16}{3}a^2$

11. The area of the region bounded by the curves $y = x^2$ and $y = |x|$ is

(A) 1/6 **(B)** 1/3

(C) 5/6 **(D)** 5/3

12. The area bounded by curves $y = \cos x$ and $y = \sin x$ and ordinates $x = 0$ and $x = \frac{\pi}{4}$ is

(A) $\sqrt{2}$ **(B)** $\sqrt{2} + 1$

(C) $\sqrt{2} - 1$ **(D)** $\sqrt{2}(\sqrt{2} - 1)$

13. The area in the first quadrant between $x^2 + y^2 = \pi^2$ and $y = \sin x$ is

(A) $\frac{(\pi^3 - 8)}{4}$ **(B)** $\frac{\pi^3}{4}$

(C) $\frac{(\pi^3 - 16)}{4}$ **(D)** $\frac{(\pi^3 - 8)}{2}$

14. The area bounded by the curves $y^2 - x = 0$ and $y - x^2 = 0$ is

(A) $\frac{7}{3}$ **(B)** $\frac{1}{3}$

(C) $\frac{5}{3}$ **(D)** 1

15. The area of region $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$ is

(A) $\frac{\pi^2}{5}$ **(B)** $\frac{\pi^2}{2}$

(C) $\frac{\pi^2}{3}$ **(D)** $\frac{\pi}{4}-\frac{1}{2}$

16. Area under the curve $y = \sin 2x + \cos 2x$ between $x = 0$ and $x=\frac{\pi}{4}$ is

(A) 2 sq. units **(B)** 1 sq. units
(C) 3 sq. units **(D)** 4 sq. units

17. Area under the curve $y=\sqrt{3x+4}$ between $x = 0$ and $x = 4$ is

(A) $\frac{56}{9}$ sq. units **(B)** $\frac{64}{9}$ sq. units

(C) 8 sq. units **(D)** None of these

18. If area bounded by the curves $y^2 = 4ax$ and $y = mx$ is $\frac{a^2}{3}$, then the value of m is

(A) 2 **(B)** –2

(C) $\frac{1}{2}$ **(D)** None of these

19. Area bounded by parabola $y^2 = x$ and straight line $2y = x$ is

(A) $\frac{4}{3}$ **(B)** 1

(C) $\frac{2}{3}$ **(D)** $\frac{1}{3}$

20. Area bounded by lines $y = 2 + x$, $y = 2 - x$ and $x = 2$ is

(A) 3 **(B)** 4
(C) 8 **(D)** 16

21. The ratio of the areas bounded by the curves $y = \cos x$ and $y = \cos 2x$ between $x = 0$, $x = \pi/3$ and x-axis is

(A) $\sqrt{2}:1$ **(B)** 1 : 1

(C) 1 : 2 **(D)** 2 : 1

22. The area bounded by the x-axis and the curve $y = \sin x$ and $x = 0$, $x = \pi$ is

(A) 1 **(B)** 2
(C) 3 **(D)** 4

23. The area bounded by the parabola $y^2 = 4ax$, its axis and two ordinates $x = 4$, $x = 9$ is

(A) $4a^2$ **(B)** $4a^2 \cdot 4$

(C) $4a^2 (9 - 4)$ **(D)** $\frac{152\sqrt{a}}{3}$

24. For $0 \le x \le \pi$, the area bounded by $y = x$ and $y = x + \sin x$ is

(A) 2 **(B)** 4
(C) 2π **(D)** 4π

25. The area of the region bounded by the x-axis and the curves defined by $y = \tan x$, $(-\pi/3 \le x \le \pi/3)$ is

(A) $\log\sqrt{2}$ **(B)** $-\log\sqrt{2}$

(C) 2log2 **(D)** 0

26. If a curve $y=a\sqrt{x}+bx$ passes through the point (1, 2) and the area bounded by the curve, line $x = 4$ and x-axis is 8 sq. units, then

(A) $a = 3, b = -1$ **(B)** $a = 3, b = 1$
(C) $a = -3, b = 1$ **(D)** $a = -3, b = -1$

27. If the area above the x-axis, bounded by the curves $y = 2^{kx}$ and $x = 0$ and $x = 2$ is $\frac{3}{\ln 2}$, then the value of k is

(A) $\frac{1}{2}$ **(B)** 1

(C) –1 **(D)** 2

28. The area bounded by the x-axis, the curve $y = f(x)$ and the lines $x = 1$, $x = b$ is equal to $\sqrt{b^2+1}-\sqrt{2}$ for all $b > 1$, then $f(x)$ is

(A) $\sqrt{x-1}$ **(B)** $\sqrt{x+1}$

(C) $\sqrt{x^2+1}$ **(D)** $\frac{x}{\sqrt{1+x^2}}$

29. The area bounded by the circle $x^2 + y^2 = 4$, line $x=\sqrt{3}y$ and x-axis lying in the first quadrant is

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ **(D)** π

30. Area bounded by the curve $y = \log x$, x-axis and the ordinates $x = 1$, $x = 2$ is

(A) log 4 sq. units **(B)** (log 4 + 1) sq. units
(C) (log 4 – 1) sq. units **(D)** None of these

31. Area bounded by the curve $y = xe^{x^2}$, x-axis and the ordinates $x = 0$, $x = a$ is

(A) $\frac{e^{a^2}+1}{2}$ sq. units **(B)** $\frac{e^{a^2}-1}{2}$ sq. units

(C) $e^{a^2}+1$ sq. units **(D)** $e^{a^2}-1$ sq. units

32. Area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$ is

(A) 2 sq. units **(B)** 4 sq. units
(C) 8 sq. units **(D)** None of these

33. Area bounded by the parabola $y = 4x^2$, y-axis and the lines $y = 1$, $y = 4$ is

(A) 3 sq. units **(B)** $\frac{7}{5}$ sq. units

(C) $\frac{7}{3}$ sq. units **(D)** None of these

34. Area bounded by the lines $y = x$, $x = -1$, $x = 2$ and x-axis is

(A) $\frac{5}{2}$ sq. units **(B)** $\frac{3}{2}$ sq. units

(C) $\frac{1}{2}$ sq. units **(D)** None of these

35. If the ordinate $x = a$ divides the area bounded by the curve $y=\left(1+\frac{8}{x^2}\right)$, x-axis and the ordinates $x = 2$, $x = 4$ into two equal parts, then $a =$

(A) 8 **(B)** $2\sqrt{2}$

(C) 2 **(D)** $\sqrt{2}$

36. Area between the curve $y = \cos x$ and x-axis when $0 \le x$ is

(A) 2 **(B)** 4
(C) 0 **(D)** 3

37. Area bounded by curve $y = x^3$, x-axis and ordinates $x = 1$ and $x = 4$ is
(A) 64 sq. units **(B)** 27 sq. units
(C) $\frac{127}{4}$ sq. units **(D)** $\frac{255}{4}$ sq. units

38. Area bounded by curve $xy = c$, x-axis between $x = 1$ and $x = 4$ is
(A) $c \log 3$ sq. units **(B)** $2\log c$ sq. units
(C) $2c \log 2$ sq. units **(D)** $2c \log 5$ sq. units

39. Area bounded by curve $y = k \sin x$ between $x = \pi$ and $x = 2\pi$ is
(A) $2k$ sq. units **(B)** 0
(C) $\frac{k^2}{2}$ sq. units **(D)** k sq. units

40. Area bounded by $y = x \sin x$ and x-axis between $x = 0$ and $x = 2\pi$ is
(A) 0 **(B)** 2π sq. units
(C) π sq. units **(D)** 4π sq. units

41. The ratio in which the area bounded by the curves $y^2 = x$ and $x^2 = y$ is divided by the line $x = \frac{1}{2}$ is
(A) $\frac{4\sqrt{2}-1}{9-4\sqrt{2}}$ **(B)** $\frac{3\sqrt{2}+3}{9-4\sqrt{2}}$
(C) $\frac{\sqrt{2}-1}{\sqrt{3}-1}$ **(D)** $\frac{2\sqrt{2}-1}{3\sqrt{3}-1}$

42. The area of the curve $x + |y| = 1$ and the y-axis is
(A) 1 sq. unit **(B)** 2 sq. units
(C) $\frac{1}{2}$ sq. units **(D)** $\sqrt{2}$ sq. units

43. The area bounded by the curve $y = e^{|x|}$, $y = e^{-|x|}$, $x \ge 0$ and $x \le 5$ is
(A) $e^5 + e^{-5} + 2$ sq. units **(B)** $e^5 + e^{-5} - 2$ sq. units
(C) $e^5 - e^{-5} + 2$ sq. units **(D)** $e^5 - e^{-5} - 2$ sq. units

44. Find the area of quadrilateral, combined equation of whose sides are $(x^2 - y^2)(x^2 - y^2 - 8x + 16)$
(A) 8 **(B)** 4
(C) $2\sqrt{2}$ **(D)** 9

45. Let f be a real valued function satisfying

$$f\left(\frac{x}{y}\right) = f(x) - f(y) \text{ and } \lim_{x\to 0}\frac{f(1+x)}{x} = 3$$

Find the area bounded by the curve $y = f(x)$, the y-axis and the line $y = 3$.

46. Let A_n be the area bounded by $y = \tan^n x$, $x = 0$, $y = 0$ and $x = \pi/4$. Prove that for $n \ge 2$.
(i) $A_n + A_{n-2} = \frac{1}{n-1}$ **(ii)** $\frac{1}{2(n+1)} < A_n < \frac{1}{2(n-1)}$

47. Let $f(x)$ be a continuous function given by

$$f(x) = \begin{cases} 2x & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

Find the area of the region in the third quadrant bounded by the curve $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

48. Let C_1 and C_2 be the graphs of the function $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$; $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P parallel to the axis, meet C_2 and C_3 at Q and R, respectively (see Fig. 24.50). If for every position of P on (C_1), the areas of shaded region OPQ and ORP are equal, determine the function $f(x)$.

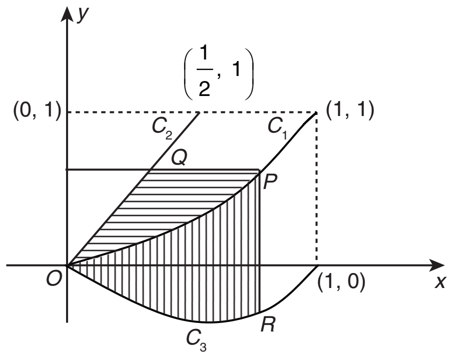


Figure 24.50

49. Let $f(x) =$ maximum $\{x^2, (1 - x)^2, 2x(1 - x)\}$ where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ and $x = 1$.

50. Let A_n be the area bounded by the curve $y = (1 + \tan x)^n$ and the lines $x = 0$, $y = 0$, and $x = \frac{\pi}{4}$. Prove that for $n \ge 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.

51. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$.

52. Sketch the region bounded by $y = x^2$ and $y = \frac{2}{(1+x^2)}$ and find its area.

53. Sketch the curves and identify the region bounded by $x = 1/2$, $x = 2$, $y = \log_e x$ and $y = 2^x$. Find area of region.

54. Compute the area of the region bounded by the curves $y = ex$ In x, and $y = \left(\frac{\ln x}{ex}\right)$.

55. Find all the maxima and minima of the function $f(x) = x(x - 1)^2$, $(0 \le x \le 2)$. Also determine the area bounded by the curve $y = x(x - 1)^2$, the y-axis and the line $y = 2$.

56. Find the area of the region bounded by the curve whose equation is $y = \tan x$, its tangent drawn at $x - \pi/4$ and the x-axis.

57. Find the area bounded by the curves $x^2 + y^2 = 4$, $x^2 = -\sqrt{2}\, y$ and $x = y$.

58. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is
(A) 1 **(B)** 2 **(C)** $2\sqrt{2}$ **(D)** 4

59. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at $(0, 0)$ and $[x, g(x)]$

is $\frac{\sqrt{3}}{4}$, then the function $g(x)$ is

(A) $g(x)=\pm\sqrt{1-x^2}$ **(B)** $g(x)=\sqrt{1-x^2}$

(C) $g(x)=-\sqrt{1-x^2}$ **(D)** $g(x)=\sqrt{1+x^2}$

60. Area bounded by $y = g(x)$, x-axis and the lines $x = -2$, $x = 3$, where $g(x)=\begin{cases}\max i:\{f(t);-2\le t\le x\}, & -2\le x<0\\ \min i:\{f(t);0\le t\le x\}, & 0\le x<3\end{cases}$

and $f(x) = x^2 - |x|$, is equal to

(A) $\frac{113}{24}$ sq. units **(B)** $\frac{111}{24}$ sq. units

(C) $\frac{117}{24}$ sq. units **(D)** $\frac{121}{24}$ sq. units

61. Area of the region that consists of all the points satisfying the conditions $|x-y|+|x+y|\le 8$ and $xy\ge 2$ is equal to

(A) 4(7 – ln 8) sq. units **(B)** 4(9 – ln 8) sq. units

(C) 2(7 – ln 8) sq. units **(D)** 2(9 – ln 8) sq. units

62. Two lines draw through the point P (4, 0) divide the area bounded by the curves $y=\sqrt{2}\sin\frac{\pi x}{4}$ and x-axis, between the line $x = 2$ and $x = 4$, in to three equal parts. Sum of the slopes of the drawn lines is equal to

(A) $-\frac{2\sqrt{2}}{\pi}$ **(B)** $-\frac{\sqrt{2}}{\pi}$ **(C)** $-\frac{2}{\pi}$ **(D)** $-\frac{4\sqrt{2}}{\pi}$

63. The area bounded by the curve $y = x(3 - x)^2$, the x-axis and the ordinates of the maximum and minimum points of the curve is

(A) 2 sq. units **(B)** 6 sq. units

(C) 4 sq. units **(D)** 8 sq. units

64. What is the area of a plane figure bounded by the points of the lines max $(x, y) = 1$ and $x^2 + y^2 = 1$?

(A) $1-\frac{\pi}{2}$ sq. units **(B)** $1-\frac{\pi}{3}$ sq. units

(C) $1-\frac{\pi}{4}$ sq. units **(D)** $1-\pi$ sq. units

65. The area bounded by the curve $y = (x - 1)(x - 2)(x - 3)$ lying between the ordinates $x = 0$ and $x = 3$ is

(A) $\frac{7}{4}$ sq. units **(B)** 4 sq. units

(C) $\frac{11}{4}$ sq. units **(D)** 3 sq. units

66. The area common to the curves $y = x^3$ and $y=\sqrt{x}$ is

(A) 2 **(B)** 4

(C) 8 **(D)** None of these

67. The area of the region consisting of points (x, y) satisfying $|x\pm y|\le 2$ and $x^2+y^2\ge 2$ is

(A) $8-2\pi$ sq. units **(B)** $4-2\pi$ sq. units

(C) $1-2\pi$ sq. units **(D)** 2π sq. units

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. For which of the following values of m, is the area of the region bounded by the curve $y=x-x^2$ and the line $y = mx$ equals $\frac{9}{2}$?

(A) – 4 **(B)** – 2 **(C)** 2 **(D)** 4

2. Three straight lines are drawn through a point M, lying in the interior of triangle ABC, parallel to its sides. The areas of the resulting three triangles are S_1, S_2 and S_3. The area of triangle ABC is

(A) $S_1+S_2+S_3$ **(B)** $(\sqrt{S_1}+\sqrt{S_2}+\sqrt{S_3})^2$

(C) $\frac{(S_1+S_2+S_3)^{3/2}}{S_1+S_2+S_3}$ **(D)** None of these

Comprehension Type Questions

Paragraph for Questions 3–5: Let $f: R \to R$ be a continuous and bijective function defined such that $f(\alpha) = 0$ $(\alpha \ne 0)$. The area bounded by $y = f(x)$, $x = \alpha$, $x = \alpha - t$ is equal to the area bounded by $y = f(x)$, $x = \alpha$, $x = \alpha + t\ \forall\ t \in R$, then

3. Graph of $y = f(x)$ is symmetrical about point

(A) (0, 0) **(B)** $(0, \alpha)$

(C) $(\alpha, 0)$ **(D)** (α, α)

4. The value of $f(2\alpha)$ is equal to

(A) $f(\alpha)$ **(B)** $-f(\alpha)$

(C) $f(0)$ **(D)** $-f(0)$

5. The value of $\int_{-\beta}^{\beta} f^{-1}(t)\,dt$ is equal to

(A) 0 **(B)** $2\alpha\beta$

(C) $\alpha\beta$ **(D)** None of these

Paragraph for Questions 6–8: Let $f(x)$ be a polynomial of degree 4 satisfying

$$\left(\int_1^x A(t)B(t)dt\right)\left(\int_1^x C(t)D(t)dt\right)-\left(\int_1^x A(t)C(t)dt\right)\left(\int_1^x B(t)D(t)dt\right)$$

$= f(x)\ \forall\ x \in R$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are non-constant continuous and differentiable functions. It is given that the leading coefficient (coefficient of x^4) of $f(x)$ is 1.

6. The area included between the line $y = x - 1$ and the curve $y = f(x)$ is

(A) $\frac{2}{5}$ sq. units **(B)** $\frac{3}{10}$ sq. units

(C) $\frac{7}{10}$ sq. units **(D)** $\frac{7}{5}$ sq. units

7. The area of the smaller region intercepted between the curve $y = f(x)$ and $x^2 + y^2 = 1$ is

(A) $\frac{\pi}{4}-\frac{1}{5}$ sq. units **(B)** $\frac{\pi}{4}$ sq. units

(C) $\frac{\pi}{4}+\frac{1}{5}$ sq. units **(D)** $\frac{\pi}{2}+\frac{1}{5}$ sq. units

8. The area included between $y = f(x)$ and $y + 2 = 0$ between the ordinates $x = 0$ and $x = 3$ is

(A) $\frac{3}{5}$ sq. units **(B)** $\frac{33}{5}$ sq. units

(C) $\frac{23}{10}$ sq. units **(D)** $\frac{63}{5}$ sq. units

Paragraph for Questions 9–11: *ABCD* is a square of side length 2 units and the centre of square is at origin. C_2 is a circle passing through vertices *A*, *B*, *C*, *D* and C_1 are the circle touching all the sides of square *ABCD*. Line L_1 is tangent at *A* line L_2 is tangent at *D* on circle C_2 who intersects at *K*, where *A*, *B*, *C*, *D* lie in 2nd, 1st, 4th and 3rd quadrant. Point *Q* is variable point on C_2, let perpendicular drawn from *Q* to cut the line L_1 and L_2 at *E* and *F*, respectively. Given that *AB*, *BC*, *CD* and *AD* are parallel to the coordinate axes.

9. The maximum area of rectangle *QEKF* is

(A) $\frac{9}{4}$ **(B)** $\frac{8}{3}$

(C) $\frac{9}{5}$ **(D)** None of these

10. Area of ΔBQC, (where *Q* is such that the area of the rectangle *QEKF* is maximum) is

(A) $\frac{5-2\sqrt{2}}{2\sqrt{2}}$ **(B)** $\frac{2+3\sqrt{2}}{2\sqrt{2}}$

(C) $\frac{5+2\sqrt{3}}{\sqrt{2}}$ **(D)** None of these

11. Locus of point which are equidistant from *Q* and line L_1 intersect the line $y = x$ at *M* (other than origin), then area of ΔOQM (where *Q* is such that the area of the rectangle *QEKF* is maximum) is

(A) 1 **(B)** 2 **(C)** 3 **(D)** None of these

Paragraph for Questions 12–14: Let a function $f(x)$ satisfies the condition $f(x+y)=\frac{f(x)+f(y)}{f(x)}$ such that $f(0) = 2$ and $f(x) \geq 0$.

12. The curve $y = f(x)$ is

(A) $y=\sqrt{2(x+1)}$ **(B)** $y=2\sqrt{(x+1)}$

(C) $y=\ln(x+1)$ **(D)** $y=\ln(x-1)$

13. Area bounded between $y = f(|x|)$ and $y = 7 - |x|$ is

(A) $\frac{23}{6}$ sq. units **(B)** $\frac{11}{6}$ sq. units

(C) $\frac{86}{6}$ sq. units **(D)** 7 sq. units

14. The number of points where $g(x) = \max\{f(x), 6, 7 - |x|\}$ is non-differentiable $\forall\, x \in [-10, 10]$ are

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8

Paragraph for Questions 15–17: Let *f* be function satisfying the condition $f(xy) = \frac{f(x)}{y}+\frac{f(y)}{x}$ $\forall\, x, y > 0$. If $f(x)$ is differentiable and $f(1) = 1$, then

15. $\lim_{x\to\infty} f(x)$ is equal to

(A) $-\infty$ **(B)** ∞

(C) 0 **(D)** 1

16. If $g(x)=\int_1^x f(t)\,dt$, then $g(n)-g\left(\frac{1}{n}\right)$; $n \in N$ is equal to

(A) -1 **(B)** 1

(C) 0 **(D)** $n+\frac{1}{n}$

17. The area bounded between the curve $y = f(x)$ and $y = ex \ln x$ is

(A) $\frac{2e-5}{8}$ **(B)** $\frac{2e^2-5e}{4}$

(C) $\frac{e^2+5}{4e}$ **(D)** None of these

Matrix Match Type Questions

18. Match the following:

Column I	Column II
(A) The area enclosed between the curves $\lvert x\rvert + \lvert y\rvert = 2$ and $x^2 = y$ in sq. units is	**(p)** $\frac{24}{5}$
(B) The maximum value of the function $f(x) = 3\sin x - 4\cos x - \frac{7}{3}$ will be given by	**(q)** $\frac{7}{3}$
(C) The length of common chord of two circles of radii 3 and 4 units which intersect orthogonally is	**(r)** $\frac{16}{3}$
(D) The length of chord intercepted by the parabola $y^2 = 4(x + 1)$ passing through its focus and inclined at 60° with positive *x*-axis is	**(s)** $\frac{8}{3}$

19. Match the following:

Column I	Column II
(A) $f(x)=\int_0^{\sin x} t^2\,dt$, then period of $f'(x)$ is	**(p)** $\frac{\pi}{14}$
(B) If area of ellipse $b^2x^2 + a^2y^2 = a^2b^2$ $(a > b)$, enclosed by *x*-axis and the ordinates $x = 0$ and $x = b$ be 1/8 th the area of entire ellipse, then $e\sqrt{1-e^2}+\sin^{-1}\sqrt{1-e^2}$ is equal to	**(q)** $\frac{\pi}{2}$
(C) Let $f(x)=\frac{\operatorname{cosec}^{-1}x+\cos^{-1}\left(\frac{1}{x}\right)}{\operatorname{cosec}x}$. Then greatest value of $f(x)$ is	**(r)** $\frac{\pi}{4}$
(D) $\cos^{-1}\left(\sin\left(\frac{46\pi}{7}\right)\right)$ is	**(s)** 2π

20. Match the following:

Column I	Column II
(A) If $y = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(b)$, $(0 < b < 1)$ and $0 < y \le \frac{\pi}{4}$, then the maximum value of b will be	**(p)** 3
(B) The number of solutions of $\sin^4 x + \cos^3 x \ge 1$ in $(0, 2\pi)$ will be	**(q)** $\frac{1}{3}$
(C) The area bounded by the loop of $4y^2 = x^2(4 - x^2)$ is	**(r)** $\frac{1}{2}$
	(s) $\frac{16}{3}$

Integer Type Question

21. Let $f(x)$ be a polynomial of degree 3 if the curve $y = f(x)$ has relative extremities at $x = \pm\frac{2}{\sqrt{3}}$ and passes through (0, 0) and (1, –2) dividing the circle $x^2 + y^2 = 4$ in two parts. Then the integral part of areas of these two parts is ______________.

Answer Key

Practice Exercise 1

1. (B)	**2.** (B)	**3.** (A)	**4.** (D)	**5.** (B)
6. (D)	**7.** (D)	**8.** (B)	**9.** (A)	**10.** (C)
11. (B)	**12.** (C)	**13.** (A)	**14.** (B)	**15.** (D)
16. (B)	**17.** (D)	**18.** (A)	**19.** (A)	**20.** (B)
21. (D)	**22.** (B)	**23.** (D)	**24.** (A)	**25.** (C)
26. (A)	**27.** (B)	**28.** (D)	**29.** (C)	**30.** (C)
31. (B)	**32.** (B)	**33.** (C)	**34.** (A)	**35.** (B)
36. (B)	**37.** (D)	**38.** (C)	**39.** (A)	**40.** (D)
41. (A)	**42.** (A)	**43.** (B)	**44.** (A)	**45.** $3e$ sq. units
47. $\frac{761}{192}$ sq. units	**48.** $x^3 - x^2$	**49.** $\frac{17}{27}$ sq. units	**51.** $\frac{121}{4}$	**52.** $\left(\pi - \frac{2}{3}\right)$ sq. units
53. $\frac{(4-\sqrt{2})}{\ln 2} + \frac{3}{2} - \left(\frac{5}{2}\right)\ln 2$	**54.** $\frac{e^2 - 5}{4e}$ sq. units	**55.** $\frac{10}{3}$ sq. units	**56.** $\left(\frac{1}{2}\right)\log 2 - \left(\frac{1}{4}\right)$	**57.** $\pi + \frac{1}{3}$
58. (B)	**59.** (A)	**60.** (A)	**61.** (A)	**62.** (A)
63. (C)	**64.** (C)	**65.** (C)	**66.** (D)	**67.** (A)

Practice Exercise 2

1. (B), (D)	**2.** (B)	**3.** (C)	**4.** (D)	**5.** (B)
6. (B)	**7.** (A)	**8.** (D)	**9.** (D)	**10.** (D)
11. (A)	**12.** (B)	**13.** (C)	**14.** (A)	**15.** (C)
16. (C)	**17.** (A)	**18.** (A) $\to$ (q), (B) $\to$ (s), (C) $\to$ (p), (D) $\to$ (r)	**19.** (A) $\to$ (s), (B) $\to$ (r), (C) $\to$ (q), (D) $\to$ (p)	
20. (A) $\to$ (q), (B) $\to$ (p), (C) $\to$ (s)		**21.** (6)		

Solutions

Practice Exercise 1

1. $y = x - 1$, if $x > 1$ and $y = -(x - 1)$, if $x < 1$

$$\text{Area } = \int_0^1 (1-x)dx + \int_1^2 (x-1)dx = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2$$

$$= \left[1 - \frac{1}{2}\right] + \left[-\left(\frac{1}{2} - 1\right)\right] = \frac{1}{2} + \frac{1}{2} = 1 \text{ sq. unit}$$

2. Required area $= 2\int_0^a \sqrt{4ax}\,dx = 4\sqrt{a} \times \frac{2}{3}[x^{3/2}]_0^a$

$$= \frac{8\sqrt{a}}{3} \cdot a\sqrt{a} = \frac{8}{3}a^2 \text{ sq. units}$$

3. Since the curve is symmetrical about x-axis, therefore,

$$\text{Required area } A = 2\int_0^a a\sqrt{\frac{a-x}{x}}\,dx$$

Put $x = a\sin^2\theta$. Then

$$dx = 2a\sin\theta \cdot \cos\theta\, d\theta$$

$$A = 2\int_0^{\pi/2} a\sqrt{\frac{a\cos^2\theta}{a\sin^2\theta}}\, a\sin 2\theta\, d\theta = 2a^2\int_0^{\pi/2} \frac{\cos\theta}{\sin\theta} 2\sin\theta\cos\theta\, d\theta$$

$$A = 4a^2\int_0^{\pi/2} \cos^2\theta\, d\theta \Rightarrow A = 4a^2 . \frac{1}{2} . \frac{\pi}{2} = \pi a^2 \text{ sq. units}$$

4. Given parabolas are $x^2 = 1 + y$, $x^2 = 1 - y$

For points of intersection, we have

$$1 + y = 1 - y \Rightarrow 2y = 0 \Rightarrow y = 0$$
$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

So,

$$\text{Required area } = 4\int_0^1 (1 - x^2)\,dx = 4\left[x - \frac{x^3}{3}\right]_0^1 = \frac{8}{3} \text{ sq. units}$$

5. Area of smaller part $I = 2\int_1^3 \sqrt{9 - x^2}\,dx$

$$= 2 \cdot \frac{1}{2}\left[x\sqrt{9 - x^2} + 9\sin^{-1}\frac{x}{3}\right]_1^3 = \left[9\frac{\pi}{2} - \sqrt{8} - 9\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \left[9\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right)\right) - \sqrt{8}\right] = \left[9\cos^{-1}\left(\frac{1}{3}\right) - \sqrt{8}\right]$$

$$= [9\sec^{-1}(3) - \sqrt{8}] \text{ sq. units}$$

6. Required area $= \int_1^3 |x - 2|\,dx = \int_1^2 (2 - x)\,dx + \int_2^3 (x - 2)dx$

$$= \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^3 = \frac{1}{2} + \frac{1}{2} = 1 \text{ sq. unit}$$

7. Equations of curves $y^2 = 4x$ and $x^2 = 4y$. The given equations may be written as $y = 2\sqrt{x}$ and $y = \dfrac{x^2}{4}$.

For points of intersection,

$$\left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x^4 = 64x$$
$$\Rightarrow x(x^3 - 64) = 0$$
$$\Rightarrow x = 0, x = 4$$

We know that area enclosed by the parabolas is

$$\int_0^4 2\sqrt{x}\,dx - \int_0^4 \frac{x^2}{4}dx = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. units}$$

8. $y^2 = 8x$ and $y = x \Rightarrow x^2 = 8x \Rightarrow x = 0, 8$

Therefore, required area is

$$\int_0^8 (2\sqrt{2}\sqrt{x} - x)dx = \left[\frac{4\sqrt{2}}{3}x^{3/2} - \frac{x^2}{2}\right]_0^8 = \frac{128}{3} - \frac{64}{2} = \frac{32}{3} \text{ sq. units}$$

9. For points of intersection,

$$\log x = (\log x)^2 \Rightarrow \log x(\log x - 1) = 0$$
$$\Rightarrow x = 1,\ x = e$$

So,

$$\text{Required area } A = \int_1^e [\log x - (\log x)^2]dx$$

$$A = \int_1^e \log x\,dx - \int_1^e (\log x)^2 dx$$

$$= [x\log x - x]_1^e - [x(\log x)^2 - 2x\log x + 2x]_1^e$$

$$= [e - e - (-1)] - [e(1)^2 - 2e + 2e - (2)]$$
$$= (1) - (e - 2) = 3 - e$$

10. For points of intersection,

$$\left(\frac{x^2}{8a}\right)^2 = 4ax \Rightarrow x^4 = 256a^2x$$
$$\Rightarrow x\left(x^3 - 256a^3\right) = 0$$
$$\Rightarrow x = 0, x = a \cdot 2^{8/3}$$

So,

$$\text{Required area, } A = \int_0^{(a2^{8/3})} \sqrt{4ax}\,dx - \int_0^{a2^{8/3}} \frac{x^2}{8a}dx = \frac{32a^2}{3}$$

11. See Fig. 24.51.

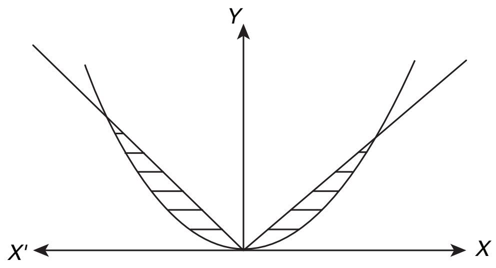


Figure 24.51

Required area $= 2 \times$ (shaded area in first quadrant)

$$= 2\int_0^1 (x - x^2)\,dx = 2 \times \frac{1}{6} = \frac{1}{3} \text{ sq. units}$$

12. Given equations of curves $y = \cos x$ and $y = \sin x$ and ordinates $x = 0$ to $x = \pi/4$.

We know that area bounded by the curves is

$$\int_{x_1}^{x_2} y\,dx = \int_0^{\pi/4} \cos x\,dx - \int_0^{\pi/4} \sin x\,dx$$

$$= [\sin x]_0^{\pi/4} - [-\cos x]_0^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4} - \sin 0\right) + \left(\cos\frac{\pi}{4} - \cos 0\right) = \left(\frac{1}{\sqrt{2}} - 0\right) + \left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \sqrt{2} - 1 \text{ sq. units}$$

13. Area of the circle in first quadrant is $\frac{\pi(\pi^2)}{4}$, that is, $\frac{\pi^3}{4}$. Also area bounded by curve $y = \sin x$ and x-axis is 2 sq. units

Hence, required area is $\frac{\pi^3}{4} - 2 = \frac{\pi^3 - 8}{4}$.

14. For points of intersection

$$(x^2)^2 - x = 0$$
$$x(x^3 - 1) = 0$$
$$\Rightarrow x = 0, x = 1$$

$$\int_0^1 (\sqrt{x} - x^2)dx = \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{3} \text{ sq. units}$$

15. $x^2 + y^2 = 1, x + y = 1$ meet when

$x^2 + (1 - x)^2 = 1 \Rightarrow x^2 + 1 + x^2 - 2x = 1$

$\Rightarrow 2x^2 - 2x = 0 \Rightarrow 2x(x - 1) = 0$

$\Rightarrow x = 0, x = 1 \Rightarrow y = 1, y = 0$, that is, $A(1, 0)$; $B(0, 1)$

$$\text{Required area } = \int_0^1 [\sqrt{1-x^2} - (1-x)]\,dx$$

$$= \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x - x + \frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} - 1 + \frac{1}{2} = \left(\frac{\pi}{4} - \frac{1}{2}\right) \text{ sq. units}$$

16. $$\text{Required area } = \int_0^{\pi/4} (\sin 2x + \cos 2x)dx$$

$$= \left[-\frac{\cos 2x}{2} + \frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[-\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \cos 0 - \sin 0\right] = 1 \text{ sq. unit}$$

17. $$\text{Area } = \int_0^4 \sqrt{3x+4}\,dx = \left|\frac{(3x+4)^{3/2}}{3.(3/2)}\right|_0^4$$

$$= \frac{2}{9} \times 56 = \frac{112}{9} \text{ sq. units}$$

18. The two curves $y^2 = 4ax$ and $y = mx$ intersect at $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$ and the area enclosed by the two curves is given by

$$\int_0^{4a/m^2} (\sqrt{4ax} - mx)\,dx$$

Therefore,

$$\int_0^{4a/m^2} (\sqrt{4ax} - mx)\,dx = \frac{a^2}{3}$$

$$\Rightarrow \frac{8}{3}\frac{a^2}{m^3} = \frac{a^2}{3} \Rightarrow m^3 = 8 \Rightarrow m = 2$$

19. $y^2 = x$ and $2y = x \Rightarrow y^2 = 2y \Rightarrow y = 0, 2$

Therefore, required area is

$$\int_0^2 (y^2 - 2y)dy = \left(\frac{y^3}{3} - y^2\right)_0^2 = \frac{4}{3} \text{ sq. units}$$

20. Obviously, triangle ACB is right angled at C. See Fig. 24.52.

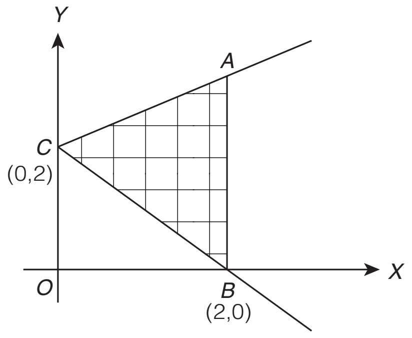


Figure 24.52

Therefore,

$$\text{Required area } = \frac{1}{2} \times AC \times BC$$

$$= \frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2} = 4 \text{ sq. units}$$

21. $A_1 = \int_0^{\pi/3} \cos x\,dx = \frac{\sqrt{3}}{2}$, $A_2 = \int_0^{\pi/3} \cos 2x\,dx = \frac{\sqrt{3}}{4}$

Therefore, $A_1 : A_2 = 2 : 1$.

22. The curve is symmetric about x-axis,

Therefore, required area is

$$2\int_0^{\pi/2} \sin x\,dx = 2[-\cos x]_0^{\pi/2} = 2 \text{ sq. units}$$

23. Required area, $A = 2\int_4^9 \sqrt{4ax}\,dx$

$$A = 4\sqrt{a} \times \frac{2}{3}[x^{3/2}]_4^9 = \frac{152\sqrt{a}}{3} \text{ sq. units}$$

24. The curves $y = x$ and $y = x + \sin x$ intersect at (0, 0) and (π, π). Hence, area bounded by the two curves is

$$\int_0^{\pi} (x + \sin x)dx - \int_0^{\pi} x\,dx = \int_0^{\pi} \sin x\,dx$$

$$= [-\cos x]_0^{\pi} = -\cos \pi + \cos 0 = -(-1) + (1) = 2 \text{ sq. units}$$

25. $\text{Required area} = 2\int_0^{\pi/3} \tan x\,dx = 2[\log \sec x]_0^{\pi/3} = 2\log(2) \text{ sq. units}$

26. Given curve $y = a\sqrt{x} + bx$. This curve passes through (1, 2).

Therefore, $2 = a + b$. (1)

Area bounded by this curve and line $x = 4$ and x-axis is 8 sq. units, then

$$\int_0^4 (a\sqrt{x} + bx)\,dx = 8$$

$$\Rightarrow \frac{2a}{3}[x^{3/2}]_0^4 + \frac{b}{2}[x^2]_0^4 = 8, \frac{2a}{3}\cdot 8 + 8b = 8$$

$\Rightarrow 2a + 3b = 3$ (2)

From equations (1) and (2), we get $a = 3$, $b = -1$.

27. $\int_0^2 2^{kx}\,dx = \dfrac{3}{\log 2} \Rightarrow 2^{2k} - 1 = 3k$. Now check from options, only (b) satisfies the above condition.

28. $\int_1^b f(x)\,dx = \sqrt{b^2+1} - \sqrt{2} = \sqrt{b^2+1} - \sqrt{1+1} = [\sqrt{x^2+1}]_1^b$

Therefore,

$$f(x) = \frac{d}{dx}\sqrt{x^2+1} = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

29. See Fig. 24.53.

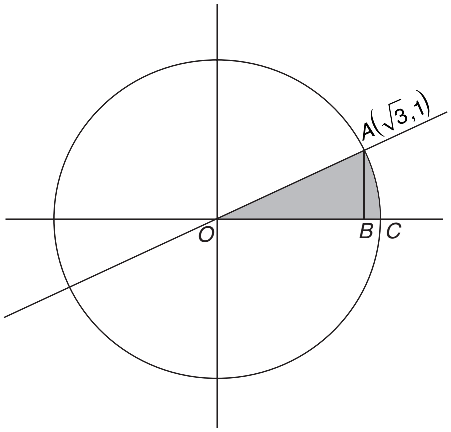


Figure 24.53

Required area = Area of $\Delta\, OBA$ + Area of BAC

$$1 = \frac{1}{2}\left(\sqrt{3}\times 1\right) + \int_{\sqrt{3}}^{2} \sqrt{4-x^2}\,dx$$

$$= \frac{\sqrt{3}}{2} + \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3}$$

$$= \frac{\pi}{3}\ \text{sq. units}$$

30. Given curve $y = \log x$ and $x = 1, x = 2$.

Hence, required area $= \int_1^2 \log x\,dx = (x\log x - x)_1^2$

$= 2\log 2 - 1 = (\log 4 - 1)$ sq. units

31. Required area is $\int_0^a y\ dx = \int_0^a xe^{x^2}\,dx$

We put $x^2 = t \Rightarrow dx = \dfrac{dt}{2x}$ as $x = 0 \Rightarrow t = 0$ and $x = a \Rightarrow t = a^2$,

then it reduces to

$$\frac{1}{2}\int_0^{a^2} e^t\,dt = \frac{1}{2}[e^t]_0^{a^2} = \frac{e^{a^2}-1}{2}\ \text{sq. units}$$

32. See Fig. 24.54.

We have $y = \sin x$

x	0	$\pi/6$	$\pi/2$	π	$3\pi/2$	2π
y	0	0.5	1	0	–1	0

Join these points with a free hand to obtain a rough sketch.

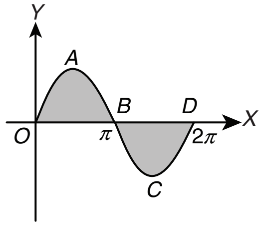


Figure 24.54

Required area = (area of OAB) + (area of BCD)

$$= \int_0^{\pi} y\,dx + \int_{\pi}^{2\pi} (-y)\,dx$$

(As area BCD is below x-axis)

$$= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx = 4\ \text{sq. units}$$

33. Required area $= \int_1^4 x\,dy = \int_1^4 \dfrac{\sqrt{y}}{2}\,dy$

$$= \frac{1}{2}\cdot\frac{2}{3}|\,y^{3/2}\,|_1^4 = \frac{7}{3}\ \text{sq. units}$$

34. See Fig. 24.55. Required area is

$$\int_{-1}^{2} y\,dx = \int_{-1}^{0} y.dx + \int_0^2 y.dx = \frac{5}{2}\ \text{sq. units}$$

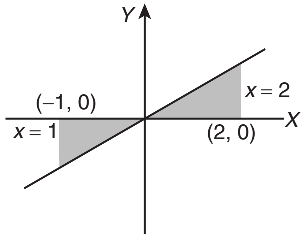


Figure 24.55

35. See Fig. 24.56. Let the ordinate at $x = a$ divide the area into two equal parts

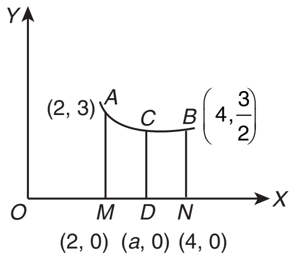


Figure 24.56

Area of $AMNB$ $= \int_2^4 \left(1 + \dfrac{8}{x^2}\right) dx = \left[x - \dfrac{8}{x}\right]_2^4 = 4$

Area of $ACDM = \int_2^a \left(1 + \frac{8}{x^2}\right) dx = 2$

On solving, we get $a = \pm 2\sqrt{2}$. Since $a > 0 \Rightarrow a = 2\sqrt{2}$.

36. $y = \cos x$, When $x \in \left[0, \frac{\pi}{2}\right]$, $\cos x \geq 0$

When $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\cos x \leq 0$

When $x \in \left[\frac{3\pi}{2}, 2\pi\right]$, $\cos x \geq 0$

Thus required area is given by,

$$\int_0^{2\pi} y dx = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/2} (-\cos x) dx + \int_{3\pi/2}^{2\pi} \cos x dx$$

$$= 1 + 2 + 1 = 4 \text{ sq. units}$$

37. Required area $= \int_1^4 x^3 dx = \left[\frac{x^4}{4}\right]_1^4 = \frac{255}{4}$ sq. units

38. Required area $= \int_1^4 y\, dx = c\int_1^4 \frac{1}{x} dx = 2c \log 2$ sq. units

39. Required area $= k\int_{\pi}^{2\pi} \sin x\ dx = k[-\cos x]_{\pi}^{2\pi} = -2k$

Hence, area $= 2k$ sq. units

40. See Fig. 24.57. Required area is

$$A_1 + A_2 = \int_0^{\pi} y\ dx + \left|\int_{\pi}^{2\pi} y\, dx\right| = 4\pi \text{ sq. units}$$

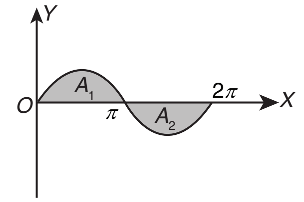


Figure 24.57

41. See Fig. 24.58.

$$A_1 = \int_0^{1/2} \sqrt{x} - x^2 = \frac{2x^{3/2}}{3} - \frac{x^3}{3}\Bigg|_0^{1/2}$$

$$= \frac{1}{3}\left[\frac{2}{2\sqrt{2}} - \frac{1}{8}\right] = \frac{1(4\sqrt{2}-1)}{3 \times 8} = \left(\frac{4\sqrt{2}-1}{24}\right)$$

$$A_2 = \int_{1/2}^{1} \sqrt{x} - x^2 = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_{1/2}^{1}$$

$$= \left[\frac{2}{3} - \frac{1}{3}\right] - \left[\frac{4\sqrt{2}-1}{24}\right] = \frac{1}{3} - \frac{\left(4\sqrt{2}-1\right)}{24}$$

$$= \frac{8 - 4\sqrt{2} + 1}{24} = \frac{9 - 4\sqrt{2}}{24}$$

Therefore, $\frac{A_1}{A_2} = \frac{4\sqrt{2}-1}{9-4\sqrt{2}}$

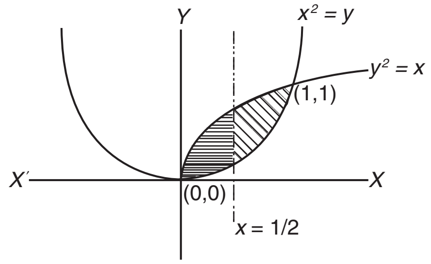


Figure 24.58

42. See Fig. 24.59.

$$x + |y| = 1 \Rightarrow |y| = 1 - x$$

$$\text{Area } = 2 \times \left(\frac{1}{2} \times 1 \times 1\right) = 1 \text{ sq. unit}$$

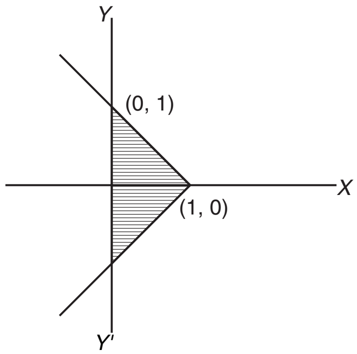


Figure 24.59

43. See Fig. 24.60.

$$A = \int_0^5 \left(e^x - e^{-x}\right) dx$$

$$= e^x + e^{-x}\Big|_0^5$$

$$= e^5 + e^{-5} - 2 \text{ sq. units}$$

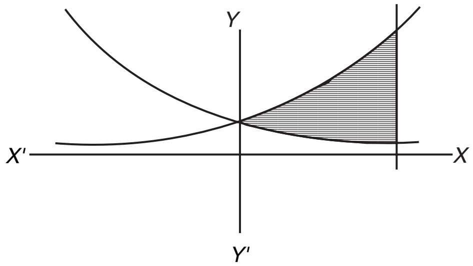


Figure 24.60

44. See Fig. 24.61.

$$(x^2 - y^2)(x^2 - y^2 - 8x + 16) = 0$$

$$(x^2 - y^2)[(x-4)^2 - y^2] = 0$$

$$y = \pm x$$

$$y = \pm (x - 4)$$

$$y = x - 4$$

$$y = -x + 4$$

Area $= \frac{1}{2} \times 2 \times 4$

Total area $= 2 \times 4 = 8$ sq. units

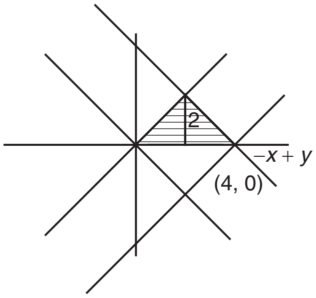


Figure 24.61

45. Given $f\left(\frac{x}{y}\right) = f(x) - f(y)$ (1)

Putting $x = y = 1$, we get $f(1) = 0$

Now, $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{h} \quad \text{(from (1))}$$

$$= \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right)}{\left(\frac{h}{x}\right)x}$$

$$\Rightarrow f'(x) = \frac{3}{x} \quad \left\{\text{since } \lim_{x\to 0} \frac{f(1+x)}{x} = 3\right\}$$

$$\Rightarrow f(x) = 3 \ \ln x + c$$

Putting $x = 1$

$$\Rightarrow c = 0$$
$$\Rightarrow f(x) = 3 \ln x = y \text{ (say)}$$

Therefore, required area $= \int\limits_{-\infty}^{3} x\,dy = \int\limits_{-\infty}^{3} e^{y/3}\,dy = 3\left[e^{y/3}\right]_{-\infty}^{3}$

$$= 3\,(e - 0) = 3e \text{ sq. units}$$

46. (i) See Fig. 24.62. Obviously, $A_n = \int\limits_0^{\pi/4} \tan^n x dx$

$$A_n + A_{n-2} = \int\limits_0^{\pi/4} (\tan^n x dx + \tan^{n-2} x)dx$$

$$= \int\limits_0^{\pi/4} (\tan^{n-2} x \sec^2 x)dx = \int\limits_0^1 t^{n-2} dt = \frac{1}{n-1}$$

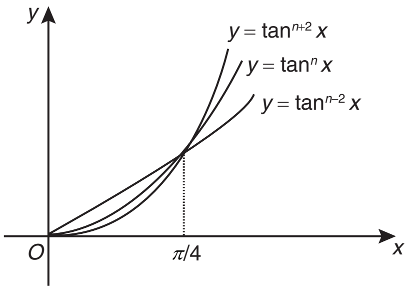


Figure 24.62

(ii) Obviously, $A_{n+2} < A_n < A_{n-2}$

Thus, $2A_n = A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$ (by part (i))

Thus, $$A_n < \frac{1}{2(n-1)} \quad (1)$$

Also $2A_n = A_n + A_n > A_n + A_{n+2} = \frac{1}{n+1}$, replacing n by $n+2$ in (i)

$$A_n > \frac{1}{2(n+1)} \quad (2)$$

From Eqs. (1) and (2), we get $\frac{1}{2(n+1)} < A_n < \frac{1}{2(n-1)}$.

47. Since $f(x)$ is continuous (Fig. 24.63).

So, it must be continuous at $x = 1, -1$, that is,

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1-} 2x = \lim_{x\to 1+} x^2 + ax + b = 2$$

$$\Rightarrow 2 = 1 + a + b = 2$$

$$\Rightarrow a + b = 1 \quad (1)$$

Also $\lim\limits_{x\to 1-} f(x) = \lim\limits_{x\to 1+} f(x) = f(-1)$

$$\Rightarrow \lim_{x\to 1-} x^2 + ax + b = \lim_{x\to 1+} 2x = -2$$

$$\Rightarrow 1 - a + b = -2$$
$$\Rightarrow -a + b = -3 \quad (2)$$

Solving Eqs. (1) and (2), we have

$$b = -1, a = 2$$

Therefore,

$$f(x) = \begin{cases} 2x & -1 \le x \le 1 \\ x^2 + ax + b, & x < 1, x > 1 \end{cases}$$

Now drawing the given curves.

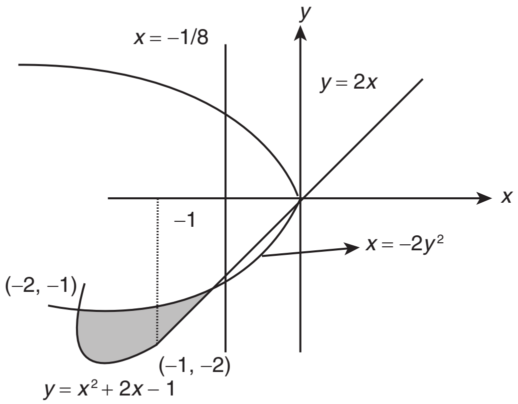


Figure 24.63

$$\text{Required area} = \int_{-2}^{-1}\left[\sqrt{\frac{-x}{2}} - (x^2+2x-1)\right]dx + \int_{-1}^{-1/8}\left[\sqrt{\frac{-x}{2}} - 2x\right]dx$$

$$= \frac{761}{192} \text{ sq. units}$$

48. See Fig. 24.64.

On the curve C_1, that is, $y = x^2$

Let P be (α, α^2). Hence, ordinate of point Q on C_2 is also α^2.

Now on C_2 $(y = 2x)$ the abscissa of Q is given by $x = \frac{y}{2} = \frac{\alpha^2}{2}$.

Therefore, Q is $\left(\frac{\alpha^2}{2}, \alpha^2\right)$ and R on C_3 is $\{\alpha, f(x)\}$.

Now area of OPQ is

$$\int_0^{\alpha^2}(x_1 - x_2)dy = \int_0^{\alpha^2}\left(\sqrt{y} - \frac{y}{2}\right)dy = \frac{2}{3}\alpha^3 - \frac{\alpha^2}{4} \quad (1)$$

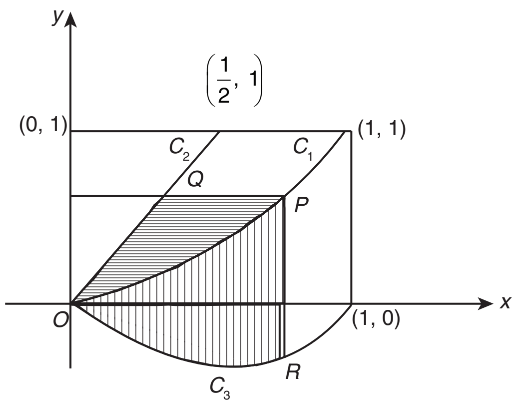


Figure 24.64

$$\text{Again, area of } \Delta ORP = \int_0^{\alpha}(y_1 - y_2)\,dx = \int_0^{\alpha}(x^2 - f(x))\,dx \quad (2)$$

From Eqs. (1) and (2)

$$\frac{2\alpha^3}{3} - \frac{\alpha^4}{4} = \int_0^{\alpha}(x^2 - f(x))\,dx$$

Differentiating both sides

$$2\alpha^2 - \alpha^3 = \alpha^2 - f(\alpha)$$
$$f(\alpha) = \alpha^3 - \alpha^2 \quad \Rightarrow f(x) = x^3 - x^2$$

49. See Fig. 24.65. $y = x^2$ is parabola (C_1) with vertex (0, 0) and passing through B (1, 1), $y = (1-x)^2$ is parabola (C_2) with vertex at A (1, 0) and passing through D (0, 1).

$y = 2x(1-x) = 2x - 2x^2 = -2\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}$ is parabola with vertex at H $\left(\frac{1}{2}, \frac{1}{2}\right)$.

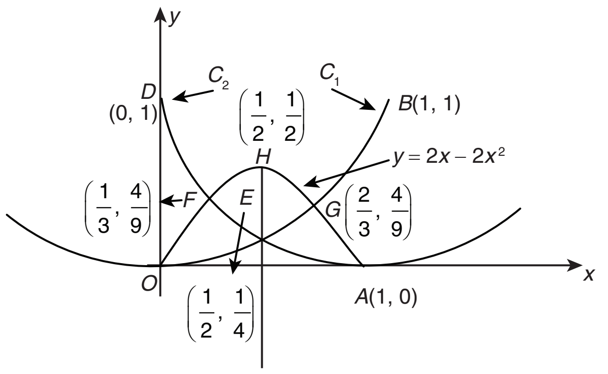


Figure 24.65

$y = x^2$ and $y = (1-x)^2$ meet at $E\left(\frac{1}{2}, \frac{1}{4}\right)$ and $y = 2x - 2x^2$ meets $y = (1-x)^2$ at $F\left(\frac{1}{3}, \frac{4}{9}\right)$ and $y = x^2$ at $G\left(\frac{2}{3}, \frac{4}{9}\right)$

$$\text{Therefore, } f(x) = \begin{cases} (1-x)^2 & 0 \le x \le \frac{1}{3} \\ 2x(1-x) & \frac{1}{3} \le x \le \frac{2}{3} \\ x^2 & 1 \ge x > \frac{2}{3} \end{cases}$$

Hence,

$$\text{Area} = \int_0^{\frac{1}{3}}(1-x)^2 dx + \int_{\frac{1}{3}}^{\frac{2}{3}}(2x - 2x^2)\,dx + \int_{\frac{2}{3}}^{1} x^2 dx$$

$$= \left[\frac{-(1-x)^3}{3}\right]_0^{\frac{1}{3}} + \left[x^2 - \frac{2x^3}{3}\right]_{\frac{1}{3}}^{\frac{2}{3}} + \left[\frac{x^3}{3}\right]_{\frac{2}{3}}^{1}$$

$$= \frac{-8}{81} + \frac{1}{3} + \frac{4}{9} - \frac{16}{81} - \frac{1}{9} + \frac{2}{81} + \frac{1}{3} - \frac{8}{81}$$

$$= 1 - \frac{30}{81} = \frac{51}{81} = \frac{17}{27} \text{ sq. units}$$

50. $(1 + \tan x)^n > (\tan x)^n$ so, we can replace

$$A_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx = \int_0^{\frac{\pi}{4}} \tan^{n-2} x(\sec^2 x - 1)\,dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - A_{n-2}$$

Therefore,

$$A_n + A_{n-2} = \frac{1}{n-1} \quad (1)$$

Again in the interval 0 to $\frac{\pi}{4}$, $\tan x$ is positive and < 1

Therefore,

$$\tan^n x \le \tan^{n-2} x$$
$$\Rightarrow A_n \le A_{n-2}$$
$$\Rightarrow A_n + A_n < A_{n-2} + A_n \left(= \frac{1}{n-1}\right) \quad (1)$$

So,

$$A_n < \frac{1}{2(n-1)} \qquad (2)$$

Now again $\quad A_{n+2} < A_n$

Therefore,

$$A_{n+2} + A_n < A_n + A_n$$

$$\Rightarrow \frac{1}{(n+2)-1} < 2A_n$$

$$\Rightarrow \frac{1}{2n+2} < A_n \qquad (3)$$

Hence, from Eqs. (2) and (3) we get

$$\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$$

51. See Fig. 24.66. $y = 4x - x^2 = 4 - (x-2)^2$

Therefore, $(x-2)^2 = -(y-4)$ is a parabola with vertex at (2, 4) and upwards.

Again $y = x^2 - x = \left[x - \frac{1}{2}\right]^2 - \frac{1}{4}$

Therefore, $\left(x - \frac{1}{2}\right)^2 = y + \frac{1}{4}$ is a parabola with vertex B at $\left(\frac{1}{2}, -\frac{1}{4}\right)$ and downwards.

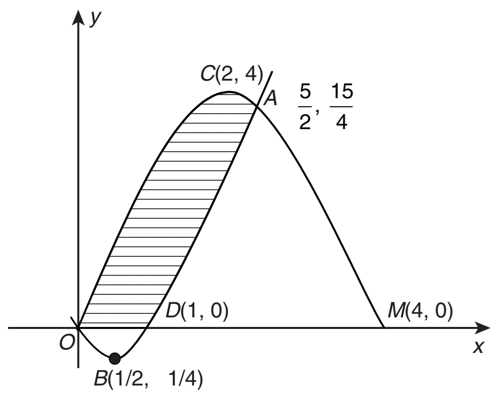


Figure 24.66

The two parabolas intersect at $O(0, 0)$ and $A\left(\frac{5}{2}, \frac{15}{4}\right)$.

$$\text{Area } OBD = \left|\int_0^1 (x^2 - x)\,dx\right| = \left|\frac{x^3}{3} - \frac{x^2}{2}\right|_0^1 = \frac{1}{6}$$

$$\text{Area } ODAC = \int_0^{\frac{5}{2}} (\underset{\text{up}}{y_1} - \underset{\text{down}}{y_2})\,dx = \int_0^{\frac{5}{2}} [(4x - x^2) - (x^2 - x)]\,dx$$

$$= \int_0^{\frac{5}{2}} (5x - 2x^2)\,dx = \left[\frac{5x^2}{2} - \frac{2x^3}{3}\right]_0^{\frac{5}{2}} = \frac{125}{8} - \frac{250}{24} = \frac{125}{24}$$

$$\text{Lined area } = \frac{125}{24} - \frac{1}{6} = \frac{121}{24}$$

Therefore, ratio $= \frac{121}{24} : \frac{1}{6} = \frac{121}{4}$.

52. Area of shaded region (Fig. 24.67) is

$$\int_{-1}^{1}\left(\frac{2}{1+x^2} - x^2\right)dx = 2\int_0^1\left(\frac{2}{1+x^2} - x^2\right)dx$$

$$= 2\left[2\tan^{-1}x - \frac{x^3}{3}\right]_0^1 = 2\left[\left(\frac{\pi}{2}\right) - \frac{1}{3}\right] = \left(\pi - \frac{2}{3}\right) \text{ sq. units}$$

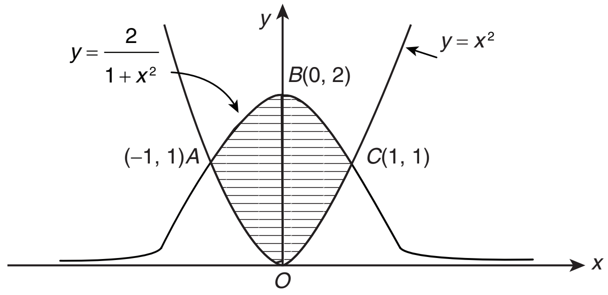


Figure 24.67

53. See Fig. 24.68. A is point of intersection of $x = \frac{1}{2}$ and $y = 2^x$.

So, A is $\left(\frac{1}{2}, \sqrt{2}\right)$.

B is point of intersection of $x = \frac{1}{2}$ and $y = \ln x$.

So, B is $\left(\frac{1}{2}, -\ln 2\right)$.

C is point of intersection of $x = 2$ and $y = \ln x$.
So, C is (2, ln2).

D is point of intersection of $x = 2$ and $y = 2^x$.
So, D is (2, 4).

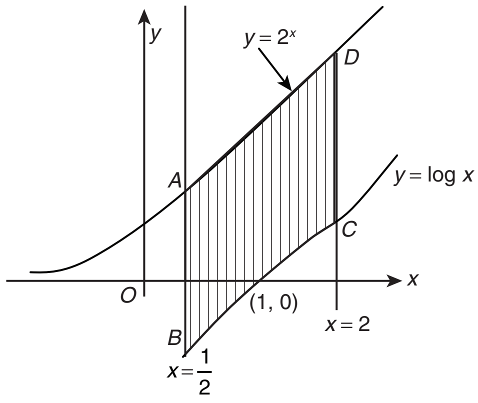


Figure 24.68

Area $ABCD = \int_{\frac{1}{2}}^{2} (y_1 - y_2)\,dx$, where $y_1 = 2^x > \ln x = y^2$ for all $x \in \left[\frac{1}{2}, 2\right]$.

$$\Rightarrow \int_{\frac{1}{2}}^{2} (2^x - \ln x)\,dx = \left[\frac{2^x}{\ln 2}\right]_{\frac{1}{2}}^{2} - [x\ln x - x]_{\frac{1}{2}}^{2}$$

$$= \frac{4}{\ln 2} - \frac{\sqrt{2}}{\ln 2} - (2\ln 2 - 2) + \frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}$$

$$= \frac{(4-\sqrt{2})}{\ln 2} + \frac{3}{2} - \left(\frac{5}{2}\right)\ln 2 \text{ sq. units}$$

54. See Fig. 24.69. Given curves are

$$y = ex\ln x \quad (1)$$

$$y = \frac{\ln x}{ex} \quad (2)$$

Points of intersection of (1) and (2) are $P\left(\frac{1}{e}, -1\right)$ and Q(1, 0)
For curve (1), $y < 0$ for $0 < x < 1$
and $y \geq 0$ for $x \geq 1$

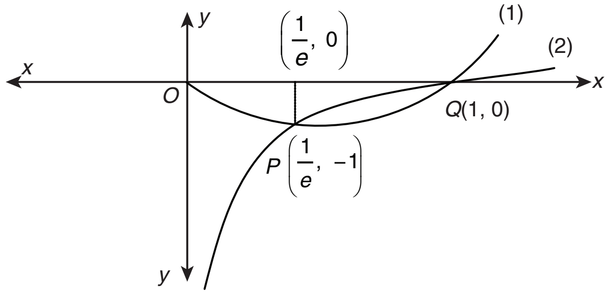


Figure 24.69

Obviously $y \to 0$ when $x \to 0$
For curve (2), $y \to -\infty$ when $x \to 0$
$y < 0$ for $0 < x < 1$, $y \geq 0$ for $x \geq 1$
Obviously $y \to 0$ when $x \to \infty$
This shape of curves is depicted in Figure 24.65.

$$\text{Required area } = \left|\int_{\frac{1}{e}}^{1} ex\ln x\, dx\right| - \left|\int_{\frac{1}{e}}^{1}\frac{\ln x}{ex}dx\right|$$

$$= \left|\frac{1}{4e}(3-e^2)\right| - \left|\frac{-1}{2e}\right| = \frac{e^2-3}{4e} - \frac{1}{2e} = \frac{e^2-5}{4e} \text{ sq. units}$$

Alternatively, area is

$$\int_{\frac{1}{e}}^{1}\left(\frac{\ln x}{ex} - ex\ln x\right)dx = \left[\frac{1}{2e}(\ln x)^2 - \frac{1}{4}ex^2(2\ln x - 1)\right]_{\frac{1}{e}}^{1}$$

$$= \frac{1}{2e} - \frac{1}{4e}(3-e^2) = \frac{e^2-5}{4e} \text{ sq. units}$$

55. See Fig. 24.70.
$f(x) = x(x-1)^2$
$f'(x) = 1\cdot(x-1)^2 + x\cdot 2(x-1) = (x-1)(3x-1) = 0$

$$\Rightarrow x = 1, \frac{1}{3}$$

$f''(x) = 1\cdot(3x-1) + (x-1)3 = 6x - 4$
At $x = 1$, $f''(1) = 2 > 0$ min. value $= 0$

At $x = \frac{1}{3}$, $f''\left(\frac{1}{3}\right) = -2 < 0$ max. value $= \frac{4}{27}$

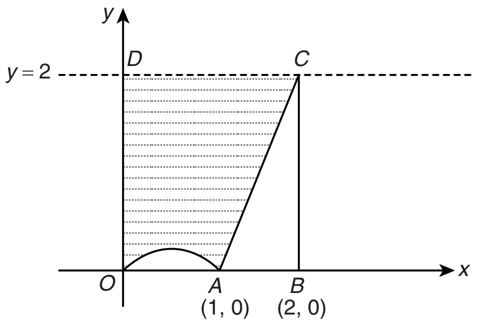


Figure 24.70

Required shaded area = Area $OBCD - \int_0^2 y\,dx$ where $y = x(x-1)^2$

$$= 2\cdot 2 - \int_0^2 x(x^2-2x+1)dx = 4 - \left[\frac{x^4}{4} - 2\left(\frac{x^3}{3}\right) + \left(\frac{x^2}{2}\right)\right]_0^2$$

$$= 4 - \left[4 - \frac{16}{3} + 2\right] = 4 - \frac{2}{3} = \frac{10}{3} \text{ sq. units}$$

56. See Fig. 24.71.
Required area (shaded) is

$$\int_0^{\frac{\pi}{4}} \tan x\, dx - \text{area of } \Delta PMQ$$

$$= [\log\sec x]_0^{\frac{\pi}{4}} - \frac{1}{2}\times QM\times MP$$

Equation of tangent at P is

$$y - 1 = \left.\frac{dy}{dx}\right|_{\frac{\pi}{4}}\left(x - \frac{\pi}{4}\right)$$

$$\left.\frac{dy}{dx}\right|_{\frac{\pi}{4}} = \left.\sec^2 x\right|_{\frac{\pi}{4}} = 2$$

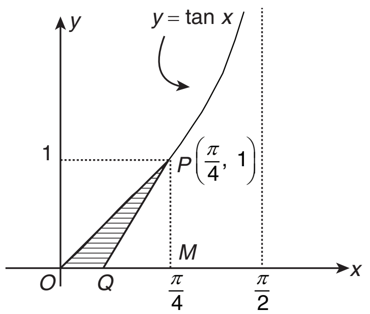


Figure 24.71

Equation of tangent is $y - 1 = 2\left(x - \frac{\pi}{4}\right)$.

When $y = 0$, $x = \frac{\pi}{4} - \frac{1}{2} = OQ$.

Therefore,

$$QM = \frac{\pi}{4} - OQ = \frac{1}{2}$$

Hence,

$$\text{Area} = \left(\frac{1}{2}\right)\log 2 - \frac{1}{2}\times\frac{1}{2}\times 1 = \left[\left(\frac{1}{2}\right)\log 2 - \left(\frac{1}{4}\right)\right] \text{ sq. units}$$

57. See Fig. 24.72.

Points of intersection of $y = x$ and $x^2 = -\sqrt{2}\,y$ are

$$x = 0, -\sqrt{2}$$

$$\text{Area} = 2\int_0^{\sqrt{2}}\left(-\frac{x^2}{\sqrt{2}} + \sqrt{4-x^2}\right)dx - \int_{-\sqrt{2}}^{0}\left(\frac{-x^2}{\sqrt{2}} - x\right)dx$$

$$= 2\left[\frac{-x^3}{3\sqrt{2}} + \frac{1}{2}x\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{2}} + \left[\frac{x^2}{2} + \frac{x^3}{3\sqrt{2}}\right]_{-\sqrt{2}}^{0}$$

$$= 2\left[\frac{-2}{3} + 1 + 2\times\frac{\pi}{4}\right] - \left[1 - \frac{2}{3}\right] = 2\left[\frac{1}{3} + \frac{\pi}{2}\right] - \frac{1}{3} = \pi + \frac{1}{3} \text{ sq. units}$$

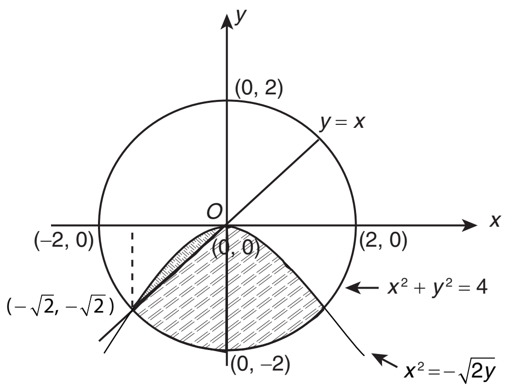


Figure 24.72

58. See Fig. 24.73.

Required area $= \frac{1}{2} \times 2 \times 2 = 2$ sq. units

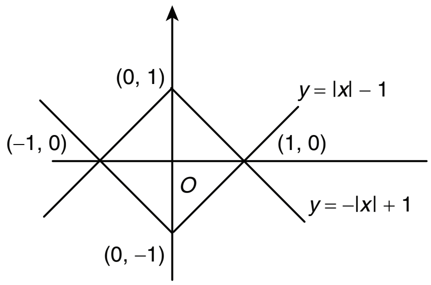


Figure 24.73

59. Area of equilateral triangle $= \frac{\sqrt{3}}{4}(\text{side})^2$.

Two vertices are at $(0, 0)$ and $[x, g(x)]$

Hence, side $= \sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$

$$\text{Area} = \left|\frac{\sqrt{3}}{4}\left[\sqrt{x^2 + (g(x))^2}\right]^2\right|$$

$$\Rightarrow \pm\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}\left\{\sqrt{x^2 + (g(x))^2}\right\}^2$$

$$\Rightarrow x^2 + (g(x))^2 = 1 \qquad (\text{since } x^2 + (g(x))^2 \neq -1)$$

$$\Rightarrow (g(x))^2 = 1 - x^2 \Rightarrow g(x) = \pm\sqrt{1-x^2}$$

60. See Fig. 24.74. Clearly

$$g(x) = \begin{cases} 2, & -2 \le x < 0 \\ x^2 - x, & 0 \le x \le \frac{1}{2} \\ -\frac{1}{4}, & \frac{1}{2} < x \le 3 \end{cases}$$

$$\text{Area} = \int_{-2}^{0} 2\,dx + \int_0^{1/2}(x - x^2)\,dx + \int_{1/2}^{3}\frac{1}{4}\,dx$$

$$= (2x)_{-2}^{0} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^{1/2} + \left(\frac{x}{4}\right)_{1/2}^{3} = \frac{113}{24} \text{ sq. units}$$

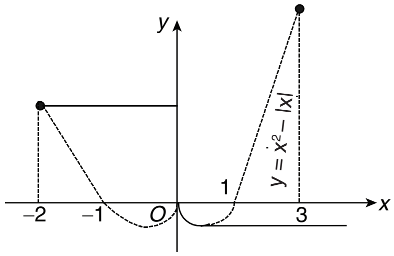


Figure 24.74

61. See Fig. 24.75. The expression $|x-y| + |x+y| \le 8$, represents the interior region of the square formed by the lines $x = \pm 4$, $y = \pm 4$ and $xy \ge 2$. Represents the region lying inside the hyperbola $xy = 2$.

Required area is

$$= 2\int_{1/2}^{4}\left(4 - \frac{2}{x}\right)dx = 2(4x - 2\ln x)_{1/2}^{4}$$

$$= 4(7 - 3\ln 2) = 4(7 - \ln 8) \text{ sq. units}$$

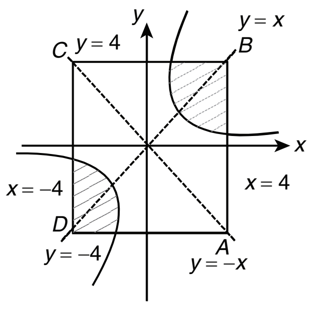


Figure 24.75

62. See Fig. 24.76. Area bounded by $y = \sqrt{2}\cdot\sin\frac{\pi x}{4}$ and x-axis between the lines $x = 2$ and $x = 4$,

$$\Delta = \sqrt{2}\int_2^4 \sin\frac{\pi x}{4}\,dx = -\frac{4\sqrt{2}}{\pi}\cdot\cos\frac{\pi x}{4}\Bigg|_2^4$$

$$=\frac{4\sqrt{2}}{\pi}\text{ sq. units}$$

Let the drawn lines are $L_1: y-m_1(x-4)=0$ and $L_2: y-m_2(x-4)=0$, meeting the line $x=2$ at the points A and B, respectively. Clearly $A=(2,-2m_1)$; $B=(2,-2m_2)$ (Fig. 24.76).

Now

$$\Delta_{ACD}=\frac{\Delta}{3}\Rightarrow\frac{4\sqrt{2}}{3\pi}=\frac{1}{2}\cdot 2\cdot -2m_1$$

$$\Rightarrow m_1=-\frac{2\sqrt{2}}{3\pi}\quad\text{Also } \Delta_{BCD}=\frac{2\Delta}{3}$$

$$\Rightarrow\frac{8\sqrt{2}}{3\pi}=\frac{1}{2}\cdot 2-2m_2$$

$$\Rightarrow m_2=\frac{-4\sqrt{2}}{3\pi}$$

$$\text{Required sum}=-\frac{2\sqrt{2}}{\pi}$$

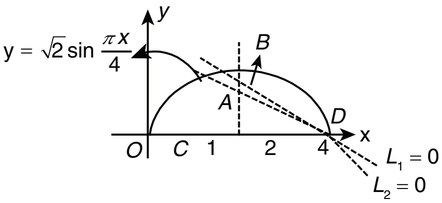


Figure 24.76

63. $$y=x(3-x)^2$$

After solving, we get $x=1$ and $x=3$ which are points of maximum and minimum, respectively.
Now the shaded region is the required region (Fig. 24.77).

Therefore, $A=\int_1^3 x(3-x)^2\,dx=4$ sq. units

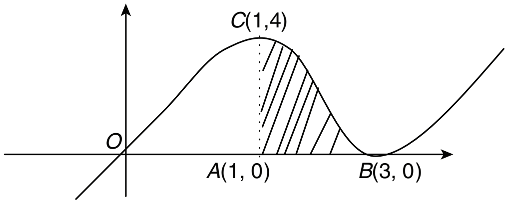


Figure 24.77

64. By definition, the lines max, $(x, y)=1$ means
$x=1$ and $y\le 1$ or $y=1$ and $x\le 1$

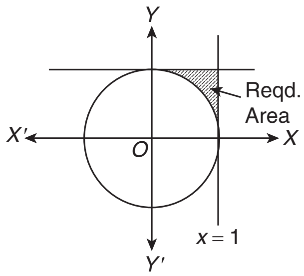


Figure 24.78

See Fig. 24.78. Required area is $\int_0^1\left[1-\sqrt{1-x^2}\right]dx$.

$$=\left[x-\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=1-0-\frac{1}{2}\left(\frac{\pi}{2}\right)=1-\frac{\pi}{4}\text{ sq.units.}$$

65. See Fig. 24.79.

$$\text{Required area}=\int_0^3|y|\,dx=\int_0^1|y|\,dx+\int_1^2|y|\,dx+\int_2^3|y|\,dx$$

$$=-\left(\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x\right)_0^1+\left(\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x\right)_1^2$$

$$-\left(\frac{x^4}{4}-2x^3+\frac{11}{2}x^2-6x\right)_2^3$$

$$=\frac{9}{4}+\frac{1}{4}+\frac{1}{4}=\frac{11}{4}\text{ sq. units}$$

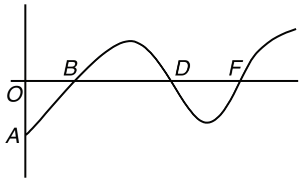


Figure 24.79

66. See Fig. 24.80.

$$A=\int_0^1(\sqrt{x}-x^3)\,dx$$

$$=\frac{2}{3}\left|x^{3/2}\right|_0^1-\left|\frac{x^4}{4}\right|_0^1$$

$$=\frac{2}{3}-\frac{1}{4}=\frac{8-3}{12}=\frac{5}{12}$$

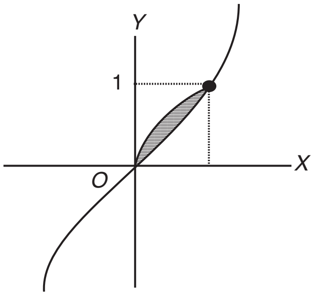


Figure 24.80

67.

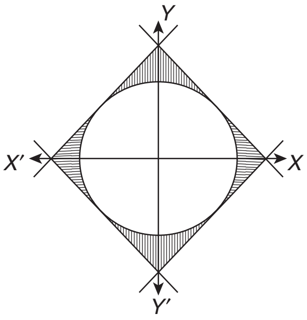


Figure 24.81

Shaded region is the required one. Therefore, required area is

$$4\times\frac{1}{2}\times 2\times 2-\pi\cdot 2=8-2\pi\text{ sq. units}$$

Practice Exercise 2

1. Solving $y = x - x^2$ and $y = mx$, we get the points of intersection (0, 0) and $[1-m, m(1-m)]$.

Case I: $m < 1$ (Fig. 24.82)

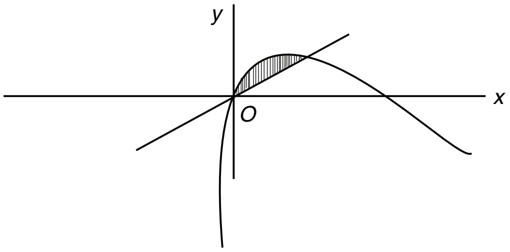


Figure 24.82

$$\text{Shaded area} = \frac{9}{2}$$

$$\Rightarrow \int_0^{1-m} \left(x - x^2\right)dx - \int_0^{1-m} mx\ dx = \frac{9}{2}$$

Solving this, we get $m = -2$

Case II: $m > 1$ (Fig. 24.83)

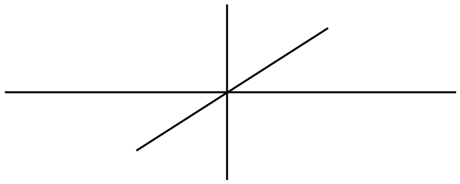


Figure 24.83

$$\text{Shaded area} = \frac{9}{2}$$

$$\Rightarrow \int_{1-m}^{0} (-mx)\ dx - \int_{1-m}^{0} (x^2 - x)\, dx = \frac{9}{2}$$

Solving this, we get $m = 4$

2. See Fig. 24.84.

$$\frac{\text{Area of } ABC}{\text{Area of } PQM} = \frac{BC^2}{QM^2}$$

$$\frac{\text{Area of } ABC}{\text{Area of } MNP} = \frac{BC^2}{NP^2}$$

$$\frac{\text{Area of } ABC}{\text{Area of } MOL} = \frac{BC^2}{ML^2}$$

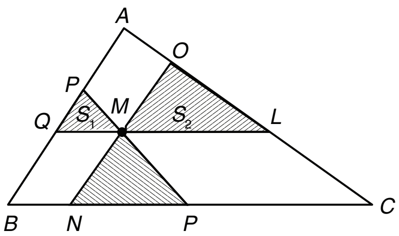


Figure 24.84

$$\sqrt{\text{Area of } \Delta ABC} = QM = \frac{BC\sqrt{S_1}}{\sqrt{S}}$$

$$PM = \frac{BC\sqrt{S_3}}{\sqrt{S}},\ ML = \frac{BC\sqrt{S_2}}{\sqrt{S}} \qquad (QM = BN \text{ and } ML = PC)$$

$$BC = BC\left(\frac{\sqrt{S_1}}{\sqrt{S}} + \frac{\sqrt{S_2}}{\sqrt{S}} + \frac{\sqrt{S_3}}{\sqrt{S}}\right)$$

$S = (\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2$, where S is the area of ΔABC.

3.
$$\int_{\alpha-t}^{\alpha} f(t)\,dt = -\int_{\alpha}^{\alpha+t} f(t)\,dt$$

$$\Rightarrow f(\alpha - t) = -f(\alpha + t)\ \forall\ t \in R$$

Therefore, $y = f(x)$ is symmetrical about point $(\alpha, 0)$.

4. Putting $x = \alpha$ in the given equation, we have

$$f(0) = -f(2\alpha) \Rightarrow -f(0) = f(2\alpha)$$

5. $\int_{\alpha-t}^{\alpha} f(t)\,dt = -\int_{\alpha}^{\alpha+t} f(t)\,dt \Rightarrow f(\alpha - t) = -f(\alpha + t)\ \forall\ t \in R.$

$$\Rightarrow f(\alpha - t) = -f(\alpha + t) = x \Rightarrow t = \alpha - f^{-1}(x) = f^{-1}(-x) - a$$

$$\Rightarrow f^{-1}(x) + f^{-1}(-x) = 2\alpha \Rightarrow \int_{-\beta}^{\beta} f^{-1}(x)\,dx = 2\alpha\beta$$

6. $x = 1$ is a root of $f(x)$ and also a root of 1st, 2nd and 3rd derivatives of $f(x)$. Hence, $f(x)$ has $x = 1$ repeated root 4 times so $f(x) = (x-1)^4$. Therefore,

$$\text{Required area} = \left|\int_1^2 (x - 1 - (x-1)^4)\,dx\right| = \frac{3}{10} \text{ sq. units}$$

7. $\text{Required area} = \left|\int_0^1 (\sqrt{1-x^2} - (x-1)^4)\,dx\right| = \frac{\pi}{4} - \frac{1}{5}$ sq. units

8. $\text{Required area} = 6 + \left|\int_0^1 (x-1)^4\,dx + \int_1^3 (x-1)^4\,dx\right| = \frac{63}{5}$ sq. units

9. Equation of circle is $x^2 + y^2 = 2$.
Equation of tangent at A is $-x + y = 2$
Equation of tangent at D is $-x - y = 2$
Form maximum area of the rectangle $QEKF$, Q is $(\sqrt{2}, 0)$

$$\text{Area of rectangle} = \frac{-\sqrt{2} - 2}{\sqrt{2}} \times \frac{-\sqrt{2} - 2}{\sqrt{2}} = \frac{6 + 4\sqrt{2}}{2} = 3 + 2\sqrt{2}$$

10. Area of $\Delta BQC = \sqrt{2} - 1$

11. Area of $\Delta OQM = \frac{1}{2}\sqrt{2} \times \sqrt{2} = 1$

12. $f(x+y) = \dfrac{f(x) + f(y)}{f(x)}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(f(x) + f(h)) - (f(x) + f(0))}{hf(x)}$$

$$= \frac{f'(0)}{f(x)} = \frac{2}{f(x)}$$

$$\Rightarrow f(x)\,f(x)\,dx = \int 2\,dx \Rightarrow f(x) = \sqrt{4(x+c)}$$

Since, $f(0) = 2 \Rightarrow c = 1$.

Therefore, $f(x) = 2\sqrt{(x+1)}$.

13. See Fig. 24.85.

The required area $= 2\int_0^3 (7 - x - 2\sqrt{x+1})\,dx$

$$= 2\left|7x - \frac{x^2}{2} - \frac{4}{3}(x+1)^{3/2}\right|_0^3$$

$$= 2\left[21 - \frac{9}{2} - \frac{4}{3}(8-1)\right] = 2\times\frac{43}{6} = \frac{43}{3} \text{ sq. units}$$

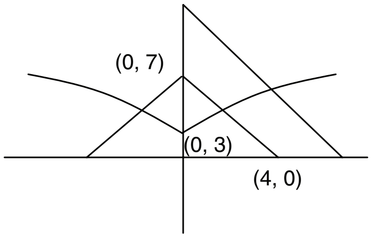


Figure 24.85

14. See Fig. 24.86.

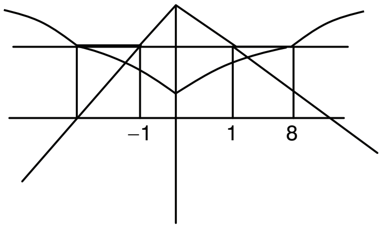


Figure 24.86

Hence, the number of points is 5.

15. $f(xy) = \frac{f(x)}{y} + \frac{f(y)}{x}$

$$f(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\left(\frac{1}{1+\frac{h}{x}} - 1\right)}{h} + \frac{f\left(1+\frac{h}{x}\right) - f(1)}{xh}$$

$$= \lim_{h\to 0}\frac{-f(x)}{x\left(1+\frac{h}{x}\right)} + \frac{f'(1)}{x^2}$$

$$\Rightarrow f(x) = \frac{-f(x)}{x} + \frac{1}{x^2} \text{ or } f'(x) + \frac{f(x)}{x} = \frac{1}{x^2}$$

This is a linear differential equation.

Solution is $f(x)\cdot x = \ln x + c$

or

$$f(x) = \frac{\ln x}{x},\ c = 0 \text{ since, } (f(1) = 0)$$

Now,

$$\lim_{x\to\infty}\frac{\ln x}{x} \qquad \left(\frac{\infty}{\infty}\right) \text{ form}$$

Applying L' Hospital's rule,

$$\lim_{x\to\infty}\frac{1}{x} = 0$$

16. $g(x) = \int_0^x f(t)\,dt = \int_0^x \frac{\ln t}{t}\,dt$

$$\Rightarrow g\left(\frac{1}{x}\right) = \int_0^{1/x}\frac{\ln t}{t}\,dt$$

Let $t = \frac{1}{u}$. Then

$$\int_0^x \frac{\ln u}{u}\,du$$

$$\Rightarrow g\left(\frac{1}{x}\right) = g(x)$$

Therefore, $g(n) - g\left(\frac{1}{n}\right) = 0$.

17. For points of intersection,

$$\frac{\ln x}{x} = ex\ln x \Rightarrow \frac{\ln x\left(1 - ex^2\right)}{x} = 0$$

$$\ln x = 0 \Rightarrow x = 1$$

$$1 - ex^2 = 0 \Rightarrow x = \frac{1}{\sqrt{e}}$$

Therefore,

$$\text{Required area} = \left|\int_{1/\sqrt{e}}^{1}\left(\frac{\ln x}{x} - ex\ln x\right)dx\right| = \frac{2e-5}{8} \text{ sq. units}$$

18. See Fig. 24.87.

(A) Required area $= 2\left[\frac{1}{2}(2+1)\times 1 - \int_0^1 x^2\,dx\right]$

$$= 2\left[\frac{3}{2} - \frac{1}{3}\right] = \frac{7}{3} \text{ sq. units}$$

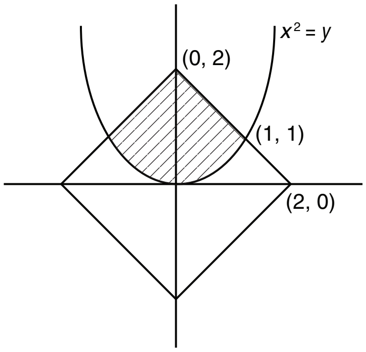


Figure 24.87

(B) $$f(x) = 3\sin x - 4\cos x - \frac{7}{3}$$

$$f'(x) = 3\cos x + 4\sin x = 0$$

$$\Rightarrow \cos x = \frac{4}{5}, \sin x = -\frac{3}{5}$$

and $\cos x = -\frac{4}{5}, \sin x = \frac{3}{5}$

Now,

$$f''(x) = -3\sin x + 4\cos x$$

At $\cos x = \frac{4}{5}, \sin x = -\frac{3}{5}$, $f''(x)$ is positive, that is, $f(x)$ has minimum value.

At $\cos x = -\frac{4}{5}, \sin x = \frac{3}{5}$, $f''(x)$ is negative, that is, $f(x)$ has minimum value. So,

$$f(x) = 3\sin x - 4\cos x - \frac{7}{3} = \frac{9}{5} + \frac{16}{5} - \frac{7}{3} = \frac{8}{3}$$

(C) Length of the chord $= \frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}} = \frac{2\times 3\times 4}{5} = \frac{24}{5}$

(D) See Fig. 24.88.

Parabola is $y^2 = 4(x+1)$ (1)

Focus is (0, 0)

Equation of AB is $\frac{x-0}{1/2} = \frac{y-0}{\sqrt{3}/2} = r$

Substituting parametric coordinates in Eq. (1), we have

$$\left(\frac{\sqrt{2}}{2}r\right)^2 = 4\left(\frac{r}{2} + 1\right)$$

$$\Rightarrow \frac{3r^2}{2} - 2r - 4 = 0$$

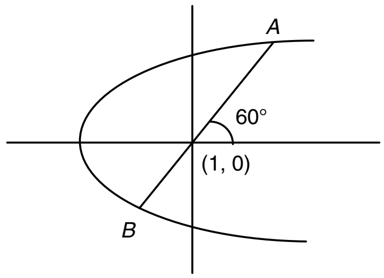


Figure 24.88

Length $AB = |PA - PB| = \sqrt{(PA+PB)^2 - 4PAPB}$

$$= \sqrt{\left(\frac{8}{3}\right)^2 - 4\times\frac{16}{3}} = \frac{16}{3}$$

19. (A) $f(x) = \sin^2 x \cos x$

$f(x) = p(x)\,q(x)$, (period of $f(x)$ will be LCM of period of $p(x)$ and $q(x)$).

LCM of π and 2π is 2π.

(B) $A = \int_0^b y\,dx = \int_0^b \frac{b}{a}\sqrt{a^2 - x^2}\,dx$

$$= \frac{ab}{2}[e\sqrt{1-e^2} + \sin^{-1}\sqrt{1-e^2}] = \frac{ab\pi}{8}$$

$$\Rightarrow e\sqrt{1-e^2} + \sin^{-1}\sqrt{1-e^2} = \frac{\pi}{4}$$

(C) $f(x) = \dfrac{\sin^{-1}\left(\frac{1}{x}\right) + \cos^{-1}\left(\frac{1}{x}\right)}{\operatorname{cosec} x} = \frac{\pi}{2}\sin x$

Greatest value of $\sin x = 0$.

So, greatest value of $f(x) = \frac{\pi}{2}$.

(D) $\cos^{-1}\cos\left(\frac{\pi}{14}\right) = \frac{\pi}{14}$

20. (A) $y = \tan^{-1}\frac{1}{2} + \tan^{-1} b$, $(0 < b < 1)$

$$\Rightarrow y = \tan^{-1}\left(\frac{1/2 + b}{1 - b/2}\right), \text{ since, } \left(\frac{1}{2}b < 1\right)$$

$$0 < \tan^{-1}\left(\frac{1+2b}{2-b}\right) \le \frac{\pi}{4} \Rightarrow 0 < \left(\frac{1+2b}{2-b}\right) \le 1$$

$$\Rightarrow 0 < (1+2b) \le (2-b), (1+2b > 0)$$

$$\Rightarrow 3b \le 1 \Rightarrow 0 \le \frac{1}{3} \Rightarrow b_{max} = \frac{1}{3}$$

(B) $\sin^4 x + \cos^3 x \ge 1$ (1)

Since $\sin^2 x + \cos^2 x = 1$ and $-1 \le \sin x, \cos x \le 1$

Eq. (1) cannot be >1

Therefore, $\sin^4 x + \cos^3 x = \pm 1$ (2)

Eq. (2) is possible if either,

$\sin x = 1$ and $\cos x = 0$ or $\sin x = 0$ and $\cos x = 1$

$$\Rightarrow x = (4n+1)\frac{\pi}{2}, x = (2n+1)\frac{\pi}{2} \text{ or } x = n\pi, x = 2n\pi$$

In $(0, 2\pi)$, $x = \frac{\pi}{2}, \frac{3\pi}{2}$ or $x = \pi$

Therefore, the number of solutions will be 3.

(C) See Fig. 24.89.

$$4y^2 = x^2(4 - x^2) \quad (1)$$

$$\Rightarrow y = \frac{1}{2}\sqrt{x^2(4-x^2)}$$

$$\Rightarrow y = \frac{x}{2}\sqrt{(4-x^2)}$$

Therefore,

$$\text{area } (A) = 4\times\int_0^2 \frac{2x}{2\times 2}\sqrt{(4-x^2)}dx$$

Let $(4-x^2)=t$. Then $-2x\,dx=dt$.

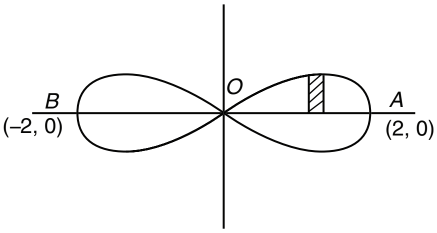


Figure 24.89

$$\Rightarrow A=\frac{-4}{4}\int_4^0\sqrt{t}\,dt=\int_0^4\sqrt{t}\,dt=\left[\frac{t^{3/2}}{3/2}\right]_0^4=\frac{2}{3}\times[\sqrt{64}-0]$$

$$\Rightarrow A=\frac{16}{3}\text{ sq. units}$$

21. Since $y=f(x)$ has relative extremities at $x=\pm\dfrac{2}{\sqrt{3}}$ these points are critical points and hence they must be roots of $f'(x)=0$ (Clearly f is differentiable everywhere). Therefore,

$$f'(x)=a\left(x-\frac{2}{\sqrt{3}}\right)\left(x+\frac{2}{\sqrt{3}}\right)=a\left(x^2-\frac{4}{3}\right)$$

$$f(x)=a\left(\frac{x^3}{3}-\frac{4x}{3}\right)+b$$

This passes through (0, 0) and (1, –2). So, $b=0$ and

$$a\left(\frac{1}{3}-\frac{4}{3}\right)=-2\Rightarrow a=2$$

Therefore, $f(x)=\dfrac{2x}{3}(x^2-4)$.

Hence, $f(x)$ meets the x-axis at (0, 0), (–2, 0) and (2, 0).

Since $f(-x)=-f(x)$, the curve $y=f(x)$ is symmetrical about the origin.

Also as $a=2$, $f'(x)>0$.

For $x<-\dfrac{2}{\sqrt{3}}$, $x>\dfrac{2}{\sqrt{3}}$ and $f'(x)<0$ for $-\dfrac{2}{\sqrt{3}}<x<\dfrac{2}{\sqrt{3}}$

Therefore,

$$\text{Area of region}=2\pi-\int_{-2}^0|f(x)|dx+\int_0^2 f(x)dx$$

$$\Rightarrow\int_{-2}^0|f(x)|dx=-\int_{-2}^0 f(x)dx=-\int_0^2 f(-t)(-1)\,dt=\int_0^2 f(t)dt$$

Required area $=2\pi$. So, integral part $=6$.

Solved JEE 2017 Questions

JEE Main 2017

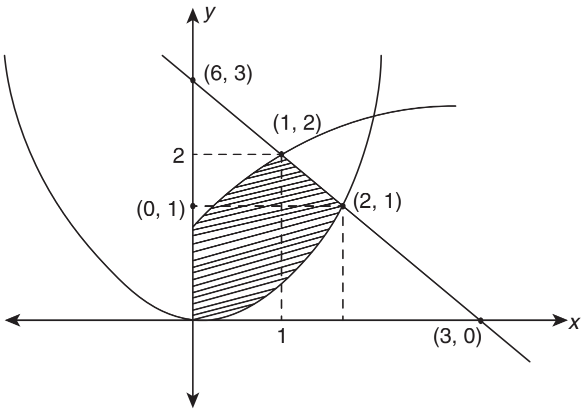
1. The area (in sq. units) of the region $\{(x, y): x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1+\sqrt{x}\}$ is

(A) $\frac{3}{2}$ **(B)** $\frac{7}{3}$

(C) $\frac{5}{2}$ **(D)** $\frac{59}{12}$

(OFFLINE)

Solution: The given situation is depicted in the following graph:



The area of the given region is

$$\int_0^1\left(1+\sqrt{x}-\frac{x^2}{4}\right)dx+\int_1^3\left(3-x-\frac{x^2}{4}\right)dx$$

$$=\left[x+\frac{x^{3/2}}{3/2}-\frac{x^3}{12}\right]_0^1+\left[3x-\frac{x^2}{2}-\frac{x^3}{12}\right]_1^2$$

$$=\frac{19}{12}+\frac{11}{12}=\frac{5}{2}$$

Hence, the correct answer is option (C).

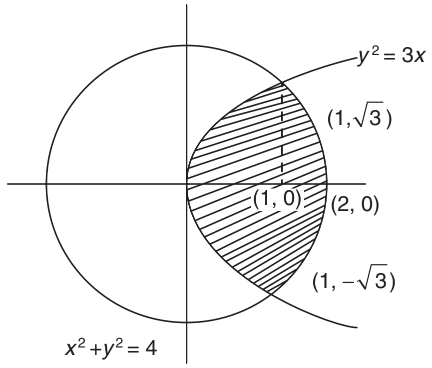
2. The area (in sq. units) of the smaller portion enclosed between the curves, $x^2 + y^2 = 4$ and $y^2 = 3x$, is

(A) $\frac{1}{2\sqrt{3}}+\frac{\pi}{3}$ **(B)** $\frac{1}{2\sqrt{3}}+\frac{2\pi}{3}$

(C) $\frac{1}{\sqrt{3}}+\frac{2\pi}{3}$ **(D)** $\frac{1}{\sqrt{3}}+\frac{4\pi}{3}$

(ONLINE)

Solution: The given equation $x^2 + y^2 = 4$ is equation of circle of radius 2 centred at origin and equation $y^2 = 3x$ is the equation of parabola.



$$x^2 + y^2 = 4 \quad (1)$$

$$y^2 = 3x \quad (2)$$

Substituting Eq. (2) in Eq. (1), we get

$$x^2 + 3x - 4 = 0$$

$$\Rightarrow x^2 + 4x - x - 4 = 0$$

$$\Rightarrow x(x + 4) - 1(x + 4) = 0$$

$$\Rightarrow (x - 1)(x + 4) = 0$$

$$\Rightarrow (x - 1) = 0 \text{ and } (x + 4) = 0$$

Therefore, $x = 1, -4$. Considering $x = 1$, then from Eq. (2), we get $y = \sqrt{3}, -\sqrt{3}$.

Thus, $(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are the points of intersection of parabola and circle.

The required area (A) is the area of the shaded region shown in the figure. Therefore,

$$A = 2\left[\int_0^1 y_2 dx + \int_1^2 y_1 dx\right]$$

From Eq. (1), we get

$$y_1 = \sqrt{4 - x^2}$$

From Eq. (2), we get

$$y_2 = \sqrt{3x}$$

Therefore,

$$A = 2\left[\int_0^1 \sqrt{3x}\, dx + \int_1^2 \sqrt{4 - x^2}\, dx\right]$$

$$= 2\left[\int_0^1 \sqrt{3}x^{1/2} dx + \int_1^2 \sqrt{2^2 - x^2}\, dx\right]$$

Using standard integral, $\int x^n dx = \frac{x^{n-1}}{n+1}$, we have

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}$$

Therefore,

$$A=2\left[\left(\sqrt{3}\,\frac{x^{3/2}}{3/2}\right)\Bigg|_0^1+\left(\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\tan^{-1}\frac{x}{\sqrt{4-x^2}}\right)\Bigg|_1^2\right]$$

$$=2\left[\sqrt{3}.\frac{1}{3/2}-0+\frac{2}{2}\sqrt{4-4}+2\tan^{-1}\frac{2}{\sqrt{4-4}}-\frac{1}{2}\sqrt{4-1}-2\tan^{-1}\frac{1}{\sqrt{4-1}}\right]$$

$$=2\left[\frac{2\sqrt{3}}{3}+2\tan^{-1}(\infty)-\frac{\sqrt{3}}{2}-2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

Now, $\tan\frac{\pi}{2}=\infty$ and $\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$. Therefore, the area of the smaller portion enclosed between the two curves is obtained as follows:

$$A=2\left[\frac{2\sqrt{3}}{3}-\frac{\sqrt{3}}{2}+2\tan^{-1}\left(\tan\frac{\pi}{2}\right)-2\tan^{-1}\left(\tan\frac{\pi}{6}\right)\right]$$

$$=2\left[\sqrt{3}\frac{1}{6}+2\frac{\pi}{2}-2\frac{\pi}{6}\right]$$

$$=2\left[\sqrt{3}\frac{1}{6}+2\frac{2\pi}{6}\right]=2\left[\frac{\sqrt{3}}{6}+\frac{4\pi}{6}\right]=\frac{\sqrt{3}}{3}+\frac{4\pi}{3}$$

$$=\left(\frac{1}{\sqrt{3}}+\frac{4\pi}{3}\right)\text{ sq.units}$$

Hence, the correct answer is option (D).

JEE Advanced 2017

1. If the line $x = \alpha$ divides the area of region $R=\{(x,y)\in\mathbb{R}^2 : x^3\le y\le x, 0\le x\le 1\}$ into two equal parts, then

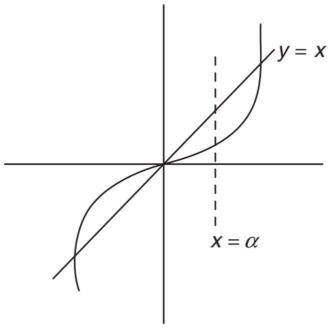
(A) $0<\alpha\le\frac{1}{2}$ **(B)** $\frac{1}{2}<\alpha<1$

(C) $2\alpha^4-4\alpha^2+1=0$ **(D)** $\alpha^4+4\alpha^2-1=0$

Solution: Let us consider $y = x^3$ and $y = x$. Then the area between these two curves in region $0\le x\le 1$ is

$$A=\int_0^1(x-x^3)dx$$

It is given that the line $x = \alpha$ divides the area under the curve into two equal parts. Therefore,



$$\int_0^\alpha(x-x^3)dx=\int_\alpha^1(x-x^3)dx$$

$$\Rightarrow \frac{x^2}{2}-\frac{x^4}{4}\Bigg|_0^\alpha=\frac{x^2}{2}-\frac{x^4}{4}\Bigg|_\alpha^1$$

$$\Rightarrow\left(\frac{\alpha^2}{2}-0\right)-\left(\frac{\alpha^4}{4}-0\right)=\left(\frac{1}{2}-\frac{\alpha^2}{2}\right)-\left(\frac{1}{4}-\frac{\alpha^4}{4}\right)$$

$$\Rightarrow\frac{\alpha^2}{2}-\frac{\alpha^4}{4}=\frac{1}{2}-\frac{\alpha^2}{2}-\frac{1}{4}+\frac{\alpha^4}{4}$$

$$\Rightarrow\frac{\alpha^2}{2}-\frac{\alpha^4}{4}-\frac{1}{2}+\frac{\alpha^2}{2}+\frac{1}{4}-\frac{\alpha^4}{4}=0$$

$$\Rightarrow\alpha^2-\frac{\alpha^4}{2}-\frac{1}{4}=0$$

$$\Rightarrow\frac{4\alpha^2-2\alpha^4-1}{4}=0\Rightarrow 4\alpha^2-2\alpha^4-1=0$$

$$\Rightarrow-(2\alpha^4-4\alpha^2+1)=0\Rightarrow 2\alpha^4-4\alpha^2+1=0$$

Now, let us consider the equation $2\alpha^4 - 4\alpha^2 + 1 = 0$.
Let $\alpha^2 = u$. Therefore,

$$2u^2-4u+1=0$$

$$\Rightarrow u=\frac{4\pm\sqrt{16-4\times 2}}{2\times 2}$$

$$\left(\begin{array}{l}\text{since for equation } ax^2+bx+c=0,\\ \text{the discriminant}=b^2-4ac=D\text{ and}\\ \text{the roots of } x \text{ are } \dfrac{-b\pm\sqrt{D}}{2a}\end{array}\right)$$

$$\Rightarrow u=\frac{4\pm\sqrt{16-8}}{4}=\frac{4\pm\sqrt{8}}{4}=\frac{4\pm\sqrt{4\times 2}}{4}$$

$$\Rightarrow u=\frac{4\pm 2\sqrt{2}}{4}=1\pm\frac{\sqrt{2}}{2}\ \Rightarrow u=1\pm\frac{1}{\sqrt{2}}$$

Substituting $u = \alpha^2$, we get

$$\alpha^2=1\pm\frac{1}{\sqrt{2}}$$

From $\alpha^2=1\pm\frac{1}{\sqrt{2}}$, we get $\frac{1}{2}<\alpha<1$.

Hence, the correct answers are options (B) and (C).

25 Differential Equations

25.1 Introduction

Generally, any equation such as

$$f(x, y, a) = 0 \tag{25.1}$$

represents a member of a family of curves for each individual value of a. Sometimes it is found necessary to represent the whole family of curves as a single unit and consider them as one for the purpose of studying a common property or characteristic which may run through the members of the family.

From Eq. (25.1), solve for a, and the equation $\phi(x, y) = a$ may be obtained; and on differentiating, 'a' gets removed. The resulting equation involving dy/dx is known as a differential equation, that is, the equation representing all the members of the family $f(x, y, a) = 0$ or alternately $\phi(x, y) = a$.

25.2 Basic Definition

An equation containing an independent variable, a dependent variable and the differential coefficients of a dependent variable with respect to an independent variable is called a *differential equation*. An ordinary differential equation is one in which there is only one independent variable. Examples:

1. $\dfrac{dy}{dx} = 1 + x + y$

2. $\dfrac{dy}{dx} = \cot x + xy$

3. $\left(\dfrac{d^4y}{dx^4}\right)^3 - 4\dfrac{dy}{dx} + 4y = 5\cos 3x$

4. $x^2\dfrac{d^2y}{dx^2} + \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = 0$

25.3 Order of a Differential Equation

The order of the highest derivative occurring in the differential equation is called the *order of the differential equation*. For example, the order of the above differential equations is 1, 1, 4 and 2, respectively.

The order of a differential equation is a positive integer. To determine the order of a differential equation, it is not required to make the equation free from radicals.

25.4 Degree of a Differential Equation

The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions. The degree of the above differential equations is 1, 1, 3 and 2, respectively.

Illustration 25.1 Find the order and degree (if defined) of the following differential equations:

(i) $\left(\dfrac{d^3y}{dx^3}\right)^{2/3} = \dfrac{dy}{dx} + 2$ **(ii)** $\dfrac{d^2y}{dx^2} = x\ln\left(\dfrac{dy}{dx}\right)$

(iii) $\dfrac{dy}{dx} + 4y = \sin x$ **(iv)** $\left(\dfrac{d^2y}{dx^2}\right)^4 + \left(\dfrac{dy}{dx}\right)^5 - y = e^x$

Solution:

(i) The given differential equations can be rewritten as

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left(\frac{dy}{dx} + 2\right)^3$$

Hence, its order and degree are 3 and 2, respectively.

(ii) $\dfrac{d^2y}{dx^2} = x\ln\left(\dfrac{dy}{dx}\right)$

Hence, its order and degree are 2 and 1, respectively.

(iii) $\dfrac{dy}{dx} + 4y = \sin x$

Hence, its order and degree are 1 and 1, respectively.

(iv) $\left(\dfrac{d^2y}{dx^2}\right)^4 + \left(\dfrac{dy}{dx}\right)^5 - y = e^x$

Hence, its order and degree are 2 and 4, respectively.

25.5 Formation of a Differential Equation

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. The equation thus obtained is the differential equation of order n for the family of the given curves.

Consider a family of curves

$$f(x, y, a_1, a_2, \ldots, a_n) = 0 \tag{25.2}$$

where $a_1, a_2, \ldots, a_n$ are n independent parameters.

Equation (25.2) is known as an n-parameter family of curves, for example, $y = mx$ is a one-parameter family of straight lines. $x^2 + y^2 + ax + by = 0$ is a two-parameter family of circles.

If we differentiate Eq. (25.2) n times w.r.t. x, we will get n more relations between x, y, a_1, a_2, …, a_n and derivatives of y w.r.t. x. By eliminating a_1, a_2, …, a_n from these n relations and Eq. (25.2), we get a differential equation.

Clearly, the order of this differential equation will be n, that is, equal to the number of independent parameters in the family of curves.

25.5.1 Steps for Formation of Differential Equations

Step 1: Write the given equation involving an independent variable x (say), a dependent variable y (say) and the arbitrary constants.

Step 2: Obtain the number of arbitrary constants in step 1. Let there be n arbitrary constants.

Step 3: Differentiate the relation in step 1, n times with respect to x.

Step 4: Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in step 3 and an equation in step 2. The equation thus obtained is the desired differential equation.

Illustration 25.2 Form the differential equation of the following relation:

(i) $x^2 + y^2 = 2ax$

(ii) $x^2 + y^2 = 2ax + b$

Solution:

(i) $$x^2 + y^2 = 2ax$$

On differentiating w.r.t. x,

$$2x + 2y\frac{dy}{dx} = 2a$$

Eliminating a,

$$x^2 + y^2 = x\left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow x^2 - y^2 + 2xy\frac{dy}{dx} = 0$$

In this case, the relation contains only one arbitrary constant and hence, the differential equation contains only dy/dx.

(ii) $$x^2 + y^2 = 2ax + b$$

On differentiating w.r.t. x,

$$2x + 2y\frac{dy}{dx} = 2a$$

On differentiating once again w.r.t. x,

$$1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$$

which is the differential equation to the given equation and since there are two arbitrary constants a and b, the differential equation contains (the second order) the derivative d^2y/dx^2.

Illustration 25.3 Find the differential equation of the family of curves $y = Ae^x + Be^{3x}$ for different values of A and B.

Solution:

$$y = Ae^x + Be^{3x} \quad (1)$$

$$y_1 = Ae^x + 3Be^{3x} \quad (2)\ \left(y_1 = \frac{dy}{dx}\right)$$

$$y_2 = Ae^x + 9Be^{3x} \quad (3)\ \left(y_2 = \frac{d^2y}{dx^2}\right)$$

Eliminating A and B from the above three equations, we get

$$\begin{vmatrix} e^x & e^{3x} & -y \\ e^x & 3e^{3x} & -y_1 \\ e^x & 9e^{3x} & -y_2 \end{vmatrix} = 0 \Rightarrow e^x e^{3x}\begin{vmatrix} 1 & 1 & -y \\ 1 & 3 & -y_1 \\ 1 & 9 & -y_2 \end{vmatrix} = 0$$

$$\Rightarrow 3y + 4y_1 - y_2 = 0 \Rightarrow 3y + 4\frac{dy}{dx} - \frac{d^2y}{dx^2} = 0$$

Illustration 25.4 Find the order and degree of the differential equation of all the parabolas whose axes are parallel to the x-axis and having a latus rectum a.

Solution: Equation of required parabolas is

$$(y - \beta)^2 = a(x - \alpha)$$

On differentiating both sides w.r.t. x, we get

$$2(y-\beta)\frac{dy}{dx} = a$$

Again differentiating w.r.t. x, we get

$$2(y-\beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \left(2(y-\beta)\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0$$

$$\Rightarrow a\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0$$

Thus, the order of the differential equation is 2 and degree is 1.

Illustration 25.5 Find the differential equation of family of curves whose tangent form an angle of $\pi/4$ with the hyperbola $xy = c^2$?

Solution: The slope of the tangent to the family of curves is $m_1 = \dfrac{dy}{dx}$

The equation of the hyperbola is

$$xy = c^2 \Rightarrow y = \frac{c^2}{x}$$

Therefore,

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

Therefore, the slope of the tangent to $xy = c^2$ is $m_2 = -\dfrac{c^2}{x^2}$.

Now,

$$\tan\frac{\pi}{4} = \frac{m_1 - m_2}{1 + m_1 m_2} \Rightarrow 1 = \frac{\dfrac{dy}{dx} + \dfrac{c^2}{x^2}}{1 - \dfrac{c^2}{x^2}\dfrac{dy}{dx}} \Rightarrow \frac{dy}{dx}\left(1 + \frac{c^2}{x^2}\right) = \left(1 - \frac{c^2}{x^2}\right)$$

Hence, the required equation is

$$\frac{dy}{dx}=\frac{x^2-c^2}{x^2+c^2}$$

Your Turn 1

1. The degree of the differential equation

$$y(x)=1+\frac{dy}{dx}+\frac{1}{1.2}\left(\frac{dy}{dx}\right)^2+\frac{1}{1.2.3}\left(\frac{dy}{dx}\right)^3+\cdots \text{ is}$$

(A) 2 **(B)** 3 **(C)** 1 **(D)** 4

Ans. (C)

2. The degree of the differential equation $\frac{d^2y}{dx^2}-\sqrt{\frac{dy}{dx}}-3=x$ is

(A) 2 **(B)** 3 **(C)** 1 **(D)** 4

Ans. (A)

3. The degree and order of the differential equation of the family of all parabolas whose axis is x-axis are, respectively

(A) 2, 1 **(B)** 1, 2 **(C)** 3, 2 **(D)** 2, 3

Ans. (B)

4. If m and n are the order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^5+4\frac{\left(\frac{d^2y}{dx^2}\right)^3}{\left(\frac{d^3y}{dx^3}\right)}+\frac{d^3y}{dx^3}=x^2-1\text{, then}$$

(A) $m=3$ and $n=5$ **(B)** $m=3$ and $n=1$
(C) $m=3$ and $n=3$ **(D)** $m=3$ and $n=2$

Ans. (D)

5. $y=4\sin 3x$ is a solution of the differential equation ______.

(A) $\frac{dy}{dx}+8y=0$ **(B)** $\frac{dy}{dx}-8y=0$

(C) $\frac{d^2y}{dx^2}+9y=0$ **(D)** $\frac{d^2y}{dx^2}-9y=0$

Ans. (A)

6. $y=\frac{x}{x+1}$ is a solution of the differential equation ______.

(A) $y^2\frac{dy}{dx}=x^2$ **(B)** $x^2\frac{dy}{dx}=y^2$

(C) $y\frac{dy}{dx}=x$ **(D)** $x\frac{dy}{dx}=y$

Ans. (B)

7. The differential equation for all the straight lines which are at a unit distance from the origin is

(A) $\left(y-x\frac{dy}{dx}\right)^2=1-\left(\frac{dy}{dx}\right)^2$ **(B)** $\left(y+x\frac{dy}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2$

(C) $\left(y-x\frac{dy}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2$ **(D)** $\left(y+x\frac{dy}{dx}\right)^2=1-\left(\frac{dy}{dx}\right)^2$

Ans. (C)

8. The differential equation whose solution is $y=c_1\cos ax+c_2\sin ax$ is (where c_1, c_2 are arbitrary constants)

(A) $\frac{d^2y}{dx^2}+y^2=0$ **(B)** $\frac{d^2y}{dx^2}+a^2y=0$

(C) $\frac{d^2y}{dx^2}+ay^2=0$ **(D)** $\frac{d^2y}{dx^2}-a^2y=0$

Ans. (B)

9. The differential equation of the family of curves $y^2=4a(x+a)$, where a is an arbitrary constant, is

(A) $y\left[1+\left(\frac{dy}{dx}\right)^2\right]=2x\frac{dy}{dx}$ **(B)** $y\left[1-\left(\frac{dy}{dx}\right)^2\right]=2x\frac{dy}{dx}$

(C) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ **(D)** $\left(\frac{dy}{dx}\right)^3+3\frac{dy}{dx}+y=0$

Ans. (B)

10. If $y=ax^{n+1}+bx^{-n}$, then $x^2\frac{d^2y}{dx^2}$ equals

(A) $n(n-1)y$ **(B)** $n(n+1)y$
(C) ny **(D)** n^2y

Ans. (B)

25.6 Solution of a Differential Equation

If we have a differential equation of order 'n' then by solving a differential equation we mean to get a family of curves with n parameters whose differential equation is the given differential equation. The solution or integral of a differential equation is a relation between the variables, not involving the differential coefficients such that this relation and the derivatives obtained from it satisfy the given differential equation. The solution of a differential equation is also called its primitive.

For example, $y=e^x$ is a solution of the differential equation $dy/dx=y$.

25.6.1 General Solution

The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation. For example, $y=A\cos x+B\sin x$ is the general solution of the differential equation $\frac{d^2y}{dx^2}+y=0$. But $y=A\cos x$ is not the general solution as it contains one arbitrary constant.

25.6.2 Particular Solution

The solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution. For example, $y=3\cos x+2\sin x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2}+y=0$.

25.7 Differential Equations of First-Order and First-Degree

A differential equation of first-order and first-degree involves x, y and dy/dx. Thus, it can be put in any one of the following forms:

$$\frac{dy}{dx}=f(x,y) \text{ or } f\left(x,y,\frac{dy}{dx}\right) \text{ or } f(x,y)dx+g(x,y)dy=0$$

where $f(x, y)$ and $g(x, y)$ are obviously the functions of x and y.

25.7.1 Geometrical Interpretation of the Differential Equations of First-Order and First-Degree

The general form of a first-order and first-degree differential equation is

$$f\left(x,y,\frac{dy}{dx}\right)=0 \tag{25.3}$$

The direction of the tangent of a curve in the Cartesian rectangular coordinates at any point is given by dy/dx, therefore Eq. (25.3) establishes the relationship between the coordinates of a point and the slope of the tangent. Solution of the differential equation given by Eq. (25.3) gives those curves for which the direction of tangent at each point coincides with the direction of the field. All the curves represented by the general solution, when taken together, will give the locus of the differential equation. The locus of the general solution can be said to be made up of single infinity of curves as there is one arbitrary constant in the general solution of the equation of first order.

25.8 Solution of First-Order and First-Degree Differential Equations

A first-order and first-degree differential equation can be written as

$$f(x,y)dx+g(x,y)dy=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$$

$$\Rightarrow \frac{dy}{dx}=\phi(x,y)$$

where $f(x, y)$ and $g(x, y)$ are obviously the functions of x and y. It is not always possible to solve this type of equations. The solution of this type of differential equations is possible only when it falls under the category of some standard forms.

The simple standard form of the differential equation of the first order and first degree is given in the following section.

25.9 Variable Separable Type Differential Equation

Differential equations of the form $\frac{dy}{dx}=f(x,y)$ can be reduced to form $\Psi(x)dx+\phi(y)dy=0$ then integrate it, i.e. find $\int\Psi(x)dx+\int\phi(y)dy=c$.

Illustration 25.6 Solve the differential equation

$$\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$$

Solution: Separating the variables

$$\frac{dy}{dx}=e^{-y}(e^x+x^2)\Rightarrow\int e^y dy=\int(e^x+x^2)dx+A$$

On integrating, the solution is

$$e^y=e^x+\frac{x^3}{3}+A\Rightarrow 3(e^y-e^x)=x^3+C$$

(C is an arbitrary constant)

Illustration 25.7 Solve $\frac{dy}{dx}=e^x y$.

Solution: Given

$$\frac{dy}{dx}=e^x y\Rightarrow\int\frac{dy}{y}=\int e^x dx+c$$

On integrating, we get $\ln y=e^x+c$.

Illustration 25.8 Solve the differential equation

$$(1+x^2)\frac{dy}{dx}=x(1+y^2)$$

Solution: Separating the variables, we can rewrite the given differential equation as

$$\frac{xdx}{1+x^2}=\frac{dy}{1+y^2}$$

$$\Rightarrow\int\frac{2xdx}{1+x^2}=2\int\frac{dy}{1+y^2}+c\Rightarrow 2\tan^{-1}y=\log_e(1+x^2)+c$$

25.10 Equation Reducible to Variable Separable Type Differential Equation

Sometimes differential equation of the first order cannot be solved directly by variable separation but by some substitution we can reduce it to a differential equation with separable variables.

A differential equation of the form

$$\frac{dy}{dx}=f(ax+by+c)$$

is solved by taking

$$ax+by+c=Z\Rightarrow a+b\frac{dy}{dx}=\frac{dZ}{dx}$$

Therefore,

$$\left(\frac{dZ}{dx}-a\right)\frac{1}{b}=f(Z)\Rightarrow\frac{dZ}{dx}=a+bf(Z)$$

This is the variable separable form which can be solved.

Illustration 25.9 Solve the differential equation

$$\frac{dy}{dx}=\cos(x+y)$$

Solution: Put $x+y=t$. Then differentiating w.r.t. x

$$1+\frac{dy}{dx}=\frac{dt}{dx}$$

Thus,

$$\frac{dt}{dx}-1=\cos t$$

$$\Rightarrow \int\frac{dt}{1+\cos t}=\int dx+c$$

On integrating both sides, we get

$$\Rightarrow \tan\left(\frac{t}{2}\right)=x+c$$

$$\Rightarrow \tan\left(\frac{x+y}{2}\right)=x+c$$

Illustration 25.10 Solve $(x-y)^2\frac{dy}{dx}=1$.

Solution: Put $z = x - y$. Then differentiating w.r.t. x

$$\frac{dz}{dx}=1-\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=1-\frac{dz}{dx}$$

Now,

$$z^2\left(1-\frac{dz}{dx}\right)=1\Rightarrow z^2\frac{dz}{dx}=z^2-1$$

$$\Rightarrow dx=\frac{z^2}{z^2-1}dz$$

which is in the form of variable separable.

Now integrating, we get

$$x=z+\frac{1}{2}\ln\frac{z-1}{z+1}+c$$

Thus, the solution is $x=(x-y)+\frac{1}{2}\ln\frac{x-y-1}{x-y+1}+c$

Illustration 25.11 The solution of the differential equation

$$\frac{dy}{dx}=(4x+y+1)^2$$

Solution: Let $4x + y + 1 = z$. Then

$$4+\frac{dy}{dx}=\frac{dz}{dx}\Rightarrow\frac{dy}{dx}=\frac{dz}{dx}-4$$

Therefore,

$$\frac{dy}{dx}=(4x+y+1)^2$$

$$\Rightarrow\frac{dz}{dx}-4=z^2$$

$$\Rightarrow\frac{dz}{dx}=z^2+4$$

$$\Rightarrow\int\frac{dz}{z^2+4}=\int dx+c$$

$$\Rightarrow\frac{1}{2}\tan^{-1}\frac{z}{2}=x+c$$

$$\Rightarrow\tan^{-1}\left(\frac{4x+y+1}{2}\right)=2x+2c$$

25.11 Homogeneous Type Differential Equation

A function $f(x, y)$ is called a homogeneous function of degree n if $f(\lambda x,\lambda y)=\lambda^n f(x,y)$.

For example, $f(x,y)=x^2-y^2+3xy$ is a homogeneous function of degree 2 because $f(\lambda x,\lambda y)=\lambda^2x^2-\lambda^2y^2+3\lambda x\cdot\lambda y=\lambda^2 f(x,y)$. A homogeneous function $f(x, y)$ of degree n can always be written as $f(x,y)=x^n f\left(\frac{y}{x}\right)$ or $f(x,y)=y^n f\left(\frac{x}{y}\right)$.

25.11.1 Steps for Solving Homogeneous Differential Equation

Step 1: Put the differential equation in the form

$$\frac{dy}{dx}=\frac{\phi(x,y)}{\psi(x,y)}$$

Step 2: Put $y = vx$ (or $x = vy$) and $\frac{dy}{dx}=v+x\frac{dv}{dx}\left(\text{or }\frac{dx}{dy}=v+y\frac{dv}{dy}\right)$ in the equation in step 1 and cancel out x (or y) from the right-hand side. The equation reduces to the form

$$v+x\frac{dv}{dx}=F(v)\left(\text{or } v+y\frac{dv}{dy}=F(v)\right)$$

Step 3: Shift v on the right-hand side and separate the variables v and x (or y).

Step 4: Integrate both sides of the equation to obtain the solution in terms of v and x (or y).

Step 5: Replace v by $\frac{y}{x}\left(\text{or }\frac{x}{y}\right)$ in the solution obtained in step 4 to obtain the solution in terms of x and y.

Illustration 25.12 Solve the differential equation $\frac{dy}{dx}=\frac{x-y}{x+y}$.

Solution: Put $y = vx$. Then differentiating w.r.t. x

$$\frac{dy}{dx}=v\cdot 1+x\frac{dv}{dx}$$

Substituting into the given equation

$$v+x\frac{dv}{dx}=\frac{x-vx}{x+vx}=\frac{1-v}{1+v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v}{1+v}-v=\frac{1-v-v-v^2}{1+v}$$

Now, separating the variables

$$\int\frac{1+v}{1-2v-v^2}dv=\int\frac{dx}{x}+A$$

$$\Rightarrow -\frac{1}{2}\log(1-2v-v^2)=\log x+A$$

$$\Rightarrow \log[(1-2v-v^2)\,x^2]=\text{constant}$$

$$\Rightarrow (1 - 2v - v^2)\, x^2 = C$$
$$\Rightarrow x^2 - 2xy - y^2 = C$$

Illustration 25.13 Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

Solution: Put $y = vx$. Then differentiating w.r.t. x

$$\frac{x^2 + v^2x^2}{2x \cdot vx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \frac{dx}{x} = \frac{2v}{1-v^2}dv$$

Integrating both sides, we set

$$C + \ln x = -\ln(1 - v^2)$$
$$\Rightarrow \ln kx + \ln(1 - v^2) = 0$$
$$\Rightarrow kx(1 - v^2) = 1 \Rightarrow k(x^2 - y^2) = x$$

Illustration 25.14 Solve the differential equation

$$x\frac{dy}{dx} = y(\log y - \log x + 1)$$

Solution: The given equation may be expressed as

$$\frac{dy}{dx} = \frac{y}{x}\left[\log\left(\frac{y}{x}\right) + 1\right] \tag{1}$$

Let $\frac{y}{x} = v$.Then

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Therefore, from Eq. (1),

$$v + x\frac{dv}{dx} = v(\log v + 1) \Rightarrow x\frac{dv}{dx} = v\log v$$

$$\frac{dv}{v\log v} = \frac{dx}{x} \Rightarrow \int\frac{1}{\log v}d(\log v) = \int\frac{dx}{x} + c$$

Therefore,

$$\log(\log v) = \log x + \log c \Rightarrow \log(\log v) = \log(cx)$$
$$\Rightarrow \log v = cx$$
$$\Rightarrow v = e^{cx} \Rightarrow \frac{y}{x} = e^{cx}$$

Hence, the required solution is $y = xe^{cx}$.

Illustration 25.15 Solve the differential equation

$$\frac{dy}{dx} = \frac{y^3 + 2x^2y}{x^3 + 2xy^2}$$

Solution: The given equation is homogeneous.
Let $y = vx$. Therefore,

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow \frac{y^3 + 2x^2y}{x^3 + 2xy^2} = v + x\frac{dv}{dx}$$

$$\Rightarrow \frac{(y/x)^3 + 2(y/x)}{1 + 2(y/x)^2} = v + x\frac{dv}{dx} \Rightarrow \frac{v^3 + 2v}{1 + 2v^2} = v + x\frac{dv}{dx}$$

$$\Rightarrow x\frac{dv}{dx} = v\left\{\frac{v^2 + 2}{1 + 2v^2} - 1\right\} = v\left\{\frac{1 - v^2}{1 + 2v^2}\right\}$$

$$\Rightarrow \frac{1 + 2v^2}{v(1 - v^2)}dv = \frac{dx}{x} \Rightarrow \frac{1 + 2v^2}{v(1-v)(1+v)}dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{A}{v} + \frac{B}{1-v} + \frac{D}{1+v}\right)dv = \frac{dx}{x}$$

where,

$$A(1-v)(1+v) + Bv(1+v) + Dv(1-v) = 1 + 2v^2$$

Putting, $v = 0$, we get $A = 1$

$v = 1$, we get $B = \frac{3}{2}$

$v = -1$, we get $D = -\frac{3}{2}$

Hence,

$$\left(\frac{1}{v} + \frac{3}{2}\frac{1}{1-v} - \frac{3}{2}\frac{1}{1+v}\right)dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\ln v + \frac{3}{2}\frac{\ln(1-v)}{-1} - \frac{3}{2}\ln(1+v) = \ln x + \ln c$$

$$\Rightarrow \ln v - \frac{3}{2}\ln(1-v) - \frac{3}{2}\ln(1+v) = \ln cx$$

$$\Rightarrow v/\{(1-v)(1+v)\}^{3/2} = cx \Rightarrow \left(\frac{v}{cx}\right)^2 = (1-v^2)^3$$

$$\Rightarrow \left(\frac{y}{cx^2}\right)^2 = \left(1 - \frac{y^2}{x^2}\right)^3 \Rightarrow (x^2 - y^2)^3 = \frac{x^2y^2}{c^2}$$

Thus, the required solution is

$$(x^2 - y^2)^3 = Bx^2y^2 \qquad \left(\frac{1}{c^2} = B\right)$$

25.12 Non-Homogeneous Type Differential Equation

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, can be reduced to homogeneous equation by putting $x = X + h$ and $y = Y + k$, where h and k are such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$. Also

$$\frac{dy}{dx} = \frac{dY}{dX}$$

Hence, the equation reduces to $\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$ (homogeneous form).

If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\lambda$, then $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}=\frac{\lambda(a_1x+b_1y)+c_1}{(a_1x+b_1y)+c_2}$ can be solved by putting $a_1x+b_1y=t$. It then reduces to equation with variable separable.

Illustration 25.16 Solve $\frac{dy}{dx}=\frac{x-y+3}{2x-2y+5}$.

Solution: Here,

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{1}{2}$$

Hence, we put $x–y=v$. Then

$$1-\frac{dy}{dx}=\frac{dv}{dx}\Rightarrow\frac{dy}{dx}=1-\frac{dv}{dx}$$

$$\Rightarrow 1-\frac{dv}{dx}=\frac{v+3}{2v+5}\Rightarrow\frac{2v+5}{v+2}dv=dx$$

$$\left(2+\frac{1}{v+2}\right)dv=dx$$

On integrating both sides, we get

$$2v+\ln(v+2)=x+C$$

Put the value of v. Therefore,

$$x-2y+\ln(x-y+2)=C$$

Illustration 25.17 Solve the differential equation

$$\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$$

Solution: Take $x=X+\ell$; $y=Y+m$ (ℓ and m are constants). So,

$$\frac{dy}{dx}=\frac{dY}{dX}$$

Therefore, the equation becomes (in X, Y)

$$\frac{dY}{dX}=\frac{X+2Y+\ell+2m-3}{2X+Y+2\ell+m-3}=\frac{X+2Y}{2X+y}$$

If ℓ, m are chosen to satisfy

$$\left.\begin{array}{l}\ell+2m-3=0\\2\ell+m-3=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}\ell=1\\m=1\end{array}\right\}$$

In X, Y the equation is homogeneous and of the first degree. Put $Y=VX$. Then

$$V+X\frac{dV}{dX}=\frac{X+2VX}{2X+VX}=\frac{1+2V}{2+V}$$

$$X\frac{dV}{dX}=\frac{1+2V-(2+V)V}{2+V}=\frac{1-V^2}{2+V}$$

Separating the variables (X, V) and integrating,

$$\int\frac{2+V}{1-V^2}dV=\int\frac{dX}{X}+A$$

where A is an arbitrary constant.

$$\int\left(\frac{1}{2}\cdot\frac{1}{1+V}+\frac{3}{2}\cdot\frac{1}{1-V}\right)dV=\int\frac{dX}{X}+A$$

$$\Rightarrow\frac{1}{2}\log(1+V)-\frac{3}{2}\log(1-V)-\log X=A$$

Now,

$$V=\frac{Y}{X}=\frac{y-m}{x-\ell}=\frac{y-1}{x-1}$$

Reverting to x and y, the solution is

$$\frac{1}{2}\log\left(1+\frac{y-1}{x-1}\right)-\frac{3}{2}\log\left(1-\frac{y-1}{x-1}\right)-\log(x-1)=A$$

which simplifies to

$$\left[\frac{x+y-2}{(x-1)(x-1)^2}\cdot\frac{(x-1)^3}{(x-y)^3}\right]=C(=e^{2A})$$

$$\Rightarrow(x+y-2)=C\,(x-y)^3$$

Illustration 25.18 Solve the differential equation

$$\frac{dy}{dx}=\frac{x+y+1}{2x+2y+1}$$

Solution: x and y coefficients in $x+y+1$ are proportional to the x and y coefficients in $2x+2y+1$.
Therefore, in this case, take $x+y=Z$. Then

$$1+\frac{dy}{dx}=\frac{dZ}{dx}$$

$$\frac{dZ}{dx}-1=\frac{Z+1}{2Z+1}$$

$$\frac{dZ}{dx}=\frac{Z+1+2Z+1}{2Z+1}=\frac{3Z+2}{2Z+1}$$

Separating the variables and integrating, we get

$$\int\frac{2Z+1}{3Z+2}dZ=\int dx+A$$

where A is an arbitrary constant.

$$\int\frac{\frac{2}{3}(3Z+2)-\frac{1}{3}}{(3Z+2)}dZ=x+A$$

$$\Rightarrow\frac{2}{3}Z-\frac{1}{9}\log(3Z+2)=x+A$$

Reverting to x and y ($x+y=Z$)

$$\frac{2}{3}(x+y)-x-\frac{1}{9}\log(3x+3y+2)=A$$

$$\Rightarrow\frac{1}{3}(-x+2y)-\frac{1}{9}\log(3x+3y+2)=A$$

$$\Rightarrow -3x+6y=\log(3x+3y+2)+C$$

Illustration 25.19 Solve the differential equation

$$\frac{dy}{dx}=\frac{x-3y+2}{3x-y+6}$$

Solution: Given equation is non-homogeneous.
Let $x=X+h$, $y=Y+k$. Then

$$\frac{dy}{dx}=\frac{dY}{dX}$$

Therefore,

$$\frac{dY}{dX}=\frac{(X+h)-3(Y+k)+2}{3(X+h)-(Y+k)+6}=\frac{X-3Y+(h-3k+2)}{3X-Y+(3h-k+6)}$$

Let us select h and k so that $h-3k+2=0$ and $3h-k+6=0$.
Solving, $k=0$ and $h=-2$
Therefore,

$$X=x-h=x+2,\ Y=y-k=y$$

Hence,

$$\frac{dY}{dX}=\frac{X-3Y}{3X-Y},\ \text{which is homogeneous.}$$

Now, let $Y=vX$. Then

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

$$\Rightarrow \frac{X-3Y}{3X-Y}=v+X\frac{dv}{dX}$$

$$\Rightarrow \frac{1-3(Y/X)}{3-(Y/X)}=v+X\frac{dv}{dX}$$

$$\Rightarrow \frac{1-3v}{3-v}=v+X\frac{dv}{dX}$$

$$\Rightarrow X\frac{dv}{dX}=\frac{1-3v}{3-v}-v=\frac{v^2-6v+1}{3-v}$$

$$\Rightarrow \frac{(3-v)dv}{v^2-6v+1}=\frac{dX}{X}$$

$$\Rightarrow \frac{2v-6}{v^2-6v+1}dv=-2\frac{dX}{X}$$

On integrating both side, we get

$$\ln(v^2-6v+1)=-2\ln X+\ln c$$
$$\Rightarrow \ln(v^2-6v+1)+\ln X^2=\ln c$$
$$\Rightarrow X^2(v^2-6v+1)=c$$
$$\Rightarrow Y^2-6XY+X^2=c$$

Thus, the required solution is

$$y^2-6(x+2)y+(x+2)^2=c$$

Your Turn 2

1. The solution of the differential equation $x\cos y\,dy=(xe^x\log x+e^x)dx$ is

(A) $\sin y=\frac{1}{x}e^x+c$ **(B)** $\sin y+e^x\log x+c=0$
(C) $\sin y=e^x\log x+c$ **(D)** None of these

Ans. (C)

2. The solution of the equation $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$ is

(A) $e^y=e^x+\frac{x^3}{3}+c$ **(B)** $e^y=e^x+2x+c$
(C) $e^y=e^x+x^3+c$ **(D)** $y=e^x+c$

Ans. (A)

3. The solution of the differential equation $\frac{dy}{dx}=e^x+\cos x+x+\tan x$ is

(A) $y=e^x+\sin x+\frac{x^2}{2}+\log\cos x+c$

(B) $y=e^x+\sin x+\frac{x^2}{2}+\log\sec x+c$

(C) $y=e^x-\sin x+\frac{x^2}{2}+\log\cos x+c$

(D) $y=e^x-\sin x+\frac{x^2}{2}+\log\sec x+c$

Ans. (B)

4. The solution of the differential equation $(\sin x+\cos x)dy+(\cos x-\sin x)dx=0$ is

(A) $e^x(\sin x+\cos x)+c=0$ **(B)** $e^y(\sin x+\cos x)=c$
(C) $e^y(\cos x-\sin x)=c$ **(D)** $e^x(\sin x-\cos x)=c$

Ans. (B)

5. The solution of the differential equation $(x^2-yx^2)\frac{dy}{dx}+y^2+xy^2=0$ is

(A) $\log\left(\frac{x}{y}\right)=\frac{1}{x}+\frac{1}{y}+c$ **(B)** $\log\left(\frac{y}{x}\right)=\frac{1}{x}+\frac{1}{y}+c$
(C) $\log(xy)=\frac{1}{x}+\frac{1}{y}+c$ **(D)** $\log(xy)+\frac{1}{x}+\frac{1}{y}=c$

Ans. (A)

6. The solution of the differential equation $x(e^{2y}-1)dy+(x^2-1)e^ydx=0$ is

(A) $e^y+e^{-y}=\log x-\frac{x^2}{2}+c$ **(B)** $e^y-e^{-y}=\log x-\frac{x^2}{2}+c$
(C) $e^y+e^{-y}=\log x+\frac{x^2}{2}+c$ **(D)** None of these

Ans. (A)

7. The solution of the differential equation $(3xy+y^2)dx+(x^2+xy)dy=0$ is

(A) $x^2(2xy+y^2)=c^2$ **(B)** $x^2(2xy-y^2)=c^2$
(C) $x^2(y^2-2xy)=c^2$ **(D)** None of these

Ans. (A)

8. The solution of the differential equation $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$ is

(A) $ay^2=e^{x^2/y^2}$ **(B)** $ay=e^{x/y}$
(C) $y=e^{x^2}+e^{y^2}+c$ **(D)** $y=e^{x^2}+y^2+c$

Ans. (A)

9. The general solution of the differential equation $(2x-y+1)dx+(2y-x+1)dy=0$ is

(A) $x^2+y^2+xy-x+y=c$ **(B)** $x^2+y^2-xy+x+y=c$
(C) $x^2-y^2+2xy-x+y=c$ **(D)** $x^2-y^2-2xy+x-y=c$

Ans. (B)

10. The solution of the equation $\dfrac{dy}{dx}=\dfrac{x}{2y-x}$ is

(A) $(x-y)(x+2y)^2=c$ **(B)** $y=x+c$

(C) $y=(2y-x)+c$ **(D)** $y=\dfrac{x}{2y-x}+c$

Ans. (A)

25.13 Exact Differential Equation

If M and N are functions of x and y, then the equation $Mdx+Ndy=0$ is called exact when there exists a function $f(x, y)$ of x and y such that

$$d[f(x,y)]=Mdx+Ndy, \quad \text{i.e.} \quad \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=Mdx+Ndy$$

where $\dfrac{\partial f}{\partial x}$ is the partial derivative of $f(x, y)$ with respect to x (treating y as a constant) and $\dfrac{\partial f}{\partial y}$ is the partial derivative of $f(x, y)$ with respect to y (treating x as a constant).

The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

that is, the partial derivative of $M(x, y)$ w.r.t. y is equal to the partial derivative of $N(x, y)$ w.r.t. x.

If the given equation is exact, then the solution of the exact differential equation is

$$\underbrace{\int Mdx}_{\text{Regarding } y \text{ as constant}}+\underbrace{\int Ndy}_{\text{Only those terms not containing } x}=c$$

25.13.1 Integrating Factor

A factor which when multiplied by a non-exact differential equation makes it exact is known as an integrating factor, for example the non-exact equation $ydx - xdy = 0$ can be made exact on multiplying by the factor $\dfrac{1}{y^2}$. Hence, $\dfrac{1}{y^2}$ is the integrating factor for this equation.

Notes: In general, such a factor exists but except in certain special cases, and it is likely to be difficult to determine.

The number of the integrating factor for equation $Mdx + Ndy = 0$ is infinite.

25.13.2 Some Useful Results

1. $d(x+y)=dx+dy$

2. $d(xy)=xdy+ydx$

3. $d\left(\dfrac{x}{y}\right)=\dfrac{ydx-xdy}{y^2}$

4. $d\left(\dfrac{y}{x}\right)=\dfrac{xdy-ydx}{x^2}$

5. $d\left(\dfrac{x^2}{y}\right)=\dfrac{2xydx-x^2dy}{y^2}$

6. $d\left(\dfrac{y^2}{x}\right)=\dfrac{2xydy-y^2dx}{x^2}$

7. $d\left(\dfrac{x^2}{y^2}\right)=\dfrac{2xy^2dx-2x^2ydy}{y^4}$

8. $d\left(\dfrac{y^2}{x^2}\right)=\dfrac{2x^2ydy-2xy^2dx}{x^4}$

9. $d\left(\tan^{-1}\dfrac{x}{y}\right)=\dfrac{ydx-xdy}{x^2+y^2}$

10. $d\left(\tan^{-1}\dfrac{y}{x}\right)=\dfrac{xdy-ydx}{x^2+y^2}$

11. $d[\ln(xy)]=\dfrac{xdy+ydx}{xy}$

12. $d\left(\ln\left(\dfrac{x}{y}\right)\right)=\dfrac{ydx-xdy}{xy}$

13. $d\left[\dfrac{1}{2}\ln(x^2+y^2)\right]=\dfrac{xdx+ydy}{x^2+y^2}$

14. $d\left[\ln\left(\dfrac{y}{x}\right)\right]=\dfrac{xdy-ydx}{xy}$

15. $d\left(-\dfrac{1}{xy}\right)=\dfrac{xdy+ydx}{x^2y^2}$

16. $d\left(\dfrac{e^x}{y}\right)=\dfrac{ye^xdx-e^xdy}{y^2}$

17. $d\left(\dfrac{e^y}{x}\right)=\dfrac{xe^ydy-e^ydx}{x^2}$

18. $d(x^my^n)=x^{m-1}y^{n-1}(mydx+nxdy)$

19. $d(\sqrt{x^2+y^3})=\dfrac{xdx+y\,dy}{\sqrt{x^2+y^2}}$

20. $d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)=\dfrac{x\,dy-y\,dx}{x^2-y^2}$

21. $\dfrac{d[f(x,y)]^{1-n}}{1-n}=\dfrac{f'(x,y)}{[f(x,y)]^n}$

Illustration 25.20 Solve the differential equation

$$(x^2-ay)dx+(y^2-ax)dy=0$$

Solution: Here,

$$M=x^2-ay$$
$$N=y^2-ax$$
$$\frac{\partial M}{\partial y}=-a$$
$$\frac{\partial N}{\partial x}=-a\Rightarrow\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Therefore, the equation is exact.

Thus,

$$\int(x^2-ay)dx+\int y^2\,dy=c$$
$$\Rightarrow\frac{x^3}{3}-ayx+\frac{y^3}{3}=c$$
$$\Rightarrow x^3-3axy+y^3=3c$$

Illustration 25.21 Solve the differential equation

$$(x+y)dx+xdy=0$$

Solution: We have

$$xdx+(ydx+xdy)=0\Rightarrow xdx+d(xy)=0$$

On integrating, we get

$$\frac{x^2}{2}+xy=\frac{c}{2}$$

Therefore, $x^2+2xy=c$.

Illustration 25.22 Solve the differential equation

$$(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$$

Solution: Comparing the given equation with $Mdx + Ndy = 0$, we get

$$M=x^2-4xy-2y^2,\ N=y^2-4xy-2x^2$$

$$\frac{\partial M}{\partial y} = -4x - 4y$$

$$\frac{\partial N}{\partial x} = -4y - 4x$$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So, the given differential equation is exact.
On integrating M w.r.t. x, treating y as a constant,

$$\int M dx = \int (x^2 - 4xy - 2y^2) dx = \frac{x^3}{3} - 2x^2 y - 2y^2 x$$

On integrating N w.r.t. y, treating x as a constant,

$$\int N dy = \int (y^2 - 4xy - 2x^2) dy = \frac{y^3}{3} - 2xy^2 - 2x^2 y = \frac{y^3}{3}$$

(omitting $-2xy^2 - 2x^2y$ which already occur in $\int M\,dx$)
Therefore, the solution of the given equation is

$$\frac{x^3}{3} - 2x^2 y - 2xy^2 + \frac{y^3}{3} = \lambda \Rightarrow x^3 + y^3 - 6xy(x+y) = 3\lambda$$

Hence, $x^3 + y^3 - 6xy(x + y) = c$ $(3\lambda = c)$

Illustration 25.23 Solve the differential equation

$$xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$$

Solution: We know that

$$d(x^2 + y^2) = 2(xdx + ydy)$$

$$d\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{1+(y/x)^2} \cdot \frac{xdy - ydx}{x^2} = \frac{xdy - ydx}{x^2 + y^2}$$

Therefore, the equation becomes

$$\frac{1}{2} d(x^2 + y^2) + d\tan^{-1}\left(\frac{y}{x}\right) = 0 \text{ (exact equation)}$$

On integrating, we get

$$\frac{1}{2}(x^2 + y^2) + \tan^{-1}\frac{y}{x} = \frac{c}{2}$$

$$\Rightarrow x^2 + y^2 + 2\tan^{-1}\frac{y}{x} = c$$

Therefore, $y = x\tan\dfrac{c - x^2 - y^2}{2}$.

25.14 Linear Differential Equation

A differential equation is a linear differential equation if it is expressible in the form

$$P_o \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_{n-1} \frac{dy}{dx} + P_n y = Q$$

where $P_0, P_1, P_2, \ldots, P_{n-1}, P_n$ and Q are either constants or functions of the independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and a dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be a linear differential equation. Otherwise, it is a non-linear differential equation.

It follows from the above definition that a differential equation will be a non-linear differential equation if

(A) its degree is more than one,
(B) any of the differential coefficient has exponent more than one,
(C) exponent of the dependent variable is more than one, and
(D) products containing a dependent variable and its differential coefficients are present.

25.14.1 Linear Differential Equation of First Order

The general form of a linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q \quad (25.4)$$

where P and Q are functions of x (or constants).

For example, $x\dfrac{dy}{dx} + 2y = x^3$, $\dfrac{dy}{dx} + 2y = \sin x$, etc. are linear differential equations. These types of differential equations are solved when they are multiplied by a factor, which is called an integrating factor, because by multiplication of this factor the left-hand side of the differential equation (25.4) becomes the exact differential of some function.

Multiplying both sides of Eq. (25.4) by $e^{\int Pdx}$, we get

$$e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int Pdx} \Rightarrow \frac{d}{dx}\left\{ye^{\int Pdx}\right\} = Qe^{\int Pdx}$$

On integrating both sides w.r.t. x, we get

$$ye^{\int Pdx} = \int Qe^{\int Pdx} \cdot dx + C \quad (25.5)$$

which is the required solution, where C is the constant of integration. $e^{\int Pdx}$ is called the integrating factor. The solution or Eq. (25.5) in short may also be written as $y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + C$.

Illustration 25.24 Solve the differential equation

$$x\frac{dy}{dx} = y - \cos\left(\frac{1}{x}\right)$$

Solution: Here,

$$x\frac{dy}{dx} - y = -\cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x} \cdot y = -\frac{1}{x}\cos\left(\frac{1}{x}\right)$$

this is in the linear form.

Integrating factor, $e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}$.

Multiplying by the integrating factor,

$$\frac{1}{x} \cdot \frac{dy}{dx} - \frac{1}{x^2} y = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{d}{dx}\left\{\frac{1}{x} \cdot y\right\} = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

$$\Rightarrow d\left(\frac{y}{x}\right) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$$

$$\Rightarrow \int d\left(\frac{y}{x}\right) = \int -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$$

Hence,

$$\frac{y}{x} = \int \cos\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{y}{x} = \sin\left(\frac{1}{x}\right) + c,$$

where c is an arbitrary constant.

Thus, $y - x\sin\left(\frac{1}{x}\right) = cx.$

Illustration 25.25 Solve the differential equation

$$\frac{dy}{dx} + 2y\tan x = \sin x$$

Solution: Comparing with $\frac{dy}{dx} + Py = Q$, we get

$$P = 2\tan x,\ Q = \sin x$$

$$\text{I.F.} = e^{\int 2\tan x dx} = e^{2\ln\sec^2 x} = e^{\ln\sec^2 x} = \sec^2 x$$

Multiplying the given equation by I.F. and integrating, we get

$$y\sec^2 x = \int \sin x\sec^2 x dx = \int \sec x\tan x dx$$

Therefore, y $\sec^2 x = \sec x + c$.

Illustration 25.26 Solve the differential equation

$$(1+y^2) + (x - e^{\tan^{-1}y})\frac{dy}{dx} = 0$$

Solution: We have

$$(x - e^{\tan^{-1}y})\frac{dy}{dx} = -(1+y^2) \Rightarrow \frac{dx}{dy} = -\left(\frac{x - e^{\tan^{-1}y}}{1+y^2}\right)$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}y}}{1+y^2} \qquad (1)$$

This is a linear differential equation of the form $\frac{dx}{dy} + R(y)\cdot x = S(y)$

$$R = \frac{1}{1+y^2},\ S = \frac{e^{\tan^{-1}y}}{1+y^2}$$

Integrating factor $= e^{\int Rdy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1}y}$

Multiplying Eq. (1) by I.F. and integrating,

$$xe^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2}\cdot e^{\tan^{-1}y}dy = \int \frac{(e^{\tan^{-1}y})^2 dy}{1+y^2} = \frac{(e^{\tan^{-1}y})^2}{2} + \frac{k}{2}$$

Therefore, $2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + k$.

25.14.2 Equation Reducible to Linear Differential Equation (Bernoulli's Differential Equation)

1. Differential equation of the form

$$R(y)\frac{dy}{dx} + P(x)S(y) = Q(x)$$

such that $\frac{dS}{dy} = R$, then put $S(y) = t$. So,

$$\frac{dt}{dx} = \frac{dS}{dx} = \frac{dS}{dy}\cdot\frac{dy}{dx} = \frac{Rdy}{dx}$$

Thus, the differential equation reduces to $\frac{dt}{dx} + P(x)t = Q(x)$, which is the linear differential equation.

2. Differential equation of the form

$$\frac{dy}{dx} + Py = Qy^n$$

P and Q are functions of x and this equation is called Bernoulli's equation.
To solve this, divide the equation by y^n, then

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{1}{y^{n-1}} = Q$$

Put $\frac{1}{y^{n-1}} = t$. Then

$$-\frac{(n-1)}{y^n}\frac{dy}{dx} = \frac{dt}{dx}$$

Differential equation reduces to

$$\frac{dt}{dx} + \left(\frac{1}{n-1}\right)P(x)t = \frac{Q(x)}{(1-n)}$$

Illustration 25.27 Solve the differential equation

$$\frac{dy}{dx} - y\tan x = -y^2\sec x$$

Solution: Rewriting the given equation,

$$y^{-2}\frac{dy}{dx} - y^{-1}\tan x = -\sec x$$

Let $y^{-1} = v$. Then

$$-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$$

Therefore,

$$\frac{dv}{dx} + \tan x\cdot v = \sec x \qquad (1)$$

$$\text{I.F.} = e^{\int \tan x} = e^{\ln\sec x} = \sec x$$

Multiplying Eq. (1) by $\sec x$ and integrating,

$$v\sec x = \int \sec^2 x dx = \tan x + c$$

Thus, the required solution is $\frac{\sec x}{y} = \tan x + c$.

Illustration 25.28 Solve the differential equation

$$\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$$

Solution:

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{x}\frac{1}{\log z} = \frac{1}{x^2}$$

Let $\frac{1}{\log z} = t$. Then

$$-\frac{1}{(\log z)^2}\cdot\frac{1}{z}\frac{dz}{dx}=\frac{dt}{dx}$$

Therefore,

$$-\frac{dt}{dx}+\frac{t}{x}=\frac{1}{x^2}$$

$$\Rightarrow \frac{dt}{dx}-\frac{t}{x}=-\frac{1}{x^2} \quad (1)$$

$$\text{I.F.}=e^{\int -\frac{dx}{x}}=e^{-\ln x}=e^{\ln\frac{1}{x}}=\frac{1}{x}$$

Multiplying Eq. (1) by $\frac{1}{x}$ and integrating, we get

$$\frac{t}{x}=\int -\frac{1}{x^3}dx=\frac{1}{2x^2}+c \Rightarrow \frac{1}{x\log z}=\frac{1}{2x^2}+c$$

Hence, the required solution is

$$\left(\frac{1}{\ln z}\right)x=\left(\frac{1}{2}\right)+cx^2$$

Illustration 25.29 Solve the differential equation

$$\frac{dy}{dx}=x^3y^3-xy$$

Solution: Here,

$$\frac{dy}{dx}+xy=x^3y^3$$

Therefore,

$$\frac{1}{y^3}\frac{dy}{dx}+x\,.\,\frac{1}{y^2}=x^3 \quad (1)$$

Put $\frac{1}{y^2}=z$. Then

$$\frac{-2}{y^3}\frac{dy}{dx}=\frac{dz}{dx}$$

Therefore, Eq. (1) becomes

$$-\frac{1}{2}\frac{dz}{dx}+xz=x^3$$

$$\Rightarrow \frac{dz}{dx}-2xz=-2x^3$$

It is in the linear form. So, the integrating factor $e^{\int -2x dx}=e^{-x^2}$. Multiplying by it,

$$e^{-x^2}\frac{dz}{dx}-2xe^{-x^2}\cdot z=-2x^3\cdot e^{-x^2}$$

$$\Rightarrow \frac{d}{dx}(ze^{-x^2})=-2x^3e^{-x^2}$$

Therefore,

$$d(ze^{-x^2})=-2x^3e^{-x^2}dx$$

$$\Rightarrow \int d\,(ze^{-x^2})=\int -2x^3.e^{-x^2}dx$$

$$\Rightarrow ze^{-x^2}=-\int te^{-t}dt \qquad (\text{putting } x^2=t)$$

$$=-\left[t\,.\,\frac{e^{-t}}{-1}-\int\frac{e^{-t}}{-1}dt\right]=te^{-t}-\int e^{-t}dt$$

$$\Rightarrow \frac{1}{y^2}e^{-x^2}=x^2.e^{-x^2}+e^{-x^2}+c$$

Thus, $(x^2+1+ce^{x^2})\,y^2=1$.

Illustration 25.30 Solve the differential equation

$$\frac{dy}{dx}=\frac{y}{y\sin y-x}$$

Solution: We have

$$\frac{dy}{dx}=\frac{y}{y\sin y-x}$$

$$\Rightarrow \frac{dx}{dy}=\frac{y\sin y-x}{y}=\sin y-\frac{x}{y}$$

$$\Rightarrow \frac{dx}{dy}+\frac{1}{y}x=\sin y$$

which is a linear differential equation in x and y.

Its integrating factor is I.F. $=e^{\int\frac{1}{y}dy}=e^{\log y}=y$.

Thus, the solution is

$$xy=\int y\sin y\ dy$$

$$\Rightarrow xy=-y\cos y+\sin y+c$$

25.15 Solution of Differential Equation of the First Order but of Higher Degree

In such differential equations, we substitute the lower degree derivative by some other variable.

Illustration 25.31 Solve the differential equation

$$x\left(\frac{dy}{dx}\right)^2+(y-x)\frac{dy}{dx}-y=0$$

Solution: Here,

$$xp^2+(y-x)p-y=0 \qquad \left(\text{where } p=\frac{dy}{dx}\right)$$

$$\Rightarrow xp^2-xp+yp-y=0$$

$$\Rightarrow xp(p-1)+y(p-1)=0$$

$$\Rightarrow (p-1)\,(xp+y)=0$$

Thus,

$$p-1=0$$

or

$$xp+y=0$$

Now,

$$p-1=0$$

$$\Rightarrow \frac{dy}{dx}=1 \Rightarrow y=x+c$$

Also,

$$xp+y=0$$

$$x\frac{dy}{dx}+y=0$$

$$\Rightarrow xdy+ydx=0$$

$$\Rightarrow d(xy) = 0$$
$$\Rightarrow xy = c$$

Hence, the general solution is $(y - x - c)(xy - c) = 0$, where c is an arbitrary constant.

Illustration 25.32 Solve the differential equation

$$1+\left(\frac{dy}{dx}\right)^2 = x\frac{dy}{dx}$$

Solution: Here,

$$xp = 1 + p^2 \qquad \left(\text{where } p = \frac{dy}{dx}\right)$$

$$\Rightarrow x = \frac{1}{p} + p \qquad (1)$$

which is of the form $x = f(y, p)$ {solvable for x}

On differentiating Eq. (1) w.r.t. y, we get

$$\frac{1}{p} = \left(\frac{-1}{p^2}+1\right)\frac{dp}{dy}$$

$$\Rightarrow dy = p\left(1-\frac{1}{p^2}\right)dp$$

Therefore,

$$y + c = \int\left(p-\frac{1}{p}\right)dp$$

where c is an arbitrary constant.

$$y + c = \frac{p^2}{2} - \log p \qquad (2)$$

Thus, the p-eliminate, obtained by eliminating p from Eqs. (1) and (2), is the general solution.

Your Turn 3

1. The solution of $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$ is

(A) $\frac{x}{y} + e^{x^3} = c$ **(B)** $\frac{x}{y} - e^{x^3} = 0$

(C) $\frac{-x}{y} + e^{x^3} = 0$ **(D)** None of these

Ans. (A)

2. The solution of $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$ is

(A) $\frac{x}{y} + \frac{1}{xy} = k$ **(B)** $\log\left(\frac{x}{y}\right) = \frac{1}{xy} + k$

(C) $\frac{x}{y} + \frac{1}{xy} = k$ **(D)** $\log\left(\frac{x}{y}\right) = xy + k$

Ans. (B)

3. Solution of $(xy \cos xy + \sin xy)dx + x^2 \cos xy\, dy = 0$ is

(A) $x \sin(xy) = k$ **(B)** $xy \sin(xy) = k$

(C) $\frac{x}{y}\sin(xy) = k$ **(D)** $x \sin(xy) = k$

Ans. (A)

4. The solution of $(x - y^3)dx + 3xy^2\,dy = 0$ is

(A) $\log x + \frac{x}{y}$ **(B)** $\log x + \frac{y^3}{x} = k$

(C) $\log x - \frac{x}{y^3} = k$ **(D)** $\log xy - y^3 = k$

Ans. (B)

5. If c is any arbitrary constant, then the general solution of the differential equation $y\,dx - x\,dy = xy\,dx$ is given by

(A) $y = cx\,e^{-x}$ **(B)** $x = cx\,e^{-x}$

(C) $y + e^x = cx$ **(D)** $ye^x = cx$

Ans. (D)

6. The solution of the equation $\frac{dy}{dx} + y\tan x = x^m \cos x$ is

(A) $(m + 1)y = x^{m+1} \cos x + c(m + 1)\cos x$

(B) $my = (x^m + c) \cos x$

(C) $y = (x^{m+1} + c)\cos x$

(D) None of these

Ans. (A)

7. An integrating factor for the differential equation $(1 + y^2)dx - (\tan^{-1} y - x)dy = 0$ is

(A) $\tan^{-1} y$ **(B)** $e^{\tan^{-1} y}$

(C) $\frac{1}{1+y^2}$ **(D)** $\frac{1}{x(1+y^2)}$

Ans. (B)

8. The solution of the differential equation

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3}y = \frac{\sin^2 x}{1+x^3} \text{ is}$$

(A) $y(1+x^3) = x + \frac{1}{2}\sin 2x + c$

(B) $y(1+x^3) = cx + \frac{1}{2}\sin 2x$

(C) $y(1+x^3) = cx - \frac{1}{2}\sin 2x$

(D) $y(1+x^3) = \frac{x}{2} - \frac{1}{4}\sin 2x + c$

Ans. (D)

9. An integrating factor of the differential equation

$$x\frac{dy}{dx} + y\log x = xe^x x^{-\frac{1}{2}\log x}, \quad (x > 0) \text{ is}$$

(A) $x(y + \cos x) = \sin x + c$

(B) $x(y - \cos x) = \sin x + c$

(C) $x(y \cdot \cos x) = \sin x + c$

(D) $x(y - \cos x) = \cos x + c$

Ans. (A)

10. An integrating factor of the differential equation

$$x\frac{dy}{dx} + y\log x = xe^x x^{-\frac{1}{2}\log x}, \quad (x > 0) \text{ is}$$

(A) $x^{\log x}$ **(B)** $(\sqrt{x})^{\log x}$

(C) $(\sqrt{e})^{\log x}$ **(D)** e^{x^2}

Ans. (B)

25.16 Applications of Differential Equation

25.16.1 Problem Based on Rate of Change

Illustration 25.33 A particle moves in a straight line with a velocity given by $\frac{dx}{dt}=(x+1)$ (x is the distance described). What is the time taken by a particle to transverse a distance of 99 metres?

Solution: We have

$$\frac{dx}{x+1}=dt$$

On integrating, we get

$$\int_0^{99}\frac{dx}{x+1}=\int_0^t dt$$

$$\Rightarrow [\ln(x+1)]_0^{99}=t$$

Therefore, time taken by the particle is

$$t=\ln 100=\log_e(10)^2=2\log_e 10$$

Illustration 25.34 The rate of cooling of a substance in moving air is proportional to the difference of temperatures of the substance and the air. A substance cools from 36°C to 34°C in 15 min. Find when the substance will have the temperature 32°C, and it is known that the constant temperature of air is 30°C.

Solution: Let the temperature of the substance at time t minutes is T.
The rate of cooling of the substance = $k(T-30)$°C/min (from the question).

But the rate of cooling = rate of decrease of temperature

$$=-\frac{dT}{dt}$$

Therefore, from the question,

$$-\frac{dT}{dt}=k(T-30)$$

or

$$-\frac{dT}{T-30}=kdt$$

Hence,

$$\int_{36}^{34}-\frac{dT}{T-30}=\int_0^{15}kdt$$

$$\Rightarrow [-\log(T-30)]_{36}^{34}=k[t]_0^{15}$$

Therefore,

$$\log\frac{6}{4}=k\cdot 15\Rightarrow k=\frac{1}{15}\log\frac{3}{2}$$

Again,

$$\int_{36}^{32}-\frac{dT}{T-30}=\int_0^t kdt$$

where t minutes is the required time.

Therefore,

$$[-\log(T-30)]_{36}^{32}=\frac{1}{15}\log\frac{3}{2}.[t]_0^t$$

$$\Rightarrow \log 3=\frac{1}{15}\log\frac{3}{2}.t$$

$$\Rightarrow t=\frac{15\log 3}{\log 3/2}\text{ minutes}$$

Illustration 25.35 A and B are two separate reservoirs of water. The capacity of A is double that of B. Both the reservoirs are filled completely with water. Water is released simultaneously from both the reservoirs. For each of the reservoirs, the rate of flow out at any instant is proportional to the quantity of water left in the reservoir. After 1h, the quantity of water in A is 1.5 times the quantity of water in B. After how many hours from the time of release of water, do both A and B have the same quantity of water?

Solution: Let at time t hours, the volume of water in A and B be u and v, respectively. From the question,

$$\frac{du}{dt}=-k_1u \quad (1)$$

and

$$\frac{dv}{dt}=-k_2v \quad (2)$$

At $t=0$, $u=2V$ and $v=V$ (from the question). Solving Eq. (1),

$$\frac{du}{u}=-k_1dt$$

Integrating both sides, we get

$$\log u=-k_1t+c_1$$

$$u=e^{-k_1t+c_1} \quad (3)$$

When $t=0$, $u=2V$. Then

$$2V=e^{c_1}$$

Hence, Eq. (3) gives

$$u=2Ve^{-k_1t} \quad (4)$$

Similarly, from Eq. (2), we get

$$v=Ve^{-k_2t} \quad (5)$$

From the question, if $v=v_0$ when $t=1$, then $u=\frac{3}{2}v_0$.

Therefore, Eq. (4) gives

$$\frac{3}{2}v_0=2Ve^{-k_1}$$

and Eq. (5) gives

$$v_0=Ve^{-k_2}$$

Dividing these, we get

$$\frac{3}{2}=\frac{2e^{-k_1}}{e^{-k_2}}=2e^{k_2-k_1}$$

Thus,

$$e^{k_2-k_1}=\frac{3}{4} \quad (6)$$

Let after T hours, the volume of water in A and B be equal. Therefore, Eqs. (4) and (5) gives

$$2Ve^{-k_1T}=Ve^{-k_2T}$$

$$\Rightarrow 2e^{-k_1T}=e^{-k_2T}\Rightarrow 2e^{(k_2-k_1)T}=1$$

Using Eq. (6), we get

$$2.\left(\frac{3}{4}\right)^T = 1$$

$$\Rightarrow \left(\frac{4}{3}\right)^T = 2$$

Therefore,

$$T\log\frac{4}{3} = \log 2 \Rightarrow T = \frac{\log 2}{\log\frac{4}{3}} = \log_{4/3}{}^2$$

Hence, after $\log_{4/3} 2$ hours, the reservoirs will have the same quantity of water.

25.16.2 Problem Based on Geometry: Some Results on Tangents and Normal

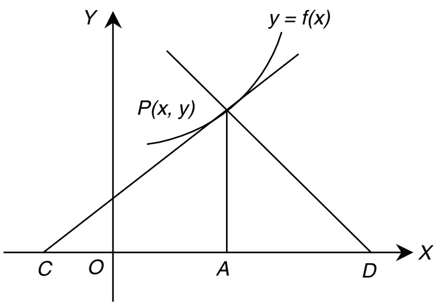


Figure 25.1

1. (See Fig. 25.1). The equation of the tangent at point $P(x, y)$ to the curve $y = f(x)$ is $Y - y = \frac{dy}{dx}(X - x)$.

2. The equation of the normal at point $P(x, y)$ to the curve $y = f(x)$ is $Y - y = -\frac{1}{\left(\frac{dy}{dx}\right)}(X - x)$.

3. The length of the tangent $= CP = y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

4. The length of the normal $= PD = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

5. The length of the Cartesian subtangent = CA = $y\frac{dx}{dy}$

6. The length of the Cartesian subnormal = AD = $y\frac{dy}{dx}$

7. The initial ordinate of the tangent = $OB = y - x\frac{dy}{dx}$

8. The tangent meets X-axis at $\left(x - \frac{y}{(dy/dx)}, 0\right)$ and Y-axis at $\left(0, y - x\frac{dy}{dx}\right)$.

9. The normal meets X-axis at $\left(x + y\frac{dy}{dx}, 0\right)$ and Y-axis at $\left(0, y + \frac{x}{(dy/dx)}\right)$.

10. The orthogonal trajectories of a family of curves form another family of curves such that each curve of one family cuts all the curves of the other family at right angles.

The differential equation of the orthogonal trajectories of the curves $f\left(x, y, \frac{dy}{dx}\right) = 0$ is the family of curves whose differential equation is $f\left(x, y, \frac{-dx}{dy}\right) = 0$.

Method: To find the orthogonal trajectories of a family of curves whose differential equation is known, put $-dx/dy$ in place of dy/dx in the equation. The resulting differential equation is the equation of the orthogonal trajectories.

Illustration 25.36 If the slope of the tangent at (x, y) to a curve passing through $\left(1, \frac{\pi}{4}\right)$ is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$, then find the equation of the curve.

Solution: We have

$$\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$$

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = v - \cos^2 v$$

$$\Rightarrow x\frac{dv}{dx} = -\cos^2 v$$

$$\Rightarrow \sec^2 v dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\tan v = -\ln x + c$$
$$\Rightarrow \tan(y/x) = -\ln x + c$$

For $x = 1$, $y = \pi/4$, the above equation becomes

$$\tan\pi/4 = -\ln 1 + c \Rightarrow 1 = 0 + c$$
$$\Rightarrow c = 1$$

Therefore,

$$\tan(y/x) = 1 - \ln x$$

$$\Rightarrow y/x = \tan^{-1}(1 - \ln x) = \tan^{-1}(\ln e - \ln x) = \tan^{-1}\left[\ln\left(\frac{e}{x}\right)\right]$$

$$\Rightarrow y = x\tan^{-1}\left[\ln\left(\frac{e}{x}\right)\right]$$

Illustration 25.37 Find the equation of the curve which is such that the portion of the axis of x cut off between the origin and the tangent at any point is proportional to the ordinate of that point (b is constant of proportionality).

Solution: Tangent at $P(x, y)$ to the curve $y = f(x)$ may be expressed as

$$Y - y = \frac{dy}{dx}(X - x)$$

Therefore,

$$Q = \left(x - y\frac{dx}{dy}, 0\right)$$

As per question, $OQ \propto y$. Therefore,

$$x - y\frac{dx}{dy} \propto y \Rightarrow x - y\frac{dx}{dy} = by \Rightarrow \frac{x}{y} - \frac{dx}{dy} = b$$

Therefore,

$$\frac{dx}{dy} = \frac{x}{y} - b$$

Let $\frac{x}{y} = v$. Then

$$x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\Rightarrow \frac{x}{y} - b = v + y\frac{dv}{dy} \Rightarrow v - b = v + y\frac{dv}{dy}$$

$$\Rightarrow -b = y\frac{dv}{dy} \Rightarrow -b\frac{dy}{y} = dv$$

On integrating, we get

$$\int dv = -b\int\frac{dy}{y} \Rightarrow v = -b\ln y + a \Rightarrow \frac{x}{y} = a - b\ln y \text{ (a, an arbitrary constant)}$$

Hence, the equation of the curve is $x = y(a - b\ln y)$.

Illustration 25.38 A curve (or line) passes through (1, 1) such that the triangle formed by the coordinate axes and the tangent at any point of the curve is in the first quadrant and has its area equal to 2. Form the differential equation and find the equations of the possible curves.

Solution: Let the curve be $y = f(x)$. Then the equation of the tangent at the point (x, y) of the curve is (see Fig. 25.2)

$$Y - y = \frac{dy}{dx}(X - x) \quad (1)$$

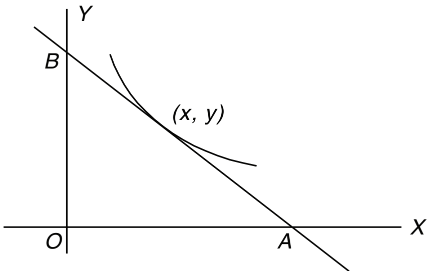


Figure 25.2

Solving Eq. (1) with $Y = 0$ and $X = 0$ successively, we get

$$-y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow X = \frac{x(dy/dx) - y}{dy/dx} \quad (= OA)$$

and

$$Y - y = \frac{dy}{dx}(-x)$$

$$\Rightarrow Y = y - x\frac{dy}{dx} \quad (= OB)$$

Therefore, the area of the triangle formed by the tangent and the axes in the first quadrant is

$$2 = \frac{1}{2}\, OA \cdot OB$$

$$\Rightarrow 2 = \frac{1}{2}\cdot\frac{x(dy/dx) - y}{dy/dx}\cdot\left(y - x\frac{dy}{dx}\right)$$

$$\Rightarrow 4\frac{dy}{dx} = \left(x\frac{dy}{dx} - y\right)\left(y - x\frac{dy}{dx}\right)$$

Taking $\frac{dy}{dx} = p$, we get

$$4p = (xp - y)(y - xp)$$

$$-(y - xp)^2 \Rightarrow y - xp = \pm 2\sqrt{-p}$$

$$\Rightarrow y = xp \pm 2\sqrt{-p} \quad (2)$$

On differentiating w.r.t. x, we get

$$p = p + x\frac{dp}{dx} \pm 2\cdot\frac{1}{2\sqrt{-p}}(-1)\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx}\left\{x \mp \frac{1}{\sqrt{-p}}\right\} = 0$$

Therefore,

$$\frac{dp}{dx} = 0 \quad \text{or} \quad x = \pm\frac{1}{\sqrt{-p}}$$

Now,

$$\frac{dp}{dx} = 0 \Rightarrow p = c$$

Therefore, putting in Eq. (2) we get, $y = cx \pm 2\sqrt{-c}$

This gives a family of lines. If it passes through (1, 1), then

$$1 = c \pm 2\sqrt{-c}$$

$$\Rightarrow (1 - c)^2 = -4c \Rightarrow (1 + c)^2 = 0 \Rightarrow c = -1$$

Hence, the line is $y = -x \pm 2$, that is, $x + y = \pm 2$.

But in the first quadrant, x, y are positive. So, $x + y = 2$.

Now,

$$x = \pm\frac{1}{\sqrt{-p}}$$

$$\Rightarrow p = -\frac{1}{x^2}$$

Putting this in Eq. (2), we get

$$y = x\cdot\left(-\frac{1}{x^2}\right) \pm 2\sqrt{\frac{1}{x^2}}$$

Therefore,

$$y = \frac{1}{x}, \frac{-3}{x}$$

$$\Rightarrow xy = 1, -3$$

But in the first quadrant, x, y are positive. So, $xy = 1$. This is the singular solution of Eq. (2), which is a curve, and it passes through (1, 1). Thus, the required curves (or lines) are $x + y = 2$ and $xy = 1$.

Illustration 25.39 A curve through (1, 2) has its slope at any point (x, y) equal to $\frac{2}{y-2}$. Find the area of the region bounded by the curve and the line $2x - y - 4 = 0$.

Solution: Here, slope is

$$\frac{dy}{dx} = \frac{2}{y-2}$$

$$\Rightarrow (y-2)dy = 2dx$$

Therefore,

$$\int (y-2)\,dy = 2\int dx$$

$$\Rightarrow \frac{y^2}{2} - 2y = 2x + c$$

It passes through (1, 2). So,

$$\frac{4}{2} - 4 = 2 + c \;\Rightarrow c = -4$$

Hence, the equation of the curve is

$$\frac{y^2}{2} - 2y = 2x - 4$$

$$\Rightarrow y^2 = 4(x + y) - 8$$

To find the area, we have to draw a rough sketch of the curve.
When $y = 0$, then $0 = 4(x + 0) - 8$, that is, $x = 2$. Therefore, the curve cuts the x-axis at $A(2, 0)$ only.
When $x = 0$, then $y^2 = 4(0 + y) - 8$ or $y^2 - 4y + 8 = 0$; so its roots are imaginary. Thus, the curve does not cut the y-axis.
Again,

$$y^2 = 4(x + y) - 8 \Rightarrow y^2 - 4y + 4 = 4x - 4$$

$$\Rightarrow (y-2)^2 = 4(x-1)$$

Hence, it is a parabola.
Its vertex C is (1, 2) and the axis is $y - 2 = 0$.

Now, the line is $2x - y - 4 = 0$, that is, $\frac{x}{2} + \frac{y}{-4} = 1$.

Hence, it passes through $A(2, 0)$ and $(0, -4)$ [see Fig. 25.3].

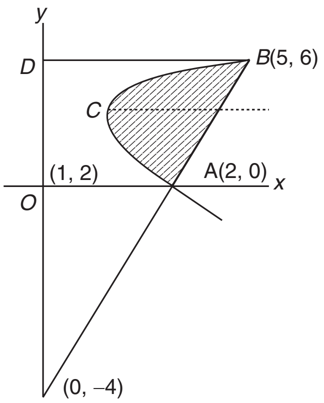


Figure 25.3

Solving $y^2 = 4(x + y) - 8$ and $2x - y - 4 = 0$, we get

$$\Rightarrow y^2 = 2(y + 4) + 4y - 8$$

$$\Rightarrow y^2 - 6y = 0$$

$$\Rightarrow y = 0, 6$$

When $y = 6$, then $x = \frac{y+4}{2} = \frac{6+4}{2} = 5$.

So, $B = (5, 6)$.

The required area = Area (OABDO) – Area (BCAODB) (1)

Now,

$$\text{Area (OABDO)} = \int_0^6 (x)_{\text{line}}\,dy = \int_0^6 \frac{y+4}{2}\,dy = \frac{1}{2}\left[\frac{y^2}{2} + 4y\right]_0^6$$

$$= \frac{1}{2}[18 + 24] = 21$$

$$\text{Area (BCAODB)} = \int_0^6 (x)_{\text{curve}}\,dy = \int_0^6 \frac{y^2 - 4y + 8}{4}\,dy$$

$$= \left[\frac{y^3}{12} - \frac{y^2}{2} + 2y\right]_0^6 = 18 - 18 + 12 = 12$$

Therefore, from Eq. (1), the required area = 21 – 12 = 9 sq. units.

Illustration 25.40 Find the orthogonal trajectories of the circles $x^2 + y^2 - ay = 0$ where a is a parameter.

Solution: Here,

$$x^2 + y^2 - ay = 0$$

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

Therefore,

$$2x + 2y\frac{dy}{dx} - \frac{x^2 + y^2}{y}\frac{dy}{dx} = 0$$

$$\left(\text{As } x^2 + y^2 - ay = 0 \;\Rightarrow a = \frac{x^2 + y^2}{y}\right)$$

$$\Rightarrow 2x + \frac{y^2 - x^2}{y}\frac{dy}{dx} = 0$$

This is the differential equation of the circles. The equation of orthogonal trajectories is

$$2x + \frac{y^2 - x^2}{y}\cdot\left(-\frac{dx}{dy}\right) = 0 \qquad \left(\text{putting } \frac{-dx}{dy} \text{ in place of } \frac{dy}{dx}\right)$$

$$\Rightarrow 2xy\,dy + (x^2 - y^2)\,dx = 0$$

It is a homogenous equation.
Put $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dy}{dx}$$

Therefore,

$$2x \cdot vx \cdot \left(v + x\frac{dv}{dx}\right) + x^2 - v^2x^2 = 0$$

$$\Rightarrow 2v\left(v + x\frac{dv}{dx}\right) + 1 - v^2 = 0 \;\Rightarrow\; 1 + v^2 + 2vx\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{dx}{x} + \frac{2v}{1+v^2}dv = 0$$

On integrating, we get

$$\log x + \log(1 + v^2) = \log c$$

Therefore,
$$x(1+v^2)=c$$
$$\Rightarrow x\left(1+\frac{y^2}{x^2}\right)=c$$
$$\Rightarrow x^2+y^2=cx$$

Your Turn 4

1. The equation of the curve which passes through the point (1, 1) and whose slope is given by $\frac{2y}{x}$ is
   (A) $y=x^2$ **(B)** $x^2-y^2=0$
   (C) $2x^2+y^2=3$ **(D)** None of these
   Ans. (A)

2. The equation of the curve that passes through the point (1, 2) and satisfies the differential equation $\frac{dy}{dx}=\frac{-2xy}{(x^2+1)}$ is
   (A) $y(x^2+1)=4$ **(B)** $y(x^2+1)+4=0$
   (C) $y(x^2-1)=4$ **(D)** None of these
   Ans. (A)

3. Equation of curve through point (1, 0) which satisfies the differential equation $(1+y^2)dx-xydy=0$ is
   (A) $x^2+y^2=1$ **(B)** $x^2-y^2=1$
   (C) $2x^2+y^2=3$ **(D)** None of these
   Ans. (B)

4. Equation of curve passing through (3, 9) which satisfies the differential equation $\frac{dy}{dx}=x+\frac{1}{x^2}$ is
   (A) $6xy=3x^2-6x+29$ **(B)** $6xy=3x^3-29x+6$
   (C) $6xy=3x^3+29x-6$ **(D)** None of these
   Ans. (C)

5. The differential equation $y\frac{dy}{dx}+x=a$ (a is any constant) represents
   (A) A set of circles having centre on the y-axis
   (B) A set of circles centre on the x-axis
   (C) A set of ellipses
   (D) None of these
   Ans. (B)

6. The equation of a curve passing through $\left(2,\frac{7}{2}\right)$ and having gradient $1-\frac{1}{x^2}$ at (x, y) is
   (A) $y=x^2+x+1$ **(B)** $xy=x^2+x+1$
   (C) $xy=x+1$ **(D)** None of these
   Ans. (B)

7. The equation of the curve through the point (1,0) and whose slope is $\frac{y-1}{x^2+x}$ is
   (A) $(y-1)(x+1)+2x=0$ **(B)** $2x(y-1)+x+1=0$
   (C) $x(y-1)(x+1)+2=0$ **(D)** None of these
   Ans. (A)

8. The slope of a curve at any point is the reciprocal of twice the ordinate at the point and it passes though the point (4, 3). The equation of the curve is
   (A) $x^2=y+5$ **(B)** $y^2=x-5$
   (C) $y^2=x+5$ **(D)** $x^2=y-5$
   Ans. (C)

9. Solution of differential equation $x\,dy-y\,dx=0$ represents
   (A) Rectangular hyperbola
   (B) Straight line passing through origin
   (C) Parabola whose vertex is at origin
   (D) Circle whose centre is at origin
   Ans. (B)

10. Integral curve satisfying $y'=\frac{x^2+y^2}{x^2-y^2}$, $y(1)=2$ has the slope at the point (1, 0) of the curve equal to
    (A) $-5/3$ **(B)** -1
    (C) 1 **(D)** 5/3
    Ans. (C)

11. A particle starts at the origin and moves along the x-axis in such a way that its velocity at the point $(x, 0)$ is given by the formula $\frac{dx}{dt}=\cos^2\pi x$. Then the particle never reaches the point on
    (A) $x=\frac{1}{4}$ **(B)** $x=\frac{3}{4}$
    (C) $x=\frac{1}{2}$ **(D)** $x=1$
    Ans. (C)

12. The slope of the tangent at (x, y) to a curve passing through a point (2, 1) is $\frac{x^2+y^2}{2xy}$. Then the equation of the curve is
    (A) $2(x^2-y^2)=3x$ **(B)** $2(x^2-y^2)=6y$
    (C) $x(x^2-y^2)=6$ **(D)** $x(x^2+y^2)=10$
    Ans. (A)

13. A function $y=f(x)$ has a second-order derivatives $f''(x)=6(x-1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y=3x-5$, then the function is
    (A) $(x+1)^3$ **(B)** $(x-1)^3$
    (C) $(x+1)^2$ **(D)** $(x-1)^2$
    Ans. (B)

Additional Solved Examples

1. Solve the differential equation:
   (i) $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$
   (ii) $\sqrt{4x-x^2}\,\frac{dy}{dx}=1+\cos 2y$
   (iii) $x\frac{dy}{dx}+\frac{y^2}{x}=y$

Solution:

(i) $$\frac{dy}{dx} = -\frac{(y^2+y+1)}{(x^2+x+1)}$$

$$\Rightarrow \frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$

On integrating,

$$\int \frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = -\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+k$$

$$\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = -\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+k$$

which may also be written as

$$\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right)+\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)=c \qquad (1)$$

where c is now arbitrary constant in which $\frac{2}{\sqrt{3}}$ is also absorbed.

We know,

$$\tan^{-1}a+\tan^{-1}b=\tan^{-1}\left(\frac{a+b}{1-ab}\right)$$

So, Eq. (1) reduces to

$$2xy + x + y + C' (x + y + 1) = 1$$

(ii) $$\frac{dy}{1+\cos 2y} = \frac{dx}{\sqrt{4x-x^2}}$$

$$\Rightarrow \frac{1}{2}\int\frac{dy}{\cos^2 y} = \int\frac{dx}{\sqrt{4-(x-2)^2}}+c$$

The solution is

$$\frac{\tan y}{2} = \frac{1}{2}\sin^{-1}\left(\frac{x-2}{2}\right)+c$$

$$\Rightarrow \tan y = \sin^{-1}\left(\frac{x-2}{2}\right)+k$$

where 'k' is an arbitrary constant.

(iii) $$\frac{dy}{dx}+\frac{y^2}{x^2}=\frac{y}{x}$$

On setting $y = vx$, the equation is $v + x\frac{dv}{dx}+v^2 = v$. Separating the variables and integrating,

$$\int\frac{dx}{x}+\int\frac{dv}{v^2}=A$$

$$\Rightarrow \log x - \frac{1}{v} = A.$$

This simplifies to the form $x = ce^{x/y}$.

2. Solve the differential equation $(1+\cos 2x)\frac{dy}{dx}-(1+e^y)\sin 2x = 0$; given that $y = 0$, when $x = \frac{\pi}{4}$.

Solution:

$$(1+\cos 2x)\,dy = (1+e^y)\sin 2x\,dx$$

Separating the variables

$$\frac{dy}{1+e^y} = \frac{\sin 2x}{1+\cos 2x}dx$$

$$\Rightarrow \int\left[\frac{(1+e^y)-e^y}{1+e^y}\right]dy = \int \tan x\,dx$$

$$\Rightarrow y-\log(1+e^y)+\log\cos x = A$$

$$\Rightarrow y+\log\left(\frac{\cos x}{1+e^y}\right) = A$$

Taking $y = 0$, when $x=\frac{\pi}{4}$ we get

$$0+\log\left(\frac{1}{2\sqrt{2}}\right) = A \Rightarrow A = -\frac{3}{2}\log 2$$

The solution (called particular solution) is

$$y+\log\left[\left(\frac{\cos x}{1+e^y}\right)2\sqrt{2}\right]=0$$

Therefore, $2\sqrt{2}\cos x = (1+e^y)e^{-y} = e^{-y}+1.$

3. Find the equation of the curve through the origin which satisfies the differential equation $\frac{dy}{dx}=(x-y)^2$.

Solution: Put $x - y = z$. Then

$$1-\frac{dy}{dx}=\frac{dz}{dx}$$

The equation in z is

$$1-\frac{dz}{dx}=z^2$$

$$\Rightarrow \frac{dz}{dx}=1-z^2$$

$$\Rightarrow \int\frac{dz}{1-z^2}=\int dx + A$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)=x+A$$

So, $\log\left(\frac{1+x-y}{1-x+y}\right)=2x+2A$ is the solution which may also be written as $\left(\frac{1+x-y}{1-x+y}\right)=Ce^{2x}$, where C is an arbitrary constant.

The curve passes through the origin. Put $x = 0$, $y = 0$, we get

$$\frac{1}{1}=C \Rightarrow C = 1$$

Therefore, the particular solution is $(1 + x - y) = (1 - x + y)\,e^{2x}$.

4. Solve the differential equation $(x^2-y^2)\frac{dy}{dx}=2xy$ given that $y = 1, x = 1$.

Solution: Taking $y = vx$ and rewriting, the equation is

$$v+x\frac{dv}{dx}=\frac{2x\cdot vx}{x^2-v^2x^2}$$

Therefore,

$$x\frac{dv}{dx}=\frac{2v}{1-v^2}-v=\frac{2v-v+v^3}{1-v^2}$$

Separating the variables,

$$\frac{1-v^2}{v+v^3}dv=\frac{dx}{x}$$

$$\Rightarrow \int\left[\frac{1}{v}-\frac{2v}{1+v^2}\right]dv=\log x+C$$

$$\Rightarrow \log v-\log(1+v^2)-\log x=C$$

$$\Rightarrow \frac{v}{x(1+v^2)}=\text{ constant}$$

Reverting to y, we get,

$$\frac{y}{x^2+y^2}=A$$

$$y=A(x^2+y^2)$$

Put $x=1$, $y=1$. Therefore,

$$1=A\cdot 2\Rightarrow A=\frac{1}{2}$$

The particular solution is $2y=x^2+y^2$.

5. Solve the differential equation $\frac{dy}{dx}+\frac{12x+5y-9}{5x+2y-4}=0$.

Solution: Putting $x=X+l$; $y=Y+m$.

$$\frac{dY}{dX}+\frac{12X+5Y}{5X+2Y}=0 \text{ where } \left.\begin{matrix}12l+5m-9=0\\5l+2m-4=0\end{matrix}\right\} \Rightarrow \left.\begin{matrix}l=2\\m=-3\end{matrix}\right\}$$

Setting $Y=VX$, we get

$$V+X\frac{dV}{dX}+\frac{12+5V}{5+2V}=0$$

$$\Rightarrow X\frac{dV}{dX}=-\left(\frac{12+5V+2V^2+5V}{5+2V}\right)$$

Separating the variables and integrating,

$$\int\frac{2V+5}{2V^2+10V+12}dV=-\int\frac{dX}{X}+A$$

$$\Rightarrow \frac{1}{2}\log(2V^2+10V+12)+\log X=A$$

$$\Rightarrow (2V^2+10V+12)X^2=e^{2A}=C$$

$$\Rightarrow 2Y^2+10XY+12X^2=C$$

$$\Rightarrow Y^2+5XY+6X^2=\frac{C}{2}=C'$$

$$\Rightarrow (y+3)^2+5(x-2)(y+3)+6(x-2)^2=C'$$

which gives the solution in the form $6x^2+5xy+y^2-9x-4y=C'$.

Alternative method: The equation may be rewritten as

$$(12x+5y-9)\,dx+(5x+2y-4)\,dy=0$$

$$(12x-9)\,dx+(2y-4)\,dy+5y\,dx+5x\,dy=0$$

$$(12x-9)\,dx+(2y-4)\,dy+5d(xy)=0$$

On integrating, $6x^2-9x+y^2-4y+5xy=C$, which agrees with the solution by the first method.

6. The tangent and a normal to a curve at any point P meet the x and y axes at A, B, C and D, respectively. Find the equation of the curve passing through (1, 0) if the centre of circle through O, C, P and B lies on the line $y=x$ (where O is origin).

Solution: Let $P(x, y)$ be a point on the curve. Then

$$C\equiv\left(x+y\frac{dy}{dx},\ 0\right)$$

$$B\equiv\left(0,\ y-x\frac{dy}{dx}\right)$$

Circle passing through O, C, P and B has its centre at mid-point of BC (Fig. 25.4).

Let the centre of the circle be (α, β). Then

$$2\alpha=x+y\frac{dy}{dx} \text{ and } 2\beta=y-x\frac{dy}{dx}$$

and since $\beta=\alpha$. Therefore,

$$y-x\frac{dy}{dx}=x+y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-x}{y+x}$$

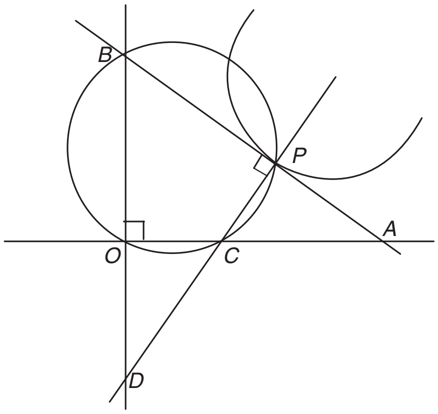


Figure 25.4

Let $y=vx$. Then

$$x\frac{dv}{dx}=-\frac{(1+v^2)}{1+v}\Rightarrow \frac{1+v}{v^2+1}dv=-\frac{dx}{x}$$

On integrating both sides, we get

$$\frac{1}{2}\int\frac{2v}{v^2+1}dv+\int\frac{dv}{v^2+1}=-\int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\log|v^2+1|+\tan^{-1}|v|=-\log x+c$$

$$\Rightarrow \log\left\{(\sqrt{v^2+1})x\right\}+\tan^{-1}v=c$$

$$\Rightarrow \log\sqrt{x^2+y^2}+\tan^{-1}\frac{y}{x}=c$$

As $x=1$ and $y=0$, we get

$$\log 1+\tan^{-1}0=c$$

$$\Rightarrow c=0$$

Therefore, the required curve is $\left(\log\sqrt{x^2+y^2}\right)+\tan^{-1}\left(\frac{y}{x}\right)=0$.

7. Solve $\frac{dy}{dx}+yf'(x)=f(x)\cdot f'(x)$ where $f(x)$ is a given function of x.

Solution: Here,

$$\frac{dy}{dx}=\{f(x)-y\}f'(x) \quad (1)$$

Put $f(x)-y=z$. Then

$$f'(x)-\frac{dy}{dx}=\frac{dz}{dx}$$

Hence, Eq. (1) becomes

$$f'(x)-\frac{dz}{dx}=zf'(x)$$

$$\frac{dz}{dx}=(1-z)f'(x) \Rightarrow \frac{dz}{1-z}=f'(x)\,dx$$

$$\Rightarrow \int\frac{dz}{1-z}=\int f'(x)\,dx \Rightarrow -\log(1-z)=f(x)-c$$

$$f(x)+\log(1-z)=c$$

Thus, $f(x)+\log\{1+y-f(x)\}=c$.

8. Find the equation of the curve passing through (1, 2) whose differential equation is $y(x+y^3)\,dx=x(y^3-x)dy$.

Solution: Here,

$$(xy+y^4)dx=(xy^3-x^2)dy$$

$$\Rightarrow y^3(ydx-xdy)+x(ydx+xdy)=0$$

$$\Rightarrow -x^2y^3.\frac{xdy-ydx}{x^2}+xd(xy)=0$$

$$\Rightarrow -\frac{y}{x}d\left(\frac{y}{x}\right)+\frac{d(xy)}{x^2y^2}=0 \quad (\text{dividing by } x^3y^2)$$

$$\Rightarrow -\int\frac{y}{x}d\left(\frac{y}{x}\right)+\int\frac{d(xy)}{(xy)^2}=c$$

Therefore,

$$-\frac{\left(\frac{y}{x}\right)^2}{2}+\frac{(xy)^{-1}}{-1}=c$$

$$\Rightarrow \frac{1}{2}\cdot\frac{y^2}{x^2}+\frac{1}{xy}+c=0$$

$$y^3+2x+2cx^2y=0$$

It passes through (1, 2). So, $2^3+2+2c\cdot 2=0$. Therefore,

$$c=\frac{-10}{4}=\frac{-5}{2}$$

Hence, the curve is $y^3+2x-5x^2y=0$.

9. If y_1, y_2 are two solutions of the differential equation $\frac{dy}{dx}+P(x)\cdot y=Q(x)$, then prove that $y=y_1+c(y_1-y_2)$ is the general solution of the equation where c is any constant. For what relation between the constants α and β will the linear combination $\alpha y_1+\beta y_2$ also be a solution?

Solution: As y_1 and y_2 are two solutions of the differential equation

$$\frac{dy}{dx}+P(x)\cdot y=Q(x) \quad (1)$$

Therefore,

$$\frac{dy_1}{dx}+P(x)\cdot y_1=Q(x) \quad (2)$$

and

$$\frac{dy_2}{dx}+P(x)\cdot y_2=Q(x) \quad (3)$$

From Eqs. (1) and (2),

$$\left(\frac{dy}{dx}-\frac{dy_1}{dx}\right)+P(x)\cdot(y-y_1)=0$$

Therefore,

$$\frac{d}{dx}(y-y_1)+P(x).(y-y_1)=0 \quad (4)$$

From Eqs. (2) and (3),

$$\frac{d}{dx}(y_1-y_2)+P(x)\cdot(y_1-y_2)=0 \quad (5)$$

From Eqs. (4) and (5),

$$\frac{\frac{d}{dx}(y-y_1)}{\frac{d}{dx}(y_1-y_2)}=\frac{y-y_1}{y_1-y_2}$$

Therefore, on integrating,

$$\log(y-y_1)=\log(y_1-y_2)+\log c$$
$$\Rightarrow \log(y-y_1)=\log[c(y_1-y_2)]$$
$$\Rightarrow y=y_1+c(y_1-y_2)$$

Now, $y=\alpha y_1+\beta y_2$ will be a solution if

$$\frac{d}{dx}(\alpha y_1+\beta y_2)+P(x)\cdot(\alpha y_1+\beta y_2)=Q(x)$$

$$\Rightarrow \alpha\left\{\frac{dy_1}{dx}+P(x)y_1\right\}+\beta\left\{\frac{dy_2}{dx}+P(x)y_2\right\}=Q(x)$$

$$\Rightarrow \alpha\cdot Q(x)+\beta\cdot Q(x)=Q(x) \quad (\text{using Eqs. (2) and (3)})$$

Therefore,

$$(\alpha+\beta)\,Q(x)=Q(x)$$

Hence, $\alpha+\beta=1$.

10. Use the methods of solving the first-order differential equation to find the general solution of $\frac{dy}{dx}\cdot\frac{d^3y}{dx^3}=3\left(\frac{d^2y}{dx^2}\right)^2$.

Solution: Put $\frac{dy}{dx}=p$. Then the equation is

$$p\frac{d^2p}{dx^2}=3\left(\frac{dp}{dx}\right)^2$$

Therefore,

$$\frac{\frac{d^2p}{dx^2}}{\frac{dp}{dx}}=\frac{3}{p}\frac{dp}{dx}$$

$$\Rightarrow \frac{1}{\frac{dp}{dx}}\cdot\frac{d}{dx}\left(\frac{dp}{dx}\right)=\frac{3}{p}\frac{dp}{dx}$$

$$\Rightarrow \frac{d}{dx}\left\{\log\frac{dp}{dx}\right\}=\frac{3}{p}\frac{dp}{dx}$$

$$\Rightarrow d\left\{\log\frac{dp}{dx}\right\} = \frac{3}{p}dp$$

On integrating, we get

$$\log\frac{dp}{dx} = \int\frac{3}{p}dp = 3\log p + \log a$$

with a being an arbitrary constant.
Therefore,

$$\frac{dp}{dx} = e^{3\log p + \log a} = e^{\log(ap^3)} = ap^3$$

or
$$\frac{dp}{p^3} = adx$$

On integrating, we get

$$-\frac{1}{2p^2} = ax + b$$

$$\Rightarrow \frac{-1}{2(ax+b)} = p^2 \Rightarrow p = \sqrt{-\frac{1}{2(ax+b)}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{-1}{2(ax+b)}}$$

Therefore,

$$y = \int\frac{1}{\sqrt{-2ax-b}}dx \Rightarrow y = \frac{\sqrt{-2ax-b}}{-a} + c$$

where a, b and c are arbitrary constants.

11. A curve that passes through (2, 4) and having subnormal of constant length of 8 units can be

(A) $y^2 = 16x - 8$ **(B)** $y^2 = -16x + 24$
(C) $x^2 = 16y - 60$ **(D)** $x^2 = -16y + 68$

Solution: Let the curve be $y = f(x)$.

Subnormal at any point $= \left|y\frac{dy}{dx}\right|$

$$y\frac{dy}{dx} = \pm 8 \Rightarrow ydy = \pm 8\,dx \Rightarrow \frac{y^2}{2} = \pm 8x + c$$

$$\Rightarrow y^2 = 16x + 2c_1 \Rightarrow c_1 = -8$$

or
$$y^2 = -16x + 2c_2 \Rightarrow c_2 = 24$$

Hence, the correct answers are options (A) and (B).

12. Equation of a curve that would cut $x^2 + y^2 - 2x - 4y - 15 = 0$ orthogonally can be

(A) $(y-2) = \lambda(x-1)$ **(B)** $(y-1) = \lambda(x-2)$
(C) $(y+2) = \lambda(x+1)$ **(D)** $(y+1) = \lambda(x+2)$

where $\lambda \in R$.

Solution: Any line passing through the centre of the given circle would meet the circle orthogonally.
Hence, the correct answer is option (A).

13. Let m and n be the order and the degree of the differential equation whose solution is $y = cx + c^2 - 3c^{3/2} + 2$, where c is a parameter. Then

(A) $m = 1, n = 4$ **(B)** $m = 1, n = 3$
(C) $m = 1, n = 2$ **(D)** None

Solution: Differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = c$$

So, the differential equation is

$$y = x\cdot\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 - 3\cdot\left(\frac{dy}{dx}\right)^{3/2} + 2$$

Clearly, its order is 1 and degree is 4.
Hence, the correct answer is option (A).

14. $y = a\sin x + b\cos x$ is the solution of differential equation

(A) $\frac{d^2y}{dx^2} + y = 0$ **(B)** $\frac{dy}{dx} + y = 0$
(C) $\frac{d^2y}{dx^2} = y$ **(D)** $\frac{dy}{dx} = y$

Solution:

$$\frac{dy}{dx} = a\cos x - b\sin x \Rightarrow \frac{d^2y}{dx^2} = -a\sin x - b\cos x = -y$$

Hence, the equation is $\frac{d^2y}{dx^2} + y = 0$.
Hence, the correct answer is option (A).

15. For any differential function $y = f(x)$, the value of

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2} \text{ is:}$$

(A) always zero **(B)** always non-zero
(C) equal to $2y^2$ **(D)** equal to x^2

Solution: For a differential equation,

$$\left(\frac{dy}{dx}\right) = \left(\frac{dx}{dy}\right)^{-1}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -1\left(\frac{dx}{dy}\right)^{-2}\frac{d}{dy}\left(\frac{dx}{dy}\right)\frac{dy}{dx}$$

$$= -\left(\frac{dx}{dy}\right)^{-2}\frac{d^2x}{dy^2}\frac{dy}{dx} = -\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3$$

$$\Rightarrow \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2} = 0$$

Hence, the correct answer is option (A).

16. The degree of differential equation $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x\log\frac{d^2y}{dx^2}$ is

(A) 1 **(B)** 2
(C) 3 **(D)** none of these

Solution: Since the equation is not a polynomial in all the differential coefficient so the degree of equation is not defined.
Hence, the correct answer is option (D).

17. The differential equation of all ellipses centred at origin is

(A) $y_2 + xy_1^2 - yy_1 = 0$ **(B)** $xyy_2 + xy_1^2 - yy_1 = 0$
(C) $yy_2 + xy_1^2 - xy_1 = 0$ **(D)** none of these

Solution: The ellipse centred at origin is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

where a and b are unknown constants.

$$\frac{2x}{a^2}+\frac{2y}{b^2}y_1=0 \Rightarrow \frac{x}{a^2}+\frac{y}{b^2}y_1=0 \qquad (2)$$

On differentiating again, we get

$$\frac{1}{a^2}+\frac{1}{b^2}(y_1^{\,2}+yy_2)=0 \qquad (3)$$

Multiplying Eq. (3) with x and then subtracting from Eq. (2) we get

$$\frac{1}{b^2}(yy_1-xy_1^2-xyy_2)=0 \Rightarrow xyy_2=xy_1^2-yy_1=0$$

Hence, the correct answer is option (B).

18. Particular solution of $y_1+3xy=x$ which passes through (0, 4) is

(A) $3y=1+11\,e^{-\frac{3x^2}{2}}$ **(B)** $y=\frac{1}{3}+11\,e^{-x^2}$

(C) $y=1+\frac{11}{3}e^{-x^2}$ **(D)** $y=\frac{1}{3}+11\,e^{\frac{3}{2}x^2}$

Solution:

$$\frac{dy}{dx}+(3x)y=x$$

$$\text{I.F.}=e^{\int 3x\,dx}=e^{\frac{3}{2}x^2}$$

Therefore, the solution of given equation is

$$ye^{\frac{3}{2}x^2}=\int x.e^{\frac{3}{2}x^2}dx+c=\frac{1}{3}e^{\frac{3}{2}x^2}+c$$

If the curve passes through (0, 4), then

$$4-\frac{1}{3}=c \Rightarrow c=\frac{11}{3}$$

$$y=\frac{1}{3}+\frac{11}{3}e^{-\frac{3}{2}x^2} \Rightarrow 3y=1+11\,e^{-\frac{3}{2}x^2}$$

Hence, the correct answer is option (A).

19. Solution of equation $\frac{dy}{dx}=\frac{3x-4y-2}{3x-4y-3}$ is

(A) $(x-y)^2+c=\log(3x-4y+1)$

(B) $x-y+c=\log(3x-4y+1)$

(C) $x-y+c=\log(3x-4y-3)$

(D) $x-y+c=\log(3x-4y-1)$

Solution: Let $3x-4y=z$. Then

$$3-4\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{1}{4}\left(3-\frac{dz}{dx}\right)$$

Therefore, the given equation becomes

$$\frac{3}{4}-\frac{1}{4}\frac{dz}{dx}=\frac{z-2}{z-3}$$

$$\Rightarrow -\left(\frac{z-3}{z+1}\right)dz=dx \Rightarrow -\left(1-\frac{4}{z+1}\right)dz=dx$$

$$\Rightarrow -z+4\log(z+1)=x+c \Rightarrow \log(3x-4y+1)=x-y+c$$

Hence, the correct answer is option (B).

20. The order of the differential equation, whose general solution is $y=c_1e^x+c_2e^{2x}+c_3e^{3x}+c_4\,e^{x+c_5}$, where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants, is

(A) 5 **(B)** 4 **(C)** 3 **(D)** None of these

Solution:

$$y=(c_1+c_4e^{c_5})e^x+c_2e^{2x}+c_3e^{3x}$$
$$\Rightarrow y=k_1e^x+k_2e^{2x}+k_3e^{3x}$$

Since there are three arbitrary constants.

Hence, the correct answer is option (C).

21. I.F. for $y\ln y\frac{dx}{dy}+x-\ln y=0$ is

(A) $\ln x$ **(B)** $\ln y$

(C) $\ln xy$ **(D)** none of these

Solution:

$$\text{I.F.}=e^{\int\frac{dy}{y\ln y}}=\ln y$$

Hence, the correct answer is option (B).

22. Which one of the following is a differential equation of the family of curves $y=Ae^{2x}+Be^{-2x}$

(A) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$ **(B)** $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-xy+x^2-2=0$

(C) $\frac{d^2y}{dx^2}=4y$ **(D)** $\left(\frac{dy}{dx}\right)^3=4y\left(x\frac{dy}{dx}-2y\right)$

Solution:

$$y=Ae^{2x}+Be^{-2x} \Rightarrow \frac{dy}{dx}=2(Ae^{2x}-Be^{-2x})$$

$$\frac{d^2y}{dx^2}=4\,(Ae^{2x}+Be^{-2x})=4\,y$$

Hence, the correct answer is option (C).

23. Solution of $\frac{dy}{dx}+\sin\left(\frac{x+y}{2}\right)=\sin\left(\frac{x-y}{2}\right)$ is

(A) $\log\tan\frac{y}{4}=c-2\sin\frac{x}{2}$ **(B)** $\log\cot\frac{y}{4}=c-2\sin\frac{x}{2}$

(C) $\log\tan\frac{y}{4}=c-2\cos\frac{x}{2}$ **(D)** None of these

Solution:

$$\frac{dy}{dx}=-2\cos\frac{x}{2}\sin\frac{y}{2} \Rightarrow -\int 2\cos\frac{x}{2}dx=\int\operatorname{cosec}\frac{y}{2}dy$$

$$\Rightarrow c-2\sin\frac{x}{2}$$

$$=\log\tan\frac{y}{4}$$

Hence, the correct answer is option (A).

24. Solution of $\frac{dy}{dx}=\frac{y}{x}+\tan\frac{y}{x}$ is

(A) $\sin\left(\frac{y}{x}\right)=kx$ **(B)** $\cos\frac{y}{x}=kx$

(C) $\tan\frac{y}{x}=kx$ **(D)** none of these

Solution:

$$\frac{dy}{dx}=\frac{y}{x}+\tan\frac{y}{x}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} = v + \tan v$$

$$\Rightarrow \cot v dv = \frac{dx}{x}$$

On integrating, we get

$$\ln \sin v = \ln x + \ln k \Rightarrow \sin \frac{y}{x} = kx$$

Hence, the correct answer is option (A).

Previous Years' Solved JEE Main/AIEEE Questions

1. The differential equation of all circles passing through the origin and having their centres on the x-axis is

(A) $x^2 = y^2 + xy\frac{dy}{dx}$ **(B)** $x^2 = y^2 + 3xy\frac{dy}{dx}$

(C) $y^2 = x^2 + 2xy\frac{dy}{dx}$ **(D)** $y^2 = x^2 - 2xy\frac{dy}{dx}$

[AIEEE 2007]

Solution: General equation of such circle is

$$(x+g)^2 + y^2 = g^2 \Rightarrow x^2 + \cancel{g^2} + 2xg + y^2 = \cancel{g^2}$$

$$\Rightarrow x^2 + y^2 + 2gx = 0 \quad (1)$$

Now differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} + 2g = 0 \Rightarrow x + y\frac{dy}{dx} + g = 0 \quad (2)$$

Therefore, from Eq. (1)

$$x^2 + y^2 + 2\left(-x - y\frac{dy}{dx}\right)x = 0$$

$$\Rightarrow x^2 + y^2 - 2x^2 - 2yx\frac{dy}{dx} = 0$$

$$\Rightarrow y^2 = x^2 + 2xy\frac{dy}{dx}$$

Hence, the correct answer is option (C).

2. The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ satisfying the condition $y(1) = 1$ is

(A) $y = \ln x + x$ **(B)** $y = x \ln x + x^2$

(C) $y = xe^{(x-1)}$ **(D)** $y = x \ln x + x$

[AIEEE 2008]

Solution: Put $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = 1 + v \Rightarrow dv = \frac{dx}{x}$$

Therefore,

$$v = \ln x + c \Rightarrow \frac{y}{x} = \ln x + c$$

As $y(1) = 1$, we have, $y = x \ln x + x$.

Hence, the correct answer is option (D).

3. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is

(A) $(x-2)y'^2 = 25 - (y-2)^2$ **(B)** $(y-2)y'^2 = 25 - (y-2)^2$

(C) $(y-2)^2 y'^2 = 25 - (y-2)^2$ **(D)** $(x-2)^2 y'^2 = 25 - (y-2)^2$

[AIEEE 2008]

Solution: The equation of the circle is

$$(x-\alpha)^2 + (y-2)^2 = 25 \quad (1)$$

On differentiating w.r.t. x,

$$(x-\alpha) + (y-2)\frac{dy}{dx} = 0 \Rightarrow x - \alpha = -(y-2)\frac{dy}{dx} \quad (2)$$

From Eqs. (1) and (2), on eliminating α,

$$(y-2)^2\left(\frac{dy}{dx}\right)^2 + (y-2)^2 = 25 \Rightarrow (y-2)^2(y')^2 = 25 - (y-2)^2$$

Hence, the correct answer is option (C).

4. The differential equation which represents the family of curves $y = c_1e^{c_2x}$ where c_1 and c_2 are arbitrary constants is

(A) $y' = y^2$ **(B)** $y'' = y'y$ **(C)** $yy'' = y'$ **(D)** $yy'' = (y')^2$

[AIEEE 2009]

Solution:

$$y = c_1e^{c_2x} \quad (1)$$

$$y' = c_2c_1e^{c_2x} \Rightarrow y' = c_2y \quad (2)$$

$$y'' = c_2y'$$

From (2), we get, $c_2 = \frac{y'}{y}$. Therefore,

$$y'' = \frac{(y')^2}{y} \Rightarrow yy'' = (y')^2$$

Hence, the correct answer is option (D).

5. Solution of the differential equation

$$\cos x\, dy = y(\sin x - y)dx, 0 < x < \frac{\pi}{2} \text{ is}$$

(A) $y \sec x = \tan x + c$ **(B)** $y \tan x = \sec x + c$

(C) $\tan x = (\sec x + c)y$ **(D)** $\sec x = (\tan x + c)y$

[AIEEE 2010]

Solution: We have

$$\cos x\, dy = y(\sin x - y)\, dx \Rightarrow \frac{dy}{dx} = y\tan x - y^2 \sec x$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

Let us consider that $\frac{1}{y} = t$. Then

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow -\frac{dt}{dx} - t\tan x = -\sec x \Rightarrow \frac{dt}{dx} + (\tan x)t = \sec x$$

So, I.F. $= e^{\int \tan x\, dx} = \sec x$.

Solution is

$$t\,(\text{I.F}) = \int (\text{I.F}) \sec x dx$$

$$\Rightarrow \frac{1}{y}\sec x = \tan x + c \Rightarrow \sec x = y(\tan x + c)$$

Hence, the correct answer is option (D).

6. Let f: $(-1, 1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0) =$

(A) -4 **(B)** 0
(C) -2 **(D)** 4

[AIEEE 2010]

Solution: We have

$$g'(x) = 2(f(2f(x)+2))\left(\frac{d}{dx}(f(2f(x)+2))\right)$$
$$= 2f(2f(x)+2)f'(2f(x)+2)\cdot(2f'(x))$$

This implies that

$$g'(0) = 2f(2f(0)+2)\cdot f'(2f(0)+2)\cdot 2(f'(0))$$
$$= 4f(0)f'(0) = 4(-1)(1) = -4$$

Hence, the correct answer is option (A).

7. If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\ln 2)$ is equal to

(A) 5 **(B)** 13
(C) -2 **(D)** 7

[AIEEE 2011]

Solution: We have

$$\frac{dy}{dx} = y + 3 \Rightarrow \frac{dy}{y+3} = dx \Rightarrow \ln(y+3) = x + c$$

when $x = 0$, $y = 2$. So,

$$\ln 5 = 0 + c$$
$$\Rightarrow c = \ln 5$$

So, $\ln(y+3) = x + \ln 5 \Rightarrow y + 3 = e^{x+\ln 5}$

$$\Rightarrow y + 3 = e^{\ln 2 + \ln 5} \Rightarrow y + 3 = 10 \Rightarrow y = 7$$

Hence, the correct answer is option (D).

8. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T-t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then the scrap value $V(T)$ of the equipment is

(A) $I - \frac{kT^2}{2}$ **(B)** $I - \frac{k(T-t)^2}{2}$

(C) e^{-kT} **(D)** $T^2 - \frac{1}{k}$

[AIEEE 2011]

Solution: We have,

$$\frac{dV}{dt} = -k(T-t) \Rightarrow dV = -k(T-t)dt$$

On integrating the above equation, we get

$$V = \frac{-k(T-t)^2}{(-2)} + c \Rightarrow V = \frac{k(T-t)^2}{2} + c$$

As $V(0) = I$, we have

$$I = \frac{kT^2}{2} + c \Rightarrow c = I - \frac{kT^2}{2}$$

Now, at $t = T$ (for scrap value)

$$V(T) = c = I - \frac{kT^2}{2}$$

Hence, the correct answer is option (A).

9. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is

(A) 3000 **(B)** 3500
(C) 4500 **(D)** 2500

[JEE MAIN 2013]

Solution: Given that

$$\frac{dP}{dx} = 100 - 12\sqrt{x} \Rightarrow dP = (100 - 12\sqrt{x})dx$$

Therefore, the new level of production of items is

$$\int_{2000}^{P} dP = \int_0^{25} (100 - 12\sqrt{x})dx$$
$$\Rightarrow (P - 2000) = 25 \times 100 - \frac{12 \times 2}{3}(25)^{3/2}$$
$$\Rightarrow P = 3500$$

Hence, the correct answer is option (B).

10. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ equals

(A) $600 - 500\, e^{t/2}$ **(B)** $400 - 300\, e^{-t/2}$
(C) $400 - 300\, e^{t/2}$ **(D)** $300 - 200\, e^{-t/2}$

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$\frac{dp(t)}{pt} = \frac{1}{2}p(t) - 200 \Rightarrow \frac{dp(t)}{\frac{1}{2}p(t) - 200} = dt$$

On integrating, we get

$$\frac{\log\left|\frac{p(t)}{2} - 200\right|}{(1/2)} = t + c_1 \Rightarrow \log\left|\frac{p(t) - 400}{2}\right| = \frac{1}{2}t + \frac{c_1}{2} \quad (1)$$

Using initial conditions

$$\log\left|\frac{100 - 400}{2}\right| = \frac{c_1}{2} \Rightarrow 2\log 150 = c_1 \quad (2)$$

From Eqs. (1) and (2), we get

$$\log\left|\frac{p(t) - 400}{2}\right| = \frac{1}{2}t + \log 150 \Rightarrow \log\left|\frac{p(t) - 400}{300}\right| = \frac{1}{2}t$$

$$\left|\frac{p(t) - 400}{300}\right| = e^{t/2} \text{ or } |p(t) - 400| = 300e^{t/2}$$

Since initially number of rabbits is 100 and is decreasing

$$-(p(t) - 400) = 300e^{t/2} \text{ or } p(t) = 400 - 300e^{t/2}$$

Hence, the correct answer is option (C).

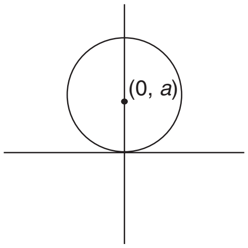
11. If the differential equation representing the family of all circles touching x-axis at the origin is $\left(x^2 - y^2\right)\frac{dy}{dx} = g(x)y$, then $g(x)$ equals

(A) $\frac{1}{2}x$ **(B)** $2x^2$

(C) $2x$ **(D)** $\frac{1}{2}x^2$

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: Equation of family is $(x - 0)^2 + (y - a)^2 = a^2$



$$x^2 + y^2 + a^2 - 2ay = a^2$$
$$\Rightarrow x^2 + y^2 - 2ay = 0 \quad (1)$$

On differentiating Eq. (1), we get

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0 \Rightarrow x = \frac{dy}{dx}(a - y)$$

$$a = \frac{x}{dy/dx} + y \quad (2)$$

From Eqs. (1) and (2), we get

$$x^2 + y^2 - 2\left(\frac{x}{dy/dx} + y\right)y = 0 \Rightarrow x^2 + y^2 - \frac{2xy}{dy/dx} - 2y^2 = 0$$

$$x^2 - y^2 - \frac{2xy}{dy/dx} = 0 \Rightarrow (x^2 - y^2)\frac{dy}{dx} = 2xy = g(x)y \Rightarrow g(x) = 2x$$

Hence, the correct answer is option (C).

12. If the general solution of the differential equation $y' = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ for some function ϕ, is given by $y \ln |cx| = x$, where c is an arbitrary constant, then $\phi(2)$ is equal to

(A) 4 **(B)** $\frac{1}{4}$ **(C)** -4 **(D)** $-\frac{1}{4}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: General solution of $\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ is

$$y \ln |cx| = x \quad (1)$$

On differentiating $y \ln |cx| = x$, we get,

$$y \times \frac{1}{|cx|} \cdot c + \ln|cx|\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{y}{x} + \frac{x}{y}\frac{dy}{dx} = 1$$

Therefore,

$$\frac{x}{y}\frac{dy}{dx} = 1 - \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2} \quad (2)$$

Now comparing with the given differential equation

$$\phi\left(\frac{x}{y}\right) = -\frac{y^2}{x^2} = -\left(\frac{y}{x}\right)^2 = -\left(\frac{1}{x/y}\right)^2$$

Therefore, $\phi(t) = -\left(\frac{1}{t}\right)^2$ and $\phi(2) = -\frac{1}{4}$.

Hence, the correct answer is option (D).

13. The general solution of the differential equation $\sin 2x\left(\frac{dy}{dx} - \sqrt{\tan x}\right) - y = 0$ is

(A) $y\sqrt{\tan x} = x + c$ **(B)** $y\sqrt{\cot x} = \tan x + c$

(C) $y\sqrt{\tan x} = \cot x + c$ **(D)** $y\sqrt{\cot x} = x + c$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: Differential equation is

$$\frac{dy}{dx} - \sqrt{\tan x} = \frac{y}{\sin 2x} \Rightarrow \frac{dy}{dx} - \frac{1}{\sin 2x}y = \sqrt{\tan x} \quad (1)$$

Therefore,

$$\text{I.F.} = e^{-\int \operatorname{cosec}(2x)dx} = e^{-\frac{1}{2}\log(\tan x)} = \sqrt{\cot x}$$

From (1), we get

$$\sqrt{\cot x}\frac{dy}{dx} - \frac{\sqrt{\cot x}}{\sin 2x}y = \sqrt{\tan x}\sqrt{\cot x}$$

$$\Rightarrow \frac{d}{dx}(y\sqrt{\cot x}) = \sqrt{\tan x}\sqrt{\cot x} = 1 \Rightarrow d(y\sqrt{\cot x}) = dx$$

On integrating, we get

$$y\sqrt{\cot x} = x + c$$

Hence, the correct answer is option (D).

14. If $\frac{dy}{dx} + y\tan x = \sin 2x$ and $y(0) = 1$, then $y(\pi)$ is equal to

(A) 1 **(B)** -1

(C) -5 **(D)** 5

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$\frac{dy}{dx} + y\tan x = \sin 2x$$

Integrating factor,

$$e^{\int \tan x dx} = e^{\ln \sec(x)} = \sec x$$

Therefore,

$$\sec x\frac{dy}{dx} + y\tan x \sec x = \sin 2x \sec x$$

$$\Rightarrow \frac{d}{dx}(y\sec x)$$

$$= 2\sin x \cos x \sec x \Rightarrow d(y\sec x) = 2\sin x\, dx$$

On integrating,

$$y\sec x = 2\int \sin x dx + c \Rightarrow y\sec x = -2\cos x + c$$

By initial condition,

$$1 \sec 0 = -2\cos 0 + c \Rightarrow 1 + 2 = c$$
$$\Rightarrow c = 3$$

Therefore,

$$y\sec x = -2\cos x + 3 \Rightarrow y = \frac{-2\cos x + 3}{\sec x}$$

$$\Rightarrow y(\pi) = \frac{-2\cos\pi + 3}{\sec\pi} = \frac{2+3}{-1} = -5$$

Hence, the correct answer is option (C).

15. Let $y(x)$ be the solution of the differential equation

$$(x \log x)\,\frac{dy}{dx} + y = 2x\log x, (x \geq 1)$$

Then, $y(e)$ is equal to

(A) 0 **(B)** 2
(C) $2e$ **(D)** e

[JEE MAIN 2015 (OFFLINE)]

Solution:

$$(x\log x)\frac{dy}{dx} + y = 2x\log x; (x \geq 1)$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x\log x}\right)y = 2$$

It is a linear differential equation of first order of the form

$$\frac{dy}{dx} + Py = Q$$

$$\Rightarrow P = \frac{1}{x\log x}, Q = 2$$

So,

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x\log x}dx}$$

$$= e^{\int \frac{dt}{t}} = e^{\ln t} = t = \log x$$

Therefore, solution of given differential equation is given by

$$y.(\text{I.F}) = \int Q.(\text{I.F})dx + C$$

$$\Rightarrow y(\log x) = \int 2\log x, dx + C$$

$$= 2\left[(\log x)x - \int \frac{1}{x}\cdot x\,dx\right] + C$$

$$= 2x\log x - 2x + C$$

When $x = 1$. Then

$$y(\log 1) = 2\log 1 - 2 + C$$
$$\Rightarrow 0 = -2 + C \Rightarrow C = 2$$

Note: Since we need to put $x = 1$ in order to find the value of the constant and at $x = 1$, P is not defined, so this whole question is conceptually incorrect. Although if we still solve it we get the general solution as

$$y(\log x) = 2x\log x - 2x + 2$$
$$\Rightarrow y(e) = 2e - 2e + 2$$
$$\Rightarrow y(e) = 2$$

Hence, the correct answer is option (B).

16. If $y(x)$ is the solution of the differential equation $(x+2)\frac{dy}{dx} = x^2 + 4x - 9, x \neq -2$ and $y(0) = 0$, then $y(-4)$ is equal to

(A) 0 **(B)** 1
(C) -1 **(D)** 2

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:

$$(x+2)\frac{dy}{dx} = x^2 + 4x - 9$$

$$\Rightarrow \int dy = \int \frac{((x+2)^2 - 13)dx}{(x+2)} \Rightarrow y = \left(\frac{x^2}{2} + 2x\right) - 13\ln|x+2| + c$$

As

$$y(0) = 0 \Rightarrow 0 = -13\ln 2 + c \Rightarrow c = 13\ln 2$$

So,

$$y = \frac{x^2}{2} + 2x + 13\ln\left(\frac{2}{|x+2|}\right)$$

$$\Rightarrow y(-4) = 8 - 8 + 13\ln\left(\frac{2}{2}\right) = 0$$

Hence, the correct answer is option (A).

17. The solution of the differential equation $ydx - (x + 2y^2)dy = 0$ is $x = f(y)$. If $f(-1) = 1$, then $f(1)$ is equal to

(A) 4 **(B)** 3
(C) 2 **(D)** 1

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: Given differential equation is

$$y\,dx - (x + 2y^2)dy = 0 \quad (1)$$

Rearranging Eq. (1), we get

$$y\frac{dx}{dy} - (x + 2y^2) = 0 \Rightarrow \frac{dx}{dy} - 2y - \frac{x}{y} = 0$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y$$

which is a linear differential equation of first order.

$$\frac{dx}{dy} + Px = Q$$

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{-1}{y}dy} = e^{-\ln y} = \frac{1}{y}$$

Therefore, solution of Eq. (1) is given by

$$x.(\text{I.F.}) = \int Q(\text{I.F.})dy + c \Rightarrow x.\frac{1}{y} = \int 2y.\frac{1}{y}dy + c \Rightarrow \frac{x}{y} = 2y + c$$

$$\Rightarrow x = 2y^2 + cy$$

we are given $f(-1) = 1$

$$\Rightarrow 1 = 2 + c(-1) \Rightarrow c = 1$$

So,

$$x = 2y^2 + y = f(y) \Rightarrow f(1) = 2 + 1 = 3$$

Hence, the correct answer is option (B).

18. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)dx = xdy$, then $f\left(-\frac{1}{2}\right)$ is equal to

(A) $\frac{4}{5}$ **(B)** $-\frac{2}{5}$

(C) $-\frac{4}{5}$ **(D)** $\frac{2}{5}$

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$$y(1 + xy)dx = xdy$$
$$\Rightarrow ydx - xdy = -xy^2dx$$
$$\Rightarrow \left(\frac{y\,dx - x\,dy}{y^2}\right) = -x\,dx$$

Therefore,

$$\int d\left(\frac{x}{y}\right) = -\int x\,dx$$
$$\Rightarrow \frac{x}{y} = -\frac{x^2}{2} + c \quad (1)$$

which passes through (1, −1).

$$-1 = -\frac{1}{2} + c$$
$$\Rightarrow c = -1 + \frac{1}{2} = -\frac{1}{2}$$

Substituting the value of c into Eq. (1), we get

$$\frac{x}{y} = -\frac{x^2}{2} - \frac{1}{2}$$
$$\Rightarrow \frac{x}{y} = -\frac{(x^2+1)}{2}$$
$$\Rightarrow y = \frac{-2x}{x^2+1} \text{ and } x = -\frac{1}{2}$$

Therefore, $$f\left(-\frac{1}{2}\right) = -\frac{-2(-1/2)}{1+(1/4)} = \frac{4}{5}$$

Hence, the correct answer is option (A).

19. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in \mathbb{R}$ meets the curve again at a point Q, then the coordinates of Q are

(A) $(16t^2 + 3, -64t^3 - 1)$ **(B)** $(4t^2 + 3, -8t^3 - 1)$
(C) $(t^2 + 3, t^3 - 1)$ **(D)** $(t^2 + 3, -t^3 - 1)$

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: We have

$$x = 4t^2 + 3 \Rightarrow \frac{dx}{dt} = 8t$$
$$y = 8t^3 - 1 \Rightarrow \frac{dy}{dt} = 24t^2$$

Therefore,

$$\frac{dy}{dx} = \frac{24t^2}{8t} = 3t$$

Equation of tangent is $(y - 8t^3 + 1) = 3t(x - 4t^2 - 3)$ which passes through point $Q(4t_1^2+3, 8t_1^3-1)$.

$$\left(8t_1^3 - 1 - 8t^3 + 1\right) = 3t\left(4t_1^2 + 3 - 4t^2 - 3\right)$$
$$\Rightarrow 8\left(t_1^3 - t^3\right) = 3t \times 4\left(t_1^2 - t^2\right)$$
$$\Rightarrow 2\left(t_1^3 - t^3\right) = 3t\left(t_1^2 - t^2\right)$$
$$\Rightarrow 2\left(t_1^2 + t^2 + t_1t\right) = 3t(t_1 + t) \qquad (t \neq t_1)$$
$$\Rightarrow 2t_1^2 + 2t^2 + 2t_1t = 3tt_1 + 3t^2$$
$$\Rightarrow 2t_1^2 + t^2 + t_1t = 0$$
$$\Rightarrow t_1^2 + t^2 + t_1^2 - t_1t = 0$$

That is,

$$(t_1 - t)(t_1 + t) + t_1(t_1 - t) = 0 \Rightarrow 2t_1 + t = 0 \Rightarrow t = -2t_1 \Rightarrow t_1 = -\frac{t}{2}$$

Hence, the coordinates of point Q are obtained as follows:

$$Q\left(4t_1^2 + 3, 8t_1^3 - 1\right) = Q(t^2 + 3, -t^3 - 1)$$

Hence, the correct answer is option (D).

20. For $x \in \mathbb{R}$, $x \neq 0$, if $y(x)$ is a differentiable function such that $x\int_1^x y(t)\,dt = (x+1)\int_1^x ty(t)dt$, then $y(x)$ equals (where C is a constant)

(A) $Cx^3 \frac{1}{e^x}$ **(B)** $\frac{C}{x^2}e^{-\frac{1}{x}}$

(C) $\frac{C}{x}e^{-\frac{1}{x}}$ **(D)** $\frac{C}{x^3}e^{-\frac{1}{x}}$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: On applying Newton–Leibniz rule, we get

$$x[y(x) - 0] + \int_1^x y(t)dt = \int_1^x ty(t)dt + (x+1)(xy(x) - 0)$$
$$xy(x) + \int_1^x y(t)dt = \int_1^x ty(t)dt + x^2y(x) + xy(x)$$
$$\int_1^x y(t)dt = \int_1^x ty(t)dt + x^2y(x)$$

On differentiating with respect to x, we get

$$y(x) - 0 = xy(x) - 0 + 2xy(x) + x^2\,y'(x)$$
$$y(x) = 3xy(x) + x^2\,y'(x)$$
$$x^2\,y'(x) + (3x - 1)\,y(x) = 0$$
$$y'(x) + \left(\frac{3}{x} - \frac{1}{x^2}\right)y(x) = 0$$

Integrating factor is

$$e^{\int\left(\frac{3}{x} - \frac{1}{x^2}\right)dx} = e^{\left(3\ln x + \frac{1}{x}\right)} = x^3 \cdot e^{1/x}$$

Therefore, $$\frac{d}{dx}[y(x) \cdot x^3e^{1/x}] = 0$$
$$y(x)x^3e^{1/x} = C$$

Hence, the correct answer is option (D).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. The differential equation $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(A) variable radii and a fixed centre at (0, 1);
(B) variable radii and a fixed centre at (0, −1);
(C) fixed radius 1 and variable centres along the x-axis;
(D) fixed radius 1 and variable centres along the y-axis.

[IIT-JEE 2007]

Solution: We have

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{y}$$

$$\Rightarrow \frac{y}{\sqrt{1-y^2}}dy = dx$$

On integrating both sides, we get

$$\int \frac{y\,dy}{\sqrt{1-y^2}} = \int dx$$

$$\Rightarrow \int \frac{-2y\,dy}{\sqrt{1-y^2}} = -2\int dx$$

$$\Rightarrow 2\sqrt{1-y^2} = -2x + 2c$$

$$\Rightarrow \sqrt{1-y^2} = -x + c$$

$$\Rightarrow 1 - y^2 = (x-c)^2$$

$$\Rightarrow (x-c)^2 + y^2 = 1$$

Hence, the correct answer is option (C).

2. Let $g(x) = \log(f(x))$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x+1) = xf(x)$. Then, for $N = 1, 2, 3, \ldots,$

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2N+1)^2}\right\}$

[IIT-JEE 2008]

Solution: Since

$g(x) = \log f(x)$
$g(x+1) = \log f(x+1)$
$= \log[xf(x)]$
$= \log x + \log f(x)$
$= \log x + g(x)$
$g(x+1) - g(x) = \log x$

On differentiating both sides, we get

$$g'(x+1) - g'(x) = \frac{1}{x}$$

Again differentiating, we get

$$g''(x+1) - g''(x) = -\frac{1}{x^2}$$

Put $x = x - \frac{1}{2}$, we get

$$g''\left(x+\frac{1}{2}\right) - g''\left(x-\frac{1}{2}\right) = -\frac{-1}{\left(x-\frac{1}{2}\right)^2}$$

$$\Rightarrow g''\left(x+\frac{1}{2}\right) - g''\left(x-\frac{1}{2}\right) = -\frac{-4}{(2x-1)^2}$$

Put $x = 1, 2, 3, \ldots, N$. Then

$$g''\left(1+\frac{1}{2}\right) - g''\left(1-\frac{1}{2}\right) = \frac{-4}{1}$$

$$g''\left(2+\frac{1}{2}\right) - g''\left(2-\frac{1}{2}\right) = \frac{-4}{9}$$

$$g''\left(3+\frac{1}{2}\right) - g''\left(3-\frac{1}{2}\right) = \frac{-4}{25}$$

$$\ldots \qquad \ldots$$

$$g''\left(N+\frac{1}{2}\right) - g''\left(N-\frac{1}{2}\right) = \frac{-4}{(2N-1)^2}$$

Hence,

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left[1+\frac{1}{9}+\frac{1}{25}+\cdots+\frac{1}{(2N-1)^2}\right]$$

Hence, the correct answer is option (A).

3. Let a solution $y = y(x)$ of the differential equation

$$x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0 \text{ satisfy } y(2) = \frac{2}{\sqrt{3}}$$

Statement 1: $y(x) = \sec\left(\sec^{-1}x - \frac{\pi}{6}\right)$

Statement 2: $y(x)$ is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$

(A) Statement 1 is true, Statement 2 is true; Statement 2 is a correct explanation for Statement 1.
(B) Statement 1 is true, Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.
(C) Statement 1 is true, Statement 2 is false.
(D) Statement 1 is false, Statement 2 is true.

[IIT-JEE 2008]

Solution: We have

$$x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$$

$$\Rightarrow \int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{dy}{y\sqrt{y^2-1}}$$

$$\sec^{-1} x = \sec^{-1} y + c$$

$$\sec^{-1} 2 = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) + c$$

$$c = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\Rightarrow \sec^{-1} x = \sec^{-1} y + \frac{\pi}{6}$$

$$y = \sec\left(\sec^{-1} x - \frac{\pi}{6}\right)$$

Now,

$$\cos^{-1}\frac{1}{x} = \cos^{-1}\frac{1}{y} + \frac{\pi}{6}$$

$$\cos^{-1}\frac{1}{y} = \cos^{-1}\frac{1}{x} + \cos^{-1}\frac{\sqrt{3}}{2}$$

$$\frac{1}{y} = \frac{1}{x}\cdot\frac{\sqrt{3}}{2} - \sqrt{1-\frac{1}{x^2}}\sqrt{1-\frac{3}{4}}$$

$$\frac{1}{y} = \frac{\sqrt{3}}{2x} - \sqrt{1-\frac{1}{x^2}}\cdot\frac{1}{2}$$

$$\frac{1}{y} = \frac{\sqrt{3}}{2x} - \sqrt{1-\frac{1}{x^2}}\cdot\frac{1}{2}$$

$$\frac{2}{y} = \frac{\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$$

Hence, the correct answer is option (C).

4. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in \mathbb{R}$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on $\mathbb{R}$ with $g(0) = g(2) = 0$. Then, the value of $y(2)$ is ______.

[IIT-JEE 2011]

Solution:

$$y'(x) + y(x)g'(x) = g(x)g'(x)$$

$$\Rightarrow e^{g(x)}y'(x) + e^{g(x)}g'(x)y(x) = e^{g(x)}g(x)g'(x)$$

$$\Rightarrow \frac{d}{dx}(y(x)e^{g(x)}) = e^{g(x)}g(x)g'(x)$$

Therefore, $y(x)\,e^{g(x)} = \int e^{g(x)}g(x)g'(x)dx$

$$= \int e^t t\,dt, \text{ where } g(x) = t$$

$$= (t-1)e^t + c$$

$$\Rightarrow y(x)e^{g(x)} = (g(x)-1)e^{g(x)} + c$$

Put $x = 0$. Then

$$0 = (0 - 1)\cdot 1 + c \Rightarrow c = 1$$

Put $x = 2$. Then

$$y(2)\cdot 1 = (0 - 1)\cdot(1) + 1$$

$$y(2) = 0$$

Hence, the correct answer is (0).

5. If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x\sec x$ and $y(0) = 0$, then

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ **(B)** $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(C) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ **(D)** $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

[IIT-JEE 2012]

Solution:

$$\frac{dy}{dx} - y\tan x = 2x\sec x$$

$$\Rightarrow \cos x\frac{dy}{dx} + (-\sin x)y = 2x$$

$$\Rightarrow \frac{d}{dx}(y\cos x) = 2x$$

$$\Rightarrow y(x)\cos x = x^2 + c, \text{ where } c = 0 \text{ since } y(0) = 0$$

When $x = \frac{\pi}{4}$. Then

$$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$$

When $x = \frac{\pi}{3}$. Then

$$y\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9}$$

When $x = \frac{\pi}{4}$. Then $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}} + \frac{\pi}{\sqrt{2}}$

When $x = \frac{\pi}{3}$. Then $y'\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{3\sqrt{3}} + \frac{4\pi}{3}$.

Hence, the correct answers are options (A) and (D).

6. The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then, $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ **(B)** $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ **(D)** $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

[IIT-JEE 2014]

Solution: Differential equation is

$$\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{1-x^2}\right)y = \frac{x^4+2x}{\sqrt{1-x^2}} \quad (1)$$

It is a linear differential equation. Therefore,

$$\text{I.F.} = e^{\int\left(\frac{-x}{1-x^2}\right)dx} = e^{\frac{1}{2}\log(1-x^2)}$$
$$= \sqrt{1-x^2}$$

Now multiplying I.F. with Eq. (1)

$$\sqrt{1-x^2}\frac{dy}{dx} - \sqrt{1-x^2}\frac{x}{(1-x^2)}y = x^4 + 2x$$
$$\Rightarrow \frac{d}{dx}(\sqrt{1-x^2}y) = x^4 + 2x$$
$$\Rightarrow \int d(\sqrt{1-x^2}y) = \int (x^4 + 2x)dx$$
$$\Rightarrow \sqrt{1-x^2}y = \frac{x^5}{5} + \frac{2x^2}{2} + c \quad (2)$$

Using initial conditions, that is, $f(0) = 0$

$$\sqrt{1-x^2}(0) = 0 + 0 + c$$

Therefore,

$$c = 0$$

Therefore, Eq. (2) gives $\sqrt{1-x^2}y = \frac{x^5}{5} + x^2$.

Hence,

$$y = \frac{\frac{x^5}{5} + x^2}{\sqrt{1-x^2}} = \frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}$$

Now,

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^5}{5\sqrt{1-x^2}}dx + \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1-x^2}}dx$$
$$= 0 + 2\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1-x^2}}dx \quad (3)$$

Since even function, now solving

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1-x^2}}dx$$

Putting $x = \sin\theta$. Therefore, $dx = \cos\theta d\theta$

$$\int_0^{\frac{\pi}{3}} \frac{\sin^2\theta\cos\theta d\theta}{\cos\theta} = \int_0^{\frac{\pi}{3}} \left(\frac{1-\cos 2\theta}{2}\right)d\theta$$
$$= \left[\frac{1}{2}\theta\right]_0^{\frac{\pi}{3}} - \frac{1}{2}\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$
$$= \frac{1}{2}\times\frac{\pi}{3} - \frac{1}{4}\left[\sin\frac{2\pi}{3}\right]$$
$$= \frac{\pi}{6} - \frac{1}{4}\frac{\sqrt{3}}{2} = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$$

Hence, from Eq. (3)

$$2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

Hence, the correct answer is option (B).

7. Let $y(x)$ be a solution of the differential equation $(1 + e^x)\, y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is (are) true?

(A) $y(-4) = 0$
(B) $y(-2) = 0$
(C) $y(x)$ has a critical point in the interval $(-1, 0)$
(D) $y(x)$ has no critical point in the interval $(-1, 0)$

[IIT-JEE 2015]

Solution: Given differential equation is

$$(1 + e^x)\, y' + ye^x = 1;\ y(0) = 2$$
$$\Rightarrow y' + \frac{e^x}{1+e^x}y = \frac{1}{1+e^x} \quad (1)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where, $P = \frac{e^x}{1+e^x}, Q = \frac{1}{1+e^x}$

$$\text{I.F.} = e^{\int pdx} \Rightarrow \text{I.F.} = e^{\int\frac{e^x}{1+e^x}dx} = e^{\ln(1+e^x)} = (1+e^x)$$

Therefore, solution of Eq. (1) is given by

$$y(1+e^x) = \int dx + c$$
$$\Rightarrow y(1+e^x) = x + c$$

Therefore,

$$y(0) = 2 \Rightarrow 2(2) = c \Rightarrow c = 4$$

So,

$$y = \frac{x+4}{1+e^x} \Rightarrow y(-4) = 0, \text{and } y(-2) = \frac{2}{1+e^{-2}} \neq 0$$

Also,

$$y' = \frac{(1+e^x)-(x+4)(e^x)}{(1+e^x)^2} = \frac{1-(x+3)e^x}{(1+e^x)^2}$$

For critical point,

$$e^x = \frac{1}{x+3} \text{ or } e^{-x} = x + 3$$

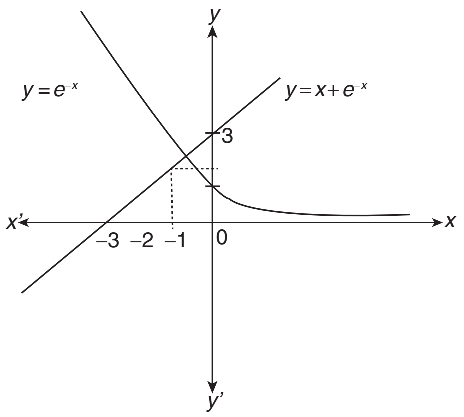


Figure 25.5

Therefore, $y(x)$ has a critical point in $(-1, 0)$. (See Fig. 25.5)

Hence, the correct answers are options (A) and (C).

8. Consider the family of all circles whose centres lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' $\left(\text{here } y' = \dfrac{dy}{dx}, y'' = \dfrac{d^2y}{dx^2}\right)$, then which of the following statements is (are) true?

(A) $P = y + x$ **(B)** $P = y - x$

(C) $P + Q = 1 - x + y + y' + (y')^2$ **(D)** $P - Q = x + y - y' - (y')^2$

[IIT-JEE 2015]

Solution: Let the family of circles be

$$(x - h)^2 + (y - h)^2 = r^2 \quad (1)$$

$$\Rightarrow x^2 + y^2 - 2xh - 2yh + 2h^2 - r^2 = 0$$

On differentiating this w.r.t. x, we get

$$2x + 2yy' - 2h - 2hy' = 0$$

$$\Rightarrow x + yy' - h - hy' = 0 \quad (2)$$

On differentiating again w.r.t. x, we get

$$1 + yy'' + (y')^2 - hy'' = 0$$

$$\Rightarrow (y - h)y'' + (y')^2 + 1 = 0 \quad (3)$$

From Eq. (2)

$$h = \frac{x + yy'}{1 + y'} \quad (4)$$

Using Eq. (4) in Eq. (3), we get

$$\left[y - \left(\frac{x + yy'}{1 + y'}\right)\right]y'' + (y')^2 + 1 = 0$$

$$\Rightarrow \left(\frac{y - x}{1 + y'}\right)y'' + (y').y' + 1 = 0$$

$$\Rightarrow (y - x)y'' + (1 + y' + y'^2)\, y' + 1 = 0$$

$$\Rightarrow P = y - x,\ Q = 1 + y' + (y')^2$$

$$\Rightarrow P + Q = 1 - x + y + y' + (y')^2$$

Hence, the correct answers are options (B) and (C).

9. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\dfrac{dy}{dx} - y^2 = 0,\ x > 0,$ passes through the point (1, 3). The solution curve

(A) intersects $y = x + 2$ exactly at one point;

(B) intersects $y = x + 2$ exactly at two points;

(C) intersects $y = (x + 2)^2$;

(D) does not intersect $y = (x + 3)^2$.

[JEE ADVANCED 2016]

Solution: The given differential equation is

$$[x^2 + 4x + 4 + y(x + 2)]\frac{dy}{dx} - y^2 = 0 \quad (x > 0)$$

which is further simplified as follows:

$$[(x + 2)^2 + y(x + 2)]\frac{dy}{dx} - y^2 = 0$$

Substituting $x + 2 = t$, we get

$$\frac{dx}{dy} = \frac{dt}{dy}$$

Now,

$$(x + 2)^2 + y(x + 2) - y^2\frac{dx}{dy} = 0$$

$$\Rightarrow t^2 + yt - y^2\frac{dt}{dy} = 0$$

$$\Rightarrow y^2\frac{dt}{dy} - yt - t^2 = 0$$

$$\Rightarrow \frac{1}{t^2}\frac{dt}{dy} - \frac{1}{yt} = \frac{1}{y^2}$$

Let $\dfrac{1}{t} = z$. Therefore,

$$\frac{dt}{dy}\left(-\frac{1}{t^2}\right) = \frac{dz}{dy}$$

Now,

$$\frac{-dz}{dy} - \frac{z}{y} = \frac{1}{y^2} \Rightarrow \frac{dz}{dy} + \frac{z}{y} = \frac{-1}{y^2}$$

$$\Rightarrow \int d(zy) = \int -\frac{1}{y}dy$$

$$\Rightarrow zy = -\ln|y| + c$$

$$\Rightarrow \frac{y}{t} = -\ln|y| + c$$

$$\Rightarrow \frac{y}{(x + 2)} = -\ln|y| + c \quad (1)$$

which passes through the point (1, 3). Therefore, from Eq. (1), we get

$$\frac{3}{3} = -\ln 3 + c \Rightarrow c = \ln 3e$$

$$\frac{y}{x + 2} = -\ln|y| + \ln 3e = \ln\left(\frac{3e}{|y|}\right)$$

$$\frac{3e}{|y|} = e^{y/(x+2)}$$

$$3e = |y|e^{y/(x+2)}$$

Substituting $y = (x + 2)$, we get

$$3e = |x + 2|e^1$$

$$|x + 2| = 3 \Rightarrow x + 2 = -3, 3 \Rightarrow x = -5, 1$$

Therefore, $x = 1$ (since $x \neq -5$).

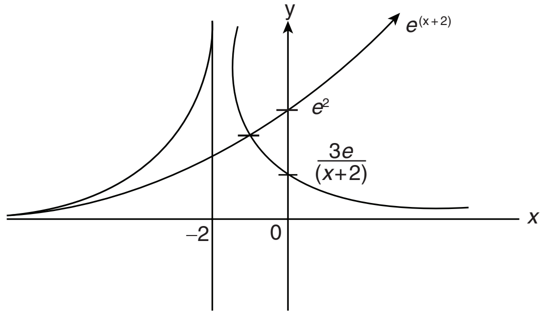


Figure 25.6

That is, the solution curve intersects $y = (x + 2)$ exactly at one point and not at two points. Therefore, option (A) is correct and option (B) is incorrect.
Checking for option (C), we have

$$\frac{3e}{|(x+2)^2|} = e^{(x+2)}$$

which meets at two points for $x < 0$ and for $x > 0$, there is no intersection point (Fig. 25.6).
Hence, option (C) is incorrect.
Checking for option (D), we have

$$\frac{3e}{(x+3)^2} = e^{\frac{(x+3)^2}{(x+2)}} = e^{\frac{(x+2)^2+1+2(x+2)}{(x+2)}} = e^{2+\frac{1}{(x+2)}+(x+2)}$$

Therefore, there is no intersection point for $x > 0$.
Hence, option (D) is correct.
Hence, the correct answers are options (A) and (D).

Practice Exercise 1

1. The differential equation of all non-vertical lines in a plane is
(A) $\frac{d^2y}{dx^2} = 0$ (B) $\frac{d^2x}{dy^2} = 0$
(C) $\frac{dy}{dx} = 0$ (D) $\frac{dx}{dy} = 0$

2. The differential equation of all non-horizontal lines in a plane is
(A) $\frac{d^2y}{dx^2} = 0$ (B) $\frac{d^2x}{dy^2} = 0$
(C) $\frac{dy}{dx} = 0$ (D) $\frac{dx}{dy} = 0$

3. A solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$ is
(A) $y = 2$ (B) $y = 2x$
(C) $y = 2x - 4$ (D) $y = 2x^2 - 4$

4. The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter, is of
(A) Order 1 (B) Order 2
(C) Degree 2 (D) Degree 1

5. The solution of $\frac{d^3y}{dx^3} - 8\frac{d^2y}{dx^2} = 0$ satisfying $y(0) = 1/8$, $y_1(0) = 0$ and $y_2(0) = 1$ is
(A) $y = \frac{1}{8}\left(\frac{e^{8x}}{8} - x + \frac{7}{8}\right)$ (B) $y = \frac{1}{8}\left(\frac{e^{8x}}{8} + x + \frac{7}{8}\right)$
(C) $y = \frac{1}{8}\left(\frac{e^{8x}}{8} + x - \frac{7}{8}\right)$ (D) None of these

6. If $y = e^{4x} + 2e^{-x}$ satisfies the relation $\frac{d^3y}{dx^3} + A\frac{dy}{dx} + By = 0$, then values of A and B respectively are
(A) −13, 14 (B) −13, −12
(C) −13, 12 (D) 12, −13

7. The function $f(\theta) = \frac{d}{d\theta}\int_0^\theta \frac{dx}{1-\cos\theta\cos x}$ satisfies the differential equation
(A) $\frac{df}{d\theta} + 2f(\theta)\cot\theta = 0$ (B) $\frac{df}{d\theta} - 2f(\theta)\cot\theta = 0$
(C) $\frac{df}{d\theta} + 2f(\theta) = 0$ (D) $\frac{df}{d\theta} - 2f(\theta) = 0$

8. If $f(x)$, $g(x)$ be twice differentiable function on [0, 2] satisfying $f''(x) = g''(x)$, $f'(1) = 4$ and $g'(1) = 6$, $f(2) = 3$, $g(2) = 9$, then $f(x) - g(x)$ at $x = 4$ equals
(A) 0 (B) −10 (C) 8 (D) 2

9. If the general solutions of a differential equation are $(y + c)^2 = cx$, where c is an arbitrary constant, then the order and degree of differential equation is
(A) 1, 2 (B) 2, 1 (C) 1, 3 (D) None of these

10. Solution of $(x^2 \sin^3 y - y^2 \cos x)\, dx + (x^3 \cos y \sin^2 y - 2y \sin x)\, dy = 0$ is
(A) $\frac{x^3\sin^3 y}{3} = c$ (B) $x^3 \sin^3 y = y^2 \sin x + c$
(C) $\frac{x^3\sin^3 y}{3} = y^2\sin x + c$ (D) None of these

11. Solution of $\frac{xdy}{x^2+y^2} = \left(\frac{y}{x^2+y^2} - 1\right)dx$ is
(A) $x - \tan^{-1}\frac{y}{x}$ (B) $\tan^{-1}\frac{y}{x} = c$
(C) $x\tan^{-1}\frac{y}{x} = c$ (D) None of these

12. Solution of $\frac{dy}{dx} + 2xy = y$ is
(A) $y = ce^{x-x^2}$ (B) $y = ce^{x^2} - x$
(C) $y = ce^x$ (D) $y = ce^{-x^2}$

13. Solution of the differential equation $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$ is
(A) $\log\left|1 + \tan\frac{(x+y)}{2}\right| = y + c$
(B) $\log\left|2 + \sec\frac{(x+y)}{2}\right| = x + c$
(C) $\log|1 + \tan(x + y)| = y + c$
(D) None of these

14. If $y = a\cos(\log x) + b\sin(\log x)$, then
(A) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ (B) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

(C) $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$ (D) None of these

15. If $y = \sin(a\sin^{-1} x)$, then

(A) $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+a^2y=0$

(B) $(1-x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-a^2y=0$

(C) $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-a^2y=0$

(D) None of these

16. If $y = A\sin x + B\cos x + x\sin x$, then

(A) $\dfrac{d^2y}{dx^2}+y=\cos x$ (B) $\dfrac{d^2y}{dx^2}+y=2\cos x$

(C) $\dfrac{d^2y}{dx^2}-y=2\sin x$ (D) $\dfrac{d^2y}{dx^2}-y=2\cos x$

17. The differential equation $y\dfrac{dy}{dx}=a-x(x\neq a,\ a\in R)$ represents

(A) A family of circles with centre on the y-axis.
(B) A family of circles with centre at the origin.
(C) A family of circles with the given radius.
(D) A family of circles with centre on the x-axis.

18. A particle, initially at the origin moves along the x-axis according to the rule $\dfrac{dx}{dt}=x+4$. The time taken by the particle to traverse a distance of 96 units is

(A) ln 5 (B) $\log_5 e$
(C) 2 ln 5 (D) $2\log_5 e$

19. The solution of the differential equation $\dfrac{dy}{dx}=\dfrac{y}{x}+\dfrac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is

(A) $\phi\left(\dfrac{y}{x}\right)=kx$ (B) $x\phi\left(\dfrac{y}{x}\right)=k$

(C) $\phi\left(\dfrac{y}{x}\right)=ky$ (D) $y\phi\left(\dfrac{y}{x}\right)=k$

20. Solution of the differential equation $\ln\left(\dfrac{dy}{dx}\right)=ax+by$ is

(A) $-\dfrac{1}{b}e^{-by}=\dfrac{1}{a}e^{ax}+c$ (B) $\dfrac{1}{b}e^{-by}=\dfrac{1}{a}e^{ax}+c$

(C) $\dfrac{1}{b}e^{-by}=-\dfrac{1}{a}e^{ax}+c$ (D) $-\dfrac{1}{b}e^{-by}=-\dfrac{1}{a}e^{ax}+c$

21. The differential equation of the family of curves $cy^2 = 2x + c$, where c is an arbitrary constant is

(A) $y\dfrac{dy}{dx}=1$ (B) $\left(\dfrac{dy}{dx}\right)^2+y\dfrac{d^2y}{dx^2}=0$

(C) $y^2=2xy\dfrac{dy}{dx}+1$ (D) None of these

22. The general solution of the differential equation $x(1+y^2)\,dx + y(1+x^2)\,dy = 0$ is

(A) $(1+x^2)(1+y^2)=0$ (B) $(1+x^2)(1+y^2)=c$
(C) $(1+y^4)=c(1+x^2)$ (D) None of these

23. Solution of $\dfrac{dy}{dx}+\sqrt{\dfrac{1-y^2}{1-x^2}}=0$ is

(A) $\sin^{-1}x-\sin^{-1}y=c$ (B) $\sin^{-1}y+\sin^{-1}x=c$
(C) $\sin^{-1}x=c\sin^{-1}y$ (D) $(\sin^{-1}x)(\sin^{-1}y)=c$

24. General solution of $\dfrac{d^2y}{dx^2}=e^{-2x}$ is

(A) $y=\dfrac{1}{4}e^{-2x}+c$ (B) $y=e^{-2x}+cx+d$

(C) $y=\dfrac{1}{4}e^{-2x}+cx+d$ (D) $y=e^{-2x}+cx^2+d$

25. Solution of $\dfrac{dy}{dx}+\dfrac{y}{x}=x^2$ is

(A) $x+y=\dfrac{x^2}{2}+c$ (B) $x-y=\dfrac{x^3}{3}+c$

(C) $xy=\dfrac{1}{4}x^4+c$ (D) $y-x=\dfrac{1}{4}x^4+c$

26. The curve satisfying $y=2x\dfrac{dy}{dx}$ is a

(A) Family of parabola (B) Family of circle
(C) Pair of straight line (D) None of these

27. The equation of the curve through the origin satisfying the equation $dy = (\sec x + y\tan x)dx$ is

(A) $y\sin x = x$ (B) $y\cos x = x$
(C) $y\tan x = x$ (D) None of these

28. The solution of $x^2\dfrac{dy}{dx}-xy=1+\cos\dfrac{y}{x}$ is

(A) $\tan\dfrac{y}{2x}=c-\dfrac{1}{2x^2}$ (B) $\tan\dfrac{y}{x}=c+\dfrac{1}{x}$

(C) $\cos\left(\dfrac{y}{x}\right)=1+\dfrac{c}{x}$ (D) $x^2=(c+x^2)\tan\dfrac{y}{x}$

29. If $\dfrac{dy}{dx}=e^{-2y}$ and $y = 0$ when $x = 5$, then value of x where $y = 3$ is given by

(A) e^5 (B) $\dfrac{e^6+9}{2}$
(C) e^6+1 (D) $\log_e 6$

30. The equation of this curve passing through (1, 3) and having slope $-[1+(y/x)]$ at (x, y) is

(A) $xy+4x^2=7$ (B) $2xy+x^2=7$
(C) $3xy+2x^2=7$ (D) $\sqrt{x}y+4x^2=7$

31. Given that $\dfrac{dy}{dx}=ye^x$ such that $x = 0$, $y = e$. The value of y $(y > 0)$ when $x = 1$ will be

(A) e **(B)** $\frac{1}{e}$ **(C)** e^e **(D)** e^2

32. The degree and order of the differential equation of all tangent lines to the parabola $x^2 = 4y$ is

(A) 2, 1 **(B)** 2, 2
(C) 1, 3 **(D)** 1, 4

33. If $f(x) = f'(x)$ and $f(1) = 2$, then $f(3) =$

(A) e^2 **(B)** $2e^2$
(C) $3e^2$ **(D)** $2e^3$

34. The equation of the curve passing through (1, 1) which satisfies the differential equation $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ is

(A) $e^y = e^x + \frac{x}{2} + \frac{1}{2}$ **(B)** $e^y = e^x + \frac{x}{3} - \frac{1}{2}$

(C) $e^y = e^x + \frac{x^3}{3} - \frac{1}{3}$ **(D)** $e^y = e^x - \frac{x^3}{3} + \frac{1}{3}$

35. Solution of $2y\sin x\frac{dy}{dx} = 2\sin x\cos x - y^2\cos x$, $x = \frac{\pi}{2}$, $y = 1$ is given by

(A) $y^2 = \sin x$ **(B)** $y = \sin^2 x$
(C) $y^2 = \cos x + 1$ **(D)** None of these

36. The general solution of the equation $\frac{dy}{dx} = 1 + xy$ is

(A) $y = ce^{-\frac{x^2}{2}}$ **(B)** $y = ce^{\frac{x^2}{2}}$

(C) $y = (x + c)e^{-\frac{x^2}{2}}$ **(D)** None of these

37. Solution of $\frac{dy}{dx} = e^{y+x} + e^{y-x}$ is

(A) $ex(x + 1) = y$ **(B)** $ex(x + 1) + 1 = y$
(C) $ex(x - 1) + 1 = y$ **(D)** None of these

38. The differential equation of the family of parabolas with focus at the origin and the x-axis as axis is

(A) $y\left(\frac{dy}{dx}\right)^2 + 4x\frac{dy}{dx} = 4y$ **(B)** $-y\left(\frac{dy}{dx}\right)^2 = 2x\frac{dy}{dx} - y$

(C) $y\left(\frac{dy}{dx}\right)^2 + y = 2xy\frac{dy}{dx}$ **(D)** $y\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + y = 0$

39. The differential equation of the family of curves for which the length of the normal is equal to a constant k is given by

(A) $y^2\frac{dy}{dx} = k^2 - y^2$ **(B)** $\left(y\frac{dy}{dx}\right)^2 = k^2 - y^2$

(C) $y\left(\frac{dy}{dx}\right)^2 = k^2 + y^2$ **(D)** $\left(y\frac{dy}{dx}\right)^2 = k^2 + y^2$

40. The solution of the differential equation $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ is

(A) $y = c(x + a)(1 + ay)$ **(B)** $y = c(x + a)(1 - ay)$
(C) $y = c(x - a)(1 + ay)$ **(D)** None of these

41. The solution of the differential equation $\sqrt{a+x}\frac{dy}{dx} + xy = 0$ is

(A) $y = Ae^{2/3(2a-x)\sqrt{x+a}}$ **(B)** $y = Ae^{-2/3(a-x)\sqrt{x+a}}$

(C) $y = Ae^{2/3(2a+x)\sqrt{x+a}}$ **(D)** $y = Ae^{-2/3(2a-x)\sqrt{x+a}}$

(where A is an arbitrary constant)

42. Solution of the equation $x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$ is

(A) $\sin\left(\frac{y}{x}\right) = \log|kx|$ **(B)** $y = x\log|kx|$

(C) $\cos\left(\frac{y}{x}\right) = \log|kx|$ **(D)** $\tan\left(\frac{y}{x}\right) = \log|kx|$

43. The general solution of $y^2dx + (x^2 - xy + y^2)\,dy = 0$ is

(A) $\tan^{-1}\left(\frac{x}{y}\right) + \log y + c = 0$

(B) $2\tan^{-1}\left(\frac{x}{y}\right) + \log x + c = 0$

(C) $\log(y + \sqrt{x^2 + y^2}) + \log y + c = 0$

(D) $\sinh^{-1}\left(\frac{x}{y}\right) + \log y + c = 0$

44. The solution of the equation $\frac{dy}{dx} = \frac{1}{x + y + 1}$ is

(A) $x = ce^y - y - 2$ **(B)** $y = x + ce^y - 2$
(C) $x + ce^y - y - 2 = 0$ **(D)** None of these

45. If integrating factor of $x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$ is $e^{\int Pdx}$, then P is equal to

(A) $\frac{2x^2 - ax^3}{x(1-x^2)}$ **(B)** $(2x^2 - 1)$

(C) $\frac{2x^2 - 1}{ax^3}$ **(D)** $\frac{(2x^2 - 1)}{x(1-x^2)}$

46. The equation of the curve passing through the point $(1, \pi/4)$ and tangent at any point of which makes an angle $\tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)$ is

(A) $y = \tan^{-1}\left[\log\left(\frac{e}{x}\right)\right]$ **(B)** $y = x\tan^{-1}\left(\frac{y}{x}\right) + 1$

(C) $y = x\tan^{-1}(1 - \log x)$ **(D)** $y - x = \tan^{-1}(1 - \log x)$

47. The equation of family of curves for which the length of the normal is equal to the radius vector is

(A) $y^2 \pm x^2 = k$ **(B)** $y \pm x = k$
(C) $y^2 = kx$ **(D)** None of these

48. A continuously differentiable function $\phi(x)$ in $(0, \pi)$ satisfying $y' = 1 + y^2$, $y(0) = 0 = y(\pi)$ is

(A) $\tan x$ **(B)** $x(x - \pi)$
(C) $(x - \pi)(1 - e^x)$ **(D)** Not possible

49. The rate of increase of bacteria in a certain culture is proportional to the number present. If it doubles in 5 h, then in 25 h its number would be

(A) 8 times the original **(B)** 16 times the original
(C) 32 times the original **(D)** 64 times the original

50. The solution of $\frac{d^2y}{dx^2} = \cos x - \sin x$ is

(A) $y = -\cos x + \sin x + c_1x + c_2$
(B) $y = -\cos x - \sin x + c_1x + c_2$
(C) $y = \cos x - \sin x + c_1x^2 + c_2x$
(D) $y = \cos x + \sin x + c_1x^2 + c_2x$

51. The solution of the differential equation $x^4\frac{dy}{dx} + x^3y + \text{cosec}(xy) = 0$ is equal to

(A) $2\cos(xy) + x^{-2} = c$ **(B)** $2\cos(xy) + y^{-2} = c$
(C) $2\sin(xy) + x^{-2} = c$ **(D)** $2\sin(xy) + y^{-2} = c$

52. The solution of the equation $\frac{x^2d^2y}{dx^2} = \ln x$, when $x = 1$, $y = 0$ and $\frac{dy}{dx} = -1$ is

(A) $\frac{1}{2}(\ln x)^2 + \ln x$ **(B)** $\frac{1}{2}(\ln x)^2 - \ln x$
(C) $-\frac{1}{2}(\ln x)^2 + \ln x$ **(D)** $-\frac{1}{2}(\ln x)^2 - \ln x$

53. The solution of the differential equation $x\frac{d^2y}{dx^2} = 1$, given that $y = 1, \frac{dy}{dx} = 0$ when $x = 1$, is

(A) $y = x\log x + x + 2$ **(B)** $y = x\log x - x + 2$
(C) $y = x\log x + x$ **(D)** $y = x\log x - x$

54. The solution of the differential equation $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$ is

(A) $y = \log x + c_1x + c_2$ **(B)** $y = -\log x + c_1x + c_2$
(C) $y = -\frac{1}{x} + c_1x + c_2$ **(D)** None of these

55. The solution of the differential equation $\cos^2 x\frac{d^2y}{dx^2} = 1$ is

(A) $y = \log\cos x + cx$ **(B)** $y = \log\sec x + c_1x + c_2$
(C) $y = \log\sec x - c_1x + c_2$ **(D)** None of these

56. The solution of $\frac{d^2y}{dx^2} = \sec^2 x + xe^x$ is

(A) $y = \log(\sec x) + (x-2)e^x + c_1x + c_2$
(B) $y = \log(\sec x) + (x+2)e^x + c_1x + c_2$
(C) $y = \log(\sec x) - (x+2)e^x + c_1x + c_2$
(D) None of these

57. The differential equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \sin y + x^2 = 0$ is of the following type

(A) Linear **(B)** Homogeneous
(C) Order two **(D)** Degree one

58. The differential equation $x\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y = x^2$ is of

(A) Degree 3 and order 2 **(B)** Degree 1 and order 1
(C) Degree 4 and order 3 **(D)** Degree 4 and order 4

59. The order and degree of the differential equation $\left[x + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = a\frac{d^2y}{dx^2}$ are , respectively

(A) 2, 2 **(B)** 2, 3
(C) 2, 1 **(D)** 2, 4

60. The elimination of the arbitrary constants *A*, *B* and *C* from $y = A + Bx + Ce^{-x}$ leads to the differential equation

(A) $y''' - y' = 0$ **(B)** $y''' - y'' + y' = 0$
(C) $y''' + y'' = 0$ **(D)** $y'' + y'' - y' = 0$

61. The differential equation obtained on eliminating *A* and *B* from the equation $y = A\cos\omega t + B\sin\omega t$ is

(A) $y'' = -\omega^2y$ **(B)** $y'' + y = 0$
(C) $y'' + y' = 0$ **(D)** $y'' - \omega^2y = 0$

62. If the substitution $x = \tan z$ is used, then find the transformed form of the equation $(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + y = 0$

(A) $\frac{d^2y}{dz^2} + 2y = 0$ **(B)** $\frac{d^2y}{dz^2} + 2\frac{dy}{dz} - y = 0$
(C) $\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + y = 0$ **(D)** $\frac{d^2y}{dz^2} + y = 0$

63. Solution of $\frac{dy}{dx} = \frac{x\log x^2 + x}{\sin y + y\cos y}$ is

(A) $y\sin y = x^2\log x + c$ **(B)** $y\sin y = x^2 + c$
(C) $y\sin y = x^2 + \log x + c$ **(D)** $y\sin y = x\log x + c$

64. If $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$, $y(0) = 1$, then $y\left(\frac{\pi}{2}\right) =$

(A) 1 **(B)** $\frac{1}{2}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{4}$

65. The solution of the differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$ is

(A) $x^2(2xy + y^2) = c^2$ **(B)** $x^2(2xy - y^2) = c^2$
(C) $x^2(y^2 - 2xy) = c^2$ **(D)** None of these

66. The solution of the equation $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$ is

(A) $\log\left(\frac{y}{x}\right) = cx$ (B) $\frac{y}{x} = \log y + c$

(C) $y = \log y + 1$ (D) $y = xy + c$

67. Solution of $x\frac{dy}{dx} + y - x + xy\cot x = 0\ (x \neq 0)$ is

(A) $xy = \frac{c}{\sin x} + 1 - x\cot x$ (B) $xy = c\sin x + x\cot x$

(C) $xy\sin x = c - \cot x$ (D) $y\sin x = cx - x\cot x$

68. The solution of $ye^{-x/y}dx - (xe^{-x/y} + y^3)dy = 0$ is

(A) $\frac{y^2}{2} + e^{-x/y} = k$ (B) $\frac{x^2}{2} + e^{-x/y} = k$

(C) $\frac{x^2}{2} + e^{x/y} = k$ (D) $\frac{y^2}{2} + e^{x/y} = k$

69. The solution of the differential equation $x\frac{dy}{dx} + y = x^2 + 3x + 2$ is

(A) $xy = \frac{x^3}{3} + \frac{3}{2}x^2 + 2x + c$ (B) $xy = \frac{x^4}{4} + x^3 + x^2 + c$

(C) $xy = \frac{x^4}{4} + \frac{x^3}{3} + x^2 + c$ (D) $xy = \frac{x^4}{4} + x^3 + x^2 + cx$

70. If y is a function of x and $y(1) = 0$, then the solution of the equation $x\frac{dy}{dx} - \frac{y}{x+1} = x$ is

(A) $y = \frac{x}{x+1}(x + \log|x|)$ (B) $y = \frac{1}{x+1}(x - 1 + \log|x|)$

(C) $y = \frac{x+1}{x}(x - 1 + \log|x|)$ (D) $y = \frac{x}{x+1}(x - 1 + \log|x|)$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. The differential equation of the system of circles touching the x-axis at origin is

(A) $(x^2 - y^2)\frac{dy}{dx} - 2xy = 0$ (B) $(x^2 - y^2)\frac{dy}{dx} + 2xy = 0$

(C) $(x^2 + y^2)\frac{dy}{dx} - 2xy = 0$ (D) $(x^2 + y^2)\frac{dy}{dx} + 2xy = 0$

2. If the general solution for the differential equation $\frac{dy}{dx} = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ is $y = \frac{x}{\ln(cx)}$, then function $\phi\left(\frac{x}{y}\right)$ is

(A) $-\frac{x^2}{y^2}$ (B) $-\frac{y^2}{x^2}$ (C) $\frac{y^2}{x^2}$ (D) $\frac{x^2}{y^2}$

3. Let a function $f(x)$ be such that $f''(x) = f'(x) + e^x$ and $f(0) = 0$ and $f'(0) = 1$, then $\ln\left\{\frac{(f(2))^2}{4}\right\}$ is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

4. The solution of differential equation $2x^3y\,dy + (1 - y^2)(x^2y^2 + y^2 - 1)dx = 0$ is

(A) $x^2y^2 = (cx + 1)(1 - y^2)$ (B) $x^2y^2 = (cx + 1)(1 + y^2)$

(C) $x^2y^2 = (cx - 1)(1 - y^2)$ (D) None of these

Comprehension Type Questions

Paragraph for Questions 5–7: If $f: R \to [0, \infty)$ be a function satisfying the property $f(x + y) - f(x - y) = f(x)[f(y) - f(-y)]$ for all real x and y, $f'(0) = \log a$, $f(0) = 1$, then

5. $f(x)$ is

(A) e^x (B) $2\ln x$ (C) $4x$ (D) a^x

6. $f'(x)$ is

(A) e^x (B) $a^x\log a$ (C) 4 (D) $5x$

7. The solution of differential equation $\frac{dy}{dx} = \frac{(\log_a f(x) + \log_a f(y))^2}{[\log_a f(x) + 2][\log_a f(y) - 2]}$ is

(A) $\frac{y-2}{x+2} - \ln\left|1 + 2\left(\frac{y-2}{x+2}\right)\right| = 2\log(x+2) + c$

(B) $\frac{y+2}{x-2} - \ln\left|1 + \left(\frac{y+2}{x-2}\right)\right| = c$

(C) $(x + 2)(y - 2) = \ln\left(\frac{x+2}{y-2}\right) + c$

(D) None of these

Paragraph for Questions 8–10: A tangent drawn to the curve $y = f(x)$ at $P(x, y)$ cuts the x-axis and the y-axis at A and B, respectively. If $BP:AP = 3:1$ and $f(1) = 1$, then

8. The differential equation of the curve is

(A) $x\frac{dy}{dx} - 3y = 0$ (B) $x\frac{dy}{dx} + 3y = 0$

(C) $3x\frac{dy}{dx} + y = 0$ (D) None of these

9. The curve passes through the point

(A) $\left(\frac{1}{8}, 2\right)$ (B) $\left(2, \frac{1}{8}\right)$ (C) $\left(8, \frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, 8\right)$

10. Normal to the curve at (1, 1) is

(A) $x + 3y = 4$ (B) $3x + y = 4$

(C) $x - 3y = 2$ (D) $-x + 3y = 2$

Paragraph for Questions 11–13: Consider a family of curves, where the ordinate is proportional to the cube of the abscissa and let A be a fixed point, which has coordinates (a, b).

11. If tangents be drawn through A to the members of the family of curves, then the locus of the points of contact is

(A) $xy + bx - 3ay = 0$ (B) $xy - 4bx + 3ay = 0$

(C) $2xy + bx - 3ay = 0$ (D) $2xy - 4bx + 3ay + 2 = 0$

12. If normals be drawn through A to the members of the family of curves, then the feet of these normals on the curves also lie on the curve

(A) $xy + bx - 3ay = 0$ **(B)** $xy - 4bx + 3ay = 0$
(C) $x^2 - 3y^2 = ax - 3by$ **(D)** $x^2 + 3y^2 = ax + 3by$

13. If the tangent through A to a curve cuts the curve again at a point B, then the locus of B is
(A) $xy - 4bx + 3ay = 0$ **(B)** $2xy + bx - 3ay = 0$
(C) $x^2 - 3y^2 = ax - 3by$ **(D)** $a^2x^2 + b^2y^2 = 1$

Paragraph for Questions 14–16: A curve passing through origin is such that slope of tangent at any point is reciprocal of sum of coordinate of point of tangents.

14. Slope of tangent at ordinate ln 3 is
(A) 1 **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** -2

15. Area bounded by curve and the abscissa $y = 0$ and $y = 1$ is
(A) $e - \frac{1}{2}$ **(B)** $e - \frac{3}{2}$ **(C)** $e - \frac{5}{2}$ **(D)** $e + 1$

16. If $I = \int_{-1}^{[|\sin\alpha|+|\cos\alpha|]} xe^{-y} d(e^y)$; (where [.] denotes the greatest integer function), then I is
(A) $e - \frac{1}{e} - \frac{1}{3}$ **(B)** $e - \frac{1}{e} - 2$
(C) $e - \frac{1}{e}$ **(D)** $e + \frac{1}{e} + \frac{1}{3}$

Paragraph for Questions 17–19: *Isogonal Trajectories:* Suppose we have a one-parameter family of curves $\phi(x, y, c) = 0$. Lines intersecting all the curves of the given family at a constant angle are called isogonal trajectories. If this angle is a right angle, they are orthogonal trajectories. Let the trajectories cut the curve of a given family at an angle α, where $\tan \alpha = k$.

The slope $\frac{dy}{dx} = \tan\phi$ of the tangent to a member of the family and the slope $\frac{dy_T}{dx} = \tan\psi$ to the isogonal trajectory are connected by the relationship

$$\tan\phi = \tan(\psi - \alpha) = \frac{\tan\psi - \tan\alpha}{1 + \tan\psi \tan\alpha}$$

That is, $$\frac{dy}{dx} = \frac{\frac{dy_T}{dx} - k}{1 + k\frac{dy_T}{dx}}$$

Eliminating $\frac{dy}{dx}$, we get the differential equation of isogonal trajectories.

17. The isogonal trajectories of a family of straight lines, $y = cx$ that cut the lines of the given family at an angle α, whose tangent is k, have differential equation
(A) $\frac{dy}{dx} = \frac{y}{x}$ **(B)** $\frac{dy}{dx} = \frac{y - kx}{y + kx}$
(C) $\frac{dy}{dx} = \frac{kx + y}{x - ky}$ **(D)** $\frac{dy}{dx} = k\frac{y}{x}$

18. The isogonal trajectories of a family of parabolas, $y^2 = 4ax$ that intersect the members of the family at an angle $\pi/4$, have differential equation
(A) $\frac{dy}{dx} = \frac{x - y}{x + y}$ **(B)** $\frac{dy}{dx} = \frac{2x + y}{2x - y}$
(C) $\frac{dy}{dx} = \frac{x - 2y}{x + 2y}$ **(D)** $\frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}$

19. Let $f(x, y) = 0$ represents a family of circles touching the axis of y at origin. The differential equation of the family of curves intersecting the above family orthogonally is
(A) $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$ **(B)** $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$
(C) $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ **(D)** Either (A) or (C)

Paragraph for Questions 20–22: Consider a polynomial $f(x)$, which satisfies the following conditions:

(i) $f(x) = \{f'(x)\}^2 \; \forall x$

(ii) $\int_0^1 f(x)dx = \frac{19}{12}$

(iii) $f'(0) > 0$

20. The function $f(x)$ can be
(A) A linear function
(B) A quadratic function
(C) A cubic function
(D) Any polynomial of even degree

21. The value of $f'(0)$ is
(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1

22. The function $f(x)$ is
(A) Even
(B) Odd
(C) Neither even nor odd
(D) May be either even or odd

Answer Key

Practice Exercise 1

1. (A)	**2.** (B)	**3.** (C)	**4.** (A)	**5.** (A)	**6.** (B)	**7.** (A)	**8.** (B)	**9.** (A)	**10.** (C)	**11.** (D)	**12.** (A)
13. (D)	**14.** (A)	**15.** (A)	**16.** (B)	**17.** (D)	**18.** (C)	**19.** (A)	**20.** (A)	**21.** (C)	**22.** (B)	**23.** (B)	**24.** (C)
25. (C)	**26.** (A)	**27.** (B)	**28.** (A)	**29.** (B)	**30.** (B)	**31.** (C)	**32.** (A)	**33.** (B)	**34.** (C)	**35.** (A)	**36.** (D)

37. (D) **38.** (B) **39.** (B) **40.** (B) **41.** (A) **42.** (C) **43.** (A) **44.** (A) **45.** (D) **46.** (C) **47.** (A) **48.** (D)
49. (C) **50.** (A) **51.** (A) **52.** (D) **53.** (B) **54.** (A) **55.** (B, C) **56.** (A) **57.** (C, D) **58.** (A) **59.** (A) **60.** (C)
61. (A) **62.** (D) **63.** (A) **64.** (C) **65.** (A) **66.** (A) **67.** (A) **68.** (A) **69.** (A) **70.** (D)

Practice Exercise 2

1. (B) **2.** (B) **3.** (D) **4.** (C) **5.** (D) **6.** (B) **7.** (A) **8.** (B) **9.** (B) **10.** (D) **11.** (C) **12.** (D) **13.** (A)
14. (C) **15.** (B) **16.** (B) **17.** (C) **18.** (B) **19.** (A) **20.** (B) **21.** (D) **22.** (C)

Solutions

Practice Exercise 1

1. The general equation of all non-vertical lines in a plane is

$$y = mx + c$$

$$\Rightarrow \frac{dy}{dx} = m \qquad \text{[differentiating w.r.t. } x\text{]}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 0 \qquad \text{[differentiating w.r.t. } x\text{]}$$

2. The general equation of all non-horizontal lines in xy-plane is

$$x = my + c$$

$$\Rightarrow \frac{dx}{dy} = m \qquad \text{[differentiating w.r.t. } y\text{]}$$

$$\Rightarrow \frac{d^2x}{dy^2} = 0 \qquad \text{[differentiating w.r.t. } y\text{]}$$

3. The given equation can be written as

$$y = x\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

If $\frac{dy}{dx} = p$, then $y = px - p^2$.

On differentiating w.r.t. x, we get

$$p = p + \frac{dp}{dx} - 2p\frac{dp}{dx} \Rightarrow \frac{dp}{dx}(x - 2p) \Rightarrow \frac{dp}{dx} = 0$$

On integrating w.r.t. x, we get $p = c$

$$\frac{dy}{dx} = c; y = cx - c^2$$

If $c = 2$, then $y = 2x - 4$.

4. We have

$$y^2 = 2c(x + \sqrt{c}) \qquad (1)$$

$$\Rightarrow 2y\,y_1 = 2c \Rightarrow yy_1 = c \qquad (2)$$

Eliminating c from Eqs. (1) and (2), we get

$$y^2 = 2yy_1(x + \sqrt{yy_1}) \Rightarrow y - 2xy_1 = 2\sqrt{y}\,y_1^{3/2} \Rightarrow (y - 2xy_1)^2 = 4yy_1^3$$

Clearly, it is a differential equation of order 1 and degree 3.

5. We have

$$\frac{y_3}{y_2} = 8 \Rightarrow \ln y_2 = 8x + c_1$$

Putting $x = 0$, we have

$$c_1 = \log y_2(0) = \log 1 = 0$$

Therefore,

$$\log y_2 = 8x \text{ or } y_2 = e^{8x}$$

Integrating both sides, we get

$$y_1 = \frac{e^{8x}}{8} + c_2$$

Again putting $x = 0$, we have

$$c_2 = -1/8$$

So,

$$y_1 = \frac{1}{8}(e^{8x} - 1) \Rightarrow y = \frac{1}{8}\left(\frac{e^{8x}}{8} - x\right) + c_3$$

Putting $x = 0$. We have

$$c_3 = \frac{1}{8} - \frac{1}{64} = \frac{7}{64}$$

Thus, $y = \frac{1}{8}\left(\frac{e^{8x}}{8} - x + \frac{7}{8}\right)$.

6. $\frac{dy}{dx} = 4e^{4x} - 2e^{-x} \Rightarrow \frac{d^2y}{dx^2} = 16e^{4x} + 2e^{-x} \Rightarrow \frac{d^3y}{dx^3} = 64e^{4x} - 2e^{-x}$

Putting these values in $\frac{d^3y}{dx^3} + A\frac{dy}{dx} + By = 0$, we have

$$(64 + 4A + B)\,e^{4x} + (-2 - 2A + 2B)e^{-x} = 0$$

Solving, we get $A = -13$ and $B = -12$.

7. We have

$$f(\theta) = \frac{d}{d\theta}\int_0^{\theta}\frac{dx}{1 - \cos\theta\cos x} = \frac{1}{1 - \cos^2\theta} = \operatorname{cosec}^2\theta$$

Therefore, $\frac{df(\theta)}{d\theta} = -2\operatorname{cosec}^2\theta\cot\theta$.

8. We have

$$f''(x) = g''(x)$$

On integrating, we get

$$f'(x) = g'(x) + c$$

Putting $x = 1$, we get

$$f'(1) = g'(1) + c \Rightarrow c = -2$$

$$\Rightarrow f'(x) = g'(x) - 2 \Rightarrow f(x) = g(x) - 2x + c_1$$

$$\Rightarrow f(2) = g(2) - 4 + c_1 \Rightarrow c_1 = -2$$

Thus, we have

$$f(x) = g(x) - 2x - 2$$
$$\Rightarrow f(4) - g(4) = -10$$

9. Hint: There will only one constant in the first-order differential equation.

Differentiating the given equation

$$(y + c)^2 = cx \quad (1)$$

$$\Rightarrow 2(y+c)\cdot\frac{dy}{dx} = c \Rightarrow 2y\cdot\frac{dy}{dx} = c - 2c\cdot\frac{dy}{dx} = c\left(1-2\cdot\frac{dy}{dx}\right)$$

$$\Rightarrow c = \frac{2y\cdot\frac{dy}{dx}}{1-2\cdot\frac{dy}{dx}}$$

Putting the value of c in Eq. (1) and simplifying we will get a first-order and second-degree equation. Hence, (A) is the correct answer.

10. Hint: $dX = dY \Rightarrow X = Y + c$

$$x^2\sin^3 y\,dx - y^2\cos x\,dx + x^3\cos y\sin^2 y\,dy - 2y\sin x\,dy = 0$$

$$\Rightarrow (x^2\sin^3 y\,dx + x^3\cos y\sin^2 y\,dy) - (y^2\cos x\,dx + 2y\sin x\,dy) = 0$$

$$\Rightarrow d\left(\frac{\sin^3 y\cdot x^3}{3}\right) - d(y^2\sin x) = 0$$

On integrating both sides, we get

$$\frac{\sin^3 y\cdot x^3}{3} - y^2\sin x = c$$

11. Hint: Put $y = vx$

$$\frac{dy}{dx} = \left(\frac{y}{x^2+y^2} - 1\right)\left(\frac{x^2+y^2}{x}\right) = \left(\frac{y}{x} - \frac{x^2+y^2}{x}\right)$$

Putting $y = vx$. Then

$$\frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$$

$$\Rightarrow v + x\cdot\frac{dv}{dx} = \left(v - \frac{x^2+v^2x^2}{x}\right)$$

$$\Rightarrow v + x\cdot\frac{dv}{dx} = v - x(1+v^2)$$

$$\Rightarrow \frac{dv}{1+v^2} = -dx$$

$$\Rightarrow \int\frac{dv}{1+v^2} = -\int dx + c$$

$$\Rightarrow \tan^{-1}v = -x + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = -x + c$$

12. Hint: Given is the linear equation

$$\frac{dy}{dx} + (2x-1)y = 0$$

$$\text{I.F.} = e^{\int(2x-1)dx} = e^{x^2-x}$$

Therefore, the solution of differential equation is

$$ye^{x^2-x} = c$$

$$\Rightarrow y = ce^{x-x^2}$$

13. Hint: Put $x + y = v$

Putting $x + y = v$. Then

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \quad\Rightarrow\quad \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\Rightarrow \frac{dv}{dx} = \sin v + (\cos v + 1) = 2\sin\frac{v}{2}\cos\frac{v}{2} + 2\cos^2\frac{v}{2}$$

$$= 2\cos\frac{v}{2}\left(\sin\frac{v}{2} + \cos\frac{v}{2}\right)$$

$$\Rightarrow \int\frac{dv}{2\cos\frac{v}{2}\left(\sin\frac{v}{2}+\cos\frac{v}{2}\right)} = \int dx + c$$

Dividing numerator and denominator by cos (v/2) and integrating, we get

$$\log\left|\tan\frac{x+y}{2} + 1\right| = x + c$$

14.

$$y = a\cos(\log x) + b\sin(\log x) \quad (1)$$

On differentiating Eq. (1) w.r.t. to x both sides, we get

$$\frac{dy}{dx} = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow x\cdot\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

$$\Rightarrow x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx} = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

$$\Rightarrow x^2\cdot\frac{d^2y}{dx^2} + x\cdot\frac{dy}{dx} = -y$$

$$\Rightarrow x^2\cdot\frac{d^2y}{dx^2} + x\cdot\frac{dy}{dx} + y = 0$$

15.

$$y = \sin(a\sin^{-1}x)$$

$$\Rightarrow \frac{dy}{dx} = \cos(a\sin^{-1}x)\cdot\frac{a}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\cdot\frac{dy}{dx} = a\cos(a\sin^{-1}x)$$

$$\Rightarrow \sqrt{1-x^2}\cdot\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = -\frac{a^2y}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\cdot\frac{d^2y}{dx^2} - x\cdot\frac{dy}{dx} + a^2y = 0$$

16.

$$y = A\sin x + B\cos x + x\sin x$$

Differentiating w.r.t. x, we get

$$y_1 = A\cos x - B\sin x + \sin x + x\cos x$$

Again, differentiating w.r.t. x, we get

$$y_2 = -A\sin x - B\cos x + \cos x + \cos x - x\sin x$$
$$y_2 = -(A\sin x + B\cos x + x\sin x) + 2\cos x$$
$$= -y + 2\cos x$$

17. $$y\cdot\frac{dy}{dx} = a - x$$
$$\Rightarrow\ y\,dy = (a-x)dx$$
$$\Rightarrow \int y dx = \int (a-x)dx + c$$
$$\Rightarrow\ \frac{y^2}{2} = ax - \frac{x^2}{2} + c$$
$$\Rightarrow\ x^2 + y^2 - 2ax - c = 0 \qquad (1)$$

Obviously, the equation is a circle with the centre on the x-axis.

18. $$\int_0^{96}\frac{dx}{x+4} = \int_0^t dt$$
$$\Rightarrow \left|\ln(x+4)\right|_0^{96} = t$$
$$\Rightarrow \ln 100 - \ln 4 = t$$
$$\Rightarrow \ln 25 = t$$
$$\Rightarrow 2\ln 5 = t$$

19. Hint: Put $y = vx$

$$\frac{dy}{dx} = v + x\cdot\frac{dv}{dx}\ \Rightarrow\ v + x\cdot\frac{dv}{dx} = v + \frac{\phi(v)}{\phi'(v)}$$
$$\Rightarrow\ x\cdot\frac{dv}{dx} = \frac{\phi(v)}{\phi'(v)}$$
$$\Rightarrow\ \int\frac{\phi'(v)dv}{\phi(v)} = \int\frac{dx}{x}$$

Solving, we get solution as

$$\phi\left(\frac{y}{x}\right) = kx$$

20. $$\frac{dy}{dx} = e^{ax+by}$$
$$\Rightarrow\ \frac{dy}{dx} = e^{ax}\cdot e^{by}$$
$$\Rightarrow\ \int e^{-by}dy = \int e^{ax}dx + c \qquad (c \text{ is a constant})$$
$$\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + c$$

21. Differentiating the given differential equation

$$cy^2 = 2x + c \qquad (1)$$
$$c\cdot 2y\cdot\frac{dy}{dx} = 2$$
$$\Rightarrow\ c = \frac{1}{y}\cdot\frac{dx}{dy}$$

Putting this value of c in Eq. (1), we get

$$\frac{1}{y}\cdot\frac{dx}{dy}\cdot y^2 = 2x + \frac{1}{y}\cdot\frac{dx}{dy}$$
$$\Rightarrow\ y^2 = 2xy\cdot\frac{dy}{dx} + 1$$

22. $$x(1+y^2)dx + y(1+x^2)dy = 0$$
$$\Rightarrow \frac{ydy}{1+y^2} = -\frac{xdx}{1+x^2}$$

Integrating both sides, we get

$$\int\frac{ydy}{1+y^2} = -\int\frac{xdx}{1+x^2}$$
$$\Rightarrow \ln(1+y^2) = -\ln(1+x^2) + \ln c$$
$$\Rightarrow \ln(1+y^2)(1+x^2) = \ln c$$
$$\Rightarrow (1+y^2)(1+x^2) = c$$

23. $$\frac{dy}{\sqrt{1-y^2}} = \frac{-dx}{\sqrt{1-x^2}} \Rightarrow \sin^{-1}y + \sin^{-1}x = c$$

24. $$\frac{d^2y}{dx^2} = e^{-2x} \Rightarrow \frac{dy}{dx} = \frac{e^{-2x}}{-2} + k_1$$

On integrating,

$$y = \frac{e^{-2x}}{4} + k_1x + k_2 \Rightarrow y = \frac{e^{-2x}}{4} + cx + d$$

25. $$\frac{dy}{dx} + \frac{y}{x} = x^2$$
$$\text{I.F.} = e^{\int\frac{dx}{x}} = x$$

Therefore, solution is

$$xy = \int x^2\cdot x\,dx = \frac{x^4}{4} + c$$

26. $$\frac{dx}{x} = \frac{2dy}{y} \Rightarrow \ln x = \ln y^2 + \ln c \Rightarrow y^2 = kx$$

It represents a family of parabola.

27. $$\frac{dy}{dx} - y\tan x = \sec x$$
$$\text{I.F.} = e^{-\int\tan x dx} = \cos x$$
$$y\cos x = \int \sec x\cos x dx = x + c$$
$$\Rightarrow y\cos x = x + c$$

At (0, 0) $c = 0$,

$$y\cos x = x$$

28. $$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x^2} + \frac{1}{x^2}\cos\frac{y}{x} \qquad (1)$$

Put $y = vx$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Therefore, Eq. (1) becomes

$$v + x\frac{dv}{dx} - v = \frac{1}{x^2} + \frac{1}{x^2}\cos v$$
$$\Rightarrow\ x^3\frac{dv}{dx} = 1 + \cos v$$

$$\Rightarrow \frac{dv}{1+\cos v} = \frac{dx}{x^3} \Rightarrow \int \frac{1}{2}\sec^2 \frac{v}{2} dv = \frac{x^{-2}}{-2} + c$$

$$\Rightarrow \tan \frac{v}{2} = -\frac{1}{2x^2} + c \Rightarrow \tan \frac{y}{2x} = c - \frac{1}{2x^2}$$

29. $e^{2y} dy = dx \Rightarrow \frac{e^{2y}}{2} + c = x$. Put $x = 5$ and $y = 0$, we get

$$c = 5 - \frac{1}{2} = \frac{9}{2}$$

At $y = 3$

$$\frac{e^6}{2} + \frac{9}{2} = x \Rightarrow x = \frac{e^6 + 9}{2}$$

30. $$\frac{dy}{dx} = -1 - \frac{y}{x} \Rightarrow \frac{dy}{dx} + \frac{y}{x} = -1$$

Put $z = y/x$. Therefore,

$$z + x\frac{dz}{dx} + z = -1$$

$$\Rightarrow x\frac{dz}{dx} = -(2z+1)$$

$$\Rightarrow \frac{dz}{2z+1} + \frac{dx}{x} = 0$$

Integrating, we get

$$\int \frac{dz}{2z+1} + \int \frac{dx}{x} = c$$

$$\Rightarrow \frac{1}{2}\log|2z+1| + \log|x| = c \Rightarrow \log|2z+1| + \log(x^2) = c$$

$$\Rightarrow |2z+1| = e^{c-\log(x^2)} = \frac{k}{x^2} \Rightarrow 2\left(\frac{y}{x}\right) + 1 = \frac{k}{x^2}$$

As the curve passes through (1, 3), so $k = 7$. Therefore, the equation of the curve is

$$2xy + x^2 = 7$$

31. $$\frac{dy}{y} = e^x\,dx \Rightarrow \ln y = e^x + c$$

At $x = 0$, $y = e$. So, $c = 0$.

$$\ln y = e^x$$

Therefore, at $x = 1$, $y = e^e$.

32. Equation of any tangent to $x^2 = 4y$ is $x = my + \frac{1}{m}$, where m is an arbitrary constant. So,

$$1 = m\frac{dy}{dx} \Rightarrow m = \frac{1}{\frac{dy}{dx}}$$

Therefore, putting this value of m in $x = my + \frac{1}{m}$, we get

$$x = \frac{y}{\frac{dy}{dx}} + \frac{dy}{dx} \Rightarrow \left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$$

which is a differential equation of order 1 and degree 2.

33. $$\frac{f'(x)}{f(x)} = 1 \Rightarrow \log f(x) = x + c$$

Since $f(1) = 2$. Therefore,

$$\log 2 = 1 + c$$

$$\log f(x) = x + \log 2 - 1$$

$$\log f(3) = 3 + \log 2 - 1 = 2 + \log 2$$

$$\Rightarrow f(3) = e^{2+\log 2} = e^2 \cdot e^{\log 2} = 2e^2$$

34. We have

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\Rightarrow e^y dy = (e^x + x^2)dx$$

Integrating, we get

$$\int e^y dy = \int (e^x + x^2)dx + c$$

$$\Rightarrow e^y = e^x + \frac{1}{3}x^3 + c$$

The curve passes through (1, 1) implies

$$e = e + \frac{1}{3} + c$$

$$\Rightarrow c = -\frac{1}{3}$$

Therefore, the equation of the curve is

$$e^y = e^x + \frac{x^3}{3} - \frac{1}{3}$$

35. On dividing by $\sin x$,

$$2y\frac{dy}{dx} + y^2 \cot x = 2\cos x$$

Put $y^2 = v$, we get

$$\frac{dv}{dx} + v\cot x = 2\cos x$$

$$\text{I.F.} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$$

Therefore, the solution is

$$v \sin x = \int \sin x(2\cos x)\, dx + c$$

$$\Rightarrow y^2 \sin x = \sin^2 x + c$$

When $x = \frac{\pi}{2}$, $y = 1$, then $c = 0$. Therefore,

$$y^2 = \sin x$$

36. $$\frac{dy}{dx} - xy = 1 \Rightarrow P = -x, Q = 1$$

$$\text{I.F.} = e^{-\int x dx} = e^{-\frac{x^2}{2}}$$

Therefore, the solution is $y \cdot e^{\frac{-x^2}{2}} = \int e^{-\frac{x^2}{2}} \cdot 1 \cdot dx + c.$

37. $$\frac{dy}{dx} = e^y \cdot e^x + e^y \cdot e^{-x} = e^y(e^x + e^{-x})$$

$$\Rightarrow e^{-y}dy = (e^x + e^{-x})dx \Rightarrow -e^{-y} = e^x - e^{-x} + c$$

38. Equation of family of parabolas with focus at (0, 0) and the x-axis as axis is $y^2 = 4a(x + a)$. (1)

On differentiating Eq. (1) with respect to x,

$$2yy_1 = 4a;\ y^2 = 2yy_1\left(x + \frac{yy_1}{2}\right)$$

$$y = 2xy_1 + yy_1^2 \Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$$

39. The length of normal is given by

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Therefore,

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = k \Rightarrow y^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = k^2$$

$$\Rightarrow y^2 + y^2\left(\frac{dy}{dx}\right)^2 = k^2 \Rightarrow y^2\left(\frac{dy}{dx}\right)^2 = k^2 - y^2$$

40. $$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right) \Rightarrow y - ay^2 = (x+a)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{y(1-ay)} = \frac{dx}{x+a}$$

On integrating both sides, we get

$$\log y - \log(1-ay) = \log(x+a) + \log c$$

$$\Rightarrow \frac{y}{(1-ay)} = c(x+a) \Rightarrow c(x+a)(1-ay) = y$$

41. $$\frac{dy}{dx} + \frac{xy}{\sqrt{a+x}} = 0 \Rightarrow \frac{dy}{y} = \frac{-xdx}{\sqrt{a+x}}$$

On integrating both sides,

$$\int\frac{dy}{y} = \int\frac{-x}{\sqrt{x+a}}\,dx$$

$$\log y = -\int\frac{x+a-a}{\sqrt{x+a}}\,dx = -\int\sqrt{x+a}\,dx + \int\frac{a}{\sqrt{x+a}}\,dx$$

$$\Rightarrow \log y = -\frac{2}{3}(x+a)^{3/2} + 2a\sqrt{x+a} + \log A$$

$$y = Ae^{-2/3(x+a)^{3/2} + 2a\sqrt{x+a}} = Ae^{\left[(\sqrt{x+a}\left(-\frac{2}{3}(x+a)+2a\right)\right]}$$

$$= Ae^{\left[\sqrt{x+a}\left(\frac{-2x-2a+6a}{3}\right)\right]} = Ae^{[-2/3\sqrt{x+a}(x-2a)]}$$

$$\Rightarrow y = Ae^{[2/3\sqrt{x+a}(2a-x)]}$$

42. We have

$$x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$$

$$\frac{dy}{dx} = \frac{y\sin(y/x) - x}{x\sin(y/2)}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v = \frac{-1}{\sin v}$$

$$\Rightarrow (\sin v)dv + \frac{dx}{x} = 0$$

Integrating, we get

$$\int(\sin v)dv + \int\frac{dx}{x} = c$$

$$\Rightarrow -\cos v + \log|x| = c \Rightarrow \cos\frac{y}{x} = \log|k\,x|$$

43. $$\frac{dx}{dy} + \frac{x^2 - xy + y^2}{y^2} = 0$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{x}{y}\right)^2 - \left(\frac{x}{y}\right) + 1 = 0$$

Put $v = x/y$. Then

$$x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$v + y\frac{dv}{dy} + v^2 - v + 1 = 0 \Rightarrow \frac{dv}{v^2+1} + \frac{dy}{y} = 0$$

$$\Rightarrow \int\frac{dv}{v^2+1} + \int\frac{dy}{y} = 0 \Rightarrow \tan^{-1}(v) + \log y + c = 0$$

$$\Rightarrow \tan^{-1}(x/y) + \log y + c = 0$$

44. $$\frac{dy}{dx} = \frac{1}{x+y+1} \Rightarrow \frac{dx}{dy} = x + y + 1 \Rightarrow \frac{dx}{dy} - x = y + 1$$

It is a linear equation, therefore

$$\text{I.F.} = e^{\int -1dy} = e^{-y}$$

Hence, the solution of the equation is

$$x.e^{-y} = \int (y+1)e^{-y}dy + c \Rightarrow x = ce^{y} - y - 2$$

45. $$x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{(2x^2-1)}{x(1-x^2)}y = \frac{ax^2}{(1-x^2)}$$

Therefore, $$P = \frac{2x^2-1}{x(1-x^2)}$$

46. We have

$$\frac{dy}{dx} = \tan\left(\tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)\right)$$

$$= \frac{y}{x} - \cos^2\frac{y}{x}$$

Put $y = vx$. Then

$$v + x\frac{dv}{dx} = v - \cos^2 v \Rightarrow (\sec^2 v)dv + \frac{dx}{x} = 0$$

Integrating, we get

$$\int(\sec^2 v)dv+\int\frac{dx}{x}=c$$

$$\Rightarrow \tan v+\log x=c\Rightarrow \tan\left(\frac{y}{x}\right)+\log x=c$$

The curve passes through (1, $\pi/4$), so $c = 1$ Therefore,

$$\tan\frac{y}{x}=1-\log x\Rightarrow\frac{y}{x}=\tan^{-1}(1-\log x)$$

47. Length of the normal $=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

It is given that $y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{x^2+y^2}$.

So, (since radius vector $=r=\sqrt{x^2+y^2}$)

$$y^2+y^2\left(\frac{dy}{dx}\right)^2=x^2+y^2\Rightarrow y^2\left(\frac{dy}{dx}\right)^2=x^2$$

$$\Rightarrow ydy\pm xdx=0\Rightarrow y^2\pm x^2=k$$

48. $$\frac{dy}{dx}=1+y^2\Rightarrow\frac{dy}{1+y^2}=dx$$

On integrating both sides

$$\int\frac{dy}{1+y^2}=\int dx\Rightarrow\tan^{-1}y=x+c$$

At $x = 0$, $y = 0$, then $c = 0$.
At $x = \pi$, $y = 0$, then $\tan^{-1} 0 = p + c \Rightarrow c = -\pi$
Therefore,

$$\tan^{-1}y = x \Rightarrow y = \tan x = \phi(x)$$

Therefore, the solution is $y = \tan x$.
But $\tan x$ is not a continuous function in $(0, \pi)$.
Hence, $\phi(x)$ is not possible in $(0, \pi)$.

49. Let P_0 be the initial population and let the population after t years be P. Then

$$\frac{dP}{dt}=kP\Rightarrow\frac{dP}{P}=k\,dt$$

On integrating, we have $\log P = kt + c$.
At $t = 0$, $P = P_0$

Therefore, $\log P_0=0+c$. So,

$$\log P=kt+\log P_0\Rightarrow\log\frac{P}{P_0}=kt$$

When $t = 5\ h$, $P = 2P_0$. Therefore,

$$\log\frac{2P_0}{P_0}=5k\Rightarrow k=\frac{\log 2}{5}$$

Therefore,

$$\log\frac{P}{P_0}=\frac{\log 2}{5}t$$

When $t = 25h$, we have

$$\log\frac{P}{P_0}=\frac{\log 2}{5}\times 25=5\log 2=\log 32$$

Hence, $P = 32P_0$.

50. $$\frac{d^2y}{dx^2}=\cos x-\sin x$$

On integrating both sides, we get

$$\frac{dy}{dx}=\sin x+\cos x+c_1$$

Integrating again, we get

$$y=-\cos x+\sin x+c_1x+c_2$$

51. $$x^4\frac{dy}{dx}+x^3y+\operatorname{cosec}(xy)=0$$

$$\Rightarrow x^4dy+x^3y\,dx+\operatorname{cosec}(xy)dx=0$$

$$\Rightarrow x^3(x\,dy+y\,dx)+\operatorname{cosec}(xy)dx=0$$

$$\Rightarrow x^3d(xy)+\operatorname{cosec}(xy)dx=0$$

$$\Rightarrow\frac{d(xy)}{\operatorname{cosec}(xy)}+\frac{dx}{x^3}=0$$

On integrating both sides, we get

$$\int\frac{d(xy)}{\operatorname{cosec}(xy)}+\int\frac{dx}{x^3}=0$$

$$\int\sin(xy)d(xy)+\int x^{-3}dx=0$$

$$-\cos(xy)+\left(\frac{x^{-2}}{-2}\right)=c\Rightarrow 2\cos(xy)+x^{-2}=c$$

52. $$\frac{d^2y}{dx^2}=\frac{\ln x}{x^2}\Rightarrow\frac{dy}{dx}=\frac{-(\ln x+1)}{x}+c$$

At $\frac{dy}{dx}=-1$, $x = 1$, $y = 0$, therefore $c = 0$. So,

$$y=-\int\frac{\ln x+1}{x}dx=-\frac{1}{2}(\ln x)^2-\ln x$$

53. $$x\frac{d^2y}{dx^2}=1\Rightarrow\frac{d^2y}{dx^2}=\frac{1}{x}\Rightarrow\frac{dy}{dx}=\log x+c_1$$

$\Rightarrow y=x\log x-x+c_1x+c_2$ (on integrating twice)

Given $y = 1$ and $\frac{dy}{dx}=0$ at $x = 1$. So, $c_1 = 0$ and $c_2 = 2$.

Therefore, the required solution is $y=x\log x-x+2$.

54. $$\frac{d^2y}{dx^2}=-\frac{1}{x^2}$$

Now on integrating both sides, we get

$$\frac{dy}{dx}=\frac{1}{x}+c_1\Rightarrow y=\log x+c_1x+c_2$$

55. $$\cos^2 x\frac{d^2y}{dx^2}=1\Rightarrow\frac{d^2y}{dx^2}=\sec^2 x$$

On integrating, we get $\frac{dy}{dx} = \tan x \pm c_1$.

Again integrating, we get $y = \log \sec x \pm c_1 x \pm c_2$.

56. $$\frac{d^2y}{dx^2} = \sec^2 x + xe^x$$

On integrating, we get

$$\frac{dy}{dx} = \tan x + xe^x - e^x + c_1$$

Integrating again, we get

$$y = \log(\sec x) + xe^x - e^x - e^x + c_1 x + c_2$$

Thus, the required solution is,

$$y = \log(\sec x) + (x-2)e^x + c_1 x + c_2$$

57. Given

$$\frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} + \sin y + x^2 = 0$$

The order of highest derivative is 2 and degree is 1.

58. Given differential equation,

$$x\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y = x^2$$

In this equation order of highest derivative is 2,

Hence, order is 2 and degree of highest derivative is 3.

59. The given equation can be written as

$$\left[x + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

Therefore, order is 2 and degree is 2.

60. $$y = A + Bx + Ce^{-x} \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = B - Ce^{-x} \quad (2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = Ce^{-x} \quad (3)$$

$$\Rightarrow \frac{d^3y}{dx^3} = -Ce^{-x} \quad (4)$$

Adding Eqs. (3) and (4), we get

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = 0 \Rightarrow y''' + y'' = 0$$

61. $$y' = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$\Rightarrow y'' = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t = -\omega^2(A\cos \omega t + B\sin \omega t)$$

Therefore, $y'' = -\omega^2 y$.

62. Given that $x = \tan z$. Therefore,

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \cdot \frac{1}{1+x^2}$$

So,

$$(1+x^2)\frac{dy}{dx} = \frac{dy}{dz}$$

Again differentiating both sides w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \left(\frac{d^2y}{dz^2}\right)\frac{1}{1+x^2}$$

Therefore,

$$(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

Hence, the equation will transform to

$$\frac{d^2y}{dz^2} + y = 0$$

63. $$\frac{dy}{dx} = \frac{x\log x^2 + x}{\sin y + y\cos y}$$

Separating the variables and integrating

$$\int(\sin y + y\cos y)dy = \int(x\log x^2 + x)dx$$

$$\Rightarrow -\cos y + y\sin y + \cos y$$

$$= \frac{x^2}{2}\log x^2 - \int \frac{x^2}{2} \cdot \frac{1}{x^2} \cdot 2x dx + \int x\,dx + c$$

$$\Rightarrow y\sin y = \frac{x^2}{2} 2\log x - \int x\,dx + \int x dx + c$$

$$\Rightarrow y\sin y = x^2 \log x + c.$$

64. The given differential equation is

$$\frac{\cos x}{2+\sin x}dx + \frac{dy}{y+1} = 0$$

$$\Rightarrow \log(2+\sin x) + \log(y+1) = \log c$$

$$\Rightarrow (y+1)(2+\sin x) = c \Rightarrow 2 \times 2 = c \Rightarrow c = 4$$

Thus,

$$y+1 = \frac{4}{2+\sin x} \Rightarrow y = \frac{2-\sin x}{2+\sin x} \Rightarrow y\left(\frac{\pi}{2}\right) = \frac{1}{3}$$

65. It can be written in the form of homogeneous equation

$$\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}$$

So now putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = -\frac{3x^2v + x^2v^2}{x^2 + x^2v} \Rightarrow x\frac{dv}{dx} = \frac{-2v(v+2)}{v+1}$$

$$\Rightarrow \frac{1}{x}dx = -\frac{v+1}{2v(v+2)}dv = -\left[\frac{1}{2(v+2)} + \frac{1}{2v(v+2)}\right]dv$$

$$\Rightarrow -\frac{2}{x}dx = \left[\frac{1}{v+2} + \frac{1}{2v} - \frac{1}{2(v+2)}\right]dv$$

On integrating, we get

$$-2\log_e x = \frac{1}{2}\log(v+2) + \frac{1}{2}\log v + \log c$$

$$\Rightarrow v(v+2)x^4 = c^2 \Rightarrow \frac{y}{x}\left(\frac{y}{x}+2\right)x^4 = c^2, \quad \left(\text{As } v = \frac{y}{x}\right)$$

Hence, the required solution is $(y^2+2xy)x^2 = c^2$.

66.
$$\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x}+1\right)$$

Put $y = vx$. Then

$$\frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$$

Therefore,

$$v + x\cdot\frac{dv}{dx} = v(\log v+1)$$

$$\Rightarrow v + x\frac{dv}{dx} = v\log v + v$$

$$\Rightarrow x\frac{dv}{dx} = v\log v$$

$$\Rightarrow \frac{dv}{v\log v} = \frac{dx}{x}$$

On integrating both sides,

$$\int\frac{dv}{v\log v} = \int\frac{dx}{x} \Rightarrow \log\ \log v = \log x + \log c$$

$$\Rightarrow \log v = xc \Rightarrow \log(y/x) = xc$$

67. We have

$$x\frac{dy}{dx} + (1+x\cot x)y = x$$

$$\Rightarrow \frac{dy}{dx} + \frac{(1+x\cot x)}{x}y = 1$$

Therefore,

$$P = \frac{1+x\cot x}{x}, Q = 1$$

The integrating factor is

$$\text{I.F.} = e^{\int P dx} = e^{\int\left(\frac{1}{x}+\cot x\right)dx} = e^{\log(x\sin x)} = x\sin x$$

Therefore, the solution is

$$y\,(x\sin x) = \int 1(x\sin x)dx + c$$
$$= \sin x - x\cos x + c$$

So,

$$xy = 1 - x\cot x + \frac{c}{\sin x}$$

68.
$$y e^{-x/y}dx - (xe^{-x/y}+y^3)dy = 0$$

$$\Rightarrow e^{-x/y}(ydx - xdy) = y^3 dy \Rightarrow e^{-x/y}\frac{(ydx-xdy)}{y^2} = ydy$$

$$\Rightarrow e^{-x/y}d\left(\frac{x}{y}\right) = ydy$$

On integrating both sides, we get

$$k - e^{-x/y} = \frac{y^2}{2} \Rightarrow \frac{y^2}{2} + e^{-x/y} = k$$

69.
$$x\frac{dy}{dx} + y = x^2+3x+2$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x+3+\frac{2}{x}$$

Here, $P = \frac{1}{x}, Q = x+3+\frac{2}{x}$, therefore

$$\text{I.F. } = e^{\int\frac{1}{x}dy} = x$$

$$\int d(xy) = \int(x^2+3x+2)dx$$

$$xy = \frac{x^3}{3} + \frac{3x^2}{2} + 2x + c$$

70. Given equation is

$$\frac{dy}{dx} - \frac{y}{x(x+1)} = 1$$

The integrating factor is

$$\text{I.F.} = e^{-\int\frac{dx}{x(x+1)}} = e^{-\int\left(\frac{1}{x}-\frac{1}{x+1}\right)dx} = e^{\log\left(\frac{x+1}{x}\right)} = \frac{x+1}{x}$$

Therefore, the solution is

$$y\left(\frac{x+1}{x}\right) = \int\frac{x+1}{x}dx + c$$
$$= x + \log|x| + c$$

Now,

$$y(1) = 0 \Rightarrow 0 = 1+0+c \Rightarrow c = -1$$

Therefore,

$$y\left(\frac{x+1}{x}\right) = x + \log|x| - 1 \Rightarrow y = \frac{x}{x+1}(x-1+\log|x|)$$

Practice Exercise 2

1.
$$(x-0)^2 + (y-k)^2 = k^2$$
$$\Rightarrow x^2 + (y-k)^2 = k^2$$
$$\Rightarrow 2x + 2\,(y-k)\,\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y-k}$$
$$\Rightarrow y - k = -\frac{xdx}{dy}$$
$$\Rightarrow k = y + \frac{xdx}{dy}$$
$$\Rightarrow x^2 + \left(y - \left(y + \frac{xdx}{dy}\right)\right)^2 = \left(y + \frac{xdx}{dy}\right)^2$$
$$\Rightarrow x^2 + x^2\left(\frac{dx}{dy}\right)^2 = y^2 + x^2\left(\frac{dx}{dy}\right)^2 + \frac{2xydx}{dy}$$
$$x^2 = y^2 + \frac{2xydx}{dy}$$
$$(x^2-y^2)\frac{dy}{dx} - 2xy = 0$$

2. Writing $y = vx$ in the given equations, we get

$$x\frac{dv}{dx} = \phi\left(\frac{1}{v}\right)$$

and $$\log c + \log x = \frac{1}{v}$$

$$\Rightarrow -v^2 = x\frac{dv}{dx}$$

$$\Rightarrow \phi\left(\frac{1}{v}\right) = -v^2$$

$$\Rightarrow \phi\left(\frac{x}{y}\right) = -\frac{y^2}{x^2}$$

3. $$f''(x) - f'(x) = e^x$$

$$\Rightarrow \frac{e^x f''(x) - e^x f'(x)}{e^{2x}} = 1$$

$$\Rightarrow \frac{d}{dx}\left(\frac{f'(x)}{e^x}\right) = 1$$

$$\Rightarrow \frac{f'(x)}{e^x} = x + c \quad \{c = 1, \text{ since, } f'(0) = 1\}$$

Therefore, $f'(x) = e^x(x + 1)$. Now,

$$f(x) = \int e^x(x+1)dx$$

$$\Rightarrow f(x) = xe^x + c$$

Since, $f(0) = 0$, therefore $c = 0$.
Therefore,

$$f(x) = xe^x \Rightarrow \ln\left\{\frac{(f(2))^2}{4}\right\} = 4$$

4. $2x^3y\,dy + (1 - y^2)(x^2y^2 + y^2 - 1)dx = 0$

$$\Rightarrow \frac{2y}{(1-y^2)^2}\frac{dy}{dx} + \frac{y^2}{1-y^2}\frac{1}{x} = \frac{1}{x^3}$$

Put $\dfrac{y^2}{1-y^2} = u$. Then

$$\frac{2y}{(1-y^2)^2}\frac{dy}{dx} = \frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} + \frac{u}{x} = \frac{1}{x^3}$$

$$u \cdot x = \int \frac{1}{x^2}dx + c \Rightarrow x^2y^2 = (cx-1)(1-y^2)$$

5. $f(x + y) - f(x - y) = f(x)[f(y) - f(-y)]$

$$f'(0) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} = \lim_{h\to 0}\frac{f(h)-1}{h} = \log a$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x-h)}{2h} = \lim_{h\to 0}\frac{f(x)[f(h) - f(-h)]}{2h}$$

$$= \lim_{h\to 0}\frac{f(x)}{2}\left[\frac{f(h)-1}{h} + \frac{f(-h)-1}{-h}\right] = \frac{f(x)}{2}2\log a$$

$$\frac{f'(x)}{f(x)} = \log a \Rightarrow \log f(x) = \log a^x + c$$

$f(0) = 1 \Rightarrow c = 0 \Rightarrow f(x) = a^x$

6. $$f(x) = a^x$$

$$\Rightarrow f'(x) = a^x \log a$$

7. $$\frac{dy}{dx} = \frac{(x+y)^2}{(x+2)(y-2)} \qquad (x + 2 = X, y - 2 = Y)$$

$$\frac{dY}{dX} = \frac{(X+Y)^2}{XY}$$

Put $Y = tX$. Then

$$t + X\frac{dt}{dX} = \frac{(1+t)^2}{t}$$

$$\Rightarrow dt - \frac{dt}{2t+1} = 2\frac{dX}{X}$$

$$\Rightarrow \frac{y-2}{x+2} - \ln\left|1 + \left(\frac{2(y-2)}{x+2}\right)\right| = 2\log(x+2) + c$$

8. See Fig. 25.7.

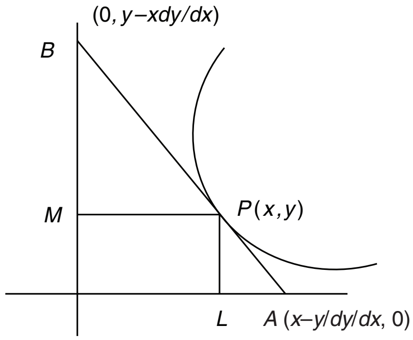


Figure 25.7

Equation of the tangent to the curve is $Y - y = \dfrac{dy}{dx}(X - x)$, so that the coordinates of A and B are, respectively, $A\left(x - \dfrac{y}{dy/dx}, 0\right)$ and $B\left(0, y - x\dfrac{dy}{dx}\right)$. Also,

$$\frac{BP}{AP} = 3 \Rightarrow \frac{MP}{AL} = 3 \Rightarrow \frac{x}{-y/(dy/dx)} = 3$$

$$\Rightarrow x\frac{dy}{dx} + 3y = 0 \qquad (1)$$

9. From Eq. (1) of Solution 8, we have

$$x^3\frac{dy}{dx} + 3x^2y = 0$$

$$\Rightarrow x^3y = c = 1$$

and the curve passes through $\left(2, \dfrac{1}{8}\right)$.

10. Putting (1, 1) in Eq. (1) of Solution 8, we have

$$\frac{dy}{dx} = -3$$

Hence, the equation of the normal to the curve at (1, 1) is

$$y - 1 = \frac{1}{3}(x - 1) \Rightarrow 3y - x = 2$$

11. $$y=\lambda x^3 \Rightarrow \frac{dy}{dx}=\frac{3y}{x} \Rightarrow \frac{y-b}{x-a}=\frac{3y}{x}$$

Let point of contact on curve is $P \equiv (h, k)$.

Slope of the curve at point P is $\frac{3k}{h}$, which is equal to slope of line $AP=\frac{k-b}{h-a}$. So,

$$\frac{k-b}{h-a}=\frac{3k}{h}$$
$$\Rightarrow 2kh-3ak+bh=0$$

Therefore, locus of P is $2xy-3ay+bx=0$.

12. Let point on the curve is $P(h, k)$. Then slope of normal at P point is

$$-\frac{h}{3k}$$

which is equal to slope of line $AP=\frac{k-b}{h-a}$

So,

$$-\frac{h}{3k}=\frac{k-b}{h-a}$$
$$\Rightarrow h^2+3k^2=ah+3bk$$

Therefore, locus of point P is $x^2+3y^2=ax+3by$.

13. Let point of contact of tangent P is $(\alpha, \lambda\alpha^3)$ through point $A(a, b)$ and let this tangent cuts the curve again at point B $(\beta, \lambda\beta^3)$.

Slope of PB = Slope of tangent at point P

$$\Rightarrow \frac{\lambda(\beta^3-\alpha^3)}{(\beta-\alpha)}=3\lambda\alpha^2$$
$$\Rightarrow \beta^2+\alpha^2+\alpha\beta=3\alpha^2$$
$$\Rightarrow \beta=-2\alpha, \beta\neq\alpha$$

Now point P is $\left(-\frac{\beta}{2}, -\lambda\frac{\beta^3}{8}\right)$.

Slope of PA = Slope of tangent at point P

$$\Rightarrow \frac{\lambda\alpha^3-b}{(\alpha-a)}=3\lambda\alpha^2$$
$$\Rightarrow 2\lambda\alpha^3-3\lambda\alpha^2a+b=0$$

Put $\beta=-2\alpha$. Therefore,

$$-2\lambda\frac{\beta^3}{8}-\frac{3\lambda\beta^2a}{2\beta}+b=0$$

Hence, locus of B is $xy-4bx+3ay=0$.

14. $$\frac{dy}{dx}=\frac{1}{x+y}$$

Let $x+y=v$. Then

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$
$$\Rightarrow \frac{dv}{dx}=\frac{1}{v}+1=\frac{1+v}{v}$$
$$\Rightarrow \frac{v}{1+v}dv=dx$$
$$\Rightarrow v-\ln(v+1)=x+c \Rightarrow y=\ln(x+y+1)+c$$

At $x=0$, $y=0$, therefore, $c=0$. So,

$$y=\ln(x+y+1)$$
$$\Rightarrow x=e^y-y-1$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_{\ln 3}=\frac{1}{2}$$

15. Area $=\int_0^1 (e^y-y-1)dy=e-\frac{3}{2}$

16. $$\int_{-1}^{[|\sin\alpha|+|\cos\alpha|]} xe^{-y}d(e^y)$$

$$1\leq|\sin\alpha|+|\cos\alpha|\leq\sqrt{2}$$
$$\Rightarrow [|\sin\alpha|+|\cos\alpha|]=1$$
$$\Rightarrow \int_{-1}^{1} xe^{-y}d(e^y)=\int_{-1}^{1} xe^{-y}e^y dy=\int_{-1}^{1} xdy$$
$$\Rightarrow \int_{-1}^{1}(e^y-y-1)dy=e-e^{-1}-2$$

17. $$y=cx \Rightarrow \frac{dy}{dx}=c=\frac{y}{x}$$

So,

$$\frac{\frac{dy_T}{dx}-k}{1+k\frac{dy_T}{dx}}=\frac{y}{x}$$

Dropping the subscript T, we get the desired differential equation $\frac{dy}{dx}=\frac{kx+y}{x-ky}$.

18. $$y^2=4ax \Rightarrow 2y\frac{dy}{dx}=4a=\frac{y^2}{x} \Rightarrow \frac{dy}{dx}=\frac{y}{2x}$$

So,

$$\frac{\frac{dy_T}{dx}-1}{1+\frac{dy_T}{dx}}=\frac{y}{2x}$$

Dropping subscript T, we get

$$\frac{dy}{dx}=\frac{2x+y}{2x-y}$$

19. The equation of family of circles is $x^2+y^2+2gx=0$. So,

$$2x+2y\frac{dy}{dx}+2g=0$$

Eliminating g, we get $x^2+y^2-x\left\{2x+2y\frac{dy}{dx}\right\}=0$

Therefore,

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

For orthogonal intersection, $\left(\frac{dy}{dx}\right)\left(\frac{dy_T}{dx}\right) = -1$. Therefore,

$$\frac{dy_T}{dx} = \frac{2xy}{x^2 - y^2}$$

So, the differential equation is $\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$.

20. On differentiating a polynomial of n^{th} degree, we get another polynomial of $(n - 1)$ degrees. So,

$$f(x) = \{f'(x)\}^2 \Rightarrow n = 2(n-1) \Rightarrow n = 2$$

21. Let $f(x) = ax^2 + bx + c$. Then $f'(0) = b > 0$.
Also,

$$f(x) = \{f'(x)\}^2$$
$$\Rightarrow ax^2 + bx + c = 4a^2x^2 + 4abx + b^2$$

Thus, $a = 4a^2$, $b = 4ab$ and $c = b^2$.

From which, we get

$$a = \frac{1}{4} \qquad (\text{since } b \neq 0)$$

Now,

$$\int_0^1 f(x)dx = \frac{19}{12}$$
$$\Rightarrow \frac{a}{3} + \frac{b}{2} + c = \frac{19}{12}$$

Therefore,

$$\frac{b}{2} + b^2 = \frac{3}{2} \Rightarrow b = 1$$

since $b > 0$ and so $c = 1$
Hence, $f'(0) = b = 1$.

22. Using the value $a = \frac{1}{4}$, $b = 1$ and $c = 1$, the function is

$$f(x) = \frac{x^2}{4} + x + 1$$
$$\Rightarrow f(x) \neq \pm f(-x)$$

Hence, $f(x)$ is neither odd nor even function.

Solved JEE 2017 Questions

JEE Main 2017

1. If $(2+\sin x)\dfrac{dy}{dx}+(y+1)\cos x=0$ and $y(0) = 1$, then $y\left(\dfrac{\pi}{2}\right)$ is equal to:

(A) $-\dfrac{2}{3}$ **(B)** $-\dfrac{1}{3}$

(C) $\dfrac{4}{3}$ **(D)** $\dfrac{1}{3}$

(OFFLINE)

Solution: It is given that

$$(2+\sin x)\frac{dy}{dx}+(y+1)\cos x=0$$

That is,

$$\frac{dy}{dx}=\frac{-(y+1)\cos x}{2+\sin x}$$

$$\int\frac{dy}{y+1}=-\int\left(\frac{\cos x}{2+\sin x}\right)dx$$

$$\log(y+1)=-\log(2+\sin x)+\log c$$

$$y+1=\frac{c}{2+\sin x} \qquad (1)$$

Given that $y(0) = 1$. Therefore,

$$1+1=\frac{c}{2}\Rightarrow c=4$$

Therefore, the equation of the curve is

$$y+1=\frac{4}{2+\sin x}$$

At $x=\dfrac{\pi}{2}$, we get

$$y+1=\frac{y}{2+1}$$

$$y=\frac{4}{3}-1=\frac{1}{3}$$

Hence, the correct answer is option (D).

2. If $y=\left[x+\sqrt{x^2-1}\right]^{15}+\left[x-\sqrt{x^2-1}\right]^{15}$, then $(x^2-1)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}$ is equal to

(A) $225\,y^2$ **(B)** $224\,y^2$

(C) $125\,y$ **(D)** $225\,y$

(ONLINE)

Solution: The given equation is

$$y=\left[x+\sqrt{x^2-1}\right]^{15}+\left[x-\sqrt{x^2-1}\right]^{15}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=15(x+\sqrt{x^2-1})^{14}\left(1+\frac{1(2x)}{2\sqrt{x^2-1}}\right)$$
$$+15(x-\sqrt{x^2-1})^{14}\left(1-\frac{1(2x)}{2\sqrt{x^2-1}}\right)$$

Here, we have used the standard differentials

$$\frac{d}{dx}x^n=n\,x^{n-1}$$

That is,

$$\frac{d}{dx}[\sqrt{f(x)}]=\frac{1}{2\sqrt{f(x)}}\times\frac{d}{dx}[f(x)]$$

Therefore,

$$\frac{dy}{dx}=\frac{15(x+\sqrt{x^2-1})^{14}(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}}+\frac{15(x-\sqrt{x^2-1})^{14}(\sqrt{x^2-1}-x)}{\sqrt{x^2-1}}$$

$$\Rightarrow\sqrt{x^2-1}\frac{dy}{dx}=15(x+\sqrt{x^2-1})^{15}-15(x-\sqrt{x^2-1})^{15}$$

Differentiating w.r.t. x, we get

$$\frac{1(2x)}{2\sqrt{x^2-1}}\frac{dy}{dx}+\sqrt{x^2-1}\frac{d^2y}{dx^2}=15\times15(x+\sqrt{x^2-1})^{14}\left(1+\frac{1(2x)}{2\sqrt{x^2-1}}\right)$$
$$-15\times15(x-\sqrt{x^2-1})^{14}\left(\frac{1-1(2x)}{2\sqrt{x^2-1}}\right)$$

$$\Rightarrow\frac{x}{\sqrt{x^2-1}}\frac{dy}{dx}+\sqrt{x^2-1}\frac{d^2y}{dx^2}=225(x+\sqrt{x^2-1})^{14}\frac{(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}}$$
$$-\frac{225(x-\sqrt{x^2-1})^{14}(\sqrt{x^2-1}-x)}{\sqrt{x^2-1}}$$

$$\Rightarrow\sqrt{x^2-1}\left[\frac{x}{\sqrt{x^2-1}}\frac{dy}{dx}+\sqrt{x^2-1}\frac{d^2y}{dx^2}\right]=225(x+\sqrt{x^2-1})^{15}$$
$$+225(x-\sqrt{x^2-1})^{15}$$

$$\Rightarrow x\frac{dy}{dx}+(x^2-1)\frac{d^2y}{dx^2}=225\left[(x+\sqrt{x^2-1})^{15}+(x-\sqrt{x^2-1})^{15}\right]$$

Substituting $(x+\sqrt{x^2-1})^{15}+(x-\sqrt{x^2-1})^{15}=y$, we get

$$(x^2-1)\frac{d^2y}{dx^2}+\frac{x\,dy}{dx}=225y$$

Hence, the correct answer is option (D).

3. If $f\left(\dfrac{3x-4}{3x+4}\right)=x+2,\ x\neq-\dfrac{4}{3}$, and $\int f(x)dx=A\log|1-x|+Bx+C$, then the ordered pair (A, B) is equal to (where C is a constant of integration)

(A) $\left(-\dfrac{8}{3},\dfrac{2}{3}\right)$ **(B)** $\left(\dfrac{8}{3},-\dfrac{2}{3}\right)$

(C) $\left(\frac{8}{3}, \frac{2}{3}\right)$ **(D)** $\left(-\frac{8}{3}, -\frac{2}{3}\right)$

(ONLINE)

Solution: It is given that

$$\int f(x)dx = A\log(1-x) + Bx + C$$

Differentiating w.r.t. x, we get

$$f(x) = \frac{-A}{1-x} + B$$

$$\Rightarrow f\left(\frac{3x-4}{3x+4}\right) = \frac{-A}{1-\left(\frac{3x-4}{3x+4}\right)} + B$$

$$\Rightarrow x+2 = \frac{-A(Bx+4)}{8} + B$$

$$\Rightarrow x+2 = \frac{-3Ax}{8} - \frac{4A}{8} + B \quad \Rightarrow \frac{-3A}{8} = 1 \quad \Rightarrow A = \frac{-8}{3}$$

$$\Rightarrow \frac{-4A}{8} + B = 2 \quad \Rightarrow B = 2 + \frac{A}{2} \quad \Rightarrow B = 2 - \frac{4}{3} \quad \Rightarrow \frac{2}{3}$$

Therefore, the ordered pair (A, B) is equal to $\left(\frac{-8}{3}, \frac{2}{3}\right)$.

Hence, the correct answer is option (A).

JEE Advanced 2017

1. If $y = y(x)$ satisfies the differential equation $8\sqrt{x}(\sqrt{9+\sqrt{x}})dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx$, $x > 0$ and $y(0) = \sqrt{7}$, then $y(256)$ = ______.

(A) 3 **(B)** 9
(C) 16 **(D)** 80

Solution: It is given that

$$8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx \qquad \text{(where } x > 0)$$

Therefore,

$$\frac{dy}{dx} = \frac{\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}}{8\sqrt{x}\sqrt{9+\sqrt{x}}} = \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{8\sqrt{x}\sqrt{9+\sqrt{x}}}$$

$$\Rightarrow dy = \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{8\sqrt{x}\sqrt{9+\sqrt{x}}} dx \qquad (1)$$

Integrating Eq. (1), we get

$$\int dy = \int \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{8\sqrt{x}} \frac{1}{\sqrt{9+\sqrt{x}}} dx$$

$$\Rightarrow y = \frac{1}{8}\int \frac{1}{\sqrt{4+\sqrt{9+\sqrt{x}}}} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{9+\sqrt{x}}} dx$$

Let $\sqrt{9+\sqrt{x}} = t$. Differentiating this equation, we get

$$2\frac{1}{\sqrt{9+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} dx = dt \Rightarrow \frac{dx}{\sqrt{x}\sqrt{9+\sqrt{x}}} = 4dt$$

$$\Rightarrow y = \frac{1}{8}\int \frac{1}{\sqrt{4+t}} 4dt = \frac{1}{2}\int \frac{1}{\sqrt{4+t}} dt$$

$$\Rightarrow y = \frac{1}{2}\frac{(4+t)^{1/2}}{1/2} + C = \sqrt{4+t} + C$$

Substituting $t = \sqrt{9+\sqrt{x}}$, we get

$$y = \sqrt{4+\sqrt{9+\sqrt{x}}} + C$$

It is also given that $y(0) = \sqrt{7}$.

$$\sqrt{7} = \sqrt{4+\sqrt{9+\sqrt{0}}} + C$$

$$\Rightarrow \sqrt{7} = \sqrt{4+\sqrt{9}} + C = \sqrt{4+3} + C$$

$$\Rightarrow \sqrt{7} = \sqrt{7} + C \Rightarrow C = 0$$

Therefore,

$$y = \sqrt{4+\sqrt{9+\sqrt{x}}}$$

$$\Rightarrow y(256) = \sqrt{4+\sqrt{9+\sqrt{256}}} = \sqrt{4+\sqrt{9+16}} = \sqrt{4+\sqrt{25}} = \sqrt{4+5}$$

$$\Rightarrow y(256) = \sqrt{9} = 3$$

Hence, the correct answer is option (A).

26 Vector Algebra

26.1 Introduction

Vectors represent one of the most important mathematical systems, which is used to handle certain types of problems in geometry, mechanics and other branches of applied mathematics, physics and engineering.

26.1.1 Scalar and Vector Quantities

Physical quantities are divided into two categories—scalar quantities and vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space are called *scalar quantities*, or briefly scalars. Examples of scalars are mass, volume, density, work, temperature, etc.

A scalar quantity is represented by a real number along with a suitable unit.

Second kind of quantities is those, which have both magnitude and direction. Such quantities are called vectors. Displacement, velocity, acceleration, momentum, weight, force, etc. are examples of vector quantities.

26.2 Representation of a Vector

A vector may be described as a quantity having both magnitude and direction.

Geometrically a directed line segment as shown in Fig. 26.1 represents a vector. A is called the initial point and B the terminal point of vector $\overrightarrow{AB} = \vec{a}$.

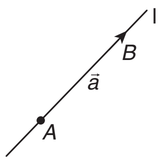


Figure 26.1

Magnitude or modulus of $\vec{a}$ is expressed as $|\vec{a}| = |\overrightarrow{AB}| = AB$. The magnitude of a vector is always a non-negative real number.

Every vector $\overrightarrow{AB}$ has the following three characteristics:

1. **Length:** The length of $\overrightarrow{AB}$ will be denoted by $|\overrightarrow{AB}|$ or AB.
2. **Support:** The line of unlimited length of which AB is a segment is called the support of the vector $\overrightarrow{AB}$.
3. **Sense:** The sense of $\overrightarrow{AB}$ is from A to B and that of $\overrightarrow{BA}$ is from B to A. Thus, the sense of a directed line segment is from its initial point to the terminal point.

26.3 Types of Vectors

1. **Zero or null vector:** A vector whose magnitude is zero is called *zero or null vector* and it is represented by $\vec{0}$. The initial and terminal points of the directed line segment representing zero vector are co-incident and its direction is arbitrary.
2. **Unit vector:** A vector whose modulus is unity is called a *unit vector*. The unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$, read as *a cap*. Thus, $|\hat{a}| = 1$.

   $$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\text{Vector } a}{\text{Magnitude of } a}$$

   Unit vectors parallel to x-axis, y-axis and z-axis are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$, respectively.
   Two unit vectors may not be equal unless they have the same direction.
3. **Like and unlike vectors:** Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.
4. **Collinear or parallel vectors:** Vectors which have the same or parallel supports are called *collinear vectors*.
5. **Coinitial vectors:** Vectors which have the same initial point are called *co-initial vectors*.
6. **Coplanar vectors:** A system of vectors is said to be coplanar, if their supports are parallel to the same plane. Two vectors having the same initial point are always coplanar but such three or more vectors may or may not be coplanar.
7. **Coterminous vectors:** Vectors which have the same terminal point are called *coterminous vectors*.
8. **Negative of a vector:** The vector which has the same magnitude as the vector $\vec{a}$ but opposite direction is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$. Thus, if $\overrightarrow{PQ} = \vec{a}$, then $\overrightarrow{QP} = -\vec{a}$.
9. **Reciprocal of a vector:** A vector having the same direction as that of a given vector $\vec{a}$ but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\vec{a}$ and is denoted by $\vec{a}^{-1}$. Thus, if $|\vec{a}| = a, |\vec{a}^{-1}| = \dfrac{1}{a}$. A unit vector is self-reciprocal.
10. **Localized and free vectors:** A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector. For example, a force acting on a rigid body is a localized vector as its effect depends on the line of action of the force. If the value of a vector depends only on its length and the direction and is independent of its position in the space, it is called a free vector.

11. **Position vectors:** We take arbitrarily any point O in space to be called the origin of reference. The position vector (PV) of any point P, with respect to the origin is the vector $\overrightarrow{OP}$. For any two points P and Q in space, the equality $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ expresses any vector $\overrightarrow{PQ}$ in terms of the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ of P and Q, respectively (Fig. 26.2).

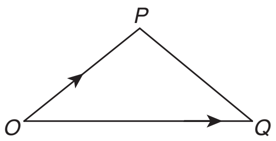


Figure 26.2

12. **Equality of vectors:** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, if
    - **(a)** $|\vec{a}| = |\vec{b}|$
    - **(b)** they have the same or parallel support and
    - **(c)** the same sense.

26.4 Rectangular Resolution of Vectors (Orthogonal System of Vectors): Resolution of a Vector in Two Dimensions

Any vector $\vec{r}$ can be expressed as a linear combination of two unit vectors $\hat{i}$ and $\hat{j}$ at right angle, that is, $\vec{r} = x\hat{i} + y\hat{j}$. The vector $x\hat{i}$ and $y\hat{j}$ are called the perpendicular component vectors of $\vec{r}$. The scalars x and y are called the components or resolved parts of $\vec{r}$ in the directions of x-axis and y-axis, respectively, and the ordered pair (x, y) is known as coordinates of point whose position vector is $\vec{r}$ (Fig. 26.3).

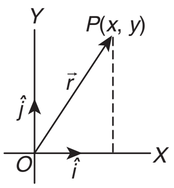


Figure 26.3

Also, the magnitude of $r = \sqrt{x^2 + y^2}$ and let θ be the inclination of $\vec{r}$ with the x-axis, then $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.

26.5 Resolution of a Vector in Three Dimensions

In the orthogonal system of vectors, we choose these vectors as three mutually perpendicular unit vectors denoted by $\hat{i}, \hat{j}$ and $\hat{k}$ directed along the positive directions of X, Y and Z axes, respectively.

If the coordinates of P are (x, y, z), then the position vector of $\vec{r}$ can be written as $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

The vectors $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the right-angled components of $\vec{r}$. The scalars x, y, z are called the components or resolved parts of $\vec{r}$ in the directions of x-axis, y-axis and z-axis, respectively, and ordered triplet (x, y, z) is known as coordinates of P whose position vector is $\vec{r}$ (Fig. 26.4).

Also, the magnitude or modulus of $\vec{r} = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

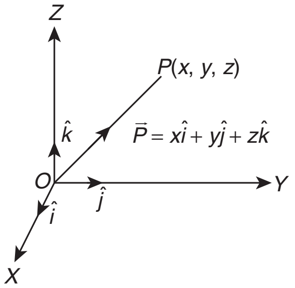


Figure 26.4

If a vector $\overrightarrow{OP}$ makes angles α, β and γ with the positive directions of X, Y and Z axes, respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines of $\overrightarrow{OP}$ (Fig. 26.5).

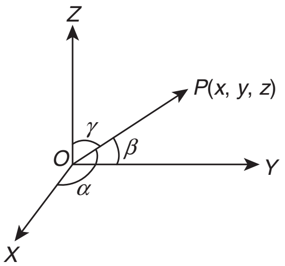


Figure 26.5

$$\cos\alpha = \frac{x}{OP},\ \cos\beta = \frac{y}{OP},\ \cos\gamma = \frac{z}{OP}$$

$$\cos\alpha = l = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{|\vec{r}|},$$

$$\cos\beta = m = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{|\vec{r}|} \text{ and } \cos\gamma = n = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{|\vec{r}|}$$

Clearly, $l^2 + m^2 + n^2 = 1$ (or $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$).

Here, $\alpha = \angle POX$, $\beta = \angle POY$ and $\gamma = \angle POZ$ and $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors along OX, OY and OZ, respectively.

Unit vector in the direction of $\overrightarrow{OP}$ is

$$\frac{\overrightarrow{OP}}{OP} = \frac{x}{OP}\hat{i} + \frac{y}{OP}\hat{j} + \frac{z}{OP}\hat{k} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

Illustration 26.1 Find the direction cosines of the vector $3\hat{i} - 4\hat{j} + 5\hat{k}$.

Solution:

$$\vec{r} = 3\hat{i} - 4\hat{j} + 5\hat{k};\ |r| = \sqrt{3^2 + (-4)^2 + 5^2} = 5\sqrt{2}$$

Hence, direction cosines are $\dfrac{3}{5\sqrt{2}}, \dfrac{-4}{5\sqrt{2}}, \dfrac{5}{5\sqrt{2}}$, that is, $\dfrac{3}{5\sqrt{2}}, \dfrac{-4}{5\sqrt{2}}, \dfrac{1}{\sqrt{2}}$.

Illustration 26.2 If a vector has direction cosines $\left(\frac{1}{2}, m, \frac{1}{2}\right)$, deduce the possible values of m.

If in addition, it is stated that the vector makes an obtuse angle θ with the y-axis, determine θ.

Solution: The direction cosine (l, m, n) of any direction has the property

$$l^2 + m^2 + n^2 = 1$$

Therefore,

$$\frac{1}{4} + m^2 + \frac{1}{4} = 1 \Rightarrow m = \pm\frac{1}{\sqrt{2}}$$

Since the vector makes an obtuse angle θ with the y-axis,

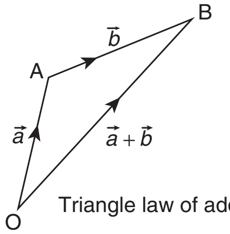
$$\cos\theta = -\frac{1}{\sqrt{2}} = m$$

and hence,

$$\theta = \frac{3\pi}{4}$$

26.6 Properties of Vectors

1. **Addition of vectors**
   - **(a)** ***Triangle law of addition*:** Given two vectors $\vec{a}$ and $\vec{b}$, their sum or resultant written as $(\vec{a} + \vec{b})$ is a vector obtained by first bringing the initial point of $\vec{b}$ to the terminal point of $\vec{a}$ and then joining the initial point of $\vec{a}$ to the terminal point of $\vec{b}$ giving a consistent direction by completing the triangle OAB (Fig. 26.6).



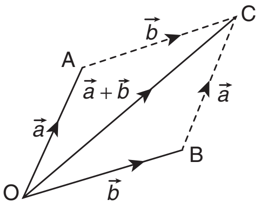
Triangle law of addition

Figure 26.6

   - **(b)** ***Parallelogram law of addition*:** The sum can also be obtained by bringing the initial points of $\vec{a}$ and $\vec{b}$ together and then completing the parallelogram OACB.

     Note that addition is commutative, that is, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

     Also, $\vec{a} + (\vec{b}+\vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$, that is, the addition of vectors obeys the associative law. If $\vec{a}$ and $\vec{b}$ are collinear, their sum is still obtained in the same manner although we do not have a triangle or a parallelogram in this case (Fig. 26.7).



Parallelogram law of addition

Figure 26.7

   - **(c)** ***Polygon law of addition*:** For adding more than two vectors, we have a polygon law of addition which is just an extension of the triangle law.

$$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} = \overrightarrow{OF}$$

     A consequence of this is that, if the terminus of the last vector coincides with the initial point of the first vector, the sum of the vectors is $\vec{0}$ (Fig. 26.8). To obtain $\vec{a} - \vec{b}$ (difference of two vectors), perform addition of $\vec{a}$ and $(-\vec{b})$. Also,

$$\vec{a} + \vec{o} = \vec{a};\ \vec{a} + (-\vec{a}) = \vec{o};$$

$$(k_1 + k_2)\,\vec{a} = k_1\vec{a} + k_2\,\vec{a};\ k((\vec{a} + \vec{b}) = k\vec{a} + k\vec{b})$$

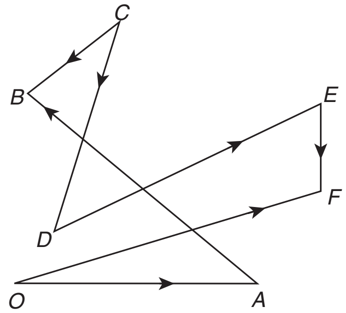


Figure 26.8

   - **(d)** ***Properties of vector addition*:**
     - **(i) Binary operation:** The sum of two vectors is always a vector.
     - **(ii) Commutativity:** For any two vectors $\vec{a}$ and $\vec{b}$,
       $$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$
     - **(iii) Associativity:** For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$,
       $$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$$
     - **(iv) Identity:** Zero vector is the identity for addition. For any vector $\vec{a}$, $|\vec{a}| = 3, |\vec{b}| = 4$.
     - **(v) Additive inverse:** For every vector $\vec{a}$ its negative vector $-\vec{a}$ exists such that $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = 0$, that is, $(-\vec{a})$ is the additive inverse of the vector $\vec{a}$.

Illustration 26.3 If the vectors $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a regular hexagon, express the other sides as vectors in terms of $\vec{a}$ and $\vec{b}$.

Solution: See Fig. 26.9. *ABCDEF* is a regular hexagon.

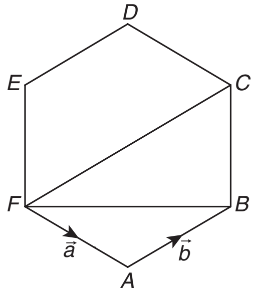


Figure 26.9

Let $\overrightarrow{FA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$. Then

$$\overrightarrow{FB} = \overrightarrow{FA} + \overrightarrow{AB} = \vec{a} + \vec{b}$$

$\overrightarrow{FC} = 2\vec{b}$ ($\overrightarrow{FC}$ is parallel to $\overrightarrow{AB}$ and lengthwise doubled)

Therefore,

$$\overrightarrow{BC} = \overrightarrow{FC} - \overrightarrow{FB} = 2\vec{b} - \vec{a} - \vec{b} = \vec{b} - \vec{a}$$

$$\overrightarrow{CD} = -\vec{a}\,;\ \overrightarrow{DE} = -\vec{b};\ \ \overrightarrow{EF} = \vec{a} - \vec{b}$$

Illustration 26.4 Prove that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

Solution: See Fig. 26.10. *ABC* is the triangle and *AD* is the median through *A*. If *AD* be produced to a length $DG = AD$, then *ACGB* is a parallelogram.

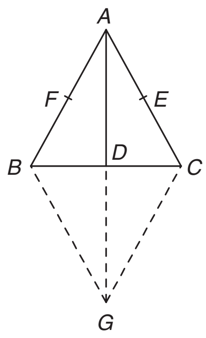


Figure 26.10

Hence, by the parallelogram law of addition of two vectors,

$$\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AG} = 2\overrightarrow{AD}$$

Similarly,

$$\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BE} \ \text{ and } \ \overrightarrow{CB} + \overrightarrow{CA} = 2\overrightarrow{CF}$$

Adding, we have

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{CB} + \overrightarrow{CA} = 2\left(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}\right)$$

But the LHS is such that $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AB} - \overrightarrow{AB} = 0$.
Similarly, the other two pairs also become zero. Hence,

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$$

Illustration 26.5 Five forces represented by $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$ and $\overrightarrow{AF}$ act at the vertex *A* of a regular hexagon *ABCDEF*. Prove that their resultant is a force represented by 6 $\overrightarrow{AO}$, where *O* is the centre of the hexagon.

Solution: See Fig. 26.11.

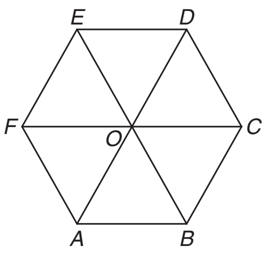


Figure 26.11

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = \overrightarrow{ED} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{CD}$$

(Since $\overrightarrow{AB} = \overrightarrow{ED}$ and $\overrightarrow{AF} = \overrightarrow{CD}$)

$$= \overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{AD}$$

$$= \overrightarrow{AD} + \overrightarrow{AD} + \overrightarrow{AD} = 3\,\overrightarrow{AD}$$

$$= 6\,\overrightarrow{AO}$$

This is the resultant required.

Illustration 26.6 *ABCD* is a parallelogram. A_1 and B_1 are the mid-points of side *BC* and *CD*, respectively. If $\overrightarrow{AA_1} + \overrightarrow{AB_1} = \lambda\overrightarrow{AC}$, then find the value of λ.

Solution: See Fig. 26.12. Let *PV* of *A*, *B* and *D* be $\vec{0}, \vec{b}$ and $\vec{d}$, respectively. Then *PV* of $C = \vec{b} + \vec{d}$.

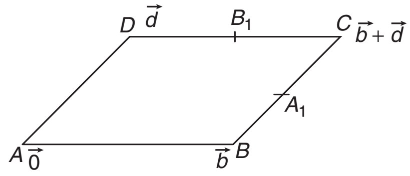


Figure 26.12

Also, *PV* of $A_1 = \vec{b} + \dfrac{\vec{d}}{2}$ and *PV* of $B_1 = \vec{d} + \dfrac{\vec{b}}{2}$. So,

$$\overrightarrow{AA_1} + \overrightarrow{AB_1} = \frac{3}{2}\left(\vec{b} + \vec{d}\right) = \frac{3}{2}\overrightarrow{AC}$$

Hence, the value of λ is 3/2.

Illustration 26.7 If *ABCDEF* is a regular hexagon, then find value of $\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC}$?

Solution: See Fig. 26.13.

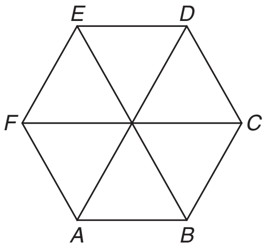


Figure 26.13

We have

$$\overrightarrow{AD} + \overrightarrow{EB} + \overrightarrow{FC} = (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}) + (\overrightarrow{ED} + \overrightarrow{DC} + \overrightarrow{CB}) + \overrightarrow{FC}$$

$$= \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CB}) + (\overrightarrow{CD} + \overrightarrow{DC}) + \overrightarrow{ED} + \overrightarrow{FC}$$

$$= \overrightarrow{AB} + \vec{O} + \vec{O} + \overrightarrow{AB} + 2\,\overrightarrow{AB} = 4\overrightarrow{AB}$$

$$(\overrightarrow{ED} = \overrightarrow{AB}, \overrightarrow{FC} = 2\,\overrightarrow{AB})$$

2. **Subtraction of vectors:** If $\vec{a}$ and $\vec{b}$ are two vectors, then their subtraction $\vec{a} - \vec{b}$ is defined as $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$, where $-\vec{b}$ is the negative of $\vec{b}$ having same magnitude and direction opposite to vector $\vec{b}$.
   If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then
   $$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(a) Properties of vector subtraction:

(i) $\vec{a}-\vec{b}\neq\vec{b}-\vec{a}$

(ii) $(\vec{a}-\vec{b})-\vec{c}\neq\vec{a}-(\vec{b}-\vec{c})$

(iii) Since any one side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors $\vec{a}$ and $\vec{b}$, we have

(a) $|\vec{a}+\vec{b}|\le|\vec{a}|+|\vec{b}|$ **(b)** $|\vec{a}+\vec{b}|\ge|\vec{a}|-|\vec{b}|$

(c) $|\vec{a}-\vec{b}|\le|\vec{a}|+|\vec{b}|$ **(d)** $|\vec{a}-\vec{b}|\ge|\vec{a}|-|\vec{b}|$

3. **Multiplication of a vector by a scalar:** If $\vec{a}$ is a vector and m is a scalar (that is, a real number), then $m\vec{a}$ is a vector whose magnitude is m times that of $\vec{a}$ and whose direction is the same as that of $\vec{a}$, if m is positive and opposite to that of $\vec{a}$, if m is negative. Therefore,
magnitude of $m\vec{a}=|m\vec{a}|\Rightarrow m$ (magnitude of $\vec{a}$) $=m|\vec{a}|$

Again, if $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, then $m\vec{a}=(ma_1)\hat{i}+(ma_2)\hat{j}+(ma_3)\hat{k}$

(a) Properties of multiplication of vectors by a scalar: The following are the properties of multiplication of vectors by scalars, for vectors $\vec{a}$, $\vec{b}$ and scalars m, n.

(i) $m(-\vec{a})=(-m)\vec{a}=-(m\vec{a})$

(ii) $(-m)(-\vec{a})=m\vec{a}$

(iii) $m(n\vec{a})=(mn)\vec{a}=n(m\vec{a})$

(iv) $(m+n)\vec{a}=m\vec{a}+n\vec{a}$

(v) $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

4. **Resultant of two forces:** See Fig. 26.14.

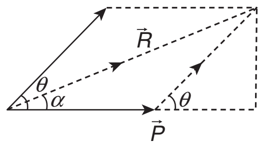


Figure 26.14

$$\vec{R}=\vec{P}+\vec{Q}$$

$$|\vec{R}|=R=\sqrt{P^2+Q^2+2PQ\cos\theta}$$

where $|\vec{P}|=P, |\vec{Q}|=Q,\ \tan\alpha=\dfrac{Q\sin\theta}{P+Q\cos\theta}$

Deduction: When $|\vec{P}|=|\vec{Q}|$, that is, $P=Q$, then

$$\tan\alpha=\frac{P\sin\theta}{P+P\cos\theta}=\frac{\sin\theta}{1+\cos\theta}=\tan\frac{\theta}{2}$$

Therefore,

$$\alpha=\frac{\theta}{2}$$

Hence, the angular bisector of two unit vectors $\vec{a}$ and $\vec{b}$ is along the vector sum $\vec{a}+\vec{b}$.

Note:

1. The internal bisector of the angle between any two vectors is along the vector sum of the corresponding unit vectors. See Fig. 26.15.
2. The external bisector of the angle between two vectors is along the vector difference of the corresponding unit vectors.

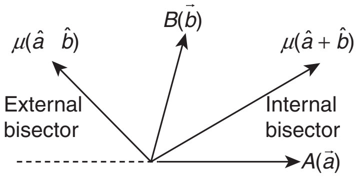


Figure 26.15

Illustration 26.8 In a quadrilateral $PQRS$, $\overrightarrow{PQ}=\vec{a}, \overrightarrow{QR}=\vec{b}, \overrightarrow{SP}=\vec{a}-\vec{b}$, M is the mid-point of $\overrightarrow{QR}$ and X is a point on SM such that $SX=\dfrac{4}{5}SM$. Then prove that P, X and R are collinear.

Solution: From the given information, we get

$$\overrightarrow{QM}=\frac{\vec{b}}{2}\Rightarrow\overrightarrow{PM}=\vec{a}+\frac{\vec{b}}{2}$$

Also,

$$\overrightarrow{SM}=\overrightarrow{PM}-\overrightarrow{PS}=2\vec{a}-\frac{\vec{b}}{2} \text{ and } \overrightarrow{SX}=\frac{4}{5}\overrightarrow{SM}=\frac{8}{5}\vec{a}-\frac{2}{5}\vec{b}$$

$$\Rightarrow\overrightarrow{PX}=\overrightarrow{PS}+\overrightarrow{SX}=-\vec{a}+\vec{b}+\frac{8}{5}\vec{a}-\frac{2}{5}\vec{b}=\frac{3}{5}(\vec{a}+\vec{b})$$

Also,

$$\overrightarrow{PR}=\overrightarrow{PQ}+\overrightarrow{QR}=(\vec{a}+\vec{b})$$

Therefore, $\dfrac{\overrightarrow{PQ}}{\overrightarrow{QR}}=\dfrac{3}{5}$, hence P, X and R are collinear.

Illustration 26.9 The sum of two forces is $18N$ and resultant force whose direction is at right angles to the smaller force is $12N$. Then find the magnitude of the two forces.

Solution: We have,

$$|\vec{P}|+|\vec{Q}|=18N;\ |\vec{R}|=|\vec{P}+\vec{Q}|=12N$$

$$\alpha=90^\circ\ \Rightarrow P+Q\cos\theta=0\Rightarrow Q\cos\theta=-P$$

Now,

$$R^2=P^2+Q^2+2PQ\cos\theta\ \Rightarrow\ R^2=P^2+Q^2+2P(-P)=Q^2-P^2$$

$$\Rightarrow 12^2=(P+Q)(Q-P)=18(Q-P)$$

$$\Rightarrow Q-P=8 \text{ and } Q+P=18\Rightarrow Q=13,\ P=5$$

Therefore, magnitudes of the two forces are $5N$, $13N$.

Illustration 26.10 Find the vector $\vec{c}$, directed along the internal bisector of the angle between the vectors $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$ and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$ with $|\vec{c}|=5\sqrt{6}$.

Solution: Let $\vec{a}=7\hat{i}-4\hat{j}-4\hat{k}$ and $\vec{b}=-2\hat{i}-\hat{j}+2\hat{k}$. Then the required vector,

$$\vec{c}=\lambda\left(\frac{\vec{a}}{|\vec{a}|}+\frac{\vec{b}}{|\vec{b}|}\right)$$

$$=\lambda\left(\frac{7\hat{i}-4\hat{j}-4\hat{k}}{9}+\frac{-2\hat{i}-\hat{j}+2\hat{k}}{3}\right)=\frac{\lambda}{9}(\hat{i}-7\hat{j}+2\hat{k})$$

$$|\vec{c}|^2=\frac{\lambda^2}{81}\times54=150\Rightarrow\lambda=\pm15\Rightarrow\vec{c}=\pm\frac{5}{3}(\hat{i}-7\hat{j}+2\hat{k})$$

26.7 Fundamental Theorems of Vectors

26.7.1 Fundamental Theorems of Vectors in Two Dimensions

If $\vec{a}$ and $\vec{b}$ be two non-zero non-collinear vectors, then any vector $\vec{r}$ in the plane of $\vec{a}$ and $\vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}$ and $\vec{b}$, that is, there exist unique $l, m \in R$ such that

$$l\vec{a} + m\vec{b} = \vec{r}$$

This also means that if $l_1\vec{a} + m_1\vec{b} = l_2\vec{a} + m_2\vec{b}$, then $l_1 = l_2$ and $m_1 = m_2$.

26.7.2 Fundamental Theorems of Vectors in Three Dimensions

If $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero, non-coplanar vectors in space, then any vector $\vec{r}$ in space can be expressed uniquely as a linear combination of $\vec{a}, \vec{b}$ and $\vec{c}$, that is, there exist unique l, m, $n \in R$ such that

$$l\vec{a} + m\vec{b} + n\vec{c} = \vec{r}$$

This also means that if $l_1\vec{a} + m_1\vec{b} + n_1\vec{c} = l_2\vec{a} + m_2\vec{b} + n_2\vec{c}$, then $l_1 = l_2$, $m_1 = m_2$ and $n_1 = n_2$.

26.8 Linear Combinations of Vectors

A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$, if there exist scalars x, y, z, etc. such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \cdots$.

Vectors $\vec{r}_1 = \vec{a} + \vec{b} + \vec{c}$, $\vec{r}_2 = \vec{a} + 8\vec{b} + 5\vec{c}$ are linear combinations of the vectors $\vec{a}, \vec{b}, \vec{c}$.

26.8.1 Collinear and Non-Collinear Vectors

Let $\vec{a}$ and $\vec{b}$ be two collinear vectors and let $\vec{x}$ be the unit vector in the direction of $\vec{a}$. Then the unit vector in the direction of $\vec{b}$ is $\vec{x}$ or $-\vec{x}$, accordingly as $\vec{a}$ and $\vec{b}$ are like or unlike parallel vectors. Now, $\vec{a} = |\vec{a}|\hat{x}$ and $\vec{b} = \pm|\vec{b}|\hat{x}$. Therefore,

$$\vec{a} = \left(\frac{|\vec{a}|}{|\vec{b}|}\right)|\vec{b}|\hat{x} \Rightarrow \vec{a} = \left(\pm\frac{|\vec{a}|}{|\vec{b}|}\right)\vec{b} \Rightarrow \vec{a} = \lambda\vec{b}, \text{ where } \lambda = \pm\frac{|\vec{a}|}{|\vec{b}|}$$

Thus, if $\vec{a}, \vec{b}$ are collinear vectors, then $\vec{a} = \lambda\vec{b}$ or $\vec{b} = \lambda\vec{a}$ for some scalar λ.

26.8.2 Relation Between Two Parallel Vectors

1. If $\vec{a}$ and $\vec{b}$ be two parallel vectors, then there exists a scalar k such that $\vec{a} = k\vec{b}$, that is, there exist two non-zero scalar quantities x and y so that $x\vec{a} + y\vec{b} = 0$.
   If $\vec{a}$ and $\vec{b}$ be two non-zero, non-parallel vectors, then

$$x\vec{a} + y\vec{b} = 0 \Rightarrow x = 0 \text{ and } y = 0$$

Obviously,
$$x\vec{a} + y\vec{b} = 0 \Rightarrow \begin{cases} \vec{a} = 0, \vec{b} = 0 \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \vec{a} \,||\, \vec{b} \end{cases}$$

2. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then from the property of parallel vectors, we have

$$\vec{a} \,||\, \vec{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

26.8.3 Test of Collinearity of Three Points

Three points with position vectors $\vec{a}, \vec{b}, \vec{c}$ are collinear iff there exist scalars x, y, z not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} = 0$, where $x + y + z = 0$. If $\vec{a} = a_1\hat{i} + a_2\hat{j}$, $\vec{b} = b_1\hat{i} + b_2\hat{j}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j}$, then the points with position vector $\vec{a}, \vec{b}, \vec{c}$ will be collinear iff

$$\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0$$

26.8.4 Test of Coplanarity of Three Vectors

Let $\vec{a}$ and $\vec{b}$ be two given non-zero, non-collinear vectors. Then any vectors $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be uniquely expressed as $\vec{r} = x\vec{a} + y\vec{b}$ for some scalars x and y.

26.8.5 Test of Coplanarity of Four Points

Four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar iff there exist scalars x, y, z, u not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = 0$, where $x + y + z + u = 0$. Four points with position vectors

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k},\ \vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$$

will be coplanar, if

$$\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0$$

26.9 Linearly Dependent and Independent Vectors

26.9.1 Linearly Independent Vectors

A system of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly independent if every relation of the type $k_1\vec{a}_1 + k_2\vec{a}_2 + \cdots + k_n\vec{a}_n = \vec{0}$ implies that $k_1 = k_2 = \cdots = k_n = 0$.

26.9.2 Linearly Dependent Vectors

A system of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly dependent if there exists a system of scalars $k_1, k_2, \ldots, k_n$ (not all zero) such that $k_1\vec{a}_1 + k_2\vec{a}_2 + \cdots + k_n\vec{a}_n = \vec{0}$.

Three vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ will be linearly dependent vectors iff

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Note:

1. Two collinear vectors are always linearly dependent.
2. Two non-collinear, non-zero vectors are always linearly independent.
3. Three coplanar vectors are always linearly dependent.
4. Three non-coplanar, non-zero vectors are always linearly independent.
5. More than three vectors are always linearly dependent.
6. Three points with position vectors $\vec{a},\vec{b},\vec{c}$ are collinear if $\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}=0$ with $\lambda_1+\lambda_2+\lambda_3=0$.
7. Four points with position vectors $\vec{a},\vec{b},\vec{c}$, $\vec{d}$ are coplanar if $\lambda_1\vec{a}+\lambda_2\vec{b}+\lambda_3\vec{c}+\lambda_4\vec{d}=0$ with $\lambda_1+\lambda_2+\lambda_3+\lambda_4=0$.

Illustration 26.11 If $\vec{a},\vec{b},\vec{c}$, are non-zero, non-coplanar vectors, determine whether the vectors: $\vec{r}_1=2\vec{a}-3\vec{b}+\vec{c}$, $\vec{r}_2=3\vec{a}-5\vec{b}+2\vec{c}$ and $\vec{r}_3=4\vec{a}-5\vec{b}+\vec{c}$ are linearly independent or dependent.

Solution: Let $\vec{r}_3=x\vec{r}_1+y\vec{r}_2$, where x and y are scalars. If the given vectors are linearly dependent then x and y will exist uniquely; otherwise not. Consider

$$\vec{r}_3=x\vec{r}_1+y\vec{r}_2$$

$$\Rightarrow (4\vec{a}-5\vec{b}+\vec{c})=x(2\vec{a}-3\vec{b}+\vec{c})+y(3\vec{a}-5\vec{b}+2\vec{c})$$

$$\Rightarrow \left(4\vec{a}-5\vec{b}+\vec{c}\right)=\vec{a}\left(2x+3y\right)+\vec{b}\left(-3x-5y\right)+\vec{c}\left(x+2y\right)$$

but $\vec{a},\vec{b},\vec{c}$ are non-zero, non-coplanar vectors. Hence,

$$2x+3y=4 \quad (1)$$

$$-3x-5y=-5 \quad (2)$$

$$x+2y=1 \quad (3)$$

Solving Eqs. (1) and (2), we get $x=5, y=-2$ which also satisfy Eq. (3). So, x and y are unique numbers. Therefore,

$$\vec{r}_3=5\vec{r}_1-2\vec{r}_2$$

Hence, $\vec{r}_1,\vec{r}_2$ and $\vec{r}_3$ are linearly dependent vectors.

Illustration 26.12 Let $\vec{a},\vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of these are collinear. If the vector $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$ (λ being some non-zero scalar), then $\vec{a}+2\vec{b}+6\vec{c}$ equals

(A) 0 **(B)** $\lambda\vec{b}$
(C) $\lambda\vec{c}$ **(D)** $\lambda\vec{a}$

Solution: As $\vec{a}+2\vec{b}$ and $\vec{c}$ are collinear, so

$$\vec{a}+2\vec{b}=\lambda\vec{c} \quad (1)$$

Again $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$. Therefore,

$$\vec{b}+3\vec{c} = \mu\vec{a} \quad (2)$$

Now,

$$\vec{a}+2\vec{b}+6\vec{c}=(\vec{a}+2\vec{b})+6\vec{c} = \lambda\vec{c}+6\vec{c} = (\lambda+6)\vec{c} \quad (3)$$

Also,

$$\vec{a}+2\vec{b}+6\vec{c}=\vec{a}+2(\vec{b}+3\vec{c}) = \vec{a}+2\mu\vec{a}=(2\mu+1)\vec{a} \quad (4)$$

From Eqs. (3) and (4),

$$(\lambda+6)\vec{c}=(2\mu+1)\vec{a}$$

But $\vec{a}$ and $\vec{c}$ are non-zero, non-collinear vectors, therefore

$$\lambda+6=0=2\mu+1$$

Therefore, $\vec{a}+2\vec{b}+6\vec{c}=0$.

Hence, the correct answer is option (A).

Illustration 26.13 If the points with position vectors $60\hat{i}+3\hat{j}$, $40\hat{i}-8\hat{j}$, $a\hat{i}-52\hat{j}$ are collinear, then find a.

Solution: As the three points are collinear, $x(60\hat{i}+3\hat{j})+y(40\hat{i}-8\hat{j})+z(a\hat{i}-52\hat{j})=0$ such that x, y, z are not all zero and $x+y+z=0$. So,

$$(60x+40y+az)\hat{i}+(3x-8y-52z)\hat{j}=0 \text{ and } x+y+z=0$$

$$\Rightarrow 60x+40y+az=0,\ 3x-8y-52z=0 \text{ and } x+y+z=0$$

For non-trivial solution,

$$\begin{vmatrix} 60 & 40 & a \\ 3 & -8 & -52 \\ 1 & 1 & 1 \end{vmatrix}=0 \Rightarrow a=-40$$

Illustration 26.14 If the position vectors of A, B, C, D are $2\hat{i}+\hat{j}$, $\hat{i}-3\hat{j}$, $3\hat{i}+2\hat{j}$ and $\hat{i}+\lambda\hat{j}$, respectively, and $\overrightarrow{AB}\,||\,\overrightarrow{CD}$, then find λ.

Solution:

$$\overrightarrow{AB}=(\hat{i}-3\hat{j})-(2\hat{i}+\hat{j})=-\hat{i}-4\hat{j};$$

$$\overrightarrow{CD}=(\hat{i}+\lambda\hat{j})-(3\hat{i}+2\hat{j})=-2\hat{i}+(\lambda-2)\hat{j};$$

$$\overrightarrow{AB}\,||\,\overrightarrow{CD} \Rightarrow \overrightarrow{AB}=x\overrightarrow{CD}$$

$$-\hat{i}-4\hat{j}=x\{-2\hat{i}+(\lambda-2)\hat{j}\}$$

$$\Rightarrow -1=-2x, -4=(\lambda-2)x$$

$$\Rightarrow x=\frac{1}{2}, \lambda=-6$$

26.10 Position Vector of a Dividing Point (Section Formulae)

1. **Internal division:** Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively, and C be a point dividing AB internally in the ratio m:n. Then the position vector of C is given by

$$\overrightarrow{OC}=\frac{m\vec{b}+n\vec{a}}{m+n}$$

Proof: See Fig. 26.16. Let O be the origin. Then $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$. Let $\vec{c}$ be the position vector of C which divides AB internally in the ratio m:n. Then

$$\frac{AC}{CB}=\frac{m}{n}$$

$$\Rightarrow n\overrightarrow{AC}=m\overrightarrow{CB}$$

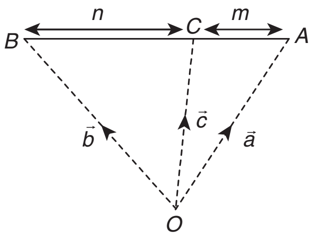


Figure 26.16

$$\Rightarrow n(\text{PV of } \vec{C} - \text{PV of } \vec{A}) = m(\text{PV of } \vec{B} - \text{PV of } \vec{C})$$

$$\Rightarrow n(\vec{c}-\vec{a}) = m(\vec{b}-\vec{c})$$

$$\Rightarrow \vec{c} = \frac{m\vec{b}+n\vec{a}}{m+n} \text{ or } \overrightarrow{OC} = \frac{m\vec{b}+n\vec{a}}{m+n}$$

2. **External division:** Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$, respectively, and let C be a point dividing $\overline{AB}$ externally in the ratio m:n (Fig. 26.17). Then the position vector of $\vec{C}$ is given by $\overline{OC} = \dfrac{m\vec{b}-n\vec{a}}{m-n}$.

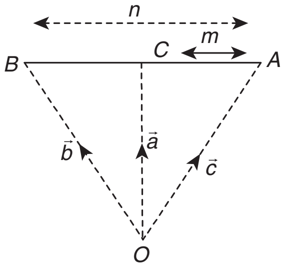


Figure 26.17

Note:

(a) If C is the midpoint of AB, then PV of C is $\dfrac{\vec{b}+\vec{a}}{2}$.

(b) We have $\vec{c} = \dfrac{m\vec{b}+n\vec{a}}{m+n}$. Hence, $\vec{c}$ is in the form of $\vec{c} = \lambda\vec{a}+\mu\vec{b}$, where, $\lambda = \dfrac{n}{m+n}$ and $\mu = \dfrac{m}{m+n}$. Thus, position vector of any point C on $\overrightarrow{AB}$ can always be taken as $\vec{c} = \lambda\vec{a}+\mu\vec{b}$ where $\lambda+\mu = 1$.

(c) If the circumcentre is the origin and vertices of a triangle have position vectors $\vec{a}, \vec{b}, \vec{c}$, then the position vector of orthocentre will be $-(\vec{a}+\vec{b}+\vec{c})$.

Illustration 26.15 ABC is a triangle. A line is drawn parallel to BC to meet AB and AC at D and E, respectively. Prove that the median through A bisects DE.

Solution: Take the vertex A of the triangle ABC as the origin. Let $\vec{b}$ and $\vec{c}$ be the position vector (PV) of B and C. The mid-point of BC has PV $= \dfrac{\vec{b}+\vec{c}}{2}$. The equation of the median is $r = t\dfrac{\vec{b}+\vec{c}}{2}$.

Let D divide AB in the ratio $1:\mu$. Then

$$\text{PV of } D = \frac{\vec{b}}{1+\mu}$$

let E divide AC in the ratio $1:\lambda$. Then

$$\text{PV of } E = \frac{\vec{c}}{1+\lambda} \Rightarrow \lambda = \mu$$

$$\text{PV of the midpoint of } DE = \frac{\vec{b}+\vec{c}}{2(1+\mu)}$$

which lies on the median. Hence, the median bisects DE.

Illustration 26.16 The median AD of the triangle ABC is bisected at E, BE meets AC in F. Then find $AF : AC$.

Solution: Let the position vector of A with respect to B is $\vec{a}$ and that of C with respect to B is $\vec{c}$. Then

$$\text{Position vector of } D \text{ wrt } B = \frac{0+\vec{c}}{2} = \frac{\vec{c}}{2}$$

$$\text{Position vector of } E = \frac{\vec{a}+\frac{\vec{c}}{2}}{2} = \frac{\vec{a}}{2}+\frac{\vec{c}}{4} \quad (1)$$

Let $AF{:}FC = \lambda:1$ and $BE:EF = \mu:1$. Then

$$\text{Position vector of } F = \frac{\lambda\vec{c}+\vec{a}}{1+\lambda}$$

Now,

$$\text{Position vector of } E = \frac{\mu\left(\frac{\lambda\vec{c}+\vec{a}}{1+\lambda}\right)+1\cdot 0}{\mu+1} \quad (2)$$

From Eqs. (1) and (2), we get

$$\frac{\vec{a}}{2}+\frac{\vec{c}}{4} = \frac{\mu}{(1+\lambda)(1+\mu)}\vec{a}+\frac{\lambda\mu}{(1+\lambda)(1+\mu)}\vec{c}$$

$$\Rightarrow \frac{1}{2} = \frac{\mu}{(1+\lambda)(1+\mu)} \text{ and } \frac{1}{4} = \frac{\lambda\mu}{(1+\lambda)(1+\mu)}$$

$$\Rightarrow \lambda = \frac{1}{2}$$

Therefore,

$$\frac{AF}{AC} = \frac{AF}{AF+FC} = \frac{\lambda}{1+\lambda} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

26.11 Bisector of the Angle Between Two Vectors

Consider two non-zero, non-collinear vectors $\vec{a}$ and $\vec{b}$. The bisector of the angle between the two vectors $\vec{a}$ and $\vec{b}$ is $k\left(\dfrac{\vec{a}}{|\vec{a}|}+\dfrac{\vec{b}}{|\vec{b}|}\right)$, where $k \in R^+$.

Illustration 26.17 If the vector $(\hat{i}-3\hat{j}+5\hat{k})$ bisects the angle between $\hat{a}$ and $(2\hat{k}+2\hat{j}-\hat{i})$, where $\hat{a}$ is a unit vector, then find $\hat{a}$.

Solution: According to the given conditions,

$$\lambda(\hat{i}-3\hat{j}+5\hat{k}) = \hat{a}+\frac{2\hat{k}+2\hat{j}-\hat{i}}{3}$$

$$\Rightarrow 3\hat{a} = 3\lambda(\hat{i}-3\hat{j}+5\hat{k})-(2\hat{k}+2\hat{j}-\hat{i})$$

$$\Rightarrow 3\hat{a} = \hat{i}(3\lambda+1) - \hat{j}(2+9\lambda) + \hat{k}(15\lambda-2)$$

$$\Rightarrow |\hat{a}| = \frac{1}{3}\sqrt{(3\lambda+1)^2 + (2+9\lambda)^2 + (15\lambda-2)^2}$$

$$\Rightarrow 9 = (3\lambda+1)^2 + (2+9\lambda)^2 + (15\lambda-2)^2$$

$$\Rightarrow 315\lambda^2 - 18\lambda = 0 \Rightarrow \lambda = 0, \frac{2}{35}$$

If $\lambda = 0$, then $\hat{a} = \hat{i} - 2\hat{j} - 2\hat{k}$ (which is not acceptable).

Therefore, for $\lambda = \frac{2}{35}, \hat{a} = \frac{1}{105}(41\hat{i} - 88\hat{j} - 40\hat{k})$.

Your Turn 1

1. If the vectors $4\hat{i}+11\hat{j}+m\hat{k}, 7\hat{i}+2\hat{j}+6\hat{k}$ and $\hat{i}+5\hat{j}+4\hat{k}$ are coplanar, then m is
   (A) 38 **(B)** 0
   (C) 10 **(D)** –10 **Ans.** (C)

2. The value of λ for which the four points $2\hat{i}+3\hat{j}-\hat{k}, \hat{i}+2\hat{j}+3\hat{k}, 3\hat{i}+4\hat{j}-2\hat{k}, \hat{i}-\lambda\hat{j}+6\hat{k}$ are coplanar is
   (A) 8 **(B)** 0
   (C) –2 **(D)** 6 **Ans.** (C)

3. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}, \vec{b} = 4\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c} = \hat{i}+\alpha\hat{j}+\beta\hat{k}$ are linearly dependent vectors and $|\vec{c}| = \sqrt{3}$, then
   (A) $\alpha = 1, \beta = -1$ **(B)** $\alpha = 1, \beta = \pm 1$
   (C) $\alpha = -1, \beta = \pm 1$ **(D)** $\alpha = \pm 1, \beta = 1$ **Ans.** (D)

4. The position vectors of the vertices A, B, and C of a triangle are $\hat{i}-\hat{j}-3\hat{k}$, $2\hat{i}+\hat{j}-2\hat{k}$ and $-5\hat{i}+2\hat{j}-6\hat{k}$, respectively. The length of the bisector AD of the angle BAC where D is on the segment BC is
   (A) $\frac{3}{4}\sqrt{10}$ **(B)** $\frac{1}{4}$
   (C) $\frac{11}{2}$ **(D)** None of these **Ans.** (A)

5. The unit vector parallel to the resultant vector of $2\hat{i}+4\hat{j}-5\hat{k}$ and $\hat{i}+2\hat{j}+3\hat{k}$ is
   (A) $\frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$ **(B)** $\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$
   (C) $\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}$ **(D)** $\frac{1}{\sqrt{69}}(-\hat{i}-\hat{j}+8\hat{k})$ **Ans.** (A)

6. If the sum of two vectors is a unit vector, then the magnitude of their difference is
   (A) $\sqrt{2}$ **(B)** $\sqrt{3}$
   (C) $\frac{1}{\sqrt{3}}$ **(D)** 1 **Ans.** (B)

7. The length of the longer diagonal of the parallelogram constructed on $5\vec{a}+2\vec{b}$ and $\vec{a}-3\vec{b}$, given that $|\vec{a}| = 2\sqrt{2}, |\vec{b}| = 3$ and angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$, is
   (A) 15 **(B)** $\sqrt{113}$
   (C) $\sqrt{593}$ **(D)** $\sqrt{369}$ **Ans.** (C)

8. If the position vector of a point A is $\vec{a}+2\vec{b}$ and $\vec{a}$ divides AB in the ratio 2:3, then the position vector of B is
   (A) $2\vec{a}-\vec{b}$ **(B)** $\vec{b}-2\vec{a}$
   (C) $\vec{a}-3\vec{b}$ **(D)** $\vec{b}$ **Ans.** (C)

26.12 Product of Two Vectors

Product of two vectors is processed by two methods. When the product of two vector results is a scalar quantity, then it is called scalar product. It is also known as dot product because we are putting a dot (·) between two vectors.

When the product of two vector results is a vector quantity, then this product is called vector product. It is also known as cross product because we are putting a cross (×) between two vectors.

26.13 Scalar or Dot Product of Two Vectors

See Fig. 26.18. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors and θ be the angle between them, then their scalar product (or dot product) is denoted by $\vec{a}\cdot\vec{b}$ and is defined as the scalar $|\vec{a}||\vec{b}|\cos\theta$, where $|\vec{a}|$ and $|\vec{b}|$ are moduli of $\vec{a}$ and $\vec{b}$, respectively, and $0 \le \theta \le \pi$.

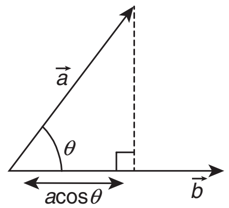


Figure 26.18

Note:
1. $\vec{a}\cdot\vec{b} \le |\vec{a}||\vec{b}|$.
2. If $\vec{a}\cdot\vec{b} > 0$, then angle between $\vec{a}$ and $\vec{b}$ is acute.
3. If $\vec{a}\cdot\vec{b} < 0$, then angle between $\vec{a}$ and $\vec{b}$ is obtuse.
4. The dot product of a zero and non-zero vector is a scalar zero.

26.13.1 Geometrical Interpretation of Scalar Product

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw $BL \perp OA$ and $AM \perp OB$.

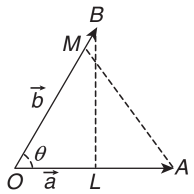


Figure 26.19

From Δs, OBL and OAM, we have $OL = OB\cos\theta$ and $OM = OA\cos\theta$. Here, OL and OM are known as projection of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$, respectively (Fig. 26.19). Now,

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$= |\vec{a}|(OB\cos\theta) = |\vec{a}|(OL)$$
$$= (\text{Magnitude of } \vec{a})(\text{Projection of } \vec{b} \text{ on } \vec{a}) \quad (1)$$

Again,

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{b}|(|\vec{a}|\cos\theta)$$
$$= |\vec{b}|(OA\cos\theta) = |\vec{b}|(OM)$$
$$\vec{a}\cdot\vec{b} = (\text{Magnitude of } \vec{b})\ (\text{Projection of } \vec{a} \text{ on } \vec{b}) \quad (2)$$

Thus, geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

26.13.2 Properties of Scalar Product

1. **Commutativity:** The scalar product of two vector is commutative, that is, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.
2. **Distributivity of scalar product over vector addition:** The scalar product of vectors is distributive over vector addition, that is
   (a) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ **(b)** $(\vec{b}+\vec{c})\cdot\vec{a} = \vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}$
3. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}\perp\vec{b}$.
   As $\hat{i},\hat{j},\hat{k}$ are mutually perpendicular unit vectors along the coordinate axes, therefore
   $$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = 0;\ \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j} = 0;\ \hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = 0$$
4. For any vector $\vec{a}$,
   $$\vec{a}\cdot\vec{a} = |\vec{a}|^2$$
   As $\hat{i},\hat{j},\hat{k}$ are unit vectors along the co-ordinate axes, therefore
   $$\hat{i}\cdot\hat{i} = |\hat{i}|^2 = 1,\ \hat{j}\cdot\hat{j} = |\hat{j}|^2 = 1 \text{ and } \hat{k}\cdot\hat{k} = |\hat{k}|^2 = 1$$
5. If m is a scalar and $\vec{a},\vec{b}$ be any two vectors, then
   $$(m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(m\vec{b})$$
6. If m, n are scalars and $\vec{a},\vec{b}$ be two vectors, then
   $$m\vec{a}\cdot n\vec{b} = mn(\vec{a}\cdot\vec{b}) = (mn\vec{a})\cdot\vec{b} = \vec{a}\cdot(mn\vec{b})$$
7. For any vectors $\vec{a}$ and $\vec{b}$, we have
   (a) $\vec{a}\cdot(-\vec{b}) = -(\vec{a}\cdot\vec{b}) = (-\vec{a})\cdot\vec{b}$ **(b)** $(-\vec{a})\cdot(-\vec{b}) = \vec{a}\cdot\vec{b}$
8. For any two vectors $\vec{a}$ and $\vec{b}$, we have
   (a) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}$
   (b) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$
   (c) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2-|\vec{b}|^2$
   (d) $|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}| \Rightarrow \vec{a}\,||\,\vec{b}$
   (e) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2 \Rightarrow \vec{a}\perp\vec{b}$
   (f) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Rightarrow \vec{a}\perp\vec{b}$
9. If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then,
   $$\vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$$
   Thus, scalar product of two vectors is equal to the sum of the products of their corresponding components. In particular,
   $$\vec{a}\cdot\vec{a} = |\vec{a}|^2 = a_1^2+a_2^2+a_3^2$$
10. If $\vec{a},\vec{b}$ be two vectors inclined at an angle θ, then
    $$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
    $$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$
    If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then angle between vector is
    $$\theta = \cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Illustration 26.18 A unit vector in the plane of the vectors $2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\hat{j}+\hat{k}$ and orthogonal to $5\hat{i}+2\hat{j}+6\hat{k}$ is

(A) $\dfrac{6\hat{i}-5\hat{k}}{\sqrt{61}}$ **(B)** $\dfrac{3\hat{j}-\hat{k}}{\sqrt{10}}$

(C) $\dfrac{2\hat{i}-5\hat{j}}{\sqrt{29}}$ **(D)** $\dfrac{2\hat{i}+\hat{j}-2\hat{k}}{3}$

Solution: Let a unit vector in the plane of $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$ be $\hat{a} = \alpha(2\hat{i}+\hat{j}+\hat{k})+\beta(\hat{i}-\hat{j}+\hat{k}) = (2\alpha+\beta)\hat{i}+(\alpha-\beta)\hat{j}+(\alpha+\beta)\hat{k}$

As $\hat{a}$ is a unit vector, we have

$$(2\alpha+\beta)^2+(\alpha-\beta)^2+(\alpha+\beta)^2 = 1$$
$$\Rightarrow 6\alpha^2+4\alpha\beta+3\beta^2 = 1 \quad (1)$$

As $\hat{a}$ is orthogonal to $5\hat{i}+2\hat{j}+6\hat{k}$, we get

$$5(2\alpha+\beta)+2(\alpha-\beta)+6(\alpha+\beta) = 0$$
$$\Rightarrow 18\alpha+9\beta = 0 \Rightarrow \beta = -2\alpha$$

From Eq. (1), we get

$$6\alpha^2-8\alpha^2+12\alpha^2 = 1 \Rightarrow \alpha = \pm\frac{1}{\sqrt{10}}$$
$$\Rightarrow \beta = \mp\frac{2}{\sqrt{10}}$$

Thus, $\hat{a} = \pm\left(\dfrac{3}{\sqrt{10}}\hat{j}-\dfrac{1}{\sqrt{10}}\hat{k}\right)$.

Hence, the correct answer is option (B).

Illustration 26.19 If three non-zero vectors are $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. If $\vec{c}$ is the unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$ and the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$, then find the value of $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$.

Solution: As $\vec{c}$ is the unit vector perpendicular to $\vec{a}$ and $\vec{b}$, we have

$$|\vec{c}| = 1,\ \vec{a}\cdot\vec{c} = 0 = \vec{b}\cdot\vec{c}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3 \\ a_1b_1+a_2b_2+a_3b_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3 \\ a_1c_1+a_2c_2+a_3c_3 & b_1c_1+b_2c_2+b_3c_3 & c_1^2+c_2^2+c_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} |\vec{a}|^2 & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{a}.\vec{b} & |\vec{b}|^2 & \vec{b}.\vec{c} \\ \vec{a}.\vec{c} & \vec{b}.\vec{c} & |\vec{c}|^2 \end{vmatrix} = \begin{vmatrix} |\vec{a}|^2 & \vec{a}.\vec{b} & 0 \\ \vec{a}.\vec{b} & |\vec{b}|^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - \left(|\vec{a}||\vec{b}|\cos\frac{\pi}{6}\right)^2 = |\vec{a}|^2|\vec{b}|^2\left(1-\frac{3}{4}\right)$$

$$= \frac{1}{4}|\vec{a}|^2|\vec{b}|^2 = \frac{1}{4}(\Sigma a_1^2)(\Sigma b_1^2)$$

26.13.3 Components of a Vector Along and Perpendicular to Another Vector

If $\vec{a}$ and $\vec{b}$ be two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Let θ be the angle between $\vec{a}$ and $\vec{b}$. Draw $BM \perp OA$. In ΔOBM, we have

$$\overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB} \Rightarrow \vec{b} = \overrightarrow{OM} + \overrightarrow{MB}$$

Thus, $\overrightarrow{OM}$ and $\overrightarrow{MB}$ are components of $\vec{b}$ along $\vec{a}$ and perpendicular to $\vec{a}$, respectively (Fig. 26.20).

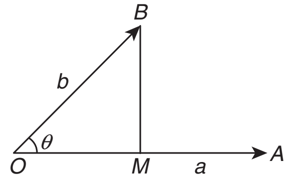


Figure 26.20

Now,

$$\overrightarrow{OM} = (OM)\,\hat{a} = (OB\cos\theta)\,\hat{a}$$

$$= (|\vec{b}|\cos\theta\,\hat{a}\left(|\vec{b}|\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}||\vec{b}|}\right)\hat{a}$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\hat{a} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Therefore,

$$\vec{b} = \overrightarrow{OM} + \overrightarrow{MB} \Rightarrow \overrightarrow{MB} = \vec{b} - \overrightarrow{OM} = \vec{b} - \left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Thus, the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{b} - \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$, respectively.

Illustration 26.20 A vector $\vec{a}$ has components $2p$ and 1 with respect to a rectangular Cartesian system. The system is rotated through a certain angle about the origin in the anticlockwise sense. If $\vec{a}$ has components $p+1$ and 1 with respect to the new system, then find p.

Solution: See Fig. 26.21. Without loss of generality, we can write

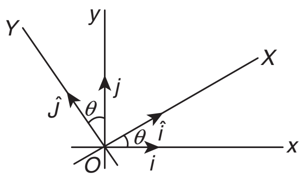


Figure 26.21

$$\vec{a} = 2p\hat{i} + \hat{j} = (p+1)\,\hat{i} + \hat{j} \qquad (1)$$

Now,

$$\hat{i} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{j} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Therefore, from Eq. (1)

$$2p\hat{i} + \hat{j} = (p+1)(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

$$\Rightarrow 2p\hat{i} + \hat{j} = \{(p+1)\cos\theta - \sin\theta\}\,\hat{i} + \{(p+1)\sin\theta + \cos\theta\}\,\hat{j}$$

$$\Rightarrow 2p = (p+1)\cos\theta - \sin\theta \qquad (2)$$

and

$$1 = (p+1)\sin\theta + \cos\theta \qquad (3)$$

Squaring and adding,

$$4p^2 + 1 = (p+1)^2 + 1$$

$$\Rightarrow (p+1)^2 = 4p^2 \Rightarrow p = 1, -\frac{1}{3}$$

26.13.4 Work Done by a Force

The work done by a force $\vec{F}$, acting on a body due to which displacement of body is $\vec{d}$, is given by

Work done (W) = (Magnitude of force in the direction of displacement) × (distance moved)

$$= \left(|\vec{F}|\cos\theta\right)\left(|\vec{d}|\right) = \vec{F}\cdot\vec{d}$$

- The work done by a force is a scalar quantity.
- If a number of forces are acting on a body, then the sum of the works done by the separate forces is equal to the work done by the resultant force (Fig. 26.22).

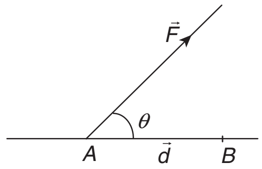


Figure 26.22

Illustration 26.21 A groove is in the form of a broken line ABC and the position vectors of the three points are, respectively, $2\hat{i} - 3\hat{j} + 2\hat{k}$, $3\hat{i} + 2\hat{j} - \hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$. A force of magnitude $24\sqrt{3}$ acts on a particle of unit mass kept at the point A and moves it along the groove to the point C. If the line of action of the force is parallel to the vector $\hat{i} + 2\hat{j} + \hat{k}$ all along, then find the number of units of work done by the force.

Solution:

$$\vec{F} = (24\sqrt{3})\,\frac{\hat{i} + 2\hat{j} + \hat{k}}{|\hat{i} + 2\hat{j} + \hat{k}|} = \frac{24\sqrt{3}}{\sqrt{6}}(\hat{i} + 2\hat{j} + \hat{k}) = 12\sqrt{2}(\hat{i} + 2\hat{j} + \hat{k})$$

Displacement $\vec{r}$ = Position vector of C – Position vector of A

$$= (\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - 3\hat{j} + 2\hat{k}) = (-\hat{i} + 4\hat{j} - \hat{k})$$

Work done by the force is $W = \vec{r}\cdot\vec{F}$

$$= (-\hat{i} + 4\hat{j} - \hat{k})\cdot 12\sqrt{2}\,(\hat{i} + 2\hat{j} + \hat{k}) = 12\sqrt{2}(-1 + 8 - 1) = 72\sqrt{2}$$

Illustration 26.22 Prove by vector method that

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1{}^2 + a_2{}^2 + a_3{}^2)\,(b_1{}^2 + b_2{}^2 + b_3{}^2)$$

Solution: Let $\vec{a} = a_1\hat{i} + a_2\,\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then

$$\vec{a}\cdot\vec{b} = a_1\,b_1 + a_2b_2 + a_3b_3 = |\vec{a}|\,|\vec{b}|\cos\theta \le |\vec{a}|\,|\vec{b}|$$

$$\Rightarrow (\vec{a}\cdot\vec{b})^2 \le |\vec{a}|^2\,|\vec{b}|^2$$

$$\Rightarrow (a_1b_1 + a_2b_2 + a_3b_3)^2 \le \left(a_1^2 + a_2^2 + a_3^2\right)\left(b_1^2 + b_2^2 + b_3^2\right)$$

Illustration 26.23 If $|\vec{a}| = 3, |\vec{b}| = 1, |\vec{c}| = 4$, and $\vec{a} + \vec{b} + \vec{c} = 0$, find the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$.

Solution: We know,

$$(\vec{a} + \vec{b} + \vec{c})^2 = (\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} + \vec{b} + \vec{c})$$

$$\Rightarrow 0 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) \text{ (Given } \vec{a} + \vec{b} + \vec{c} = 0)$$

$$\Rightarrow 0 = (3)^2 + (1)^2 + (4)^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

$$\Rightarrow \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\frac{26}{2} = -13$$

Illustration 26.24 In a ΔABC, prove by vector method that

$$\cos 2A + \cos 2B + \cos 2C \ge -3/2$$

Solution: As we know

$$(\overline{OA} + \overline{OB} + \overline{OC})^2 \ge 0 \qquad (1)$$

and

$$|\overline{OA}|^2 = |\overline{OB}|^2 = |\overline{OC}|^2 = R^2 \qquad (2)$$

Now using Eq. (1), we get

$$|\overline{OA}|^2 + |\overline{OB}|^2 + |\overline{OC}|^2 + 2\left(\overline{OA}\cdot\overline{OB} + \overline{OB}\cdot\overline{OC} + \overline{OC}\cdot\overline{OA}\right) \ge 0$$

$$\Rightarrow 3R^2 + 2R^2\,(\cos 2A + \cos 2B + \cos 2C) \ge 0$$

$$\Rightarrow \cos 2A + \cos 2B + \cos 2C \ge -3/2$$

Illustration 26.25 A particle is acted upon by constant forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ which displace it from a point $\hat{i} + 2\hat{j} + 3\hat{k}$ to the point $5\hat{i} + 4\hat{j} + \hat{k}$. Then find the work done in standard units by the force.

Solution:

Total force $\vec{F} = (4\hat{i} + \hat{j} - 3\hat{k}) + (3\hat{i} + \hat{j} - \hat{k}) = 7\hat{i} + 2\hat{j} - 4\hat{k}$

Displacement $\vec{d} = (5\hat{i} + 4\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} + 2\hat{j} - 2\hat{k}$

Work done $= \vec{F}\cdot\vec{d} = (7\hat{i} + 2\hat{j} - 4\hat{k}).(4\hat{i} + 2\hat{j} - 2\hat{k}) = 28 + 4 + 8 = 40$

Illustration 26.26 Find the work done by the force $\vec{F} = \hat{i} + \hat{j} + 2\hat{k}$ acting on a particle, if the particle is displaced from the point with position vector $\hat{i} + 2\hat{j} + 2\hat{k}$ to the point with position vector $2\hat{i} + 3\hat{j} + 3\hat{k}$.

Solution: Here, $\vec{F} = \hat{i} + \hat{j} + 2\hat{k}$ and displacement,

$$\vec{d} = (2\hat{i} + 3\hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j} + 2\hat{k}) = \hat{i} + \hat{j} + \hat{k}$$

Therefore,

$$\text{Work done} = \vec{F}\cdot\vec{d} = (\hat{i} + \hat{j} + 2\hat{k})\cdot(\hat{i} + \hat{j} + \hat{k})$$

$$= (1)(1) + (1)(1) + (2)(1) = 1 + 1 + 2 = 4 \text{ units}$$

Illustration 26.27 A particle acted on by constant forces $4\hat{i} + \hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} - \hat{k}$ is displaced from the point $\hat{i} + 2\hat{j} + 3\hat{k}$ to the point $5\hat{i} + 4\hat{j} + \hat{k}$. Find the total work done by the forces.

Solution: Let $\vec{F}$ be the resultant of the forces and $\vec{d}$, the displacement. Then

$$\vec{F} = (4\hat{i} + \hat{j} - 3\hat{k}) + (3\hat{i} + \hat{j} - \hat{k}) = 7\hat{i} + 2\hat{j} - 4\hat{k}$$

$$\vec{d} = \left(5\hat{i} + 4\hat{j} + \hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 4\hat{i} + 2\hat{j} - 2\hat{k}$$

Therefore,

$$\text{Total work done} = \vec{F}\cdot\vec{d} = (7\hat{i} + 2\hat{j} - 4\hat{k})\cdot(4\hat{i} + 2\hat{j} - 2\hat{k}) = 40 \text{ units}$$

Your Turn 2

1. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be vectors with magnitudes 3, 4 and 5, respectively, and $\vec{a} + \vec{b} + \vec{c} = 0$. Then find the values of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$.
Ans. –25

2. Let $\vec{b} = 3\hat{j} + 4\hat{k}$, $\vec{a} = \hat{i} + \hat{j}$ and let $\vec{b}_1$ and $\vec{b}_2$ be component vectors of $\vec{b}$ parallel and perpendicular to $\vec{a}$. If $\vec{b}_1 = \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j}$, then $\vec{b}_2 =$ ______.
[MP PET 1989]
(A) $\frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} + 4\hat{k}$ **(B)** $-\frac{3}{2}\hat{i} + \frac{3}{2}\hat{j} + 4\hat{k}$
(C) $-\frac{3}{2}\hat{i} + \frac{3}{2}\hat{j}$ **(D)** None of these **Ans.** (B)

3. $(\vec{a}\cdot\hat{i})\hat{i} + (\vec{a}\cdot\hat{j})\,\hat{j} + (\vec{a}\cdot\hat{k})\hat{k} =$ ______.
(A) $\vec{a}$ **(B)** $2\vec{a}$
(C) $3\vec{a}$ **(D)** 0 **Ans.** (A)

4. If $|\vec{a}| = 3, |\vec{b}| = 4$, then a value of λ for which $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{a} - \lambda\vec{b}$ is
(A) 9/16 **(B)** 3/4
(C) 3/2 **(D)** 4/3 **Ans.** (B)

5. The vectors $\vec{a} = 2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + \lambda\hat{k}$ make an obtuse angle, whereas the angle between $\vec{b}$ and $\vec{k}$ is acute and less than $\pi/6$. Then domain of λ is
(A) $0 < \lambda < \frac{1}{2}$ **(B)** $\lambda > \sqrt{159}$
(C) $-\frac{1}{2} < \lambda < 0$ **(D)** Null set **Ans.** (D)

26.14 Vector or Cross-Product of Two Vectors

Let $\vec{a}, \vec{b}$ be two non-zero, non-parallel vectors. Then the vector product $\vec{a}\times\vec{b}$, in that order, is defined as a vector whose magnitude is $|\vec{a}||\vec{b}|\sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}, \vec{b}$ and this direction constitute a right-handed system.

In other words, $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\ \sin\theta\,\hat{\eta}$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $\hat{\eta}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right-handed system.

26.14.1 Geometrical Interpretation of the Vector Product

If $\vec{a}, \vec{b}$ be two non-zero, non-parallel vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, respectively, and let θ be the angle between them. Complete the parallelogram *OACB*. Draw $BL \perp OA$.

In ΔOBL,

$$\sin\theta = \frac{BL}{OB} \Rightarrow BL = OB\sin\theta = |\vec{b}|\sin\theta \tag{1}$$

Now,

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\ \hat{\eta} = (OA)(BL)\ \hat{\eta}$$

$$= (\text{Base} \times \text{Height})\hat{\eta} = (\text{area of parallelogram } OACB)\hat{\eta}$$

$$= \text{Vector area of the parallelogram } OACB$$

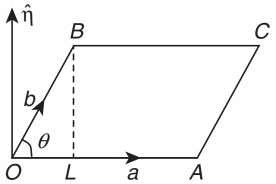


Figure 26.23

Thus, $\vec{a}\times\vec{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its adjacent sides and whose direction $\hat{\eta}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right-handed system. Hence $\vec{a}\times\vec{b}$ represents the vector area of the parallelogram having adjacent sides along $\vec{a}$ and $\vec{b}$ (Fig. 26.23). Thus, area of parallelogram $OACB = |\vec{a}\times\vec{b}|$.

Also,

$$\text{area of } \Delta OAB = \frac{1}{2}\ \text{area of parallelogram } OACB$$

$$= \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}|$$

26.14.2 Properties of Vector Product

1. Vector product is not commutative, that is, if $\vec{a}$ and $\vec{b}$ are any two vectors, then $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$, however, $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$.
2. If $\vec{a}, \vec{b}$ are two vectors and m is a scalar, then
$$m\vec{a}\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times m\vec{b}$$
3. If $\vec{a}, \vec{b}$ are two vectors and m, n are scalars, then
$$m\vec{a}\times n\vec{b} = mn(\vec{a}\times\vec{b}) = m(\vec{a}\times n\vec{b}) = n(m\vec{a}\times\vec{b})$$
4. Distributivity of vector product over vector addition.
Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors. Then
(a) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (Left distributivity)
(b) $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times a$ (Right distributivity)
5. For any three vectors $\vec{a},\vec{b},\vec{c}$, we have $\vec{a}\times(\vec{b}-\vec{c}) = \vec{a}\times\vec{b}-\vec{a}\times\vec{c}$.
6. The vector product of two non-zero vectors is zero vector if they are parallel (collinear), that is, for non-zero vectors $\vec{a},\vec{b}$
$$\vec{a}\times\vec{b} = 0 \Leftrightarrow \vec{a}\,||\,\vec{b}$$
It follows from the above property that $\vec{a}\times\vec{a} = 0$ for every non-zero vector $\vec{a}$, which in turn implies that $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$.
7. Vector product of orthonormal triad of unit vectors $\hat{i},\hat{j},\hat{k}$ using the definition of the vector product, we obtain
$$\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j},\ \hat{j}\times\hat{i} = -\hat{k},\ \hat{k}\times\hat{j} = -\hat{i},\ \hat{i}\times\hat{k} = -\hat{j}$$
8. **Lagrange's identity:** If $\vec{a}, \vec{b}$ are any two vectors, then
$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 \text{ or } |\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$$
9. **Vector product in terms of components:** If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Then,
$$\vec{a}\times\vec{b} = (a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
10. **Angle between two vectors:** If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$.
Expression for $\sin\theta$: If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and θ be angle between $\vec{a}$ and $\vec{b}$, then
$$\sin^2\theta = \frac{(a_2b_3-a_3b_2)^2+(a_1b_3-a_3b_1)^2+(a_1b_2-a_2b_1)^2}{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}$$
11. **(a) Right-handed system of vectors:** Three mutually perpendicular vectors $\vec{a},\vec{b},\vec{c}$, form a right-handed system (Fig. 26.24) of vector iff,
$$\vec{a}\times\vec{b} = \vec{c},\ \vec{b}\times\vec{c} = \vec{a},\ \vec{c}\times\vec{a} = \vec{b}$$

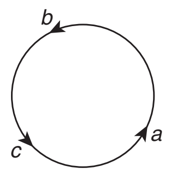


Figure 26.24

Example: The unit vectors $\hat{i},\hat{j},\hat{k}$ form a right-handed system as shown in Fig. 26.25.

$$\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$$

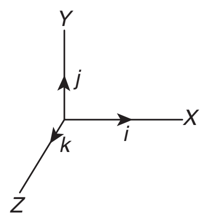


Figure 26.25

(b) Left-handed system of vectors: The vectors $\vec{a}, \vec{b}, \vec{c}$, mutually perpendicular to one another form a left handed system (Fig. 26.26) of vector iff

$$\vec{c}\times\vec{b} = \vec{a},\ \vec{a}\times\vec{c} = \vec{b},\ \vec{b}\times\vec{a} = \vec{c}$$

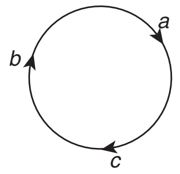


Figure 26.26

26.14.3 Vector Normal to the Plane of Two Given Vectors

If $\vec{a},\vec{b}$ be two non-zero, non-parallel vectors and let θ be the angle between them, then $(\vec{a}\times\vec{b})=|\vec{a}||\vec{b}|\sin\theta\,\hat{\eta}$, where $\hat{\eta}$ is a unit vector $\perp$ to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a},\vec{b},\hat{\eta}$ from a right-handed system. So,

$$(\vec{a}\times\vec{b})=|\vec{a}\times\vec{b}|\,\hat{\eta}\Rightarrow\hat{\eta}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

Thus, $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is a unit vector $\perp$ to the plane of $\vec{a}$ and $\vec{b}$.

Note that $-\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is also a unit vector $\perp$ to the plane of $\vec{a}$ and $\vec{b}$.

Vectors of magnitude 'λ' normal to the plane of $\vec{a}$ and $\vec{b}$ are given by $\pm\dfrac{\lambda(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.

Illustration 26.28 If $\vec{a}$ is any vector, then find the value of

$$(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2$$

Solution: Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then

$$\vec{a}\times\hat{i}\ =\ (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{i}\ =\ -a_2\hat{k}+a_3\hat{j}$$

$$(\vec{a}\times\hat{i})^2=(\vec{a}\times\hat{i})\cdot(\vec{a}\times\hat{i})=(-a_2\hat{k}+a_3\hat{j})\cdot(-a_2\hat{k}+a_3\hat{j})=a_2^2+a_3^2$$

Similarly,

$$(\vec{a}\times\hat{j})^2=a_3^2+a_1^2 \text{ and } (\vec{a}\times\hat{k})^2=a_1^2+a_2^2$$

Therefore,

$$(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2=2(a_1^2+a_2^2+a_3^2)=2|\vec{a}|^2$$

26.14.4 Area of Parallelogram and Triangle

1. The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.
2. The area of a parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$.
3. The area of a plane quadrilateral $ABCD$ is $\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$, where AC and BD are its diagonals.
4. The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$.
5. The area of a triangle ABC is
$$\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| \text{ or } \frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}| \text{ or } \frac{1}{2}|\overrightarrow{CB}\times\overrightarrow{CA}|$$
6. If $\vec{a},\vec{b},\vec{c}$ are position vectors of vertices of a ΔABC, then its
$$\text{area}=\frac{1}{2}|(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})|$$
7. Three points with position vectors $\vec{a},\vec{b},\vec{c}$ are collinear, if
$$(\vec{a}\times\vec{b})+(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})=0$$

Illustration 26.29 Find the area of a triangle whose vertices are $A(1,-1,2)$, $B(2,1,-1)$ and $C(3,-1,2)$.

Solution:

$$\overrightarrow{AB}=(2\hat{i}+\hat{j}-\hat{k})-(\hat{i}-\hat{j}+2\hat{k})=\hat{i}+2\hat{j}-3\hat{k},$$
$$\overrightarrow{AC}=(3\hat{i}-\hat{j}+2\hat{k})-(\hat{i}-\hat{j}+2\hat{k})=2\hat{i}$$

Area of the triangle $ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

$$=\frac{1}{2}|(\hat{i}+2\hat{j}-3\hat{k})\times 2\hat{i}|=\frac{1}{2}|-4\hat{k}-6\hat{j}|=|-3\hat{j}-2\hat{k}|=\sqrt{13}$$

Illustration 26.30 The position vectors of the vertices of a quadrilateral $ABCD$ are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, respectively. Area of the quadrilateral formed by joining the middle points of its sides is

(A) $\frac{1}{4}|\vec{a}\times\vec{b}+\vec{b}\times\vec{d}+\vec{d}\times\vec{a}|$

(B) $\frac{1}{4}|\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{a}\times\vec{d}+\vec{b}\times\vec{a}|$

(C) $\frac{1}{4}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{d}\times\vec{a}|$

(D) $\frac{1}{4}|\vec{b}\times\vec{c}+\vec{c}\times\vec{d}+\vec{d}\times\vec{b}|$

Solution: See Fig. 26.27. Let P, Q, R, S be the middle points of the sides of the quadrilateral $ABCD$.

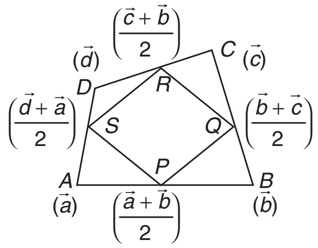


Figure 26.27

Position vector of

$$P=\frac{\vec{a}+\vec{b}}{2},\ Q=\frac{\vec{b}+\vec{c}}{2},\ R=\frac{\vec{c}+\vec{d}}{2} \text{ and } S=\frac{\vec{d}+\vec{a}}{2}$$

Mid-point of diagonal $SQ\equiv\left(\frac{\vec{d}+\vec{a}}{2}+\frac{\vec{b}+\vec{c}}{2}\right)\frac{1}{2}=\frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$

Similarly,

$$\text{mid-point of } PR\equiv\frac{1}{4}(\vec{a}+\vec{b}+\vec{c}+\vec{d})$$

As the diagonals bisect each other, $PQRS$ is a parallelogram.

$$\overrightarrow{SP}=\frac{\vec{a}+\vec{b}}{2}-\frac{\vec{d}+\vec{a}}{2}=\frac{\vec{b}-\vec{d}}{2};$$
$$\overrightarrow{SR}=\frac{\vec{c}+\vec{d}}{2}-\frac{\vec{d}+\vec{a}}{2}=\frac{\vec{c}-\vec{a}}{2}$$

Area of parallelogram $PQRS=|\overrightarrow{SP}\times\overrightarrow{SR}|=\left|\left(\frac{\vec{b}-\vec{d}}{2}\right)\times\left(\frac{\vec{c}-\vec{a}}{2}\right)\right|$

$$= \frac{1}{4}\left|\vec{b}\times\vec{c} - \vec{b}\times\vec{a} - \vec{d}\times\vec{c} + \vec{d}\times\vec{a}\right|$$

$$= \frac{1}{4}\left|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{a}\right|$$

Hence, the correct answer is option (C).

26.14.5 Moment of a Force

1. **About a point:** Let a force $\vec{F}$ be applied at a point P. The moment of force $\vec{F}$ about a point O is defined (Fig. 26.28) as

$$\vec{M} = \overrightarrow{OP} \times \vec{F}$$

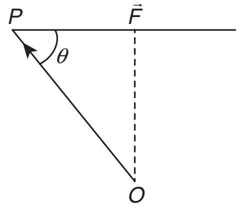


Figure 26.28

   - **(a)** Moment of force about a point is vector quantity.
   - **(b)** Moment is independent of selection of point P, in fact P can be any point on the line of action of force $\vec{F}$.
   - **(c)** If several forces are acting through the point P, then the vector sum of the moments of the separate forces about O is equal to the moment of their resultant force about O.
   - **(d)** The moment of $\vec{F}$ about a point O measures the amount of $\vec{F}$ to turn the body about point O. If tendency of rotation is in the anticlockwise direction, the moment is positive, otherwise it is negative.

2. **About a line:** Let $\vec{F}$ be any given force, acting at a point P and L be any directed line segment. The moment of force $\vec{F}$ about line L is defined as

$$M_a = (\overrightarrow{OP} \times \vec{F}) \cdot \hat{a}$$

where $\hat{a}$ is a unit vector in the direction of line and O is any point on the line.

   - **(a)** Moment about a line is a scalar quantity.
   - **(b)** Moment of $\vec{F}$ about the line L is the projection along L, of the vector moment of the force $\vec{F}$ about any point on the L.

26.14.6 Moment of a Couple

A system consisting of a pair of equal unlike parallel forces is called a couple. The vector sum of two forces of a couple is always zero vector.

See Fig. 26.29. The moment of a couple is a vector perpendicular to the plane of couple and its magnitude is the product of the magnitude of either force with the perpendicular distance between the lines of the forces.

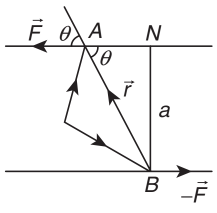


Figure 26.29

$$\vec{M} = \vec{r} \times \vec{F}, \text{ where } \vec{r} = \overrightarrow{BA}$$

$$\Rightarrow |\vec{M}| = |\overrightarrow{BA} \times \vec{F}| = |\vec{F}||\overrightarrow{BA}|\sin\theta,$$

where θ is the angle between $\overrightarrow{BA}$ and $\vec{F}$

$$|\vec{M}| = |\vec{F}|(BN) = |\vec{F}|a$$

where $a = BN$ is the arm of the couple and +ve or –ve sign is to be taken according as the forces indicate a counter-clockwise rotation or clockwise rotation.

Illustration 26.31 Find the moment about the point $\hat{i} + 2\hat{j} + 3\hat{k}$ of a force represented by $\hat{i} + \hat{j} + \hat{k}$ acting through the point $-2\hat{i} + 3\hat{j} + \hat{k}$.

Solution: See Fig. 26.30. Let O be the point $\hat{i} + 2\hat{j} + 3\hat{k}$ and P the point $-2\hat{i} + 3\hat{j} + \hat{k}$. Then

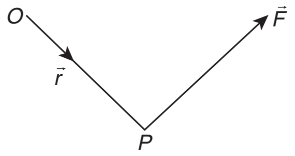


Figure 26.30

$$\overrightarrow{OP} = (PV \text{ of } P) - (PV \text{ of } O)$$

$$\Rightarrow \vec{r} = -3\hat{i} + \hat{j} - 2\hat{k}$$

Let $\vec{M}$ be the vector moment of $\vec{F}$ acting at P about point O. Then

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = 3\hat{i} + \hat{j} - 4\hat{k}$$

Illustration 26.32 Forces $2\hat{i} + 7\hat{j}$, $2\hat{i} - 5\hat{j} + 6\hat{k}$, $-\hat{i} + 2\hat{j} - \hat{k}$ act at a point P whose position vector is $4\hat{i} - 3\hat{j} - 2\hat{k}$. Find the vector moment of the resultant of three forces acting at P about the point Q, whose position vector is $6\hat{i} + \hat{j} - 3\hat{k}$.

Solution: Let $\vec{F}_1 = 2\hat{i} + 7\hat{j}$, $\vec{F}_2 = 2\hat{i} - 5\hat{j} + 6\hat{k}$, $\vec{F}_3 = -\hat{i} + 2\hat{j} - \hat{k}$. Then the resultant force $\vec{F}$ is given by

$$\vec{F} = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

Let $\vec{r} = \overrightarrow{QP}$. Then

$$\vec{r} = PV \text{ of } P - PV \text{ of } Q$$

$$= (4\hat{i} - 3\hat{j} - 2\hat{k}) - (6\hat{i} + \hat{j} - 3\hat{k})$$

$$= -2\hat{i} - 4\hat{j} + \hat{k}$$

Let $\vec{M}$ be the moment of the resultant force $\vec{F}$ about Q. Then

$$\vec{M} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -4 & 1 \\ 3 & 4 & 5 \end{vmatrix} = -24\hat{i} + 13\hat{j} + 4\hat{k}$$

Your Turn 3

1. A force given by $3\hat{i}+2\hat{j}-4\hat{k}$ is applied at the point (1, –1, 2). Find the moment of the force about the point (2, –1, 3).
Ans. $2\hat{i}-7\hat{j}-2\hat{k}$

2. $\vec{a}$ and $\vec{c}$ are unit vectors and $|\vec{b}|=4$. If angle between $\vec{b}$ and $\vec{c}$ is $\cos^{-1}\left(\frac{1}{4}\right)$ and $\vec{a}\times\vec{b}=2\vec{a}\times\vec{c}$, then show that $\vec{b}$ can be written as $\vec{b}=\lambda\vec{a}+2\vec{c}$, also find the value of λ. **Ans.** $\lambda \pm 4$

3. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{a}$, then show that $|\vec{b}|=1, \vec{a}=\vec{c}$.

4. $(\vec{a}\times\vec{b})^2+(\vec{a}\cdot\vec{b})^2$ is equal to
(A) $\vec{a}^2+\vec{b}^2$ **(B)** $\vec{a}^2\vec{b}^2$
(C) $2\vec{a}\cdot\vec{b}$ **(D)** 1
Ans. (B)

5. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{b}=\hat{i}+3\hat{j}+5\hat{k}$ and $\vec{c}=7\hat{i}+9\hat{j}+11\hat{k}$, then the area of the parallelogram having diagonals $\vec{a}+\vec{b}$ and $\vec{b}+\vec{c}$ is
(A) $4\sqrt{6}$ **(B)** $\frac{1}{2}\sqrt{21}$
(C) $\frac{\sqrt{6}}{2}$ **(D)** $\sqrt{6}$
Ans. (A)

6. Three forces $\hat{i}+2\hat{j}-3\hat{k}, 2\hat{i}+3\hat{j}+4\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$ are acting on a particle at the point (0, 1, 2). The magnitude of the moment of the forces about the point (1, –2, 0) is
(A) $2\sqrt{35}$ **(B)** $6\sqrt{10}$
(C) $4\sqrt{17}$ **(D)** None of these
Ans. (B)

26.15 Scalar Triple Product

For any two vectors $\vec{b}$ and $\vec{c}$, $\vec{b}\times\vec{c}$ is a vector. This can be scalarly multiplied with a third vector $\vec{a}$ to give the scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$. This is a scalar whose value is the volume of a box having $\vec{a}, \vec{b}, \vec{c}$ as coterminous edges. Hence, it is also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and in this sense is called the box product.

Similarly, other scalar triple products can be defined as $(\vec{b}\times\vec{c})\cdot\vec{a}, (\vec{c}\times\vec{a})\cdot\vec{b}$. By the property of scalar product of two vectors we can say, $\vec{a}\cdot(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\cdot\vec{c}$.

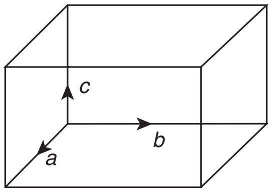


Figure 26.31

See Fig. 26.31. The value of scalar triple product depends on the cyclic order of the vectors and is independent of the position of the dot and cross. These may be interchanged at pleasure. However, anticyclic permutation of the vectors changes the value of triple product in sign but not in magnitude.

26.15.1 Geometrical Interpretation of Scalar Triple Product

The scalar triple product of three vectors is equal to the volume of the parallelepiped whose three coterminous edges are represented by the given vectors. $\vec{a}, \vec{b}, \vec{c}$ form a right–handed system of vectors. Therefore, $(\vec{a}\times\vec{b})\cdot\vec{c}=[\vec{a}\ \vec{b}\ \vec{c}]$ = volume of the parallelepiped, whose coterminous edges are $\vec{a}, \vec{b}$, and $\vec{c}$.

26.15.2 Properties of Scalar Triple Product

1. If $\vec{a},\vec{b},\vec{c}$ are cyclically permuted, then the value of scalar triple product remains the same, that is,
$$(\vec{a}\times\vec{b})\cdot\vec{c}=(\vec{b}\times\vec{c})\cdot\vec{a}=(\vec{c}\times\vec{a})\cdot\vec{b} \text{ or } [\vec{a}\vec{b}\vec{c}]=[\vec{b}\vec{c}\vec{a}]=[\vec{c}\vec{a}\vec{b}]$$
2. The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude, that is
$$[\vec{a}\vec{b}\vec{c}]=-[\vec{b}\vec{a}\vec{c}]=-[\vec{c}\vec{b}\vec{a}]=-[\vec{a}\vec{c}\vec{b}].$$
3. In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same, that is, $(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})$.
4. The scalar triple product of three vectors is zero if any two of them are equal.
5. For any three vectors $\vec{a},\vec{b},\vec{c}$ and scalar λ, $[\lambda\vec{a}\vec{b}\vec{c}]=\lambda[\vec{a}\vec{b}\vec{c}]$.
6. The scalar triple product of three vectors is zero if any two of them are parallel or collinear.
7. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, then $[(\vec{a}+\vec{b})\ \vec{c}\ \vec{d}]=[\vec{a}\vec{c}\vec{d}]+[\vec{b}\vec{c}\vec{d}]$.
8. The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a}, \vec{b}, \vec{c}$ to be coplanar is that $[\vec{a}\vec{b}\vec{c}]=0$, that is, $\vec{a}, \vec{b}, \vec{c}$ are coplanar.
9. Four points with position vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ will be coplanar, if
$$[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{d}\ \vec{c}\ \vec{a}]+[\vec{d}\ \vec{a}\ \vec{b}]=[\vec{a}\ \vec{b}\ \vec{c}]$$
10. Scalar triple product in terms of components:
(a) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three vectors then,
$$[\vec{a}\vec{b}\vec{c}]=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
(b) If $\vec{a}=a_1\vec{l}+a_2\vec{m}+a_3\vec{n}, \vec{b}=b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ and $\vec{c}=c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$, then
$$[\vec{a}\vec{b}\vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\vec{l}\ \vec{m}\vec{n}]$$
(c) For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$
(i) $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$
(ii) $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0$
(iii) $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]^2$

26.15.3 Tetrahedron

A tetrahedron is a three-dimensional figure formed by four triangle *OABC* is a tetrahedron with ΔABC as the base. *OA*, *OB*, *OC*, *AB*, *BC* and *CA* are known as edges of the tetrahedron. *OA*, *BC*; *OB*, *CA* and *OC*, *AB* are known as the pairs of opposite edges. A tetrahedron in which all edges are equal, is called a regular tetrahedron (Fig. 26.32).

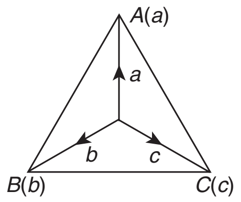


Figure 26.32

26.15.4 Properties of a Tetrahedron

1. If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair are also perpendicular to each other.
2. In a tetrahedron, the sum of the squares of two opposite edges is the same for each pair.
3. Any two opposite edges in a regular tetrahedron are perpendicular.

26.15.5 Volume of a Tetrahedron

1. The volume of a tetrahedron is

$$\frac{1}{3}\text{(area of the base) (corresponding altitude)}$$

$$=\frac{1}{3}\cdot\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}||\overrightarrow{ED}|=\frac{1}{6}|\overrightarrow{AB}\times\overrightarrow{AC}||\overrightarrow{ED}|\cos 0^\circ \text{ for } \overrightarrow{AB}\times\overrightarrow{AC}\,||\,\overrightarrow{ED}$$

$$=\frac{1}{6}(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{ED}=\frac{1}{6}[\overrightarrow{AB}\;\overrightarrow{AC}\;\overrightarrow{EA}+\overrightarrow{AD}]=\frac{1}{6}[\overrightarrow{AB}\;\overrightarrow{AC}\;\overrightarrow{AD}]$$

Because $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{EA}$ are coplanar, so $[\overrightarrow{AB}\;\overrightarrow{AC}\;\overrightarrow{EA}]=0$

2. If $\vec{a},\vec{b},\vec{c}$ are position vectors of vertices *A*, *B* and *C* with respect to *O*, then volume of tetrahedron $OABC=\frac{1}{6}[\vec{a}\;\vec{b}\;\vec{c}]$.
3. If $\vec{a},\vec{b},\vec{c},\vec{d}$ are position vectors of vertices *A*, *B*, *C*, *D* of a tetrahedron *ABCD*, then its volume $=\frac{1}{6}[\vec{b}-\vec{a}\;\vec{c}-\vec{a}\;\vec{d}-\vec{a}]$.

26.15.6 Reciprocal System of Vectors

Let $\vec{a},\vec{b},\vec{c}$ be three non-coplanar vectors, and let

$$\vec{a'}=\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]},\;\vec{b'}=\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]},\;\vec{c'}=\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

Then $\vec{a'},\vec{b'},\vec{c'}$ are said to form a reciprocal system of vectors for the vectors $\vec{a},\vec{b},\vec{c}$.

If $\vec{a},\vec{b},\vec{c}$ and $\vec{a'},\vec{b'},\vec{c'}$ form a reciprocal system of vectors, then

1. $\vec{a}\cdot\vec{a'}=\vec{b}\cdot\vec{b'}=\vec{c}\cdot\vec{c'}=1$
2. $\vec{a}\cdot\vec{b'}=\vec{a}\cdot\vec{c'}=0;\;\vec{b}\cdot\vec{c'}=\vec{b}\cdot\vec{a'}=0;\;\vec{c}\cdot\vec{a'}=\vec{c}\cdot\vec{b'}=0$
3. $[\vec{a'}\,\vec{b'}\,\vec{c'}]=\frac{1}{[\vec{a}\;\vec{b}\;\vec{c}]}$
4. $\vec{a},\vec{b},\vec{c}$ are non-coplanar iff so are $\vec{a'},\vec{b'},\vec{c'}$

Illustration 26.33 If $\vec{u},\vec{v}$ and $\vec{w}$ are three non-coplanar vectors, then find the value of $(\vec{u}+\vec{v}-\vec{w})\cdot[(\vec{u}-\vec{v})\times(\vec{v}-\vec{w})]$.

Solution:

$$(\vec{u}+\vec{v}-\vec{w})\cdot[\vec{u}-\vec{v}\times(\vec{v}-\vec{w})]$$

$$=(\vec{u}+\vec{v}-\vec{w})\cdot[(\vec{u}\times\vec{v})-(\vec{u}\times\vec{w})-0+(\vec{v}\times\vec{w})]$$

$$=[\vec{u}\,\vec{u}\,\vec{v}]+[\vec{v}\,\vec{u}\,\vec{v}]-[\vec{w}\,\vec{u}\,\vec{v}]-[\vec{u}\,\vec{u}\,\vec{w}]$$

$$-[\vec{v}\,\vec{u}\,\vec{w}]+[\vec{w}\,\vec{u}\,\vec{w}]+[\vec{u}\,\vec{v}\,\vec{w}]+[\vec{v}\,\vec{v}\,\vec{w}]-[\vec{w}\,\vec{v}\,\vec{w}]$$

$$=0+0-[\vec{u}\,\vec{v}\,\vec{w}]-0+[\vec{u}\,\vec{v}\,\vec{w}]+0+[\vec{u}\,\vec{v}\,\vec{w}]+0-0=[\vec{u}\,\vec{v}\,\vec{w}]$$

$$=\vec{u}.(\vec{v}\times\vec{w})$$

Illustration 26.34 Find the value of '*a*' so that the volume of parallelepiped formed by $\hat{i}+a\hat{j}+\hat{k};\;\hat{j}+a\hat{k}$ and $a\hat{i}+\hat{k}$ becomes minimum.

Solution: Volume of the parallelepiped,

$$V=[\hat{i}+a\hat{j}+\hat{k}\quad\hat{j}+a\hat{k}\quad a\hat{i}+\hat{k}]$$

$$=(\hat{i}+a\hat{j}+\hat{k})\cdot\{(\hat{j}+a\hat{k})\times(a\hat{i}+\hat{k})\}$$

$$=(\hat{i}+a\hat{j}+\hat{k})\cdot\{\hat{i}+a^2\hat{j}-a\hat{k}\}$$

$$=1+a^3-a$$

Now,

$$\frac{dV}{da}=3a^2-1\Rightarrow\frac{d^2V}{da^2}=6a$$

$$\frac{dV}{da}=0\Rightarrow 3a^2-1=0\Rightarrow a=\pm\frac{1}{\sqrt{3}}$$

At $a=\frac{1}{\sqrt{3}}$,

$$\frac{d^2V}{da^2}=\frac{6}{\sqrt{3}}>0$$

Therefore, *V* is minimum at $a=\frac{1}{\sqrt{3}}$.

Illustration 26.35 *x*, *y*, *z* are distinct scalars such that $[x\vec{a}+y\vec{b}+z\vec{c},x\vec{b}+y\vec{c}+z\vec{a},x\vec{c}+y\vec{a}+z\vec{b}]=0$, where $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors. Then

(a) $x+y+z=0$

(b) $x^3+y^3+z^3=0$

(c) $xy+yz+zx=0$

(d) $x^2+y^2+z^2=0$

Solution: $\vec{a},\vec{b},\vec{c}$ are non-coplanar. Therefore,

$$[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$$

Now,

$$[x\vec{a}+y\vec{b}+z\vec{c},x\vec{b}+y\vec{c}+z\vec{a},x\vec{c}+y\vec{a}+z\vec{b}]=0$$

$$\Rightarrow(x\vec{a}+y\vec{b}+z\vec{c})\cdot\{(x\vec{b}+y\vec{c}+z\vec{a})\times(x\vec{c}+y\vec{a}+z\vec{b})\}=0$$

$$\Rightarrow(x\vec{a}+y\vec{b}+z\vec{c})\cdot\{(x^2-yz)(\vec{b}\times\vec{c})+$$
$$(z^2-xy)(\vec{a}\times\vec{b})+(y^2-zx)(\vec{c}\times\vec{a})\}=0$$

$$\Rightarrow x(x^2-yz)[\vec{a}\vec{b}\vec{c}]+y(y^2-zx)[\vec{b}\,\vec{c}\,\vec{a}]+z(z^2-xy)[\vec{c}\,\vec{a}\,\vec{b}]=0$$

$$\Rightarrow(x^3-xyz)[\vec{a}\,\vec{b}\,\vec{c}]+(y^3-xyz)[\vec{a}\vec{b}\vec{c}]+(z^3-xyz)[\vec{a}\vec{b}\vec{c}]=0$$

As $[\vec{a}\vec{b}\vec{c}]\neq 0$, so

$$x^3+y^3+z^3-3xyz=0$$

$$\Rightarrow (x+y+z)(x^2+y^2+z^2-xy-yz-zx)=0$$

$$\Rightarrow \frac{1}{2}(x+y+z)\{(x-y)^2+(y-z)^2+(z-x)^2\}=0$$

$$\Rightarrow x+y+z=0 \text{ or } x=y=z$$

But x, y, z are distinct. Therefore,

$$x+y+z=0$$

26.16 Vector Triple Product

For three vectors $\vec{a},\vec{b},\vec{c}$ a product of the form $\vec{a}\times(\vec{b}\times\vec{c})$ or $(\vec{a}\times\vec{b})\times\vec{c}$ is called a vector triple product.

This is a vector, and the value depends upon the placement of the brackets. In fact $\vec{a}\times(\vec{b}\times\vec{c})$ is a vector in the plane of b and c (the two placed in the brackets).

In value $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$ and $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{c}\cdot\vec{a})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}$

26.16.1 Properties of Vector Triple Product

1. The vector triple product $\vec{a}\times(\vec{b}\times\vec{c})$ is a linear combination of those two vectors which are within brackets.
2. The vector $\vec{r}=\vec{a}\times(\vec{b}\times\vec{c})$ is perpendicular to $\vec{a}$ and lies in the plane of $\vec{b}$ and $\vec{c}$.
3. The formula $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$ is true only when the vector outside the bracket is on the left most side. If it is not, we first shift on left by using the properties of cross product and then apply the same formula.
   Thus,
   $$(\vec{b}\times\vec{c})\times\vec{a}=-\{\vec{a}\times(\vec{b}\times\vec{c})\}=-\{(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}\}=(\vec{a}\cdot\vec{b})\vec{c}-(\vec{a}\cdot\vec{c})\vec{b}$$
4. If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$,
   then $\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_2c_3-b_3c_2 & b_3c_1-b_1c_3 & b_1c_2-b_2c_1\end{vmatrix}$.
5. $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$

Illustration 26.36 Show that

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$$

Solution:

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$

$$\vec{b}\times(\vec{c}\times\vec{a})=(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}$$

$$\vec{c}\times(\vec{a}\times\vec{b})=(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}$$

Adding the three results,

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})$$

$$=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{c}\cdot\vec{a})\vec{b}+(\vec{b}\cdot\vec{a})\vec{c}-(\vec{a}\cdot\vec{b})\vec{c}+(\vec{c}\cdot\vec{b})\vec{a}-(\vec{b}\cdot\vec{c})\vec{a}$$
$$=0$$

Illustration 26.37 Prove that a necessary and sufficient condition that $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$ is $(\vec{a}\times\vec{c})\times\vec{b}=\vec{0}$.

Solution: Necessary part: Given that

$$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$$

$$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=(\vec{c}\cdot\vec{a})\vec{b}-(\vec{c}\cdot\vec{b})\vec{a}$$

$$\Rightarrow (\vec{c}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{b})\vec{c}=0$$

$$\Rightarrow (\vec{a}\times\vec{c})\times\vec{b}=\vec{0}$$

Therefore, the condition is necessary.

Sufficient part: Let $(\vec{a}\times\vec{c})\times\vec{b}=0$. Then

$$(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}=0$$

$$\Rightarrow (\vec{b}\cdot\vec{a})\vec{c}=(\vec{b}\cdot\vec{c})\vec{a}\,\overrightarrow{a}$$

$$\Rightarrow -(\vec{b}\cdot\vec{a})\vec{c}=-(\vec{b}\cdot\vec{c})\vec{a}$$

$$\Rightarrow (\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{a})\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$$

Adding $(\vec{a}\cdot\vec{c})\vec{b}$ to both the sides

$$\Rightarrow \vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$$

which is the required condition.

Your Turn 4

1. If $\vec{a},\vec{b},\vec{c}$ be any three non-zero, non-coplanar vectors, then any vector $\vec{r}$ is equal to
   (A) $z\vec{a}+x\vec{b}+y\vec{c}$ **(B)** $x\vec{a}+y\vec{b}+z\vec{c}$
   (C) $y\vec{a}+z\vec{b}+x\vec{c}$ **(D)** None of these
   where $x=\frac{[\vec{r}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $y=\frac{[\vec{r}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $z=\frac{[\vec{r}\,\vec{a}\,\vec{b}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$.
   Ans. (B)
2. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\vec{a}+2\vec{b}+3\vec{c},\lambda\vec{b}+4\vec{c}$ and $(2\lambda-1)\vec{c}$ are non-coplanar for
   (A) No value of λ **(B)** All except one value of λ
   (C) All except two values of λ **(D)** All values of λ
   Ans. (C)
3. Let $\vec{a},\vec{b}$ and $\vec{c}$ be non-zero vectors such that $(\vec{a}\times\vec{b})\times\vec{c}=\frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$ and θ is the acute angle between the vectors $\vec{b}$ and $\vec{c}$. Then $\sin\theta$ equals
   (A) $\frac{2\sqrt{2}}{3}$ **(B)** $\frac{\sqrt{2}}{3}$
   (C) $\frac{2}{3}$ **(D)** $\frac{1}{3}$ **Ans.** (A)
4. If $\vec{a}=\hat{i}+\hat{j}+\hat{k},\vec{b}=\hat{i}+\hat{j},\vec{c}=\hat{i}$ and $(\vec{a}\times\vec{b})\times\vec{c}=\lambda\vec{a}+\mu\vec{b}$, then $\lambda+\mu=$
   (A) 0 **(B)** 1
   (C) 2 **(D)** 3 **Ans.** (A)
5. If $\vec{a},\vec{b},\vec{c}$ and $\vec{p},\vec{q},\vec{r}$ are reciprocal system of vectors, then $\vec{a}\times\vec{p}+\vec{b}\times\vec{q}+\vec{c}\times\vec{r}$ equals

(A) $[\vec{a}\,\vec{b}\,\vec{c}]$ **(B)** $(\vec{p}+\vec{q}+\vec{r})$

(C) 0 **(D)** $\vec{a}+\vec{b}+\vec{c}$ **Ans.** (C)

26.17 Scalar or Vector Product of Four Vectors

26.17.1 Scalar Product

$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$ is a scalar product of four vectors. It is the dot product of the vectors $\vec{a}\times\vec{b}$ and $\vec{c}\times\vec{d}$. It is a scalar triple product of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}\times\vec{d}$ as well as scalar triple product of the vectors $\vec{a}\times\vec{b}$, $\vec{c}$ and $\vec{d}$.

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}$$

26.17.2 Vector Product

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ is a vector product of four vectors. It is the cross product of the vectors $\vec{a}\times\vec{b}$ and $\vec{c}\times\vec{d}$.

$\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\},\{(\vec{a}\times\vec{b})\times\vec{c}\}\times\vec{d}$ are also different vector products of four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

Illustration 26.38 Let $\vec{a},\vec{b},\vec{c}$ be three mutually perpendicular vectors of the same magnitude and the vector $\vec{x}$ satisfy the equation $\vec{a}\times\{(\vec{x}-\vec{b})\times\vec{a}\}+\vec{b}\times\{(\vec{x}-\vec{c})\times\vec{b}\}+\vec{c}\times\{(\vec{x}-\vec{a})\times\vec{c}\}$. Then find $\vec{x}$.

Solution: Here

$$(\vec{a}\cdot\vec{a})(\vec{x}-\vec{b})-\{\vec{a}\cdot(\vec{x}-\vec{b})\}\cdot\vec{a}+(\vec{b}\cdot\vec{b})(\vec{x}-\vec{c})-\{\vec{b}\cdot(\vec{x}-\vec{c})\}\cdot$$
$$\vec{b}+(\vec{c}\cdot\vec{c})(\vec{x}-\vec{a})-\{\vec{c}\cdot(\vec{x}-\vec{a})\}\vec{c}=0$$

or

$$\lambda^2(\vec{x}-\vec{b}+\vec{x}-\vec{c}+\vec{x}-\vec{a})=\{\vec{a}\cdot(\vec{x}-\vec{b})\}\vec{a}+\{\vec{b}\cdot(\vec{x}-\vec{c})\}\vec{b}+\{\vec{c}\cdot(\vec{x}-\vec{a})\}\vec{c}$$

where $|\vec{a}|=|\vec{b}|=|\vec{c}|=\lambda$

$$\lambda^2\{3\vec{x}-(\vec{a}+\vec{b}+\vec{c})\}=(\vec{a}.\vec{x})\vec{a}+(\vec{b}.\vec{x})\vec{b}+(\vec{c}.\vec{x})\vec{c}$$

Let $\vec{x}=\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}$. Then

$$\vec{a}.\vec{x}=\alpha|\vec{a}|^2=\alpha\lambda^2,\ \vec{b}.\vec{x}=\beta\lambda^2 \text{ and } \vec{c}\cdot\vec{x}=\gamma\lambda^2$$

$$\Rightarrow \lambda^2\{3\vec{x}-(\vec{a}+\vec{b}+\vec{c})\}=\lambda^2\vec{x} \Rightarrow 3\vec{x}-(\vec{a}+\vec{b}+\vec{c})=\vec{x}$$

Hence,

$$\vec{x}=\frac{\vec{a}+\vec{b}+\vec{c}}{2}$$

26.18 Method to Prove Collinearity

1. Two vectors $\vec{p}$ and $\vec{q}$ are collinear if there exists $k\in R$ such that $\vec{p}=k\vec{q}$.
2. If $\vec{p}\times\vec{q}=\vec{o}$, then $\vec{p},\vec{q}$ are collinear.
3. Three points $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ are collinear if there exists $k\in R$ such that $\overrightarrow{AB}=k(\overrightarrow{BC})$, that is, $\vec{b}-\vec{a}=k(\vec{c}-\vec{b})$.
4. If $(\vec{b}-\vec{a})\times(\vec{c}-\vec{b})=\vec{o}$, then A, B, C are collinear.
5. $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ are collinear if there exists scalars l, m, n, (not all zero) such that $l\vec{a}+m\vec{b}+n\vec{c}=\vec{o}x$ where $l+m+n=0$.

Illustration 26.39 Let $\vec{a},\vec{b},\vec{c}$ be three non-zero vectors such that any two of them are non-collinear. If $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$, then prove that $\vec{a}+2\vec{b}+6\vec{c}=0$.

Solution: It is given that $\vec{a}+2\vec{b}$ is collinear with $\vec{c}$, so

$$\vec{a}+2\vec{b}=\lambda\vec{c} \text{ (for some scalar } \lambda) \quad (1)$$

Also $\vec{b}+3\vec{c}$ is collinear with $\vec{a}$, so

$$\vec{b}+3\vec{c}=\mu\vec{a} \text{ (for some scalar } \mu) \quad (2)$$

From Eqs. (1) and (2), we get

$$(1+2\mu)\,\vec{b}+(3-\mu\lambda)\,\vec{c}=0$$

$\Rightarrow 1+2\mu=0$ and $3-\mu\lambda=0$ {$\vec{b}$ and $\vec{c}$ are non-collinear vectors}

$\Rightarrow \mu=-1/2$ and $\lambda=-6$

Substituting the values of λ and μ in Eqs. (1) and (2), we get

$$\vec{a}+2\vec{b}+6\vec{c}=0$$

Illustration 26.40 Prove that

$$[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{b}]$$

Solution:

$$[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}]$$
$$=(\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$$
$$=(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a})$$
$$=\vec{a}\cdot b\times\vec{c}+\vec{b}\cdot\vec{c}\times\vec{a}$$
$$=[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\,\vec{b}\ \vec{c}] \qquad (\text{As } \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{a}\cdot(\vec{b}\times\vec{c}))$$
$$=2[\vec{a}\,\vec{b}\,\vec{c}]$$

Illustration 26.41 Find λ if $\lambda\hat{i}+\hat{j}+2\hat{k}$; $\hat{i}+\lambda\hat{j}-\hat{k}$ and $2\hat{i}-\hat{j}+\lambda\hat{k}$ are coplanar.

Solution: The condition for coplanarity is

$$\begin{vmatrix}\lambda & 1 & 2\\ 1 & \lambda & -1\\ 2 & -1 & \lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda(\lambda^2-1)-1(\lambda+2)+2(-1-2\lambda)=0$$

$$\Rightarrow \lambda^3-6\lambda-4=0$$

By inspection it is seen that $\lambda=-2$ is a root. Therefore,

$$\lambda^3-6\lambda-4=(\lambda+2)(\lambda^2-2\lambda-2)$$

and $\lambda^2-2\lambda-2=0$ for $\lambda=1\pm\sqrt{3}$

The required value of λ are

$$\lambda_1=-2;\ \lambda_2=1+\sqrt{3};\ \lambda_3=1-\sqrt{3}$$

Illustration 26.42 If four points, A, B, C and D with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are coplanar, prove that $[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{b}\ \vec{a}\ \vec{d}]+[\vec{c}\ \vec{a}\ \vec{d}]+[\vec{a}\ \vec{b}\ \vec{d}]$.

Solution: If the points A, B, C, D are coplanar, then the vectors

$$\overrightarrow{DA}=\overrightarrow{OA}-\overrightarrow{OD}=-\vec{d}+\vec{a};$$

$$\overrightarrow{DB}=\overrightarrow{OB}-\overrightarrow{OD}=-\vec{d}+\vec{b};$$

$$\overrightarrow{DC}=\overrightarrow{OC}-\overrightarrow{OD}=-\vec{d}+\vec{c} \text{ are coplanar.}$$

Hence,

$$(-\vec{d}+\vec{a})\cdot((-\vec{d}+\vec{b})\times(-\vec{d}+\vec{c}))=0$$

$$(-\vec{d}+\vec{a})\cdot(\vec{b}\times\vec{c}-\vec{b}\times\vec{d}-\vec{d}\times\vec{c})=0 \text{ (As } \vec{d}\times\vec{d}=0)$$

$$\Rightarrow -[\vec{d}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{d}]-[\vec{a}\ \vec{d}\ \vec{c}]=0$$

Therefore,

$$[\vec{a}\ \vec{b}\ \vec{c}]=[\vec{d}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\ \vec{d}\ \vec{c}]$$

$$=[\vec{b}\ \vec{c}\ \vec{d}]+[\vec{a}\ \vec{d}\ \vec{c}]+[\vec{c}\ \vec{a}\ \vec{d}]$$

and this is the desired result.

Illustration 26.43 Prove that if $\cos\alpha\neq 1$, $\cos\beta\neq 1$ and $\cos\gamma\neq 1$, then the vectors $\vec{a}=\hat{i}\cos\alpha+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+\hat{j}\cos\beta+\hat{k}$, $\vec{c}=\hat{i}+\hat{j}+\hat{k}\cos\gamma$ can never be coplanar.

Solution: Suppose that $\vec{a},\vec{b},\vec{c}$ are coplanar. So,

$$\begin{vmatrix}\cos\alpha & 1 & 1\\ 1 & \cos\beta & 1\\ 1 & 1 & \cos\gamma\end{vmatrix}=0$$

$$(R_2\to R_2-R_1 \text{ and } R_3\to R_3-R_1)$$

$$\Rightarrow\begin{vmatrix}\cos\alpha & 1 & 1\\ 1-\cos\alpha & \cos\beta-1 & 0\\ 1-\cos\alpha & 0 & \cos\gamma-1\end{vmatrix}=0$$

or

$\cos\alpha(\cos\beta-1)(\cos\gamma-1)-(1-\cos\alpha)(\cos\gamma-1)-(1-\cos\alpha)(\cos\beta-1)=0$

Dividing throughout by $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)$, we get

$$\frac{\cos\alpha}{1-\cos\alpha}+\frac{1}{1-\cos\beta}+\frac{1}{1-\cos\gamma}=0$$

$$\Rightarrow -1+\frac{1}{1-\cos\alpha}+\frac{1}{1-\cos\beta}+\frac{1}{1-\cos\gamma}=0$$

$$\Rightarrow \frac{1}{1-\cos\alpha}+\frac{1}{1-\cos\beta}+\frac{1}{1-\cos\gamma}=1$$

$$\Rightarrow \operatorname{cosec}^2\frac{\alpha}{2}+\operatorname{cosec}^2\frac{\beta}{2}+\operatorname{cosec}^2\frac{\gamma}{2}=2,$$

which is not possible as

$$\operatorname{cosec}^2\frac{\alpha}{2}\geq 1, \operatorname{cosec}^2\frac{\beta}{2}\geq 1, \operatorname{cosec}^2\frac{\gamma}{2}\geq 1$$

Hence, they cannot be coplanar.

26.19 Vector Equation

Generally, to solve a vector equation, we express the unknown vector as a linear combination of three known non-coplanar vectors and then we determine the coefficients from the given conditions.

If $\vec{a},\vec{b}$ are two known non-collinear vectors, then $\vec{a},\vec{b},\vec{a}\times\vec{b}$ are three non-coplanar vectors.

Thus, any vector $\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$ where x,y,z are unknown scalars.

Illustration 26.44 Let $\vec{a}=\hat{i}-\hat{j}$, $\vec{b}=\hat{j}-\hat{k}$, $\vec{c}=\hat{k}-\hat{i}$ and $\vec{d}$ is a unit vector such that $\vec{a}\cdot\vec{d}=0=[\vec{b}\,\vec{c}\,\vec{d}]$. Then find $\vec{d}$.

Solution: Let $\vec{d}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$. Then

$$\vec{a}\cdot\vec{d}=0\Rightarrow(\hat{i}-\hat{j})\cdot(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})=0\Rightarrow\alpha-\beta=0\Rightarrow\alpha=\beta$$

$$[\vec{b}\,\vec{c}\,\vec{d}]=0\Rightarrow(\vec{b}\times\vec{c})\cdot\vec{d}=0\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & -1\\ -1 & 0 & 1\end{vmatrix}\cdot(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})=0$$

$$\Rightarrow(\hat{i}+\hat{j}+\hat{k})\cdot(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})=0\Rightarrow\alpha+\beta+\gamma=0$$

$$\Rightarrow\gamma=-(\alpha+\beta)=-2\alpha;\ (\beta=\alpha)$$

$$|\vec{d}|=1\Rightarrow\alpha^2+\beta^2+\gamma^2=1\Rightarrow\alpha^2+\alpha^2+4\alpha^2=1$$

$$\Rightarrow\alpha=\pm\frac{1}{\sqrt{6}}=\beta \text{ and } \gamma=\mp\frac{2}{\sqrt{6}}$$

Therefore, $\vec{d}=\pm\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}-2\hat{k})$.

Illustration 26.45 Let the unit vectors $\vec{a}$ and $\vec{b}$ be perpendicular and the unit vector $\vec{c}$ be inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$, then

(A) $\alpha=\beta=\cos\theta,\gamma^2=\cos 2\theta$
(B) $\alpha=\beta=\cos\theta,\gamma^2=-\cos 2\theta$
(C) $\alpha=\cos\theta,\beta=\sin\theta,\gamma^2=\cos 2\theta$
(D) None of these

Solution: We have,

$$|\vec{a}|=|\vec{b}|=1$$

$$\vec{a}\cdot\vec{b}=0;\ (\text{as } \vec{a}\perp\vec{b})$$

$$\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b}) \quad (1)$$

Taking dot product by $\vec{a}$, we get

$$\vec{a}\cdot\vec{c}=\alpha|\vec{a}|^2+\beta(\vec{a}\cdot\vec{b})+\gamma[\vec{a}\ \vec{a}\ \vec{b}]$$

$$\Rightarrow|\vec{a}|\cdot|\vec{c}|\cos\theta=\alpha\cdot 1+0+0$$

$$\Rightarrow 1\cdot|\vec{c}|\cdot\cos\theta=\alpha$$

As $|\vec{c}|=1$. Therefore,

$$\alpha=\cos\theta$$

Taking dot product of Eq. (1) by $\vec{b}$, we get

$$\vec{b}\cdot\vec{c}=\vec{b}\cdot\vec{a}+\beta|\vec{b}|^2+\gamma[\vec{b}\,\vec{a}\,\vec{b}]\Rightarrow|\vec{b}||\vec{c}|\cos\theta=0+\beta\cdot 1+0$$

Therefore,

$$\beta=1\cdot 1\cdot\cos\theta=\cos\theta$$

$$|\vec{c}|^2=1\Rightarrow\alpha^2+\beta^2+\gamma^2=1\Rightarrow\cos^2\theta+\cos^2\theta+\gamma^2=1$$

Therefore,

$$\gamma^2=1-2\cos^2\theta=-\cos 2\theta$$

So,

$$\alpha = \beta = \cos\theta, \gamma^2 = -\cos 2\theta$$

Hence, the correct answer is option (B).

Your Turn 5

1. $\vec{a}\times[\vec{a}\times(\vec{a}\times\vec{b})]$ is equal to
(A) $(\vec{a}\times\vec{a})\cdot(\vec{b}\times\vec{a})$ **(B)** $\vec{a}\cdot(\vec{b}\times\vec{a})-\vec{b}\cdot(\vec{a}\times\vec{b})$
(C) $[\vec{a}\cdot(\vec{a}\times\vec{b})]\vec{a}$ **(D)** $(\vec{a}\cdot\vec{a})(\vec{b}\times\vec{a})$ **Ans.** (D)

2. $[\vec{b}\times\vec{c}\ \vec{c}\times\vec{a}\ \vec{a}\times\vec{b}]$ is equal to
(A) $\vec{a}\times(\vec{b}\times\vec{c})$ **(B)** $2[\vec{a}\,\vec{b}\,\vec{c}]$
(C) $[\vec{a}\,\vec{b}\,\vec{c}]^2$ **(D)** $[\vec{a}\,\vec{b}\,\vec{c}]$ **Ans.** (C)

3. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{a}\cdot\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$, then $\vec{b}=$
(A) $\hat{i}$ **(B)** $\hat{i}-\hat{j}+\hat{k}$
(C) $2\hat{j}-\hat{k}$ **(D)** $2\hat{i}$ **Ans.** (A)

4. The point of intersection of $\hat{r}\times\vec{a}=\vec{b}\times\vec{a}$ and $\hat{r}\times\vec{b}=\vec{a}\times\vec{b}$ where $\vec{a}=\hat{i}+\hat{j}$ and $\vec{b}=2\hat{i}-\hat{k}$ is
(A) $3\hat{i}+\hat{j}-\hat{k}$ **(B)** $3\hat{i}-\hat{k}$
(C) $3\hat{i}+2\hat{j}+\hat{k}$ **(D)** None of these **Ans.** (A)

5. Let the vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ be such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$. Let P_1 and P_2 be planes determined by pair of vectors $\vec{a},\vec{b}$ and $\vec{c},\vec{d}$, respectively. Then the angle between P_1 and P_2 is
(A) $0°$ **(B)** $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$ **Ans.** (A)

Additional Solved Examples

1. In Cartesian coordinates, the point A is (x_1, y_1), where $x_1 = 1$ on the curve $y = x^2 + x + 10$. The tangent at A cuts the x-axis at B. Then find the value of the dot product $\overrightarrow{OA}\cdot\overrightarrow{AB}$.

Solution: Given curve is

$$y = x^2 + x + 10 \quad (1)$$

When $x = 1$. Then

$$y = 1^2 + 1 + 10 = 12$$

Therefore,

$$A \equiv (1, 12)$$

$$\Rightarrow \overrightarrow{OA} = \hat{i} + 12\hat{j}$$

From Eq. (1), we get

$$\frac{dy}{dx} = 2x + 1$$

Equation of tangent at A is

$$y - 12 = \left(\frac{dy}{dx}\right)_{(1,12)}(x-1)$$

$$\Rightarrow y - 12 = (2\times 1 + 1)(x-1) \Rightarrow y - 12 = 3x - 3$$

Therefore,

$$y = 3(x+3)$$

This tangent cuts x-axis (that is, $y = 0$) at $(-3, 0)$. Therefore,

$$B \equiv (-3, 0)$$

$$\overrightarrow{OB} = -3\hat{i} + 0.\hat{j} = -3\hat{i};$$

$$\overrightarrow{OA}\cdot\overrightarrow{AB} = \overrightarrow{OA}\cdot(\overrightarrow{OB}-\overrightarrow{OA}) = (\hat{i}+12\hat{j})\cdot(-3\hat{i}-\hat{i}-12\hat{j})$$

$$= (\hat{i}+12\hat{j})\cdot(-4\hat{i}-12\hat{j}) = -4 - 144 = -148$$

2. Let $\vec{u},\vec{v},\vec{w}$ be such that $|\vec{u}|=1, |\vec{v}|=2, |\vec{w}|=3$. If the projection $\vec{v}$ along $\vec{u}$ is equal to that of $\vec{w}$ along $\vec{u}$ and $\vec{v},\vec{w}$ are perpendicular to each other, then find $|\vec{u}-\vec{v}+\vec{w}|$.

Solution: Without loss of generality, we can assume

$$\vec{v} = 2\hat{i} \text{ and } \vec{w} = 3\hat{j}$$

Let $\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$. Then

$$|\vec{u}| = 1 \Rightarrow x^2 + y^2 + z^2 = 1 \quad (1)$$

Projection of $\vec{v}$ along $\vec{u}$ = Projection of $\vec{w}$ along $\vec{u}$

$$\Rightarrow \vec{v}\cdot\vec{u} = \vec{w}\cdot\vec{u} \Rightarrow 2\hat{i}\cdot(x\hat{i}+y\hat{j}+2\hat{k}) = 3\hat{j}\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\Rightarrow 2x = 3y \Rightarrow 3y - 2x = 0$$

Now,

$$|\vec{u}-\vec{v}-\vec{w}| = |x\hat{i}+y\hat{j}+z\hat{k}-2\hat{i}+3\hat{j}|$$

$$= |(x-2)\hat{i}+(y+3)\hat{j}+z\hat{k}| = \sqrt{(x-2)^2+(y-3)^2+z^2}$$

$$= \sqrt{(x^2+y^2+z^2)+2(3y-2x)+13} = \sqrt{1+2\times 0+13} = \sqrt{14}$$

3. If $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$, then which of the following is correct?
(A) $\vec{a}\cdot\vec{c}\neq 0$ **(B)** $\vec{a}\cdot\vec{c}\neq\vec{b}\cdot\vec{c}$
(C) $|\vec{a}|=|\vec{c}|$ **(D)** None of these

Solution:

$$\vec{a}\cdot(\vec{a}\times\vec{b}) = \vec{a}\cdot\vec{c} = 0$$

Also,

$$\vec{b}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot\vec{c} = 0$$

and $\quad \vec{b}(\vec{b}\times\vec{c}) = \vec{a}\cdot\vec{b} = 0$

So, $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular vectors, therefore

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}| = |\vec{c}| \text{ and } |\vec{b}||\vec{c}| = |\vec{a}| \Rightarrow |\vec{b}| = 1$$

$$\Rightarrow |\vec{a}| = |\vec{c}|$$

Hence, the correct answer is option (C).

4. Let $\vec{A},\vec{B},\vec{C}$ are three vectors respectively given by $2\hat{i}+\hat{k}, \hat{i}+\hat{j}+\hat{k}$ and $4\hat{i}-3\hat{j}+7\hat{k}$. Then vector $\vec{R}$, which satisfies the relation $\vec{R}\times\vec{B}=\vec{C}\times\vec{B}$ and $\vec{R}\cdot\vec{A}=0$ is
(A) $2\hat{i}-5\hat{j}+2\hat{k}$ **(B)** $-\hat{i}+4\hat{j}+2\hat{k}$
(C) $-\hat{i}-8\hat{j}+2\hat{k}$ **(D)** None of these

Solution: We have

$$\vec{R}\times\vec{B}=\vec{C}\times\vec{B} \text{ and } \vec{R}\cdot\vec{A}=0$$

Therefore,

$$\vec{A}\times(\vec{R}\times\vec{B})=\vec{A}\times(\vec{C}\times\vec{B}) \Rightarrow (\vec{A}\cdot\vec{B})\vec{R}-(\vec{A}\cdot\vec{R})\vec{B}=(\vec{A}\cdot\vec{B})\vec{C}-(\vec{A}\cdot\vec{C})\vec{B}$$

$$\Rightarrow (2+1)\vec{R}=3\vec{C}-(8+7)\vec{B} \Rightarrow \vec{R}=\vec{C}-5\vec{B}=-\hat{i}-8\hat{j}+2\hat{k}$$

Hence, the correct answer is option (C).

5. The points with position vectors $60\hat{i}+3\hat{j}$, $40\hat{i}-8\hat{j}$ and $a\hat{i}-52\hat{j}$ are collinear if

(A) $a=-40$ **(B)** $a=40$
(C) $a=20$ **(D)** None of these

Solution: The points are collinear, therefore

$$\lambda(60\hat{i}+3\hat{j})+\mu(40\hat{i}-8\hat{j})+\gamma(a\hat{i}-52\hat{j})=0$$

with $\lambda+\mu+\gamma=0$,

$$3\lambda-8\mu-52\gamma=0$$
$$60\lambda+40\mu+\gamma a=0$$

For non-zero set (λ, μ, γ),

$$\begin{vmatrix} 60 & 40 & a \\ 3 & -8 & -52 \\ 1 & 1 & 1 \end{vmatrix}=0 \Rightarrow a=-40$$

Hence, the correct answer is option (A).

6. Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are two vectors such that $|\overrightarrow{OA}+\overrightarrow{OB}|=|\overrightarrow{OA}+2\overrightarrow{OB}|$. Then

(A) $\angle BOA=90°$ **(B)** $\angle BOA>90°$
(C) $\angle BOA<90°$ **(D)** $60°\le\angle BOA\le 90°$

Solution: Given

$$|\overrightarrow{OA}+\overrightarrow{OB}|=|\overrightarrow{OA}+2\overrightarrow{OB}|$$

On squaring, we get

$$(OA)^2+(OB)^2+2\overrightarrow{OA}\cdot\overrightarrow{OB}=(OA)^2+4(OB)^2+4\overrightarrow{OA}\cdot\overrightarrow{OB}$$
$$\Rightarrow \cos\theta<0 \Rightarrow \theta>90° \Rightarrow \angle BOA>90°$$

Hence, the correct answer is option (B).

7. A circle is inscribed in an n-sided regular polygon $A_1A_2\ldots A_n$ having each side of a unit length. For any arbitrary point P on the circle, prove that $\sum_{i=1}^{n}(PA_i)^2=n\dfrac{a^2}{4}\left(\dfrac{1+\cos^2\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}\right)$.

Solution: See Figs. 26.33 and 26.34.

Let the centre of the incircle be the reference point. Then

$$\overrightarrow{PA_i}=\overrightarrow{OA_i}-\overrightarrow{OP}$$
$$\overrightarrow{PA_i}\cdot\overrightarrow{PA_i}=(\overrightarrow{OA_i}-\overrightarrow{OP})(\overrightarrow{OA_i}-\overrightarrow{OP})$$
$$(PA_i)^2=(OA_i)^2+(OP)^2-2\overrightarrow{OP}\cdot\overrightarrow{OA_i}$$

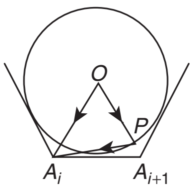


Figure 26.33

$$\sum_{i=1}^{n}(PA_i)^2=\sum_{i=1}^{n}(OA_i)^2+\sum_{i=1}^{n}(OP)^2-\sum_{i=1}^{n}2\overrightarrow{OP}\cdot\overrightarrow{OA_i}$$
$$=nR^2+nr^2-2\overrightarrow{OP}\cdot\sum_{i=1}^{n}\overrightarrow{OA_i}$$
$$=n(R^2+r^2)-2\overrightarrow{OP}\cdot(\vec{0})$$

Now

$$R=\frac{a}{2}\operatorname{cosec}\frac{\pi}{n},\ r=\frac{a}{2}\cot\frac{\pi}{n}$$

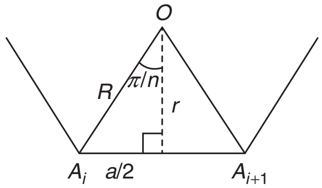


Figure 26.34

$$R^2+r^2=\frac{a^2}{4}\left(\operatorname{cosec}^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}\right)=\frac{a^2}{4}\left(\frac{1+\cos^2\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}\right)$$

$$\Rightarrow \sum_{i=1}^{n}(PA_i)^2=\frac{na^2}{4}\left(\frac{1+\cos^2\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}\right)$$

8. In a triangle PQR, S and T are points on QR and PR, respectively, such that $QS=3SR$ and $PT=4RT$. Let M be the point of intersection of PS and QT. Then determine the ratio $QM{:}MT$ using vector methods.

Solution: See Fig. 26.35.

Let $QM{:}MT=\lambda{:}1$ and $PM{:}MS=\mu{:}1$ and $\overrightarrow{QP}=\vec{a}$ and $\overrightarrow{QR}=\vec{b}$. Then

$$\overrightarrow{QM}=\frac{\lambda}{\lambda+1}\left(\frac{4\vec{b}+\vec{a}}{5}\right) \quad (1)$$

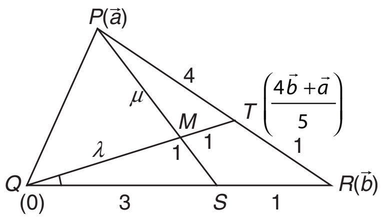


Figure 26.35

Also,

$$\overrightarrow{QS}=\frac{3}{4}\vec{b}$$

So,

$$\overrightarrow{QM}=\frac{\mu\frac{3}{4}\vec{b}+\vec{a}}{\mu+1} \quad (2)$$

From Eqs. (1) and (2), we get

$$\frac{\lambda}{\lambda+1}\left(\frac{4\vec{b}+\vec{a}}{5}\right)=\frac{3\mu\vec{b}+4\vec{a}}{4(\mu+1)}$$

On comparing, we get

$$\frac{1}{\mu+1}=\frac{\lambda}{5(\lambda+1)} \quad (3)$$

and

$$\frac{4\lambda}{5(\lambda+1)}=\frac{3\mu}{4(\mu+1)} \quad (4)$$

So, $\mu=\frac{16}{3}$ and $\lambda=\frac{15}{4}$. Hence, $QM:MT = 15:4$

9. Two systems of forces P, Q, R and P', Q', R' are along the side BC, CA, AB of a ΔABC. Prove that the resultant will be parallel if

$$\begin{vmatrix} \sin A & \sin B & \sin C \\ P & Q & R \\ P' & Q' & R' \end{vmatrix} = 0$$

Solution: See Fig. 26.36.

Unit vector along $\vec{Q} = \cos(\pi - C)\,\hat{i} + \sin(\pi - C)\,\hat{j}$

$$= -\cos C\,\hat{i} + \sin C\,\hat{j}$$

Unit vector along $\vec{R} = \cos(\pi + B)\,\hat{i} + \sin(\pi + B)\,\hat{j}$

$$= -\cos B\,\vec{i} - \sin B\,\vec{j}$$

If S and S' be resultant in the two cases, then

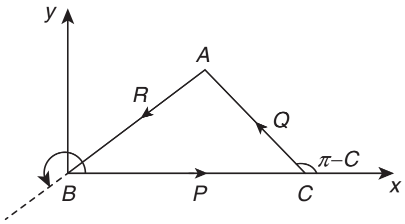


Figure 26.36

$$\vec{S} = P\hat{i} + Q(-\cos C\,\hat{i} + \sin C\,\hat{j}) + R(-\cos B\,\hat{i} - \sin B\,\hat{j}) \quad (1)$$

Similarly,

$$S' = (P' - Q'\cos C - R'\cos B)\,\hat{i} + (Q'\sin C - R'\sin B)\,\hat{j} \quad (2)$$

If θ and θ' be the angles made by the resultant with x-axis, then

$$\tan\theta = \frac{Q\sin C - R\sin B}{P - Q\cos C - R\cos B}$$

and

$$\tan\theta' = \frac{Q'\sin C - R'\sin B}{P' - Q'\cos C - R'\cos B}$$

If the resultant are parallel, then $\theta = \theta'$. Therefore,

$$\tan\theta = \tan\theta' \Rightarrow \frac{Q\sin C - R\sin B}{P - Q\cos C - R\cos B} = \frac{Q'\sin C - R'\sin B}{P' - Q'\cos C - R'\cos B}$$

On solving we get

$$(PQ' - P'Q)\sin C + (RP' - R'P)\sin B + (QR' - Q'R)\sin A = 0$$

$$\begin{vmatrix} \sin A & \sin B & \sin C \\ P & Q & R \\ P' & R' & Q' \end{vmatrix} = 0$$

10. Let ABC and PQR be any two triangles in the same plane. Assume that the perpendiculars from the points A, B, C to the sides QR, RP, PQ, respectively, are concurrent. Using vector methods or otherwise, prove that the perpendiculars from P, Q, R to BC, CA, AB, respectively, are also concurrent.

Solution: ABC and PQR are the given triangles. Let the perpendicular from A, B, C to the sides QR, PR and PQ intersect at O. Take O as the initial point. Let $\vec{a}, \vec{b}, \vec{c}, \vec{p}, \vec{q}, \vec{r}$ be the position vector of A, B, C, P, Q and R, respectively. Since OA, OB and OC are perpendicular to QR, RP and PQ, so

$$\vec{a}\cdot(\vec{r}-\vec{q}) = 0,\ \vec{b}\cdot(\vec{p}-\vec{r}) = 0 \text{ and } \vec{c}\cdot(\vec{q}-\vec{p}) = 0$$

Let the perpendicular from P and Q on BC and CA, respectively, intersect at the point X whose position vector is taken as $\vec{x}$. It implies

$$(\vec{p}-\vec{x})\cdot(\vec{c}-\vec{b}) = 0 \text{ and } (\vec{q}-\vec{x})\cdot(\vec{a}-\vec{c}) = 0$$

$$\Rightarrow \vec{p}\cdot(\vec{c}-\vec{b}) = \vec{x}\cdot(\vec{c}-\vec{b}) \text{ and } \vec{q}\cdot(\vec{a}-\vec{c}) = \vec{x}\cdot(\vec{a}-\vec{c})$$

Adding, we have

$$\vec{x}\cdot(\vec{a}-\vec{b}) = \vec{p}\cdot\vec{c} - \vec{p}\cdot\vec{b} + \vec{q}\cdot\vec{a} - \vec{q}\,\vec{c} = \vec{c}\cdot(\vec{p}-\vec{q}) - \vec{p}\cdot\vec{b} + \vec{q}\cdot\vec{a}$$

$$= -\vec{p}\cdot\vec{b} + \vec{q}\cdot\vec{a}$$

$$= -\vec{b}\cdot\vec{r} + \vec{a}\cdot\vec{r} = \vec{r}\cdot(\vec{a}-\vec{b}) \ \Rightarrow (\vec{r}-\vec{x})\cdot(\vec{a}-\vec{b}) = 0$$

Therefore, XR is perpendicular to AB.
Hence, perpendicular from R to AB passes through X.

Previous Years' Solved JEE Main/AIEEE Questions

1. The resultant of two forces P N and 3 N is a force of 7 N. If the direction of 3 N force were reversed, the resultant would be $\sqrt{19}$ N. The value of P is

(A) 5 N **(B)** 6 N
(C) 3N **(D)** 4N **[AIEEE 2007]**

Solution: See Fig. 26.37. Originally,

$$7^2 = P^2 + 3^2 + 2\times 3\times P\cos\theta \quad (1)$$

Later,

$$(\sqrt{19})^2 = P^2 + (-3)^2 + 2\times(-3)\times P\cos\theta \quad (2)$$

Adding we get,

$$68 = 2P^2 + 18 \Rightarrow P = 5$$

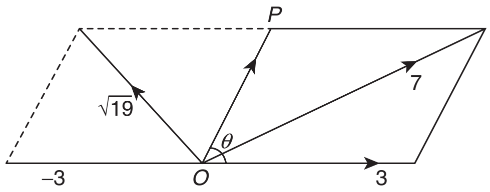


Figure 26.37

Hence, the correct answer is option (A).

2. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$. If the vector $\vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$, then x equals

(A) 0 **(B)** 1
(C) – 4 **(D)** –2 **[AIEEE 2007]**

Solution:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = \hat{i} - \hat{j} + 2\hat{k} \text{ and } \vec{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$$

$$\begin{vmatrix} x & x-2 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 0 \Rightarrow 3x+2-x+2=0 \Rightarrow 2x=-4 \Rightarrow x=-2$$

Hence, the correct answer is option (D).

3. If $\hat{u}$ and $\hat{v}$ are unit vectors and θ is the acute angle between them, then $2\hat{u}\times 3\hat{v}$ is a unit vector for
(A) exactly two values of θ
(B) more than two values of θ
(C) no value of θ
(D) exactly one value of θ **[AIEEE 2007]**

Solution:

$$|2\hat{u}\times 3\hat{v}|=1 \Rightarrow 6|\hat{u}||\hat{v}||\sin\theta|=1 \Rightarrow 6(1)(1)\sin\theta\,\hat{w}=1$$

For $6\sin\theta\,\hat{w}$ to be unit,

$$|6\sin\theta|=1 \Rightarrow |\sin\theta|=\frac{1}{6}$$

At only one value of θ. Hence, there is exactly one value of θ for which $2\hat{u}\times 3\hat{v}$ is a unit vector.
Hence, the correct answer is option (D).

4. The vector $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$ lies in the plane of the vectors $\vec{b}=\hat{i}+\hat{j}$ and $\vec{c}=\hat{j}+\hat{k}$ and bisects the angle between $\vec{b}$ and $\vec{c}$. Then which one of the following gives possible values of α and β?
(A) $\alpha=2, \beta=2$ **(B)** $\alpha=1, \beta=2$
(C) $\alpha=2, \beta=1$ **(D)** $\alpha=1, \beta=1$
[AIEEE 2008]

Solution:

$$\vec{a}=\lambda(\hat{b}+\hat{c}) \Rightarrow \alpha\hat{i}+2\hat{j}+\beta\hat{k}=\lambda\left(\frac{\hat{i}+2\hat{j}+\hat{k}}{\sqrt{2}}\right)$$

$$\Rightarrow \lambda=\sqrt{2}\alpha,\ \lambda=\sqrt{2} \text{ and } \lambda=\sqrt{2}\beta$$

$$\Rightarrow \alpha=1 \text{ and } \beta=1$$

Hence, the correct answer is option (D).

5. The non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are related by $\vec{a}=8\vec{b}$ and $\vec{c}=-7\vec{b}$. Then the angle between $\vec{a}$ and $\vec{c}$ is
(A) 0 **(B)** $\pi/4$
(C) $\pi/2$ **(D)** π **[AIEEE 2008]**

Solution: As $\vec{a}=8\vec{b}$, we have $\vec{c}=-7\vec{b}$. Therefore, $\vec{a}$ and $\vec{b}$ are like vectors and $\vec{b}$ and $\vec{c}$ are unlike. This implies that $\vec{a}$ and $\vec{c}$ will be unlike. Hence, angle between $\vec{a}$ and $\vec{c}$ is equal to π.

Hence, the correct answer is option (D).

6. Let $\vec{a}=\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$. Then vector $\vec{b}$ satisfying $\vec{a}\times\vec{b}+\vec{c}=\vec{0}$ and $\vec{a}\cdot\vec{b}=3$ is
(A) $2\hat{i}-\hat{j}+2\hat{k}$ **(B)** $\hat{i}-\hat{j}-2\hat{k}$
(C) $\hat{i}+\hat{j}-2\hat{k}$ **(D)** $-\hat{i}+\hat{j}-2\hat{k}$ **[AIEEE 2010]**

Solution: Let $\vec{b}=x\hat{i}+y\hat{j}+z\hat{k}$. It is given that

$$\vec{a}\times\vec{b}+\vec{c}=0 \Rightarrow \vec{c}=-\vec{a}\times\vec{b}=\vec{b}\times\vec{a}$$

Also, $\vec{c}$ is normal to $\vec{b}$, so

$$\vec{b}\cdot\vec{c}=0 \Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}-\hat{j}-\hat{k})=0$$

$$\Rightarrow x-y-z=0 \qquad (1)$$

It is given that $\vec{a}\cdot\vec{b}=3$, so

$$(\hat{j}-\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})=3$$

$$\Rightarrow y-z=3 \qquad (2)$$

And, $\vec{c}=\vec{b}\times\vec{a}$

$$\hat{i}-\hat{j}-\hat{k}=(x\hat{i}+y\hat{j}+z\hat{k})\times(\hat{j}-\hat{k})$$

$$=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & 1 & -1 \end{vmatrix}$$

$$=(-y-z)\hat{i}+x\hat{j}+x\hat{k}$$

From this, we get, $(y+z)=-1$. (3)
From Eqs. (2) and (3) we get $x=-1$, $y=1$ and $z=-2$. So,

$$\vec{b}=-\hat{i}+\hat{j}-2\hat{k}$$

Hence, the correct answer is option (D).

7. If the vectors $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$, $\vec{b}=2\hat{i}+4\hat{j}+\hat{k}$ and $\vec{c}=\lambda\hat{i}+\hat{j}+\mu\hat{k}$ are mutually orthogonal, then $(\lambda,\mu)=$
(A) $(2, -3)$ **(B)** $(-2, 3)$
(C) $(3, -2)$ **(D)** $(-3, 2)$ **[AIEEE 2010]**

Solution:

$$\vec{a}\cdot\vec{b}=0,\ \vec{b}\cdot\vec{c}=0,\ \vec{c}\cdot\vec{a}=0 \Rightarrow 2\lambda+4+\mu=0,\ \lambda-1+2\mu=0$$

Solving, we get $\lambda=-3$, $\mu=2$.
Hence, the correct answer is option (D).

8. If $\vec{a}=\frac{1}{\sqrt{10}}(3\hat{i}+\hat{k})$ and $\vec{b}=\frac{1}{7}(2\hat{i}+3\hat{j}-6\hat{k})$, then the value of $(2\vec{a}-\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}+2\vec{b})]$ is
(A) -3 **(B)** 5
(C) 3 **(D)** -5 **[AIEEE 2011]**

Solution:

$$(2\vec{a}-\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}+2\vec{b})]=(2\vec{a}-\vec{b})\cdot[(\vec{a}\times\vec{b})\times\vec{a}+2(\vec{a}\times\vec{b})\times\vec{b}]$$

$$=(2\vec{a}-\vec{b})\cdot[-\vec{a}\times(\vec{a}\times\vec{b})+2(-b\times(\vec{a}\times\vec{b}))]$$

$$=(2\vec{a}-\vec{b})\cdot[\{-(\vec{a}\cdot\vec{b})\vec{a}-(\vec{a}\cdot\vec{a})\vec{b}\}+2\{-(\vec{b}\cdot\vec{b})\vec{a}-(\vec{b}\cdot\vec{a})\vec{b}\}]$$

$$=(2\vec{a}-\vec{b})\cdot[-(\vec{a}\cdot\vec{b})\vec{a}+a^2\vec{b}-2b^2a-2(\vec{b}\cdot\vec{a})\vec{b}]$$

$$=(2\vec{a}-\vec{b})\cdot[-(\vec{a}\cdot\vec{b})\vec{a}+\vec{b}-2\vec{a}-2(\vec{a}\cdot\vec{b})\vec{b}]=(2\vec{a}-\vec{b})\cdot[\vec{b}-2\vec{a}]$$

$$=-(2\vec{a}-\vec{b})\cdot(2\vec{a}-\vec{b})=-\{4a^2+b^2-2\vec{a}\cdot\vec{b}\}$$

$$=-\{4+1-0\}=-5$$

Hence, the correct answer is option (D).

9. The vector $\vec{a}$ and $\vec{b}$ are not perpendicular and $\vec{c}$ and $\vec{d}$ are two vectors satisfying, $\vec{b}\times\vec{c}=\vec{b}\times\vec{d}$ and $\vec{a}\cdot\vec{d}=0$. Then the vector $\vec{d}$ is equal to
(A) $\vec{c}+\left(\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{b}$ **(B)** $\vec{b}+\left(\frac{\vec{b}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{c}$

(C) $\vec{c}-\left(\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{b}$ **(D)** $\vec{b}-\left(\frac{\vec{b}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{c}$

[AIEEE 2011]

Solution:

$$\vec{b}\times\vec{c}=\vec{b}\times\vec{d}$$

$$\Rightarrow \vec{a}\times(\vec{b}\times\vec{c})=\vec{a}\times(\vec{b}\times\vec{d})\Rightarrow(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=(\vec{a}\cdot\vec{d})\vec{b}-(\vec{a}\cdot\vec{b})\vec{d}$$

$$\Rightarrow(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=-(\vec{a}\cdot\vec{b})\vec{d}$$

Therefore, $\vec{d}=\vec{c}-\left(\frac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b}}\right)\vec{b}$.

Hence, the correct answer is option (C).

10. Let $\hat{a}$ and $\hat{b}$ be two unit vectors. If the vectors $\vec{c}=\hat{a}+2\hat{b}$ and $\vec{d}=5\hat{a}-4\hat{b}$ are perpendicular to each other, then the angle between $\hat{a}$ and $\hat{b}$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{2}$

(C) $\frac{\pi}{3}$ **(D)** $\frac{\pi}{4}$ **[AIEEE 2012]**

Solution:

$$\vec{c}\cdot\vec{d}=\vec{0}\Rightarrow 5|\vec{a}|^2+6\vec{a}\cdot\vec{b}-8|\vec{b}|^2=0$$

$$\Rightarrow 6\vec{a}\cdot\vec{b}=3\Rightarrow\vec{a}\cdot\vec{b}=\frac{1}{2}$$

So, angle between $\hat{a}$ and $\hat{b}$ is $\frac{\pi}{3}$.

Hence, the correct answer is option (C).

11. Let $ABCD$ be a parallelogram such that $\overrightarrow{AB}=\vec{q}$, $\overrightarrow{AD}=\vec{p}$ and $\angle BAD$ be an acute angle. If $\vec{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then $\vec{r}$ is given by

(A) $\vec{r}=3\vec{q}-\frac{3(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{q})}\vec{p}$ **(B)** $\vec{r}=-\vec{q}+\left(\frac{\vec{p}\cdot\vec{q}}{\vec{p}\cdot\vec{p}}\right)\vec{p}$

(C) $\vec{r}=\vec{q}-\left(\frac{\vec{p}\cdot\vec{q}}{\vec{p}\cdot\vec{p}}\right)\vec{p}$ **(D)** $\vec{r}=-3\vec{q}+\frac{3(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}\vec{p}$

[AIEEE 2012]

Solution: See Fig. 26.38. $\overrightarrow{AE}$ is the vector component of $\vec{q}$ on $\vec{p}$. So,

$$\overrightarrow{AE}=\frac{(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}\vec{p}$$

Therefore, from ΔABE;

$$\overrightarrow{AB}+\overrightarrow{BE}=\overrightarrow{AE}$$

$$\Rightarrow\vec{q}+\vec{r}=\frac{(\vec{p}\cdot\vec{q})\vec{p}}{(\vec{p}\cdot\vec{p})}\Rightarrow\vec{r}=-\vec{q}+\frac{(\vec{p}\cdot\vec{q})}{(\vec{p}\cdot\vec{p})}\vec{p}$$

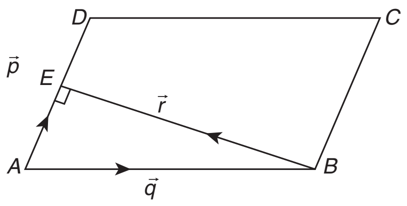


Figure 26.38

Hence, the correct answer is option (B).

12. If the vectors $\overrightarrow{AB}=3\hat{i}+4\hat{k}$ and $\overrightarrow{AC}=5\hat{i}-2\hat{j}+4\hat{k}$ are the sides of a triangle ABC, then the length of the median through A is

(A) $\sqrt{72}$ **(B)** $\sqrt{33}$

(C) $\sqrt{45}$ **(D)** $\sqrt{18}$ **[JEE MAIN 2013]**

Solution: From the following figure, we see that (Fig. 26.39),

$$\overrightarrow{AM}=\frac{\overrightarrow{AB}+\overrightarrow{AC}}{2}\Rightarrow\overrightarrow{AM}=4\hat{i}-\hat{j}+4\hat{k}$$

Therefore,

$$|\overrightarrow{AM}|=\sqrt{16+16+1}=\sqrt{33}$$

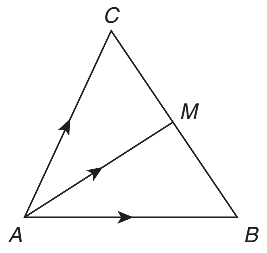


Figure 26.39

Hence, the correct answer is option (B).

13. If $[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]=\lambda[\vec{a}\,\vec{b}\,\vec{c}]^2$, then λ is equal to

(A) 0 **(B)** 1

(C) 2 **(D)** 3

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$[\vec{a}\times\vec{b}\ \vec{b}\times\vec{c}\ \vec{c}\times\vec{a}]=(\vec{a}\times\vec{b})\cdot[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]=(\vec{a}\times\vec{b})\cdot[(\vec{b}\times\vec{c})\times\vec{c}\times\vec{a}]$$

$$=(\vec{a}\times\vec{b})\cdot[(\vec{b}\times\vec{c}\cdot\vec{a})\vec{c}-(\vec{b}\times\vec{c}\cdot\vec{c})\vec{a}]$$

$$=(\vec{a}\times\vec{b})\cdot[(\vec{b}\times\vec{c}\cdot\vec{a})\vec{c}-0]$$

$$=(\vec{a}\times\vec{b}\cdot\vec{c})(\vec{a}\times\vec{b}\cdot\vec{c})=(\vec{a}\times\vec{b}\cdot\vec{c})=[\vec{a}\vec{b}\,\vec{c}]^2$$

Therefore, $\lambda=1$.

Hence, the correct answer is option (B).

14. If $|\vec{a}|=2$, $|\vec{b}|=3$ and $|2\vec{a}-\vec{b}|=5$, then $|2\vec{a}+\vec{b}|$ equals

(A) 17 **(B)** 7

(C) 5 **(D)** 1

[JEE MAIN 2014 (ONLINE SET-1)]

Solution:

$$|2\vec{a}-\vec{b}|^2=25\Rightarrow 4a^2+b^2-4\vec{a}\cdot\vec{b}=25$$

$$\Rightarrow 16+9-4\vec{a}\cdot\vec{b}=25\Rightarrow\vec{a}\cdot\vec{b}=0$$

Now,

$$|2\vec{a}+\vec{b}|^2=(2\vec{a}+\vec{b})\cdot(2\vec{a}+\vec{b})=4a^2+b^2+4\vec{a}\cdot\vec{b}=4(4)+9+4(0)=25$$

Therefore, $|2\vec{a}+\vec{b}|=5$.

Hence, the correct answer is option (C).

15. Two ships A and B are sailing straight away from a fixed point O along routes such that $\angle AOB$ is always 120°. At a certain instance, OA = 8 km, OB = 6 km and the ship A is sailing at the rate of 20 km/h while the ship B sailing at the rate of 30 km/h. Then the distance between A and B is changing at the rate (in km/h):

(A) $\dfrac{260}{\sqrt{37}}$ (B) $\dfrac{260}{37}$

(C) $\dfrac{80}{\sqrt{37}}$ (D) $\dfrac{80}{37}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 26.40.

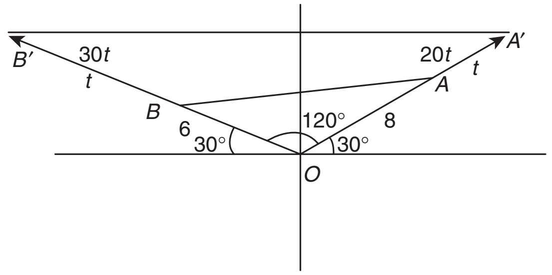


Figure 26.40

After the instance of being at A and B, when we switch on the clocks, ship A reaches A' and ship B reaches B' in time t. A travels $20t$ and B travels $30t$.
Now in triangle $OA'B'$, at the instant of t time

$$\cos 120° = \frac{OA'^2 + OB'^2 - |A'B'|^2}{2|OA'||OB'|}$$

$$\Rightarrow -\frac{1}{2} = \frac{(20t+8)^2 + (30t+6)^2 - |A'B'|^2}{2(20t+8)(30t+6)}$$

$$\Rightarrow |A'B'|^2 = 400\,t^2 + 64 + 320\,t + 900\,t^2 + 36 + 360\,t + 600\,t^2 + 120\,t + 240\,t + 48$$
$$= 1900\,t^2 + 1040\,t + 148 = 4\{475\,t^2 + 260\,t + 37\}$$

Therefore,

$$|A'B'| = 2\sqrt{475t^2+260t+37} \Rightarrow \frac{d|A'B'|}{dt} = \frac{\cancel{2}\times(950t+260)}{\cancel{2}\sqrt{475t^2+260t+37}}$$

Now, rate of change of distance $AB = \left.\dfrac{d|A'B'|}{dt}\right|_{t=0} = \dfrac{260}{\sqrt{37}}$

Hence, the correct answer is option (A).

16. If $|\vec{c}|^2 = 60$ and $\vec{c}\times(\hat{i}+2\hat{j}+5\hat{k}) = \vec{0}$, then a value of $\vec{c}\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$ is

(A) $4\sqrt{2}$ (B) 12

(C) 24 (D) $12\sqrt{2}$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$|\vec{c}|^2 = 60, \vec{c}\times(\hat{i}+2\hat{j}+5\hat{k}) = 0$$

Therefore, $\vec{c}$ is parallel to $\hat{i}+2\hat{j}+5\hat{k}$. So,

$$\vec{c} = \lambda(\hat{i}+2\hat{j}+5\hat{k})$$

Now,

$$|\vec{c}|^2 = 60$$
$$\Rightarrow \lambda^2|(\hat{i}+2\hat{j}+5\hat{k})|^2 = 60$$
$$\Rightarrow \lambda^2(1+4+25) = 60$$
$$\Rightarrow \lambda^2 = 2 \Rightarrow \lambda = \pm\sqrt{2}$$

Therefore,

$$\vec{c} = \sqrt{2}(\hat{i}+2\hat{j}+5\hat{k}) \text{ or } -\sqrt{2}(\hat{i}+2\hat{j}+5\hat{k})$$

Now,

$$\vec{c}\cdot(-7\hat{i}+2\hat{j}+3\hat{k}) = \sqrt{2}(\hat{i}+2\hat{j}+5\hat{k})\,(-7\hat{i}+2\hat{j}+3\hat{k}$$
$$= \sqrt{2}(-7+4-15) = -12\sqrt{2}$$

Hence, the correct answer is option (D).

17. If $\vec{x} = 3\hat{i}-6\hat{j}-\hat{k}$, $\vec{y} = \hat{i}+4\hat{j}-3\hat{k}$ and $\vec{z} = 3\hat{i}-4\hat{j}-12\hat{k}$, then the magnitude of the projection of $\vec{x}\times\vec{y}$ on $\vec{z}$ is

(A) 12 (B) 15

(C) 14 (D) 13

[JEE MAIN 2014 (ONLINE SET-4)]

Solution:

$$\hat{x}\times\hat{y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -6 & -1 \\ 1 & 4 & -3 \end{vmatrix} = \hat{i}(22) - \hat{j}(-8) + \hat{k}(18) = 22\hat{i}+8\hat{j}+18\hat{k}$$

$$\Rightarrow \hat{z} = \frac{3\hat{i}-4\hat{j}-12\hat{k}}{\sqrt{3^2+4^2+12^2}}$$

Therefore, projection is

$$(\vec{x}\times\vec{y})\cdot\hat{z} = (22\hat{i}+8\hat{j}+18\hat{k})\cdot\frac{3\hat{i}-4\hat{j}-12\hat{k}}{\sqrt{9+16+144}} = \frac{66-32-216}{13} = \frac{-182}{13}$$

Therefore, magnitude of projection = 14.
Hence, the correct answer is option (C).

18. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors such that no two of them are collinear and $(\vec{a}\times\vec{b})\times\vec{c} = \dfrac{1}{3}|\vec{b}||\vec{c}|\vec{a}$. If θ is the angle between vectors $\vec{b}$ and $\vec{c}$, then a value of $\sin\theta$ is

(A) $\dfrac{-\sqrt{2}}{3}$ (B) $\dfrac{2}{3}$

(C) $\dfrac{-2\sqrt{3}}{3}$ (D) $\dfrac{2\sqrt{2}}{3}$

[JEE MAIN 2015 (OFFLINE)]

Solution:

$$(\vec{a}\times\vec{b})\times\vec{c} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a} \Rightarrow -\vec{c}\times(\vec{a}\times\vec{b}) = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$$

$$\Rightarrow -(\vec{c}\cdot\vec{b})\vec{a} + (\vec{c}\cdot\vec{a})\vec{b} = \frac{1}{3}|\vec{b}||\vec{c}|\vec{a}$$

$$\Rightarrow \left[\frac{1}{3}|\vec{b}||\vec{c}| + (\vec{c}\cdot\vec{b})\right]\vec{a} = (\vec{c}\cdot\vec{a})\vec{b}$$

Since $\vec{a}$ and $\vec{b}$ are not collinear,

$$\frac{1}{3}|\vec{b}||\vec{c}| + (\vec{c}\cdot\vec{b}) = 0 \text{ and } \vec{c}\cdot\vec{a} = 0$$

$$\Rightarrow \cos\theta + \frac{1}{3} = 0 \Rightarrow \cos\theta = -\frac{1}{3} \Rightarrow \sin\theta = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

Hence, the correct answer is option (D).

19. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $|\vec{a}+\vec{b}| = \sqrt{3}$. If $\vec{c} = \vec{a}+2\vec{b}+3(\vec{a}\times\vec{b})$, then $2|\vec{c}|$ is equal to

(A) $\sqrt{55}$ **(B)** $\sqrt{51}$

(C) $\sqrt{43}$ **(D)** $\sqrt{37}$

[JEE MAIN 2015 (ONLINE SET-1)]

Solution:

$$|\vec{a}|=1,\ |\vec{b}|=1,\ |\vec{a}+\vec{b}|=\sqrt{3},\ \vec{c}=\vec{a}+2\vec{b}+3(\vec{a}\times\vec{b})$$

Since,

$$|\vec{a}+\vec{b}|^2=3 \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=3$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=3 \Rightarrow \vec{a}\cdot\vec{b}=\frac{1}{2} \Rightarrow 1+1+2\cos\theta=3$$

$$\Rightarrow \cos\theta=\frac{1}{2}$$

$$\Rightarrow \theta=60^\circ = \text{Angle between } \vec{a} \text{ and } \vec{b}$$

$$\Rightarrow \vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\cdot\hat{x},\ \hat{x}=\text{unit vector}$$

Vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b}=\frac{\sqrt{3}}{2}\hat{x}$$

Therefore,

$$|\vec{c}|^2=\left(\vec{a}+2\vec{b}+\frac{3\sqrt{3}}{2}\hat{x}\right)\cdot\left(\vec{a}+2\vec{b}+\frac{3\sqrt{3}}{2}\hat{x}\right)$$

$$=1+2\times(2\vec{a}\cdot\vec{b})+4+\left(\frac{9\times 3}{4}\right) \quad (\text{Since, } \hat{n}\vec{a}=\hat{x}\cdot\vec{b}=0)$$

$$=1+4\left(\frac{1}{2}\right)+4+\frac{27}{4}=\frac{55}{4} \Rightarrow |\vec{c}|=\frac{\sqrt{55}}{2} \Rightarrow 2|\vec{c}|=\sqrt{55}$$

Hence, the correct answer is option (A).

20. In a parallelogram $ABCD$, $|\overrightarrow{AB}|=a$, $|\overrightarrow{AD}|=b$ and $|\overrightarrow{AC}|=c$, then $\overrightarrow{DB}\cdot\overrightarrow{AB}$ has the value

(A) $\frac{1}{2}(a^2-b^2+c^2)$ **(B)** $\frac{1}{4}(a^2+b^2-c^2)$

(C) $\frac{1}{3}(b^2+c^2-a^2)$ **(D)** $\frac{1}{2}(a^2+b^2+c^2)$

[JEE MAIN 2015 (ONLINE SET-2)]

Solution: See Fig. 26.41.

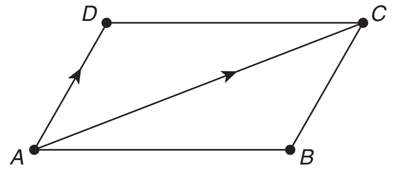


Figure 26.41

$$|\overrightarrow{AB}|=a,\ |\overrightarrow{AD}|=b \text{ and } |\overrightarrow{AC}|=c,$$

$$\overrightarrow{AB}+\overrightarrow{AD}=\overrightarrow{AC} \Rightarrow |\overrightarrow{AB}|^2+|\overrightarrow{AD}|^2+2\overrightarrow{AB}\cdot\overrightarrow{AD}=|\overrightarrow{AC}|^2$$

$$\Rightarrow a^2+b^2+2\overrightarrow{AB}(\overrightarrow{AB}+\overrightarrow{BD})=c^2$$

$$\Rightarrow a^2+b^2+2a^2+2\overrightarrow{AB}\cdot\overrightarrow{BD}=c^2 \Rightarrow 3a^2+b^2-c^2=2\overrightarrow{AB}\cdot\overrightarrow{DB}$$

$$\Rightarrow \overrightarrow{AB}\cdot\overrightarrow{DB}=\frac{1}{2}(3a^2+b^2-c^2)$$

There is no correct option for this question.

21. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$. If $\vec{b}$ is not parallel to $\vec{c}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{5\pi}{6}$ **(B)** $\frac{3\pi}{4}$

(C) $\frac{\pi}{2}$ **(D)** $\frac{2\pi}{3}$

[JEE MAIN 2016 (OFFLINE)]

Solution: It is given that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three unit vectors such that

$$\vec{a}\times(\vec{b}\times\vec{c})=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$$

Therefore,

$$(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}=\frac{\sqrt{3}}{2}(\vec{b}+\vec{c})$$

comparing both sides, we get

$$\vec{a}\cdot\vec{c}=\frac{\sqrt{3}}{2} \text{ and } \vec{a}\cdot\vec{b}=-\frac{\sqrt{3}}{2}$$

Here, $\cos\alpha=\frac{\sqrt{3}}{2}$ and $\cos\beta=-\frac{\sqrt{3}}{2}$, where α is angle between $\vec{a}$ and $\vec{c}$; β is the angle between $\vec{a}$ and $\vec{b}$. Therefore,

$$\beta=\frac{5\pi}{6}$$

Hence, the correct answer is option (A).

22. In a triangle ABC, right-angled at the vertex A, if the position vectors of A, B and C are, respectively, $3\hat{i}+\hat{j}-\hat{k}$, $-\hat{i}+3\hat{j}+p\hat{k}$ and $5\hat{i}+q\hat{j}-4\hat{k}$, then the point (p, q) lies on a line

(A) making an obtuse angle with the positive direction of x-axis.

(B) parallel to x-axis.

(C) parallel to y-axis.

(D) making an acute angle with the positive direction of x-axis.

[JEE MAIN 2016 (ONLINE SET-1)]

Solution: See Figs. 26.42 and 26.43. From the triangle ABC as shown here, we can write as

$$\overrightarrow{AB}=-4\hat{i}+2\hat{j}+(p+1)\hat{k}$$
$$\overrightarrow{AC}=2\hat{i}+(q-1)\hat{j}-3\hat{k}$$

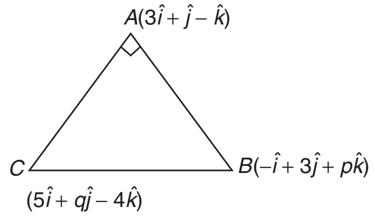


Figure 26.42

As $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$. Therefore,

$$\overrightarrow{AB}\cdot\overrightarrow{AC}=0$$

$$\Rightarrow -8+2(q-1)-3(p+1)=0$$

$$\Rightarrow -8+2q-2-3p-3=0$$
$$\Rightarrow 3p-2q+13=0$$
$$\Rightarrow \frac{p}{(-13/3)}+\frac{q}{(13/2)}=1$$

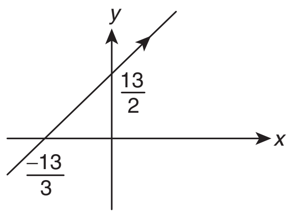


Figure 26.43

Therefore, the point (p, q) makes acute angle with positive direction of x-axis.

Hence, the correct answer is option (D).

23. Let ABC be a triangle whose circumcentre is at P. If the position vectors of A, B, C and P are $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\frac{\vec{a}+\vec{b}+\vec{c}}{4}$, respectively, then the position vector of the orthocentre of this triangle is

(A) $-\left(\frac{\vec{a}+\vec{b}+\vec{c}}{2}\right)$ **(B)** $\vec{a}+\vec{b}+\vec{c}$

(C) $\frac{(\vec{a}+\vec{b}+\vec{c})}{2}$ **(D)** $\vec{0}$

[JEE MAIN 2016 (ONLINE SET-2)]

Solution: See Figs. 26.44 and 26.45. Let ABC be a triangle whose circumcentre is at point P.

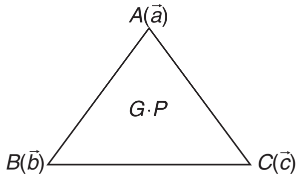


Figure 26.44

$$P=\frac{\vec{a}+\vec{b}+\vec{c}}{4} \qquad \text{(circumcentre)}$$

$$G=\frac{\vec{a}+\vec{b}+\vec{c}}{3} \qquad \text{(centroid)}$$

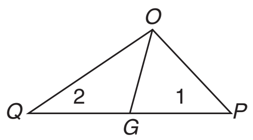


Figure 26.45

The orthocentre (Q) is

$$\frac{2\overrightarrow{OP}+1\overrightarrow{OQ}}{3}=\overline{OG}$$

Therefore,

$$\overline{OQ}=\frac{\vec{a}+\vec{b}+\vec{c}}{2}$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. The number of distinct real values of λ, for which the vectors $-\lambda^2\hat{i}+\hat{j}+\hat{k}$, $\hat{i}-\lambda^2\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}-\lambda^2\hat{k}$ are coplanar, is

(A) zero **(B)** one

(C) two **(D)** three **[IIT-JEE 2007]**

Solution: If the vectors are coplanar

$$\begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix}=0$$

$$\Rightarrow \lambda^6-3\lambda^2-2=0$$
$$\Rightarrow (\lambda^2+1)^2(\lambda^2-2)=0$$
$$\Rightarrow \lambda=\pm\sqrt{2}$$

Hence, the correct answer is option (C).

2. Let the vectors $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{ST}$, $\overline{TU}$ and $\overline{UP}$ represent the sides of a regular hexagon.

Statement-1: $\overrightarrow{PQ}\times(\overrightarrow{RS}+\overrightarrow{ST})\neq\vec{0}$.

because

Statement-2: $\overrightarrow{PQ}\times\overrightarrow{RS}=\vec{0}$ and $\overrightarrow{PQ}\times\overrightarrow{ST}\neq\vec{0}$.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True **[IIT-JEE 2007]**

Solution: See Fig. 26.46.

Statement-1: $\overrightarrow{PQ}\times(\overrightarrow{RS}+\overrightarrow{ST})\neq 0$. Since $\overrightarrow{PQ}$ is not parallel to $\overrightarrow{RT}$, we get

$$\overrightarrow{PQ}\times(\overrightarrow{RT})$$

That is, Statement-1 is true.

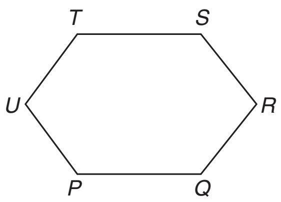


Figure 26.46

$$\overrightarrow{PQ}\times(\overrightarrow{RT})\neq 0$$

Statement-2: $\overrightarrow{PQ}\times\overrightarrow{RS}=\vec{0}$ and $\overrightarrow{PQ}\times\overrightarrow{ST}\neq\vec{0}$. Since $\overrightarrow{PQ}$ is not parallel to $\overrightarrow{RS}$, Statement-2 is false.

Hence, the correct answer is option (C).

3. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Which one of the following is correct?

(A) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=\vec{0}$

(B) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq\vec{0}$

(C) $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{a}\times\vec{c}\neq\vec{0}$
(D) $\vec{a}\times\vec{b},\ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}$ are mutually perpendicular

[IIT-JEE 2007]

Solution: We have

$$\vec{a}+\vec{b}+\vec{c}=0 \quad (1)$$

Taking cross-product by $\vec{a}$ on both sides of Eq. (1), we get

$$\vec{a}\times(\vec{a}+\vec{b}+\vec{c})=0$$
$$\Rightarrow \vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=0$$
$$0+\vec{a}\times\vec{b}-\vec{c}\times\vec{a}=0$$
$$\Rightarrow \vec{a}\times\vec{b}=\vec{c}\times\vec{a} \quad (2)$$

Taking cross-product by $\vec{b}$ on both sides of Eq. (1), we get

$$\vec{b}\times(\vec{a}+\vec{b}+\vec{c})=0$$
$$\Rightarrow \vec{b}\times\vec{a}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}=0$$
$$\Rightarrow -\vec{a}\times\vec{b}+0+\vec{b}\times\vec{c}=0$$
$$\Rightarrow \vec{a}\times\vec{b}=\vec{b}\times\vec{c} \quad (3)$$

From Eqs. (2) and (3), we get

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}\neq 0$$

Alternative Method:

Since $|\vec{a}|=|\vec{b}|=|\vec{c}|$ and $\vec{a}+\vec{b}+\vec{c}=0$, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar unit vectors. Let $\hat{n}$ be a unit normal vector in the plane of $\vec{a}$, $\vec{b}$ and $\vec{c}$, we get

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}=\hat{n}$$

Hence, the correct answer is option (B).

4. The edges of a parallelepiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that $\hat{a}\cdot\hat{b}=\hat{b}\cdot\hat{c}=\hat{c}\cdot\hat{a}=1/2$. Then the volume of the parallelepiped is

(A) $\dfrac{1}{\sqrt{2}}$ **(B)** $\dfrac{1}{2\sqrt{2}}$

(C) $\dfrac{\sqrt{3}}{2}$ **(D)** $\dfrac{1}{\sqrt{3}}$ **[IIT-JEE 2008]**

Solution: We have

$$\hat{a}\cdot\hat{a}=\hat{b}\cdot\hat{b}=\hat{c}\cdot\hat{c}=1$$
$$\hat{a}\cdot\hat{b}=\hat{b}\cdot\hat{c}=\hat{c}\cdot\hat{a}=\frac{1}{2}$$
$$[\hat{a}\ \ \hat{b}\ \ \hat{c}]^2=\begin{vmatrix}\hat{a}\cdot\hat{a} & \hat{a}\cdot\hat{b} & \hat{a}\cdot\hat{c}\\ \hat{b}\cdot\hat{a} & \hat{b}\cdot\hat{b} & \hat{b}\cdot\hat{c}\\ \hat{c}\cdot\hat{a} & \hat{c}\cdot\hat{b} & \hat{c}\cdot\hat{c}\end{vmatrix}$$
$$=\begin{vmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & 1\end{vmatrix}=\frac{1}{2}$$

So,

$$\text{Volume}=[\hat{a}\ \ \hat{b}\ \ \hat{c}]=\frac{1}{\sqrt{2}}$$

Hence, the correct answer is option (A).

5. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector $\overrightarrow{OP}$ (where O is the origin) is given by $\hat{a}\cos t+\hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\overrightarrow{OP}$ and $\hat{u}$ be the unit vector along $\overrightarrow{OP}$. Then

(A) $\hat{u}=\dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

(B) $\hat{u}=\dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+\hat{a}\cdot\hat{b})^{1/2}$

(C) $\hat{u}=\dfrac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$

(D) $\hat{u}=\dfrac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M=(1+2\hat{a}\cdot\hat{b})^{1/2}$ **[IIT-JEE 2008]**

Solution: We have

$$\overrightarrow{OP}=\hat{a}\cos t+\hat{b}\sin t$$
$$|\overrightarrow{OP}|^2=\cos^2 t\,|\hat{a}|^2+\sin^2 t\,|\hat{b}|^2+2(\hat{a}\cdot\hat{b})\sin t\cos t$$
$$=\sin^2 t+\cos^2 t+\sin 2t(\hat{a}\cdot\hat{b})$$
$$=1+\sin 2t(\hat{a}\cdot\hat{b})$$
$$|\overrightarrow{OP}|=\sqrt{1+(\hat{a}\cdot\hat{b})\sin 2t}$$
$$\overrightarrow{OP}=\hat{a}\cos t+\hat{b}\sin t$$
$$\text{Max.}\,|OP|=\sqrt{1+(\hat{a}\cdot\hat{b})}\ \text{ at }\ t=\frac{\pi}{4}$$

Therefore,

$$\overrightarrow{OP}=\frac{1}{\sqrt{2}}(\hat{a}+\hat{b})\Rightarrow \hat{u}=\frac{\frac{1}{\sqrt{2}}(\hat{a}+\hat{b})}{\frac{1}{\sqrt{2}}|\hat{a}+\hat{b}|}$$

and $$M=[1+(\hat{a}\cdot\hat{b})]^{\frac{1}{2}}$$

Hence, the correct answer is option (A).

6. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are unit vectors such that $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=1$ and $\vec{a}\cdot\vec{c}=\dfrac{1}{2}$, then

(A) $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar
(B) $\vec{b}, \vec{c}, \vec{d}$ are non-coplanar
(C) $\vec{b}, \vec{d}$ are non-parallel
(D) $\vec{a}, \vec{d}$ are parallel and $\vec{b}, \vec{c}$ are parallel **[IIT-JEE 2009]**

Solution: The given equation, $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=1$, is possible only when

$$\left|\vec{a}\times\vec{b}\right|=\left|\vec{c}\times\vec{d}\right|=1$$

and $$(\vec{a}\times\vec{b})\,||\,(\vec{c}\times\vec{d})$$

Since $\vec{a}\cdot\vec{c}=1/2$ and $\vec{b}\,||\,\vec{d}$, we get

$$|\vec{c}\times\vec{d}|\neq 1$$

Therefore, we conclude that the vectors $\vec{b}$ and $\vec{d}$ are non-parallel.
Hence, the correct answer is option (C).

7. Let $P(3, 2, 6)$ be a point in space and Q be a point on the line $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(-3\hat{i}+\hat{j}+5\hat{k})$. Then the value of μ for which the vector $\overrightarrow{PQ}$ is parallel to the plane $x-4y+3z=1$ is

(A) $\dfrac{1}{4}$ **(B)** $-\dfrac{1}{4}$

(C) $\dfrac{1}{8}$ **(D)** $-\dfrac{1}{8}$ **[IIT-JEE 2009]**

Solution: Any point on the line can be taken as

$$Q\equiv\{(1-3\mu),\ (\mu-1),\ (5\mu+2)\}$$
$$\overrightarrow{PQ}=\{-3\mu-2,\ \mu-3,\ 5\mu-4\}$$

Now,
$$1(-3\mu-2)-4(\mu-3)+3(5\mu-4)=0$$
$$\Rightarrow -3\mu-2-4\mu+12+15\mu-12=0$$
$$8\mu=2\Rightarrow \mu=1/4$$

Hence, the correct answer is option (A).

8. Match the statements/expressions in Column I with the values given in Column II.

Column I	Column II
(A) Root(s) of the expression $2\sin^2\theta+\sin^2 2\theta=2$	**(P)** $\dfrac{\pi}{6}$
(B) Points of discontinuity of the function $f(x)=\left[\dfrac{6x}{\pi}\right]\cos\left[\dfrac{3x}{\pi}\right]$, where $[y]$ denotes the largest integer less than or equal to y	**(Q)** $\dfrac{\pi}{4}$
(C) Volume of the parallelopiped with its edges represented by the vectors $\hat{i}+\hat{j},\ \hat{i}+2\hat{j}$ and $\hat{i}+\hat{j}+\pi\hat{k}$	**(R)** $\dfrac{\pi}{3}$
(D) Angle between vectors $\vec{a}$ and $\vec{b}$ where $\vec{a},\ \vec{b}$ and $\vec{c}$ are unit vectors satisfying $\vec{a}+\vec{b}+\sqrt{3}\vec{c}=\vec{0}$	**(S)** $\dfrac{\pi}{2}$
	(T) π

[IIT-JEE 2009]

Solution:
(A) $2\sin^2\theta+4\sin^2\theta\cos^2\theta=2$

$$\sin^2\theta+2\sin^2\theta(1-\sin^2\theta)=1$$
$$3\sin^2\theta-2\sin^4\theta-1=0\Rightarrow\sin\theta=\pm\frac{1}{\sqrt{2}},\ \pm1$$
$$\Rightarrow\theta=\frac{\pi}{4},\frac{\pi}{2}$$

(B) Let $y=\dfrac{3x}{\pi}$. Then

$$\frac{1}{2}\le y\le 3\quad\forall x\in\left[\frac{\pi}{6},\ \pi\right]$$

Now,
$$f(y)=[2y]\cos[y]$$

Critical points are $y=\dfrac{1}{2},\ y=1,\ y=\dfrac{3}{2},\ y=3.$

So, points of discontinuity $\left\{\dfrac{\pi}{6},\dfrac{\pi}{3},\dfrac{\pi}{2},\pi\right\}$.

(C) $\begin{vmatrix}1&1&0\\1&2&0\\1&1&\pi\end{vmatrix}=\pi\Rightarrow$ volume of parallelepiped $=\pi$

(D) $|\vec{a}+\vec{b}|=\sqrt{3}$

$$\Rightarrow\sqrt{2+2\cos\alpha}=\sqrt{3}$$
$$\Rightarrow 2+2\cos\alpha=3$$
$$\Rightarrow\alpha=\frac{\pi}{3}$$

Hence, the correct matches are (A)→(Q, S), (B)→(P, R, S, T), (C)→(T), (D)→(R).

9. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i}-\hat{j},\ 4\hat{i},\ 3\hat{i}+3\hat{j}$ and $-3\hat{i}+2\hat{j}$ respectively. The quadrilateral $PQRS$ must be a

(A) parallelogram, which is neither a rhombus nor a rectangle
(B) square
(C) rectangle, but not a square
(D) rhombus, but not a square **[IIT-JEE 2010]**

Solution: See Fig. 26.47. Evaluating mid-point of PR and QS which gives $M=\left[\dfrac{\hat{i}}{2}+\hat{j}\right]$, same for both.

$$\overrightarrow{PQ}=\overrightarrow{SR}=6\hat{i}+\hat{j}$$
$$\overrightarrow{PS}=\overrightarrow{QR}=-i+3\hat{j}$$
$$\Rightarrow\overrightarrow{PQ}\cdot\overrightarrow{PS}\neq 0$$
$$\overrightarrow{PQ}\,||\,\overrightarrow{SR},\ \overrightarrow{PS}\,||\,\overrightarrow{QR}\ \text{and}\ |\overrightarrow{PQ}|=|\overrightarrow{SR}|,\ |\overrightarrow{PS}|=|\overrightarrow{QR}|$$

Hence, $PQRS$ is a parallelogram but not rhombus or rectangle.

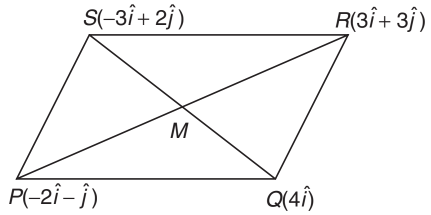


Figure 26.47

Hence, the correct answer is option (A).

10. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a}=\dfrac{\hat{i}-2\hat{j}}{\sqrt{5}}$ and $\vec{b}=\dfrac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$, then the value of $(2\vec{a}+\vec{b})\cdot[(\vec{a}\times\vec{b})\times(\vec{a}-2\vec{b})]$ is **[IIT-JEE 2010]**

Solution:
$$E=(2\vec{a}+\vec{b})\cdot[2|\vec{b}|^2\,\vec{a}-2(\vec{a}\cdot\vec{b})\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}+|\vec{a}|^2\,\vec{b}]$$

$$\vec{a}\cdot\vec{b}=\frac{2-2}{\sqrt{70}}=0$$

$$|\vec{a}|=1$$

$$|\vec{b}|=1$$

$$E=(2\vec{a}+\vec{b})\cdot[2|\vec{b}|^2\,\vec{a}+|\vec{a}|^2\,\vec{b}]$$
$$=4|\vec{a}|^2|\vec{b}|^2+2|\vec{a}|^2\,(\vec{a}\cdot\vec{b})+2|\vec{b}|^2\,(\vec{b}\cdot\vec{a})+|\vec{a}|^2|b|^2$$
$$=5|\vec{a}|^2|\vec{b}|^2=5$$

Hence, the correct answer is (5).

11. Two adjacent sides of a parallelogram *ABCD* are given by $\overrightarrow{AB}=2\hat{i}+10\hat{j}+11\hat{k}$ and $\overrightarrow{AD}=-i+2\hat{j}+2\hat{k}$. The side *AD* is rotated by an acute angle α in the plane of the parallelogram so that *AD* becomes *AD*′. If *AD*′ makes a right angle with the side *AB*, then the cosine of the angle α is given by

(A) $\frac{8}{9}$ **(B)** $\frac{\sqrt{17}}{9}$

(C) $\frac{1}{9}$ **(D)** $\frac{4\sqrt{5}}{9}$ **[IIT-JEE 2010]**

Solution:

$$\overrightarrow{AD}=\overrightarrow{AB}\times(\overrightarrow{AB}\times\overrightarrow{AD})=5(61\hat{i}-10\hat{j}-21\text{k})$$

$$\Rightarrow\cos\alpha=\frac{|\overrightarrow{AD'}\cdot\overrightarrow{AD}|}{|\overrightarrow{AD'}||\overrightarrow{AD}|}=\frac{\sqrt{17}}{9}$$

Hence, the correct answer is option (B).

12. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

(A) $\hat{i}-3\hat{j}+3\hat{k}$ **(B)** $-3\hat{i}-3\hat{j}+\hat{k}$

(C) $3\hat{i}-\hat{j}+3\hat{k}$ **(D)** $\hat{i}+3\hat{j}-3\hat{k}$

[IIT-JEE 2011]

Solution:

$$\vec{v}=\lambda\vec{a}+\mu\vec{b}$$
$$=\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}-\hat{j}+\hat{k})$$
$$=(\lambda+\mu)\hat{i}+(\lambda-\mu)\hat{j}+(\lambda+\mu)\hat{k}$$

Projection of $\vec{v}$ on $\vec{c}$,

$$\frac{\vec{v}\cdot\vec{c}}{|\vec{c}|}=\frac{1}{\sqrt{3}}\Rightarrow\frac{[(\lambda+\mu)\hat{i}+(\lambda-\mu)\hat{j}+(\lambda+\mu)\hat{k}]\cdot(\hat{i}-\hat{j}-\hat{k})}{\sqrt{3}}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow\lambda+\mu-\lambda+\mu-\lambda-\mu=1\Rightarrow\mu-\lambda=1\Rightarrow\lambda=\mu-1$$

$$\vec{v}=(\mu-1)(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}-\hat{j}+\hat{k})=\mu(2\hat{i}+2\hat{k})-\hat{i}-\hat{j}-\hat{k}$$

$$\vec{v}=(2\mu-1)\hat{i}-\hat{j}+(2\mu-1)\hat{k}$$

At $\mu=2$,

$$\vec{v}=3\hat{i}-\hat{j}+3\hat{k}$$

Hence, the correct answer is option (C).

13. The vector(s) which is/are coplanar with vectors $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$ is/are

(A) $\hat{j}-\hat{k}$ **(B)** $-\hat{i}+\hat{j}$

(C) $\hat{i}-\hat{j}$ **(D)** $-\hat{j}+\hat{k}$ **[IIT-JEE 2011]**

Solution: Let $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$.

Any vector in the plane of $\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ is

$$\vec{r}=\lambda\vec{a}+\mu\vec{b}$$
$$=\lambda(\hat{i}+\hat{j}+2\hat{k})+\mu(\hat{i}+2\hat{j}+\hat{k})$$
$$=(\lambda+\mu)\hat{i}+(\lambda+2\mu)\hat{j}+(2\lambda+\mu)\hat{k}$$

Also, $\vec{r}\cdot\vec{c}=0$

$$\Rightarrow(\lambda+\mu)\cdot1+(\lambda+2\mu)\cdot1+(2\lambda+\mu)\cdot1=0$$
$$\Rightarrow4\lambda+4\mu=0$$
$$\Rightarrow\lambda+\mu=0$$

Now,

$$[\vec{r}\quad\vec{a}\quad\vec{b}]=0$$

So, vectors $\hat{j}-\hat{k}$ and $-\hat{j}+\hat{k}$ satisfy this.

Hence, the correct answers are options (A) and (D).

14. Let $\vec{a}=-\hat{i}-\hat{k}$, $\vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{r}\cdot\vec{a}=0$, then the value of $\vec{r}\cdot\vec{b}$ is ______. **[IIT-JEE 2011]**

Solution:

$$\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$$

Taking cross with *a*

$$\vec{a}\times(\vec{r}\times\vec{b})=\vec{a}\times(\vec{c}\times\vec{b})$$
$$(\vec{a}\cdot\vec{b})\vec{r}-(\vec{a}\cdot\vec{r})\vec{b}=\vec{a}\times(\vec{c}\times\vec{b})$$

Using $\vec{a}\cdot\vec{r}=0$ and $\vec{a}\cdot\vec{b}=1$, we get

$$\vec{r}=-3\hat{i}+6\hat{j}+3\hat{k}$$
$$\vec{r}\cdot\vec{b}=3+6=9$$

Hence, the correct answer is (9).

15. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors satisfying $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$, then $|2\vec{a}+5\vec{b}+5\vec{c}|$ is ______. **[IIT-JEE 2012]**

Solution:

$$|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=3(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2)-|\vec{a}+\vec{b}+\vec{c}|^2$$

$$\Rightarrow3\times3-|\vec{a}+\vec{b}+\vec{c}|^2=9$$
$$\Rightarrow|\vec{a}+\vec{b}+\vec{c}|=0\Rightarrow\vec{a}+\vec{b}+\vec{c}=0$$
$$\Rightarrow\vec{b}+\vec{c}=-\vec{a}$$
$$\Rightarrow|2\vec{a}+5(\vec{b}+\vec{c})|=|-3\vec{a}|=3|\vec{a}|=3$$

Hence, the correct answer is (3).

16. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}+\vec{b}|=\sqrt{29}$ and $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k})=(2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$, then a possible value of $(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$ is

(A) 0 **(B)** 3
(C) 4 **(D)** 8 **[IIT-JEE 2012]**

Solution:

$$\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k})=(2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$$

$$\Rightarrow(\vec{a}+\vec{b})\times(2\hat{i}+3\hat{j}+4\hat{k})=\vec{0}$$

$$\Rightarrow\vec{a}+\vec{b}=\pm(2\hat{i}+3\hat{j}+4\hat{k}) \qquad (\text{as } |\vec{a}+\vec{b}|=\sqrt{29})$$

$$\Rightarrow(\vec{a}+\vec{b})\cdot(-7\hat{i}+2\hat{j}+3\hat{k})$$

$$=\pm(-14+6+12)=\pm4$$

Hence, the correct answer is option (C).

17. Let $\overrightarrow{PR}=3\hat{i}+\hat{j}-2\hat{k}$ and $\overrightarrow{SQ}=\hat{i}-3\hat{j}-4\hat{k}$ determine diagonals of a parallelogram $PQRS$ and $\overrightarrow{PT}=\hat{i}+2\hat{j}+3\hat{k}$ be another vector. Then, the volume of the parallelepiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is

(A) 5 **(B)** 20
(C) 10 **(D)** 30

[JEE ADVANCED 2013]

Solution: See Fig. 26.48. Let $\vec{a}$ and $\vec{b}$ be the sides of the parallelogram whose diagonals be $\overrightarrow{PR}$ and $\overrightarrow{SQ}$, as shown in the following figure. Therefore,

$$\overrightarrow{PR}=\vec{a}+\vec{b}=3\hat{i}+\hat{j}-2\hat{k}$$

$$\overrightarrow{SQ}=\vec{a}-\vec{b}=\hat{i}-3\hat{j}-4\hat{k}$$

These imply that

$$\vec{a}=2\hat{i}-\hat{j}-3\hat{k};$$

$$\vec{b}=\hat{i}+2\hat{j}+\hat{k}$$

Therefore, the volume of the parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\overrightarrow{PT}$ is

$$\begin{vmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix}=10$$

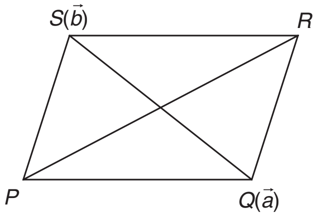


Figure 26.48

Hence, the correct answer is option (C).

18. Consider the set of eight vectors $V=\{a\hat{i}+b\hat{j}+c\hat{k}: a,b,c\in\{-1,1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is ______. **[JEE ADVANCED 2013]**

Solution: See Fig. 26.49. The eight vectors are as shown in the following figure.

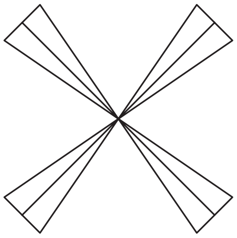


Figure 26.49

The total number of vectors is given by

$$^8C_3=56$$

The total number of coplanar vectors is given by

$$2\times(6\times2)=24$$

That is,

$$56-24=32=2^5$$

Hence, $p=5$.

Hence, the correct answer is (5).

19. Match List I with List II and select the correct answer using the code given below the list:

List I	List II
P. Volume of parallelepiped determined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelepiped determined by vectors $2(\vec{a}\times\vec{b})$, $3(\vec{b}\times\vec{c})$ and $(\vec{c}\times\vec{a})$ is	**1.** 100
Q. Volume of parallelepiped determined by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelepiped determined by vectors $3(\vec{a}+\vec{b})$, $(\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is	**2.** 30
R. Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is	**3.** 24
S. Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a}+\vec{b})$ and $\vec{a}$ is	**4.** 60

Codes:

	P	Q	R	S
(A)	4	2	3	1
(B)	2	3	1	4
(C)	3	4	1	2
(D)	1	4	3	2

[JEE ADVANCED 2013]

Solution:

(P) We have $[\vec{a}\,\vec{b}\,\vec{c}]=2$. Therefore,

$$V=[2(\vec{a}\times\vec{b})\;3(\vec{b}\times\vec{c})\;\vec{c}\times\vec{a}]$$

$$=6[\vec{a}\,\vec{b}\,\vec{c}]^2$$

$$=24$$

(Q) We have $[\vec{a}\,\vec{b}\,\vec{c}]=5$. Therefore,

$$V=[3(\vec{a}+\vec{b})\;\vec{b}+\vec{c}\;2(\vec{c}+\vec{a})]$$

$$=6[\vec{a}+\vec{b}\;\vec{b}+\vec{c}\;\vec{c}+\vec{a}]$$

$$=12[\vec{a}\,\vec{b}\,\vec{c}]$$

$$=60$$

(R) We have $|\vec{a}\times\vec{b}| = 40$. Therefore,

$$\begin{aligned}A &= \frac{1}{2}|(2\vec{a}+3\vec{b})\times(\vec{a}-\vec{b})|\\ &= \frac{1}{2}\cdot 5|\vec{a}\times\vec{b}|\\ &= \frac{5}{2}\times 40\\ &= 100\end{aligned}$$

(S) We have $|\vec{a}\times\vec{b}| = 30$. Therefore,

$$\begin{aligned}A &= |(\vec{a}+\vec{b})\times\vec{a}|\\ &= |\vec{b}\times\vec{a}|\\ &= 30\end{aligned}$$

Hence, the correct answer is option (C).

20. Let $\vec{x}$, $\vec{y}$ and $\vec{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\pi/3$. If $\vec{a}$ is a non-zero vector perpendicular to $\vec{x}$ and $\vec{y}\times\vec{z}$ and $\vec{b}$ is a non-zero vector perpendicular to $\vec{y}$ and $\vec{z}\times\vec{x}$, then

(A) $\vec{b} = (\vec{b}\cdot\vec{z})(\vec{z}-\vec{x})$ **(B)** $\vec{a} = (\vec{a}\cdot\vec{y})(\vec{y}-\vec{z})$

(C) $\vec{a}\cdot\vec{b} = -(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})$ **(D)** $\vec{a} = (\vec{a}\cdot\vec{y})(\vec{z}-\vec{y})$

[JEE ADVANCED 2014]

Solution: See Fig. 26.50.

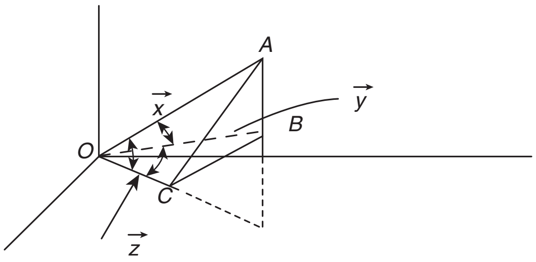


Figure 26.50

$$\angle AOB = \angle BOC = \angle COA = \frac{\pi}{3}$$

According to the question

$$\begin{aligned}\vec{a} &= \lambda\{(\vec{x}\cdot\vec{z})\vec{y} - (\vec{x}\cdot\vec{y})\vec{z}\}\\ &= \lambda\left\{\left(2\cos\frac{\pi}{3}\right)\vec{y} - \left(2\cos\frac{\pi}{3}\right)\vec{z}\right\} = \lambda(\vec{y}-\vec{z})\end{aligned}$$

(Since, $\vec{\alpha}\times(\vec{\beta}\times\vec{\gamma}) = (\vec{\alpha}\cdot\vec{\gamma})\vec{\beta} - (\vec{\alpha}\cdot\vec{\beta})\vec{\gamma}$)

$$\begin{aligned}\vec{b} &= \mu\{(\vec{y}\cdot\vec{x})\vec{z} - (\vec{y}\cdot\vec{z})\vec{x}\}\\ &= \mu\{\vec{z}-\vec{x}\}\end{aligned}$$

Now

$$\vec{a}\cdot\vec{y} = \lambda\{2-1\}$$

Therefore,

$$\lambda = \vec{a}\cdot\vec{y}$$

Therefore,

$$\vec{a} = \vec{a}\cdot\vec{y}(\vec{y}-\vec{z}) \qquad (1)$$

Similarly

$$\vec{b} = \vec{b}\cdot\vec{z}(\vec{z}-\vec{x}) \qquad (2)$$

Now

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})\{\vec{y}\cdot\vec{z} - \vec{y}\cdot\vec{x} - \vec{z}\cdot\vec{z} + \vec{z}\cdot\vec{x}\}\\ &= (\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z})\{1-\cancel{1}-2+\cancel{1}\}\\ &= -(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{z}) \qquad (3)\end{aligned}$$

Therefore, from Eqs. (1), (2) and (3), we can conclude that the correct options are (A), (B) and (C).

Hence, the correct answers are options (A), (B) and (C).

21. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\pi/3$. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c} = p\vec{a}+q\vec{b}+r\vec{c}$ where p, q and r are scalars, then the values of $\dfrac{p^2+2q^2+r^2}{q^2}$ is ______.

[JEE ADVANCED 2014]

Solution: See Fig. 26.51. Given $\vec{a}\times\vec{b}+\vec{b}\times\vec{c} = p\vec{a}+q\vec{b}+r\vec{c}$ (1)

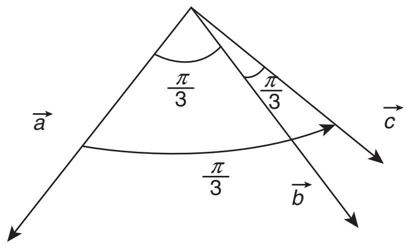


Figure 26.51

Taking dot product with $\vec{a}$. Therefore,

$$0+\vec{a}\cdot\vec{b}\times\vec{c} = p(1\cdot 1\cdot\cos 0) + q\left(1\cdot 1\cdot\cos\frac{\pi}{3}\right) + r\left(1\cdot 1\cdot\cos\frac{\pi}{3}\right)$$

$$\Rightarrow \vec{a}\cdot\vec{b}\times\vec{c} = p+\frac{q}{2}+\frac{r}{2} \qquad (2)$$

Taking the dot product of (1) with $\vec{b}$.

$$0+0 = \frac{p}{2}+q+\frac{r}{2} \qquad (3)$$

Taking the dot product of Eq. (1) with $\vec{c}$

$$\vec{c}\cdot\vec{a}\times\vec{b}+0 = \frac{p}{2}+\frac{q}{2}+r \qquad (4)$$

From Eqs. (2) and (4), we get

$$p+\cancel{\frac{q}{2}}+\frac{r}{2} = \frac{p}{2}+\cancel{\frac{q}{2}}+r$$

$$\frac{p}{2} = \frac{r}{2} \Rightarrow p = r$$

Now from Eq. (3)

$$\begin{aligned}0 &= \frac{r}{2}+q+\frac{r}{2}\\ &\Rightarrow q = -r\end{aligned}$$

Now

$$\frac{p^2+2q^2+r^2}{q^2} = \frac{r^2+2(-r)^2+r^2}{(-r)^2} = \frac{4\cancel{r^2}}{\cancel{r^2}} = 4$$

Hence, the correct answer is (4).

22. Let ΔPQR be a triangle. Let $\vec{a}=\overrightarrow{QR}$, $\vec{b}=\overrightarrow{RP}$, and $\vec{c}=\overrightarrow{PQ}$. If $|\vec{a}|=12$, $|\vec{b}|=4\sqrt{3}$ and $\vec{b}\cdot\vec{c}=24$, then which of the following is (are) true?

(A) $\frac{|\vec{c}|^2}{2}-|\vec{a}|=12$ **(B)** $\frac{|\vec{c}|^2}{2}+|\vec{a}|=30$

(C) $|\vec{a}\times\vec{b}+\vec{c}\times\vec{a}|=48\sqrt{3}$ **(D)** $\vec{a}\cdot\vec{b}=-72$

[JEE ADVANCED 2015]

Solution: See Fig. 26.52.

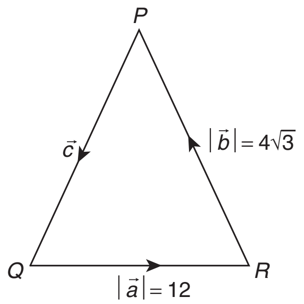


Figure 26.52

For a triangle, we have $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$$\Rightarrow \vec{a}=-(\vec{b}+\vec{c}) \quad (1)$$

$$\Rightarrow |\vec{a}|^2=|\vec{b}|^2+|\vec{c}|^2+2\vec{b}\cdot\vec{c}$$

$$\Rightarrow |\vec{a}|^2=48+|\vec{c}|^2+48$$

$$\Rightarrow |\vec{c}|^2=|\vec{a}|^2-96=144-96$$

$$\Rightarrow |\vec{c}|^2=48$$

$$\Rightarrow |\vec{c}|=\sqrt{48}=4\sqrt{3}$$

Therefore,

$$\frac{|\vec{c}|^2}{2}-|\vec{a}|=24-12=12$$

So, option (A) is correct.

$$\frac{|\vec{c}|^2}{2}+|\vec{a}|=24+12=36$$

Also

$$\vec{a}+\vec{b}=-\vec{c}\Rightarrow \vec{a}\cdot\vec{b}=-72$$

From Eq. (1), we get

$$(\vec{a}\times\vec{a})=-(\vec{a}\times\vec{b}+\vec{a}\times\vec{c})\Rightarrow \vec{a}\times\vec{b}=\vec{c}\times\vec{a}$$

$$|\vec{a}\times\vec{b}+\vec{a}\times\vec{c}|=|2(\vec{a}\times\vec{b})|=2|\vec{a}||\vec{b}|\sin\theta=96\sqrt{3}\sqrt{1-\left(\frac{-72}{48\sqrt{3}}\right)^2}$$

$$=96\sqrt{3}\sqrt{1-\left(\frac{\sqrt{3}}{12}\right)^2}=48\sqrt{3}$$

Hence, the correct answers are options (A), (C) and (D).

23. Suppose that $\vec{p},\vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}$, $\vec{q}$ and $\vec{r}$ be 4, 3 and 5, respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r})$, $(\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x, y and z, respectively, then the value of $2x+y+z$ is ______.

[JEE ADVANCED 2015]

Solution:

$$\vec{s}=4\vec{p}+3\vec{q}+5\vec{r}$$

Also,

$$\vec{s}=x(-\vec{p}+\vec{q}+\vec{r})+y(\vec{p}-\vec{q}+\vec{r})+z(-\vec{p}-\vec{q}+\vec{r})$$

$$\Rightarrow \vec{s}=(-x+y-z)\vec{p}+(x-y-z)\vec{q}+(x+y+z)\vec{r}$$

$$\left.\begin{aligned}-x+y-z&=4\\ \Rightarrow x-y-z&=3\\ x+y+z&=5\end{aligned}\right\}$$

$$\Rightarrow z=\frac{-7}{2},\ x=4,\ y=\frac{9}{2}\Rightarrow 2x+y+z=9$$

Hence, the correct answer is (9).

24. Let $\hat{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ be a unit vector in $\mathbb{R}^2$ and $\hat{\omega}=\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})$. Given that there exists a vector $\vec{v}$ in $\mathbb{R}^3$ such that $|\hat{u}\times\hat{v}|=1$ and $\hat{\omega}\cdot(\hat{u}\times\vec{v})=1$. Which of the following statement(s) is(are) correct?

(A) There is exactly one choice for such $\vec{v}$.
(B) There are infinitely many choices for such $\vec{v}$.
(C) If $\hat{u}$ lies in the xy-plane, then $|u_1|=|u_2|$.
(D) if $\hat{u}$ lies in the xz-plane, then $2|u_1|=|u_3|$.

[JEE ADVANCED 2016]

Solution: We have

$$\hat{u}=u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$$

$$\Rightarrow |\hat{u}|=1=\sqrt{u_1^2+u_2^2+u_3^2}$$

$$\Rightarrow u_1^2+u_2^2+u_3^2=1$$

Also, it is given that

$$\hat{\omega}=\frac{1}{\sqrt{6}}(\hat{i}+\hat{j}+2\hat{k})\Rightarrow |\hat{\omega}|=1$$

Now,

$$|\hat{u}\times\vec{v}|=1$$

$$\Rightarrow |\hat{u}||\vec{v}|\sin\theta=1\Rightarrow |\vec{v}|=\frac{1}{\sin\theta}$$

which shows that there are infinitely many possible values exist for $\vec{v}$ (here θ is angle between the vectors $\vec{v}$ and $\hat{u}$.
Hence, option (B) is correct.
Now,

$$\hat{\omega}\cdot(\hat{u}\times\vec{v})=1$$

$$|\hat{\omega}\cdot(\hat{u}\times\vec{v})|=1$$

$$\Rightarrow |\hat{\omega}||\hat{u}\times\vec{v}|\cos\alpha=1$$

where α is the angle between $\hat{\omega}$ and $\hat{u}\times\vec{v}$. Therefore,

$$(1)(1)\cos\alpha = 1$$
$$\Rightarrow \alpha = 0$$

which means that $\hat{\omega}$ and $\hat{u}\times\vec{v}$ are parallel vector or $\hat{\omega}$ is perpendicular vector to $\hat{u}$ and $\vec{v}$.

$$\hat{u}\cdot\hat{\omega} = 0$$
$$(u_1\hat{i}+u_2\hat{j}+u_3\hat{k})\cdot\left(\frac{(\hat{i}+\hat{j}+2\hat{k})}{\sqrt{6}}\right) = 0$$
$$\Rightarrow u_1+u_2+2u_3 = 0$$

If $\hat{u}$ lies in xy-plane then $u_3 = 0$. Therefore,

$$u_1+u_2 = 0 \Rightarrow u_1 = -u_2 \Rightarrow |u_1| = |u_2|$$

Hence, option (C) is correct.

Hence, the correct answers are options (B) and (C).

Practice Exercise 1

1. Any four non-zero vector will always be

(A) Linearly dependent **(B)** Linearly independent
(C) Either 'A' or 'B' **(D)** None of these

2. If $\vec{a}$ and $\vec{b}$ are reciprocal vectors, then

(A) $\vec{a}\cdot\vec{b} = 0$ **(B)** $\vec{a}\cdot\vec{b} = -1$
(C) $\vec{a}\cdot\vec{b} = 1$ **(D)** None of these

3. If $\vec{a}+\vec{b} = \vec{p}$ and $\vec{a}-\vec{b} = \vec{q}$, then

(A) $|\vec{a}|^2+|\vec{b}|^2 = |\vec{p}|^2+|\vec{q}|^2$
(B) $|\vec{a}|^2-|\vec{b}|^2 = |\vec{p}|^2-|\vec{q}|^2$
(C) $2(|\vec{a}^2|+|\vec{b}|^2) = |\vec{p}|^2+|\vec{q}|^2$
(D) $2(|\vec{a}|^2-|\vec{b}|^2) = |\vec{p}|^2-|\vec{q}|^2$

4. If three unit vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ satisfy $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, then angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{3}$ **(B)** $\frac{2\pi}{3}$
(C) $\frac{\pi}{6}$ **(D)** $\frac{5\pi}{6}$

5. Projection of $\hat{i}+2\hat{j}+3\hat{k}$ on $\hat{i}-2\hat{j}-2\hat{k}$ is equal to

(A) 3 **(B)** −3
(C) 9 **(D)** −9

6. If $\vec{a}$ and $\vec{b}$ are two non-collinear unit vectors, then projection of $\vec{a}+\vec{b}$ on $\vec{a}-\vec{b}$ is equal to

(A) 2 **(B)** −2
(C) 1 **(D)** None of these

7. $ABCD$ is a parallelogram with $\overrightarrow{AC} = \hat{i}-2\hat{j}+\hat{k}$ and $\overrightarrow{BD} = -\hat{i}+2\hat{j}-5\hat{k}$. Area of this parallelogram is equal to

(A) $\sqrt{5}/2$ sq. units **(B)** $2\sqrt{5}$ sq. units
(C) $4\sqrt{5}$ sq. units **(D)** $\sqrt{5}$ sq. units

8. If $\vec{a} = x\hat{i}+(x-1)\hat{j}+\hat{k}$ and $\vec{b} = (x+1)\hat{i}+\hat{j}+a\hat{k}$ always make an acute angle with each other for every value of $x\in R$, then

(A) $a\in(-\infty, 2)$ **(B)** $a\in(2,\infty)$
(C) $a\in(-\infty, 1)$ **(D)** $a\in(1,\infty)$

9. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$. Then $\lambda(\vec{b}\times\vec{a})\times\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = \vec{0}$, where λ is equal to

(A) 1 **(B)** 2
(C) −1 **(D)** −2

10. If $\vec{a}\times\vec{b} = \vec{c}$, $\vec{b}\times\vec{c} = \vec{a}$ where $\vec{c}\neq\vec{0}$, then

(A) $|\vec{a}| = |\vec{c}|$, $|\vec{b}| = 1$ **(B)** $|\vec{a}| = |\vec{b}|$, $|\vec{c}| = 1$
(C) $|\vec{b}| = |\vec{c}|$, $|\vec{a}| = 1$ **(D)** $|\vec{a}| = |\vec{b}|$, $|\vec{c}| = 1$

11. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be unit vectors such that $\vec{a}+\vec{b}+\vec{c} = \vec{x}$, $\vec{a}\cdot\vec{x} = 1$, $\vec{b}\cdot\vec{x} = \frac{3}{2}$, $|\vec{x}| = 2$. Then angle between $\vec{c}$ and $\vec{x}$ is

(A) $\cos^{-1}\left(\frac{1}{4}\right)$ **(B)** $\cos^{-1}\left(\frac{3}{4}\right)$
(C) $\cos^{-1}\left(\frac{3}{8}\right)$ **(D)** $\cos^{-1}\left(\frac{5}{8}\right)$

12. $ABCD$ is a parallelogram A_1 and B_1 are the mid-points of side BC and CD, respectively. If $\overrightarrow{AA_1}+\overrightarrow{AB_1} = \lambda\overrightarrow{AC}$, then λ is equal to

(A) 1/2 **(B)** 1
(C) 3/2 **(D)** 2

13. If $\vec{a} = \vec{b}+\vec{c}$, $\vec{b}\times\vec{d} = \vec{0}$, $\vec{c}\cdot\vec{d} = 0$, then the vector $\frac{\vec{d}\times(\vec{a}\times\vec{d})}{|\vec{d}|^2}$ is always equal to

(A) $\vec{a}$ **(B)** $\vec{d}$
(C) $\vec{b}$ **(D)** $\vec{c}$

14. For any two vectors $\vec{a}$ and $\vec{b}$, the expression $(\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i})+(\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j})+(\vec{a}\times\hat{k})$, $(\vec{b}\times\hat{k})$ is always equal to

(A) $\vec{a}\cdot\vec{b}$ **(B)** $2\vec{a}\cdot\vec{b}$
(C) Zero **(D)** None of these

15. Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three non-zero vectors such that $|\vec{c}| = 1$, angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$ and $\vec{c}$ is perpendicular to $\vec{a}$ and $\vec{b}$. Then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \lambda(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$ where λ is equal to

(A) 1/2 **(B)** 1/4
(C) 1 **(D)** 2

16. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors such that $\vec{a}\times\vec{b} = \vec{c}$, $\vec{b}\times\vec{c} = \vec{a}$, $\vec{c}\times\vec{a} = \vec{b}$. Then

(A) $|\vec{a}| = |\vec{b}| = |\vec{c}|$ **(B)** $|\vec{a}|\neq|\vec{b}| = |\vec{c}|$
(C) $|\vec{a}| = |\vec{b}|\neq|\vec{c}|$ **(D)** $|\vec{a}|\neq|\vec{b}|\neq|\vec{c}|$

17. Let $\vec{a}$ and $\vec{b}$ be unit vectors such that $|\vec{a}+\vec{b}| = \sqrt{3}$. Then the value of $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b})$ is equal to

(A) $\frac{11}{2}$ **(B)** $\frac{13}{2}$

(C) $\frac{39}{2}$ (D) $\frac{23}{2}$

18. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three unit vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$. If the angle between $\vec{b}$ and $\vec{c}$ is $\pi/4$ then $\vec{a}=\lambda(\vec{b}\times\vec{c})$, where '$\lambda$' is equal to
(A) ± 1 (B) $\pm\sqrt{2}$
(C) ± 2 (D) None of these

19. $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$, $\vec{c}=\hat{i}+\hat{j}-2\hat{k}$. A vector coplanar with $\vec{b}$ and $\vec{c}$, whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$ is
(A) $2\hat{i}+3\hat{j}-3\hat{k}$ (B) $-2\hat{i}-\hat{j}+\hat{k}$
(C) $2\hat{i}+3\hat{j}+3\hat{k}$ (D) $2\hat{i}+\hat{j}+5\hat{k}$

20. Let $\hat{a}$ and $\hat{b}$ be unit vector that are mutually perpendicular. Then for any arbitrary $\vec{r}$
(A) $\vec{r}=(\vec{r}\cdot\hat{a})\hat{a}+(\vec{r}\cdot\hat{b})\hat{b}+(\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$
(B) $\vec{r}=(\vec{r}\cdot\hat{a})\hat{a}-(\vec{r}\cdot\hat{b})\hat{b}-(\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$
(C) $\vec{r}=(\vec{r}\cdot\hat{a})\hat{a}-(\vec{r}\cdot\hat{b})\hat{b}+(\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$
(D) None of these

21. The line $\vec{r}=\vec{a}\times\lambda\vec{b}$ will not meet the plane $\vec{r}\times\vec{n}=q$, provided
(A) $\vec{b}\times\vec{n}=0,\ \vec{a}\cdot\vec{n}=q$ (B) $\vec{b}\times\vec{n}\neq 0,\ \vec{a}\cdot\vec{n}=q$
(C) $\vec{b}\times\vec{n}=0,\ \vec{a}\cdot\vec{n}\neq q$ (D) $\vec{b}\times\vec{n}\neq 0,\ \vec{a}\cdot\vec{n}=q$

22. If the projection of point $P(\vec{p})$ on the plane $\vec{r}\cdot\vec{n}=q$ is the point $S(\vec{s})$, then
(A) $\vec{s}=\frac{(q-\vec{p}\cdot\vec{n})}{|\vec{n}|^2}$ (B) $\vec{s}=\vec{p}+\frac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$
(C) $\vec{s}=\vec{p}-\frac{(\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$ (D) $\vec{s}=\vec{p}-\frac{(q-\vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$

23. Let $\vec{a}$ and $\vec{b}$ be unit vectors that are perpendicular to each other. Then $[\vec{a}+(\vec{a}\times\vec{b}),\ \vec{b}+(\vec{a}\times\vec{b}),\ \vec{a}\times\vec{b}]$ will always be equal to
(A) 1 (B) Zero
(C) –1 (D) None of these

24. A, B, C and D are any four points in the space. If $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|=\lambda\Delta_{ABC}$, where Δ_{ABC} is the area of triangle ABC, then λ is equal to
(A) 2 (B) 1/2
(C) 4 (D) 1/4

25. If $\sec^2 A\,\hat{i}+\hat{j}+\hat{k}$, $i+\sec^2 B\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\sec^2 C\hat{k}$ are coplanar, then $\cot^2 A+\cot^2 B+\cot^2 C$ is
(A) equal to 1 (B) equal to 2
(C) equal to 0 (D) not defined

26. If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors and p, q, r are vectors defined by the relations $\vec{p}=\frac{\vec{b}\times\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$, $\vec{q}=\frac{\vec{c}\times\vec{a}}{[\vec{a}\vec{b}\vec{c}]}$, $\vec{r}=\frac{\vec{a}\times\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$, then the value of expression $(\vec{a}+\vec{b})\,\vec{p}+(\vec{b}+\vec{c})\,\vec{q}+(\vec{c}+\vec{a})\cdot\vec{r}$ is equal to
(A) 0 (B) 1
(C) 2 (D) 3

27. The value of $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2$ is
(A) a^2 (B) $2a^2$
(C) $3a^2$ (D) None of these

28. If $\vec{A},\vec{B},\vec{C}$ are non-coplanar vectors, then $\frac{\vec{A}\cdot\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}\cdot\vec{B}}+\frac{\vec{B}\cdot\vec{A}\times\vec{C}}{\vec{C}\cdot\vec{A}\times\vec{B}}$ is equal to
(A) 3 (B) 0
(C) 1 (D) None of these

29. Consider ΔABC and $\Delta A_1B_1C_1$ in such a way that $\overrightarrow{AB}=\overrightarrow{A_1B_1}$ and M, N, M_1, N_1 be the mid-points of AB, BC, A_1B_1 and B_1C_1, respectively. Then
(A) $\overrightarrow{MM_1}=\overrightarrow{NN_1}$ (B) $\overrightarrow{CC_1}=\overrightarrow{MM_1}$
(C) $\overrightarrow{CC_1}=\overrightarrow{NN_1}$ (D) $\overrightarrow{MM_1}=\overrightarrow{BB_1}$

30. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=x_1\hat{i}+x_2\hat{j}+x_3\hat{k}$, where $x_1, x_2, x_3\in\{-3,-2,-1,0,1,2\}$. Then number of possible vectors $\vec{b}$ such that $\vec{a}$ and $\vec{b}$ are mutually perpendicular is
(A) 25 (B) 28
(C) 22 (D) None of these

31. Let a, b, c be distinct and non-negative. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is
(A) AM of a and b (B) GM of a and b
(C) HM of a and b (D) equal to zero

32. If $|\vec{a}|=2$, $|\vec{b}|=5$ and $\vec{a}\cdot\vec{b}=0$, then $\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times(\vec{a}\times\vec{b})))))$ is equal to
(A) $320\vec{a}$ (B) $320\vec{b}$
(C) $-320\hat{b}$ (D) $-320\,\hat{a}$

33. If $\vec{b}$ is the vector whose initial point divides the joining of $5\hat{i}$ and $5\hat{j}$ in the ratio k:1 and terminal point is origin. Also, $|\vec{b}|\leq\sqrt{37}$, then the interval in which k lies
(A) $(-\infty,-6]\cup[-1/6,\infty)$ (B) $(-\infty,-6]\cup[1/6,\infty)$
(C) $(-\infty,6]\cup[-1/6,\infty)$ (D) $(\infty,6]\cup[-1/6,\infty)$

34. If "a" is real constant and A, B, C are variable angles and, $\sqrt{a^2-4}\tan A+a\tan B+\sqrt{a^2+4}\tan C=6a$, then the least value of $\tan^2 A+\tan^2 B+\tan^2 C$ is
(A) 10 (B) 11
(C) 12 (D) 13

35. If $a_m\hat{i}+b_m\hat{j}+c_m\hat{k}$, $m=1, 2, 3$, are pairwise perpendicular unit vectors, then $\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$ is equal to
(A) 0 (B) 1 or –1
(C) 3 or –3 (D) 4 or –4

36. If $\hat{a}$, $\hat{b}$, $\hat{c}$ are three non-coplanar unit vectors, then $[\hat{a}\,\vec{p}\,\vec{q}]\hat{a}+[\hat{b}\,\vec{p}\,\vec{q}]\hat{b}+[\hat{c}\,\vec{p}\,\vec{q}]\hat{c}$ is equal to
(A) $(\hat{a}+\hat{b}+\hat{c})\times(\vec{p}\times\vec{q})$ (B) $\hat{a}+\hat{b}+\hat{c}+\vec{q}+\vec{q}$
(C) $\vec{p}+\vec{q}$ (D) $\vec{p}\times\vec{q}$

37. The vector $\vec{a}(x)=\cos x\hat{i}+\sin x\hat{j}$ and $\vec{b}(x)=x\hat{i}+\sin x\hat{j}$ are collinear for

(A) Unique value of x, $0 < x < \pi/6$
(B) Unique value of x, $\pi/6 < x < \pi/3$
(C) No value of x
(D) Infinity many value of x, $0 < x < \pi/2$

38. The vectors $2\hat{i}+3\hat{j}$, $5\hat{i}+6\hat{j}$ and $8\hat{i}+\lambda\hat{j}$ have their initial points at (1, 1), the value of λ so that the vectors terminate on one straight line is

(A) 0 **(B)** 3
(C) 6 **(D)** 9

39. Given that $\vec{a}$ is a perpendicular to $\vec{b}$ and p is a non-zero scalar, then a vector $\vec{r}$ satisfying $p\vec{r}+(\vec{r}\cdot\vec{b})\vec{a}=\vec{c}$ is given by

(A) $\vec{r}=\dfrac{\vec{c}}{p}-\dfrac{\vec{c}\cdot\vec{r}}{p^2}\vec{a}$ **(B)** $\vec{r}=\dfrac{\vec{c}}{p}+\dfrac{\vec{c}\cdot\vec{r}}{p^2}\vec{a}$
(C) $\vec{r}=\dfrac{\vec{c}}{p}-\dfrac{\vec{c}\cdot\vec{b}}{p^2}\vec{a}$ **(D)** None of these

40. Let P is any arbitrary point on the circumcircle of a given equilateral triangle of side length 'ℓ' units. Then $|\overrightarrow{PA}|^2+|\overrightarrow{PB}|^2+|\overrightarrow{PC}|^2$ is always equal to

(A) $2\ell^2$ **(B)** $2\sqrt{3}\ell^2$
(C) ℓ^2 **(D)** $3\ell^2$

41. Let $\vec{a}$ and $\vec{b}$ are two non-collinear vector such that $|\vec{a}|=1$. The angle of a triangle whose two sides are represented by the vector $\sqrt{3}(\vec{a}\times\vec{b})$ and $\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}$ are

(A) $\dfrac{\pi}{2},\dfrac{\pi}{4},\dfrac{\pi}{4}$ **(B)** $\dfrac{\pi}{2},\dfrac{\pi}{3},\dfrac{\pi}{6}$
(C) $\dfrac{\pi}{2},\dfrac{\pi}{12},\dfrac{\pi}{12}$ **(D)** None of these

42. E and F are the interior points on the sides BC and CD of a parallelogram $ABCD$. Let $\overrightarrow{BE}=4\overrightarrow{EC}$ and $\overrightarrow{CF}=4\overrightarrow{FD}$. If the line EF meets the diagonal AC in G, then $\overrightarrow{AG}=\lambda\overrightarrow{AC}$ where λ is equal to

(A) $\dfrac{21}{25}$ **(B)** $\dfrac{1}{3}$
(C) $\dfrac{7}{13}$ **(D)** $\dfrac{21}{5}$

43. If $\begin{vmatrix}(x-a)^2 & (x-b)^2 & (x-c)^2\\(y-a)^2 & (y-b)^2 & (y-c)^2\\(z-a)^2 & (z-b)^2 & (z-c)^2\end{vmatrix}=0$ and the vector $X=(x^2,x,1)$, $Y=(y^2,y,1)$ and $Z=(z^2,z,1)$ are non-coplanar, then the vectors $(a^2,a,1)$, $(b^2,b,1)$ and $(c^2,c,1)$ are

(A) Coplanar **(B)** Non-coplanar
(C) Collinear **(D)** Non-collinear

44. If b and c are any two perpendicular unit vectors and a is any vector, then

$$(a\cdot b)b+(a\cdot c)c+\frac{a\cdot(b\times c)}{|b\times c|}(b\times c)=$$

(A) b **(B)** a
(C) c **(D)** $b+c$

45. If the lines $\vec{r}=\vec{a}+t(\vec{b}\times\vec{c})$ and $\vec{r}=\vec{b}+s(\vec{c}\times\vec{a})$ intersect (t and s are scalars), then

(A) $\vec{a}\cdot\vec{c}=0$ **(B)** $\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}$
(C) $\vec{b}\cdot\vec{c}=0$ **(D)** None of these

46. The position vector of a point P is $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ where $x, y, z \in N$ and $\vec{a}=\hat{i}+\hat{j}+\hat{k}$. If $\vec{r}\cdot\vec{a}=10$, then the number of possible positions of P is

(A) 30 **(B)** 72
(C) 66 **(D)** ${}^9C_2$

47. If vectors $ax\hat{i}+3\hat{j}-5\hat{k}$ and $x\hat{i}+2\hat{j}+2ax\hat{k}$ make an acute angle with each other, for all $x\in R$, then a belongs to the interval

(A) $\left(-\dfrac{1}{4},0\right)$ **(B)** (0, 1)
(C) $\left(0,\dfrac{6}{25}\right)$ **(D)** $\left(-\dfrac{3}{25},0\right)$

48. If $\vec{a}$ and $\vec{b}$ are two unit vectors and θ is the angle between them, then the unit vector along the angular bisector of $\vec{a}$ and $\vec{b}$ will be given by

(A) $\dfrac{\vec{a}-\vec{b}}{2\cos\dfrac{\theta}{2}}$ **(B)** $\dfrac{\vec{a}+\vec{b}}{2\cos\dfrac{\theta}{2}}$
(C) $\dfrac{\vec{a}-\vec{b}}{2\cos\dfrac{\theta}{4}}$ **(D)** None of these

49. If $\sum_{i=1}^{n}\vec{a}_i=\vec{0}$ where $|\vec{a}_i|=1\ \forall i\in n\in N$, then the value of $\sum_{i=1}^{n}\sum_{j=1}^{n}\vec{a}_i\cdot\vec{a}_j$ is

(A) $-n/2$ **(B)** $-n$
(C) $n/2$ **(D)** n

50. If $4\vec{a}+5\vec{b}+9\vec{c}=0$, then $(\vec{a}\times\vec{b})\times[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]$ is equal to

(A) A vector perpendicular to plane of $\vec{a},\vec{b}$ and $\vec{c}$
(B) A scalar
(C) $\vec{0}$
(D) None of these

51. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. If $\vec{c}$ is a vector such that $\vec{a}\cdot\vec{c}=|\vec{c}|$, $|\vec{c}-\vec{a}|=2\sqrt{2}$ and the angle between $(\vec{a}\times\vec{b})$ and $\vec{c}$ is 30°, then the value of $|(\vec{a}\times\vec{b})\times\vec{c}|$ is

(A) 2/3 **(B)** 3/2
(C) 2 **(D)** 3

52. If r, a, b, c are the three non-zero vectors such that $\vec{r}\cdot\vec{a}=\vec{r}\cdot\vec{b}=\vec{r}\cdot\vec{c}=0$, then $[\vec{a}\ \vec{b}\ \vec{c}]$

(A) Is equal to 1 **(B)** Cannot be evaluated
(C) Is equal to zero **(D)** None of these

53. If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are non-zero, non-collinear vectors such that $\vec{d}$ is perpendicular to $\vec{a}$, $\vec{b}$ and $\vec{c}$ and $(\vec{a}\cdot\vec{c})\vec{a}=\vec{c}$, then $\vec{a}\cdot(\vec{b}\times\vec{d})$ is equal to

(A) $\dfrac{\vec{c}\cdot\vec{d}}{|\vec{c}|^2}$ **(B)** $|\vec{c}|^2$
(C) $\dfrac{\vec{a}\cdot\vec{d}}{|\vec{a}|^2}$ **(D)** None of these

54. Prove by vector methods, that the altitudes of a triangle are concurrent.

55. If $\vec{a}$ and $\vec{b}$ are unit vectors, θ is the angle between them, then show that $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a} - \vec{b}|$.

56. Show that the area of the triangle formed by joining the extremities of an oblique side of a trapezium to the mid-point of opposite side is half that of the trapezium.

57. The position vectors of the vertices A, B and C of a tetrahedron $ABCD$ are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}$ and $3\hat{i}$, respectively. The altitude from the vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, then find the position vector of the point E for all its possible positions (Fig. 26.53).

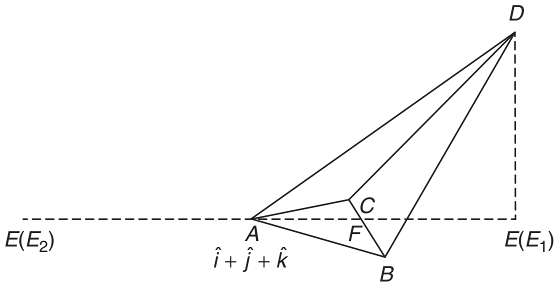


Figure 26.53

58. If the vectors $\vec{b}$, $\vec{c}$, $\vec{d}$ are non-coplanar, then prove that the vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is a vector parallel to $\vec{a}$.

59. Show that the solution of the equation $k\vec{x} + \vec{x}\times\vec{a} = \vec{b}$ where $k \neq 0$ is a scalar and $\vec{a}$ and $\vec{b}$ are any two vectors is given by

$$\vec{x} = \frac{1}{(k^2+a^2)}\left\{\frac{\vec{a}\cdot\vec{b}}{k}\vec{a} + k\vec{b} + \vec{a} + \vec{b}\right\}.$$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If the shortest distance between lines $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda_1(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{r} = 2\hat{i} + 4\hat{j} + 5\hat{k} + \lambda_2(3\hat{i} + 4\hat{j} + 5\hat{k})$ is x, then $\cos^{-1}\cos\sqrt{6}x$ is equal to

(A) $\frac{1}{2}$ **(B)** 0
(C) 1 **(D)** π

2. If $\vec{r} = \lambda(\vec{a}\times\vec{b}) + \mu(\vec{b}\times\vec{c}) + \gamma(\vec{c}\times\vec{a})$ and $[\vec{a}\ \vec{b}\ \vec{c}] = \frac{1}{8}$, then $\lambda + \mu + \gamma$ is

(A) $8(\vec{r}\cdot\vec{a})$ **(B)** $8(\vec{r}\cdot\vec{b})$
(C) $8(\vec{r}\cdot\vec{c})$ **(D)** $8\vec{r}\cdot(\vec{a}+\vec{b}+\vec{c})$

3. In a triangle OAB, E is the mid-point of OB and D is a point on AB such that AD:DB = 2:1. If OD and AE intersect at P, then ratio of $\frac{OP}{PD}$ is equal to

(A) 3:2 **(B)** 2:3
(C) 3:4 **(D)** 4:3

4. If $\vec{x}$ and $\vec{y}$ be unit vectors and $|\vec{z}| = \frac{2}{\sqrt{7}}$ such that $\vec{z} + \vec{z}\times\vec{x} = \vec{y}$, then the angle θ between $\vec{x}$ and $\vec{z}$ is

(A) 30° **(B)** 60°
(C) 90° **(D)** None of these

5. If a non-zero vector $\vec{a}$ is parallel to the line of intersection of the plane P_1 determined by $\hat{i}+\hat{j}$ and $\hat{i}-2\hat{j}$ and plane P_2 determined by vector $2\hat{i}+\hat{j}$ and $3\hat{i}+2\hat{k}$, then angle between $\vec{a}$ and vector $\hat{i}-2\hat{j}+2\hat{k}$ is

(A) $\frac{\pi}{4}$ **(B)** $\frac{\pi}{2}$
(C) $\frac{\pi}{3}$ **(D)** None of these

6. $\vec{a}$ and $\vec{b}$ are non-zero, non-collinear vectors such that $|\vec{a}| = 2$, $\vec{a}\cdot\vec{b} = 1$ and angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$. If $\vec{r}$ is any vector satisfying $\vec{r}\cdot\vec{a} = 2$, $\vec{r}\cdot\vec{b} = 8$, $(\vec{r} + 2\vec{a} - 10\vec{b})(\vec{a}\times\vec{b}) = 4\sqrt{3}$ and is equal to $\vec{r} + 2\vec{a} - 10\vec{b} = \lambda(\vec{a}\times\vec{b})$, then $\lambda =$

(A) $\frac{1}{2}$ **(B)** 2
(C) $\frac{1}{4}$ **(D)** 4

7. Let $\hat{a}$, $\hat{b}$, $\hat{c}$ be unit vectors such that $\hat{a}\times\hat{b} = \hat{c}$ and $\hat{a}\cdot\hat{b} = 0$. Also, $\vec{x}$ is any vectors such that $[\vec{x}\ \hat{b}\ \hat{c}] = 3$, $[\vec{x}\ \hat{c}\ \hat{a}] = 4$ and $[\vec{x}\ \hat{a}\ \hat{b}] = 2$. Then $\vec{x}$ is equal to

(A) $2\hat{a} + 3\hat{b} + \hat{c}$ **(B)** $3\hat{a} + 4\hat{b} + 2\hat{c}$
(C) $\hat{a} + 2\hat{b} + 3\hat{c}$ **(D)** None of these

8. If A, B, C, D be four points in a plane with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, respectively, such that $(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c}) = (\vec{b}-\vec{d})\cdot(\vec{c}-\vec{a}) = 0$, then

(A) D is the circumcentre of the triangle ABC
(B) D is the orthocentre of the triangle ABC
(C) A is the orthocentre of the triangle DBC
(D) A is the circumcentre of the triangle DBC

9. $[\vec{a}\times(\vec{b}+\vec{c}),\ \vec{b}\times(\vec{c}-2\vec{a}),\ \vec{c}\times(\vec{a}+3\vec{b})] = ?$

(A) $[a, b, c]^2$ **(B)** $7[a, b, c]^2$
(C) $-5[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}]$ **(D)** None of these

10. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar unimodular vectors, each inclined with other at an angle 30°, then volume of tetrahedron whose edges are $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

(A) $\frac{\sqrt{3\sqrt{3}-5}}{12}$ **(B)** $\frac{3\sqrt{3}+5}{12}$
(C) $\frac{5\sqrt{2}+3}{12}$ **(D)** None of these

11. Let $\vec{\alpha} = a\hat{i} + b\hat{j} + c\hat{k}$, $\vec{\beta} = b\hat{i} + c\hat{j} + a\hat{k}$ and $\vec{\gamma} = c\hat{i} + a\hat{j} + b\hat{k}$ be three coplanar vectors with $a \neq b$ and $\vec{v} = \hat{i} + \hat{j} + \hat{k}$. Then $\vec{v}$ is perpendicular to

(A) $\vec{\alpha}$ (B) $\vec{\beta}$

(C) $\vec{\gamma}$ (D) None of these

12. Let $\vec{a}$ and $\vec{b}$ are two given perpendicular vectors, which are non-zero. A vector $\vec{r}$ satisfying the equation $\vec{r}\times\vec{b}=\vec{a}$, can be

(A) $\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$ (B) $2\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$

(C) $|\vec{a}|\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$ (D) $|\vec{b}|\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$

13. Let unit vectors $\vec{a}$ and $\vec{b}$ are perpendicular and unit vector $\vec{c}$ be inclined at an angle θ to both $\vec{a}$ and $\vec{b}$. If $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\gamma(\vec{a}\times\vec{b})$, then

(A) $\alpha=\beta$ (B) $1-2\alpha^2=\gamma^2$

(C) $\alpha^2=\dfrac{1+\cos 2\theta}{2}$ (D) $\alpha^2-\beta^2=\gamma^2$

14. If $\vec{a}$ and $\vec{b}$ are two vectors and angle between $\vec{a}$ and $\vec{b}$ is θ, then

(A) $|\vec{a}\times\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$

(B) $|\vec{a}\times\vec{b}|=(\vec{a}\cdot\vec{b})$, if $\theta=\dfrac{\pi}{4}$

(C) $\vec{a}\times\vec{b}=(\vec{a}\cdot\vec{b})\hat{n}$, ($\hat{n}$ is normal unit vector), if $\theta=\dfrac{\pi}{4}$

(D) $(\vec{a}\times\vec{b})\cdot(\vec{a}+\vec{b})=0$

15. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{a}\cdot\vec{b}=2$ and $\vec{a}\times\vec{b}=2\hat{i}+\hat{j}-3\hat{k}$, then

(A) $\vec{a}+\vec{b}=5\hat{i}-4\hat{j}+2\hat{k}$ (B) $\vec{a}+\vec{b}=3\hat{i}+2\hat{k}$

(C) $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$ (D) $\vec{b}=\hat{i}-2\hat{j}-3\hat{k}$

Comprehension Type Questions

Paragraph for Questions 16–18: The vertices of a triangle *ABC* are $A=(2, 0, 2)$, $B=(-1, 1, 1)$ and $C=(1, -2, 4)$. The point *D* and *E* divide the sides *AB* and *CA* in the ratio 1:2, respectively. Another point *F* is taken in space such that perpendicular drawn from *F* on ΔABC meets the triangle at the point of intersection of the line segment *CD* and *BE*, say *P*. If the distance of *F* from the plane of the ΔABC is $\sqrt{2}$ units, then

16. The *PV* of *P* is

(A) $\hat{i}+\hat{j}-3\hat{k}$ (B) $\hat{i}-\hat{j}+3\hat{k}$

(C) $2\hat{i}-\hat{j}-3\hat{k}$ (D) $\hat{i}+\hat{j}+3\hat{k}$

17. The vector $\overrightarrow{PF}$ is

(A) $7\hat{j}+7\hat{k}$ (B) $\dfrac{7}{\sqrt{2}}(\hat{j}+\hat{k})$

(C) $\hat{j}+\hat{k}$ (D) None of these

18. The volume of tetrahedron *ABCF* is

(A) 7 cubic units (B) $\dfrac{3}{5}$ cubic units

(C) $\dfrac{7}{3}$ cubic units (D) $\dfrac{7}{5}$ cubic units

Integer Type Questions

19. Let $A(2\hat{i}+3\hat{j}+5\hat{k})$, $B(-\hat{i},+3\hat{j}+2\hat{k})$ and $C(\lambda\hat{i}+5\hat{j}+\mu\hat{k})$ are vertices of a triangle and its median through *A* is equally inclined to the positive directions of the axes. The value of $2\lambda-\mu$ is equal to ________.

20. Find the value of the expression

$$\frac{\{(\vec{a}-\vec{b})\times(\vec{a}-\vec{b}-\vec{c})\}\cdot(\vec{a}+2\vec{b}-\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Answer Key

Practice Exercise 1

1. (A)	**2.** (C)	**3.** (C)	**4.** (B)	**5.** (B)	**6.** (D)
7. (B)	**8.** (B)	**9.** (D)	**10.** (A)	**11.** (B)	**12.** (C)
13. (D)	**14.** (B)	**15.** (A)	**16.** (A)	**17.** (C)	**18.** (B)
19. (A)	**20.** (A)	**21.** (C)	**22.** (B)	**23.** (A)	**24.** (C)
25. (C)	**26.** (D)	**27.** (B)	**28.** (B)	**29.** (D)	**30.** (A)
31. (B)	**32.** (C)	**33.** (A)	**34.** (C)	**35.** (B)	**36.** (D)
37. (B)	**38.** (D)	**39.** (C)	**40.** (A)	**41.** (B)	**42.** (D)
43. (A)	**44.** (B)	**45.** (B)	**46.** (D)	**47.** (C)	**48.** (B)
49. (A)	**50.** (C)	**51.** (B)	**52.** (C)	**53.** (D)	**57.** $E_1=3\hat{i}-\hat{j}-\hat{k}$, $E_2=-\hat{i}+3\hat{j}+3\hat{k}$

Practice Exercise 2

1. (C)	**2.** (D)	**3.** (A)	**4.** (B)	**5.** (B)	**6.** (B)
7. (B)	**8.** (B)	**9.** (B)	**10.** (A)	**11.** (A), (B), (C)	**12.** (A), (B), (C), (D)
13. (A), (B), (C)	**14.** (A), (B), (C), (D)	**15.** (B), (C)	**16.** (B)	**17.** (C)	**18.** (C)
19. 2	**20.** 3				

Solutions

Practice Exercise 1

1. Four or more than four non-zero vectors are always linearly dependent. Hence, (A) is correct answer.

2. If $\vec{a}$ and $\vec{b}$ are reciprocal, then

$$\vec{a}=\lambda\vec{b}, \lambda\in R^{+} \text{ and } |\vec{a}||\vec{b}|=1$$

$$\Rightarrow |\vec{a}|=|\lambda||\vec{b}|$$

$$\Rightarrow |\lambda|=\frac{|\vec{a}|}{|\vec{b}|}=\frac{1}{|\vec{b}|^2}$$

$$\Rightarrow |\vec{a}|=\frac{1}{|\vec{b}|^2}|\vec{b}|$$

$$\Rightarrow \vec{a}\cdot\vec{b}=\frac{1}{|\vec{b}|^2}|\vec{b}||\vec{b}|\cos 0=1$$

3. $\vec{a}+\vec{b}=\vec{p}$

$$\Rightarrow |\vec{a}+\vec{b}|^2=|\vec{p}|^2$$

$$\Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=|\vec{p}|^2$$

$$\Rightarrow |\vec{a}|^2+|b|^2+2\cdot\vec{a}\cdot\vec{b}=|\vec{p}|^2$$

Also,

$$\vec{a}-\vec{b}=\vec{q}$$

$$\Rightarrow |\vec{a}-\vec{b}|^2=|\vec{q}|^2$$

$$\Rightarrow (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})=|\vec{q}|^2$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2-2\cdot\vec{a}\cdot\vec{b}=|\vec{q}|^2$$

Thus,

$$2(|\vec{a}|^2+|\vec{b}|^2)=|\vec{p}|^2+|\vec{q}|^2$$

4. $\vec{a}+\vec{b}=-\vec{c}$

$$\Rightarrow |\vec{a}+\vec{b}|^2=|\vec{c}|^2=1$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=1$$

$$\Rightarrow \vec{a}\cdot\vec{b}=-\frac{1}{2}$$

$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta=-\frac{1}{2}$$

$$\Rightarrow \cos\theta=-\frac{1}{2}$$

$$\Rightarrow \theta=-\frac{2\pi}{3}$$

5. Projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

Thus, required projection is

$$\frac{(\hat{i}+2\hat{j}+3\hat{k})\cdot(\hat{i}-2\hat{j}-2\hat{k})}{\sqrt{1+4+4}}$$

$$=\frac{1-4-6}{3}=-3$$

6. $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=|\vec{a}|^2-|\vec{b}|^2=0$

Thus, projection of $\vec{a}+\vec{b}$ on $\vec{a}-\vec{b}$ is zero. Hence, (D) is the correct answer.

7. Area vector of parallelogram is

$$\frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{BD}) =\frac{1}{2}\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ -1 & 2 & -5 \end{vmatrix}$$

$$=\frac{1}{2}(8\hat{i}+4\hat{j})$$

$$=4\hat{i}+2\hat{j}$$

Therefore, area of the parallelogram is

$$|4\hat{i}+2\hat{j}|=2\sqrt{5} \text{ sq. units}$$

8. $\vec{a}\cdot\vec{b}=(x\hat{i}+(x-1)\hat{j}+\hat{k})\cdot((x+1)\hat{i}+\hat{j}+a\hat{k})$

$$=x(x+1)+x-1+a$$

$$=x^2+2x+a-1$$

We must have

$$\vec{a}\cdot\vec{b}>0 \forall x\in R$$

$$\Rightarrow x^2+2x+a-1>0 \quad \forall x\in R$$

$$\Rightarrow 4-4(a-1)<0$$

$$\Rightarrow a>2$$

9. See Fig. 26.54. Clearly, $\vec{a}$, $\vec{b}$, and $\vec{c}$ represent the sides of a triangle.

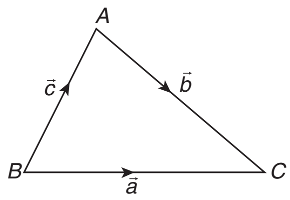


Figure 26.54

It is an area vector, such that

$$\frac{1}{2}\vec{a}\times\vec{b}-\frac{1}{2}\vec{c}\times\vec{d}=\frac{1}{2}\vec{a}\times\vec{c}$$

Thus,

$$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=-\vec{a}\times\vec{b}$$

$$\Rightarrow 2(\vec{a}\times\vec{b})+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=0$$

10. $\vec{a}\times\vec{b}=\vec{c}, \vec{b}\times\vec{c}=\vec{a}$

Taking cross with $\vec{b}$ in first equation, we get

$$\vec{b}\times(\vec{a}\times\vec{b})=\vec{b}\times\vec{c}=\vec{a}$$

$$\Rightarrow |\vec{b}|^2\vec{a}-(\vec{a}\cdot\vec{b})\vec{b}=\vec{a}$$

$$\Rightarrow |\vec{b}|^2=1 \text{ and } \vec{a}\cdot\vec{b}=0$$

Also,

$$|\vec{a}\times\vec{b}|=|\vec{c}|$$

$$\Rightarrow |\vec{a}||\vec{b}|\sin\frac{\pi}{2} = |\vec{c}|$$

$$\Rightarrow |\vec{a}| = |\vec{c}|$$

11. $$\vec{a}+\vec{b}+\vec{c}=\vec{x}$$

Taking dot with $\vec{x}$ on both sides, we get

$$\vec{x}\cdot\vec{a}+\vec{x}\cdot\vec{b}+\vec{x}\cdot\vec{c}=|\vec{x}|^2=4$$

$$\Rightarrow \vec{x}\cdot\vec{c}=\frac{3}{2}$$

If 'θ' be the angle between $\vec{c}$ and $\vec{x}$, then

$$|\vec{x}||\vec{c}|\cos\theta=\frac{3}{2}$$

$$\Rightarrow \cos\theta=\frac{3}{4}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{3}{4}\right)$$

12. See Fig. 26.55. Let PV of A, B, D be $\vec{0}, \vec{b},$ and $\vec{d},$ respectively. Then PV of $C=\vec{b}+\vec{d}$.

Also, PV of $A_1=\vec{b}+\dfrac{\vec{d}}{2}$

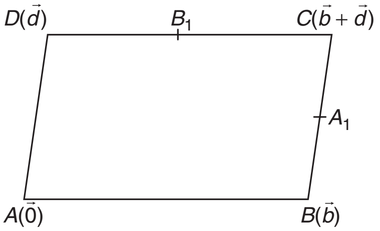


Figure 26.55

and PV of $B_1=\vec{d}+\dfrac{\vec{b}}{2}$. Therefore

$$\overline{AA_1}+\overline{AB_1}=\frac{3}{2}(\vec{b}+\vec{d})=\frac{3}{2}\overline{AC}$$

13. $\vec{a}=\vec{b}+\vec{c}, \vec{b}\times\vec{d}=0, \vec{c}\cdot\vec{d}=0$

$$\Rightarrow \vec{a}\times\vec{d}=\vec{b}\times\vec{d}+\vec{c}\times\vec{d}=\vec{c}\times\vec{d}$$

$$\Rightarrow \vec{d}\times(\vec{a}\times\vec{d})=\vec{d}\times(\vec{c}\times\vec{d})$$

$$=(\vec{d}\cdot\vec{d})\vec{c}-(\vec{c}\cdot\vec{d})\vec{d}$$

$$=|\vec{d}|^2\,\vec{c}$$

$$\Rightarrow \frac{\vec{d}\times(\vec{a}\times\vec{d})}{|\vec{d}|^2}=\vec{c}$$

14. $$(\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i})=\begin{vmatrix}\vec{a}\cdot\vec{b} & \vec{a}\cdot\hat{i}\\ \vec{b}\cdot\hat{i} & \hat{i}\cdot\hat{i}\end{vmatrix}$$

$$=(\vec{a}\cdot\vec{b})-(\vec{a}\cdot\hat{i})(\vec{b}\cdot\hat{i})$$

Similarly,

$$(\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j})=(\vec{a}\cdot\vec{b})-(\vec{a}\cdot\hat{j})(\vec{b}\cdot\hat{j})$$

and $$(\vec{a}\times\hat{k})\cdot(\vec{b}\times\hat{k})=\vec{a}\cdot\vec{b}-(\vec{a}\cdot\hat{k})(\vec{b}\cdot\hat{k})$$

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Then

$$(\vec{a},\hat{i})=a_1,(\vec{a}\cdot\hat{j})=a_2,(\vec{a}\cdot\hat{k})=a_3$$

$$(\vec{b}\cdot\hat{i})=b_1,(\vec{b}\cdot\hat{j})=b_2,(\vec{b}\cdot\hat{k})=b_3$$

$$\Rightarrow (\vec{a}\times\hat{i})\cdot(\vec{b}\times\hat{i})+(\vec{a}\times\hat{j})\cdot(\vec{b}\times\hat{j})+(\vec{a}\times\hat{k})\cdot(\vec{b}\cdot\hat{k})$$

$$3\vec{a}\cdot\vec{b}-(a_1b_1+a_2b_2+a_3b_3)$$

$$=3\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{b}$$

$$=2\vec{a}\cdot\vec{b}$$

15. $$\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}^2=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}^2$$

$$=|\vec{a}\cdot(\vec{b}\times\vec{c})|^2$$

$$=|(\vec{a}\times\vec{b})\cdot\vec{c}|^2$$

$$=(|\vec{a}\times\vec{b}||\vec{c}|)^2$$

$$=|\vec{a}|^2|\vec{b}|^2\sin^2\frac{\pi}{4}$$

$$=\frac{1}{2}\left(a_1^2+a_2^2+a_3^2\right)\left(b_1^2+b_2^2+b_3^2\right)$$

16. $\vec{a}\times\vec{b}=\vec{c},\ \vec{b}\times\vec{c}=\vec{a},\ \vec{c}\times\vec{a}=\vec{b}$

$$\Rightarrow \vec{c}\cdot(\vec{a}\times\vec{b})=\vec{c}\cdot\vec{c},\ \vec{a}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot\vec{a},\ \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{b}\cdot\vec{b}$$

$$\Rightarrow [\vec{c}\vec{a}\vec{b}]=|\vec{c}|^2,\ [\vec{a}\vec{b}\vec{c}]=|\vec{a}|^2,[\vec{b}\vec{c}\vec{a}]=|\vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2=|\vec{b}|^2=|\vec{c}|^2$$

$$\Rightarrow |\vec{a}|=|\vec{b}|=|\vec{c}|$$

17. $(2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b})=6\vec{a}\cdot\vec{a}+17\vec{a}\cdot\vec{b}+5\vec{b}\cdot\vec{b}$

$$=11+17\vec{a}\cdot\vec{b}$$

Now,

$$|\vec{a}+\vec{b}|=\sqrt{3}\Rightarrow|\vec{a}+\vec{b}|^2=3$$

$$\Rightarrow|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=3$$

$$\Rightarrow \vec{a}\cdot\vec{b}=\frac{1}{2}$$

$$\Rightarrow (2\vec{a}+5\vec{b})\cdot(3\vec{a}+\vec{b}+\vec{a}\times\vec{b})=11+\frac{17}{2}=\frac{39}{2}$$

18. $\vec{a}=\lambda(\vec{b}\times\vec{c})\ \Rightarrow|\vec{a}|=|\lambda||\vec{b}\times\vec{c}|=|\lambda||\vec{b}||\vec{c}|\sin\dfrac{\pi}{4}$

$$\Rightarrow|\lambda|=\sqrt{2}$$

$$\Rightarrow \lambda=\pm\sqrt{2}$$

19. Let the required vector be $\vec{r}$. Then

$$\vec{r}=x_1\vec{b}+x_2\vec{c}\text{ and }\vec{r}\cdot\hat{a}=\pm\sqrt{\frac{2}{3}}$$

$$\Rightarrow \vec{r}\cdot\vec{a}=\pm\sqrt{\frac{2}{3}}|\vec{a}|=\pm 2$$

Now,

$$\vec{r}\cdot\vec{a} = x_1\vec{a}\cdot\vec{b} + x_2\vec{a}\cdot\vec{c}$$

$$\Rightarrow \pm 2 = x_1(2-2-1) + x_2(2-1-2)$$

$$\Rightarrow x_1 + x_2 = -2 \text{ or } 2$$

If $x_1 + x_2 = -2$, then

$$\vec{r} = x_1(\hat{i}+2\hat{j}-\hat{k}) + x_2(\hat{i}+\hat{j}-2\hat{k})$$
$$= \hat{i}(x_1+x_2) + \hat{j}(2x_1+x_2) - \hat{k}(x_1+2x_2)$$
$$= -2\hat{i} + \hat{j}(x_1-2) - \hat{k}(x_2-2)$$
$$= -2\hat{i} + \hat{j}(x_1-2) - \hat{k}(-4-x_1)$$

where $x_1 \in R$.

If $x_1 + x_2 = 2$, then

$$\vec{r} = x_1(\hat{i}+2\hat{j}-\hat{k}) + x_2(\hat{i}+\hat{j}-2\hat{k})$$
$$= \hat{i}(x_1+x_2) + \hat{j}(2x_1+x_2) - \hat{k}(x_1+2x_2)$$
$$= 2\hat{i} + \hat{j}(x_1+2) - \hat{k}(4-x_1)$$

20. Let $\vec{r} = x_1\hat{a} + x_2\hat{b} + x_3(\hat{a}\times\hat{b})$. Then

$$\vec{r}\cdot\hat{a} = x_1 + x_2\hat{a}\cdot\hat{b} + x_3\hat{a}\cdot(\hat{a}\times\hat{b}) = x_1$$

Also,

$$\vec{r}\cdot\hat{b} = x_1\hat{a}\cdot\hat{b} + x_2 + x_3\hat{b}\cdot(\hat{a}\times\hat{b}) = x_2$$

and

$$\vec{r}\cdot(\hat{a}\times\hat{b}) = x_1\hat{a}\cdot(\hat{a}\times\hat{b}) + x_2\hat{b}\cdot(\hat{a}\times\hat{b}) + x_3(\hat{a}\times\hat{b})\cdot(\hat{a}\times\hat{b}) = x_3$$

$$\Rightarrow \vec{r} = (\vec{r}\cdot\hat{a})\hat{a} + (\vec{r}\cdot\hat{b})\hat{b} + (\vec{r}\cdot(\hat{a}\times\hat{b}))(\hat{a}\times\hat{b})$$

21. We must have $\vec{b}\cdot\vec{n} = 0$ and $\vec{a}\cdot\vec{n} \neq q$.
Hence, (C) is the correct answer.

22. We have

$$\vec{s} - \vec{p} = \lambda\vec{n} \text{ and } \vec{s}\cdot\vec{n} = q$$
$$\Rightarrow (\lambda\vec{n} + \vec{p})\cdot\vec{n} = q$$
$$\Rightarrow \vec{s} = \vec{p} + \frac{(q - \vec{p}\cdot\vec{n})\vec{n}}{|\vec{n}|^2}$$

23. $[\vec{a}+(\vec{a}\times\vec{b}), \vec{b}+(\vec{a}\times\vec{b}), \vec{a}\times\vec{b}]$

$$= (\vec{a}+(\vec{a}\times\vec{b}))\cdot((\vec{b}+(\vec{a}\times\vec{b}))\times(\vec{a}\times\vec{b})$$
$$(\vec{a}+(\vec{a}\times\vec{b}))\cdot(\vec{b}\times(\vec{a}\times\vec{b}))$$
$$(\vec{a}+(\vec{a}\times\vec{b}))\cdot(\vec{a}-(\vec{a}\cdot\vec{b})\vec{b})$$
$$= \vec{a}\cdot\vec{a} + \vec{a}\cdot(\vec{a}\times\vec{b})$$
$$= 1 \text{ (as } \vec{a}\cdot\vec{b} = 0,\ \vec{a}\cdot(\vec{a}\times\vec{b}) = 0)$$

24. Let PV of A, B, C, and D be $\vec{a}, \vec{b}, \vec{c}$, and $\vec{0}$. Then

$$\overrightarrow{AB}\times\overrightarrow{CD} = (\vec{b}-\vec{a})\times -\vec{c},$$
$$\overrightarrow{BC}\times\overrightarrow{AD} = (\vec{c}-\vec{b})\times -\vec{a}$$

and

$$\overrightarrow{CA}\times\overrightarrow{BD} = (\vec{a}-\vec{c})\times -\vec{b}$$

$$\Rightarrow \overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD}$$
$$= \vec{c}\times\vec{b} + \vec{a}\times\vec{c} + \vec{a}\times\vec{c} + \vec{b}\times\vec{a} - \vec{a}\times\vec{b} + \vec{c}\times\vec{b}$$
$$= 2(\vec{c}\times\vec{b} + \vec{b}\times\vec{a} + \vec{a}\times\vec{c})$$
$$= 2(\vec{c}\times(\vec{b}-\vec{a}) - \vec{a}\times(\vec{b}-\vec{a}))$$
$$= 2((\vec{c}-\vec{a})\times(\vec{b}-\vec{a}))$$
$$= 2(\overrightarrow{AC}\times\overrightarrow{AB})$$
$$\Rightarrow |\overrightarrow{AB}\times\overrightarrow{CD} + \overrightarrow{BC}\times\overrightarrow{AD} + \overrightarrow{CA}\times\overrightarrow{BD}| = 4\left|\frac{1}{2}(\overrightarrow{AC}\times\overrightarrow{AB})\right| = 4\Delta_{ABC}$$

25. The vectors are coplanar, therefore

$$\begin{vmatrix} \sec^2 A & 1 & 1 \\ 1 & \sec^2 B & 1 \\ 1 & 1 & \sec^2 C \end{vmatrix} = 0$$

So, $\cot^2 A + \cot^2 B + \cot^2 C + 1 = 0$ which is not possible.
Hence, (C) is the correct answer.

26. $(\vec{a}+\vec{b})\cdot\vec{p} = \dfrac{(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c})}{[\vec{a}\vec{b}\vec{c}]} = \dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{[\vec{a}\vec{b}\vec{c}]} = 1$

Hence, the given scalar expression = 1 + 1 + 1 = 3.

27. $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2$

$$\Rightarrow |a|^2\sin^2\alpha + |a|^2\sin^2\beta + |a|^2\sin^2\gamma$$
$$= 3|a^2| - |a^2|(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$
$$= 2|a^2| = 2a^2$$

28. $\dfrac{[\vec{A}\vec{B}\vec{C}]}{[\vec{C}\vec{A}\vec{B}]} + \dfrac{[\vec{B}\vec{A}\vec{C}]}{[\vec{C}\vec{A}\vec{B}]} = 1 - 1 = 0$

29. See Figs. 26.56 and 26.57.

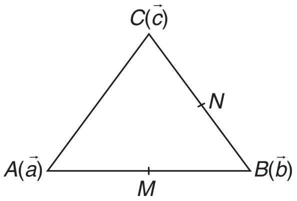


Figure 26.56

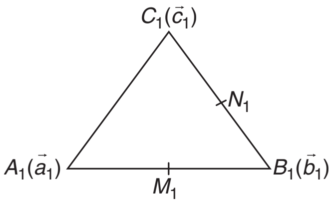


Figure 26.57

$$\overrightarrow{AB} = \overrightarrow{A_1B_1} \Rightarrow \vec{b} - \vec{a} = \vec{b_1} - \vec{a_1}$$
$$\Rightarrow \vec{b} - \vec{b_1} = \vec{a} - \vec{a_1} \Rightarrow \overrightarrow{B_1B} = \overrightarrow{A_1A} \Rightarrow \overrightarrow{AA_1} = \overrightarrow{BB_1}$$
$$\Rightarrow \overrightarrow{NN_1} = \frac{\vec{b_1}+\vec{c_1}}{2} - \frac{\vec{b}+\vec{c}}{2} \Rightarrow \overrightarrow{NN_1} = \frac{\vec{b_1}+\vec{c_1}-\vec{b}-\vec{c}}{2}$$
$$\Rightarrow 2\overrightarrow{NN_1} = \overrightarrow{BB_1} + \overrightarrow{CC_1}$$
$$\Rightarrow \overrightarrow{MM_1} = \frac{\vec{b_1}-\vec{b}+\vec{a_1}-\vec{a}}{2} \Rightarrow 2\overrightarrow{MM_1} = \overrightarrow{BB_1} + \overrightarrow{AA_1} = 2\overrightarrow{BB_1} = 2\overrightarrow{AA_1}$$
$$\Rightarrow \overrightarrow{MM_1} = \overrightarrow{BB_1} = \overrightarrow{AA_1}$$

30. $\vec{a}\cdot\vec{b} = 0 \Rightarrow x_1 + x_2 + x_3 = 0$

Thus, we have to obtain the number of integral solution of this equation.

Coefficient of $x^\circ \mid (x^{-3} + x^{-2} + x^{-1} + x^0 + x + x^2)^3$

$$= x^\circ \left|\left(\frac{1 + x + x^2 + x^3 + x^4 + x^5}{x^3}\right)^3\right.$$
$$= x^9 \,|(1-x^6)^3\,(1-x)^{-3} = {}^{11}C_9 - 3.{}^5C_3 = 25$$

31. $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

$$C_2 \to C_2 - C_1$$
$$-1(ab - c^2) = 0 \Rightarrow c^2 = ab$$

32. $\vec{a} \times (\vec{a} \times \vec{b}) = (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b} = -4\vec{b}$

$\vec{a} \times (\vec{a} \times (-4\vec{b})) = 16\vec{b}$ and process gives $-320\hat{b}$

33. The point that divides $5\hat{i}$ and $5\hat{j}$ in the ratio of k:1 is given by $\vec{b} = \dfrac{5\hat{i} + 5k\hat{j}}{k+1}$ $\left\{\text{as terminal position is origin and initials } \dfrac{(5\hat{j})k + 5\hat{i}}{k+1}\right\}$

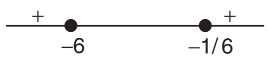
Also,

$|\vec{b}| \le \sqrt{37} \Rightarrow \dfrac{1}{k+1}\sqrt{25 + 25k^2} \le \sqrt{37} \Rightarrow 5\sqrt{1+k^2} \le \sqrt{37}(k+1)$

On squaring both sides, we get

$$25(1 + k^2) \le 37(k^2 + 2k + 1)$$

or $\quad 12k^2 + 74k + 12 \ge 0 \Rightarrow (6k + 1)(k + 6) \ge 0$



Hence, $k \in (-\infty, -6] \cup [-1/6, \infty)$.

34. The given relation can be rewritten as

$(\sqrt{a^2 - 4}\hat{i} + a\hat{j} + \sqrt{a^2 + 4}\hat{k}) \cdot (\tan A\hat{i} + \tan B\hat{j} + \tan C\hat{k}) = 6a$

$\Rightarrow \sqrt{(a^2 - 4) + a^2 + (a^2 + 4)} \cdot \sqrt{\tan^2 A + \tan^2 B + \tan^2 C} \cdot \cos\theta = 6a$

(as, $a \cdot b = |a|\,|b| \cos\theta$)

$\Rightarrow \sqrt{3}a \cdot \sqrt{\tan^2 A + \tan^2 B + \tan^2 C} \cdot \cos\theta = 6a$

$\Rightarrow \tan^2 A + \tan^2 B + \tan^2 C = 12\sec^2\theta \quad (1)$

also,

$$12\sec^2\theta \ge 12 \qquad (\text{as, } \sec^2\theta \ge 1) \quad (2)$$

From Eqs. (1) and (2), we get

$$\tan^2 A + \tan^2 B + \tan^2 C \ge 12$$

Hence, least value of $\tan^2 A + \tan^2 B + \tan^2 C$ is 12.

35. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \pm 1$

36. $[\hat{a}\vec{p}\vec{q}]$ = projection of $\vec{p} \times \vec{q}$ in the direction of $\hat{a}$.

Hence, the given vector is $\vec{p} \times \vec{q}$.

37. Since $\vec{a}$ and $\vec{b}$ are collinear, for some λ, we can write

$$\vec{a} = \lambda\vec{b}$$
$$\Rightarrow \cos x\hat{i} + \sin x\hat{j} = \lambda(x\hat{i} + \sin x\hat{j})$$
$$\Rightarrow \cos x = x\lambda \text{ and } \lambda = 1 \Rightarrow \cos x = x$$

Here, we will get only one unique value of x which belongs to $\left(\dfrac{\pi}{6}, \dfrac{\pi}{3}\right)$.

38. Since initial point of $2\hat{i} + 3\hat{j}, 5\hat{i} + 6\hat{j}$ and $8\hat{i} + \lambda\hat{j}$ is $\hat{i} + \hat{j}$, their terminal points will be $3\hat{i} + 4\hat{j}, 6\hat{i} + 7\hat{j}$, and $9\hat{i} + (\lambda + 1)\hat{j}$. Now given all the vectors terminate on one straight line. Hence,

$$3\hat{i} + 3\hat{j} = \lambda_1(3\hat{i} + (\lambda + 1 - 7)\hat{j}) \Rightarrow \lambda_1 = 1 \text{ and } \lambda = 9$$

39. We have

$$p\vec{r} + (\vec{r} \cdot \vec{b})\vec{a} = \vec{c}$$

Taking dot by vector $\vec{b}$, we get

$$p\vec{r} \cdot \vec{b} + (\vec{r} \cdot \vec{b})\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{b} \Rightarrow p\vec{r} \cdot \vec{b} + 0 = \vec{c} \cdot \vec{b} \Rightarrow \vec{r} \cdot \vec{b} = \frac{\vec{c} \cdot \vec{b}}{p}$$

$$\Rightarrow p\vec{r} + \frac{\vec{c} \cdot \vec{b}}{p}\vec{a} = \vec{c} \Rightarrow \vec{r} = \frac{\vec{c}}{p} - \frac{\vec{c} \cdot \vec{b}}{p^2}\vec{a}$$

40. Let PV of P, A, B, and C are $\vec{p}, \vec{a}, \vec{b}$, and $\vec{c}$, respectively, and $O(\vec{0})$ be the circumcentre of the equilateral triangle ABC. Then

$$|\vec{p}| = |\vec{b}| = |\vec{a}| = |\vec{c}| = \frac{\ell}{\sqrt{3}}$$

Now

$$|\overrightarrow{PA}|^2 = |\vec{a} - \vec{p}|^2 = |\vec{a}|^2 + |\vec{p}|^2 - 2\vec{p} \cdot \vec{a}$$

Similarly,

$$|\overrightarrow{PB}|^2 = |\vec{b}|^2 + |\vec{p}|^2 - 2\vec{p} \cdot \vec{b}$$

and $\quad |\overrightarrow{PC}|^2 = |\vec{c}|^2 + |\vec{p}|^2 - 2\vec{p} \cdot \vec{c}$

So,

$$|\overrightarrow{PA}|^2 + |\overrightarrow{PB}|^2 + |\overrightarrow{PC}|^2 = 6 \cdot \frac{\ell^2}{3} - 2p \cdot (\vec{a} + \vec{b} + \vec{c}) = 2\ell^2$$

$$\left(\text{as } \frac{\vec{a} + \vec{b} + \vec{c}}{3} = 0\right)$$

41. Let $\vec{r}_1 = \sqrt{3}(\vec{a} \times \vec{b})$, $\vec{r}_2 = \vec{b} - (\vec{a} \cdot \vec{b})\vec{a}$, clearly $\vec{r}_1$ and $\vec{r}_2$ are mutually perpendicular as $\vec{r}_2$ is coplanar with $\vec{a}$ and $\vec{b}$ and $\vec{r}_1$ is at right angle to the plane of $\vec{a}$ and $\vec{b}$.

And $\quad |\vec{r}_1| = \sqrt{3}|\vec{a} \times \vec{b}| \Rightarrow |\vec{r}_1|^2 = 3(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$

$$= 3((\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2) = 3(|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2)$$

Also,

$$|\vec{r}_2|^2 = (\vec{b} - (\vec{a} \cdot \vec{b})\vec{a}) \cdot (\vec{b} - (\vec{a} \cdot \vec{b})\vec{a}) = |\vec{b}|^2 - 2(\vec{a} \cdot \vec{b})^2 + (\vec{a} \cdot \vec{b})^2(\vec{a} \cdot \vec{a})$$

$$= |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 \Rightarrow \frac{|r_1|}{|r_2|} = \sqrt{3}$$

Thus, angles are $\dfrac{\pi}{2}, \dfrac{\pi}{3}, \dfrac{\pi}{6}$.

42. See Fig. 26.58. Let PV of A, B and D be $\vec{0}, \vec{b}, \vec{d}$. Then,

$$\overrightarrow{AC} = \vec{b} + \vec{d}$$
$$\Rightarrow \vec{E} = \frac{\vec{b} + 4(\vec{b} + \vec{d})}{5} = \vec{b} + \frac{4}{5}\vec{d}$$

and $$\vec{F} = \frac{\vec{b}+\vec{d}+4\vec{d}}{5} = \vec{d} + \frac{1}{5}\vec{b}$$

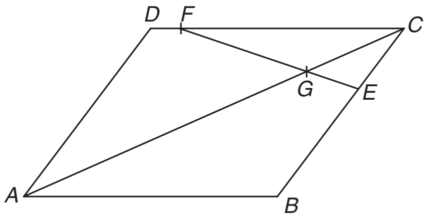


Figure 26.58

Equation of EF: $\vec{r} = \vec{b} + \frac{4}{5}\vec{d} + \lambda\left(\frac{4}{5}\vec{b} - \frac{1}{5}\vec{d}\right)$

Equation of AC: $\vec{r} = \lambda_1(\vec{b}+\vec{d})$

For point G we must have,

$$\vec{b} + \frac{4}{5}\vec{d} + \frac{\lambda}{5}(4\vec{b}-\vec{d}) = \lambda_1(\vec{b}+\vec{d})$$

$$\Rightarrow \lambda_1 = \frac{21}{5}, \quad \lambda = -\frac{1}{5} \Rightarrow \overrightarrow{AG} = \frac{21}{5}\overrightarrow{AC}$$

43. Given

$$\begin{vmatrix}(x-a)^2 & (x-b)^2 & (x-c)^2\\(y-a)^2 & (y-b)^2 & (y-c)^2\\(z-a)^2 & (z-b)^2 & (z-c)^2\end{vmatrix} = 0 \Rightarrow 2\begin{vmatrix}x^2 & x & 1\\y^2 & y & 1\\z^2 & z & 1\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\-a & -b & -c\\a^2 & b^2 & c^2\end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}1 & 1 & 1\\-a & -b & -c\\a^2 & b^2 & c^2\end{vmatrix} = 0 \text{ [since, } X, Y, Z \text{ are non-coplanar]}$$

Hence, $(a^2, a, 1)$, $(b^2, b, 1)$ and $(c^2, c, 1)$ are coplanar.

44. Consider three non-coplanar vectors b, c and $b \times c$. Any vector a can be written as

$$a = xb + yc + z(b \times c) \qquad (1)$$

Taking dot product with $b \times c$ in Eq. (1), we get

$$a \cdot (b \times c) = z|b \times c|^2 \Rightarrow z = \frac{a \cdot (b \times c)}{|b \times c|^2}$$

Taking dot product with b in Eq. (1), we get

$$a \cdot b = xb \cdot b + yc \cdot b + z \cdot 0 = x$$

Taking dot product with c in Eq. (1), we get

$$a \cdot c = y$$

Thus,

$$a = (a \cdot b)b + (a \cdot c)c + \frac{a \cdot (b \times c)}{|b \times c|^2}(b \times c)$$

45. For the point of intersection of the lines

$$\vec{a} + t(\vec{b} \times \vec{c}) = \vec{b} + s(\vec{c} + \vec{a}) \Rightarrow \vec{a}\vec{c} + t(\vec{b} \times \vec{c})\vec{c} = \vec{b} \cdot \vec{c} + s(\vec{c} \times \vec{a})\vec{c}$$

$$\Rightarrow \vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$$

46. Given

$$\vec{r} \cdot \vec{a} = 10 \Rightarrow x + y + z = 10, x, y, x \geq 1$$

The number of possible positions of P is coefficient of x^{10} in $(x + x^2 + x^3 + \cdots)^3$

$$= \text{coefficient of } x^7 \text{ in } (1-x)^{-3}$$
$$= {}^{3+7-1}C_7 = {}^9C_7 = {}^9C_2 = 36$$

47. Since vectors make an acute angle with each other, so their dot product must be positive, that is,

$$ax^2 - 10\,ax + 6 > 0 \ \forall\, x \in R$$

$$\Rightarrow -ax^2 + 10ax - 6 < 0 \ \forall\, x \in R \Rightarrow -a < 0 \text{ and } 100a^2 < 24a$$

48. See Fig. 26.59.

Vector in the direction of angular bisector of $\vec{a}$ and $\vec{b} = \frac{\vec{a}+\vec{b}}{2}$.

The vector $\frac{\vec{a}+\vec{b}}{2}$ have magnitude $\cos(\theta/2)$.

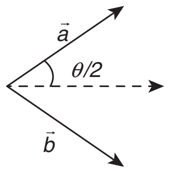


Figure 26.59

So, the unit vector in this direction will have magnitude

$$\frac{\vec{a}+\vec{b}}{2\cos\frac{\theta}{2}}$$

49. $\sum_{i=1}^{n} \vec{a}_i = \vec{0}$

$$\left(\sum_{i=1}^{n} \vec{a}_j\right) \cdot \left(\sum_{j=1}^{n} \vec{a}_j\right) = \sum_{i=1}^{n} |\vec{a}_i|^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{a}_j \cdot \vec{a}_j$$

$$\Rightarrow 0 = n + 2\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{a}_i \cdot \vec{a}_j$$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{a}_i . \vec{a}_j = -\frac{n}{2}$$

50. We have

$$4\vec{a} + 5\vec{b} + 9\vec{c} = 0$$

Therefore, vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar. So,

$$\vec{b} \times \vec{c} \text{ and } \vec{c} \times \vec{a} \text{ are collinear}$$

$$\Rightarrow (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = \vec{0}$$

51. $|\vec{c} - \vec{a}|^2 = 8$

$$\Rightarrow |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c} \cdot \vec{a} = 8 \Rightarrow |\vec{c}|^2 - 2|\vec{c}| + 1 = 0 \Rightarrow |\vec{c}| = 1$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k\\ 2 & 1 & -2\\ 1 & 1 & 0\end{vmatrix} = -2i + 2j + k \Rightarrow |(\vec{a} \times \vec{b})| = 3$$

$$|(\vec{a} \times \vec{b}) \times \vec{c}| = |(\vec{a} \times \vec{b})||\vec{c}|\sin 30^\circ \Rightarrow |(\vec{a} \times \vec{b}) \times \vec{c}| = 3 \times 1 \times \frac{1}{2} = \frac{3}{2}$$

52. Since $\vec{r} \cdot \vec{a} = 0$, $\vec{r} \cdot \vec{b} = 0$ and $\vec{r} \cdot \vec{c} = 0$; $\vec{r}$ must be perpendicular to all the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Hence $\vec{a}, \vec{b}$ and $\vec{c}$ must be coplanar. So,

$$[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

53. Vector $\vec{d}$ is perpendicular to $\vec{a},\vec{b}$ and $\vec{c}$ which is possible only when $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar and then $\vec{a}\cdot\vec{d}=\vec{c}\cdot\vec{d}=\vec{b}\cdot\vec{d}=0$ and $[\vec{a}\vec{b}\vec{c}]=0$. Given

$$\vec{b}\times(\vec{d}\times\vec{c})=\vec{d}\times(\vec{b}\times\vec{c})+\vec{a}+\vec{d}$$

$$\Rightarrow[\vec{a}\vec{b}\vec{d}]+[\vec{a}\vec{b}\vec{c}]=(\vec{c}\cdot\vec{d})(\vec{a}\vec{b})-(\vec{d}\cdot\vec{b})(\vec{a}\cdot\vec{b})+\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{d}$$

$$\Rightarrow[\vec{a}\vec{b}\vec{d}]=[\vec{a}]^2$$

So,

$$(\vec{a}\cdot\vec{c})\vec{a}=\vec{c}\Rightarrow|\vec{a}|=1$$

54. See Fig. 26.60. Let the altitudes *AD* and *BE* intersect at *O*. Join *CO* and produce to meet *AB* in *F*.

Let $\overrightarrow{OA}=\vec{a};\overrightarrow{OB}=\vec{b};\overrightarrow{OC}=\vec{c}$

The vector $\vec{a}$ is perpendicular to

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\vec{c}-\vec{b}$$

Therefore,

$$\vec{a}(c-\vec{b})=0,\ \Rightarrow\vec{a}\cdot\vec{c}=\vec{a}\cdot\vec{b} \quad (1)$$

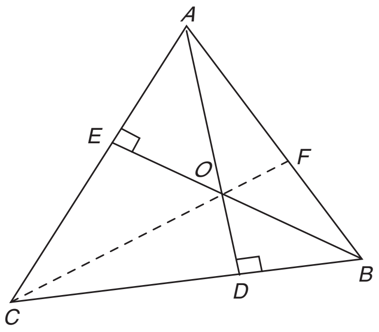


Figure 26.60

Also the direction *OB* is perpendicular to *AC*

Therefore,

$$\vec{b}\cdot(\vec{c}-\vec{a})=0\Rightarrow\vec{b}\cdot\vec{c}=\vec{a}\cdot\vec{b} \quad (2)$$

From Eqs. (1) and (2)

$$\vec{a}\cdot\vec{c}=\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}$$

That is,

$$\vec{c}\cdot\vec{a}-\vec{c}\cdot\vec{b}=0\Rightarrow\vec{c}\cdot(\vec{a}-\vec{b})=0 \quad (3)$$

$$\vec{a}-\vec{b}=\overrightarrow{OA}-\overrightarrow{OB}=\overrightarrow{BA},$$

and by virtue of Eq. (3), $\vec{c}$ is perpendicular to $\overrightarrow{BA}$; but $\vec{c}$ is a vector in the direction of $\overrightarrow{OC}$. Hence, $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$, that is, *CF* is the third altitude of the triangle through *C*. Hence, the three altitudes are concurrent at *O*.

55. $\vec{a}\cdot\vec{a}=\vec{a}^2=|\vec{a}|^2=1;\quad \vec{b}\cdot\vec{b}=\vec{b}^2=|\vec{b}|^2=1$

Consider

$$|\vec{a}-\vec{b}|^2=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-2\vec{a}\cdot\vec{b}$$

$$=1+1-2\cos\theta=2(1-\cos\theta)=4\sin^2\frac{\theta}{2}$$

Therefore,

$$|\vec{a}-\vec{b}|=2\sin\frac{\theta}{2}$$

$$\Rightarrow\sin\left(\frac{\theta}{2}\right)=\frac{1}{2}|\vec{a}-\vec{b}|$$

56. See Fig. 26.61. Let *ABCD* be the trapezium and *E* be the mid-point of *BC*. Let *A* be the initial point and let $\vec{b}$ be the PV of *B* and $\vec{d}$ that of *D*. Since *DC* is parallel to *AB*, $t\vec{b}$ is a vector along *DC*, so that the PV of *C* is $\vec{d}+t\vec{b}$. Therefore, the PV of *E* is

$$\frac{\vec{b}+\vec{d}+t\vec{b}}{2}=\frac{\vec{d}+(1+t)\vec{b}}{2}$$

$$\text{Area of }\Delta AED=\frac{1}{2}\left|\frac{\vec{d}+(1+t)\vec{b}}{2}\times\vec{d}\right|=\frac{1}{4}(1+t)|\vec{b}\times\vec{d}|$$

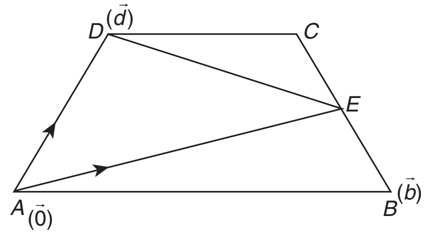


Figure 26.61

Area of the trapezium = Area (ΔACD) + Area (ΔABC)

$$=\frac{1}{2}|\vec{b}\times(\vec{d}+t\vec{b})|+\frac{1}{2}|(\vec{d}+t\vec{b})\times\vec{d}|=\frac{1}{2}|\vec{b}\times\vec{d}|+\frac{1}{2}t|\vec{b}\times\vec{d}|$$

$$=\frac{1}{2}(1+t)|\vec{b}\times\vec{d}|=2\Delta AED$$

57. *ABCD* is the tetrahedron.

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=i-(\hat{i}+\hat{j}+\hat{k})=-\hat{j}-\hat{k}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=3i-(\hat{i}+\hat{j}+\hat{k})=2\hat{i}-\hat{j}-\hat{k}$$

Area of ΔABC magnitude is equal to

$$\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}|(2\hat{i}-\hat{j}-\hat{k})\times(-\hat{j}-\hat{k})|$$

$$=|\hat{j}-\hat{k}|=\sqrt{2}$$

Therefore,

$$\text{volume of }ABCD=\frac{1}{3}\text{ (Area of the base)}\times\text{Height}$$

$$=\frac{1}{3}\sqrt{2}\cdot DE=\frac{2\sqrt{2}}{3}$$

$$\Rightarrow DE=2$$

$$AE=\sqrt{AD^2-DE^2}=\sqrt{16-4}=2\sqrt{3}$$

Therefore, *E* falls outside of *AF* and lies on *AF* produced and such that *AF* = *FE*.

Therefore, the position vector of *E* is

$$2(2\hat{i})-(\hat{i}+\hat{j}+\hat{k})=3\hat{i}-\hat{j}-\hat{k}$$

This is one position of *E* (taken as E_1).

Another position of *E*, taken as E_2, lies on *FA* produced, and for that position, position vector of $E(E_2)$ which is an external division such that $\frac{FA}{FE}=\frac{2}{3}$. Therefore,

$$\text{PV of }E(E_2)=\frac{2(2\hat{i})-3(\hat{i}+\hat{j}+\hat{k})}{2-3}=-i+3\hat{j}+3\hat{k}$$

This is a second possible position for *E*.

58. $\vec{b}, \vec{c}, \vec{d}$ are given to be non-coplanar and hence $[\vec{b}\ \vec{c}\ \vec{d}] \neq 0$ Any other vector $\vec{a}$ can be expressed as a linear combination of $\vec{b}, \vec{c}$, and $\vec{d}$ so that we have

$$\vec{a} = \lambda\vec{b} + \mu\vec{c} + \gamma\vec{d}$$
$$\vec{a}\times\vec{d} = \lambda\vec{b}\times\vec{d} + \mu\vec{c}\times\vec{d} \quad (\text{since}, \vec{d}\times\vec{d} = 0)$$
$$(\vec{a}\times\vec{d})\cdot\vec{c} = \lambda(\vec{b}\times\vec{d})\cdot\vec{c}$$

Therefore,
$$\lambda = \left\{[\vec{a}\ \vec{d}\ \vec{c}]/[\vec{b}\ \vec{d}\ \vec{c}]\right\}$$

Similarly,
$$\mu = [\vec{a}\ \vec{b}\ \vec{d}]/[\vec{c}\ \vec{b}\ \vec{d}]$$

Therefore,
$$\gamma = [\vec{a}\ \vec{c}\ \vec{b}]/[\vec{d}\ \vec{c}\ \vec{b}]$$

Now,
$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$
$$(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) = [\vec{a}\ \vec{c}\ \vec{b}]\vec{d} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b}$$
$$(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c}) = [\vec{a}\ \vec{d}\ \vec{c}]\vec{b} - [\vec{a}\ \vec{d}\ \vec{b}]\vec{c}$$

and $\quad [\vec{a}\ \vec{b}\ \vec{d}] = -[\vec{a}\ \vec{d}\ \vec{b}];\ [\vec{a}\ \vec{c}\ \vec{b}] = -[\vec{a}\ \vec{b}\ \vec{c}]$

and $\quad [\vec{a}\ \vec{d}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{d}]$

Hence, by adding the three results and using above information, we get

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) + (\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) + (\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$
$$= 2\{[\vec{a}\vec{d}\vec{c}]\vec{b} + [\vec{a}\vec{b}\vec{d}]\vec{c} + [\vec{a}\vec{c}\vec{b}]\vec{d}\}$$
$$= 2[\vec{b}\vec{d}\vec{c}]\left\{\frac{[\vec{a}\vec{d}\vec{c}]}{[\vec{b}\vec{d}\vec{c}]}\vec{b} + \frac{[\vec{a}\vec{b}\vec{d}]}{[\vec{b}\vec{d}\vec{c}]}\vec{c} + \frac{[\vec{a}\vec{c}\vec{b}]}{[\vec{b}\vec{d}\vec{c}]}\vec{d}\right\} = 2[\vec{b}\vec{d}\vec{c}]\vec{a}$$

This is certainly in the direction of $\vec{a}$.

59. $k\vec{x} + \vec{x}\times\vec{a} = \vec{b} \Rightarrow k\vec{a}\cdot\vec{x} = \vec{a}\cdot\vec{b}$

$$\Rightarrow \vec{a}\cdot\vec{x} = \frac{\vec{a}\cdot\vec{b}}{k}$$

Also,
$$k\,\vec{a}\times\vec{x} + \vec{a}\times(\vec{x}\times\vec{a}) = \vec{a}\times\vec{b}$$

$$k(\vec{a}\times\vec{x}) + (\vec{a}\cdot\vec{a})\vec{x} - (\vec{a}\cdot\vec{x})\vec{a} = \vec{a}\times\vec{b} \quad (\text{using that } \vec{a}\times\vec{x} = k\,\vec{x} - \vec{b})$$

Therefore,
$$k^2\vec{x} - k\vec{b} + a^2\vec{x} - \frac{\vec{a}\cdot\vec{b}}{k}\vec{a} = \vec{a}\times\vec{b}$$
$$(\text{substituting } \vec{a}\cdot\vec{x} = \vec{a}\cdot\vec{b}/k)$$

Therefore,
$$\vec{x}(k^2 + a^2) = \frac{\vec{a}\cdot\vec{b}}{k}\vec{a} + k\,\vec{b} + \vec{a}\times\vec{b}$$
$$\Rightarrow \vec{x} = \frac{1}{(k^2 + a^2)}\left\{\frac{\vec{a}\cdot\vec{b}}{k}\vec{a} + k\,\vec{b} + \vec{a}\times\vec{b}\right\}$$

Practice Exercise 2

1. $x = \left|\frac{\overrightarrow{AB}\cdot\vec{b}_1\times\vec{b}_2}{|\vec{b}_1\times\vec{b}_2|}\right| = \left|\frac{(\hat{i}+2\hat{j}+2\hat{k})\cdot(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{1+4+1}}\right| = \frac{1}{\sqrt{6}}$

Now,
$$\cos^{-1}\cos\sqrt{6}x = \cos^{-1}\cos 1 = 1$$

2. $[\vec{a}\vec{b}\vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D$

Given
$$\vec{r} = \lambda(\vec{a}\times\vec{b}) + \mu(\vec{b}\times\vec{c}) + \gamma(\vec{c}\times\vec{a}), [\vec{a}, \vec{b}, \vec{c}] = \frac{1}{8}$$

Multiply the above relation with a, b, c, we have
$$\vec{r}\cdot\vec{a} = \lambda\cdot 0 + \mu[\vec{a}\cdot\vec{b}\cdot\vec{c}] + 0 = \frac{\mu}{8}$$
$$\Rightarrow \mu = 8(\vec{r}\cdot\vec{a}), \gamma = 8(\vec{r}\cdot\vec{b}), \lambda = 8(\vec{r}\cdot\vec{c})$$

Therefore,
$$\lambda + \mu + \gamma = 8\vec{r}\cdot(\vec{a} + \vec{b} + \vec{c})$$

3. See Fig. 26.62. Let $A(\vec{a}), B(\vec{b})$. Then

$$PV \text{ of } D = \frac{2\vec{b} + \vec{a}}{3}$$
$$PV \text{ of } E = \frac{\vec{b}}{2}$$

Let $\frac{OP}{PD} = t, \frac{AP}{PE} = \lambda$. Then

$$PV \text{ of } P = \frac{t(2\vec{b} + \vec{a})}{3(t+1)} = \frac{\frac{\lambda\vec{b}}{2} + \vec{a}}{\lambda + 1}$$

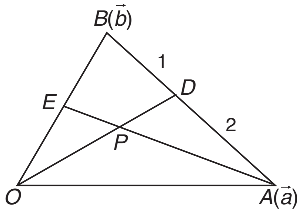


Figure 26.62

Comparing coefficients of $\vec{a}$ and $\vec{b}$, we get

$$\frac{2t}{3(t+1)} = \frac{\lambda}{2(\lambda+1)} \quad (1)$$

and
$$\frac{t}{3(t+1)} = \frac{1}{\lambda+1} \quad (2)$$
$$\Rightarrow \frac{2}{\lambda+1} = \frac{\lambda}{2(\lambda+1)} \Rightarrow \lambda = 4$$
$$\Rightarrow \frac{t}{3(t+1)} = \frac{1}{5} \Rightarrow 3t + 3 = 5t \Rightarrow 2t = 3 \Rightarrow t = \frac{3}{2}$$

4. $\vec{z}+\vec{z}\times\vec{x}=\vec{y}\Rightarrow|\vec{z}+\vec{z}\times\vec{x}|^2=|\vec{y}|^2$

$$\Rightarrow (\vec{z}+\vec{z}\times\vec{x})\cdot(\vec{z}+\vec{z}\times\vec{x})=|\vec{y}|^2=1$$

$$\Rightarrow |z^2|+|z|^2|x|^2\sin^2\theta=1$$

$$\Rightarrow |\vec{z}|=\frac{1}{\sqrt{1+\sin^2\theta}}=\frac{2}{\sqrt{7}}$$

$$\Rightarrow \sin\theta=\frac{\sqrt{3}}{2}\ \Rightarrow\ \theta=\frac{\pi}{3}=60^\circ$$

5. Normal vector to plane P_1, $-3\hat{k}$.

Normal vector to plane P_2, $2\hat{i}-4\hat{j}-3\hat{k}$. So,

$$\vec{a}=\lambda(2\hat{i}+\hat{j})$$

Angle between $\vec{a}$ and vector $\hat{i}-2\hat{j}+2\hat{k}$ is given by

$$\cos\theta=\frac{\lambda(2\hat{i}+\hat{j})\cdot(\hat{i}-2\hat{j}+2\hat{k})}{\lambda\sqrt{5}\sqrt{9}}=0$$

$$\Rightarrow \theta=\frac{\pi}{2}$$

6. Let $\vec{r}=\alpha\,\vec{a}+\beta\,\vec{b}+\lambda(\vec{a}\times\vec{b})$. Then

$$4\alpha+\beta=2$$

Also,

$$\alpha+\beta=8$$

Therefore,

$$\alpha=-2,\ \beta=10$$

Also, $\lambda=2$

Now,

$$\vec{r}+2\vec{a}-10\vec{b}=2(\vec{a}\times\vec{b})=2$$

7. $\hat{a}\times\hat{b}=\hat{c}$ and $\hat{a}\cdot\hat{b}=0$

So, $\hat{a}$, $\hat{b}$, $\hat{c}$ are mutually perpendicular vectors.

$[\vec{x}\ \hat{b}\ \hat{c}]=3\Rightarrow\vec{x}\cdot(\hat{b}\times\hat{c})=3\Rightarrow\vec{x}\cdot\hat{a}=3$

$[\vec{x}\ \hat{c}\ \hat{a}]=4\Rightarrow\vec{x}\cdot\hat{b}=4$

$[\vec{x}\ \hat{a}\ \hat{b}]=2\Rightarrow\vec{x}\cdot\hat{c}=2$

$\vec{x}=(\vec{x}\cdot\hat{a})\hat{a}+(\vec{x}\cdot\hat{b})\hat{b}+(\vec{x}\cdot\hat{c})\hat{c}=3\hat{a}+4\hat{b}+2\hat{c}$

8. $\overrightarrow{DA}\perp\overrightarrow{CB}$ and $\overrightarrow{DB}\perp\overrightarrow{AC}$

Therefore, D is the orthocentre of the triangle ABC.

9. Let $\vec{a}\times\vec{b}=\ell$, $\vec{b}\times\vec{c}=m$, $\vec{c}\times\vec{a}=n$. Then

$$[\ell-n,\ m+2\ell,\ n-3m]=\begin{vmatrix}1 & 0 & -1\\ 2 & 1 & 0\\ 0 & -3 & 1\end{vmatrix}[\ell,\ m,\ n]=7[a,b,c]^2$$

10. $$V^2=\frac{1}{36}\begin{vmatrix}a\cdot a & a\cdot b & a\cdot c\\ b\cdot a & b\cdot b & b\cdot c\\ c\cdot a & c\cdot b & c\cdot c\end{vmatrix}=\frac{1}{36}\left(\frac{3\sqrt{3}}{4}-\frac{5}{4}\right)$$

$$\Rightarrow V=\frac{1}{12}\left(\sqrt{3\sqrt{3}-5}\right)$$

11. Given vectors are coplanar, so

$$\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=0$$

$$\Rightarrow a^3+b^3+c^3-3abc=0=(a+b+c)[(a-b)^2+(b-c)^2$$
$$+(c-a)^2]=0$$
$$\Rightarrow a+b+c=0 \text{ as } a\neq b$$
$$\Rightarrow \vec{v}\cdot(a\hat{i}+b\hat{j}+c\hat{k})=\vec{v}\cdot(b\hat{i}+c\hat{j}+a\hat{k})=\vec{v}\cdot(c\hat{i}+a\hat{j}+b\hat{k})=0$$

12. Since $\vec{a}$, $\vec{b}$ and $\vec{a}\times\vec{b}$ are non-coplanar. So,

$$\vec{r}=x\vec{a}+y\vec{b}+z(\vec{a}\times\vec{b})$$

Therefore,

$$\vec{r}\times\vec{b}=\vec{a}\Rightarrow x\vec{a}\times\vec{b}+z\{(\vec{a}\cdot\vec{b})\vec{b}-(\vec{b}\cdot\vec{b})\vec{a}\}=\vec{a}$$

$$\Rightarrow -(1+z|\vec{b}|^2)\vec{a}+x\vec{a}\times\vec{b}=0 \qquad [\text{since } \vec{a}\cdot\vec{b}=0]$$

Therefore,

$$x=0 \text{ and } z=-\frac{1}{|\vec{b}|^2}$$

Thus, $\vec{r}=y\vec{b}-\dfrac{\vec{a}\times\vec{b}}{|\vec{b}|^2}$, where y is the parameter.

13. $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$

$$\cos\theta=\frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|}=\frac{\vec{b}\cdot\vec{c}}{|\vec{b}|\cdot|\vec{c}|}=\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}$$

$$\vec{a}\cdot\vec{b}=0$$

$$\vec{a}\cdot\vec{c}=\alpha\cdot\vec{a}\cdot\vec{a}+\beta\cdot\vec{a}\cdot\vec{b}+\gamma\cdot\vec{a}\cdot(\vec{a}\times\vec{b})\ =\alpha=\cos\theta$$

Similarly,

$$\vec{b}\cdot\vec{c}=\cos\theta=\beta$$

So,

$$\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}=\cos\theta$$
$$\Rightarrow\alpha=\beta$$

Now

$$1=\vec{c}\cdot\vec{c}=2\alpha^2+\gamma^2|\vec{a}\times\vec{b}|$$

$$1=2\alpha^2+\gamma^2[|\vec{a}|^2\cdot|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2]=2\alpha^2+\gamma^2$$

14. $$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

$$\Rightarrow |\vec{a}\times\vec{b}|=|\vec{a}|\,|\vec{b}|\sin\theta\ \Rightarrow\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} \qquad (1)$$

$$\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\cos\theta\ \Rightarrow\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \qquad (2)$$

From Eqs. (1) and (2),

$$\sin^2\theta+\cos^2\theta=1$$

$$\Rightarrow |\vec{a}\times\vec{b}|+\left(\vec{a}\cdot\vec{b}\right)^2=|\vec{a}|^2|\vec{b}|^2$$

If $\theta = \dfrac{\pi}{4}$, then

$$\sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$$

Therefore,

$$|\vec{a}\times\vec{b}| = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}} \text{ and } \vec{a}\cdot\vec{b} = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}}$$

Hence,

$$|\vec{a}\times\vec{b}| = \vec{a}\cdot\vec{b}$$

Thus,

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n} = \frac{|\vec{a}||\vec{b}|}{\sqrt{2}}\hat{n}$$

or

$$\vec{a}\times\vec{b} = (\vec{a}\cdot\vec{b})\hat{n}$$

15. We have

$$(\vec{a}\times\vec{b})\times\vec{a} = (\vec{a}\cdot\vec{a})\vec{b} - (\vec{b}\cdot\vec{a})\vec{a} = |a|^2\,\vec{b} - 2\vec{a}$$

Therefore,

$$\vec{b} = \frac{(\vec{a}\times\vec{b})\times\vec{a} + 2\vec{a}}{|\vec{a}|^2}$$

Now,

$$(\vec{a}\times\vec{b})\times\vec{a} = 4\hat{i} - 5\hat{j} + \hat{k} \text{ and } |\vec{a}|^2 = 3$$

Therefore,

$$\vec{b} = \frac{(4\hat{i} - 5\hat{j} + \hat{k}) + 2(\hat{i} + \hat{j} + \hat{k})}{3} = 2\hat{i} - \hat{j} + \hat{k}$$

16. Clearly, vector equations of CD and BE are

$$\vec{r} = \hat{i} - 2\hat{j} + 4\hat{k} + \frac{\lambda}{3}(7\hat{j} - 7\hat{k}) \quad (1)$$

$$\vec{r} = -\hat{i} + \hat{j} + \hat{k} + \frac{\mu}{3}(7\hat{i} - 7\hat{j} + 7\hat{k}) \quad (2)$$

For point of intersection, equating $\vec{r}$ in Eqs. (1) and (2), we get

$$\mu = \frac{6}{7}, \lambda = \frac{3}{7} \Rightarrow \text{PV of } \vec{P} = \hat{i} - \hat{j} + 3\hat{k}$$

17. We have $\overline{AB}\times\overline{AC} = 7\hat{j} + 7\hat{k}$

Since PF is parallel to $\overline{AB}\times\overline{AC}$ and $PF = \sqrt{2}$

Therefore,

$$\overline{PF} = \sqrt{2}\,\frac{7\hat{j} + 7\hat{k}}{\sqrt{49+49}} = \hat{j} + \hat{k}$$

18. Δ = Area of $\Delta ABC = \dfrac{1}{2}|\overline{AB}\times\overline{AC}|$

$$= \frac{1}{2}\left|(-3\hat{i} + \hat{j} - \hat{k})\times(-\hat{i} - 2\hat{j} + 2\hat{k})\right| = \frac{7\sqrt{2}}{2} \text{ sq. units}$$

Therefore,

$$\text{Volume of tetrahedron} = \frac{1}{3}\times\text{area of base}\times\text{height}$$

$$= \frac{7}{3} \text{ cubic units}$$

19. See Fig. 26.63. PV of $D = \dfrac{\lambda-1}{2}\hat{i} + 4\hat{j} + \dfrac{\mu+2}{2}\hat{k}$

DR of $AD = \dfrac{\lambda-4}{2}, 1, \dfrac{\mu-8}{2}$

But direction ratios of AD should be $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$. So,

$$\frac{\lambda-4}{2} = 1 = \frac{\mu-8}{2}$$

$$\lambda = 6, \mu = 10$$

$$2\lambda - \mu = 2$$

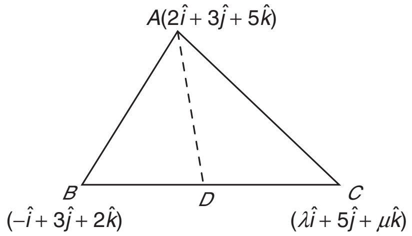


Figure 26.63

20. The given expression reduces to

$$\frac{(-\vec{a}\times\vec{b} + \vec{c}\times\vec{a} - \vec{b}\times\vec{a} + \vec{b}\times\vec{c})\cdot(\vec{a} + 2\vec{b} - \vec{c})}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

$$= \frac{[\vec{a}\;\vec{b}\;\vec{c}] + 2[\vec{a}\;\vec{b}\;\vec{c}] + [\vec{a}\;\vec{b}\;\vec{c}] - [\vec{a}\;\vec{b}\;\vec{c}]}{[\vec{a}\;\vec{b}\;\vec{c}]} = 3$$

Solved JEE 2017 Questions

JEE Main 2017

1. Let $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. Let $\vec{c}$ be a vector such that $|\vec{c}-\vec{a}|=3$, $|(\vec{a}\times\vec{b})\times\vec{c}|=3$ and the angle between $\vec{c}$ and $\vec{a}\times\vec{b}$ be 30°. Then $\vec{a}\cdot\vec{c}$ is equal to

(A) 2 **(B)** 5
(C) $\frac{1}{8}$ **(D)** $\frac{25}{8}$

(OFFLINE)

Solution: Let us find the value of $|\vec{a}\times\vec{b}|$:

$$|\vec{a}\times\vec{b}|=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -2\\ 1 & 1 & 0\end{vmatrix}=2\hat{i}-2\hat{j}+\hat{k}=3$$

Now, it is given that $|(\vec{a}\times\vec{b})\times\vec{c}|=3$. That is,

$$|\vec{a}\times\vec{b}||\vec{c}|\sin 30^\circ=3 \quad =3|\vec{c}|\cdot\frac{1}{2}=3 \quad =|\vec{c}|=2$$

Now,

$$|\vec{c}-\vec{a}|=3$$

Therefore,

$$|\vec{c}|^2+|\vec{a}|^2-2\vec{a}\cdot\vec{c}=9$$

$$4+9-2\vec{a}\cdot\vec{c}=9$$

Therefore, $\vec{a}\cdot\vec{c}=2$.

Hence, the correct answer is option (A).

2. The area (in sq. units) of the parallelogram, whose diagonals are along the vectors $8\hat{i}-6\hat{j}$ and $3\hat{i}+4\hat{j}-12\hat{k}$, is

(A) 65 **(B)** 52
(C) 26 **(D)** 20

(ONLINE)

Solution: The area of parallelogram when diagonals are given is

$$A=\frac{1}{2}|\vec{d}_1\times\vec{d}_2| \qquad (1)$$

The given vectors are $d_1=8\hat{i}-6\hat{j}$ and $d_2=3\hat{i}+4\hat{j}-12\hat{k}$.
Therefore, from Eq. (1), we get

$$\vec{d}_1\times\vec{d}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 8 & -6 & 0\\ 3 & 4 & -12\end{vmatrix}=\hat{i}(72-0)-\hat{j}(-96-0)+\hat{k}(32+18)$$

$$=72\hat{i}+96\hat{j}+50\hat{k}$$

$$=\sqrt{(72)^2+(96)^2+(50)^2}=\sqrt{5184+9216+2500}=\sqrt{16{,}900}=130$$

Therefore, the area of the parallelogram is

$$A=\frac{1}{2}|\vec{d}_1\times\vec{d}_2|=\frac{1}{2}\times 130= 65 \text{ sq. units}$$

Hence, the correct answer is option (A).

3. If the vector $\vec{b}=3\hat{j}+4\hat{k}$ is written as the sum of a vector $\vec{b}_1$, parallel to $\vec{a}=\hat{i}+\hat{j}$ and a vector $\vec{b}_2$, perpendicular to $\vec{a}$, then $\vec{b}_1\times\vec{b}_2$ is equal to

(A) $3\hat{i}-3\hat{j}+9\hat{k}$ **(B)** $-3\hat{i}+3\hat{j}-9\hat{k}$
(C) $-6\hat{i}+6\hat{j}-\frac{9}{2}\hat{k}$ **(D)** $6\hat{i}-6\hat{j}+\frac{9}{2}\hat{k}$

(ONLINE)

Solution: Given $\vec{a}=\hat{i}+\hat{j}$.
Since $\vec{b}_1$ is parallel to $\vec{a}$, we can say that $\vec{b}_1=k(\hat{i}+\hat{j})$
Let $\vec{b}_2=p\hat{i}+q\hat{j}+r\hat{k}$.
Since $\vec{a}\perp b_2$, we have $\vec{a}\cdot\vec{b}_2=0$. That is,

$$(p\hat{i}+q\hat{j}+r\hat{k})\cdot(\hat{i}+\hat{j})=0$$

Therefore,

$$p+q=0$$

Since

$$\vec{b}=\vec{b}_1+\vec{b}_2$$

we have

$$3\hat{j}+4\hat{k}=k(\hat{i}+\hat{j})+(p\hat{i}-p\hat{j}+r\hat{k})$$

Comparing the components, we get

$$0=k+p \qquad (1)$$
$$3=k-p \qquad (2)$$
$$4=r \qquad (3)$$

Adding Eqs. (1) and (2), we get

$$2k=3 \Rightarrow k=\frac{3}{2}$$

From Eq. (1), we get

$$p=-k=\frac{-3}{2}$$

$$\Rightarrow k=\frac{3}{2} \text{ and } p=\frac{-3}{2}$$

Hence,

$$\vec{b}_1=\frac{3}{2}\hat{i}+\frac{3}{2}\hat{j}$$

and

$$\vec{b}_2=\frac{-3}{2}\hat{i}+\frac{3}{2}\hat{j}+4\hat{k}$$

Therefore, $\vec{b}_1 \times \vec{b}_2$ is obtained as follows:

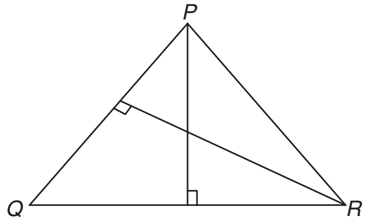
$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3/2 & 3/2 & 0 \\ -3/2 & 3/2 & 4 \end{vmatrix} = \hat{i}(6) - \hat{j}(6) + \hat{k}\left(\frac{9}{4} + \frac{9}{4}\right) = 6\hat{i} - 6\hat{j} + \frac{9}{2}\hat{k}$$

Hence, the correct answer is option (D).

JEE Advanced 2017

1. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that $\overrightarrow{OP}\cdot\overrightarrow{PQ} + \overrightarrow{OR}\cdot\overrightarrow{OS} = \overrightarrow{OR}\cdot\overrightarrow{OP} + \overrightarrow{OQ}\cdot\overrightarrow{OS} = \overrightarrow{OQ}\cdot\overrightarrow{OR} + \overrightarrow{OP}\cdot\overrightarrow{OS}$. Then the triangle PQR has S as its
   (A) centroid. **(B)** circumcenter.
   (C) incentre. **(D)** orthocenter.

Solution: The given geometrical situation for the triangle is depicted in the following figure:



It is given that

$$\overrightarrow{OP}\cdot\overrightarrow{OQ} + \overrightarrow{OR}\cdot\overrightarrow{OS} = \overrightarrow{OR}\cdot\overrightarrow{OP} + \overrightarrow{OQ}\cdot\overrightarrow{OS} = \overrightarrow{OQ}\cdot\overrightarrow{OR} + \overrightarrow{OP}\cdot\overrightarrow{OS}$$

Let us consider that

$$\overrightarrow{OP}\cdot\overrightarrow{OQ} + \overrightarrow{OR}\cdot\overrightarrow{OS} = \overrightarrow{OR}\cdot\overrightarrow{OP} + \overrightarrow{OQ}\cdot\overrightarrow{OS}$$

That is,

$$\overrightarrow{OP}\cdot\overrightarrow{OQ} + \overrightarrow{OR}\cdot\overrightarrow{OS} - \overrightarrow{OR}\cdot\overrightarrow{OP} - \overrightarrow{OQ}\cdot\overrightarrow{OS} = 0$$

$$\Rightarrow \overrightarrow{OP}\cdot(\overrightarrow{OQ} - \overrightarrow{OR}) + \overrightarrow{OS}\cdot(\overrightarrow{OR}\cdot\overrightarrow{OQ}) = 0$$

$$\Rightarrow (\overrightarrow{OP} - \overrightarrow{OS})\cdot(\overrightarrow{OQ} - \overrightarrow{OR}) = 0$$

Therefore, $\overrightarrow{SP}\cdot\overrightarrow{RQ} = 0 \;\Rightarrow\; \overrightarrow{SP} \perp \overrightarrow{RQ}$.
Similarly, let us consider

$$\overrightarrow{OR}\cdot\overrightarrow{OP} + \overrightarrow{OQ}\cdot\overrightarrow{OS} = \overrightarrow{OQ}\cdot\overrightarrow{OR} + \overrightarrow{OP}\cdot\overrightarrow{OS}$$

$$\Rightarrow \overrightarrow{OR}\cdot\overrightarrow{OP} + \overrightarrow{OQ}\cdot\overrightarrow{OS} - \overrightarrow{OQ}\cdot\overrightarrow{OR} - \overrightarrow{OP}\cdot\overrightarrow{OS} = 0$$

$$\Rightarrow \overrightarrow{OR}\cdot(\overrightarrow{OP} - \overrightarrow{OQ}) + \overrightarrow{OS}\cdot(\overrightarrow{OQ} - \overrightarrow{OP}) = 0$$

$$\Rightarrow (\overrightarrow{OR} - \overrightarrow{OS})\cdot(\overrightarrow{OP} - \overrightarrow{OQ}) = 0$$

$$\Rightarrow \overrightarrow{SR}\cdot\overrightarrow{QP} = 0$$

$$\Rightarrow \overrightarrow{SR} \perp \overrightarrow{QP}$$

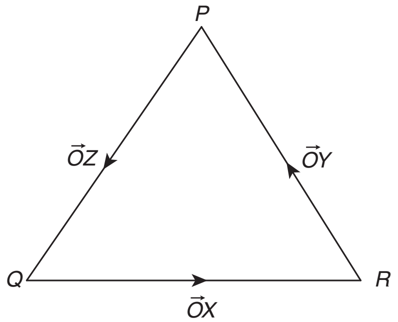
Thus, point S is orthocentre of the triangle PQR.

Hence, the correct answer is option (D).

Paragraph for Questions 2 and 3: Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR.

2. $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| =$ _____.
   (A) $\sin(P+Q)$ **(B)** $\sin 2R$
   (C) $\sin(P+R)$ **(D)** $\sin(Q+R)$

Solution: The given geometrical situation is depicted in the following figure:



Now,

$$\overrightarrow{OX} = \frac{\overrightarrow{QR}}{QR}$$

and

$$\overrightarrow{OY} = \frac{\overrightarrow{RP}}{RP}$$

Therefore,

$$(\overrightarrow{OX} \times \overrightarrow{OY}) = \frac{\overrightarrow{QR}}{QR} \times \frac{\overrightarrow{RP}}{RP} = \frac{\overrightarrow{QR} \times \overrightarrow{RP}}{PQ}$$

$$= \frac{PQ \sin R}{PQ} = \sin R = \sin(\pi - (P+Q) = \sin(P+Q)$$

Hence, the correct answer is option (A).

3. If the triangle PQR varies, then the minimum value of cos $(P + Q) + \cos(Q + R) + \cos(R + P)$ is
   (A) $-\frac{5}{3}$ **(B)** $-\frac{3}{2}$
   (C) $\frac{3}{2}$ **(D)** $\frac{5}{3}$

Solution: It is given that

$$\cos(P+Q) + \cos(Q+R) + \cos(R+P) = \cos R + \cos P + \cos Q$$

In the given triangle, the maximum value of

$$\cos P + \cos Q + \cos R = \frac{3}{2}$$

Therefore, the minimum value of

$$\cos P + \cos Q + \cos R = \frac{-3}{2}$$

$$\Rightarrow \cos(P+Q) + \cos(Q+R) + \cos(R+P) = \frac{-3}{2}$$

Hence, the correct answer is option (B).

27 Three-Dimensional Geometry

27.1 Rectangular Coordinate System in Space

Let 'O' be any point in space and $\overleftrightarrow{X'OX}$, $\overleftrightarrow{Y'OY}$ and $\overleftrightarrow{Z'OZ}$ be three lines perpendicular to each other (Fig. 27.1). These lines are known as coordinate axes and *O* is called the origin. The planes *XY*, *YZ* and *ZX* are known as the coordinate planes.

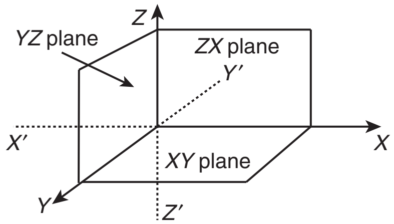


Figure 27.1

27.1.1 Coordinates of a Point in Space

Consider a point *P* in space. The position of the point *P* is given by triad (*x*, *y*, *z*), where *x*, *y* and *z* are perpendicular distances from *YZ*-plane, *ZX*-plane and *XY*-plane, respectively (Fig. 27.2).

If we assume $\hat{i}, \hat{j}, \hat{k}$ unit vectors along *OX*, *OY* and *OZ*, respectively, then the position vector of point *P* is $x\hat{i} + y\hat{j} + z\hat{k}$ or simply (*x*, *y*, *z*).

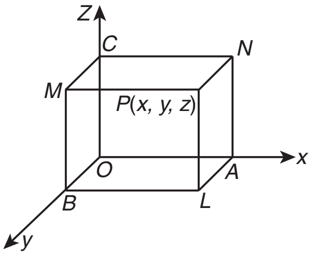


Figure 27.2

Note:

1. x-axis $= \{(x, y, z) \mid y = z = 0\}$
2. y-axis $= \{(x, y, z) \mid x = z = 0\}$
3. z-axis $= \{(x, y, z) \mid x = y = 0\}$
4. xy-plane $= \{(x, y, z) \mid z = 0\}$
5. yz-plane $= \{(x, y, z) \mid x = 0\}$
6. zx-plane $= \{(x, y, z) \mid y = 0\}$
7. $OP = \sqrt{x^2 + y^2 + z^2}$

27.1.2 Signs of Coordinates of a Point

The signs of the coordinates of a point in three dimensions follow the convention that all distances measured along or parallel to *OX*, *OY* and *OZ* will be positive and distances moved along or parallel to *OX*′, *OY*′ and *OZ*′ will be negative.

27.2 Other Methods of Defining the Position of Any Point *P* in Space

27.2.1 Cylindrical Coordinates

See Fig. 27.3. If the rectangular Cartesian coordinates of *P* are (*x*, *y*, *z*), then those of *N* are (*x*, *y*, 0), and we can easily have the following relations: $x = u\cos\phi$, $y = u\sin\phi$ and $z = z$.
Hence,

$$u^2 = x^2 + y^2$$

and

$$\phi = \tan^{-1}(y/x)$$

Cylindrical coordinates of $P \equiv (u, \phi, z)$.

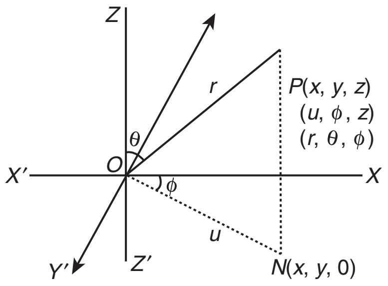


Figure 27.3

27.2.2 Spherical Polar Coordinates

The measures of quantities *r*, θ and ϕ are known as spherical or three-dimensional polar coordinates of the point *P*. If the rectangular Cartesian coordinates of *P* are (*x*, *y*, *z*), then $z = r\cos\theta$ and $u = r\sin\theta$

Therefore,

$$x = u\cos\phi = r\sin\theta\cos\phi,$$

$$y = u\sin\phi = r\sin\theta\sin\phi$$

and

$$z = r\cos\theta$$

Also,

$$r^2 = x^2 + y^2 + z^2$$

and

$$\tan\theta = \frac{u}{z} = \frac{\sqrt{x^2 + y^2}}{z},$$

$$\tan\phi = \frac{y}{x}$$

27.3 Shifting the Origin

Shifting the origin to another point without changing the directions of the axes is called the translation of axes.

Let the origin O be shifted to another point $O'(x', y', z')$ without changing the direction of axes (Fig. 27.4). Let the new coordinate frame be $O'X'Y'Z'$. Let $P(x, y, z)$ be a point with respect to the coordinate frame $OXYZ$.

Then, the coordinates of point P w.r.t. the new coordinate frame $O'X'Y'Z'$ are (x_1, y_1, z_1), where $x_1 = x - x'$, $y_1 = y - y'$ and $z_1 = z - z'$

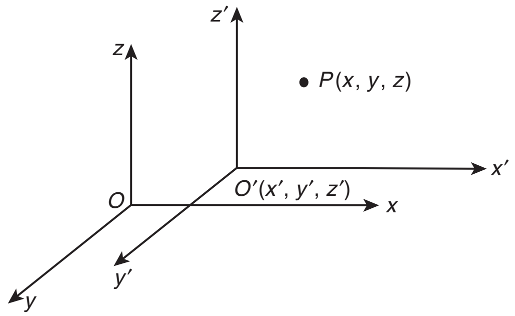


Figure 27.4

Illustration 27.1 If the origin is shifted (1, 2, –3) without changing the directions of the axes, then find the new coordinates of the point (0, 4, 5) with respect to the new frame.

Solution:

$$x' = x - x_1$$

where (x_1, y_1, z_1) is the shifted origin

$$y' = y - y_1$$

$$z' = z - z_1$$

$$x' = 0 - 1 = -1$$

$$y' = 4 - 2 = 2$$

$$z' = 5 + 3 = 8$$

Therefore, the coordinates of the point w.r.t. to the new coordinate frame are (–1, 2, 8).

27.4 Distance Formula

The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

The distance between the origin and point $P(x, y, z)$ is given by

$$OP = \sqrt{x^2 + y^2 + z^2}$$

27.4.1 Distance of a Point from Coordinate Axes

Let $P(x, y, z)$ be any point in the space. Let PA, PB and PC be the perpendiculars drawn from P to the axes OX, OY and OZ, respectively. Then

$$PA = \sqrt{(y^2 + z^2)}, \qquad PB = \sqrt{(z^2 + x^2)}$$

and

$$PC = \sqrt{(x^2 + y^2)}$$

Illustration 27.2 Prove that points (5, –4, 2), (4, –3, 1), (7, –6, 4) and (8, –7, 5) are the vertices of a parallelogram.

Solution: Let the points be $A(5, -4, 2)$, $B(4, -3, 1)$, $C(7, -6, 4)$ and $D(8, -7, 5)$.

$$AB = \sqrt{1+1+1} = \sqrt{3},$$

$$CD = \sqrt{1+1+1} = \sqrt{3},$$

$$BC = \sqrt{9+9+9} = 3\sqrt{3}$$

and

$$AD = \sqrt{9+9+9} = 3\sqrt{3}$$

Length of diagonals

$$AC = \sqrt{4+4+4} = 2\sqrt{3}$$

and

$$BD = \sqrt{16+16+16} = 4\sqrt{3}$$

That is,

$$AC \neq BD.$$

Hence, A, B, C and D are vertices of a parallelogram.

27.5 Section Formula

27.5.1 Internal Division

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Let R be a point on the line segment joining P and Q such that it divides the joining of P and Q internally in the ratio $m_1 : m_2$.

Then, the coordinates of R are

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

27.5.2 External Division

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let R be a point on PQ produced, dividing it externally in the ratio $m_1 : m_2$ $(m_1 \uparrow m_2)$.

Then the coordinates of R are

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$$

27.5.3 Coordinates of the Mid-Point

When division point is the mid-point of PQ, then the ratio will be 1:1, hence coordinates of the mid-point of PQ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

27.5.4 Coordinates of the General Point

The coordinates of any point lying on the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as $\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$, which divides PQ in the ratio k:1.

This is called the general point on the line PQ.

Note: The coordinates of centroid of a triangle having vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Illustration 27.3 If the x-coordinate of a point P on the join of $Q(2, 2, 1)$ and $R(5, 1, -2)$ is 4, then find its z-coordinate.

Solution: Let the point P be

$$\left(\frac{5k+2}{k+1}, \frac{k+2}{k+1}, \frac{-2k+1}{k+1}\right)$$

Given that

$$\frac{5k+2}{k+1} = 4 \Rightarrow k = 2$$

Therefore,

$$z\text{-coordinate of } P = \frac{-2(2)+1}{2+1} = -1$$

Illustration 27.4 Find the coordinates of the point which divides the line joining the points (2, 3, 4) and (3, –4, 7) in the ratio 3:5.

Solution: Let the coordinates of the required point be (x, y, z). Then

$$x = \frac{2(5)+3(3)}{3+5} = \frac{19}{8}$$

$$y = \frac{3(5)-4(3)}{3+5} = \frac{3}{8}$$

and

$$z = \frac{4(5)+7(3)}{3+5} = \frac{41}{8}$$

Hence, the required point is $\left(\frac{19}{8}, \frac{3}{8}, \frac{41}{8}\right)$.

Illustration 27.5 Prove that the three points $A(3, -2, 4)$, $B(1, 1, 1)$ and $C(-1, 4, -2)$ are collinear.

Solution: The general coordinates of a point R, which divides the line joining $A(3, -2, 4)$ and $B(1, 1, 1)$ in the ratio μ:1 are

$$\left(\frac{\mu+3}{\mu+1}, \frac{\mu-2}{\mu+1}, \frac{\mu+4}{\mu+1}\right) \quad (1)$$

If $C(-1, 4, -2)$ lies on the line AB, then for some value of μ, the coordinates of the point R will be the same as those of C.
Let x-coordinate of point R = x-coordinate of point C.
Then,

$$\frac{\mu+3}{\mu+1} = -1 \Rightarrow \mu = -2$$

Putting $\mu = -2$ in Eq. (1), the coordinates of R are (–1, 4, –2), which are also the coordinates of C.
Hence, the points A, B and C are collinear.

27.6 Triangle and Tetrahedron

27.6.1 Coordinates of the Centroid

1. If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) are the vertices of a triangle, then coordinates of its centroid are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

2. If (x_r, y_r, z_r), $r = 1, 2, 3, 4$, are vertices of a tetrahedron, then the coordinates of its centroid are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

27.6.2 Area of a Triangle

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of a triangle. Then

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \quad \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$$

and

$$\Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Area of ΔABC is given by the relation

$$\Delta = \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$$

Also,

$$\Delta = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left\|\begin{matrix} i & j & k \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{matrix}\right\|$$

27.6.3 Volume of a Tetrahedron

Volume of a tetrahedron with vertices (x_r, y_r, z_r), $r = 1, 2, 3, 4$, is

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

27.6.4 Condition of Collinearity

Points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are collinear if

$$\frac{x_1-x_2}{x_2-x_3} = \frac{y_1-y_2}{y_2-y_3} = \frac{z_1-z_2}{z_2-z_3}$$

Illustration 27.6 If the centroid of a tetrahedron $OABC$, where A, B and C are given by $(a, 2, 3)$, $(1, b, 2)$ and $(2, 1, c)$, respectively, be (1, 2, –1), then find the distance of $P(a, b, c)$ from the origin.

Solution: (1, 2, –1) is the centroid of the tetrahedron. Therefore,

$$1 = \frac{0+a+1+2}{4} \Rightarrow a = 1,$$

$$2 = \frac{0+2+b+1}{4} \Rightarrow b = 5,$$

$$-1 = \frac{0+3+2+c}{4} \Rightarrow c = -9$$

Therefore,

$$(a, b, c) = (1, 5, -9)$$

Its distance from the origin $= \sqrt{1+25+81} = \sqrt{107}$ units.

27.7 Direction Cosines of a Line

If α, β and γ be the angles that a given directed line makes with the positive directions of the coordinate axes, then cos α, cos β and cos γ are called the direction cosines of the given line and are generally denoted by l, m and n, respectively (Fig. 27.5).

Thus, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

By the definition it follows that the direction cosines of the axis of x are respectively cos 0°, cos 90° and cos 90°, that is, (1, 0, 0).

Similarly, direction cosines of the axes of y and z are, respectively, (0, 1, 0) and (0, 0, 1).

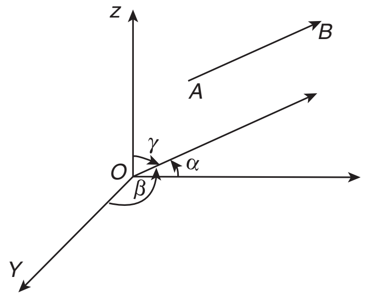


Figure 27.5

27.7.1 Relation Between the Direction Cosines

Let OP be any line through the origin O, which has direction cosines l, m and n (Fig. 27.6)

Let $P \equiv (x, y, z)$ and $OP = r$. Then

$$OP^2 = x^2 + y^2 + z^2 = r^2 \tag{27.1}$$

From P, draw PA, PB and PC perpendicular on the coordinate axes, so that

$$OA = x,\ OB = y \text{ and } OC = z$$

Also,

$$\angle POA = \alpha,\ \angle POB = \beta \text{ and } \angle POC = \gamma$$

From the triangle AOP,

$$l = \cos\alpha = \frac{x}{r} \Rightarrow x = lr$$

Similarly, $y = mr$ and $z = nr$.

Hence, from Eq. (27.1)

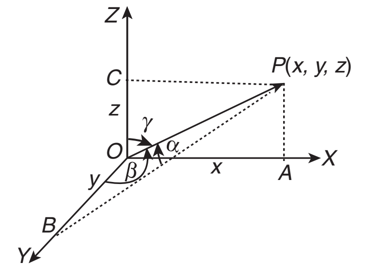


Figure 27.6

$$r^2(l^2 + m^2 + n^2) = x^2 + y^2 + z^2 = r^2 \Rightarrow l^2 + m^2 + n^2 = 1$$

Note:

1. If the coordinates of any point P be (x, y, z) and l, m and n be the direction cosines of the line OP, O being the origin, then lr, mr, nr will give us the coordinates of a point on the line OP which is at a distance r from (0, 0, 0).
2. If $OP = r$ and the coordinates of point P be (x, y, z), then dc's of the line OP are x/r, y/r and z/r.
3. Direction cosines of $\vec{r} = ai + bj + ck$ are $\dfrac{a}{|\vec{r}|}, \dfrac{b}{|\vec{r}|}, \dfrac{c}{|\vec{r}|}$.
4. Since $-1 \le \cos x \le 1\ \forall x \in R$, hence values of l, m and n are such real numbers that are not less than −1 and not greater than 1. Hence, direction cosines $\in [-1, 1]$.
5. The direction cosines of a line parallel to any coordinate axis are equal to the direction cosines of the coordinate axis.
6. The number of lines, which are equally inclined to the coordinate axes, is 4.
7. If l, m and n are the dc's of a line, then the maximum value of $lmn = \dfrac{1}{3\sqrt{3}}$.
8. The angles α, β and γ are called the direction angles of the line AB.
9. The direction cosines of the line BA are $\cos(\pi - \alpha)$, $\cos(\pi - \beta)$ and $\cos(\pi - \gamma)$, that is, $-\cos\alpha$, $-\cos\beta$ and $-\cos\gamma$.
10. Angles α, β and γ are not coplanar.
11. $\alpha + \beta + \gamma$ is not equal to 360° as these angles do not lie in same plane.
12. If $P(x, y, z)$ be a point in space such that $\vec{r} = \overrightarrow{OP}$ has dc's l, m and n, then $x = l|\vec{r}|, y = m|\vec{r}|, z = n|\vec{r}|$.
13. Projections of a vector r on the coordinate axes are $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$.
14. $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$ and $\vec{r} = (l\hat{i} + m\hat{j} + n\hat{k})$.

27.8 Direction Ratios

If a, b and c are three numbers proportional to the direction cosines l, m and n of a straight line, then a, b and c are called its direction ratios. They are also called the direction numbers or direction components.

Hence, by definition we have

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k \text{ (say)} \Rightarrow l = ak$$

and

$$m = bk$$
$$n = ck$$

$$\Rightarrow k^2(a^2 + b^2 + c^2) = l^2 + m^2 + n^2 = 1$$

or

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

or

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

$$\Rightarrow k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{1}{\sqrt{\sum a^2}}$$

Therefore,

$$l = \pm\frac{a}{\sqrt{\sum a^2}}$$

Similarly,

$$m = \pm\frac{b}{\sqrt{\sum a^2}}$$

and

$$n = \pm\frac{c}{\sqrt{\sum a^2}}$$

where the same sign, either positive or negative, is to be chosen throughout.

Note:

1. Direction cosines of a line are unique but direction ratios (dr's) of a line is in no way unique but can be infinite, i.e. $a^2+b^2+c^2 \neq 1$.
2. Let $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$ be a vector. Then, its dr's are a, b and c.
   If a vector $\vec{r}$ has dr's a, b and c, then
   $$\vec{r}=\frac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i}+b\hat{j}+c\hat{k})$$
   For example, if $\hat{r}=2\hat{i}-3\hat{j}+10\hat{k}$, then its direction ratios are 2, –3 and 10 or 4, –6 and 20 or any positive multiple of the components or direction cosines of $\vec{r}$.

27.8.1 Direction Cosine and Direction Ratio of a Line joining Two Given Points

See Fig. 27.7. The direction ratios of the line PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$x_2-x_1=a \text{ (say)}$$
$$y_2-y_1=b \text{ (say)}$$
and
$$z_2-z_1=c \text{ (say)}$$

Then direction cosines are

$$l=\frac{(x_2-x_1)}{\sqrt{\sum(x_2-x_1)^2}}$$
$$m=\frac{(y_2-y_1)}{\sqrt{\sum(x_2-x_1)^2}}$$
and
$$n=\frac{(z_2-z_1)}{\sqrt{\sum(x_2-x_1)^2}}$$

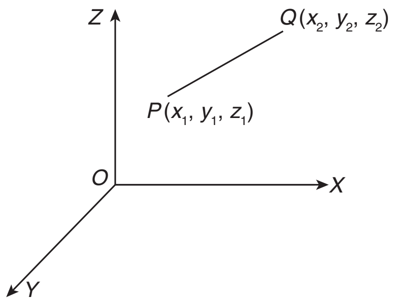


Figure 27.7

Illustration 27.7 A vector $\vec{r}$ has length 21 and direction ratios 2, –3 and 6. Find the vector $\vec{r}$.

Solution: The direction cosines of $\vec{r}$ are

$$\pm\frac{2}{\sqrt{2^2+(-3)^2+6^2}}, \pm\frac{-3}{\sqrt{2^2+(-3)^2+6^2}}, \pm\frac{6}{\sqrt{2^2+(-3)^2+6^2}}$$

Since $\vec{r}$ makes an acute angle with x-axis, therefore cos $\alpha > 0$, i.e. $l > 0$.

So, direction cosines of $\vec{r}$ are $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$.

Therefore,

$$\vec{r}=21\left(\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right) \text{ [using } \vec{r}=|\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k})\text{]}$$

or

$$\vec{r}=6\hat{i}-9\hat{j}+18\hat{k}$$

So, the components of $\vec{r}$ along OX, OY and OZ are $6\hat{i}, -9\hat{j}$ and $18\hat{k}$ respectively.

Illustration 27.8 Find the angle between the vectors with direction ratios 4, –3 and 5 and 3, 4 and 5.

Solution: Let $\vec{a}$ be a vector parallel to the vector having direction ratios 4, –3 and 5

$$\vec{a}=4\hat{i}-3\hat{j}+5\hat{k}$$

and $\vec{b}$ be a vector parallel to the vector having direction ratios 3, 4 and 5

$$\vec{b}=3\hat{i}+4\hat{j}+5\hat{k}$$

Let θ be the angle between the given vectors.

Then,

$$\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{12-12+25}{\sqrt{16+9+25}\sqrt{9+16+25}}=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$$

Thus, the angle between the vectors with direction ratios 4, –3, 5 and 3, 4, 5 is 60°.

Illustration 27.9 Find the direction ratios and direction cosines of the line joining the points $A(6, -7, -1)$ and $B(2, -3, 1)$.

Solution: Direction ratios of AB are

$$(4, -4, -2)=(2, -2, -1)$$
$$a^2+b^2+c^2=9$$

Direction cosines are $\left(\pm\frac{2}{3}, \mp\frac{2}{3}, \mp\frac{1}{3}\right)$.

Illustration 27.10 A line makes the same angle θ with each of the x- and z-axis. If the angle β, which it makes with y-axis, is such that $\sin^2\beta=3\sin^2\theta$, then $\cos^2\theta$ equals.

Solution: We know that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

Since line makes angle θ with x- and z-axis and angle β with y-axis.

$$\cos^2\theta+\cos^2\beta+\cos^2\theta=1$$
$$\Rightarrow -(2\cos^2\theta-1)=\cos^2\beta \quad (1)$$

Given that

$$\sin^2\beta=3\sin^2\theta \quad (2)$$

From Eqs. (1) and (2),

$$1=3\sin^2\theta-2\cos^2\theta+1$$
$$\Rightarrow 0=3(1-\cos^2\theta)-2\cos^2\theta$$
$$\Rightarrow 5\cos^2\theta=3$$
$$\Rightarrow \cos^2\theta=3/5$$

Illustration 27.11 Find the direction cosines of the line that makes equal angles with the three axes in a space.

Solution: Since,

$$l^2+m^2+n^2=1$$
$$\Rightarrow \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$$

Now,

$$\alpha=\beta=\gamma$$

$$\Rightarrow 3\cos^2\alpha = 1$$
$$\Rightarrow \cos\alpha = \pm 1/\sqrt{3}$$

That is,

$$l = m = n = \pm 1/\sqrt{3}$$

Hence, required dc's are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$.

Your Turn 1

1. If the vertices of a triangle are $A(1, -1, 2)$, $B(2, 0, -1)$ and $C(0, 2, 1)$, then find the area of the triangle. **Ans.** $2\sqrt{6}$
2. If the points (5, 2, 4), (6, –1, 2) and (8, –7, k) are collinear, then find k. **Ans.** –2
3. A line that makes angle 60° with y-axis and z-axis, then the angle, which it makes with x-axis is
   (A) 45° **(B)** 60° **(C)** 75° **(D)** 30°
   Ans. (A)
4. A line passes through the points (6, –7, –1) and (2, –3, 1). The direction cosines of the line are directed so that the angle made by it with the positive direction of x-axis is acute are
   (A) $\frac{2}{3}, \frac{-2}{3}, \frac{-1}{3}$ **(B)** $\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}$
   (C) $\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}$ **(D)** $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$ **Ans.** (A)
5. If the direction cosines of a line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$, then
   (A) $c > 0$ **(B)** $c = \pm\sqrt{3}$
   (C) $0 < c < 1$ **(D)** $c > 2$ **Ans.** (B)
6. If $\vec{r}$ is a vector of magnitude 21 and has dr's 2, –3 and 6. Then $\vec{r}$ is equal to
   (A) $6i - 9\phi + 18k$ **(B)** $6i + 9\phi + 18k$
   (C) $6i - 9\phi - 18k$ **(D)** $6i + 9\phi - 18k$ **Ans.** (A)

27.9 Projection of a Line

See Fig. 27.8. Projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m and n is

$$AB = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

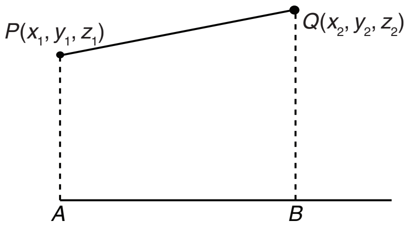


Figure 27.8

27.9.1 Perpendicular Distance of a Point from a Line

Let AB be a straight line passing through the point $A(a, b, c)$ and having direction cosines l, m and n (Fig. 27.9).

AN = projection of the line AP on the straight line

$$AN = l(x - a) + m(y - b) + n(z - c)$$

and

$$AP = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$$

Therefore, the perpendicular distance of point P

$$PN = \sqrt{AP^2 - AN^2}$$

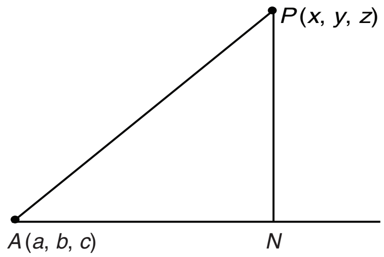


Figure 27.9

Illustration 27.12 Find out the perpendicular distance of point $P(0, -1, 3)$ from a straight line passing through $A(1, -3, 2)$ and having direction ratios 1, 2 and 2.

Solution: Direction cosines of the line are

$$\frac{1}{\sqrt{1^2+2^2+2^2}}, \frac{2}{\sqrt{1^2+2^2+2^2}}, \frac{2}{\sqrt{1^2+2^2+2^2}}, \text{ that is, } \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$$

Therefore,

$$PN = l(x - a) + m(y - b) + n(z - c)$$
$$= \frac{1}{3}(0-1) + \frac{2}{3}(-1+3) + \frac{2}{3}(3-2) = \frac{5}{3}$$

$$AP = \sqrt{(0-1)^2 + (-1+3)^2 + (3-2)^2} = \sqrt{6}$$

Therefore, the perpendicular distance

$$PN = \sqrt{AP^2 - PN^2} = \sqrt{6 - \frac{25}{9}} = \frac{\sqrt{29}}{3}$$

Illustration 27.13 If A, B, C and D are the points (3, 4, 5), (4, 6, 3), (–1, 2, 4) and (1, 0, 5), then the projection of CD on AB is

(A) $\frac{3}{4}$ **(B)** $\frac{-4}{3}$ **(C)** $\frac{3}{5}$ **(D)** None of these

Solution: Let l, m and n be the direction cosines of AB. Then

$$l = \frac{4-3}{\sqrt{(4-3)^2 + (6-4)^2 + (3-5)^2}} = \frac{1}{3}$$

and

$$m = \frac{6-4}{3} = \frac{2}{3}$$

Similarly,

$$n = \frac{-2}{3}$$

Therefore,

$$\text{Projection of } CD \text{ on } AB = [1-(-1)]\left(\frac{1}{3}\right) + [0-2]\left(\frac{2}{3}\right) + [5-4]\left(-\frac{2}{3}\right)$$
$$= \frac{2}{3} - \frac{4}{3} + \left(-\frac{2}{3}\right) = -\frac{4}{3}$$

Hence, the correct answer is option (B).

Illustration 27.14 The projections of a line on coordinate axes are 2, 3 and 6. Then find the length of the line.

Solution: Let AB be the line and its direction cosines be $\cos\alpha$, $\cos\beta$ and $\cos\gamma$. Then the projections of the line AB on the coordinate axes are $AB\cos\alpha$, $AB\cos\beta$ and $AB\cos\gamma$.

Therefore,

$$AB\cos\alpha = 2,$$
$$AB\cos\beta = 3$$

and
$$AB\cos\gamma = 6$$

So,

$$AB^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2^2 + 3^2 + 6^2 = 49$$
$$\Rightarrow AB^2(1) = 49 \Rightarrow AB = 7$$

27.10 Equation of a Straight Line in Space

Every equation of the first degree represents a plane. Two equations of the first degree are satisfied by the coordinates of every point on the line of intersection of the planes represented by them. Therefore, the two equations together represent that line. Therefore, $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ together represent a straight line.

27.10.1 Vector Equation of a Line Passing Through a Given Point and Parallel to a Given Vector

Let A be a fixed point having position vector $\vec{a}$ and the line is parallel to the vector $\vec{b}$. P is an arbitrary point having position vector $\vec{r}$ on the line (Fig. 27.10).

From ΔOAP,

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} \Rightarrow \vec{r} = \vec{a} + \lambda\vec{b}$$

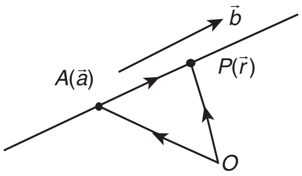


Figure 27.10

This is the required equation of the line. λ is an arbitrary real number.

27.10.2 Cartesian Equation of a Line Passing Through a Given Point and Given Direction Ratios

Let $A(a_1, a_2, a_3)$ be the fixed point and the line has direction ratios b_1, b_2 and b_3.

Taking $\vec{r}$ as $x\hat{i} + y\hat{j} + z\hat{k}$ in the vector equation, we see that

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \lambda$$

This is the Cartesian equation of the line, also called the symmetrical form of a line. Any point on this line can be taken as $(a_1 + b_1\lambda, a_2 + b_2\lambda, a_3 + b_3\lambda)$.

Direction ratios b_1, b_2 and b_3 can also be replaced by the direction cosines l, m and n of vector $\vec{b}$.

Note:

1. The parametric equations of the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ are $x = x_1 + a\lambda, y = y_1 + b\lambda, z = z_1 + c\lambda$, where λ is the parameter.
2. The coordinates of any point on the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ are $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where $\lambda \in$ R.
3. Since the direction cosines of a line are also direction ratios, therefore equation of a line passing through (x_1, y_1, z_1) and having direction cosines l, m and n is
$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$
4. Since x-, y- and z-axis pass through the origin and have direction cosines 1, 0, 0; 0, 1 , 0; and 0, 0, 1, respectively. Therefore, the equations are
$$x\text{-axis: } \frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0} \text{ or } y = 0 \text{ and } z = 0,$$
$$y\text{-axis: } \frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0} \text{ or } x = 0 \text{ and } z = 0$$
and
$$z\text{-axis: } \frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1} \text{ or } x = 0 \text{ and } y = 0.$$
5. In the symmetrical form of the equation of a line, the coefficients of x, y and z are unity.

27.10.3 Vector Equation of a Line Passing Through Two Given Points

Let A and B be two fixed points having position vectors $\vec{a}$ and $\vec{b}$. P is a variable point on the line (Fig. 27.11).

From ΔOPA again,

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

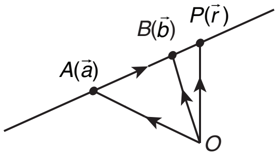


Figure 27.11

$$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \lambda(\overrightarrow{AB})$$
$$\Rightarrow \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

This is the required equation.

27.10.4 Cartesian Equation of a Line Passing Through Two Given Points

If the coordinates of A and B are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the Cartesian equation is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Illustration 27.15 Find the vector equation of a line passing through a point with position vector $2\hat{i} - \hat{j} + \hat{k}$, and parallel to the line joining the points $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$. Also find the Cartesian equivalent of this equation.

Solution: Let A, B and C be the points with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $-\hat{i}+4\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+2\hat{k}$, respectively.

We have to find the equation of a line passing through the point A and parallel to $\overrightarrow{BC}$.
Now,

$$\overrightarrow{BC} = \text{position vector of } C - \text{position vector of } B$$
$$= (\hat{i}+2\hat{j}+2\hat{k})-(-\hat{i}+4\hat{j}+\hat{k}) = 2\hat{i}-2\hat{j}+\hat{k}$$

We know that the equation of a line passing through a point $\vec{a}$ and parallel to $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$.
Here,

$$\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$$

So, the equation of the required line

$$\vec{r} = (2\hat{i}-\hat{j}+\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k}) \quad (1)$$

Reduction to Cartesian form by putting

$$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

We obtain

$$x\hat{i}+y\hat{j}+z\hat{k} = (2+2\lambda)\hat{i}+(-1-2\lambda)\hat{j}+(1+\lambda)\hat{k}$$
$$\Rightarrow x = 2+2\lambda, y = -1-2\lambda, z = 1+\lambda$$
$$\Rightarrow \frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$$

which is the Cartesian equivalent of Eq. (1).

Illustration 27.16 The Cartesian equation of a line is $6x-2=3y+1=2z-2$. Find its direction ratios and also find vector equation of the line.

Solution: Recall that in the symmetrical form of a line, coefficients of x, y and z are unity. Therefore, to put the given line in a symmetric form, we must make the coefficients of x, y and z as unity. The given line is

$$6x-2 = 3y+1 = 2z-2$$
$$\Rightarrow 6\left(x-\frac{1}{3}\right) = 3\left(y+\frac{1}{3}\right) = 2(z-1)$$
$$\Rightarrow \frac{x-\frac{1}{3}}{1} = \frac{y+\frac{1}{3}}{2} = \frac{z-1}{3}$$

This shows that the given line passes through (1/3, –1/3, 1), and has direction ratios 1, 2 and 3. In vector form, this means that the line passes through the point having position vector $\vec{a}=\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}$ and is parallel to the vector $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$.
Therefore, its vector equation is

$$\vec{r} = \left(\frac{1}{3}\hat{i}-\frac{1}{3}\hat{j}+\hat{k}\right)+\lambda(\hat{i}+2\hat{j}+3\hat{k})$$

27.11 Angle Between Two Lines

27.11.1 Cartesian Form

See Fig. 27.12. Let θ be the angle between two straight lines AB and AC, whose direction cosines are l_1, m_1 and n_1 and l_2, m_2 and n_2, respectively, and is given by

$$\cos\theta = l_1l_2+m_1m_2+n_1n_2$$

If the direction ratios of two lines a_1, b_1, c_1, and a_2, b_2, c_2 are given, then the angle between the two lines is given by

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$$

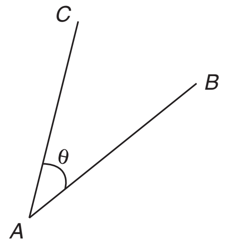


Figure 27.12

27.11.1.1 Particular Results

We have

$$\sin^2\theta = 1-\cos^2\theta$$
$$= \left(l_1^2+m_1^2+n_1^2\right)\left(l_2^2+m_2^2+n_2^2\right)-(l_1l_2+m_1m_2+n_1n_2)^2$$
$$= (l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2$$
$$\Rightarrow \sin\theta = \pm\sqrt{\sum(l_1m_2-l_2m_1)^2}$$

When dr's of the two lines are given as a_1, b_1, c_1 and a_2, b_2, c_2, then angle θ between them is given by

$$\sin\theta = \frac{\sqrt{\Sigma(a_1b_2-a_2b_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

27.11.1.2 Condition of Perpendicularity

If the given lines are perpendicular, then $\theta = 90°$.
That is,

$$\cos\theta = 0 \Rightarrow l_1l_2+m_1m_2+n_1n_2 = 0$$

or

$$a_1a_2+b_1b_2+c_1c_2 = 0$$

27.11.1.3 Condition of Parallelism

If the given lines are parallel, then $\theta = 0°$
That is,

$$\sin\theta = 0 \Rightarrow (l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2 = 0,$$

which is true, only when

$$l_1m_2-l_2m_1 = 0$$
$$m_1n_2-m_2n_1 = 0$$

and

$$n_1l_2-n_2l_1 = 0$$
$$\Rightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

Similarly,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Note:

1. The angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

2. The angle between a diagonal of a cube and the diagonal of the faces of the cube is $\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$.
3. If a straight line makes angles α, β, γ and δ with the diagonals of a cube, then

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$$

4. If the edges of a rectangular parallelepiped be a, b and c, then the angles between the two diagonals are $\cos^{-1}\left[\frac{\pm a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right]$

27.11.2 Vector Form

Let the vector equations of two lines be $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$.

As the lines are parallel to the vectors $\vec{b}_1$ and $\vec{b}_2$, respectively, therefore the angle between the lines is same as the angle between the vectors $\vec{b}_1$ and $\vec{b}_2$.

Thus, if θ is the angle between the given lines, then

$$\cos\theta=\frac{\vec{b}_1\cdot\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$$

If the lines are perpendicular, then $\vec{b}_1\cdot\vec{b}_2=0$ and if the lines are parallel, then $\vec{b}_1$ and $\vec{b}_2$ are parallel, therefore $\vec{b}_1=\lambda\vec{b}_2$ for some scalar λ.

Illustration 27.17 Find the angle between the lines $\frac{x-2}{3}=\frac{y+1}{-2}, z=2$ and $\frac{x-1}{1}=\frac{2y+3}{3}=\frac{z+5}{2}$.

Solution: The given equations are not in the standard form. The equations of the given lines can be written as

$$\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-2}{0} \quad (1)$$

and

$$\frac{x-1}{1}=\frac{y+3/2}{3/2}=\frac{z+5}{2} \quad (2)$$

Let $\vec{b}_1$ and $\vec{b}_2$ be vectors parallel to Eqs. (1) and (2), respectively, then,

$$\vec{b}_1=3\hat{i}-2\hat{j}+0\hat{k}$$

and

$$\vec{b}_2=\hat{i}+\frac{3}{2}\hat{j}+2\hat{k}$$

If θ is the angle between the given lines, then

$$\cos\theta=\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1|\vec{b}_2|}=\frac{(3)(1)+(-2)(3/2)+(0)(2)}{\sqrt{3^2+(-2)^2}\sqrt{1^2+(3/2)^2+2^2}}=0$$

$$\Rightarrow \theta=\pi/2$$

Your Turn 2

1. If dc's of two lines are proportional to (2, 3, –6) and (3, –4, 5), then the acute angle between them is

(A) $\cos^{-1}\left(\frac{49}{36}\right)$ **(B)** $\cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$

(C) 90° **(D)** $\cos^{-1}\left(\frac{18}{35}\right)$

Ans. (B)

2. If the direction ratio of two lines are given by $3lm-4ln+mn=0$ and $l+2m+3n=0$, then the angle between the lines is

(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{3}$ **(C)** $\frac{\pi}{4}$ **(D)** $\frac{\pi}{6}$

Ans. (A)

3. If a line makes angles α, β, γ and δ with four diagonals of a cube, then the value of $\sin^2\alpha+\sin^2\beta+\sin^2\gamma+\sin^2\delta$ is

(A) $\frac{4}{3}$ **(B)** 1 **(C)** $\frac{8}{3}$ **(D)** $\frac{7}{3}$ **Ans.** (C)

4. If l_1, m_1, n_1 and l_2, m_2, n_2 are dc's of two lines inclined to each other at an angle θ, then the dc's of the internal bisectors of angle between these lines are

(A) $\frac{l_1+l_2}{2\sin\theta/2}, \frac{m_1+m_2}{2\sin\theta/2}, \frac{n_1+n_2}{2\sin\theta/2}$

(B) $\frac{l_1+l_2}{2\cos\theta/2}, \frac{m_1+m_2}{2\cos\theta/2}, \frac{n_1+n_2}{2\cos\theta/2}$

(C) $\frac{l_1-l_2}{2\sin\theta/2}, \frac{m_1-m_2}{2\sin\theta/2}, \frac{n_1-n_2}{2\sin\theta/2}$

(D) $\frac{l_1-l_2}{2\cos\theta/2}, \frac{m_1-m_2}{2\cos\theta/2}, \frac{n_1-n_2}{2\cos\theta/2}$

Ans. (B)

5. The angle between the lines $\vec{r}=(4\hat{i}-\hat{j})+s(2\hat{i}+\hat{j}-3\hat{k})$ and $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+t(\hat{i}-3\hat{j}+2\hat{k})$ is

(A) $\frac{3\pi}{2}$ **(B)** $\frac{\pi}{3}$ **(C)** $\frac{2\pi}{3}$ **(D)** $\frac{\pi}{6}$ **Ans.** (B)

27.12 Intersections of Two Lines

Two lines in space can have the following three positions:

1. They are parallel.
2. They are intersecting.
3. They are neither intersecting nor parallel. Such lines are called skew lines.

In case of intersecting line,

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \quad (27.2)$$

and

$$\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2} \quad (27.3)$$

The coordinates of general points on Eqs. (27.2) and (27.3), respectively, are $(a_1\lambda+x_1, b_1\lambda+y_1, c_1\lambda+z_1)$ and $(a_2\mu+x_2, b_2\mu+y_2, c_2\mu+z_2)$.

If the lines intersect, then equate the corresponding coordinates $a_1\lambda+x_1=a_2\mu+x_2, b_1\lambda+y_1=b_2\mu+y_2$ and $c_1\lambda+z_1=c_2\mu+z_2$.

Calculate λ and μ using any two equation above. If the values of λ and μ satisfy the third equation, then the lines (27.2) and (27.3) intersect, otherwise they do not intersect.

By substituting the value of λ (or μ) in the coordinates of general point(s) we will get the intersecting point.

Illustration 27.18 Show that the lines $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ and $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$ intersect. Find the point of intersection.

Solution: The position vectors of arbitrary points on the given lines are

$$(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})=(3\lambda+1)\hat{i}+(1-\lambda)\hat{j}-\hat{k}$$

and

$$(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})=(2\mu+4)\hat{i}+0\hat{j}+(3\mu-1)\hat{k}$$

If the lines intersect, then they have a common point. So, for some values of λ and μ, we must have

$$(3\lambda+1)\ i+(1-\lambda)\hat{j}-\hat{k}=(2\mu+4)\hat{i}+0\hat{j}+(3\mu-1)\hat{k}$$

$$\Rightarrow 3\lambda+1=2\mu+4,\ 1-\lambda=0 \text{ and } -1=3\mu-1$$

Solving the last two of these three equations, we get $\lambda = 1$ and $\mu = 0$. These values of λ and μ satisfy the first equation. So, the given lines intersect. Putting $\lambda = 1$ in first line, we get

$$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+(3\hat{i}-\hat{j})=4\hat{i}+0\hat{j}-\hat{k}$$

which is the position vector of the point of intersection. Thus, the coordinates of the point of intersection are (4, 0, −1).

Illustration 27.19 If the lines $\dfrac{x-1}{2}=\dfrac{y+1}{3}=\dfrac{z-1}{4}$ and $\dfrac{x-3}{1}=\dfrac{y-k}{2}=\dfrac{z}{1}$ intersect, then find k.

Solution: We have

Let
$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}=r_1$$

$$x=2r_1+1,\ y=3r_1-1,\ z=4r_1+1$$

That is, the point is $(2r_1+1, 3r_1-1, 4r_1+1)$.

and
$$\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}=r_2$$

That is, the point is $(r_2+3, 2r_2+k, r_2)$.

If the lines are intersecting, then they have a common point. So,

$$2r_1+1=r_2+3,\ 3r_1-1=2r_2+k,\ 4r_1+1=r_2$$

On solving

$$r_1=-3/2,\ r_2=-5$$

Hence,

$$k=9/2$$

Illustration 27.20 A line with direction cosines proportional to 2, 1 and 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The coordinates of each of the points of intersection are given by

(A) $(2a, 3a, 3a)$ $(2a, a, a)$
(B) $(3a, 2a, 3a)$ (a, a, a)
(C) $(3a, 2a, 3a)$ $(a, a, 2a)$
(D) $(3a, 3a, 3a)$ (a, a, a)

Solution: Given lines are

$$\frac{x}{1}=\frac{y+a}{1}=\frac{z}{1}=\lambda \text{ (say)}$$

Therefore point is $P(\lambda, \lambda - a, \lambda)$ and

$$\frac{x+a}{1}=\frac{y}{1/2}=\frac{z}{1/2}$$

That is,

$$\frac{x+a}{2}=\frac{y}{1}=\frac{z}{1}=\mu \text{ (say)}$$

Therefore, the point is $Q(2\mu - a, \mu, \mu)$. Since direction ratios of the given lines are 2, 1 and 2, and direction ratios of

$$PQ=(2\mu-a-\lambda,\ \mu-\lambda+a,\ \mu-\lambda)$$

According to the question,

$$\frac{2\mu-a-\lambda}{2}=\frac{\mu-\lambda+a}{1}=\frac{\mu-\lambda}{2}$$

Then,

$$\lambda=3a$$

and

$$\mu=a$$

Therefore, points of intersection are $P(3a, 2a, 3a)$ and $Q(a, a, a)$.

Hence, the correct answer is option (B).

27.13 Shortest Distance Between Two Non-intersecting Lines

Two lines are called non-intersecting lines if they do not lie in the same plane. The straight line that is perpendicular to each of the non-intersecting lines is called the line of the shortest distance. And the length of the shortest distance line intercepted between the two lines is called the length of the shortest distance.

27.13.1 Vector Form

If $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ are two skew lines, the shortest distance between them is the perpendicular distance (Fig. 27.13). It is obtained as

$$d = PQ = \text{projection of } \overrightarrow{AB} \text{ on } \overrightarrow{PQ} = \overrightarrow{AB}.\hat{e} = \pm\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}$$

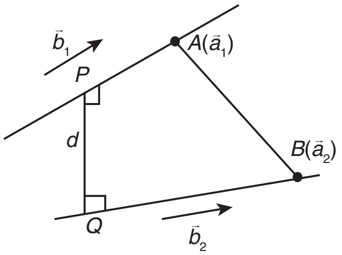


Figure 27.13

More appropriately (Fig. 27.14),

$$d=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|=\left|\frac{[\vec{b}_1\vec{b}_2(\vec{a}_2-\vec{a}_1)]}{|\vec{b}_1\times\vec{b}_2|}\right|$$

Clearly, two lines intersect if $[\vec{b}_1\vec{b}_2(\vec{a}_2-\vec{a}_1)]=0$.

If the lines are parallel, $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}$, the formula to calculate shortest distance becomes

$$d=\frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{b}|}$$

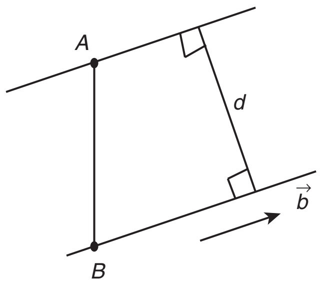


Figure 27.14

27.13.2 Cartesian Form

Let the equation of two non-intersecting lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}=r_1 \text{ (say)} \qquad (27.4)$$

and

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}=r_2 \text{ (say)} \qquad (27.5)$$

Any point on line (27.4) is $P(x_1+l_1r_1, y_1+m_1r_1, z_1+n_1r_1)$ and on line (27.5) is $Q(x_2+l_2r_2, y_2+m_2r_2, z_2+n_2r_2)$.

Let PQ be the line of the shortest distance. Its direction ratios will be $[(l_1r_1+x_1-x_2-l_2r_2), (m_1r_1+y_1-y_2-m_2r_2), (n_1r_1+z_1-z_2-n_2r_2)]$.

This line is perpendicular to both the given lines. By using condition of perpendicularity, we obtain two equations in r_1 and r_2.

So by solving these, values of r_1 and r_2 can be found. And subsequently points P and Q can be found. The distance PQ is the shortest distance.

The shortest distance can be found by

$$PQ=\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

Note: If any straight line is given in general form, then it can be transformed into symmetrical form and we can further proceed.

Illustration 27.21 Find the shortest distance between the lines $\vec r=(4\hat i-\hat j)+\lambda(\hat i+2\hat j-3\hat k)$ and $\vec r=(\hat i-\hat j+2\hat k)+\mu(2\hat i+4\hat j-5\hat k)$.

Solution: We know that the shortest distance between the lines $\vec r=\vec a_1+\lambda\vec b_1$ and $\vec r=\vec a_2+\mu\vec b_2$ is given by

$$d=\left|\frac{(\vec a_2-\vec a_1)\cdot(\vec b_1\times\vec b_2)}{|\vec b_1\times\vec b_2|}\right|$$

Comparing the given equations with the equations $\vec r=\vec a_1+\lambda\vec b_1$ and $\vec r=\vec a_2+\mu\vec b_2$, respectively, we have

$$\vec a_1=4\hat i-\hat j,\ \vec a_2=\hat i-\hat j+2\hat k, \vec b_1=i+2j-3\hat k$$

and

$$\vec b_2=2\hat i+4\hat j-5\hat k$$

Now,

$$\vec a_2-\vec a_1=-3\hat i+0\hat j+2\hat k$$

and

$$\vec b_1\times\vec b_2=\begin{vmatrix} \hat i & \hat j & \hat k \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}=2\hat i-\hat j+0\hat k$$

Therefore,

$$(\vec a_2-\vec a_1)\cdot(\vec b_1\times\vec b_2)=(-3\hat i+0\hat j+2\hat k)\cdot(2\hat i-\hat j+0\hat k)=-6+0+0=-6$$

and $$|\vec b_1\times\vec b_2|=\sqrt{4+1+0}=\sqrt5$$

Therefore,

$$d=\left|\frac{(\vec a_2-\vec a_1)\cdot(\vec b_2\times\vec b_1)}{|\vec b_2\times\vec b_1|}\right|=\left|\frac{-6}{\sqrt5}\right|=\frac{6}{\sqrt5}$$

Illustration 27.22 Find the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$.

Also find the equation of line of the shortest distance.

Solution: Given lines are

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}=r_1 \text{ (say)} \qquad (1)$$

and

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=r_2 \text{ (say)} \qquad (2)$$

Any point on line (1) is $P(3r_1+3, 8-r_1, r_1+3)$ and on line (2) is $Q(-3-3r_2, 2r_2-7, 4r_2+6)$.

If PQ is the line of the shortest distance, then direction ratios of $PQ=(3r_1+3)-(-3-3r_2), (8-r_1)-(2r_2-7), (r_1+3)-(4r_2+6)$.

That is,

$$3r_1+3r_2+6, -r_1-2r_2+15, r_1-4r_2-3$$

As PQ is perpendicular to lines (1) and (2), therefore

$$3(3r_1+3r_2+6)-1(-r_1-2r_2+15)+1(r_1-4r_2+3)=0$$
$$\Rightarrow 11r_1+7r_2=0 \qquad (3)$$

and

$$-3(3r_1+3r_2+6)+2(-r_1-2r_2+15)+4(r_1-4r_2+3)=0$$
$$\Rightarrow 7r_1+11r_2=0 \qquad (4)$$

On solving Eqs. (3) and (4), we get $r_1=r_2=0$.

So, points $P(3, 8, 3)$ and $Q(-3, -7, 6)$. Therefore, length of the shortest distance

$$PQ=\sqrt{\{(-3-3)^2+(-7-8)^2+(6-3)^2\}}=3\sqrt{30}$$

Direction ratios of shortest distance line are 2, 5 and –1.

Therefore equation of the shortest distance line is

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$$

27.14 Point and Line

27.14.1 Foot of Perpendicular from a Given Point to the Given Line

27.14.1.1 Cartesian Form

See Fig. 27.15. Point $A(\alpha, \beta, \gamma)$ and equation line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r$$

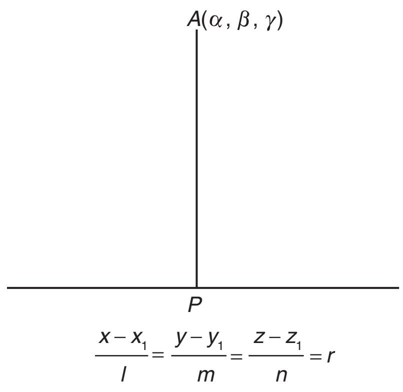


Figure 27.15

If P be the foot of perpendicular, then P is $(lr+x_1, mr+y_1, nr+z_1)$.

Find the direction ratios of AP and apply the condition of perpendicularity of AP and the given line. This will give the value of r and hence the point P that is the foot of perpendicular.

27.14.1.2 Vector Form

See Fig. 27.16. Let L be the foot of a perpendicular drawn from $P(\vec{\alpha})$ on the line $\vec{r}=\vec{a}+\lambda\vec{b}$. Since $\vec{r}$ denotes the position vector of any point on the line $\vec{r}=\vec{a}+\lambda\vec{b}$. So, let the position vector of L be $\vec{a}+\lambda\vec{b}$.

Then,

$$\overrightarrow{PL}=\vec{a}-\vec{\alpha}+\lambda\vec{b}=(\vec{a}-\vec{\alpha})-\left(\frac{(\vec{a}-\vec{\alpha})\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

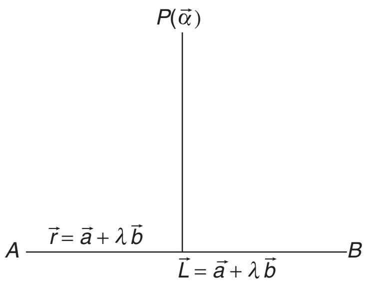


Figure 27.16

The length PL is the magnitude of $\overrightarrow{PL}$, and the required length of a perpendicular.

27.14.2 Reflection or Image of a Point in a Straight Line

See Fig. 27.17. If the perpendicular PL from point P on the given line be produced to Q such that $PL = QL$, then Q is known as the image or reflection of P in the given line. Also, L is the foot of the perpendicular or the projection of P on the line.

Let $Q(\vec{\beta})$ is the image of P in $\vec{r}=\vec{a}+\lambda\vec{b}$

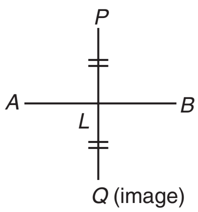


Figure 27.17

Then

$$\vec{\beta}=2\vec{a}-\left(\frac{2(\vec{a}-\vec{\alpha}).\vec{b}}{|\vec{b}|^2}\right)\vec{b}.\alpha$$

Illustration 27.23 Find the foot of the perpendicular drawn from the point $2\hat{i}-\hat{j}+5\hat{k}$ to the line $\vec{r}=(11\hat{i}-2\hat{j}-8\hat{k})+\lambda(10\hat{i}-4\hat{j}-11\hat{k})$. Also find the length of the perpendicular.

Solution: See Fig. 27.18. Let L be the foot of the perpendicular drawn from $P(2\hat{i}-\hat{j}+5\hat{k})$ on the line $\vec{r}=11\hat{i}-2\hat{j}-8\hat{k}+\lambda(10\hat{i}-4\hat{j}-11\hat{k})$. Let the position vector of L be

$$11\hat{i}-2\hat{j}-8\hat{k}+\lambda(10\hat{i}-4\hat{j}-11\hat{k})$$
$$=(11+10\lambda)\hat{i}+(-2-4\lambda)\hat{j}+(-8-11\lambda)\hat{k}$$

Then,

$$\overrightarrow{PL} = \text{position vector of } L - \text{position vector of } P$$
$$=[(11+10\lambda)\hat{i}+(-2-4\lambda)\hat{j}+(-8-11\lambda)\hat{k}]-[2\hat{i}-\hat{j}+5\hat{k}]$$
$$=(9+10\lambda)\hat{i}+(-1-4\lambda)\hat{j}+(-13-11\lambda)\hat{k}$$

Since PL is perpendicular to the given line and the given line is parallel to $\vec{b}=10\hat{i}-4\hat{j}-11\hat{k}$.

Therefore,

$$\overrightarrow{PL}\perp\vec{b}$$

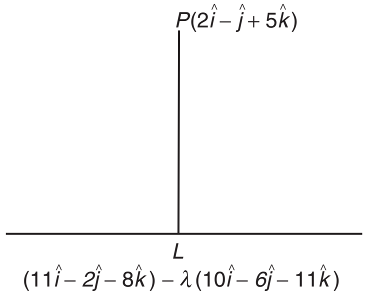


Figure 27.18

$$\Rightarrow \overrightarrow{PL}\cdot\vec{b}=0$$
$$\Rightarrow [(9+10\lambda)\hat{i}+(-1-4\lambda)\hat{j}+(-13-11\lambda)\hat{k}]\cdot(10\hat{i}-4\hat{j}-11\hat{k})=0$$
$$\Rightarrow 10(9+10\lambda)-4(-1-4\lambda)-11(-13-11\lambda)=0$$
$$\Rightarrow 90+100\lambda+4+16\lambda+143+121\lambda=0$$
$$\Rightarrow 237\lambda=-237 \Rightarrow \lambda=-1$$

Putting the value of λ, we obtain the position vector of L as $\hat{i}+2\hat{j}+3\hat{k}$

Now,

$$\overrightarrow{PL} = (i+2\hat{j}+3\hat{k})-(2\hat{i}-\hat{j}+5\hat{k})=-\hat{i}+3\hat{j}-2\hat{k}$$

Hence, the length of the perpendicular from P on the given line is

$$|\overrightarrow{PL}|=\sqrt{1+9+4}=\sqrt{14}$$

Illustration 27.24 Find the image of the point (1, 6, 3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

Solution: Let $P(1, 6, 3)$ be the given point, and let L be the foot of the perpendicular from P to the given line (Fig. 27.19). The coordinates of a general point on the given line are given by

$$\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{3}=\lambda$$

That is,

$$x=\lambda,$$
$$y=2\lambda+1$$

and

$$z=3\lambda+2$$

Let the coordinates of L be $(\lambda, 2\lambda+1, 3\lambda+2)$.

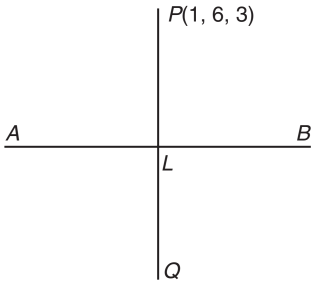


Figure 27.19

So, direction ratios of PL are $\lambda-1$, $2\lambda+1-6$ and $3\lambda+2-3$, i.e. $\lambda-1$, $2\lambda-5$ and $3\lambda-1$. Direction ratios of the given line are 1, 2 and 3, which is perpendicular to PL.

Therefore,

$$(\lambda-1)1+(2\lambda-5)2+(3\lambda-1)3=0$$
$$\Rightarrow 14\lambda-14=14 \Rightarrow \lambda=1$$

So, coordinates of L are (1, 3, 5).

Let $Q(x_1, y_1, z_1)$ be the image of $P(1, 6, 3)$ in the given line.

Then, L is the mid-point of PQ. Therefore,

$$\frac{x_1+1}{2}=1, \frac{y_1+6}{2}=3 \text{ and } \frac{z_1+3}{2}=5$$
$$\Rightarrow x_1=1, y_1=0 \text{ and } z_1=7$$

Hence, the image of $P(1, 6, 3)$ in the given line is (1, 0, 7).

27.15 The Plane

Consider the locus of a point $P(x, y, z)$. If x, y and z are allowed to vary without any restriction for their different combinations, we have a set of points like P. The surface on which these points lie is called the locus of P. It may be a plane or any curved surface. If Q is any other point on its locus and all points of the straight line PQ lie on it, it is a plane. In other words, if the straight line PQ is, however, small and in whatever direction it may be, lies completely on the locus, it is a plane, otherwise any curved surface.

27.16 Equation of Plane in Different Forms

27.16.1 General Equation of Plane

Every equation of first degree of the form

$$Ax+By+Cz+D=0$$

represents the equation of a plane. The coefficients of x, y and z, that is, A, B and C are the direction ratios of the normal to the plane. Equation of parallel plane is given by

$$Ax+By+Cz+E=0$$

If this plane is passing through the origin, then equation of plane is

$$Ax+By+Cz=0$$

27.16.2 Equation of Coordinate Planes

Equation of YZ plane is $x=0$, equation of plane parallel to YZ plane is $x=d$.

Equation of ZX plane is $y=0$, equation of plane parallel to ZX plane is $y=d$.

Equation of XY plane is $z=0$, equation of plane parallel to XY plane is $z=d$.

27.16.3 Equation of a Plane in Vector Form

Following are the four useful ways of specifying a plane:

1. See Fig. 27.20. A plane at a perpendicular distance d from the origin and normal to a given direction $(\hat{n})$ has the equation $(\hat{r}-d\hat{n})\cdot d\hat{n}=0$ or $\vec{r}\cdot\hat{n}=d$ ($\hat{n}$ is a unit vector).

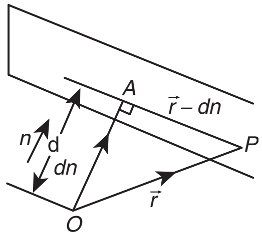


Figure 27.20

2. See Fig. 27.21. A plane passing through the point $A(\vec{a})$ and normal to $\hat{n}$ has the equation $(\vec{r}-\vec{a})\cdot\hat{n}=0$

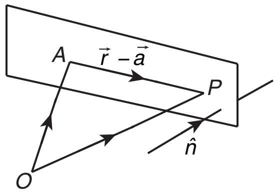


Figure 27.21

3. Parametric equation of the plane passing through $A(\vec{a})$ and parallel to the plane of vectors $(\vec{b})$ and $(\vec{c})$ is given by

$$\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c} \Rightarrow \vec{r}\cdot(\vec{b}\times\vec{c})=[\vec{a}\,\vec{b}\,\vec{c}]$$

4. Parametric equation of the plane passing through $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ (A, B and C non-collinear) is given by

$$\vec{r}=(1-\lambda-\mu)\vec{a}+\lambda\vec{b}+\mu\vec{c} \Rightarrow \vec{r}\cdot[\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}]=[\vec{a}\,\vec{b}\,\vec{c}]$$

27.16.4 Equation of Plane in Various Forms

27.16.4.1 Intercept Form

If the plane cuts the intercepts of length a, b and c on coordinate axes, then its equation is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

27.16.4.2 Normal Form

Normal form of the equation of plane is

$$lx+my+nz=p$$

where l, m and n are the dc's of the normal to the plane and p is the length of the perpendicular from the origin.

27.16.5 Equation of Plane Parallel to Coordinate Plane or Perpendicular to Coordinates Axis

1. Equation of plane parallel to YOZ-plane (or perpendicular to x-axis) at a distance 'a' from it is $x=a$.
2. Equation of plane parallel to ZOX-plane (or perpendicular to y-axis) at a distance 'b' from it is $y=b$.

3. Equation of plane parallel to XOY-plane (or perpendicular to z-axis) at a distance 'c' from it is $z = c$.

27.16.6 Equation of Plane Perpendicular to Coordinate Plane or Parallel to Coordinates Axis

1. Equation of plane perpendicular to YOZ-plane or parallel to x-axis is $By + Cz + D = 0$.
2. Equation of plane perpendicular to ZOX-plane or parallel to y-axis is $Ax + Cz + D = 0$.
3. Equation of plane perpendicular to XOY-plane or parallel to z-axis is $Ax + By + D = 0$.

27.16.7 Equation of Plane Passing Through a Point and Having Given Direction Ratio

The equation to the plane passing through $P(x_1, y_1, z_1)$ and having direction ratios (a, b, c) for its normal is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

27.16.8 Equation of Plane Passing Through Three Non-Collinear Points

The equation of the plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} (x - x_1) & (y - y_1) & (z - z_1) \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \end{vmatrix} = 0$$

Illustration 27.25 Reduce the equation $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 12\hat{k}) = 5$ to normal form and hence find the length of perpendicular from the origin to the plane.

Solution: The given equation is

$$\vec{r}.(3\hat{i} - 4\hat{j} + 12\hat{k}) = 5 \text{ or } \vec{r} \cdot \vec{n} = 5$$

where $\vec{n} = 3\hat{i} - 4\hat{j} + 12\hat{k}$.

Since

$$|\vec{n}| = \sqrt{3^2 + (-4)^2 + 12^2} = 13 \neq 1$$

Therefore, the given equation is not in the normal form. To reduce it to normal form, we divide both sides by $|\vec{n}|$, that is,

$$\vec{r} . \frac{\vec{n}}{|\vec{n}|} = \frac{5}{|\vec{n}|}$$

or

$$\vec{r} . \left(\frac{3}{13}\hat{i} - \frac{4}{13}\hat{j} + \frac{12}{13}\hat{k} \right) = \frac{5}{13}$$

This is the normal form of the equation of the given plane. The length of the perpendicular from the origin is $\frac{5}{13}$.

Illustration 27.26 Find the equation in the Cartesian form of the plane passing through the point (3, –3, 1) and normal to the line joining the points (3, 4, –1) and (2, –1, 5).

Solution: We know that the vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to $\vec{n}$ is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

or

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \quad (1)$$

Since the given plane passes through the point (3, –3, 1) and is normal to the line joining the points $A(3, 4, -1)$ and $B(2, -1, 5)$.

Therefore,

$$\vec{a} = 3\hat{i} - 3\hat{j} + \hat{k}$$

and

$$\vec{n} = \vec{AB} = P.V. \text{ of } B - P.V. \text{ of } A$$

$$= (2\hat{i} - \hat{j} + 5\hat{k}) - (3\hat{i} + 4\hat{j} - \hat{k}) = -\hat{i} - 5\hat{j} + 6\hat{k}$$

Substituting $\vec{a} = 3\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{n} = -\hat{i} - 5\hat{j} + 6\hat{k}$ in Eq. (1), we obtain

$$\vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = (3\hat{i} - 3\hat{j} + \hat{k}) \cdot (-\hat{i} - 5\hat{j} + 6\hat{k})$$

or

$$\vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = -3 + 15 + 6 \text{ or } \vec{r} \cdot (-\hat{i} - 5\hat{j} + 6\hat{k}) = 18$$

This is the vector equation of the required plane. The Cartesian equation is

$$(x\hat{i} + y\hat{j} + z\hat{k}).\ (-\hat{i} - 5\hat{j} + 6\hat{k}) = 18$$

$$\Rightarrow -x - 5y + 6z = 18$$

or

$$x + 5y - 6z + 18 = 0$$

Illustration 27.27 Write the equation of the plane whose intercepts on the coordinate axes are –4, 2 and 3.

Solution: We know that the equation of a plane whose intercepts on the coordinate axes are a, b and c, respectively, is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Here, $a = -4$, $b = 2$ and $c = 3$.
So, the equation of the required plane is

$$\frac{x}{-4} + \frac{y}{2} + \frac{z}{3} = 1$$

or

$$-3x + 6y + 4z = 12$$

27.17 Point and Plane

27.17.1 Position of Two Points w.r.t the Plane

Two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same or opposite sides of a plane $ax + by + cz + d = 0$ according to $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same or opposite signs. The plane divides the line joining the points P and Q externally or internally according to P and Q lying on same or opposite sides of the plane.

27.17.2 Perpendicular Distance

27.17.2.1 Cartesian Form

The length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

27.17.2.2 Vector Form

The perpendicular distance of a point having position vector $\vec{a}$ from the plane $\vec{r}\cdot\vec{n} = d$ is given by

$$p = \frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$$

Note: Distance between two parallel planes $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$ is

$$\frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}$$

27.17.3 Image of a Point About Plane Mirror

Let point P is the image of point Q in the plane. Then

1. Line PQ is perpendicular to the plane, and
2. Mid-point of PQ lies on the plane.

   Then either of the point is the image of the other in the plane. To find the image of a point in a given plane, we proceed as follows:

1. Write the equations of the line passing through P and normal to the given plane as $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.
2. Write the coordinates of image Q as $(x_1 + ar, y_1, + br, z_1 + cr)$.
3. Find the coordinates of the mid-point R of PQ.
4. Obtain the value of r by putting the coordinates of R in the equation of the plane.
5. Put the value of r in the coordinates of Q.

Illustration 27.28 Find the distance of the point (2, 1, 0) from the plane $2x + y + 2z + 5 = 0$.

Solution: We know that the distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

So,

$$\text{Required distance } = \frac{|2\times 2 + 1 + 2\times 0 + 5|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{10}{3}$$

Illustration 27.29 Find the distance between the parallel planes $2x - y + 2z + 3 = 0$ and $4x - 2y + 4z + 5 = 0$.

Solution: Let $P(x_1, y_1, z_1)$ be any point on $2x - y + 2z + 3 = 0$. Then

$$2x_1 - y_1 + 2z_1 + 3 = 0 \qquad (1)$$

The length of the perpendicular from $P(x_1, y_1, z_1)$ to $4x - 2y + 4z + 5 = 0$ is

$$\left|\frac{4x_1 - 2y_1 + 4z_1 + 5}{\sqrt{4^2 + (-2)^2 + 4^2}}\right| = \left|\frac{2(2x_1 - y_1 + 2z_1) + 5}{\sqrt{36}}\right| = \frac{|2(-3) + 5|}{6} = \frac{1}{6}$$

Therefore, the distance between the two given parallel planes is $\frac{1}{6}$.

Illustration 27.30 Find the image of the point (3, –2, 1) in the plane $3x - y + 4z = 2$.

Solution: Let Q be the image of the point $P(3, -2, 1)$ in the plane $3x - y + 4z = 2$. Then PQ is normal to the plane. Therefore direction ratios of PQ are 3, –1 and 4.

Since PQ passes through $P(3, -2, 1)$ and has direction ratios 3, –1 and 4. Therefore equation of PQ is

$$\frac{x - 3}{3} = \frac{y + 2}{-1} = \frac{z - 1}{4} = r \text{ (say)}$$

Let the coordinates of Q be $(3r + 3, -r - 2, 4r + 1)$. Let R be the mid-point of PQ. Then R lies on the plane $3x - y + 4z = 2$. The coordinates of R are

$$\left(\frac{3r + 3 + 3}{2}, \frac{-r - 2 - 2}{2}, \frac{4r + 1 + 1}{2}\right) \quad \text{or} \quad \left(\frac{3r + 6}{2}, \frac{-r - 4}{2}, 2r + 1\right)$$

Since R lies on $3x - y + 4z = 2$. Therefore,

$$3\left(\frac{3r + 6}{2}\right) - \left(\frac{-r - 4}{2}\right) + 4(2r + 1) = 2$$

$$\Rightarrow 13r = -13 \Rightarrow r = -1$$

Therefore, the image is (0, –1, –3).

27.18 Angle Between Two Planes

27.18.1 Cartesian Form

Angle θ between the planes is defined as angle between normals to the planes drawn from any point. Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

27.18.2 Vector Form

Angle θ between the planes $\vec{r}_1\cdot\vec{n}_1 = d_1$ and $\vec{r}_2\cdot\vec{n}_2 = d_2$ is given by

$$\cos\theta = \pm\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Obviously, two planes are perpendicular if $\vec{n}_1\cdot\vec{n}_2 = 0$.

They are parallel if $\vec{n}_1 = \lambda\vec{n}_2$, where λ is a scalar.

Illustration 27.31 Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$.

Solution: We know that the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_1 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Therefore, angle between $x+y+2z=9$ and $2x-y+z=15$ is given by

$$\cos\theta=\frac{(1)(2)+(1)(-1)+(2)(1)}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+(-1)^2+1^2}}=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{3}$$

27.19 Angle Bisectors of Two Planes

In general, there are two angles between two planes. One is acute and other obtuse.

27.19.1 Cartesian Form

If the equations of planes are $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$, then equation of angle bisectors of plane is given by

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Note: If $a_1a_2+b_1b_2+c_1c_2=0$, then the planes are perpendicular to each other.

If $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, then the planes are parallel to each other.

27.19.2 Vector Form

If the equations of planes are $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$, then the equation of angle bisectors of the plane is given by

$$\vec{r}\cdot(\hat{n}_1\pm\hat{n}_2)=\frac{d_1}{|\vec{n}_1|}\pm\frac{d_2}{|\vec{n}_2|}$$

Illustration 27.32 Find the equations of the bisector planes of the angles between the planes $x+2y+2z=19$ and $4x-3y+12z+3=0$, and specify the plane that bisects the acute angle and the plane that bisects the obtuse angle.

Solution: The two given planes are

$$\vec{r}\cdot(\hat{i}+2\hat{j}+2\hat{k})=19 \tag{1}$$

and

$$\vec{r}\cdot(4\hat{i}-3\hat{j}+12\hat{k})+3=0 \tag{2}$$

The equations of the planes bisecting the angles between (1) and (2) are

$$\frac{r\cdot(i+2j+2k)-19}{\sqrt{1^2+2^2+2^2}}=\pm\frac{r\cdot(4i-3j+12k)+3}{\sqrt{4^2+(-3)^2+(12)^2}}$$

or

$$\frac{\vec{r}\cdot(\hat{i}+2\hat{j}+2\hat{k})-19}{3}=\pm\frac{(\vec{r}\cdot(4\hat{i}-3\hat{j}+12\hat{k})+3)}{13}$$

or

$$\vec{r}\cdot(13\hat{i}+26\hat{j}+26\hat{k})-247=\pm[\vec{r}\cdot(12\hat{i}-9\hat{j}+36\hat{k})+9]$$

Taking positive sign on the right-hand side (RHS), we get

$$\vec{r}\cdot(13\hat{i}+26\hat{j}+26\hat{k})-247=\vec{r}\cdot(12\hat{i}-9\hat{j}+36\hat{k})+9$$

or

$$\vec{r}\cdot(\hat{i}+35\hat{j}-10\hat{k})-256=0 \tag{3}$$

and taking negative sign on the right-hand side, we obtain

$$\vec{r}(25\hat{i}+17\hat{j}+62\hat{k})-238=0 \tag{4}$$

Hence, the two bisector planes are

$$\vec{r}\cdot(\hat{i}+35\hat{j}-10\hat{k})=256$$

and

$$\vec{r}\cdot(25\hat{i}+17\hat{j}+62\hat{k})=238$$

Now, to obtain the angle bisector bisecting the acute angle between Eqs. (1) and (2), we find the angle between one of the given planes and one of the angle bisectors. Let θ be the angle between Eqs. (1) and (3), then

$$\cos\theta=\frac{(\hat{i}+2\hat{j}+2\hat{k}).(\hat{i}+35\hat{j}-10\hat{k})}{\sqrt{1^2+2^2+2^2}\sqrt{1^2+(35)^2+(-10)^2}}=\sqrt{\frac{17}{78}}$$

Therefore,

$$\sin\theta=\sqrt{1-\frac{17}{78}}=\sqrt{\frac{61}{78}}$$

So,

$$\tan\theta=\sqrt{\frac{61}{17}}>1$$

Thus, Eq. (3) bisects the obtuse angle between Eqs. (1) and (2) and hence Eq. (4) bisects the acute angle between the given planes.

27.20 Family of Plane

Two planes intersect in a line if they are not parallel. Any plane through the line of intersection of two planes can be written as

In Cartesian form: $(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$

In Vector form: $(\vec{r}\cdot\vec{n}_1-d_1)+\lambda(\vec{r}\cdot\vec{n}_2-d_2)=0$, where λ is a real number.

Illustration 27.33 Find the equation of the plane containing the line of intersection of the planes $x+y+z-6=0$ and $2x+3y+4y+5=0$ and passing through the point (1, 1, 1)

Solution: The equation of a plane through the line of intersection of the given plane is

$$(x+y+z-6)+\lambda(2x+3y+4z+5)=0 \tag{1}$$

If Eq. (1) passes through (1, 1, 1), we have

$$-3+14\lambda=0\Rightarrow\lambda=3/14$$

Putting $\lambda=\frac{3}{14}$ in Eq. (1), we obtain the equation of the required plane as

$$(x+y+z-6)+\frac{3}{14}(2x+3y+4z+5)=0$$
$$\Rightarrow 20x+23y+26z-69=0$$

27.21 Line and Plane

27.21.1 Conversion of Unsymmetrical Form of Line to Symmetrical Form

The unsymmetrical form of a line

$$ax+by+cz+d=0=a'x+b'y+c'z+d'$$

can be changed to symmetrical form as follows:

$$\frac{x-\frac{bd'-b'd}{ab'-a'b}}{bc'-b'c}=\frac{y-\frac{da'-d'a}{ab'-a'b}}{ca'-c'a}=\frac{z}{ab'-a'b}$$

We can understand this with the help of given example.

Illustration 27.34 Find in symmetrical form the equations of the line $3x+2y-z-4=0=4x+y-2z+3$.

Solution: The equations of the line in general form are

$$3x+2y-z-4=0$$

and

$$4x + y - 2z + 3 = 0 \qquad (1)$$

Let l, m and n be the direction cosines of the line. Since the line is common to both the planes, it is perpendicular to the normals to both the planes. Hence,

$$3l + 2m - n = 0$$

and

$$4l + m - 2n = 0$$

Solving these, we get

$$\frac{l}{-4+1} = \frac{m}{-4+6} = \frac{n}{3-8}$$

that is,

$$\frac{l}{-3} = \frac{m}{2} = \frac{n}{-5} = \frac{1}{\sqrt{(-3)^2 + 2^2 + (-5)^2}} = \frac{1}{\sqrt{38}}$$

So, direction cosines of the line are $-\frac{3}{\sqrt{38}}$, $\frac{2}{\sqrt{38}}$ and $-\frac{5}{\sqrt{38}}$.

Now to find the coordinates of a point on a line, let us find out the point where it meets the plane $z = 0$.
Putting $z = 0$ in the equation given by (1), we have

$$3x + 2y - 4 = 0$$

and

$$4x + y + 3 = 0$$

Solving these, we get $x = -2$ and $y = 5$.

So, one point of the line is (–2, 5, 0).

Therefore equation of the line in symmetrical form is

$$\frac{x+2}{-\frac{3}{\sqrt{38}}} = \frac{y-5}{\frac{2}{\sqrt{38}}} = \frac{z-0}{-\frac{5}{\sqrt{38}}}$$

That is,

$$\frac{x+2}{-3} = \frac{y-5}{2} = \frac{z}{-5}$$

Illustration 27.35 Reduce in symmetrical form, the equation of the line of intersection of the two planes $x - y + 2z = 5$ and $3x + y + z = 6$.

Solution: Let a, b and c be the direction ratios of the required line. Since the required line lies in both the given planes, we must have

$$a - b + 2c = 0$$

and

$$3a + b + c = 0$$

Solving these two equations by cross-multiplication, we get

$$\frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3} \quad \text{or} \quad \frac{a}{-3} = \frac{b}{5} = \frac{c}{4}$$

In order to find a point on the required line, we put $z = 0$ in the two given equations to obtain $x - y = 5$ and $3x + y = 6$.

Solving these two equations, we obtain $x = \frac{11}{4}, y = \frac{-9}{4}$.

Therefore, coordinates of a point on the required line are (11/4, –9/4, 0).

Hence, the equation of the required line is

$$\frac{x - \frac{11}{4}}{-3} = \frac{y - \left(-\frac{9}{4}\right)}{5} = \frac{z-0}{4}$$

or

$$\frac{4x-11}{-12} = \frac{4y+9}{20} = \frac{z-0}{4}$$

or

$$\frac{4x-11}{-3} = \frac{4y+9}{5} = \frac{z-0}{1}.$$

27.21.2 Angle Between Line and Plane

27.21.2.1 Cartesian Form

If the plane is $ax + by + cz + d = 0$ and line is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

Then,

$$\sin\theta = \frac{al + bm + cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}.$$

27.21.2.2 Vector Form

Let the line be $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane be $\vec{r}\cdot\vec{n} = d$ (Fig. 27.22). If θ is the angle between them, then

$$\cos(90^\circ - \theta) = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

$$\Rightarrow \sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}$$

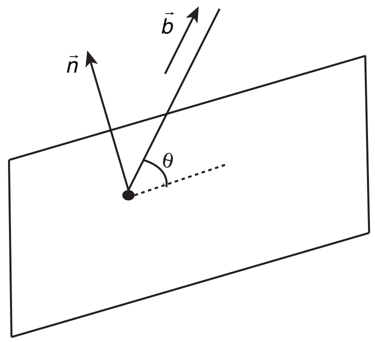


Figure 27.22

Note:

1. If the line is parallel to the plane, then $\vec{b}\cdot\vec{n} = 0$ or $al + bm + cn = 0$.
2. If the line is perpendicular to the plane, then $\vec{b} = \lambda\vec{n}$ or
$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$
3. The line lies in the plane if and only if $al + bm + cn = 0$ (or $\vec{b}\cdot\vec{n} = 0$) and $a\alpha + b\beta + c\gamma + d = 0$ (or $\vec{a}\cdot\vec{n} = d$).

Illustration 27.36 Find the angle between the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-2}{4}$ and the plane $2x + y - 3z + 4 = 0$.

Solution: The given line is parallel to the vector $\vec{b} = 3\hat{i} + 2\hat{j} + 4\hat{k}$ and the given plane is normal to the vector $\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$.

Therefore, the angle θ between the given line and given plane is given by

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|} = \frac{(3\hat{i}+2\hat{j}+4\hat{k})\cdot(2\hat{i}+\hat{j}-3\hat{k})}{\sqrt{3^2+2^2+4^2}\sqrt{2^2+(1)^2+(-3)^2}}$$

$$\Rightarrow \sin\theta = \frac{6+2-12}{\sqrt{29}\sqrt{14}} = \frac{-4}{\sqrt{406}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{-4}{\sqrt{406}}\right)$$

27.21.3 Intersection of Line and Plane

To find the point of intersection of the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$, the coordinates of any point on the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ are given by

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r \text{ (say)}$$

or

$$(x_1+lr, y_1+mr, z_1+nr) \quad (1)$$

If it lies on the plane $ax+by+cz+d=0$, then

$$a(x_1+lr)+b(y_1+mr)+c(z_1+nr)+d=0$$

$$(ax_1+by_1+cz_1+d)+r(al+bm+cn)=0$$

Therefore,

$$r = -\frac{(ax_1+by_1+cz_1+d)}{al+bm+cn}$$

Substituting the value of r in Eq. (1), we obtain the coordinates of the required point of intersection.

Your Turn 3

1. Find the vector equation of the following plane in scalar product form:
   $$\vec{r} = (\hat{i}-\hat{j})+\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}-2\hat{j}+3\hat{k}).$$
   Ans. $\vec{r}\cdot(5\hat{i}-2\hat{j}-3\hat{k})=7$
2. Find the vector equation of the plane passing through the points $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$.
   Ans. $\vec{r}\cdot(9\hat{i}+3\hat{j}-\hat{k})=14$
3. Find the equation of the plane through the points $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$.
   Ans. $5x+2y-3z=17$
4. Find the distance between the point with position vector $-\hat{i}-5\hat{j}-10\hat{k}$ and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ with the plane $x-y+z=5$.
   Ans. 13

27.21.4 Coplanarity of Two Lines

27.21.4.1 Cartesian Form

In Cartesian form, if the lines are

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

and

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2},$$

then the condition of coplanarity is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of plane containing them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

27.21.4.2 Vector Form

If the lines $\vec{r} = \vec{a}_1+\lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2+\lambda\vec{b}_2$ are coplanar, then $[\vec{a}_1\,\vec{b}_1\,\vec{b}_2] = [\vec{a}_2\,\vec{b}_1\,\vec{b}_2]$ and the equation of plane containing them is $[\vec{r}\,\vec{b}_1\,\vec{b}_2] = [\vec{a}_1\,\vec{b}_1\,\vec{b}_2]$ or $[\vec{r}\,\vec{b}_1\,\vec{b}_2] = [\vec{a}_2\,\vec{b}_1\,\vec{b}_2]$.

Note:

1. Every pair of parallel lines is coplanar.
2. Two coplanar lines are either parallel or intersecting.
3. The three sides of a triangle are coplanar.

27.21.5 Image of a Line in Plane

If P be the point of intersection of given line and plane and Q be the foot of the perpendicular from any point on the line to the plane, then PQ is called the projection of given line on the given plane.

Let the line be $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and plane be

$$a_2x+b_2y+c_2z+d=0$$

Find point of intersection (say P) of line and plane. Find image (say Q) of point (x_1, y_1, z_1) in the plane. Line PQ is the reflected line.

Illustration 27.37 Prove that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$ are coplanar. Also find the plane containing these two lines.

Solution: We know that the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing these two lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Here, $x_1 = -1, y_1 = -3, z_1 = -5, x_2 = 2, y_2 = 4, z_2 = 6, l_1 = 3, m_1 = 5, n_1 = 7$, $l_2 = 1$, $m_2 = 4$ and $n_2 = 7$.
Therefore,

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

So, the given lines are coplanar. The equation of the plane containing the line is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

or $$(x+1)(35-28) - (y+3)(21-7) + (z+5)(12-5) = 0$$

or $$x - 2y + z = 0$$

27.22 Sphere

A sphere is a locus of a point that moves in space such that its distance from a fixed point is constant. Fixed point is called the centre of the sphere and constant distance is called the radius of the sphere.

27.22.1 Equation of Sphere in Different Forms

27.22.1.1 Centre Radius Form of Sphere

Cartesian Form

If centre of sphere is (a, b, c) and radius is r, then the equation of sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

If the centre of sphere is the origin and the radius is r, then

$$x^2 + y^2 + z^2 = r^2$$

Vector Form

The equation of the sphere with the centre at $C(\vec{c})$ and radius 'a' is

$$|\vec{r} - \vec{c}| = a$$

27.22.1.2 General Form of Sphere

The general equation of a sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Centre of the sphere $= (-u, -v, -w)$.
That is,

$(-(1/2)$ coeff. of x, $-(1/2)$ coeff. of y and $-(1/2)$ coeff. of $z)$

and

$$\text{radius} = \sqrt{u^2 + v^2 + w^2 - d}$$

From the above equation, we note the following characteristics of the equation of a sphere:

1. It is a second degree equation in x, y and z.
2. The coefficients of x^2, y^2, z^2 are all equal.
3. The terms containing the products xy, yz and zx are absent.

Note: The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents

1. A real sphere, if $u^2 + v^2 + w^2 - d > 0$.
2. A point sphere, if $u^2 + v^2 + w^2 - d = 0$.
3. An imaginary sphere, if $u^2 + v^2 + w^2 - d < 0$.

27.22.1.3 Diametric Form of the Sphere

Cartesian Form

Equation of a sphere whose extremities of diameter are $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$$

Vector Form

If the position vectors of the extremities of a diameter of a sphere are $\vec{a}$ and $\vec{b}$, then its equation is

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$$

or

$$|\vec{r}|^2 - \vec{r}(\vec{a} - \vec{b}) + \vec{a}\vec{b} = 0$$

27.22.1.4 Equation of Concentric Sphere

Any sphere concentric with the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + \lambda = 0$,
where λ is some real number that makes it a sphere.

Illustration 27.38 Find the equation of the sphere that passes through the points (1, –3, 4), (1, –5, 2) and (1, –3, 0) and whose centre is on the plane $x + y + z = 0$.

Solution: Let equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Its centre is $(-u, -v, -w)$ that is on $x + y + z = 0$

$$\Rightarrow u + v + w = 0 \tag{1}$$

It passes through

$$(1, -3, 4) \Rightarrow 2u - 6v + 8w + d = -26 \tag{2}$$
$$(1, -5, 2) \Rightarrow 2u - 10v + 4w + d = -30 \tag{3}$$
and $$(1, -3, 0) \Rightarrow 2u - 6v + d = -10 \tag{4}$$

Solving these four equations, we get $u = -1$, $v = 3$, $w = -2$ and $d = 10$.
Therefore, the required equation of the sphere is

$$x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0$$

Illustration 27.39 Find the equation of the sphere whose centre is (2, –3, 4) and which passes through the point (1, 2, –1).

Solution:

Radius of the sphere $= \sqrt{\{(2-1)^2 + (-3-2)^2 + (4+1)^2} = \sqrt{51}$

Therefore, equation of the sphere is

$$(x-2)^2 + (y+3)^2 + (z-4)^2 = (\sqrt{51})^2$$

That is,

$$x^2 + y^2 + z^2 - 4x + 6y - 8z - 22 = 0$$

27.22.1.5 Condition of Tangency of a Plane to Sphere

A plane touches a given sphere if the perpendicular distance from the centre of the sphere to the plane is equal to the radius of the sphere.

Cartesian Form

The plane $lx+my+nz=p$ touches the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$, if

$$(ul+vm+wn-p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$$

Vector Form

The plane $\vec{r}\cdot\vec{n}=d$ touches the sphere $|\vec{r}-\vec{a}|=R$ if

$$\frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}=R$$

27.22.1.6 Two Spheres Touch Each Other

Two spheres S_1 and S_2 with centres C_1 and C_2 and radii r_1 and r_2, respectively.

1. If $|C_1C_2|>r_1+r_2$, then spheres are non-intersecting.
2. If $|C_1C_2|=|r_1-r_2|$, then spheres are touching each other internally.
3. If $|C_1C_2|=r_1+r_2$, then spheres are touching each other externally.
4. Cut in a circle, if $|r_1-r_2|<|C_1C_2|<r_1+r_2$.
5. One lies within the other, if $|C_1C_2|<|r_1-r_2|$.

   When two spheres touch each other, the common tangent plane is $S_1-S_2=0$, and when they cut in a circle, the plane of the circle is $S_1-S_2=0$; coefficients of x^2,y^2,z^2 being unity in both the cases.

27.22.1.7 Angle of Intersection of Two Spheres

The angle of intersection of two spheres is the angle between the tangent planes to them at their point of intersection. As the radii of the spheres at this common point are normal to the tangent planes, this angle is also equal to the angle between the radii of the spheres at their point of intersection.

If the angle of intersection of two spheres is a right angle, the spheres are said to be orthogonal.

27.22.1.8 Condition for Orthogonality of Two Spheres

Let the equation of the two spheres be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \tag{27.6}$$

and

$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0 \tag{27.7}$$

If the spheres (i) and (ii) cut orthogonally, then

$$2uu'+2vv'+2ww'=d+d'$$

this is the required condition.

Note: If the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ cut orthogonally, then $d=a^2$.

Two spheres of radii r_1 and r_2 cut orthogonally, then the radius of the common circle is $\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$.

Illustration 27.40 The equation $|\vec{r}|^2-\vec{r}(2\hat{i}+4\hat{j}-2\hat{k})-10=0$ represents a

(A) Plane **(B)** Sphere of radius 4
(C) Sphere of radius 3 **(D)** None of these

Solution: The given equation is

$$|\vec{r}|^2-\vec{r}(2\hat{i}+4\hat{j}-2\hat{k})-10=0$$

$$\Rightarrow x^2+y^2+z^2-2x-4y+2z-10=0,$$

which is the equation of the sphere whose centre is (1, 2 –1) and

$$\text{radius }=\sqrt{1+4+1+10}=4$$

Hence, the correct answer is option (B).

Illustration 27.41 The intersection of the spheres $x^2+y^2+z^2+7x-2y-z=13$ and $x^2+y^2+z^2-3x+3y+4z=8$ is the same as the intersection of one of the sphere and the plane

(A) $2x-y-z=1$ **(B)** $x-2y-z=1$
(C) $x-y-2z=1$ **(D)** $x-y-z=1$

Solution: We have the spheres

$$x^2+y^2+z^2+7x-2y-z-13=0$$

and

$$x^2+y^2+z^2-3x+3y+4z-8=0$$

Required plane is

$$S_1-S_2=0$$

Therefore,

$$(7x+3x)-(2y+3y)-(z+4z)-5=0$$

That is,

$$10x-5y+(-5z)-5=0 \Rightarrow 2x-y-z=1$$

Hence, the correct answer is option (A).

Additional Solved Examples

1. The ratio in which *yz*-plane divides the line joining (2, 4, 5) and (3, 5, 7) is

   (A) –2:3 **(B)** 2:3 **(C)** 3:2 **(D)** –3:2

Solution: Let the ratio be λ:1, then *x*-coordinate is

$$\frac{3\lambda+2}{\lambda+1}=0 \Rightarrow \lambda=-\frac{2}{3}$$

Hence, the correct answer is option (A).

2. The plane $ax+by+cz=d$ meets the coordinate axes at the points, *A*, *B* and *C*, respectively. Area of triangle *ABC* is equal to

   (A) $\frac{d^2\sqrt{a^2+b^2+c^2}}{|abc|}$ **(B)** $\frac{d^2\sqrt{a^2+b^2+c^2}}{2|abc|}$
   (C) $\frac{d^2\sqrt{a^2+b^2+c^2}}{4|abc|}$ **(D)** None of these

Solution:

$$A=\left(\frac{d}{a},0,0\right),B=\left(0,\frac{d}{b},0\right),c=\left(0,0,\frac{d}{c}\right)$$

$$\text{Area of triangle OAB}=\Delta_1=\frac{1}{2}\frac{d^2}{|ab|}$$

$$\text{Area of triangle OBC}=\Delta_2=\frac{1}{2}\frac{d^2}{|bc|}$$

$$\text{Area of triangle OAC}=\Delta_3=\frac{1}{2}\frac{d^2}{|ac|}$$

If area of triangle ABC be Δ, then

$$\Delta^2=\Delta_1^2+\Delta_2^2+\Delta_3^2=\frac{d^4}{4}\left(\frac{a^2+b^2+c^2}{a^2b^2c^2}\right)$$

$$\Rightarrow \Delta=\frac{d^2\sqrt{a^2+b^2+c^2}}{2|abc|}$$

Hence, the correct answer is option (B).

3. Equation of the plane passing through (–1, 1, 4) and containing the line $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z}{5}$ is

(A) $9x-22y+2z+23=0$ **(B)** $x+22y+z=25$

(C) $9x+22y-3z=1$ **(D)** $22y-9x+z=35$

Solution: Equation of any plane containing the line $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z}{5}$ will be

$$a(x-1)+b(y-2)+cz=0,$$

where

$$3a+b+5c=0 \quad (1)$$

It is given that plane passes through (–1, 1, 4). Therefore,

$$-2a-b+4c=0 \quad (2)$$

From Eqs. (1) and (2), we get

$$\frac{a}{-9}=\frac{b}{22}=\frac{c}{1}$$

Thus, the equation of required plane is

$$-9(x-1)+22(y-2)+z=0$$
$$\Rightarrow 22y-9x+z=35$$

Hence, the correct answer is option (D).

4. Equation of the plane containing the lines $\frac{x}{1}=\frac{y-2}{3}=\frac{z+4}{-1}$ and $\frac{x-4}{2}=\frac{y}{3}=\frac{z}{1}$ is

(A) $x+y-4z=6$ **(B)** $x-y+4z=6$

(C) $x+y+4z=6$ **(D)** None of these

Solution: Equation of any plane containing the line $\frac{x}{1}=\frac{y-2}{3}=\frac{z+4}{-1}$ is

$$ax+b(y-2)+c(z+4)=0,$$

where

$$a+3b-c=0$$

This plane will also contain the second line if

$$2a+3b+c=0$$

and

$$4a+b(0-2)+c(0+4)=0$$

Solving this equation, we get $a=0$, $b=0$ and $c=0$.

That means the given lines are non-coplanar.

Hence, the correct answer is option (D).

5. Reflection of the line $\frac{x-1}{-1}=\frac{y-2}{3}=\frac{z-4}{1}$ in the plane $x+y+z=7$ is

(A) $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-4}{1}$ **(B)** $\frac{x-1}{-3}=\frac{y-2}{-1}=\frac{z-4}{1}$

(C) $\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-4}{-1}$ **(D)** $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-4}{1}$

Solution: Given that the line passes through (1, 2, 4) and this point also lies on the given plane.

Thus, required line will be in the form of $\frac{x-1}{\ell}=\frac{y-2}{m}=\frac{z-4}{n}$

Any point on the given line is $(-r_1+1, 3r_1+2, r_1+4)$.
If $r_1=1$, this point becomes $P=(0, 5, 5)$.
Let $Q=(a, b, c)$ be the reflection of 'P' in the given plane. Then

$$\frac{a}{2}\cdot 1+\frac{b+5}{2}\cdot 1+\frac{5+c}{2}\cdot 1=7$$

That is,

$$a+b+c=4$$

and

$$\frac{a}{1}=\frac{b-5}{1}=\frac{c-5}{1}=\lambda\,(\text{say})$$

$$\Rightarrow a=\lambda,\ b=5+\lambda,\ c=5+\lambda$$

$$\Rightarrow 10+3\lambda=4 \Rightarrow \lambda=-2$$

Thus, $Q=(-2, 3, 3)$

Hence, direction ratios of reflected line are –3, 1 and –1.
Thus, its equation is

$$\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-4}{-1}$$

Hence, the correct answer is option (C).

6. If a variable plane cuts the coordinate axes in A, B and C and is at a constant distance p from the origin, then find the locus of the centroid of the tetrahedron $OABC$.

Solution: See Fig. 27.23. Let $A\equiv(a, 0, 0)$, $B\equiv(0, b, 0)$ and $C\equiv(0, 0, c)$. Therefore, equation of plane ABC is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad (1)$$

Now,

p = length of perpendicular from O to plane (1)

$$=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

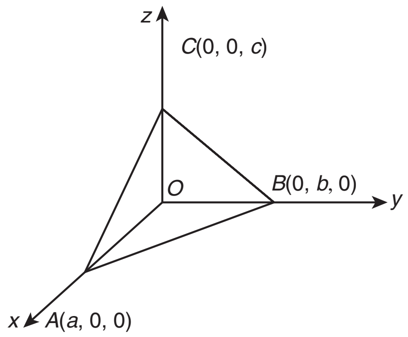


Figure 27.23

or

$$p^2=\frac{1}{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2} \quad (2)$$

Let $G(\alpha, \beta, \gamma)$ be the centroid of the tetrahedron $OABC$. Then

$$\alpha = \frac{a}{4}, \beta = \frac{b}{4}, \gamma = \frac{c}{4} \qquad \left[\begin{array}{l}\text{Since,} \\ \alpha = \dfrac{a+0+0+0}{4} = \dfrac{a}{4}\end{array}\right]$$

or $\quad a = 4\alpha, \quad b = 4\beta \quad$ and $\quad c = 4\gamma$

Putting these values of α, β and γ in Eq. (2), we get

$$p^2 = \frac{16}{\left(\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}\right)}$$

or
$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{16}{p^2}$$

Therefore, locus of $\beta(\alpha, \beta, \gamma)$ is

$$x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$$

7. Through a point $P(h, k, l)$, a plane is drawn at right angles to OP to meet the coordinate axes in A, B and C. If $OP = p$, show that the area of ΔABC is $\dfrac{p^5}{2hkl}$.

Solution:

$$OP = \sqrt{h^2 + k^2 + l^2} = p$$

Direction cosines of OP are

$$\frac{h}{\sqrt{h^2+k^2+l^2}}, \frac{k}{\sqrt{h^2+k^2+l^2}}, \frac{l}{\sqrt{h^2+k^2+l^2}}$$

Since OP is normal to the plane, therefore, equation of the plane will be

$$\frac{h}{\sqrt{h^2+k^2+l^2}}x + \frac{k}{\sqrt{h^2+k^2+l^2}}y + \frac{l}{\sqrt{h^2+k^2+l^2}}z = \sqrt{h^2+k^2+l^2}$$

or
$$hx + ky + lz = h^2 + k^2 + l^2 = p^2 \qquad (1)$$

Therefore,

$$A \equiv \left(\frac{p^2}{h}, 0, 0\right), B \equiv \left(0, \frac{p^2}{k}, 0\right), C \equiv \left(0, 0, \frac{p^2}{l}\right)$$

Now,
$$\text{Area of } \Delta ABC = A_{xy}^2 + A_{yz}^2 + A_{zx}^2$$

Now,
A_{xy} = area of projection of ΔABC on xy-plane = area of ΔAOB

$$= \text{Mod of } \frac{1}{2}\begin{vmatrix} \dfrac{p^2}{h} & 0 & 1 \\ 0 & \dfrac{p^2}{k} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\frac{p^4}{|hk|}$$

Similarly, $\quad A_{yz} = \dfrac{1}{2}\dfrac{p^4}{|kl|}$ and $A_{zx} = \dfrac{1}{2}\dfrac{p^4}{|lh|}$

Therefore,

$$\Delta^2 = \frac{1}{4}\frac{p^8}{h^2k^2} + \frac{1}{4}\frac{p^8}{k^2l^2} + \frac{1}{4}\frac{p^8}{h^2l^2} = \frac{p^{10}}{4h^2k^2l^2} \Rightarrow \Delta = \frac{p^5}{2hkl}$$

8. Find the coordinates of these points on the line $\dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{z-3}{6}$ that is at a distance of 3 units from the point $(1, -2, 3)$.

Solution: Given that line is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6} \qquad (1)$$

Let $P \equiv (1, -2, 3)$.

Direction ratios of line (1) are 2, 3 and 6.

Therefore direction cosines of line (1) are $\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{6}{7}$.

Equation of line (1) may be written as

$$\frac{x-1}{2/7} = \frac{y+2}{3/7} = \frac{z-3}{6/7} = r \qquad (2)$$

Coordinates of any point on line (2) may be taken as

$$\left(\frac{2}{7}r + 1, \frac{3}{7}r - 2, \frac{6}{7}r + 3\right)$$

Let
$$Q \equiv \left(\frac{2}{7}r + 1, \frac{3}{7}r - 2, \frac{6}{7}r + 3\right)$$

Distance of Q from $P = |r|$

According to question, $|r| = 3$.

Therefore, $r = \pm 3$.

Putting the value of r, we have

$$Q \equiv \left(\frac{13}{7}, -\frac{5}{7}, \frac{39}{7}\right)$$

or
$$Q \equiv \left(\frac{1}{7}, -\frac{23}{7}, \frac{3}{7}\right)$$

9. If the planes $y = az + cx$, $x = cy + bz$ and $z = bx + ay$ meet in a line, show that the line of intersection of these planes is

$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{a}{\sqrt{1-c^2}}$$

Solution: Given that planes are

$$y = az + cx \qquad (1)$$
$$x = cy + bz \qquad (2)$$
$$z = bx + ay \qquad (3)$$

Clearly point (0, 0, 0) lies on all the three planes.

Therefore line in which the three planes meet will pass through (0, 0, 0).

Equation of any line through (0, 0, 0) may be taken as

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} \qquad (4)$$

Therefore line (4) lies on planes (2) and (1).

Therefore,
$$l - cm - bn = 0 \qquad (5)$$

and
$$cl - m + an = 0 \qquad (6)$$

Solving Eqs. (5) and (6), we get

$$\frac{l}{-ac-b} = \frac{m}{-bc-a} = \frac{n}{-1+c^2}$$

or
$$\frac{l}{ac+b} = \frac{m}{bc+a} = \frac{n}{1-c^2}$$

Substituting the values of l, m, n in Eq. (4), we get

$$\frac{x}{ac+b} = \frac{y}{bc+a} = \frac{z}{1-c^2} \qquad (5)$$

Now, equation of any plane passing through the line of intersection of planes (1) and (2) may be taken as

$$x - cy - bz + k(y - az - cx) = 0$$

or

$$x(1-ck)+y(-c+k)+z(-b-ak)=0 \quad (6)$$

If planes (3) and (6) are the same, then

$$\frac{1-ck}{b}=\frac{-c+k}{a}=\frac{-b-ak}{-1}$$

From first two equations,

$$\frac{1-ck}{b}=\frac{-c+k}{a} \Rightarrow k=\frac{a+bc}{b+ac} \quad (A)$$

From last two equations,

$$\frac{-c+k}{a}=\frac{-b-ak}{-1}$$

or

$$c-k=-ab-a^2k$$

$$\Rightarrow k=\frac{c+ab}{1-a^2} \quad (B)$$

From (A) and (B), we have

$$\frac{a+bc}{b+ac}=\frac{c+ab}{1-a^2}$$

or

$$a+bc-a^3-a^2bc=bc+ac^2+ab^2+a^2bc$$

or

$$a^2+b^2+c^2+2abc=1$$

or

$$b^2+2abc+a^2c^2=1-a^2-c^2+a^2c^2$$

or

$$(b+ac)^2=(1-a^2)(1-c^2)$$

or

$$b+ac=\sqrt{(1-a^2)(1-c^2)}$$

Similarly,

$$bc+a=\sqrt{(1-b^2)(1-c^2)} \quad (7)$$

From Eqs. (5) and (7), required line of intersection is

$$\frac{x}{\sqrt{(1-a^2)}\sqrt{(1-c^2)}}=\frac{y}{\sqrt{(1-b^2)}\sqrt{(1-c^2)}}=\frac{z}{1-c^2}$$

or

$$\frac{x}{\sqrt{1-a^2}}=\frac{y}{\sqrt{1-b^2}}=\frac{z}{\sqrt{1-c^2}}$$

10. The direction cosines of the shortest distance between the lines $\frac{x-2}{1}=\frac{y-3}{4}=\frac{z+1}{3}$ and $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-1}{2}$ are $\frac{-1}{\sqrt{42}},\frac{4}{\sqrt{42}},\frac{-5}{\sqrt{42}}$. Find **(i)** its equation and **(ii)** the points where it intersects the lines.

Solution: Let

$$AB: \frac{x-2}{1}=\frac{y-3}{4}=\frac{z+1}{3} \quad (1)$$

and

$$CD: \frac{x-3}{2}=\frac{y-4}{3}=\frac{z-1}{2} \quad (2)$$

Let PQ be the shortest distance between the lines AB and CD.

Now,

(i) Equation of the plane containing AB and PQ is

$$\begin{vmatrix} x-2 & y-3 & z+1 \\ 1 & 4 & 3 \\ -1 & 4 & -5 \end{vmatrix}=0$$

or

$$-32(x-2)+2(y-3)+8(z+1)=0$$

or $\quad -32x+2y+8z+66=0$

or $\quad -16x+y+4z+33=0$

The equation of the plane containing CD and PQ is

$$\begin{vmatrix} x-3 & y-4 & z-1 \\ 2 & 3 & 2 \\ -1 & 4 & -5 \end{vmatrix}=0$$

or $\quad -23(x-3)+8(y-4)+11(z-1)=0$

or $\quad -23x+8y+11z+26=0$

The equations of the shortest distance are

$$-16x+y+4z+33=0$$

and $\quad -23x+8y+11z+26=0$

(ii) See Fig. 27.24. Since P and Q lie on lines (1) and (2), let

$$P\equiv(r+2, 4r+3, 3r-1)$$

and $\quad Q\equiv(2r'+3, 3r'+4, 2r'+1)$

The direction ratios of PQ are $(x_1-x_2, y_1-y_2, z_1-z_2)$.

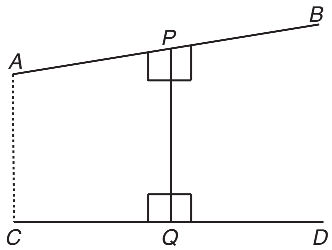


Figure 27.24

That is, $(r-2r'-1, 4r-3r'-1, 3r-2r'-2)$.

But PQ is perpendicular to both AB and CD

Therefore,

$$1(r-2r'-1)+4(4r-3r'-1)+3(3r-2r'-2)=0$$

and $\quad 2(r-2r'-1)+3(4r-3r'-1)+2(3r-2r'-2)=0$

or $\quad 26r-20r'=11 \quad (3)$

and $\quad 20r-17r'=9 \quad (4)$

Solving Eqs. (3) and (4), we get

$$r=\frac{1}{6}$$

and $\quad r'=-\frac{1}{3}$

Hence, $P\equiv\left(\frac{13}{6},\frac{11}{3},-\frac{1}{2}\right)$ and $Q\equiv\left(\frac{7}{3},3,\frac{1}{3}\right)$.

Previous Years' Solved JEE Main/AIEEE Questions

1. Let L be the line of intersection of the planes $2x+3y+z=1$ and $x+3y+2z=2$. If L makes an angle α with the positive x-axis, then $\cos\alpha$ equals

(A) $\frac{1}{\sqrt{3}}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** $\frac{1}{\sqrt{2}}$

[AIEEE 2007]

Solution: If direction cosines of L be l, m and n, then

$$2l + 3m + n = 0$$
$$l + 3m + 2n = 0$$

On solving, we get

$$\frac{l}{3} = \frac{m}{-3} = \frac{n}{3}$$

Therefore,

$$l : m : n = \frac{1}{\sqrt{3}} : -\frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}} \Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$$

Hence, the correct answer is option (A).

2. If (2, 3, 5) is one end of a diameter of the sphere $x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$, then the coordinates of the other end of the diameter are

(A) (4, 9, –3) **(B)** (4, –3, 3)
(C) (4, 3, 5) **(D)** (4, 3, –3)

[AIEEE 2007]

Solution: Coordinates of the centre is

$$\left(\frac{-(-6)}{2}, \frac{-(-12)}{2}, \frac{-(-2)}{2}\right) = (3, 6, 1)$$

Let the coordinates of the other end of the diameter be (α, β, γ). (See Fig. 27.25.)
Therefore,

$$\frac{\alpha + 2}{2} = 3; \frac{\beta + 3}{2} = 6; \frac{\gamma + 5}{2} = 1$$

Hence, $\alpha = 4$, $\beta = 9$ and $\gamma = -3$.

Hence, the correct answer is option (A).

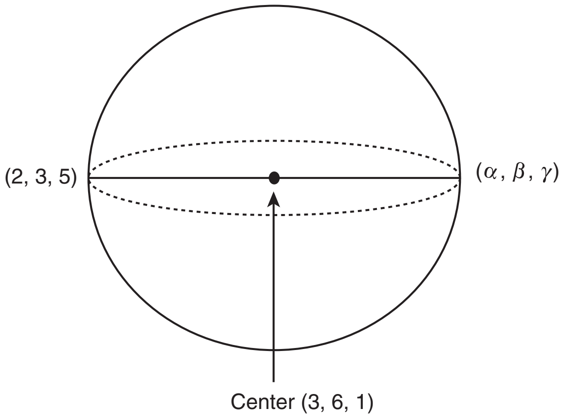


Figure 27.25

3. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$. Then (α, β) equals

(A) (6, –17) **(B)** (–6, 7)
(C) (5, –15) **(D)** (–5, 15)

[AIEEE 2009]

Solution: See Fig. 27.26.

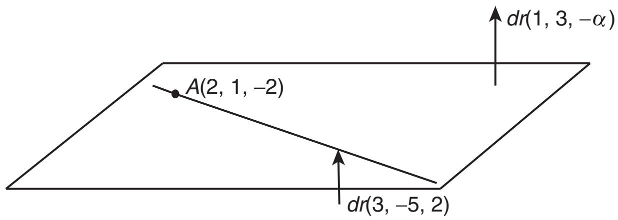


Figure 27.26

$$\text{dr's of line} = (3, -5, 2)$$

and

$$\text{dr's of normal to the plane} = (1, 3, -\alpha)$$

The line is perpendicular to the normal.
Therefore,

$$3(1) - 5(3) + 2(-\alpha) = 0 \Rightarrow 3 - 15 - 2\alpha = 0 \Rightarrow 2\alpha = -12 \Rightarrow \alpha = -6$$

Also, (2, 1, –2) lies on the plane

$$2 + 3 + 6(-2) + \beta = 0 \Rightarrow \beta = 7$$

Therefore, $(\alpha, \beta) = (-6, 7)$.

Hence, the correct answer is option (B).

4. The projections of a vector on the three coordinate axis are 6, –3 and 2, respectively. The direction cosines of the vector are

(A) 6, –3, 2 **(B)** $\frac{6}{5}, -\frac{3}{5}, \frac{2}{5}$
(C) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$ **(D)** $-\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

[AIEEE 2009]

Solution: The projections of a vector on coordinate axis are

$$x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$$
$$x_2 - x_1 = 6,\ y_2 - y_1 = -3,\ z_2 - z_1 = 2$$
$$\Rightarrow \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{36 + 9 + 4} = 7$$

The direction cosines of the vector are $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$.

Hence, the correct answer is option (C).

5. If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane $x + 2y + 3z = 4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals

(A) $\frac{3}{2}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{5}{3}$ **(D)** $\frac{2}{3}$

[AIEEE 2011]

Solution: Direction ratios of the line are 1, 2 and λ and direction ratios of normal to the plane are 1, 2 and 3.
Therefore angle between line and plane is given by

$$\cos(90^\circ - \theta) = \frac{1 \times 1 + 2 \times 2 + 3\lambda}{\sqrt{1^2 + 2^2 + \lambda^2}\sqrt{1^2 + 2^2 + 3^2}}$$

$$\Rightarrow \sin\theta = \frac{1 + 4 + 3\lambda}{\sqrt{1 + 4 + \lambda^2}\sqrt{1 + 4 + 9}} \Rightarrow \sin\theta = \frac{5 + 3\lambda}{\sqrt{5 + \lambda^2}\sqrt{14}}$$

Given that

$$\theta=\cos^{-1}\sqrt{\frac{5}{14}}=\sin^{-1}\sqrt{1-\frac{5}{14}}=\sin^{-1}\frac{3}{\sqrt{14}}$$

Therefore,

$$\frac{5+3\lambda}{\sqrt{5+\lambda^2}\sqrt{14}}=\frac{3}{\sqrt{14}}\Rightarrow\lambda=\frac{2}{3}$$

Hence, the correct answer is option (D).

6. **Statement-1:** The point $A(1, 0, 7)$ is the mirror image of the point $B(1, 6, 3)$ in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

 Statement-2: The line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ bisects the line segment joining $A(1, 0, 7)$ and $B(1, 6, 3)$.

 (A) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
 (B) Statement-1 is true, Statement-2 is false
 (C) Statement-1 is false, Statement-2 is true
 (D) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

[AIEEE 2011]

Solution: Figure 27.27 is used to infer the following two statements:

Statement-1 AB is perpendicular to given line and mid-point of AB lies on the line.
Statement-2 is true but it is not correct explanation as it is a bisector only.
Statement-2 is the correct explanation only when it is a perpendicular bisector, but it is not a perpendicular bisector.

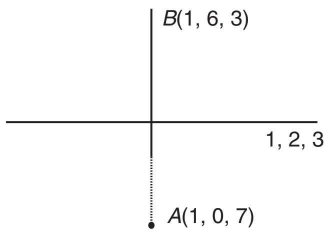


Figure 27.27

Hence, the correct answer is option (A).

7. If the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, then k can have

 (A) Exactly one value **(B)** Exactly two values
 (C) Exactly three values **(D)** Any value

[JEE MAIN 2013]

Solution: For lines to be coplanar, scalar triple product of vectors joining the two given points of the lines and the parallel vectors to the line must be 0.
We have

$$\begin{vmatrix}1 & -1 & -1\\ 1 & 1 & -k\\ k & 2 & 1\end{vmatrix}=0\Rightarrow 1(1+2k)+1(1+k^2)-1(2-k)=0$$

$$\Rightarrow k^2+1+2k+1-2+k=0$$

$$\Rightarrow k^2+3k=0\Rightarrow(k)(k+3)=0$$

Therefore, there are two values of k.

Hence, the correct answer is option (B).

8. Distance between two parallel planes $2x + y + 2z = 8$ and $4x+2y+4z+5=0$ is

 (A) $\frac{5}{2}$ **(B)** $\frac{7}{2}$ **(C)** $\frac{9}{2}$ **(D)** $\frac{3}{2}$

[JEE MAIN 2013]

Solution: The two parallel planes can be written as

$$4x+2y+4z=16; 4x+2y+4z=-5$$

Let a point on the first plane be (0, 0, 4)
Therefore, its distance from the other plane is obtained as

$$\frac{|0+0+16+5|}{\sqrt{4^2+2^2+4^2}}=\frac{21}{\sqrt{36}}=\frac{21}{6}=\frac{7}{2}$$

Hence, the correct answer is option (B).

9. The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane $2x-y+z+3=0$ is the line

 (A) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ **(B)** $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$
 (C) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ **(D)** $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$

[JEE MAIN 2014 (OFFLINE)]

Solution: Since,

$$3\,(2)+1\,(-1)+(-5)\,(1)=6-1-5=0$$

Therefore, line is parallel to the plane.

Now finding the image of point (1, 3, 4) in the plane. Let the image be (x_1, y_1, z_1).
Therefore,

$$\frac{x_1-1}{2}=\frac{y_1-3}{-1}=\frac{z_1-4}{1}=\frac{-2(2(1)-(3)+(4)+3)}{2^2+1^2+1^2}=\frac{-2\times\cancel{6}}{\cancel{6}}$$

Therefore,

$$x_1=-3, y_1=5, z_1=2$$

Therefore, required image of the line is

$$\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$$

Hence, the correct answer is option (C).

10. The angle between the lines whose direction cosines satisfy the equations $l+m+n=0$ and $l^2=m^2+n^2$ is

 (A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{4}$

[JEE MAIN 2014 (OFFLINE)]

Solution: Since,

$$l=-m-n$$

Therefore,

$$(-m-n)^2=m^2+n^2 \text{ or } 2\,mn=0\Rightarrow m=0 \text{ or } n=0$$

If $m=0$, then

$$l^2=n^2\Rightarrow l=n$$

Therefore,

$$n^2+0^2+n^2=1 \Rightarrow n^2=\frac{1}{2} \Rightarrow n=\pm\frac{1}{\sqrt{2}}$$

Therefore, directions are $\left(\frac{-1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$.

If $n=0$, then

$$l=-m \Rightarrow m^2+m^2+0=1$$

Thus,

$$m^2=\frac{1}{2} \Rightarrow m=\pm\frac{1}{\sqrt{2}}$$

Therefore, directions are $\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$.

$$\cos\theta=\frac{1}{2}+0+0 \Rightarrow \theta=\frac{\pi}{3}$$

Hence, the correct answer is option (C).

11. Equation of the plane that passes through the point of intersection of lines $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-3}{2}$ and $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and has the largest distance from the origin is

(A) $7x+2y+4z=54$ **(B)** $3x+4y+5z=49$
(C) $4x+3y+5z=50$ **(D)** $5x+4y+3z=57$

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: Writing general points

$$x=3\lambda+1,\ y=\lambda+2 \text{ and } 2\lambda+3$$
$$x=\mu+3,\ y=2\mu+1 \text{ and } 3\mu+2$$

Lines intersect. Therefore

$$3\lambda+1=\mu+3$$

and

$$\lambda+2=2\lambda+1$$

$$\begin{array}{ccc} 1 & 2 & 3 \\ 3\lambda-\mu & -2=0 & \\ \lambda & -2\mu+1=0 & \end{array} \qquad \begin{array}{cccc} 2 & 3 & 1 & 2 \\ -1 & -2 & 3 & -1 \\ -2 & 1 & 1 & -2 \end{array}$$

$$\frac{\lambda}{-1-4}=\frac{\mu}{-2-3}=\frac{1}{-6+1} \Rightarrow \lambda=1 \text{ and } \mu=1$$

Therefore, point of intersection is (4, 3, 5).

Now plane passing through (4, 3, 5) and at maximum distance from the origin must have directions of the normal as 4 – 0, 3 – 0 and 5 – 0.

Therefore, equation of required plane is

$$(x-4)4+(y-3)3+(z-5)5=0$$

or

$$4x+3y+5z=16+9+25 \Rightarrow 4x+3y+5z=50$$

Hence, the correct answer is option (C).

12. A line in the three-dimensional space makes an angle $\theta\left(0<\theta\le\frac{\pi}{2}\right)$ with both the x-and y-axis. Then the set of all values of θ is the interval

(A) $\left(0,\frac{\pi}{4}\right]$ **(B)** $\left[\frac{\pi}{6},\frac{\pi}{3}\right]$ **(C)** $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ **(D)** $\left(\frac{\pi}{3},\frac{\pi}{2}\right]$

[JEE MAIN 2014 (ONLINE SET-1)]

Solution: Sum of squares of dc's = 1. Therefore,

$$\cos^2\theta+\cos^2\theta+\cos^2\alpha=1 \Rightarrow 2\cos^2\theta=\sin^2\alpha$$

Thus,

$$0\le 2\cos^2\theta\le 1 \Rightarrow |\cos\theta|\le\frac{1}{\sqrt{2}}$$

$$\theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$$

Note: Interval in question should be $0\le\theta\le\frac{\pi}{2}$ for, third given option to be true, which in the only apparent possibility!

Hence, the correct answer is option (C).

13. Let $A(2, 3, 5)$, $B(-1, 3, 2)$ and $C(\lambda, 5, \mu)$ be the vertices of a ΔABC. If the median through A is equally inclined to the coordinate axes, then

(A) $5\lambda-8\mu=0$ **(B)** $8\lambda-5\mu=0$
(C) $10\lambda-7\mu=0$ **(D)** $7\lambda-10\mu=0$

[JEE MAIN 2014 (ONLINE SET-2)]

Solution: See Fig. 27.28.

Dr's of AD are $\frac{\lambda-1}{2}-2,\ 4-3,\ \frac{\mu+2}{2}-5$.

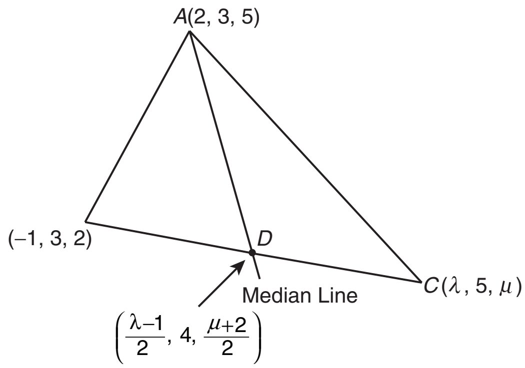


Figure 27.28

Therefore, dc's are

$$\cos\theta=\frac{\frac{\lambda-1}{2}-2}{|AD|} \Rightarrow \cos\theta=\frac{1}{|AD|} \Rightarrow \cos\theta=\frac{\frac{\mu+2}{2}-5}{|AD|}$$

$$\frac{\lambda-5}{2}=1 \Rightarrow \lambda=7$$

and

$$\frac{\mu+2-10}{2}=1 \Rightarrow \mu=10$$

It satisfies $10\lambda-7\mu=0$.

Hence, the correct answer is option (C).

14. The plane containing the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and parallel to the line $\frac{x}{1}=\frac{y}{1}=\frac{z}{4}$ passes through the point

(A) (1, –2, 5) **(B)** (1, 0, 5)
(C) (0, 3, –5) **(D)** (–1, –3, 0)

[JEE MAIN 2014 (ONLINE SET-2)]

Solution:

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3} \quad (1)$$

$$\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-0}{4}$$

(1, 2, 3) lie on required plane. Let required plane be

$$(x-1)\alpha+(y-2)\beta+(z-3)\chi=0$$

Therefore,

$$1\alpha+2\beta+3\chi=0$$

Since 1, 2 and 3 are the dr's of Eq. (1).

and $$1\alpha+1\beta+4\chi=0$$

Since 1, 1 and 1 are the dr's of Eq. (2).

On solving,

$$\frac{\alpha}{8-3}=\frac{\beta}{3-4}=\frac{\chi}{1-2}$$

Therefore, dr's of the required plane are 5, –1 and –1.

Thus, required plane is

$$(x-1)\,5+(y-2)\,(-1)+(z-3)\,(-1)=0$$
$$\Rightarrow (x-1)\,5-(y-2)\,1-(z-3)\,1=0$$

Now (1, 0, 5) satisfies it.

Hence, the correct answer is option (B).

15. A symmetrical form of the line of intersection of the planes $x=ay+b$ and $z=cy+d$ is

(A) $\frac{x-b}{a}=\frac{y-1}{1}=\frac{z-d}{c}$ **(B)** $\frac{x-b-a}{a}=\frac{y-1}{1}=\frac{z-d-c}{c}$

(C) $\frac{x-a}{b}=\frac{y-0}{1}=\frac{z-c}{d}$ **(D)** $\frac{x-b-a}{b}=\frac{y-1}{0}=\frac{z-d-c}{d}$

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: Planes are $x=ay+b$ and $z=cy+d$. Re-writing

1 2 3	2	3	1	2
$1x-ay-b=0$	$-a$	$-b$	1	$-a$
$0x-cy+z-d=0$	$-c$	$z-d$	0	$-c$

On solving by considering only x and y as variables and z like constant,

$$\frac{x}{-az+ad-bc}=\frac{y}{0-z+d}=\frac{1}{-c}$$

Taking first and third, therefore,

$$x=\frac{-az+ad-bc}{-c}\Rightarrow +\frac{a}{c}z-\frac{a}{c}d+b$$

$$\Rightarrow x-b=\frac{a}{c}(z-d)\Rightarrow \frac{x-b}{a}=\frac{z-d}{c} \quad (1)$$

Again taking second and third,

$$y=\frac{-z+d}{-c}=\frac{z}{c}-\frac{d}{c} \Rightarrow \frac{y-0}{1}=\frac{z-d}{c} \quad (2)$$

From Eqs. (1) and (2), required equation is

$$\frac{x-b}{a}=\frac{y-0}{1}=\frac{z-d}{C}=\lambda$$

$$\Rightarrow \frac{x-b-a}{a}=\frac{y-1}{1}=\frac{z-d-c}{c}=\lambda-1=\mu$$

Hence, the correct answer is option (B).

16. If the distance between planes $4x-2y-4z+1=0$ and $4x-2y-4z+d=0$ is 7, then d is

(A) 41 or –42 **(B)** 42 or –43
(C) –41 or 43 **(D)** –42 or 44

[JEE MAIN 2014 (ONLINE SET-3)]

Solution: Let a point of plane (1) is (0, 0, α).

Therefore,

$$4(0)-2(0)-4\alpha+1=0\Rightarrow 0-0-4\alpha=-1,$$
$$\Rightarrow \alpha=\frac{1}{4}$$

Now distance of point $\left(0,0,\frac{1}{4}\right)$ from plane (2) is 7.

Therefore,

$$7=\frac{\left|0-0-4\left(\frac{1}{4}\right)+d\right|}{\sqrt{4^2+2^2+4^2}}\Rightarrow |d-1|=7\times\sqrt{36}=7\times 6\Rightarrow d-1=\pm 42$$

Therefore, $d=43$ and -41.

Hence, the correct answer is option (C).

17. If $\hat{x},\hat{y}$ and $\hat{z}$ are three unit vectors in three-dimensional space, then the minimum value of $|\hat{x}+\hat{y}|^2+|\hat{y}+\hat{z}|^2+|\hat{z}+\hat{x}|^2$ is

(A) $\frac{3}{2}$ **(B)** 3 **(C)** $3\sqrt{3}$ **(D)** 6

[JEE MAIN 2014 (ONLINE SET-3)]

Solution:

$$|\hat{x}+\hat{y}|^2+|\hat{y}+\hat{z}|^2+|\hat{z}+\hat{x}|^2=1+1+2\hat{x}\cdot\hat{y}+1+1+2\hat{y}\cdot\hat{z}+1+1+2\hat{z}\cdot\hat{x}$$
$$=1+1+2\cos\alpha+1+1+2\cos\beta+1+1+2\cos\gamma$$
$$=6+2(\cos\alpha+\cos\beta+\cos\gamma) \quad (1)$$

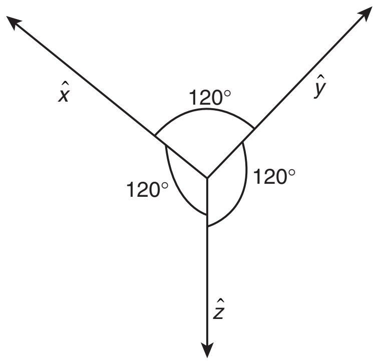


Figure 27.29

Now the minimum value of $|\hat{x}+\hat{y}|^2+|\hat{y}+\hat{z}|^2+|\hat{z}+\hat{x}|^2$ is the value when $\cos\alpha+\cos\beta+\cos\gamma$ is minimum, which is possible when $\alpha=\beta=\gamma=120°$ (Fig. 27.29).

Therefore,

$$-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}=-\frac{3}{2}$$

Therefore, the required minimum value is (Fig. 27.30)

$$6-2\left(\frac{3}{2}\right)=3 \qquad \text{[using (1)]}$$

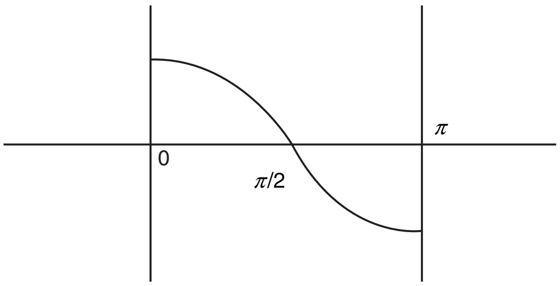


Figure 27.30

Hence, the correct answer is option (B).

18. Equation of the line of the shortest distance between the lines $\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$ and $\frac{x-1}{0}=\frac{y+1}{-2}=\frac{z}{1}$ is

(A) $\frac{x}{1}=\frac{y}{-1}=\frac{z}{-2}$ **(B)** $\frac{x-1}{1}=\frac{y+1}{-1}=\frac{z}{-2}$

(C) $\frac{x-1}{1}=\frac{y+1}{-1}=\frac{z}{1}$ **(D)** $\frac{x}{-2}=\frac{y}{1}=\frac{z}{2}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: See Fig. 27.31.

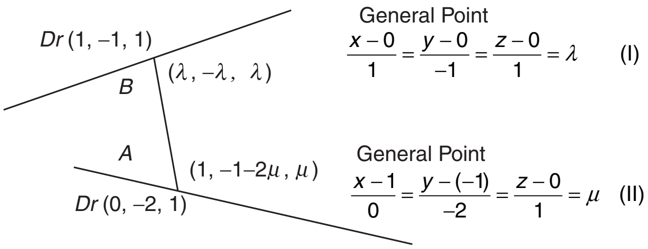


Figure 27.31

Dr's of $AB(\lambda-1, -\lambda+1+2\mu, \lambda-\mu)$. Now AB will be shortest distance, if

$$(1)(\lambda-1)+(-1)(-\lambda+1+2\mu)+(1)(\lambda-\mu)=0 \qquad (1)$$

That is, $AB\perp$ Line (1)

and $$0(\lambda-1)+(-2)(-\lambda+1+2\mu)+1(\lambda-\mu)=0 \qquad (2)$$

That is, $AB\perp$ Line (2)

Now,

$$\Rightarrow \lambda-1+\lambda-1-2\mu+\lambda-\mu=0\Rightarrow 3\lambda-3\mu-2=0 \qquad (3)$$

$$\Rightarrow 2\lambda-2-4\mu+\lambda-\mu=0\Rightarrow 3\lambda-5\mu-2=0 \qquad (4)$$

Solving Eqs. (3) and (4), we get

$$2\mu=0\Rightarrow\mu=0$$

From Eq. (3), $\lambda=\frac{2}{3}$ and B is $\frac{2}{3},\frac{-2}{3},\frac{2}{3}$.

Dr's of AB is

$$\left(\frac{2}{3}-1,-\frac{2}{3}+1,\frac{2}{3}\right)\Rightarrow\left(-\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)\Rightarrow -1,1,2$$

Therefore, equation of line of the shortest distance is

$$\frac{x-2/3}{-1}=\frac{y+2/3}{1}=\frac{z-2/3}{2}=k$$

And if we take the point A of required line, then the equation of the shortest distance is

$$\frac{x-1}{-1}=\frac{y+1}{1}=\frac{z-0}{2} \quad \text{or} \quad \frac{x-1}{1}=\frac{y+1}{-1}=\frac{z}{-2}$$

Hence, the correct answer is option (B).

19. If the angle between the line $2(x+1)=y=z+4$ and the plane $2x-y+\sqrt{\lambda}z+4=0$ is $\frac{\pi}{6}$, then the value of λ is

(A) $\frac{135}{7}$ **(B)** $\frac{45}{11}$ **(C)** $\frac{45}{7}$ **(D)** $\frac{135}{11}$

[JEE MAIN 2014 (ONLINE SET-4)]

Solution: See Fig. 27.32.

Line is $\frac{x-(-1)}{1/2}=\frac{y-0}{1}=\frac{z-(-4)}{1}$.

Plane is $2x-y+\sqrt{\lambda}z+4=0$.

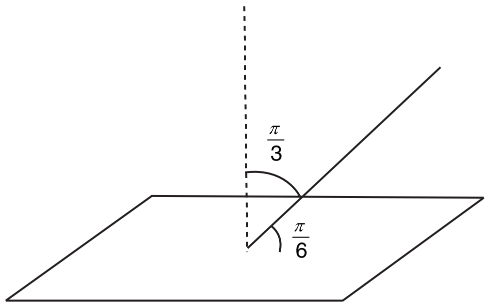


Figure 27.32

Dr's of normal to the plane $(2,-1,\sqrt{\lambda})$; Dr's of line $\frac{1}{2},1,1$

Therefore,

$$\cos\frac{\pi}{3}=\frac{2\left(\frac{1}{2}\right)+(-1)(1)+\sqrt{\lambda}(1)}{\sqrt{4+1+\lambda}\sqrt{\frac{1}{4}+1+1}}\Rightarrow\frac{1}{2}=\frac{1-1+\sqrt{\lambda}}{\sqrt{5+\lambda}\sqrt{\frac{9}{4}}}$$

Squaring on both the sides, we get

$$\frac{1}{4}=\frac{\lambda}{(5+\lambda)\frac{9}{4}}\Rightarrow(5+\lambda)\frac{9}{4}=4\lambda\Rightarrow 45+9\lambda=16\lambda$$

$$\Rightarrow 7\lambda=45\Rightarrow\lambda=\frac{45}{7}$$

Hence, the correct answer is option (C).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. Consider the following linear equations:

$$ax+by+cz=0,$$
$$bx+cy+az=0$$

and

$$cx+ay+bz=0$$

Match Column I and Column II.

Column I	Column II
(A) $a+b+c\neq 0$ and $a^2+b^2+c^2=ab+bc+ca$	**(p)** The equations represent planes meeting only at a single point.
(B) $a+b+c=0$ and $a^2+b^2+c^2\neq ab+bc+ca$	**(q)** The equations represent the line $x=y=z$.
(C) $a+b+c\neq 0$ and $a^2+b^2+c^2\neq ab+bc+ca$	**(r)** The equations represent identical planes.
(D) $a+b+c=0$ and $a^2+b^2+c^2=ab+bc+ca$	**(s)** The equations represent the whole of the three-dimensional space.

[IIT-JEE 2007]

Solution: We have

$$ax+by+cz=0$$
$$bx+cy+az=0$$
$$cx+ay+bz=0$$

$$\Delta=\begin{vmatrix} a & b & c\\ b & c & a\\ c & a & b\end{vmatrix}=3abc-a^3-b^3-c^3$$
$$=-(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$
$$=\frac{-1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

$$a+b+c\neq 0$$
$$a^2+b^2+c^2=ab+bc+ca$$
$$\Rightarrow a=b=c\neq 0,$$

which implies that the equations represent identical planes.

$$(A)\rightarrow(r)$$
$$a+b+c=0$$

and

$$a^2+b^2+c^2\neq ab+bc+ca$$
$$\Rightarrow \Delta=0,$$

which implies that the equations has infinitely many solutions and $x=y=z=\lambda\in\mathbb{R}$ satisfy the given equation.

$$(B)\rightarrow(q)$$
$$a+b+c\neq 0$$
$$a^2+b^2+c^2\neq ab+bc+ca$$
$$\Rightarrow \Delta\neq 0$$

That is, the equations have a unique solution.

$$(C)\rightarrow(p)$$
$$a+b+c=0$$
$$a^2+b^2+c^2=ab+bc+ca$$
$$\Rightarrow a=b=c=0$$

That is, all values of (x, y, z) satisfy the given equations and the equations represent whole of the three-dimensional space.

$$(D)\rightarrow(s)$$

Hence, the correct matches are (A)→(r), (B)→(q), (C)→(p) and (D)→(s).

2. Consider the planes $3x-6y-2z=15$ and $2x+y-2z=5$.

Statement-1: The parametric equations of the line of intersection of the given planes are $x=3+14t$, $y=1+2t$ and $z=15t$.

and

Statement-2: The vector $14\hat{i}+2\hat{j}+15\hat{k}$ is parallel to the line of intersection of given planes.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(C) Statement-1 is true, Statement-2 is false

(D) Statement-1 is false, Statement-2 is true

[IIT-JEE 2007]

Solution: Let $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ be the equation of line of intersection of planes

$$3x-6y-2z=15$$
$$2x+y-2z=5$$

Since, the line of intersection lies in both planes, we get

$$3a-6b-2c=0$$
$$2a+b-2c=0$$

That is,

$$\frac{a}{14}=\frac{b}{2}=\frac{c}{15}\Rightarrow a=14, b=2, c=15$$

Let $(x_1, y_1, 0)$ be a point that lies on the line of intersection; therefore,

$$3x_1-6y_1=15$$
$$2x_1+y_1=5$$
$$\Rightarrow (x_1, y_1, z_1)=(3, -1, 0)$$

Hence, the equation of line of intersection is

$$\frac{x-3}{14}=\frac{y+1}{2}=\frac{z}{15}=t$$

That is,

$$x=3+14t$$
$$y=2t-1$$
$$z=15t$$

Therefore, Statement-1 is false and Statement-2 is true.
Hence, the correct answer is option (D).

3. Consider three planes

$$P_1: x-y+z=1,$$
$$P_2: x+y-z=-1$$

and

$$P_3: x - 3y + 3z = 2$$

Let L_1, L_2 and L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1 and P_1 and P_2, respectively.

Statement-1: At least two of the lines L_1, L_2 and L_3 are non-parallel

and

Statement-2: The three planes do not have a common point.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(C) Statement-1 is true, Statement-2 is false
(D) Statement-1 is false, Statement-2 is true

[IIT-JEE 2008]

Solution: We have

$$P_1: x - y + z = -1$$
$$P_2: x + y - z = -1$$
$$P_3: x - 3y + 3z = 2$$

Let dr's of the lines L_1, L_2 and L_3 are a_1, b_1, c_1; a_2, b_2, c_2; and a_3, b_3, c_3, respectively.

Therefore,

$$a_1 + b_1 - c_1 = 0$$
$$a_1 - 3b_1 + 3c_1 = 0$$
$$\Rightarrow \frac{a_1}{0} = \frac{b_1}{-4} = \frac{c_1}{-4}$$
$$\Rightarrow a_1, b_1, c_1 = 0, 1, 1$$

Again

$$a_2 - b_2 + c_2 = 0$$
$$a_2 - 3b_2 + 3c_2 = 0$$
$$\Rightarrow \frac{a_2}{0} = \frac{b_2}{-2} = \frac{c_2}{-2}$$
$$\Rightarrow a_2, b_2, c_2 = 0, 1, 1$$

Again

$$a_3 - b_3 + c_3 = 0$$
$$a_3 + b_3 - c_3 = 0$$
$$\Rightarrow \frac{a_3}{0} = \frac{b_3}{2} = \frac{c_3}{2}$$
$$\Rightarrow a_3, b_3, c_3 = 0, 1, 1$$

Clearly L_1, L_2 and L_3 are parallel.

Hence, the correct answer is option (D).

4. Match the statements/expressions in **Column I** with the values given in **Column II.**

Column I	Column II
(A) The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$	**(P)** 1
(B) Value(s) of k for which the planes $kx + 4y + z = 0$, $4x + ky + 2z = 0$ and $2x + 2y + z = 0$ intersect in a straight line	**(Q)** 2
(C) Value(s) of k for which $\|x - 1\| + \|x - 2\| + \|x + 1\| + \|x + 2\| = 4k$ has integer solution(s)	**(R)** 3
(D) If $y' = y + 1$ and $y(0) = 1$, then value(s) of $y(\ln 2)$	**(S)** 4
	(T) 5

[IIT-JEE 2009]

Solution:

(A)
$$f'(x) > 0, \forall x \in (0, \pi/2)$$
$$f(0) < 0 \text{ and } f(\pi/2) > 0$$

So, one solution.

(B) Let (a, b, c) be the direction ratio of the intersected line.

Then

$$ak + 4b + c = 0$$
$$4a + kb + 2c = 0$$
$$\frac{a}{8-k} = \frac{b}{4-2k} = \frac{c}{k^2 - 16}$$

We must have

$$2(8 - k) + 2(4 - 2k) + (k^2 - 16) = 0 \Rightarrow k = 2 \text{ and } 4$$

(C) Let $f(x) = |x + 2| + |x + 1| + |x - 1| + |x - 2|$. Then k can take value 2, 3, 4 and 5 (Fig. 27.33).

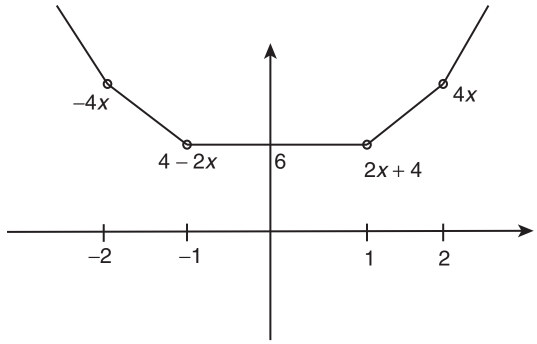


Figure 27.33

(D)
$$\int \frac{dy}{y+1} = \int dx$$
$$\Rightarrow f(x) = 2e^x - 1$$
$$\Rightarrow f(\ln 2) = 3$$

Hence, the correct matches are (A)→(P), (B)→(Q), (S), (C)→(Q), (R), (S), (T) and (D)→(R).

5. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

(A) $x + 2y - 2z = 0$ **(B)** $3x + 2y - 2z = 0$
(C) $x - 2y + z = 0$ **(D)** $5x + 2y - 4z = 0$

[IIT-JEE 2010]

Solution: Plane 1: $ax + by + cz = 0$ contains line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$. Therefore,

$$2a + 3b + 4c = 0 \quad (1)$$

Plane 2: $a'x + b'y + c'z = 0$ contains lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$. Hence,

$$3a' + 4b' + 2c' = 0$$

and $$4a' + 2b' + 3c' = 0$$

$$\Rightarrow \frac{a'}{12-4} = \frac{b'}{8-9} = \frac{c'}{6-16}$$

So, equation of the plane 2 is $8x - y - 10z = 0$. Now plane 1 is perpendicular to plane 2, therefore

$$8a - b - 10c = 0 \quad (2)$$

From Eqs. (1) and (2), we get

$$\frac{a}{-30+4} = \frac{b}{32+20} = \frac{c}{-2-24}$$

Therefore, equation of required plane is

$$x - 2y + z = 0$$

Hence, the correct answer is option (C).

6. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is.

[IIT-JEE 2010]

Solution:

$$2l + 3m + 4n = 0$$
$$3l + 4m + 5n = 0$$
$$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

Equation of plane will be

$$a(x-1) + b(y-2) + c(z-3) = 0$$
$$-1(x-1) + 2(y-2) - 1(z-3) = 0$$
$$-x + 1 + 2y - 4 - z + 3 = 0$$
$$-x + 2y - z = 0$$
$$x - 2y + z = 0$$

Now,

$$\frac{|d|}{\sqrt{6}} = \sqrt{6}$$
$$d = 6$$

Hence, the correct answer is (6).

7. If the distance of the point $P(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is

(A) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ **(B)** $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$

(C) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ **(D)** $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

[IIT-JEE 2010]

Solution: Distance of point $(1, -2, 1)$ from plane $x + 2y - 2z = \alpha$ is 5. So,

$$\alpha = 10$$

Equation of PQ is

$$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2} = t$$
$$Q \equiv (t+1, 2t-2, -2t+1)$$

and

$$PQ = 5 \Rightarrow t = \frac{5+\alpha}{9} = \frac{5}{3} \Rightarrow Q \equiv \left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$$

Hence, the correct answer is option (A).

Paragraph for Questions 8 to 10: Let a, b and c be three real numbers satisfying $[a \quad b \quad c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0 \quad 0 \quad 0]$ (E)

[IIT-JEE 2011]

8. If the point $P(a, b, c)$, with reference to (E), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

(A) 0 **(B)** 12 **(C)** 7 **(D)** 6

Solution:

$$a + 8b + 7c = 0$$
$$9a + 2b + 3c = 0$$
$$a + b + c = 0$$

Solving these, we get

$$b = 6a \Rightarrow c = -7a$$

Now,

$$2x + y + z = 1 \Rightarrow 2a + 6a + (-7a) = 1 \Rightarrow a = 1, b = 6, c = -7$$

So,

$$7a + b + c = 7(1) + 6 - 7 = 6$$

Hence, the correct answer is option (D).

9. Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfying (E), then the value of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to

(A) -2 **(B)** 2 **(C)** 3 **(D)** -3

Solution: $a = 2$, b and c satisfies (E), therefore

$$b = 12 \text{ and } c = -14$$
$$\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}} = -2$$

Hence, the correct answer is option (A).

10. Let $b = 6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is

(A) 6 **(B)** 7 **(C)** $\frac{6}{7}$ **(D)** ∞

Solution:

$$ax^2 + bx + c = 0 \Rightarrow x^2 + 6x - 7 = 0 \Rightarrow \alpha = 1, \beta = -7$$
$$\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{1} - \frac{1}{7}\right)^n = 7$$

Hence, the correct answer is option (B).

11. The point P is the intersection of the straight line joining the points $Q(2, 3, 5)$ and $R(1, -1, 4)$ with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point $T(2, 1, 4)$ to QR, then the length of the line segment PS is

(A) $\frac{1}{\sqrt{2}}$ **(B)** $\sqrt{2}$ **(C)** 2 **(D)** $2\sqrt{2}$

[IIT-JEE 2012]

Solution: See Fig. 27.34. Dr's of QR are 1, 4 and 1and its equation is

$$\frac{x-1}{1} = \frac{y+1}{4} = \frac{z-4}{1} = r$$
$$\Rightarrow x = r+1, y = 4r-1, z = r+4$$

Putting the values of x, y and z in

$$5x-4y-z=1$$
$$\Rightarrow 5(r+1)-4(4r-1)-(r+4)=1$$
$$\Rightarrow r=\frac{1}{3}$$

Therefore,

$$\text{coordinate of } P\equiv\left(\frac{4}{3},\frac{1}{3},\frac{13}{3}\right)$$

Dr's of PT are 2, 2 and –1.

Angle between QR and PT is 45° and $PT=1$, therefore,

$$PS=TS=\frac{1}{\sqrt{2}}$$

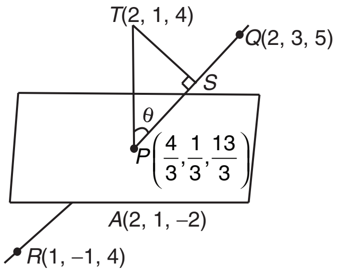


Figure 27.34

Hence, the correct answer is option (A).

12. The equation of a plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1) is

(A) $5x-11y+z=17$ **(B)** $\sqrt{2}x+y=3\sqrt{2}-1$

(C) $x+y+z=\sqrt{3}$ **(D)** $x-\sqrt{2}y=1-\sqrt{2}$

[IIT-JEE 2012]

Solution: Equation of required plane is

$$P\equiv(x+2y+3z-2)+\lambda(x-y+z-3)=0$$
$$\Rightarrow(1+\lambda)x+(2-\lambda)y+(3+\lambda)z-(2+3\lambda)=0$$

Its distance from (3, 1, –1) is $\frac{2}{\sqrt{3}}$, therefore,

$$\frac{2}{\sqrt{3}}=\frac{|3(1+\lambda)+(2-\lambda)-(3+\lambda)-(2+3\lambda)|}{\sqrt{(\lambda+1)^2+(2-\lambda)^2+(3+\lambda)^2}}$$

$$\Rightarrow\frac{4}{3}=\frac{(-2\lambda)^2}{3\lambda^2+4\lambda+14}\Rightarrow 3\lambda^2+4\lambda+14=3\lambda^2$$

$$\Rightarrow\lambda=-\frac{7}{2}\Rightarrow-\frac{5}{2}x+\frac{11}{2}y-\frac{z}{2}+\frac{17}{2}=0$$

$$-5x+11y-z+17=0$$

Hence, the correct answer is option (A).

13. If the straight lines $\frac{x-1}{2}=\frac{y+1}{k}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is(are)

(A) $y+2z=-1$ **(B)** $y+z=-1$

(C) $y-z=-1$ **(D)** $y-2z=-1$

[IIT-JEE 2012]

Solution: For given lines to be coplanar, we get

$$\begin{vmatrix}2 & k & 2\\ 5 & 2 & k\\ 2 & 0 & 0\end{vmatrix}=0\Rightarrow k^2=4\Rightarrow k=\pm 2$$

For $k=2$, obviously the plane $y+1=z$ is common in both lines.

For $k=-2$, family of plane containing first line is

$$x+y+\lambda(x-z-1)=0$$

Point (–1, –1, 0) must satisfy it

$$-2+\lambda(-2)=0\Rightarrow\lambda=-1\Rightarrow y+z+1=0$$

Hence, the correct answers are options (B) and (C).

14. Consider the lines $L_1:\frac{x-1}{2}=\frac{y}{-1}=\frac{z+3}{-1}, L_2:\frac{x-4}{1}=\frac{y+3}{1}=\frac{z+3}{2}$ and the planes $P_1:7x+y+2z=3, P_2=3x+5y-6z=4$. Let $ax+by+cz=d$ be the equation of the plane passing through the point of intersection of the lines L_1 and L_2 and perpendicular to planes P_1 and P_2. Match List I with List II and select the correct answer using the code given below the lists.

List I	List II
P. $a=$	**1.** 13
Q. $b=$	**2.** –3
R. $c=$	**3.** 1
S. $d=$	**4.** –2

	P	Q	R	S
(A)	3	2	4	1
(B)	1	3	4	2
(C)	3	2	1	4
(D)	2	4	1	3

[JEE ADVANCED 2013]

Solution: We have

$$L_1\equiv\frac{x-1}{2}=\frac{y}{-1}=\frac{z+3}{1}=t_1;$$
$$L_2\equiv\frac{x-4}{1}=\frac{y+3}{1}=\frac{z+3}{2}=t_2$$

For finding point of intersection, we have

$$1+2t_1=4+t_2 \quad (1)$$

and

$$-t_1=-3+t_2 \quad (2)$$

Solving, we get $t_1=2$ and $t_2=1$. The point of intersection is (5, –2, –1). The equation of plane P will be as follows:

$$\begin{vmatrix}x-5 & y+2 & z+1\\ 7 & 1 & 2\\ 3 & 5 & -6\end{vmatrix}=0$$

$$\Rightarrow(-16)(x-5)+48(y+2)+32(z+1)=0$$
$$\Rightarrow(x-5)-3(y+2)-2(z+1)=0$$
$$\Rightarrow x-3y-2z=13$$

Therefore, $a=1, b=-3, c=-2$ and $d=13$.

Hence, the correct answer is option (A).

15. In R^3, consider the planes $P_1: y=0$ and $P_2: x+z=1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point (0, 1, 0) from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relations is (are) true?

(A) $2\alpha+\beta+2\gamma+2=0$ **(B)** $2\alpha-\beta+2\gamma+4=0$
(C) $2\alpha+\beta-2\gamma-10=0$ **(D)** $2\alpha-\beta+2\gamma-8=0$

[JEE ADVANCED 2014]

Solution: Equation of plane P_3 through the intersection of P_1 and P_2 is

$$P_3: P_1+\lambda P_2=0$$

That is,

$$y+\lambda(x+z-1)=0$$

Distance of P_3 from (0, 1, 0) is 1 and from (α, β, γ) is 2.
Therefore,

$$\frac{|-\lambda+1|}{\sqrt{\lambda^2+1+\lambda^2}}=1$$
$$\Rightarrow \lambda^2+1-2\lambda=2\lambda^2+1$$
$$\Rightarrow \lambda^2+2\lambda=0$$
$$\Rightarrow \lambda=0$$

or

$$\lambda=-2$$

But P_3 is distinct from P_1 and P_2 (given), so

$$\lambda\neq 0$$
$$\Rightarrow \lambda=-2$$

Therefore,

$$P_3: y-2(x+z-1)=0$$
$$\Rightarrow \frac{|-2\alpha+\beta-2\gamma+2|}{\sqrt{4+1+4}}=2$$
$$\Rightarrow -2\alpha+\beta-2\gamma+2=\pm 6$$
$$\Rightarrow 2\alpha-\beta+2\gamma=2\mp 6$$
$$\Rightarrow 2\alpha-\beta+2\gamma=-4$$
$$\Rightarrow 2\alpha-\beta+2\gamma=8$$

Hence, the correct answers are options (B) and (D).

16. In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1: x+2y-z+1=0$ and $P_2: 2x-y+z-1=0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M?

(A) $\left(0,-\frac{5}{6},-\frac{2}{3}\right)$ **(B)** $\left(-\frac{1}{6},-\frac{1}{3},\frac{1}{6}\right)$
(C) $\left(-\frac{5}{6},0,\frac{1}{6}\right)$ **(D)** $\left(-\frac{1}{3},0,\frac{2}{3}\right)$

[JEE ADVANCED 2014]

Solution: Let the equation of the line L through the origin be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=k \quad (1)$$

Therefore any point on L is (lk, mk, nk).

Since the distance of every point on L is constant from the planes

$$P_1: x+2y-z+1=0 \quad (2)$$

and

$$P_2: 2x-y+z-1=0 \quad (3)$$

Therefore, L is parallel to the line of intersection of Eqs. (2) and (3), i.e. L is perpendicular to the normal to (2) and (3)

$$l+2m-n=0 \quad (4)$$

and

$$2l-m+n=0 \quad (5)$$

On solving, we get equation of L as

$$\frac{l}{1}=\frac{m}{-3}=\frac{n}{-5} \quad (6)$$

Therefore equation of L is

$$\frac{x}{1}=\frac{y}{-3}=\frac{z}{-5}=k \Rightarrow \text{any point on } L \text{ is } (k, -3k, -5k)$$

Therefore, foot of perpendicular M from point $(k, -3k, -5k)$ on P_1 is given by

$$\frac{x-k}{1}=\frac{y+3k}{2}=\frac{z+5k}{-1}=\frac{-(k-6k+5k+1)}{(1^2+4+1^2)}$$
$$\Rightarrow x=\frac{-1}{6}+k, y=\frac{-1}{3}-3k, z=\frac{1}{6}-5k$$

For $x=0$,

$$k=\frac{1}{6}\Rightarrow y=\frac{-1}{3}-\frac{1}{2}=\frac{-5}{6}, z=\frac{-2}{3},$$

For $y=0$,

$$k=\frac{-1}{9}, x=\frac{-5}{18}, z=\frac{13}{18},$$

For $k=0$,

$$x=\frac{-1}{6}, y=\frac{-1}{3}, z=\frac{1}{6}$$

Therefore, $\left(0,\frac{-5}{6},\frac{-2}{3}\right)$ and $\left(\frac{-1}{6},\frac{-1}{3},\frac{1}{6}\right)$ are the required points.

Hence, the correct answers are options (A) and (B).

17. Consider a pyramid $OPQRS$ located in the first octant ($x\geq 0$, $y\geq 0$, $z\geq 0$) with O as the origin, and OP and OR along the x-axis and the y-axis, respectively. The base $OPQR$ of the pyramid is a square with $OP=3$. The point S is directly above the mid-point T of diagonal OQ such that $TS=3$. Then,

(A) The acute angle between OQ and OS is $\frac{\pi}{3}$.

(B) The equation of the plane containing the triangle OQS is $x-y=0$.

(C) The length of the perpendicular from P to the plane containing the triangle OQS is $\frac{3}{\sqrt{2}}$.

(D) The perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$.

[JEE ADVANCED 2016]

Solution: Let us consider a pyramid $OPQRS$ in the first octant with O as the origin as shown in Fig. 27.35.

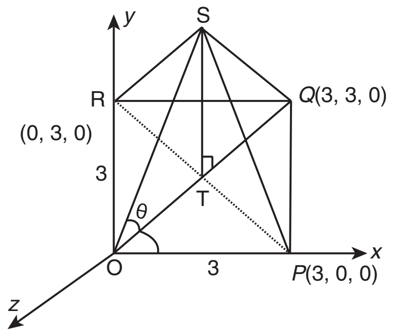


Figure 27.35

Now, the mid-point of OQ is $T\left(\frac{3}{2},\frac{3}{2},0\right)$.

Also, it is obvious that the point S is given by $S\left(\frac{3}{2},\frac{3}{2},3\right)$.

Now,

$$OT=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2+0}=\frac{3}{2}\sqrt{2}$$

$$OS=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)^2+3^2}=\frac{3}{2}\sqrt{6}$$

Therefore,

$$\cos\theta=\frac{3\sqrt{2}\cdot 2}{2\cdot 3\sqrt{6}}=\frac{1}{\sqrt{3}}$$

where θ is the angle between OQ and OS.
The ratio of the plane containing ΔOQS can be written as

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 3 & 0\\ 3/2 & 3/2 & 3\end{vmatrix}=\hat{i}(9)-\hat{j}(9)+\hat{k}(0)=9(\hat{i}-\hat{j})$$

Therefore, the ratio is (1, –1, 0). Now, the equation of the plane is

$$1(x-0)-1(y-0)+0(z-0)=0$$
$$x-y=0$$

Hence, option (B) is correct.
The coordinates of the length of perpendicular from point P to the plane $x-y=0$ is (3, 0, 0).
Hence, the length of perpendicular from point P to the plane containing the triangle OQS is

$$\frac{3-0}{\sqrt{2}}=\frac{3}{\sqrt{2}}$$

Hence, option (C) is correct.
Now, points R and S, respectively, are $R(0, 3, 0)$ and $S\left(\frac{3}{2},\frac{3}{2},3\right)$.
The equation of the line RS is

$$\bar{r}=3\hat{j}+\lambda\left(\frac{3}{2}\hat{i}+\left(\frac{3}{2}-3\right)\hat{j}+3\hat{k}\right)=3\hat{j}+\lambda\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$$

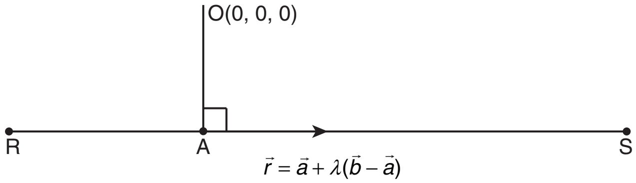


Figure 27.36

Let point A be the feet of perpendicular from O to line RS (Fig. 27.36)

$$A\left(3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)\right)$$

Now,

$$\overline{OA}=3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$$

That is,

$$(\vec{b}-\vec{a})=\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)$$

Therefore,

$$\overline{OA}\cdot(\vec{b}-\vec{a})=0$$

$$\left(3\hat{j}+\mu\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)\right)\cdot\left(\frac{3}{2}\hat{i}-\frac{3}{2}\hat{j}+3\hat{k}\right)=0$$

$$\frac{-9}{2}+\mu\left(\frac{9}{4}+\frac{9}{4}+9\right)=0$$

$$\frac{-9}{2}+\mu\left(\frac{9}{2}+9\right)=0$$

$$\Rightarrow -\frac{1}{2}+\mu\left(\frac{3}{2}\right)=0\Rightarrow \mu=\frac{1}{3}$$

Therefore,

$$A\left(3\hat{j}+\frac{\hat{i}}{2}-\frac{\hat{j}}{2}+\hat{k}\right)=A\left(\frac{\hat{i}}{2}+\frac{5\hat{j}}{2}+\hat{k}\right)$$

and hence, the perpendicular distance from point O to the straight line containing the line RS is

$$|\overline{OA}|=\sqrt{\frac{1}{4}+\frac{25}{4}+\frac{4}{4}}=\sqrt{\frac{30}{4}}=\sqrt{\frac{15}{2}}$$

Hence, option (D) is correct.
Hence, the correct answers are options (B), (C) and (D).

18. Let P be the image of the point (3, 1, 7) with respect to the plane $x-y+z=3$. Then, the equation of the plane passing through P and containing the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$ is

(A) $x+y-3z=0$ **(B)** $3x+z=0$
(C) $x-4y+7z=0$ **(D)** $2x-y=0$

[JEE ADVANCED 2016]

Solution: Let $P(x_1, y_1, z_1)$ be the image of $Q(3, 1, 7)$ w.r.t. the plane $x-y+z=3$.

Let R be the point on plane, which is mid-point of P and Q (Fig. 27.37).

The equation of the line PQ is

$$\frac{x-3}{1}=\frac{y-1}{-1}=\frac{z-7}{1}=\lambda$$

Therefore, $(x, y, z)=(3+\lambda, 1-\lambda, 7+\lambda)$ lies on the plane

$$3+\lambda-1+\lambda+7+\lambda=3$$
$$3\lambda+6=0\Rightarrow\lambda=-2$$

The point R is

$$(3-2, 3, 7-2)=(1, 3, 5)$$

Now,

$$\frac{x_1+3}{2}=1\Rightarrow x_1=-1$$
$$\frac{y_1+1}{2}=3\Rightarrow y_1=5$$
$$\frac{z_1+7}{2}=5\Rightarrow z_1=3$$

That is, the point P is (–1, 5, 3).

Now, the equation of the plane passing through P is

$$a(x+1)+b(y-5)+c(z-3)=0$$

This plane contains the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{1}$.
That is,

$$a(0+1)+b(0-5)+c(0-3)=0\Rightarrow a=5b+3c$$
$$a+2b+c=0\Rightarrow 7b+4c=0\Rightarrow b=\frac{-4c}{7}$$
$$a=-\frac{20c}{7}+3c=\frac{c}{7}$$

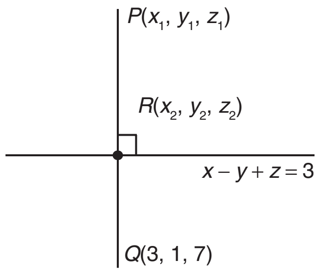


Figure 27.37

Now, the equation of the plane is obtained as follows:

$$\frac{c}{7}(x+1)-\frac{4c}{7}(y-5)+c(z-3)=0$$
$$(x+1)-4(y-5)+7(z-3)=0$$
$$x+1-4y+20+7z-21=0$$
$$x-4y+7z=0$$

Hence, the correct answer is option (C).

Practice Exercise 1

1. The angle between straight lines whose direction cosines are $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ is

(A) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **(B)** $\cos^{-1}\left(\frac{1}{\sqrt{6}}\right)$

(C) $\cos^{-1}\left(-\frac{1}{\sqrt{6}}\right)$ **(D)** None of these

2. Which one of the following is best condition for the plane $ax+by+cz+d=0$ to intersect the x-and y-axis at equal angle?

(A) $|a|=|b|$ **(B)** $a=-b$

(C) $a=b$ **(D)** $a^2+b^2=1$

3. The equation of a straight line parallel to the x-axis is given by

(A) $\frac{x-a}{1}=\frac{y-b}{1}=\frac{z-c}{1}$ **(B)** $\frac{x-a}{0}=\frac{y-b}{1}=\frac{z-c}{1}$

(C) $\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}$ **(D)** $\frac{x-a}{1}=\frac{y-b}{0}=\frac{z-c}{0}$

4. If P(2, 3, –6) and Q(3, –4, 5) are two points, the direction cosines of the line PQ are

(A) $-\frac{1}{\sqrt{171}}, -\frac{7}{\sqrt{171}}, -\frac{11}{\sqrt{171}}$ **(B)** $\frac{1}{\sqrt{171}}, -\frac{7}{\sqrt{171}}, \frac{11}{\sqrt{171}}$

(C) $\frac{1}{\sqrt{171}}, \frac{7}{\sqrt{171}}, -\frac{11}{\sqrt{171}}$ **(D)** $-\frac{7}{\sqrt{171}}, -\frac{1}{\sqrt{171}}, \frac{11}{\sqrt{171}}$

5. The ratio in which yz-plane divides the line joining the points $A(3, 1, -5)$ and $B(1, 4, -6)$ is

(A) –3:1 **(B)** 3:1 **(C)** –1:3 **(D)** 1:3

6. A straight line is inclined to the axes of x and z at angles 45° and 60°, respectively, then the inclination of the line to the y-axis is

(A) 30° **(B)** 45° **(C)** 60° **(D)** 90°

7. The angle between two diagonals of a cube is

(A) $\cos\theta = \sqrt{3}/2$ **(B)** $\cos\theta = 1/\sqrt{2}$

(C) $\cos\theta = 1/3$ **(D)** None of these

8. Given that $A(3, 2, -4)$, $B(5, 4, -6)$ and $C(9, 8, -10)$ are collinear. The ratio in which B divides AC is

(A) 1:2 **(B)** 2:1 **(C)** –1:2 **(D)** –2:1

9. If P_1P_2 is perpendicular to P_2P_3, then the value of k, where $P_1(k, 1, -1)$, $P_2(2k, 0, 2)$ and $P_3(2+2k, k, 1)$, is

(A) 3 **(B)** –3 **(C)** 2 **(D)** –2

10. A is the point (3, 7, 5) and B is the point (–3, 2, 6). The projection of AB on the line that joins the points (7, 9, 4) and (4, 5, –8) is

(A) 26 **(B)** 2 **(C)** 13 **(D)** 4

11. The shortest distance of the point from $P(x_1, y_1, z_1)$ on the x-axis is equal to

(A) $\sqrt{x_1^2+y_1^2}$ **(B)** $\sqrt{x_1^2+z_1^2}$

(C) $\sqrt{y_1^2+z_1^2}$ **(D)** None of these

12. The point of intersection of the xy-plane and the line passing through the points $A \equiv (3,4,1)$ and $B \equiv (5,1,6)$ are

(A) $\left(-\frac{13}{5}, \frac{23}{5}, 0\right)$ **(B)** $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

(C) $\left(\frac{13}{5}, -\frac{23}{5}, 0\right)$ **(D)** $\left(-\frac{13}{5}, -\frac{23}{5}, 0\right)$

13. The equation of the plane containing the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is $a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$, where $al+bm+cn$ is equal to

(A) 1 **(B)** –1 **(C)** 2 **(D)** 0

14. The shortest distance between the two straight lines $\frac{x-4/3}{2}=\frac{y+6/5}{3}=\frac{z-3/2}{4}$ and $\frac{5y+6}{8}=\frac{2z-3}{9}=\frac{3x-4}{5}$ is

(A) $\sqrt{29}$ **(B)** 3 **(C)** 0 **(D)** $6\sqrt{10}$

15. A straight line passes through the point (2, –1, –1). It is parallel to the plane $4x+y+z+2=0$ and is perpendicular to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z-5}{1}$. The equation of the straight line is

(A) $\frac{x-2}{4}=\frac{y+1}{1}=\frac{z+1}{1}$ **(B)** $\frac{x+2}{4}=\frac{y-1}{1}=\frac{z-1}{1}$

(C) $\frac{x-2}{-1}=\frac{y+1}{1}=\frac{z+1}{3}$ **(D)** $\frac{x+2}{-1}=\frac{y-1}{1}=\frac{z-1}{3}$

16. If centre of a sphere is (1, 4, –3) and the radius is 3 units, then the equation of the sphere is

(A) $x^2+y^2+z^2-2x-8y+6z+17=0$

(B) $2(x^2+y^2+z^2)-2x-8y+6z+17=0$

(C) $x^2+y^2+z^2-4x+16y+12z+17=0$

(D) $x^2+y^2+z^2+2x+8y-6z-17=0$

17. If equation of a sphere is $2(x^2+y^2+z^2)-4x-8y+12z-7=0$ and one extremity of its diameter is (2, –1, 1), then the other extremity of diameter of the sphere will be

(A) (2, 9, –13) **(B)** (0, 9, 7)

(C) (0, 5, 7) **(D)** (2, 5, –13)

18. The direction cosines of the line that is perpendicular to the lines with direction cosines proportional to (1, –2, –2), (0, 2, 1) are

(A) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ **(B)** $\left(\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)$

(C) $\left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$ (D) $\left(\frac{-2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)$

19. The points (4, 7, 8), (2, 3, 4), (–1, –2, 1) and (1, 2, 5) are
(A) The vertices of a parallelogram
(B) Collinear
(C) The vertices of a trapezium
(D) Concyclic

20. The equation of the plane parallel to the plane $4x - 3y + 2z + 1 = 0$ and passing through the point (5, 1, – 6) is
(A) $4x - 3y + 2z - 5 = 0$ **(B)** $3x - 4y + 2z - 5 = 0$
(C) $4x - 3y + 2z + 5 = 0$ **(D)** $3x - 4y + 2z + 5 = 0$

21. A plane is passed through the middle point of the segment $A(-2, 5, 1)$ and $B(6, 1, 5)$ and is perpendicular to this line. Its equation is
(A) $2x - y + z = 4$ **(B)** $2x + y + z = 4$
(C) $x - 3y + z = 5$ **(D)** $x - 4y + 2z = 5$

22. A plane meets the coordinates axes in A, B and C such that the centroid of the triangle ABC is (a, b, c). The equation of the plane is
(A) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$ **(B)** $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$
(C) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=2$ **(D)** None of these

23. The radius of the sphere $(x + 1)(x + 3) + (y - 2)(y - 4) + (z + 1)(z + 3) = 0$ is
(A) $\sqrt{2}$ **(B)** 2 **(C)** $\sqrt{3}$ **(D)** 3

24. The sum of the direction cosines of a straight line is
(A) Zero **(B)** One
(C) Constant **(D)** None of these

25. The angle between two lines whose direction cosines are given by the equation $l + m + n = 0$, $l^2 + m^2 + n^2 = 0$ is
(A) $\frac{\pi}{3}$ **(B)** $\frac{2\pi}{3}$
(C) $\frac{\pi}{4}$ **(D)** None of these

26. The equation of the plane that contains the line of intersection of the planes $x + y + z - 6 = 0$ and $2x + 3y + z + 5 = 0$ and perpendicular to the xy-plane is
(A) $x - 2y + 11 = 0$ **(B)** $x + 2y + 11 = 0$
(C) $x + 2y - 11 = 0$ **(D)** $x - 2y - 11 = 0$

27. The coordinates of the foot of the perpendicular drawn from the origin to the plane $3x + 4y - 6z + 1 = 0$ are
(A) $\left(\frac{-3}{61}, \frac{4}{61}, \frac{6}{61}\right)$ **(B)** $\left(\frac{3}{61}, \frac{-4}{61}, \frac{6}{61}\right)$
(C) $\left(\frac{-3}{61}, \frac{-4}{61}, \frac{6}{61}\right)$ **(D)** $\left(\frac{3}{61}, \frac{4}{61}, \frac{6}{61}\right)$

28. The distance of the point (1, –2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ is
(A) 7 units **(B)** 4 units
(C) 1 unit **(D)** 2 units

29. The image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$ is
(A) (–3, 5, 2) **(B)** (3, 2, 5)
(C) (–5, 3, –2) **(D)** (–2, 5, 3)

30. The equation of the sphere that passes through the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) and has its radius as small as possible is
(A) $3(x^2 + y^2 + z^2) + 2(x + y + z) - 1 = 0$
(B) $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$
(C) $(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$
(D) $(x^2 + y^2 + z^2) + 2(x + y + z) - 1 = 0$

31. A point moves so that the ratio of its distances from two fixed points is constant. Its locus is a
(A) Plane **(B)** Straight lines
(C) Circle **(D)** Sphere

32. A line makes angles α, β, γ and δ with the four diagonals of a cube, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta =$
(A) 1 **(B)** 4/3
(C) 3/4 **(D)** 4/5

33. The points (0, –1, –1), (–4, 4, 4), (4, 5, 1) and (3, 9, 4) are
(A) Collinear **(B)** Coplanar
(C) Forming a square **(D)** None of these

34. A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes in A, B and C. The locus of the point common to the plane through A, B and C parallel to the coordinate planes is
(A) $ayz + bzx + cxy = xyz$ **(B)** $axy + byz + czx = xyz$
(C) $axy + byz + czx = abc$ **(D)** $bcx + acy + abz = abc$

35. Consider the following statements:

Assertion (A): the plane $y + z + 1 = 0$ is parallel to x-axis.

Reason (R): normal to the plane is parallel to x-axis.

Of these statements:
(A) Both A and R are true and R is the correct explanation of A
(B) Both A and R are true and R is not a correct explanation of A
(C) A is true but R is false
(D) A is false but R is true

36. The projections of the line segment AB on the coordinate axes are –9, 12 and –8, respectively. The direction cosines of the line segment AB are
(A) $-\frac{9}{17}, \frac{12}{17}, -\frac{8}{17}$ **(B)** $-\frac{9}{289}, \frac{12}{289}, -\frac{8}{289}$
(C) $-\frac{9}{\sqrt{17}}, \frac{12}{\sqrt{17}}, -\frac{8}{\sqrt{17}}$ **(D)** None of these

37. The direction cosines of two mutually perpendicular lines are ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2. The direction cosines of the line perpendicular to both the given lines will be
(A) $\ell_1 + \ell_2,\ m_1 + m_2,\ n_1 + n_2$
(B) $\ell_1 - \ell_2,\ m_1 - m_2,\ n_1 - n_2$
(C) $\ell_1\ell_2,\ m_1m_2,\ n_1n_2$
(D) $m_1n_2 - m_2n_1,\ n_1\ell_2 - n_2\ell_1,\ \ell_1m_2 - \ell_2m_1$

38. A directed line segment angles α, β, γ with the coordinate axes. The value of $\sum\cos 2\alpha$ is always equal to
(A) –1 **(B)** 1 **(C)** –2 **(D)** 2

39. The locus represented by $xy + yz = 0$ is
(A) A pair of perpendicular lines
(B) A pair of parallel lines
(C) A pair of parallel planes
(D) A pair of perpendicular planes

40. The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(a)$ with x-axis. The value of 'a' is equal to
(A) $\frac{\sqrt{3}}{2}$ **(B)** $\frac{\sqrt{2}}{3}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{3}{7}$

41. The planes $x+y=0$, $y+z=0$ and $x+z=0$
(A) Meet in a unique point
(B) Meet in a unique line
(C) Are mutually perpendicular
(D) None of these

42. The equation of a plane passing through (1, 2, –3) and (0, 0, 0) and perpendicular to the plane $3x - 5y + 2z = 11$ is
(A) $3x+y+\frac{5}{3}z=0$ **(B)** $4x+y+2z=0$
(C) $3x-y+\frac{z}{3}$ **(D)** $x+y+z=0$

43. The direction ratios of a normal to the plane passing through (1, 0, 0) and (0, 1, 0) and making an angle $\frac{\pi}{4}$ with the plane $x+y=3$ are
(A) $(1, \sqrt{2}, 1)$ **(B)** $(1,1, \sqrt{2})$
(C) (1,1, 2) **(D)** $(\sqrt{2},1, 1)$

44. The equation of a plane passing through the line of intersection of the planes $x+y+z=5$ and $2x-y+3z=1$ and parallel to the line $y=z=0$ is
(A) $3x-z=9$ **(B)** $3y-z=9$
(C) $x-3z=9$ **(D)** $y-3z=9$

45. The angle between lines whose direction cosines are given by $\ell+m+n=0$ and $\ell^2+m^2-n^2=0$ is
(A) $\frac{\pi}{2}$ **(B)** $\frac{\pi}{3}$
(C) $\frac{\pi}{6}$ **(D)** None of these

46. Centroid of the tetrahedron $OABC$, where $A \equiv (a, 2, 3)$, $B \equiv (1, b, 2)$, $C \equiv (2, 1, c)$ and O is the origin of (1, 2, 3). The value of $a^2 + b^2 + c^2$ is equal to
(A) 75 **(B)** 80
(C) 121 **(D)** None of these

47. The equation of the plane passing through the points (2, –1, 0) and (3, –4, 5) and parallel to the line $2x=3y=4z$ is
(A) $125x-90y-79z=340$
(B) $32x-21y-36z=85$
(C) $73x+61y-22z=85$
(D) $29x-27y-22z=85$

48. The shortest distance of the plane $12+4y+3z=327$, from the sphere $x^2+y^2+z^2+4x-2y-6z=155$, is equal to
(A) 39 units **(B)** 26 sq. units
(C) 13 units **(D)** None of these

49. The equation of the straight line through the origin parallel to the line $(b+c)x+(c+a)y+(a+b)z=k=(b-c)x+(c-a)y+(a-b)z$ is
(A) $\frac{x}{b^2-c^2}=\frac{y}{c^2-a^2}=\frac{z}{a^2-b^2}$ **(B)** $\frac{x}{b}=\frac{y}{c}=\frac{z}{a}$
(C) $\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{a^2-ab}$ **(D)** None of these

50. The vertices of a triangle are $A(5, 4, 6)$, $B(1, -1, 3)$ and $C(4, 3, 2)$. The internal bisector of $\angle BAC$ meets BC in D. Find AD.

51. If the direction cosines of a variable line in two adjacent positions be l, m and n, and $l+\delta l$, $m+\delta m$ and $n+\delta n$, show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

52. If a line makes angles α, β, γ and δ with the diagonals of a cube, prove that
$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos 2\delta=-\frac{4}{3}$$

53. A triangle is placed so that the middle points of its sides are on the axes. If a, b and c be the length of its sides, show that the equation to its plane is $\frac{x}{x_1}+\frac{y}{y_1}+\frac{z}{z_1}=1$, where $8x_1^2=b^2+c^2-a^2$, $8y_1^2=c^2+a^2-b^2$ and $8z_1^2=a^2+b^2-c^2$.

54. The plane $ax+by=0$ is rotated through an angle α about its line of intersection with the plane $z=0$. Show that the equation to the plane in new position is $ax+by\pm z\sqrt{a^2+b^2}\tan\alpha=0$.

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If the system of equations $x-ky-z=0$, $kx-y-z=0$, $x+y-z=0$ has a non-zero solution, then the possible values of k are
(A) –1, 2 **(B)** 0, 1 **(C)** 1, 2 **(D)** –1, 1

2. A plane $2x+3y+5z=1$ has a point P which is at minimum distance from line joining $A(1, 0, -3)$ and $B(1, -5, 7)$ then distance AP is equal to
(A) $3\sqrt{5}$ **(B)** $2\sqrt{5}$
(C) $4\sqrt{5}$ **(D)** None of these

3. If $A=(p, q, r)$ and $B=(p', q' r')$ are two points on the line $\lambda x=\mu y=\gamma z$, such that $OA=3$, $OB=4$, then $pp'+qq'+rr'$ is equal to
(A) 7 **(B)** 12 **(C)** 5 **(D)** None of these

4. If the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ intersects the line $3\beta^2x+3(1-2\alpha)y+z=3=-\frac{1}{2}\{6\alpha^2x+3(1-2\beta)y+2z\}$, then point $(\alpha, \beta, 1)$ lie on the plane
(A) $2x-y+z=4$ **(B)** $x+y-z=2$
(C) $x-2y=0$ **(D)** $2x-y=0$

5. Let PM be the perpendicular from the point $P(1, 2, 3)$ to xy plane. If $\overrightarrow{OP}$ makes an angle θ with the positive direction of z-axis and $\overrightarrow{OM}$ makes an angle ϕ with the positive direction of x-axis, where O is the origin and θ and ϕ are acute angles, then
(A) $\tan\theta=\frac{\sqrt{5}}{3}$ **(B)** $\sin\theta\sin\phi=\frac{2}{\sqrt{14}}$
(C) $\tan\phi=2$ **(D)** $\cos\theta\cos\phi=\frac{1}{\sqrt{14}}$

6. Let a plane passes through origin and is parallel to line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z+1}{-2}$ such that the distance between plane and line is 5/3, then the equation of plane is

(A) $2x+2y-z=0$ **(B)** $x-2y-2z=0$

(C) $x+2y+2z=0$ **(D)** $2x-2y+z=0$

Comprehension Type Questions

Paragraph for Questions 7–9: The vertices of a triangle ABC are $A=(2, 0, 2)$, $B=(-1, 1, 1)$ and $C=(1, -2, 4)$. The point D and E divide the sides AB and CA in the ratio 1:2, respectively. Another point F is taken in space such that perpendicular drawn from F on ΔABC meets the triangle at the point of intersection of the line segment CD and BE, say P. If the distance of F from the plane of the ΔABC is $\sqrt{2}$ units, then

7. The PV of P is

(A) $\hat{i}+\hat{j}-3\hat{k}$ **(B)** $\hat{i}-\hat{j}+3\hat{k}$

(C) $2\hat{i}-\hat{j}-3\hat{k}$ **(D)** $\hat{i}+\hat{j}+3\hat{k}$

8. The vector $\overrightarrow{PF}$ is

(A) $7\hat{j}+7\hat{k}$ **(B)** $\frac{7}{\sqrt{2}}(\hat{j}+\hat{k})$

(C) $\hat{j}+\hat{k}$ **(D)** None of these

9. The volume of tetrahedron $ABCF$ is

(A) 7 cubic units **(B)** $\frac{3}{5}$ cubic units

(C) $\frac{7}{3}$ cubic units **(D)** $\frac{7}{5}$ cubic units

Paragraph for Questions 10–12: Consider a three-dimensional Cartesian system with origin at O and three rectangular coordinate axes x, y and z-axis. Suppose that the distance between two points P and Q in the space having their coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively, be defined by the following formula:

$$d(P, Q)=|x_2-x_1|+|y_2-y_1|+|z_2-z_1|$$

Although the formula of distance between two points has been defined in a new way, yet the other definitions remain same (like section formula, direction cosines, etc.). So, in general equations of straight line in space, plane in space remain unchanged.

10. If l, m, n represent direction cosines (if we can call it) of a vector $\overrightarrow{OP}$, then which of the following relations holds?

(A) $l^2+m^2+n^2=1$ **(B)** $l+m+n=1$

(C) $|l+m+n|=1$ **(D)** $|l|+|m|+|n|=1$

11. Locus of point P if $d(O, P)=k$, where k is a positive constant number, represents

(A) A sphere of radius k.

(B) A set of eight planes forming an octahedron.

(C) A set of eight planes forming hexagonal prism.

(D) An infinite cylinder of radius k.

12. Let A be a point (5, 2, 3) in the given reference system. Then locus of the point P in the first octant satisfying the equation $d(O, P)=d(A, P)$ does not contain

(A) Any of the coordinates axes.

(B) Any of the coordinate planes.

(C) Any plane parallel to coordinates axes.

(D) Any plane parallel to coordinates planes.

Paragraph for Questions 13–15: A ray of light is coming along the line $L=0$ and strikes the plane mirror kept along the plane $P=0$ at B. $A(2, 1, 6)$ is a point on the line $L=0$ whose image about $P=0$ is A'. It is given that $L=0$ is $\frac{x-2}{3}=\frac{y-1}{4}=\frac{z-6}{5}$ and $P=0$ is $x+y-2z=3$.

13. Co-ordinates of A' are

(A) (6, 5, 2) **(B)** (6, 5, –2)

(C) (6, –5, –2) **(D)** None of these

14. Co-ordinates of B are

(A) (5, 10, 6) **(B)** (10, 15, 11)

(C) (–10, –15, –14) **(D)** None of these

15. If $L_1=0$ is the reflected ray, then its equation is

(A) $\frac{x+10}{4}=\frac{y-5}{4}=\frac{z+2}{3}$ **(B)** $\frac{x+10}{3}=\frac{y+15}{5}=\frac{z+14}{5}$

(C) $\frac{x+10}{4}=\frac{y+15}{5}=\frac{z+14}{3}$ **(D)** None of these

Matrix Match Type Questions

16. A variable plane cuts the x-axis, y-axis and z-axis at the points A, B and C, respectively, such that the volume of the tetrahedron $OABC$ remains constant equal to 32 cubic unit and O is the origin of the coordinate system.

Column I	Column II
(A) The locus of the centroid of the tetrahedron is	**(p)** $xyz=24$
(B) the locus of the point equidistant from O, A, B and C is	**(q)** $(x^2+y^2+z^2)^3=192xyz$
(C) The length of the foot of perpendicular from origin to the plane is	**(r)** $xyz=3$
(D) If PA, PB and PC are mutually perpendicular then the locus of P is	**(s)** $(x^2+y^2+z^2)^3=1536xyz$

17. Match the following:

List I	List II
(A) If the line $\frac{x-1}{1}=\frac{y+1}{-2}=\frac{z+1}{\lambda}$ lies in the plane $3x-2y+5z=0$, then x is equal to	**(p)** $\sin^{-1}\sqrt{\frac{6}{25}}$
(B) If $(3, \lambda, \mu)$ is a point on the line $2x+y+z-3=0=x-2y+z-1$, then $\lambda+\mu$ is equal to	**(q)** $-\frac{7}{5}$
(C) The angle between the line $x=y=z$ and the plane $4x-3y+5z=2$ is	**(r)** -3
(D) The angle between the planes $x+y+z=0$ and $3x-4y+5z=0$ is	**(s)** $\cos^{-1}\sqrt{\frac{8}{75}}$

Answer Key

Practice Exercise 1

1. (C)	**2.** (A)	**3.** (D)	**4.** (B)	**5.** (A)	**6.** (C)
7. (C)	**8.** (A)	**9.** (A)	**10.** (B)	**11.** (C)	**12.** (B)
13. (D)	**14.** (C)	**15.** (C)	**16.** (A)	**17.** (C)	**18.** (A)
19. (A)	**20.** (A)	**21.** (A)	**22.** (A)	**23.** (C)	**24.** (C)
25. (A,B)	**26.** (B)	**27.** (C)	**28.** (C)	**29.** (A)	**30.** (B)
31. (D)	**32.** (B)	**33.** (B)	**34.** (A)	**35.** (C)	**36.** (A)
37. (D)	**38.** (A)	**39** . (D)	**40.** (C)	**41.** (B)	**42.** (D)
43. (B)	**44.** (B)	**45.** (D)	**46.** (A)	**47.** (D)	**48.** (C)
49. (C)	**50.** $\frac{\sqrt{1530}}{8}$ units				

Practice Exercise 2

1. (D)	**2.** (B)	**3.** (B)	**4.** (A), (B), (C)	**5.** (A), (B), (C)	**6.** (A)
7. (B)	**8.** (C)	**9.** (C)	**10.** (D)	**11.** (B)	**12.** (D)
13. (B)	**14.** (C)	**15.** (C)	**16.** (A) $\rightarrow$ (r), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (s)	**17.** (A) $\rightarrow$ (q), (B) $\rightarrow$ (r), (C) $\rightarrow$ (p), (D) $\rightarrow$ (s)	

Solutions

Practice Exercise 1

1. $\theta = \cos^{-1}\left(\dfrac{\frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}} - \frac{1}{\sqrt{6}}}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{2}}\sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}}\right) = \cos^{-1}\left(\dfrac{-1}{\sqrt{6}}\right)$

2. The plane $ax+by+cz+d=0$ intersects x- and y-axis at equal angles. Therefore,

$$|\cos\alpha| = |\cos\beta| \Rightarrow |l| = |m| \Rightarrow |a| = |b|$$

3. Equation of straight line parallel to x-axis is

$$\frac{x-a}{1} = \frac{y-b}{0} = \frac{z-c}{0}$$

Because $l = \cos\alpha = 1$, $m = \cos\beta = \cos\frac{\pi}{2} = 0$ and $n = \cos\gamma = 0$.

4. $P \equiv (2, 3, -6)$ and $Q \equiv (3, -4, 5)$

Direction ratios $= <1, -7, 11>$

$$\text{Direction cosines} = \left(\frac{1}{\sqrt{1+49+121}}, \frac{-7}{\sqrt{1+49+121}}, \frac{11}{\sqrt{1+49+121}}\right)$$

$$= \left(\frac{1}{\sqrt{171}}, \frac{-7}{\sqrt{171}}, \frac{11}{\sqrt{171}}\right)$$

5. $A \equiv (3, 1, -5)$ and $B \equiv (1, 4, -6)$.

Therefore,

$$\frac{3+\lambda}{\lambda+1} = 0 \Rightarrow \lambda = -3$$

Therefore required ratio is –3:1.

6. A straight line is inclined to the axes of x and z at angles 45° and 60°

$l^2+m^2+n^2 = 1 \Rightarrow m^2 = \frac{1}{4} \Rightarrow m = \frac{1}{2} \Rightarrow \text{angle} = 60°$

7. $\cos\theta = \dfrac{-a^2+a^2+a^2}{\sqrt{a^2+a^2+a^2}\sqrt{a^2+a^2+a^2}}$ of side a is $\frac{1}{3}$.

8. $\dfrac{9\lambda+3}{\lambda+1} = 5 \Rightarrow 9\alpha - 5\alpha = 2 \Rightarrow \lambda = \dfrac{1}{2}$

9. Direction ratios of $P_1P_2 = <k, -1, 3>$

Direction ratios of $P_2P_3 = <2, k, -1>$

Therefore,

$$2k - k - 3 = 0 \Rightarrow k = 3$$

10. Distances of the line joining (7, 9, 4) and (4, 5, –8) is

$$<\frac{3}{13}, \frac{4}{13}, \frac{12}{13}>$$

Therefore, required projection is $\frac{26}{13} = 2$.

11. Foot of perpendicular drawn from P to x-axis will have its coordinates as (x, 0, 0).

Therefore,

$$\text{Required distance} = \sqrt{y_1^2 + z_1^2}$$

12. Direction ratios of AB are 2, –3 and 5.

Thus equation of AB is

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}$$

For the point of intersection of this line with xy-plane, we have

$$Z=0 \Rightarrow \frac{x-3}{2}=\frac{y-4}{-3}=\frac{-1}{5} \Rightarrow x=3-\frac{2}{5}=\frac{13}{5}, y=4+\frac{3}{5}=\frac{23}{5}$$

Hence, the required point is $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

13. Since straight line lies in the plane, so it will be perpendicular to the normal at the given plane. Since direction cosines of straight line are l, m and n and direction ratios of normal to the plane are a, b and c.

So, $al + bm + cn = 0.$

14. By looking at the equation of given lines we infer that these two lines are intersecting so shortest distance between the lines will be 0. Hence, (C) is the correct answer.

15. Let direction cosines of straight line be l, m and n

Therefore,

$$4l + m + n = 0$$
$$l - 2m + n = 0 \Rightarrow \frac{l}{3}=\frac{m}{-3}=\frac{n}{-9} \Rightarrow \frac{l}{-1}=\frac{m}{+1}=\frac{n}{3}$$

Therefore equation of straight line is

$$\frac{x-2}{-1}=\frac{y+1}{1}=\frac{z+1}{3}$$

16. Equation of the sphere will be $(x-1)^2+(y-4)^2+(z+3)^2=9$

17. The centre of the sphere is (1, 2, –3) so if other extremity of diameter is (x_1, y_1, z_1),

then

$$\frac{x_1+2}{2}=1, \frac{y_1-1}{2}=2, \frac{z_1+1}{2}=-3$$

Therefore, the required point is (0, 5, 7).

18. Let direction ratios of the required line be $<a, b, c>$.

Therefore,

$$a-2b-2c=0$$

and

$$2b+c=0 \Rightarrow c=-2b$$
$$a-2b+4b=0 \Rightarrow a=-2b$$

Therefore direction ratios of the required line are

$$<-2b, b, -2b> = <2, -1, 2>$$

Direction cosines of the required line

$$\left(\frac{2}{\sqrt{2^2+1^2+2^2}}, \frac{-1}{\sqrt{2^2+1^2+2^2}}, \frac{2}{\sqrt{2^2+1^2+2^2}}\right)=\left(\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\right)$$

19. Let $A \equiv (4, 7, 8)$, $B \equiv (2, 3, 4)$, $C \equiv (-1, -2, 1)$ and $D \equiv (1, 2, 5)$.

Direction cosines of $AB \equiv \left(\frac{2}{6}, \frac{4}{6}, \frac{4}{6}\right)=\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

Direction cosines of $CD \equiv \left(\frac{-2}{6}, \frac{-4}{6}, \frac{-4}{6}\right)=\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

So, AB is parallel to CD.

Direction cosines of

$$AD \equiv \left(\frac{3}{\sqrt{43}}, \frac{5}{\sqrt{43}}, \frac{3}{\sqrt{43}}\right)$$

Direction cosines of

$$BC \equiv \left(\frac{-3}{\sqrt{43}}, \frac{-5}{\sqrt{43}}, \frac{-3}{\sqrt{43}}\right)=\left(\frac{3}{\sqrt{43}}, \frac{5}{\sqrt{43}}, \frac{3}{\sqrt{43}}\right)$$

So, AD is parallel to BC.

Therefore, $ABCD$ is a parallelogram.

20. Equation of the plane parallel to the plane $4x - 3y + 2z + 1 = 0$ is of the form of $4x - 3y + 2z + k = 0$, again it passes through (5, 1, –6). So,

$$20-3-12+k=0 \Rightarrow k=-5$$

Therefore required equation is

$$4x-3y+2z-5=0$$

21. Mid–point of $A(-2, 5, 1)$ and $B(6, 1, 5)$ is (2, 3, 3).

Direction ratios of the line joining A and B are $<2, -1, 1>$.

Therefore equation of the line perpendicular to AB and passing through (2, 3, 3) is

$$2(x-2)-1(y-3)+1(z-3)=0 \Rightarrow 2x-y+z=4$$

22. The plane meets the coordinate axes at A, B and C such that centroid of the triangle ABC is (a, b, c).

So, the plane cuts x-axis at $(3a, 0, 0)$, so x-intercept $= 3a$

The plane cuts y-axis at $(0, 3b, 0) \Rightarrow y$-intercept $= 3b$.

The plane cuts z-axis at $(0, 0, 3c) \Rightarrow z$-intercept $= 3c$

Therefore required equation is

$$\frac{x}{3a}+\frac{y}{3b}+\frac{z}{3c}=1 \Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$$

23. $(x + 1)(x + 3) + (y - 2)(y - 4) + (z + 1)(z + 3) = 0$ is the given equation of sphere.

So, end points of the diameter are (–1, 2, –1) and (–3, 4, –3)

$$\text{Radius} = \sqrt{(-2+1)^2+(3-2)^2+(-2+1)^2}=\sqrt{3}$$

24. $\cos\alpha = l$, $\cos\beta = m$ and $\cos\gamma = n$.

Sum of direction cosines $\cos\alpha + \cos\beta + \cos\gamma = l + m + n$, which is constant.

25. Eliminating n between the two relations, we have

$$l^2+m^2-(l+m)^2=0$$

or $2lm = 0 \Rightarrow$ either $l = 0$ or $m = 0$

If $l = 0$, then

$$m+n=0$$

That is,

$$m=-n \Rightarrow \frac{l}{0}=\frac{m}{1}=\frac{n}{-1}$$

giving the direction ratios of one line.

If $m = 0$, then

$$l+n=0$$

That is,

$$l=-n \Rightarrow \frac{l}{1}=\frac{m}{0}=\frac{n}{-1}$$

giving direction ratios of the other lines.

The angles between these lines is

$$\cos^{-1}\left\{\pm\frac{0\cdot1+1\cdot0+(-1)(-1)}{\sqrt{0^2+1^2+(-1)^2}\sqrt{1^2+0^2+(-1)^2}}\right\}\left(\pm\frac{1}{2}\right)$$

$$=\cos^{-1}=\frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

26. Equation of the required plane is

$$(x+y+z-6)+\lambda(2x+3y+z+5)=0$$

That is,

$$(1+2\lambda)x+(1+3\lambda)y+(1+\lambda)z+(-6+5\lambda)=0$$

This plane is perpendicular to xy-plane whose equation is $z = 0$.

That is,

$$0 \cdot x + 0 \cdot y + z = 0$$

Therefore, by condition of perpendicularity,

$$0 \cdot (1 + 2\lambda) + 0 \cdot (1 + 3\lambda) + (1 + \lambda) \cdot 1 = 0 \Rightarrow \lambda = -1$$

Therefore equation of required plane is

$(1 - 2)x + (1 - 3)y + (1 - 1)z + (-6 - 5) = 0$ or $x + 2y + 11 = 0$

27. The equation of the plane is

$$3x + 4y - 6z + 1 = 0(1)$$

The direction ratios of the normal to the plane (1) are 3, 4 and –6. So equation of the line through (0, 0, 0) and perpendicular to the plane (1) is

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{-6} = r \text{ (say)}(2)$$

The coordinates of any point P on (2) are $(3r, 4r, -6r)$. If this point lies on the plane (1), then

$$3(3r) + 4(4r) - 6(-6r) + 1 = 0$$

That is,

$$r = -\frac{1}{61}$$

Putting the value of r, coordinates of the foot of the perpendicular P are $\left(\frac{-3}{61}, \frac{-4}{61}, \frac{6}{61}\right)$.

28. Here, we are not finding perpendicular distance of the point from the plane but the distance measured along with the given line. The method is as follow:

The equation of the line through the point (1, –2, 3) and parallel to given line is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r \text{ (say)}$$

The coordinate of any point on it is $(2r + 1, 3r - 2, -6r + 3)$.

If this point lies in the given plane, then

$$2r + 1 - (3r - 2) + (-6r + 3) = 5 \Rightarrow -7r = -1$$

or $$r = \frac{1}{7}$$

Therefore point of intersection is $\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$.

Therefore, required distance = the distance between the points (1, –2, 3) and $\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$,

$$= \sqrt{\left(1-\frac{9}{7}\right)^2 + \left(-2+\frac{11}{7}\right)^2 + \left(3-\frac{15}{7}\right)^2} = \frac{1}{7}\sqrt{49} = 1 \text{ unit}$$

29. As it is clear from Fig. 27.38 that PQ will be perpendicular to the plane and foot of this perpendicular is mid-point of PQ, that is, N.

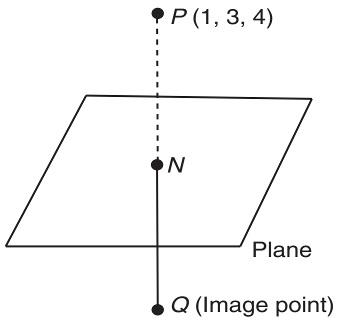


Figure 27.38

So, direction ratios of the line PQ are 2, –1 and 1.

Equation of the line PQ is

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = r \text{ (say)}$$

Any point on line PQ is $(2r + 1, -r + 3, r + 4)$.

If this point lies on the plane, then

$$2(2r + 1) - (-r + 3) + (r + 4) + 3 = 0 \Rightarrow r = -1$$

Therefore coordinate of foot of perpendicular $N = (-1, 4, 3)$.

As N is middle point of PQ, therefore

$$-1 = \frac{1 + x_1}{2}, 4 = \frac{3 + y_1}{2} \text{ and } 3 = \frac{4 + z_1}{2} \Rightarrow x_1 = -3, y_1 = 5, z_1 = 2$$

Therefore image of point $P(1, 3, 4)$ is the point $Q(-3, 5, 2)$.

30. Let equation of the sphere be given by

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0(1)$$

As the sphere passes through points (1, 0, 0), (0, 1, 0) and (0, 0, 1).

So, we have

$$1 + 2u + d = 0, 1 + 2v + d = 0$$

and $$1 + 2w + d = 0$$

On solving,

$$u = v = w = -\frac{1}{2}(d + 1)$$

If r is the radius of the sphere, then

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

$$r^2 = \frac{3}{4}(d + 1)^2 - d = \mu \text{ (say)}$$

For r to be minimum

$$\frac{d\mu}{dd} = 0 \Rightarrow \frac{3}{4} \cdot 2(d + 1) - 1 = 0$$

or

$$d = -\frac{1}{3}$$

Also,

$$\frac{d^2\mu}{dd^2} = \frac{3}{2} = \text{positive at } d = -\frac{1}{3}$$

Hence, μ is minimum at $d = -\frac{1}{3}$.

So, substituting value of d, we have

$$u = v = w = -\frac{1}{3}$$

Therefore equation of the sphere

$$x^2 + y^2 + z^2 - \frac{2}{3}(x + y + z) - \frac{1}{3} = 0$$

$$\Rightarrow 3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$$

31. Let the coordinates of moving point P be (x, y, z). Let $A(a, 0, 0)$ and $B(-a, 0, 0)$ be two fixed points. According to given condition,

$$\frac{AP}{BP} = \text{constant} = k \text{ (say)} \Rightarrow AP^2 = k^2BP^2$$

or

$$(x - a)^2 + (y - 0)^2 + (z - 0)^2 = k^2\{(x + a)^2 + (y - 0)^2 + (z - 0)^2\}$$

$\Rightarrow (1-k^2)(x^2+y^2+z^2) - 2ax(1+k^2) + a^2(1-k^2) = 0$

Therefore required locus is

$$x^2+y^2+z^2-\frac{2a(1+k^2)}{(1-k^2)}x+a^2=0,$$

which is a sphere.

32. See Fig. 27.39. The direction ratios of the diagonal $\overrightarrow{OR}$ are (1, 1, 1).

Direction cosine are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Similarly, direction cosine of $\overrightarrow{AS}$ are $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

and $\overrightarrow{BP}$ are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

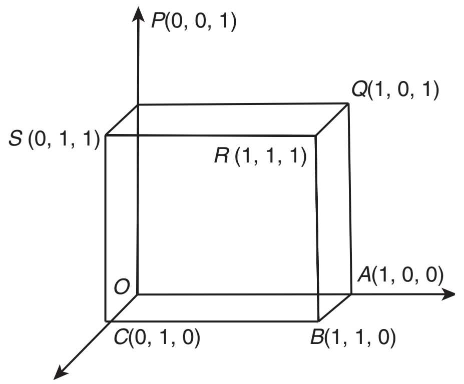


Figure 27.39

$$\overrightarrow{CQ} \text{ are } \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Let l, m and n be direction cosines of the lines. Then

$$\cos\alpha = \frac{l+m+n}{\sqrt{3}},$$

$$\cos\beta = \frac{l-m-n}{\sqrt{3}},$$

$$\cos\gamma = \frac{l+m-n}{\sqrt{3}}$$

and

$$\cos\delta = \frac{l-m+n}{\sqrt{3}}$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = \frac{4(l^2+m^2+n^2)}{3} = \frac{4}{3}$$

(Since, $l^2+m^2+n^2=1$)

33. Equation of the plane passing through the points (0, –1, –1), (–4, 4, 4) and (4, 5, 1)

$$\begin{vmatrix} x & (y+1) & (z+1) \\ -4 & 5 & 5 \\ 4 & 6 & 2 \end{vmatrix} = 0$$

$$5x - 7y + 11z + 4 = 0 \qquad (1)$$

The point (3, 9, 4) satisfies Eq. (1).

34. Let the equation to the plane be

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \Rightarrow \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$$

(since the plane passes through a, b and c).

Now the points of intersection of the plane with the coordinate axes are $A(\alpha, 0, 0)$, $B(0, \beta, 0)$ and $C(0, 0, \gamma)$.

Equations to planes parallel to the coordinate planes and passing through A, B and C are $x=\alpha$, $y=\beta$ and $z=\gamma$. Therefore, the locus of the common point is

$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$ (by eliminating α, β and γ from above equation)

35. Given that plane $y + z + 1 = 0$ is parallel to the x-axis as $0.1+1.0+1.0=0$ but normal to the plane will be perpendicular to the x-axis.

36. Length of the segment $AB=\sqrt{81+144+64}=17$.

Thus direction cosines of AB are $-\frac{9}{17}, \frac{12}{17}, \frac{-8}{17}$.

37. Let the direction cosine of the required line be ℓ, m and n.

We must have,

$$\ell\ell_1+mm_1+nn_1=0,\ \ \ell\ell_2+mm_2+nn_2=0$$

$$\Rightarrow \frac{\ell}{m_1n_2-n_1m_2}=\frac{m}{n_1\ell_2-n_2\ell_1}=\frac{n}{\ell_1m_2-\ell_2m_1}$$

$$=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{\sum(m_1n_2-n_2m_2)^2}}$$

We have

$$\ell_1\ell_2+m_1m_2+n_1n_2=0$$

Thus

$$\sum(m_1n_2-n_1m_2)^2=(\sum\ell_1)^2(\sum\ell_2)^2-\sum\ell_1\ell_2=1$$

$$\Rightarrow\ \ \ell=m_1n_2-n_1m_2,\ m=n_1\ell_2-n_2\ell_1, n=\ell_1m_2-\ell_2m_1$$

38. $\sum\cos 2\alpha=\sum(2\cos^2\alpha-1)=2\sum\ell^2-3=-1$

39.

$$xy+yz=0$$

$$\Rightarrow x(y+z)=0$$

$$\Rightarrow x=0,\ \ y+z=0$$

Thus it represents a pair of planes $x=0$, $y+z=0$ that are clearly mutually perpendicular.

40. If 'θ' be the angle between the plane and x-axis, then

$$\sin\theta=\frac{2}{\sqrt{4+9+36}}=\frac{2}{7}$$

$$\Rightarrow \theta=\sin^{-1}\left(\frac{2}{7}\right) \Rightarrow a=\left(\frac{2}{7}\right)$$

41. Clearly, given planes have a common line of intersection namely the z-axis. Hence, (B) is the correct answer.

42. Let the required plane be $ax+by+cz=0$.

We have

$$3a-5b+2c=0,\ a+2b-3c=0$$

$$\Rightarrow \frac{a}{15-4}=\frac{b}{2+9}=\frac{c}{6+5}$$

$$\Rightarrow \; a : b : c = 11 : 11 : 11$$

Thus plane is $x + y + z = 0$.

43. Let the plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow \frac{1}{a}=1, \frac{1}{b}=1 \Rightarrow a=b=1$$

Also,

$$\sin\frac{\pi}{4}=\frac{\left|\frac{1}{a}+\frac{1}{b}\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\sqrt{1+1}} \Rightarrow c=\pm\frac{1}{\sqrt{2}}$$

Thus direction rations are $(1, 1, \sqrt{2})$ or $(1, 1, -\sqrt{2})$.

44. Plane will be in the form

$$(x + y + z - 5) + a(2x - y + 3z) - 1) = 0$$

That is,

$$x(1 + 2a) + y(1 - a) + z(1 + 3a) = 5 + a$$

It is parallel to the line $y = z = 0$. Since,

$$(1 + 2a) = 0$$

Therefore,

$$a=-\frac{1}{2}$$

Thus, the required plane is

$$\frac{3}{2}y-\frac{1}{2}z=\frac{9}{2} \Rightarrow 3y - z = 9$$

45. $$\ell+m+n=0,\ \ell^2+m^2-n^2=0$$

We also have

$$\ell^2+m^2+n^2=1 \Rightarrow 2n^2 = 1 \Rightarrow n=\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$$

Also,

$$\ell^2 + m^2 = n^2 = (-(\ell+m))^2 \Rightarrow \ell m = 0 \text{ and } \ell+m=\pm\frac{1}{\sqrt{2}}$$

Hence, direction cosines are lines

$$\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right),$$

$$\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Angle between these lines in both cases is zero.

46. We have

$$4 = a + 1 + 2 + 0,$$
$$\Rightarrow a=1,$$
$$8 = 2 + b + 1 + 0$$
$$\Rightarrow b=5,$$
$$12 = 3 + 2 + c + 0$$
$$\Rightarrow c=7.$$

Therefore,

$$a^2 + b^2 + c^2 = 1 + 25 + 49 = 75$$

47. Given line is

$$2x = 3y = 4z, \Rightarrow \frac{x}{6}=\frac{y}{4}=\frac{z}{3}$$

Let the plane be $ax + by + cz = 1$.

We have

$$6a + 4b + 3c = 0$$
$$2a - b = 1$$
$$3a - 4b + 5c = 1$$
$$\Rightarrow a=\frac{29}{85}, b=\frac{27}{85}, c=-\frac{22}{85}$$

Thus, equation of plane is $29x - 27y - 22z = 85$.

48. Centre and radius of the given sphere are (–2, 1, 3) and 13 unit respectively.

Now, distance of centre of the sphere from the given plane

$$\frac{|-24+4+9-327|}{\sqrt{12^2+4^2+3^2}}=26 \text{ units}$$

Therefore, shortest distance = (26 – 13) = 13 units.

49. Equation of straight line through origin is

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n},$$

where

$$l(b + c) + m(c + a) + n(a + b) = 0$$

and

$$l(b - c) + m(c - a) + n(a - b) = 0$$

On solving,

$$\frac{l}{2(a^2-bc)}=\frac{m}{2(b^2-ca)}=\frac{n}{2(c^2-ab)}$$

Therefore, equation of the straight line is

$$\frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}$$

50. $$AB = \sqrt{4^2+5^2+3^2} = 5\sqrt{2}$$

and $$AC = \sqrt{1^2+1^2+4^2} = 3\sqrt{2}$$

Since AD is the internal bisector of ∠BAC (Fig. 27.40),

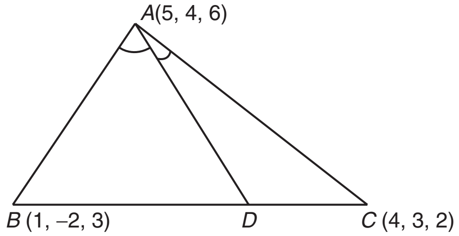


Figure 27.40

Now,

$$\frac{BD}{DC}=\frac{AB}{AC}=\frac{5}{3}$$

Since D divides BC internally in the ratio 5:3.

Therefore,

$$D \equiv \left(\frac{5\times4+3\times1}{5+3}, \frac{5\times+3(-1)}{5+3}, \frac{5\times2+3\times3}{5+3}\right)$$

or

$$D \equiv \left(\frac{23}{8}, \frac{12}{8}, \frac{19}{8}\right)$$

Hence,

$$AD=\sqrt{\left(5-\frac{23}{8}\right)^2+\left(4+\frac{12}{8}\right)^2+\left(6-\frac{19}{8}\right)^2}$$

$$=\sqrt{\frac{17^2+20^2+29^2}{8^2}}=\frac{\sqrt{1530}}{8}\text{ units}$$

51. We have

$$l^2+m^2+n^2=1$$

and $\quad (l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1$

or

$$l^2+m^2+n^2+\delta l^2+\delta m^2+\delta n^2+2\,(l\delta l+m\delta m+n\delta n)=1$$

or

$$\delta l^2+\delta m^2+\delta n^2$$
$$=-2\,(l\delta l+m\delta m+n\delta n)\ [\text{since } l^2+m^2+n^2=1] \qquad (1)$$
$$\cos\delta\theta=l\,(l+\delta l)+m\,(m+\delta m)+n\,(n+\delta n)$$

or

$$1-2\sin^2\frac{\delta\theta}{2}=l^2+m^2+n^2+(l\delta l+m\delta m+n\delta n)$$

or

$$-2\left(\frac{\delta\theta}{2}\right)^2=l\delta l+m\delta m+n\delta n$$

$$\left[\begin{array}{l}\text{Therefore,}\\ \sin\left(\dfrac{\delta\theta}{2}\right)=\left(\dfrac{\delta\theta}{2}\right)\text{ as } \delta\theta \text{ is very small and } l^2+m^2+n^2=1\end{array}\right]$$

or $\quad (\delta\theta)^2=-2\,(l\delta l+m\delta m+n\delta n)$

or $\quad (\delta\theta)^2=\delta l^2+\delta m^2+\delta n^2$ [from Eq. (1)]

52. Let the length of each side of the cube be a.

The coordinates of the corners are as shown in Fig. 27.41.

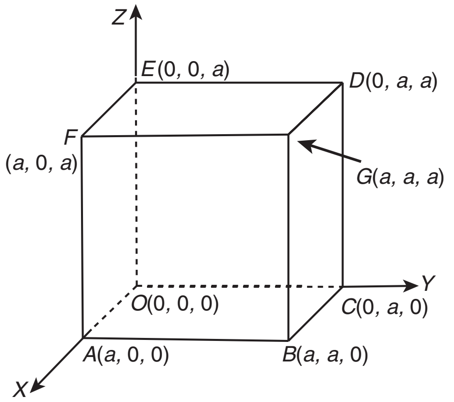


Figure 27.41

The four diagonals are OG, AD, CF and BE.

The direction ratios of OG are $a-0$, $a-0$ and $a-0$ or a, a and a.

Direction cosines of OG are $\dfrac{a}{\sqrt{3}\,a},\dfrac{a}{\sqrt{3}\,a},\dfrac{a}{\sqrt{3}\,a}$ or $\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}$

Similarly, direction cosines of AD are $\dfrac{-1}{\sqrt{3}},\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}$.

Direction cosines of CF are $\dfrac{1}{\sqrt{3}},\dfrac{-1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}$ and direction cosines of BE are $\dfrac{-1}{\sqrt{3}},\dfrac{-1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}$.

Let l, m and n be the direction cosines of the given line that makes angles α, β, γ and δ with OG, AD, CF, BE, respectively. Then

$$\cos\alpha=\frac{l}{\sqrt{3}}+\frac{m}{\sqrt{3}}+\frac{n}{\sqrt{3}}=\frac{l+m+n}{\sqrt{3}}$$

$$\cos\beta=\frac{-l}{\sqrt{3}}+\frac{m}{\sqrt{3}}+\frac{n}{\sqrt{3}}=\frac{-l+m+n}{\sqrt{3}}$$

$$\cos\gamma=\frac{l}{\sqrt{3}}-\frac{m}{\sqrt{3}}+\frac{n}{\sqrt{3}}=\frac{l-m+n}{\sqrt{3}}$$

$$\cos\delta=\frac{-l}{\sqrt{3}}-\frac{m}{\sqrt{3}}+\frac{n}{\sqrt{3}}=\frac{-l-m+n}{\sqrt{3}}$$

Therefore,

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$$

$$=\frac{1}{3}[l+m+n)^2+(-l+m+n)^2+(l-m+n)^2+(l+m+n)^2]$$

$$=\frac{1}{3}[4\,(l^2+m^2+n^2)+2lm+2ln-2lm-2ln+2mn$$
$$-2lm-2mm+2ln+2lm-2ln-2mn]$$

$$=\frac{4}{3} \qquad [\text{Therefore, } l^2+m^2+n^2=1]$$

Now,

$$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos 2\delta=2\cos^2\alpha$$
$$=1+2\cos^2\beta-1+2\cos^2\gamma-1+2\cos^2\delta-1$$
$$=2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos^2\delta)-4=2\cdot\frac{4}{3}-4=-\frac{4}{3}$$

53. Let the mid-points of BC, CA and AB be E, F and G, respectively. Let the points E, F and G be on x, y and z-axis, respectively (Fig. 27.42).

Let $E\equiv(x_1, 0, 0)$, $F=(0, y_1, 0)$ and $G=(0, 0, z_1)$.

Equation of the plane ABC is $\dfrac{x}{x_1}+\dfrac{y}{y_1}+\dfrac{z}{z_1}=1$

We know that

$$GF=\frac{1}{2}\,BC \text{ or } 4GF^2=BC^2 \text{ or } 4\left(y_1^2+z_1^2\right)=a^2 \qquad (1)$$

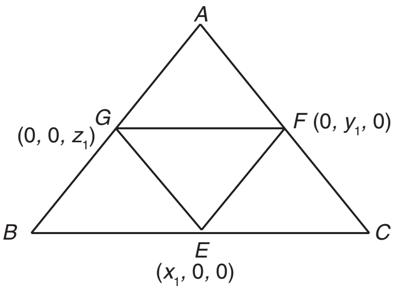


Figure 27.42

Similarly,

$$4\left(x_1^2+y_1^2\right)=c^2 \qquad (2)$$

and

$$4\left(zy_1^2+x_1^2\right)=b^2 \qquad (3)$$

Adding Eqs. (1), (2) and (3), we get

$$8\left(x_1^2+y_1^2+z_1^2\right)=a^2+b^2+c^2 \qquad (4)$$

Putting $y_1^2+z_1^2=\dfrac{a^2}{4}$ from Eq. (1) to Eq. (4), we get

$$8x_1^2+\frac{8a^2}{4}=a^2+b^2+c^2$$

or $$8x_1^2=b^2+c^2-a^2$$

Similarly,

$$8y_1^2=c^2+a^2-b^2$$

and $$8z_1^2=a^2+b^2-c^2$$

54. Given planes are

$$ax+by=0 \quad (1)$$

and $$z=0 \quad (2)$$

Equation of any plane passing through the line of intersection of planes (1) and (2) may be taken as

$$ax+by+kz=0 \quad (3)$$

The direction cosines of the normal to the plane (3) are

$$\frac{a}{\sqrt{a^2+b^2+k^2}},\ \frac{b}{\sqrt{a^2+b^2+k^2}},\ \frac{k}{\sqrt{a^2+b^2+k^2}}$$

The direction cosines of the normal to the plane given by Eq. (1) are

$$\frac{a}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}},0$$

Since the angle between planes given in Eqs. (1) and (2) is α. Therefore,

$$\cos\alpha=\frac{a.a+b.b+k.0}{\sqrt{a^2+b^2+k^2}\sqrt{a^2+b^2}}=\sqrt{\frac{a^2+b^2}{a^2+b^2+k^2}}$$

or $$(a^2+b^2+k^2)\cos^2\alpha=a^2+b^2$$

or $$k^2\cos^2\alpha=a^2(1-\cos^2\alpha)+b^2(1-\cos^2\alpha)$$

or $$k^2=\frac{(a^2+b^2)\sin^2\alpha}{\cos^2\alpha}$$

or $$k=\pm\sqrt{(a^2+b^2)}\tan\alpha$$

Putting this value of k in Eq. (3), we get the required equation of plane.

Practice Exercise 2

1. Set of homogeneous equation will have non-trivial solution if $\Delta=0$.

$$\begin{vmatrix}1 & -k & -1\\ k & -1 & -1\\ 1 & 1 & -1\end{vmatrix}=0$$

Apply c_1+c_3 to make two zeroes, we have

$$\begin{vmatrix}1 & -k & 0\\ k & -1 & k-1\\ 1 & 1 & 0\end{vmatrix}=0$$

$$\Rightarrow k^2-1=0$$
$$\Rightarrow k=\pm 1$$

2. Let line joining AB meet plane $2x+3y+5z=1$ at P. Let

$$P=\left(\frac{\lambda+1}{\lambda+1},\frac{-5\lambda}{\lambda+1},\frac{7\lambda-3}{\lambda+1}\right)\left[\frac{AP}{PB}=\lambda\right]$$

We have

$$2\cdot 1+3\left(\frac{-5\lambda}{\lambda+1}\right)+5\left(\frac{7\lambda-3}{\lambda+1}\right)=1$$

$$\Rightarrow 2(\lambda+1)-15\lambda+35\lambda-15=\lambda+1\Rightarrow\lambda=\frac{2}{3}$$
$$\Rightarrow P\equiv(1,-2,1)\Rightarrow AP=2\sqrt{5}$$

3. $\lambda p=\mu q=\gamma r$ and $\lambda p'=\mu\theta'=\gamma r'$

Therefore,

$$pp'+qq'+rr'=\frac{4}{3}(p^2+q^2+r^2)=\frac{4}{3}\cdot 3^2=3\cdot 4=12$$

4. Intersection of line when both the planes are same.

$$\frac{3}{3\beta^2+6(1-2\alpha)+3}=\frac{-6}{6\alpha^2+6(1-2\beta)+6}$$

$$\Rightarrow 2(\beta-1)^2+3(\alpha-2)^2=0\Rightarrow\alpha=2,\beta=1$$

5. See Fig. 27.43. If P be (x, y, z) then from the figure,

$$x=r\sin\theta\cos f, y=r\sin\theta\sin f, z=r\cos\theta$$
$$\Rightarrow 1=r\sin\theta\cos f, 2=r\sin\theta\sin f, 3=r\cos\theta$$
$$\Rightarrow 1^2+2^2+3^2=r^2\Rightarrow r=\pm\sqrt{14}$$

Therefore,

$$\sin\theta\cos f=\frac{1}{\sqrt{14}}\sin\theta\sin\phi=\frac{2}{\sqrt{14}},\cos\theta=\frac{3}{\sqrt{14}}$$

(neglecting negative sign as θ and ϕ are acute)

Therefore,

$$\frac{\sin\theta\sin\phi}{\sin\theta\cos\phi}=\frac{2}{1}\Rightarrow\tan\phi=2$$

Also,

$$\tan\theta=\frac{\sqrt{5}}{3}$$

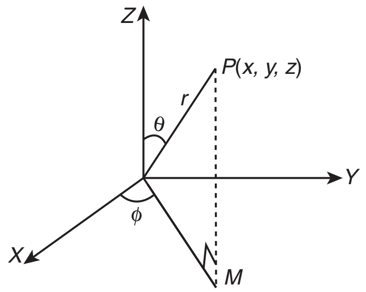


Figure 27.43

6. Let the equation of plane passing through origin is

$$lx+my+nz=0$$

where l, m, n are direction cosines, then

$$l^2+m^2+n^2=1 \quad (1)$$

The given line is parallel to the plane, then

$$2l-m-2n=0 \quad (2)$$

The distance between line and plane is 5/3, then

$$l-3m-n=5/3 \quad (3)$$

From Eqs. (1), (2) and (3), we get equation of plane are

$$x-2y+2z=0 \text{ and } 2x+2y-z=0$$

7. Clearly, vector equations of *CD* and *BE* are

$$\vec{r}=\hat{i}-2\hat{j}+4\hat{k}+\frac{\lambda}{3}(7\hat{j}-7\hat{k}) \quad (1)$$

$$\vec{r}=-\hat{i}+\hat{j}+\hat{k}+\frac{\mu}{3}(7\hat{i}-7\hat{j}+7\hat{k}) \quad (2)$$

For point of intersection equating $\vec{r}$ in Eqs. (1) and (2), we get

$$m=\frac{6}{7}, l=\frac{3}{7}$$

$$\Rightarrow PV \text{ of } \vec{P}=\hat{i}-\hat{j}+3\hat{k}$$

8. We have

$$\overrightarrow{AB}\times\overrightarrow{AC}=7\hat{j}+7\hat{k}$$

Since *PF* is parallel to $\overrightarrow{AB}\times\overrightarrow{AC}$ and $PF=\sqrt{2}$

Therefore,

$$\overrightarrow{PF}=\sqrt{2}\,\frac{7\hat{j}+7\hat{k}}{\sqrt{49+49}}=\hat{j}+\hat{k}$$

9.

$$\Delta=\text{Area of }\Delta ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$

$$=\frac{1}{2}\left|(-3\hat{i}+\hat{j}-\hat{k})\times(-\hat{i}-2\hat{j}+2\hat{k})\right|=\frac{7\sqrt{2}}{2}\text{ sq. units}$$

Therefore,

$$\text{Volume of tetrahedron}=\frac{1}{3}\times\text{area of base}\times\text{height}$$

$$=\frac{7}{3}\text{ cubic units}$$

10. In the new definition, $l=\dfrac{x}{|x|+|y|+|z|}$, etc.

Therefore, (D) is the correct answer.

11.

$$d(O,P)=k$$

$$\Rightarrow |x|+|y|+|z|=k$$

which represents a set of 9 planes making intercepts of lengths *k* on positive as well as negative sides of all three axes. See Fig. 27.44.

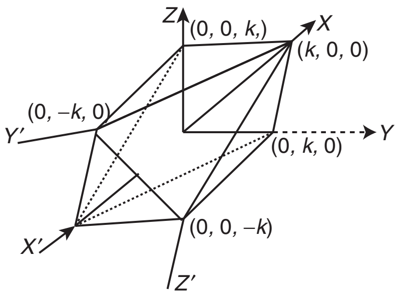


Figure 27.44

12.

$$d(O,P)=d(A,P)$$

$$\Rightarrow |x|+|y|+|z|=|x-1|+|y-2|+|z-3|$$

$$\Rightarrow x+y+z=|x-1|+|y-2|+|z-3|$$

Now the sign-scheme is

$x-1$	$y-2$	$z-3$	Resulting equation	Inference
+	+	+	$0=-6$	No solution
−	+	+	$x=-2$	No solution
+	−	+	$y=-1$	No solution
+	+	−	$z=0$	*xy*-plane
−	−	+	$x+y=0$	$x=0, y=0 \Rightarrow z$-axis
−	+	−	$x+z=1$	Plane || to *y*-axis
+	−	−	$y+z=2$	Plane || to *x*-axis
−	−	−	$x+y+z=3$	Part of an oblique plane

13. Let $Q(x_2, y_2, z_2)$ be image of $A(2, 1, 6)$ about mirror $x+y-2z=3$. Then

$$\frac{x_2-2}{1}=\frac{y_2-1}{1}=\frac{z_2-6}{-2}=\frac{-2(2+1-12-3)}{1^2+1^2+2^2}=4$$

$$\Rightarrow (x_2, y_2, z_2)\equiv(6, 5, -2)$$

14. Let

$$\frac{x-2}{3}=\frac{y-1}{4}=\frac{z-6}{5}=\lambda$$

Then

$$x=2+3\lambda, y=1+4\lambda, z=6+5\lambda$$

Lies on plane $x+y-2z=3$, so

$$2+3\lambda+1+4\lambda-2(6+5\lambda)=3$$

$$\Rightarrow 3+7\lambda-12-10\lambda=3$$

$$\Rightarrow -3\lambda=12 \Rightarrow \lambda=-4$$

Therefore, point $B\equiv(-10, -15, -14)$.

15. Equation of reflected ray $L_1=0$ is line joining $Q(x_2, y_2, z_2)$ and $B(-10, -15, -14)$. That is,

$$\frac{x+10}{16}=\frac{y+15}{20}=\frac{z+14}{12}$$

$$\Rightarrow \frac{x+10}{4}=\frac{y+15}{5}=\frac{z+14}{3}$$

16. Given $\dfrac{abc}{6}=32$, where *A*, *B*, *C* are, respectively, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

(A) Centroid of tetrahedron $(x, y\ z)\equiv\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)$

$$\Rightarrow a=4x, b=4y, c=4z$$

Therefore,

$$64xyz=32\times 6 \Rightarrow xyz=3$$

(B) Equidistant point $(x, y, z)\equiv\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

$$\Rightarrow a=2x, b=2y, c=2z$$

Therefore,

$$8abg=32\times 6 \Rightarrow xyz=24$$

(C) The equation of the plane is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$

Therefore, foot of perpendicular from the origin $\equiv(x, y, z)$

$$\equiv\left(\frac{1/a}{\sum\frac{1}{a^2}}, \frac{1/b}{\sum\frac{1}{a^2}}, \frac{1/c}{\sum\frac{1}{a^2}}\right)$$

$$\Rightarrow \frac{1}{ax} = \frac{1}{by} = \frac{1}{cz} = t, \quad \text{where } t = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \sum \frac{1}{a^2}$$

or $$t = (x^2 + y^2 + z^2)t^2 \Rightarrow t = \frac{1}{x^2 + y^2 + y^2} \quad \text{and}$$

$$a = \frac{x^2 + y^2 + z^2}{x}, \; b = \frac{x^2 + y^2 + z^2}{y}, \; c = \frac{x^2 + y^2 + z^2}{z}$$

Now,

$$abc = 6 \times 32 \Rightarrow (x^2 + y^2 + z^2)^3 = 192\, xyz$$

(D) Let P be (x, y, z). Then $PA \perp PB$.

$$x(x - a) + y(y - b) + zz = 0$$
$$\Rightarrow xx + yy = x^2 + y^2 + z^2$$

Now, $PB \perp PC$

$$xx + y(y - b) + z(z - c) = 0$$
$$\Rightarrow by + cz = x^2 + y^2 + z^2$$

Therefore,

$$\frac{a}{1/x} = \frac{b}{1/y} = \frac{c}{1/z}$$

$$\Rightarrow a = \frac{x^2 + y^2 + z^2}{2x}, \; b = \frac{x^2 + y^2 + z^2}{2y}, \; c = \frac{x^2 + y^2 + z^2}{2z}$$

Therefore,

$$xyz = 6 \times 32$$
$$\Rightarrow (x^2 + y^2 + z^2)^3 = 192 \times 8xyz = 1536xyz$$

Hence, (A)–(r), (B)–(p), (C)–(q), (D)–(s)

17. (A) We have

$$3 \cdot 1 - 2(-2) + 5(l) = 0$$
$$\Rightarrow l = -\frac{7}{5}$$

(B) Point $(3, l, m)$ lies on $2x + y + z - 3 = x - 2y + z - 1$

$$3 \cdot 2 + l + m - 3 = 0 \text{ and } 3 - 2l + m - 1 = 0$$
$$\Rightarrow l + m + 3 = 0 \text{ and } 2l - m - 2 = 0$$

So, $l + m = -3$.

(C) $$\sin\theta = \frac{1 \cdot 4 + 1(-3) + 1 \cdot 5}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{16 + 9 + 25}} = \frac{6}{\sqrt{3}\sqrt{50}}$$

$$\Rightarrow \theta = \sin^{-1}\sqrt{\frac{6}{25}}$$

(D) $$\cos\theta = \frac{1 \cdot 3 + 1(-4) + 1 \cdot 5}{\sqrt{3}\sqrt{16 + 9 + 25}} = \frac{4}{\sqrt{3}\sqrt{50}}$$

$$\Rightarrow \theta = \cos^{-1}\sqrt{\frac{8}{75}}$$

Hence, (A)–(q), (B)–(r), (C)–(p), (D)–(s)

Solved JEE 2017 Questions

JEE Main 2017

1. The distance of the point (1, 3, –7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines $\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$, is:

(A) $\frac{10}{\sqrt{83}}$ **(B)** $\frac{5}{\sqrt{83}}$

(C) $\frac{10}{\sqrt{74}}$ **(D)** $\frac{20}{\sqrt{74}}$

(OFFLINE)

Solution: We have

$$\hat{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 3\\ 2 & -1 & -1\end{vmatrix}=5\hat{i}-\hat{j}(-7)+3\hat{k}$$

Equation of the given plane is

$$5(x-1)+7(y-1)+3(z+1)=0$$
$$5x+7y+3z+5=0$$

Therefore, the perpendicular distance of plane from the point (1, 3, –7) is

$$\frac{|5+21-21+5|}{\sqrt{25+49+9}}=\frac{10}{\sqrt{83}}$$

Hence, the correct answer is option (A).

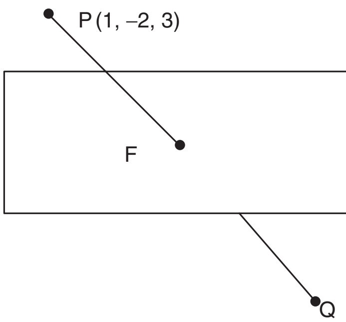
2. If the image of the point P(1, –2, 3) in the plane, $2x+3y-4z+22=0$ measured parallel to the line, $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ is Q, then PQ is equal to:

(A) $2\sqrt{42}$ **(B)** $\sqrt{42}$

(C) $6\sqrt{5}$ **(D)** $3\sqrt{5}$

(OFFLINE)

Solution: The given situation is depicted in the following figure:



Line PQ: $\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-3}{5}$

Let us consider $(\lambda+1, 4\lambda-2, 5\lambda+3)$ and point F lies on the plane. Therefore,

$$2(\lambda+1)+3(4\lambda-2)-4(5\lambda+3)+22=0$$
$$-6\lambda+6=0 \Rightarrow \lambda=1$$

That is, point F is obtained as F(2, 2, 8).

Therefore, $PQ=2PF=2\sqrt{42}$.

Hence, the correct answer is option (A).

3. The line of intersection of the planes, $\vec{r}\cdot(3\hat{i}-\hat{j}+\hat{k})=1$ and $\vec{r}\cdot(\hat{i}+4\hat{j}-2\hat{k})=2$, is

(A) $\frac{x-\frac{4}{7}}{2}=\frac{y}{7}=\frac{z-\frac{5}{7}}{13}$ **(B)** $\frac{x-\frac{6}{13}}{2}=\frac{y-\frac{5}{13}}{7}=\frac{z}{-13}$

(C) $\frac{x-\frac{4}{7}}{2}=\frac{y}{-7}=\frac{z+\frac{5}{7}}{13}$ **(D)** $\frac{x-\frac{6}{13}}{2}=\frac{y-\frac{5}{13}}{-7}=\frac{z}{-13}$

(ONLINE)

Solution: It is given that

$$\vec{r}\cdot(3\hat{i}-\hat{j}+\hat{k})=1 \quad (1)$$
$$\vec{r}\cdot(\hat{i}+4\hat{j}-2\hat{k})=2 \quad (2)$$

When $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, from Eq. (1), we get

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j}+\hat{k})=1\Rightarrow 3x-y+z=1 \quad (3)$$

and from Eq. (2), we get

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+4\hat{j}-2\hat{k})=2\Rightarrow x+4y-2z=2 \quad (4)$$

Solving Eqs. (3) and (4): Eq. (3) – {3[Eq. (4)]} gives

$$\begin{array}{l} 3x-y\quad+z=1\\ 3x+12y-6z=6\\ -\quad-\quad\quad+\\ \hline -13y+7z=-5\Rightarrow y=\frac{-5-7z}{-13}\end{array}$$

$$y=\frac{5+7z}{13} \quad (5)$$

Substituting $z=t$ and $y=\frac{5+7t}{13}$ in Eq. (4), we get

$$x+4\left(\frac{5+7t}{13}\right)-2t=2$$
$$\Rightarrow x+\frac{20+28t}{13}-2t=2$$
$$\Rightarrow x=2+2t-\left(\frac{20+28t}{13}\right)$$

$$\Rightarrow x = \frac{26+26t-20-28t}{13}$$

$$\Rightarrow x = \frac{6-2t}{13} \quad (6)$$

That is, $x = \frac{6-2t}{13}$; $y = \frac{5+7t}{13}$; $z = t$. Therefore,

$$t = \frac{13x-6}{-2};\ t = \frac{13y-5}{7};\ t = z$$

Thus, the equation of line that intersects the planes is obtained as

$$\frac{13x-6}{-2} = \frac{13y-5}{7} = z$$

$$\Rightarrow \frac{x-\frac{6}{13}}{-2} = \frac{y-\frac{5}{13}}{-7} = \frac{z}{-13}$$

Hence, the correct answer is option (D).

4. The coordinates of the foot of the perpendicular from the point (1, –2, 1) on the plane containing the lines, $\frac{x+1}{6} = \frac{y-1}{7} = \frac{z-3}{8}$ and $\frac{x-1}{3} = \frac{y-2}{5} = \frac{z-3}{7}$, is

(A) (0, 0, 0) **(B)** (2, –4, 2)
(C) (–1, 2, –1) **(D)** (1, 1, 1)

(ONLINE)

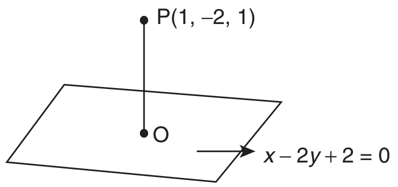
Solution: The equation of plane containing the two given lines is written as

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \quad (1)$$

From the given data, we have $l_1 = 6, m_1 = 7, n_1 = 8; l_2 = 3, m_2 = 5, n_2 = 7$. Substituting the values in Eq. (1), we get

$$\begin{vmatrix} x+1 & y-1 & z-3 \\ 6 & 7 & 8 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

$$(x+1)(49-40) - (y-1)(42-24) + (z-3)(30-21) = 0$$
$$9(x+1) - 18(y-1) + 9(z-3) = 0$$
$$9x + 9 - 18y + 18 + 9z - 27 = 0$$
$$9x - 18y + 9z = 0 \quad \text{or} \quad x - 2y + z = 0 \quad (2)$$



Let the coordinates of the 'foot of perpendicular' be O(x, y, z), then the direction ratio of OP are

$$(x_1 - 1, y_1 + 2, z_1 = 1) \quad (3)$$

Writing equation of plane in normal form is

$$\frac{x}{\sqrt{6}} - \frac{2}{\sqrt{6}}y + \frac{2}{\sqrt{6}} = 0$$

The direction cosines of given plane are

$$\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \quad (4)$$

Since the direction cosines and the direction ratios of a line are proportional to each other, we have

$$\frac{x_1-1}{1/\sqrt{6}} = \frac{y_1+2}{-2/\sqrt{6}} = \frac{z_1-1}{1/\sqrt{6}} = k$$

$$x_1 = \frac{1}{\sqrt{6}}k+1,\ y_1 = \frac{-2k}{\sqrt{6}}-2,\ z_1 = \frac{1}{\sqrt{6}}k+1 \quad (5)$$

Substituting the value of x_1, y_1, z_1 in the equation of plane [i.e. Eq. (2)], we get

$$\left(\frac{1}{\sqrt{6}}k+1\right) - 2\left(\frac{-2}{\sqrt{6}}k-2\right) + \left(\frac{1}{\sqrt{6}}k+1\right) = 0$$

$$\Rightarrow \frac{1}{\sqrt{6}}k+1+\frac{4}{\sqrt{6}}k+4+\frac{1}{\sqrt{6}}k+1 = 0$$

$$\Rightarrow \frac{6k}{\sqrt{6}}+6 = 0$$

$$\Rightarrow k = -\sqrt{6}$$

Substituting the value of k in Eq. (5), we get the following coordinates:

$$x_1 = \frac{1}{\sqrt{6}}(-\sqrt{6})+1 = -1+1 = 0$$

$$y_1 = \frac{-2}{\sqrt{6}}(-\sqrt{6})-2 = 2-2 = 0$$

$$z_1 = \frac{1}{\sqrt{6}}(-\sqrt{6})+1 = -1+1 = 0$$

Therefore, the coordinates of the foot of perpendicular is (0, 0, 0).
Hence, the correct answer is option (A).

5. If a variable plane, at a distance of 3 units from the origin, intersects the coordinate axes at A, B and C, then the locus of the centroid of ΔABC is

(A) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = 1$ **(B)** $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = 3$

(C) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = 9$ **(D)** $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = \frac{1}{9}$

(ONLINE)

Solution: Let the plane equation be $ax + by + cz + d = 0$. The distance of the plane from the origin is 3. Therefore,

$$\frac{d}{\sqrt{a^2+b^2+c^2}} = 3$$

That is,

$$d^2 = 9(a^2+b^2+c^2) \quad (1)$$

Now, the plane intersects x-axis at point $A\left(\frac{-d}{a}, 0, 0\right)$; y-axis at point $B\left(0, \frac{-d}{a}, 0\right)$ and z-axis at point $C\left(0, 0, \frac{-d}{c}\right)$.

Let $h = \frac{-d}{3a}$, $k = \frac{-d}{3b}$, $l = \frac{-d}{3c}$. Therefore, from Eq. (1), we get

$$\frac{1}{9h^2} + \frac{1}{9k^2} + \frac{1}{9l^2} = \frac{1}{9} = \frac{1}{h^2} + \frac{1}{k^2} + \frac{1}{l^2} = 1$$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 1$$

Hence, the correct answer is option (A).

6. If $x = a, y = b, z = c$ is a solution of the system of linear equations

$$x + 8y + 7z = 0$$
$$9x + 2y + 3z = 0$$
$$x + y + z = 0$$

such that the point (a, b, c) lies on the plane $x + 2y + z = 6$, then $2a + b + c$ equals

(A) 1 **(B)** 2
(C) –1 **(D)** 0

(ONLINE)

Solution: The given system of equations is

$$x + 8y + 7z = 0 \quad (1)$$
$$9x + 2y + 3z = 0 \quad (2)$$
$$x + y + z = 0 \quad (3)$$

Multiplying Eq. (3) by 3 and subtracting from Eq. (2), we get the following:

$$\begin{array}{r} 9x + 2y + 3z = 0 \\ 3x + 3y + 3z = 0 \\ - \quad - \quad - \quad \\ \hline 6x - y = 0 \\ \hline \end{array}$$

That is,

$$y = 6x \quad (4)$$

Substituting the value of y in in Eq (1), we get

$$x + 8(6x) + 7z = 0$$
$$\Rightarrow 49x + 7z = 0$$
$$\Rightarrow 7x + z = 0$$
$$\Rightarrow z = -7x \quad (5)$$

It is given that $x = a, y = b, z = c$ is the solution of the given system of equations; therefore,

$$b = 6a \text{ and } c = -7a$$

Also (a, b, c) lies on the plane $x + 2y + z = 6$. Therefore,

$$a + 2b + c = 6 \quad (6)$$

Substituting the values of b and c in Eq. (6), we get

$$a + 2(6a) - 7a = 6$$
$$\Rightarrow a + 12a - 7a = 6$$
$$\Rightarrow 6a = 6 \Rightarrow a = 1$$

Hence, we get $b = 6$ and $c = -7$. Hence, the value of $2a + b + c$ becomes 1 as follows:

$$2a + b + c = 2(1) + 6 - 7 = 1$$

Hence, the correct answer is option (A).

7. If the line $\frac{x-3}{1} = \frac{y+2}{-1} = \frac{z+\lambda}{-2}$ lies in the plane, $2x - 4y + 3z = 2$, then the shortest distance between this line and the line $\frac{x-1}{12} = \frac{y}{9} = \frac{z}{4}$ is

(A) 0 **(B)** 3
(C) 1 **(D)** 2

(ONLINE)

Solution: The given line is

$$\frac{x-3}{1} = \frac{y+2}{-1} = \frac{z+\lambda}{-2} = k$$

Let P be any point on the line, which is given by

$$P = (k + 3, -k -2, -2k - \lambda)$$

lies on the given plane $2x - 4y + 3z = 2$. Therefore,

$$2(k + 3) - 4(-k - 2) + 3(-2k - \lambda) = 2$$
$$\Rightarrow 2k + 6 + 4k + 8 - 6k - 3\lambda = 2$$
$$\Rightarrow 14 - 3\lambda = 2$$
$$\Rightarrow 3\lambda = 12 \Rightarrow \lambda = 4$$

Therefore, using the value of λ, the given Line 1 becomes

$$\frac{x-3}{1} = \frac{y+2}{-1} = \frac{z+4}{-2}$$

Here, the shortest distance is expressed as

$$d^2 = \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

where l, m, n are DRs of the line. Substituting the values in the above equation, we get

$$\begin{vmatrix} 3-1 & -2-0 & -4-0 \\ 1 & -1 & -2 \\ 12 & 9 & 4 \end{vmatrix}$$

$$\Rightarrow 2(14) + 2(28) - 4(21) = |28 + 56 - 84| = 0$$

Therefore, the shortest distance between the two given lines is $d^2 = 0 \Rightarrow d = 0$

Hence, the correct answer is option (A).

JEE Advanced 2017

1. The equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$, is

(A) $14x + 2y - 15z = 1$ **(B)** $14x - 2y + 15z = 27$
(C) $14x + 2y + 15z = 31$ **(D)** $-14x + 2y + 15z = 3$

Solution: Let the equation of the plane passing through point (1, 1, 1) be

$$a(x - 1) + b(y - 1) + c(z - 1) = 0$$

The normal to the plane is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = \hat{i}(-2-12) - \hat{j}(-4+6) + \hat{k}(-12-3) = -14\hat{i} - 2\hat{j} - 15\hat{k}$$

Thus, the equation of plane is

$$-14(x - 1) - 2(y - 1) - 15(z - 1) = 0$$

$$-14x + 14 - 2y + 2 - 15z + 15 = 0$$

$$-14x - 2y - 15z + 31 = 0 \Rightarrow 14x + 2y + 15z = 31$$

Hence, the correct answer is option (C).

28 Probability

28.1 Introduction

Often experiments are performed in order to produce observations or measurements that assist us in arriving at conclusions. These recorded information in its original collected form are referred to as 'raw data'. Mathematicians define experiment as any process or operation that generates raw data. If a chemist runs an analysis several times under the same experimental conditions, he will not get concurrent result, which indicates an element of chance in the experimental procedure. It is these chance outcomes that occur around us with which this chapter is basically concerned.

28.2 Concept of Probability in Set Theoretic Language

28.2.1 Random Experiment

An experiment, whose all possible outcomes are known in advance but the outcome of any specific performance cannot be predicted before the completion of the experiment, is known as random experiment.

For example, some of the random experiments are:

1. Toss of a coin, which can result in either a head or a tail.
2. Throw of a dice which can result in any one of the six faces.
3. Drawing a card from a pack of 52 cards which can result in any one of the 52 cards.

28.2.2 Sample Space and Sample Points

A set whose elements represent all possible outcomes of a random experiment is called the *sample space* and is usually represented by 'S'.

An element of a sample space is called a *sample point*.

Consider the experiment of tossing a dice. If we are interested in the number that shows on the top face, then sample space would be $S_1 = \{1, 2, 3, 4, 5, 6\}$.

If we are interested only in whether the number is even or odd, then sample space is simply $S_2 = \{\text{even, odd}\}$.

Clearly, more than one sample space can be used to describe the outcomes of an experiment. In this case 'S_1' provides more information than 'S_2'. If we know which element in S_1 occurs, we can tell which outcome in S_2 occurs; however, a knowledge of what happens in S_2 in no way helps us to know which element in S_1 occurs.

In general, it is desirable to use a sample space that gives the maximum information concerning the outcomes of the experiment.

Suppose three items are selected at random from a manufacturing process. Each item is inspected and classified as defective or non-defective. The sample providing the maximum information would be S_1 = {NNN, NDN, DNN, NND, DDN, DND, NDD, DDD}.

A second sample space, although it provides, less information, might be $S_2 = \{0, 1, 2, 3\}$, where the elements represent no defectives, one defective, two defectives, or three defectives in our random selection of three items. For example:

1. In toss of a coin, $S = \{H, T\}$ where H and T are sample points representing a head and a tail, respectively.
2. In a throw of dice, $S = \{1, 2, 3, 4, 5, 6\}$ where the numbers are the sample points representing the six faces.

28.2.3 Trial

When an experiment is repeated under similar conditions and it does not give the same result each time but may result in any one of the several possible outcomes, the experiment is called a *trial* and the outcomes are called *cases*. The number of times the experiment is repeated is called the *number of trials*. For example:

1. One toss of coin is a trial when coin is tossed 5 times.
2. One throw of a dice is a trial when the dice is thrown 4 times.

28.2.4 Event

A subset of sample space, that is, a set of some of the possible outcomes of a random experiment is called event.

1. **Simple event:** Each sample point in the sample space is called an elementary event or simple event. For example, occurrence of head in throw of a coin is simple event.
2. **Sure event:** The set containing all sample points is a sure event as in the throw of a dice the occurrence of natural number less than 7, is a sure event.
3. **Null event:** The set which does not contain any sample point.
4. **Mixed/Compound event:** A subset of sample space S containing more than one element is called a mixed event or a compound event.
5. **Compliment of an event:** Let S be the sample space and E be an event, then E^c or $\overline{E}$ represents complement of event E which is a subset containing all sample points in S which are not in E. It refers to the non–occurrence of event E.

28.2.5 Algebra of Events

In connection with basic probability laws, we shall need the following concepts and facts about events (subsets) A, B, C, $\cdots$ of a given sample space S.

The union $A \cup B$ of A and B consists of all points in A or B or both.

The intersection $A \cap B$ of A and B consists of all points that are in both A and B.

If A and B have no points in common, we write

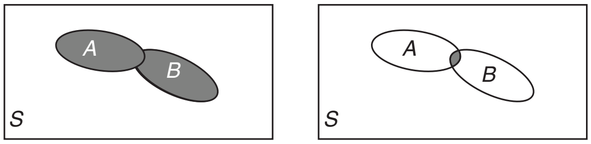
$$A \cap B = \phi$$

where ϕ is the empty set (set with no elements) and we call A and B mutually exclusive (or disjoint) because the occurrence of A excludes that of B (and conversely) if your dice turns up an odd number, it cannot turn up an even number in the same trial. Similarly, a coin cannot turn up Head and Tail at the same time.

The complement A^c of A consists of all the points of S not in A. Thus,

$$A \cap A^c = \phi, \quad A \cup A^c = S$$

Working with events can be illustrated and facilitated by Venn diagrams for showing union, intersections and complements, as shown in Fig. 28.1



Union $A \cup B$ Intersection $A \cap B$

Figure 28.1 Venn diagrams showing two events A and B in a sample space S and their union $A \cup B$ (coloured) and intersection $A \cap B$ (coloured).

Union and intersections of more events are defined similarly. The union

$$\bigcup_{j=1}^{m} A_j = A_1 \cup A_2 \cup \cdots \cup A_m$$

of events $A_1, \cdots, A_m$ consists of all points that are in at least one A_j. Similarly for the union $A_1 \cup A_2 \cup \cdots$ of infinitely many subsets A_1, A_2, $\cdots$ of an infinite sample space S (that is, S consists of infinitely many points). The intersection

$$\bigcap_{j=1}^{m} A_j = A_1 \cap A_2 \cap \cdots \cap A_m$$

of $A_1, \cdots, A_m$ consists of the points of S that are in each of these events. Similarly for the intersection $A_1 \cap A_2 \cap \cdots$ of infinitely many subsets of S.

28.2.6 Equally Likely Events

The events are said to be equally likely if none of them is expected to occur in preference to the other one. For example

In throw of a fair coin occurrence of a head or a tail have equal chances. Hence, event that a head appears and event that a tail appears are equally likely events.

28.2.7 Mutually Exclusive Events

A set of events is said to be mutually exclusive if the occurrence of one of them precludes the occurrence of any of the other events. For instance, when a pair of dice is tossed, the events 'a sum of 4 occurs', 'a sum of 10 occurs' and 'a sum of 12 occurs' are mutually exclusive. Simply speaking, if two events are mutually exclusive they cannot occur simultaneously. Using set theoretic notation, if A_1, $A_2, \cdots, An$ be the set of mutually exclusive events, then $A_i \cap A_j = \phi$ for $i \neq j$ and $1 \le i, j \le n$. For example:

1. In throw of a dice, the event of occurrence of an even number and the event of occurrence of an odd number are mutually exclusive.
2. In throw of a fair coin, occurrence of a head or a tail are mutually exclusive.

> A set of events is said to be mutually exclusive if the occurrence of one of them precludes the occurrence of any of the other events. Using set theoretic notation, if $A_1, A_2, \cdots, A_n$ be the set of mutually events then $A_i \cap A_j = \phi$ for $i \neq j$ and $1 \le i, j \le n$. Also, $P(A_i \cup A_j) = P(A_i) + P(A_j)$.

28.2.8 Exhaustive Events

A set of events is said to be exhaustive if the performance of random experiment always result in the occurrence of at least one of them. For instance, consider an ordinary pack of cards then the events 'drawn card is heart', 'drawn card is diamond', 'drawn card is club' and 'drawn card is spade' form a set of exhaustive events. In other words, all sample points put together (i.e., sample space itself) would give us an exhaustive event.

If $E_1, E_2, \cdots, En$ form a set of exhaustive events, then

$$P(E_1) + P(E_2) + \cdots + P(E_n) = 1$$

For example:

1. In a throw of dice, the event of occurrence of an even number and the event of occurrence of an odd number are exhaustive.
2. In a throw of a fair coin, occurrence of a head or a tail is exhaustive.

Exhaustive events cover the whole of the sample space. Their union is equal to S.

> A set of events is said to be exhaustive if the performance of random experiment always result in the occurrence of at least one of them. In other words, all sample points put together (i.e., sample space itself) would give us an exhaustive event. If 'E' be an exhaustive event then $P(E) = 1$.

> **Set Theoretic Principles**
>
> Let us introduce a few notations, which would be frequently used:
>
> If 'A' and 'B' be any two events of the sample space, then
>
> 1. $A \cup B$ would stand for occurrence of at least one of them.
> 2. $A \cap B$ stands for simultaneous occurrence of A and B.
> 3. $\overline{A}$ (or A') stands for non–occurrence of A.
> 4. $\overline{A} \cap \overline{B}$ (or $A' \cap B'$) stands for non–occurrence of both A and B.
> 5. $A \subseteq B$ stands for 'the occurrence of A implies the occurrence of B'.
> 6. If A and B are any two events, then
>
> $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

continued

continued

7. If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$.
8. $P(A') = 1 - P(A)$
9. $P(A \cap B') = P(A) - P(A \cap B)$
10. $P(A' \cup B') = 1 - P(A \cap B)$
11. P(exactly one of A, B occurs) $= P(A) + P(B) - 2P(A \cap B)$
12. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
13. P (Exactly one of A, B, C occurs) $= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(A \cap C) - 2P(B \cap C) + 3P(A \cap B \cap C)$
14. P(Exactly two of A, B, C occur) $= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$
15. P (at least two of A, B, C occur) $= P(A \cap B) + P(B \cap C) + P(A \cap C) - 2P(A \cap B \cap C)$
16. If $A_1, A_2 \cdots, A_n$ are 'n' events, then $P(A_1 \cup A_2 \cdots \cup A_n)$
$$= \sum_{r=1}^{n} P(A_r) - \sum_{1 \le r_1 < r_2 \le n} (P(A_{r_1} \cup A_{r_2}) + \sum_{1 \le r_1 < r_2 < r_3 \le n} P(A_{r_1} \cap A_{r_2} \cap A_{r_3}) - \cdots (-1)^{n-1} P(A_1 \cap A_2 \cdots \cap A_n)$$
17. $P(A \cup B) \le \max [P(A), P(B), P(A) + P(B) - 1]$
18. If out of $m + n$ equally likely, mutually exclusive and exhaustive cases, m cases are favourable to an A event and n are not favourable to the an event A, $m : n$ is called odds in favour of A, $n : m$ is called odds against the event A.

28.3 Definition of Probability with Discrete Sample Space

If the sample space S of an experiment consists of finitely many outcomes (points) that are equally likely, then the probability of occurrence of an event A is

$$P(A) = \frac{\text{Number of sample points in } A}{\text{Number of sample points in } S}$$

$$P(A) = \frac{n(A)}{n(S)}$$

In particular $P(S) = 1$ and $0 \le P(A) \le 1$.

Illustration 28.1 Ten items out of a set of 100 are defective. What is the probability that three out of any four chosen are defective?

Solution: Probability is

$$= \frac{{}^{90}C_1 \cdot {}^{10}C_3}{{}^{100}C_4} = \frac{144}{52283}$$

Illustration 28.2 Seven persons are to be seated on one side of a straight table. What is the probability that two particular persons will be seated next to each other?

Solution: Total number of ways of seven persons being seated is

$${}^7P_7 = 7! \text{ ways}$$

If two are to be seated next to each other, treat them as one unit and this one unit with the remaining five can be seated in 6! ways and in each one of these 6! ways, the two persons can be interchanged in 2 ways. Therefore,

$$\text{Probability} = \frac{2 \cdot 6!}{7!} = \frac{2}{7}$$

28.4 Axiomatic Definition

Given a sample space S, with each event A of S, there is associated a number $P(A)$, called the probability of A, such that the following axioms of probability are satisfied.

1. For every A in S, $0 \le P(A) \le 1$
2. The entire sample space has the probability $P(S) = 1$
3. For mutually exclusive events A and B ($A \cap B = \phi$), $P(A \cup B) = P(A) + P(B)$.

28.5 Basic Theories

1. For an event A and its compliment A^c in sample space S

$$P(A^c) = 1 - P(A) \text{ as } A \cap A^c = \phi \text{ and } A \cup A^c = S$$

and
$$P(A \cup A^c) = P(A) + P(A^c)$$
$$\Rightarrow P(S) = P(A) + P(A^c)$$
$$\Rightarrow 1 = P(A) + P(A^c)$$

Illustration 28.3 A cricket club has 15 members, among whom only 5 can bowl. What is the probability of forming a team of 11 to consist of at least 3 bowlers?

Solution: Total number of ways of forming the team is

$${}^{15}C_{11} = {}^{15}C_4$$

Of these, number of ways of formation of the team

$$\text{With one bowler } = {}^5C_1 \cdot {}^{10}C_{10} = 5$$

$$\text{With two bowlers } = {}^5C_2 \cdot {}^{10}C_9 = 100$$

Probability that at least three bowlers are in the team is

$$1 - \frac{105}{{}^{15}C_4} = \frac{12}{13}$$

Illustration 28.4 Five coins are tossed simultaneously. Find the probability of the event that at least one head turns up. (Assume that wins are fair).

Solution: Let A be the event that 'at least one head turns up' since each coin turns up on either a head or a tail hence, the sample space consists of $2^5 = 32$ outcomes. Each outcome having a probability of occurrence as $\frac{1}{32}$. Then A^c is the event that 'No head turns up'. Thus, A^c consists of only one outcome. Hence,

$$P(A^c) = \frac{1}{32}$$

$$P(A) = 1 - \frac{1}{32} = \frac{31}{32}$$

2. **Addition rule of probability:** For events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In general for n events $A_1, A_2, \cdots, A_n$ of sample space S

$$P(A_1 \cup A_2 \cup \cdots A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j < n} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k} P(A_i \cap A_j \cap A_k) \cdots + (-1)^{n-1} P(A_1 \cap \cdots \cap A_n)$$

Illustration 28.5 In tossing a fair dice, what is the probability of getting an odd number or a number less than 4?

Solution:

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A be the event that odd number occurs. Then

$$P(A) = \frac{3}{6} = \frac{1}{2} \text{ (as } A = \{1, 3, 5\})$$

Let B be the event that a number less than 4 occurs. Then

$$B = \{1, 2, 3\} \text{ and } P(B) = \frac{3}{6} = \frac{1}{2}$$

Now

$$A \cap B = \{1, 3\} \text{ (odd number less than 4)}$$

$$\Rightarrow P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$$

3. For exhaustive events $A_1, A_2, \cdots, A_n$ in sample spaces.

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = 1$$

4. For mutually exclusive events $A_1 A_2 \ldots \cdots .. A_n$ in a sample space S

$$P(A_1 \cup A_2 \cup A_3 \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Illustration 28.6 If the probability that on any workday, a garage will get 10–20, 21–30, 31–40 over 40 cars to service is 0.20, 0.35, 0.25, 0.12, respectively. What is the probability that on a given workday the garage gets at least 21 cars to service?

Solution: Since these are mutually exclusive events. Hence, required probability is

$$0.35 + 0.25 + 0.12 = 0.72$$

5. For mutually exclusive and exhaustive events $A_1, A_2, \cdots, A_n$ in a sample space.

$$P(A_1 \cup A_2 \cup A_3 \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = 1$$

6. For an event A in sample space S,

$$\textbf{'The odds in favour of } \boldsymbol{A}\textbf{'} \text{ are } \frac{P(A)}{P(A^c)}$$

where A^c is the complement of the event A is S. Also

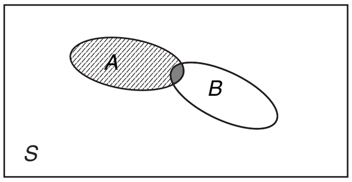
$$\textbf{'The odds against } \boldsymbol{A}\textbf{'} \text{ are } \frac{P(A^c)}{P(A)}$$

For example, in a throw of dice the odds in favour of 'a multiple of 3 occurs' is 2:4, that is, 1:2.

28.6 Conditional Probability

Often it is required to find the probability of an event B under the condition that an event A occurs. This probability is called the conditional probability of B given A and is denoted by $P(B/A)$. In this case, A serves as a new (reduced) sample space, and the probability is the fraction of that part of set A which corresponds to $A \cap$ B (see Fig. 28.2). Thus,

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{where } P(A) \neq 0$$



The shaded portion shows the favourable region and lined portion shows the reduced sample space.

Figure 28.2

Similarly, the conditional probability of A given B is

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

From the above two expressions, we can state the probability of intersection of two events A and B where $P(A) \neq 0$ and $P(B) \neq 0$ as

$$P(A \cap B) = P(A) \cdot P\left(\frac{A}{B}\right) \text{ or } P(B)P\left(\frac{A}{B}\right)$$

(Multiplication theorem)

> The probability of occurrence of an event B when it is known that some event A has occurred is called a condition probability and is denoted by $P(B/A)$.
>
> $$P(B/A) = \frac{P(A \cap B)}{P(A)},\ 0 < P(A) \leq 1,\ P(A \cap B) = \begin{cases} P(A) \cdot P(B/A), & P(A) > 0 \\ P(B) \cdot P(A/B), & P(B) > 0 \end{cases}$$

Illustration 28.7 In producing screws, let A mean 'screw too slim' and B 'screw too short'.

Let $P(A) = 0.1$ and let the conditional probability that a slim screw is also too short be $P(B/A) = 0.2$. Then what is the probability that a screw that we pick randomly from the lot produced will be both too slim and too short?

Solution: We require the probability of occurrence of both the events together, which can be given as

$$P(A \cap B) = P(A)\, P(B/A) = 0.1 \,.\, 0.2 = 0.02 = 2\%$$

28.7 Independent Events

If events A and B are such that

$$P(A \cap B) = P(A)\, P(B)$$

they are called independent events. Assuming $P(A) \neq 0$, $P(B) \neq 0$, in this case

$$P(A/B) = P(A), \qquad P(B/A) = P(B)$$

This means that the probability of A does not depend on the occurrence or non–occurrence of B, and conversely. This justifies the term independent.

Similarly, m events $A_1, \cdots, A_m$ are called independent if

$$P(A_1 \cap \cdots \cap A_m) = P(A_1) \cdots P(A_m)$$

as well as for every k different events $A_{j_1}, A_{j_2}, \cdots, A_{j_k}$

$$P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_k}) = P(A_{j_1})\, P(A_{j_2}) \cdots P(A_{j_k})$$

where, $k = 2, 3, \cdots, m - 1$.

Accordingly, three events A, B, C are independent if

$$P(A \cap B) = P(A)\, P(B)$$
$$P(B \cap C) = P(B)\, P(C)$$
$$P(C \cap A) = P(C)\, P(A)$$
$$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

Remarks:

1. If $P(A) = 0$, then for event 'B', $0 \leq P(A \cap B) \leq P(A)$, that is, $P(A \cap B) = 0$. Thus,

$$P(A \cap B) = P(A) \cdot P(B) = 0$$

Hence, an impossible event would be independent of any other event.

2. Distinction between independent and mutually exclusive events must be carefully made. If A and B are two mutually exclusive and possible events of sample space 'S' then $P(A) > 0$, $P(B) > 0$ and $P(A \cap B) = 0 \neq P(A) \cdot P(B)$ so that A and

B cannot be independent. In fact, $P(A/B) = 0$, similarly $P(B/A) = 0$. Consequently, mutually exclusive events are strongly dependent.

3. Two events A and B are independent if and only if A and B' are independent or A' and B are independent or A' and B' are independent.

We have

$$P(A \cap B) = P(A) \cdot P(B)$$

Now

$$P(A \cap B') = P(A) - P(A \cap B) = P(A) - P(A) \cdot P(B)$$
$$= P(A)\,[1 - P(B)] = P(A) \cdot P(B')$$

Thus, A and B' are independent.

Similarly,

$$P(A' \cap B) = P(B) - P(A \cap B)$$
$$= P(B) \cdot P(A')$$

Finally,

$$P(A' \cap B') = 1 - P(A \cup B)$$
$$= 1 - P(A) - P(B) + P(A \cap B)$$
$$= 1 - P(A) - P(B) + P(A) \cdot P(B)$$
$$= [1 - P(A)] - P(B)\,[1 - P(A)]$$
$$= [1 - P(A)]\,[1 - P(B)]$$
$$= P(A') \cdot P(B')$$

Thus, A' and B' are also independent.

> Two events A and B are said to be independent if occurrence or non–occurrence of one does not affect the occurrence or non–occurrence of the other, that is,
>
> $P(B/A) = P(B)$, $P(A) \neq 0$ similarly, $P(A/B) = P(A)$, $P(B) \neq 0$
>
> $$\Rightarrow P(B/A) = \frac{P(A \cap B)}{P(A)} = P(B) \quad \Rightarrow P(A \cap B) = P(A) \times P(B)$$
>
> If the events are not independent, they are said to be dependent.

Illustration 28.8 A and B are two independent witnesses in a case. The probability that A will speak the truth is 3/4; and that of B is 4/5. In what percent of cases are they likely to contradict each other in stating the same fact?

Solution: If E is the event of their contradicting each other, then $E = (A \cap \bar{B}) \cup (\bar{A} \cap B)$ also $(A \cap \bar{B})$ and $(\bar{A} \cap B)$ are two mutually exclusive events.

$$P(E) = P(A \cap \bar{B}) + P(\bar{A} \cap B) = P(A) \cdot P(\bar{B}) + P(\bar{A}) \cdot P(B)$$
$$= \frac{3}{4} \cdot \frac{1}{5} + \frac{4}{5} \cdot \frac{1}{4} = \frac{7}{20}$$

Therefore, in 35% of the cases they are likely to contradict each other.

Illustration 28.9 A sportsman's chance of shooting an animal at a distance $r\,(> a)$ is $\frac{a^2}{r^2}$. He fires when $r = 2a$ and if he misses, he reloads and fires when $r = 3a, 4a, 5a, \cdots$. If he misses at a distance na, the animal escapes. What are the odds against the sportsman?

Solution:

$$P(r) = \frac{a^2}{r^2}$$

$$P(2a) = \frac{1}{4}, P(3a) = \frac{1}{9}, P(4a) = \frac{1}{16}, \text{etc.}$$

The sportsman succeeds if

(a) He hits the first time or
(b) Misses the first time but succeeds at the second or
(c) Misses the first and second time but succeeds in the third and so on.

Hence, Probability of success is

$$\frac{1}{4} + \frac{3}{4} \times \frac{1}{9} + \frac{3}{4} \times \frac{8}{9} \times \frac{1}{16} + \frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \frac{1}{25} + \cdots (n-1) \text{ terms}$$

$$p = \frac{1}{4} + \frac{1}{12} + \frac{1}{24} + \frac{1}{40} + \cdots \text{upto } (n-1) \text{ terms}$$

$$2p = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots \text{upto } (n-1) \text{ terms}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= 1 - \frac{1}{n} = \frac{n-1}{n}$$

$$\Rightarrow P = \frac{n-1}{2n}$$

Now, probability that the animal would escape is

$$q = 1 - P = 1 - \left(\frac{n-1}{2n}\right)$$
$$= \frac{n+1}{2n}$$

Therefore, odds against the sportsman is

$$= \frac{q}{p} = \frac{n+1}{n-1}$$

Illustration 28.10 Of the three independent events, the chance that only the first occurs is a, that only the second occurs is b, and that only the third occurs is c. Show that the probabilities of occurrence of these three events are respectively $\frac{a}{a+x}, \frac{b}{b+x}, \frac{c}{c+x}$ where x is a root of the equation $(a+x)(b+x)(c+x) = x^2$.

Solution: Let E_1, E_2, E_3 be three independent events and E_1', E_2', E_3' be their complements. Then

$$P(E_1 \cap E_2' \cap E_3') = P(E_1) \cdot P(E_2') \cdot P(E_3') = a \quad (1)$$

since E_1, E_2, E_3 are independent.

$$P(E_1' \cap E_2 \cap E_3') = P(E_1')\,P(E_2)\,P(E_3') = b \quad (2)$$

and $$P(E_1' \cap E_2' \cap E_3) = P(E_1')\,P(E_2')\,P(E_3) = c \quad (3)$$

Denote $P(E_1')\,P(E_2')\,P(E_3')$ by x

Then

$$\frac{P(E_1)}{P(E_1')} = \frac{a}{x}, \text{ this implies } \frac{P(E_1)}{1 - P(E_1)} = \frac{a}{x} \quad \text{or} \quad P(E_1) = \frac{a}{a+x}$$

Similarly, we get

$$P(E_2) = \frac{b}{b+x} \text{ and } P(E_3) = \frac{c}{c+x} \quad (4)$$

Multiplying Eqs. (1), (2) and (3), we get

$$\frac{abc}{(a+x)(b+x)(c+x)}x^2 = abc \text{ or } (a+x)(b+x)(c+x) = x^2$$

Illustration 28.11 The independent probabilities that A, B and C solve a mathematical problem are $\frac{1}{3}, \frac{1}{3}$ and $\frac{1}{4}$, respectively. Find the probability that just two of them only solve the problem.

Solution: Given that

$$P(A) = \frac{1}{3}, P(B) = \frac{1}{3}, P(C) = \frac{1}{4}$$

and A, B, C are independent events.

The problem gets solved by any two of them solving but the third one fails.

Required probability $= P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C)$

$$= \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{4} + \frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{4}$$

$$= \frac{3}{36} + \frac{2}{36} + \frac{2}{36} = \frac{7}{36}$$

28.8 Total Probability

See Fig. 28.3. Consider a sample space S, let A_i, $i = 1$ to n be the set of n mutually exclusive and exhaustive set of sample space S.

Thus $A_i \cap A_j = \phi$ for $1 \le i < j \le n$ and $\sum_{i=1}^{n} P(A_i) = 1$ as $\bigcup_{i=1}^{n} A_i = S$

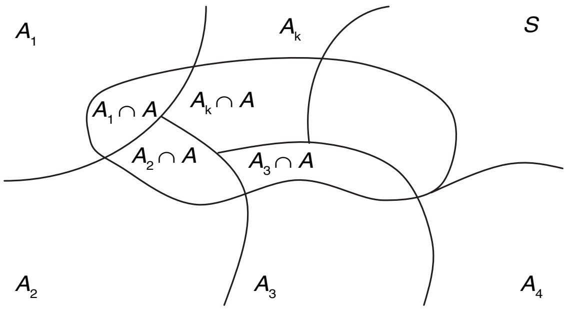


Figure 28.3

Let A be any event of S. Then total probability of the event A is given by

$$P(A) = \sum_{i=1}^{n} P(A_i)P(A/A_i)$$

where $P(A/A_i)$ gives us the contribution of A_i in the occurrence of A.

This result is obtained as

$$A = (A_1 \cap A) \cup (A_2 \cap A) \cup \cdots \cup (A_n \cap A)$$
$$\Rightarrow P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \cdots + P(A_n \cap A)$$

$$= P(A_1)P\left(\frac{A}{A_1}\right) + P(A_2)P\left(\frac{A}{A_2}\right) + \cdots + P(A_n)P\left(\frac{A}{A_n}\right) = \sum_{i=1}^{n} P(A_i)P\left(\frac{A}{A_i}\right)$$

Remark:

1. $P(A/A_i)$ gives up the contribution of A_i in the occurrence of A.

Let 'A' be any event of S. Then we can write

$$A = (A_1 \cap A) \cup (A_2 \cap A) \cdots \cup (A_n \cap A)$$

As $A_1, A_2 \cdots A_n$ are mutually exclusive, $(A_1 \cap A)$, $(A_2 \cap A)$, $\cdots$, $(A_n \cap A)$ would also be mutually exclusive. So,

$$P(A) = P(A_1 \cap A) + P(A_2 \cap A) + \cdots + P(A_n \cap A)$$

$$= P(A_1) \times P(A/A_1) + P(A_2) \times P(A/A_2) + \cdots + P(A_n) \times P(A/A_n)$$

$$\Rightarrow P(A) = \sum_{i=1}^{n} P(A_i) \cdot P(A/A_i)$$

This is known as the total probability of the event A.

Illustration 28.12 An unbiased coin is tossed. If the result is head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of eleven cards numbered 2, 3, $\cdots$, 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?

Solution: Let E_1 be the event 'toss results in a head',

E_2 be the event 'toss results in a tail',

A be the event 'the noted number is 7 or 8'

We have

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Also

$$P(A/E_1) = P(7) + P(8) = \frac{6}{36} + \frac{5}{36} = \frac{11}{36}$$

Since

$$7 = \{1+6, 2+5, 3+4, 4+3, 5+2, 6+1\} \text{ and}$$
$$8 = \{2+6, 3+5, 4+4, 5+3, 6+2\}$$

$$P(A/E_2) = \frac{2}{11}$$

We know that

$$P(A) = P(A/E_1) \times P(E_1) + P(A/E_2) \times P(E_2)$$
$$= \frac{11}{36} \times \frac{1}{2} + \frac{2}{11} \times \frac{1}{2} = \frac{193}{792}$$

Illustration 28.13 A lot contains 20 articles. The probability that the lot contains exactly two defective articles is 0.4; and the probability that it contains exactly three defective articles is 0.6. Articles are drawn from the lot at random one by one without replacement until all the defective are found. What is the probability that the testing procedure stops at the twelfth testing?

Solution: Suppose A is the event that the testing procedure ends at the twelfth testing.

A_1 = {the event that the lot contains 2 defective}

A_2 = {the event that the lot contains 3 defective}

$$P(A_1) = 0.4; P(A_2) = 0.6, P(A) = P(A_1)P(A/A_1) + P(A_2)P(A/A_2)$$

$$= 0.4\left\{\frac{{}^{18}C_{10} \cdot {}^{2}C_1}{{}^{20}C_{11}} \cdot \frac{1}{9}\right\} + 0.6\left\{\frac{{}^{17}C_9 \cdot {}^{3}C_2}{{}^{20}C_{11}} \cdot \frac{1}{9}\right\}$$

$$= \frac{1}{9}\left\{\frac{4}{10} \cdot \frac{18!}{10!8!} \cdot \frac{11!9!}{20!} \cdot 2 + \frac{6}{10} \cdot \frac{17!}{9!8!} \cdot \frac{11!9!}{20!} \cdot 3\right\}$$

$$= \frac{1}{9}\left\{\frac{4}{10}\times\frac{11\times 9}{19\times 20}\cdot 2+\frac{6}{10}\times\frac{9\times 10\times 11}{18\cdot 19\cdot 20}\cdot 3\right\}$$
$$= \frac{44}{1900}+\frac{55}{1900}=\frac{99}{1900}$$

Your Turn 1

1. Two dices are thrown. Find the probability that a total of 8 occurs.
   Ans. $\frac{5}{36}$
2. Find the probability of drawing 4 white balls and 2 black balls from a bag containing 6 white balls, 4 black balls and 1 red ball.
   Ans. $\frac{15}{77}$
3. In shuffling a pack of cards, four are accidentally dropped. What is the probability that the four are one from each suit?
   Ans. $\frac{({}^{13}C_1)^4}{{}^{52}C_4}$
4. A four digit number is formed by 1, 2, 3, 5 with no repetitions. Find the chance that (i) the number is divisible by 5 and (ii) the number is odd.
   Ans. (i) $\frac{1}{4}$ (ii) $\frac{3}{4}$
5. Determine the probability of drawing 4 white balls and 2 black balls without replacement, from a bag containing 1 red, 4 black, and 6 white balls.
   Ans. $\frac{15}{77}$
6. There are 10 tickets in a lottery. 5 wins and 5 losses. 2 tickets are taken. What is the probability of a win? **Ans.** $\frac{7}{9}$
7. Two guns fire simultaneously at the same target. The probability of a hit from the first is 0.7 and from the second one is 0.6. What is the probability that the target is hit? **Ans.** 0.88
8. A problem in mathematics is given to three students *A*, *B* and *C*. Their respective chances of solving the problem are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. What is the probability that the problem is solved?
   Ans. $\frac{3}{4}$
9. If $\frac{1+3p}{3}, \frac{1-p}{4}$, and $\frac{1+2p}{2}$ are the probabilities of three mutually exclusive events, then show that $\frac{1}{3}\le p\le\frac{1}{2}$.
10. If *k* objects are distributed at random among *k* persons, then find the probability that at least one of them will not get anything. **Ans.** $1-\frac{(k-1)!}{k^k-1}$
11. One bag contains 3 white and 2 black balls. A second bag contains 5 white and 3 black balls. If a bag is chosen at random and a ball is taken from it, what is the chance that it is white?
    Ans. 5/36

28.9 Bayes' Theorem or Inverse Probability

Bayes' theorem gives probability of occurrence of an event when the outcome of experiment is known.

See Fig. 28.4. Consider a sample space *S*, let A_i, $i = 1$ to n be the set of n mutually exclusive and exhaustive set of sample space *S*.

Thus, $A_i \cap A_j = \phi$ for $1 \le i < j \le n$ and $\sum_{i=1}^{n} P(A_i) = 1$ as $\bigcup_{i=1}^{n} A_i = S$

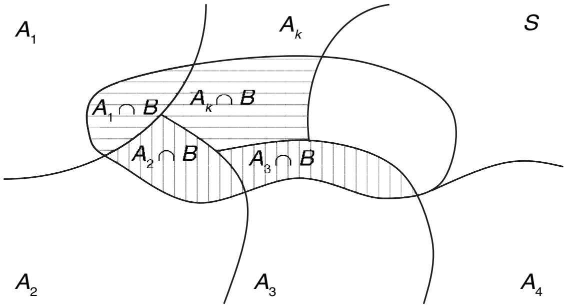


Figure 28.4

Let *B* be an event of *S* which has already occurred, then conditional probability of occurrence of any one of the event say A_k out of the A_i, $i = 1, 2, \cdots, n$ events is

$$P\left(\frac{A_k}{B}\right)=\frac{P(A_k\cap B)}{P(B)}=\frac{P(A_k)\,P(B/A_k)}{P(B)}$$

Now using the concept of total probability, we get Baye's theorem as follows:

$$P\left(\frac{A_k}{B}\right)=\frac{P(A_k)\,P(B/A_k)}{\sum_{i=1}^{n}P(A_i)\,P(B/A_i)}$$

Value of Testimony:

The theory of probability can be used to estimate the value of testimony of witnesses. Such an application rests on the following two basic assumptions:

1. That to each witness there pertains a constant *P*(his credibility), which measures the average frequency with which he speaks the truth.
2. That the statements of witnesses are independent of one another in the sense required in the theory of probability.

This theorem at times is also called inverse probability theorem. Let us consider any event '*A*' of sample space '*S*' (as in the previous section). Let us say that event *A* is found to have occurred and we have to find the probability that it has occurred to the occurrence of cause, say A_i.

From total probability theorem, we get

$$\text{P(A)} = \sum_{i=1}^{n} P(A_i)\cdot P(A/A_i)$$

Also,

$$P(A_i/A)=\frac{P(A_i\cap A)}{P(A)}=\frac{P(A_i)\cdot P(A/A_i)}{P(A)}$$

$$\Rightarrow P(A_i/A)=\frac{P(A_i)\cdot P(A/A_i)}{\sum_{i=1}^{n}P(A_i)\cdot P(A/A_i)}$$

This result is known as Bayes' theorem.

Illustration 28.14 In a factory manufacturing bolts; machines *A*, *B* and *C* manufacture respectively 20%, 30% and 50% of the total production. Of their outputs 2%, 3% and 5% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine C?

Solution: Let A, B, C be the events that a bolt selected at random is manufactured by machines A, B and C, respectively so that $P(A)=2/10$; $P(B)=3/10$ and $P(C)=5/10$.

Let D be the event that the bolt selected at random is defective, so that

$$P(D/A)=2/100$$
$$P(D/B)=3/100$$

and
$$P(D/C)=5/100$$

$$P(C/D)=\frac{P(D/C)\cdot P(C)}{P(D/A)\cdot P(A)+P(D/B)\cdot P(B)+P(D/C)\cdot P(C)}$$
$$=\frac{5\times5}{4+9+25}=\frac{25}{38}$$

28.10 Random Variable and Probability Distribution

A random variable is generally described as a variable whose values are the result of some changing conditions. Consider a simultaneous throw of two coins. The sample space is

$$S=\{HH,HT,TH,TT\}$$

Let X denote the number of heads in a point of the sample space S. Then

$$X(HH)=2,\ X(HT)=1,\ X(TH)=1,\ X(TT)=0$$

Thus, X takes values 0, 1, 2 only and no more. Here we say that X is a random variable or a stochastic variable. We have the following definition.

A random variable is a real valued function X defined over the sample space of an experiment, that is, a random variable is a function which associates to each point of a sample space, a unique real number.

1. Random variables are denoted by X, Y, Z.
2. More than one random variables can be defined on the same sample space.

For example, let Y denote the number of heads minus the number of tails for each outcome of the sample space S.

Then
$$Y(HH)=2-0=2$$
$$Y(HT)=1-1=0$$
$$Y(TH)=1-1=0$$
$$Y(TT)=0-2=-2$$

Thus, X and Y are two different random variables defined on the same sample space S.

Illustration 28.15 A bag contains 2 white and 1 red ball. One ball is drawn at random and then put back in the box after noting its colour. The process is repeated again. If X denotes the number of red balls recorded in the two draws, describe X.

Solution: Let the balls in the bag be denoted by W_1, W_2, R. Then

$$S=\{W_1W_2, W_1W_1, W_2W_1, W_2W_2, W_1R, W_2R, RW_1, RW_2, RR\}$$

Now for $W\in S$,

$$X(W)=\text{ Number of red balls}$$

Therefore,

$$X(\{W_1W_1\})=X(\{W_1W_2\})=X(\{W_2W_1\})=X(\{W_2W_2\})=0$$
$$X(\{W_1R\})=X(\{W_2R\})=X(\{RW_1\})=X(\{RW_2\})=1$$

and
$$X(\{RR\})=2$$

28.10.1 Probability Distribution of Random Variable

A distribution, in which values of the random variable and their corresponding probabilities are given is called the probability distribution of the random variable.

Let us suppose that a discrete variable X assumes values $x_1, x_2, \ldots, x_n$ with probability $p_1, p_2, \ldots, p_n$ respectively, where $p_1+p_2+\cdots+p_n=1$ and $0\le p_i 1$ for all $i=1,2,\ldots,n$.

Then the following table describes the probability distribution:

X	x_1	x_2	x_3	x_4	$\cdots$	x_n
$P(X)$	p_1	p_2	p_3	p_4	$\cdots$	p_n

Example:

Let X be a random variable denoting the number of tails in a simultaneous throw of two coins. Then clearly X can take the values 0, 1, 2.

$$X(TT)=2,\ X(HT)=1,\ X(TH)=1 \text{ and } X(HH)=0$$

Let $P(X)=$ Probability of the variable X. Then

$$P(X=0)=P\text{ (no tail)}=\frac{1}{4}$$
$$P(X=1)=P\text{ (one tail)}=\frac{2}{4}=\frac{1}{2}$$
$$P(X=2)=P\text{ (two tails)}=\frac{1}{4}$$

Note:

$$P(X=0)+P(X=1)+P(X=2)=1\ \left[\because \frac{1}{4}+\frac{1}{2}+\frac{1}{4}=1\right]$$

We can write the above result in the following form:

X	0	1	2
$P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Clearly, each of the probability is a non–negative fraction (never greater than 1) and their sum is 1.

The above form is the probability distribution of the random variable X.

Illustration 28.16 An urn contains 5 white and 3 red balls. Find the probability distribution of the number of red balls, with replacements, in three draws.

Solution: Let R be the event of drawing a red ball.

Let X denote the discrete random variable 'no. of red balls' in a draw of three balls. Then

$X=0,1,2,3$. Here, $P(R)=\frac{3}{8}$ and $P(\bar{R})=\frac{5}{8}$.

$$P(X=0)=P(\bar{R}_1\bar{R}_2\bar{R}_3)=P(\bar{R}_1)P(\bar{R}_2)P(\bar{R}_3)=\frac{5}{8}\times\frac{5}{8}\times\frac{5}{8}=\frac{125}{512}$$

$$P(X=1)=P(R_1\bar{R}_2\bar{R}_3)+P(\bar{R}_1R_2\bar{R}_3)+P(\bar{R}_1\bar{R}_2R_3)=3\left(\frac{3}{8}\times\frac{5}{8}\times\frac{5}{8}\right)$$

$$=3\left(\frac{75}{512}\right)=\frac{225}{512}$$

$$P(X=2)=P(\bar{R}_1R_2R_3)+P(R_1\bar{R}_2R_3)+P(R_1R_2\bar{R}_3)=3\left(\frac{5}{8}\times\frac{3}{8}\times\frac{3}{8}\right)=\frac{135}{512}$$

$$P(X=3)=P(R_1R_2R_3)=\frac{3}{8}\times\frac{3}{8}\times\frac{3}{8}=\frac{27}{512}.$$

Hence, the required probability distribution is

X	0	1	2	3
$P(X)$"	$\frac{125}{512}$	$\frac{225}{512}$	$\frac{135}{512}$	$\frac{27}{512}$

Illustration 28.17 Find the probability distribution of number of doublets in three throws of a pair of dice.

Solution: Let X denote the number of doublets. Possible doublets are

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

Clearly, X can take the values 0, 1, 2 or 3.

$$\text{Probability of getting a doublet } =\frac{6}{36}=\frac{1}{6}$$

$$\text{Probability of not getting a doublet } =1-\frac{1}{6}=\frac{5}{6}$$

Now,

$$P(X=0)=P\ \text{(no doublet)}\ =\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{125}{216}$$

$$P(X=1)=P\ \text{(one doublet and two non-doublets)}$$

$$=\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=3\left(\frac{1}{6}\times\frac{5^2}{6^2}\right)=\frac{75}{216}$$

$$P(X=2)=P\ \text{(two doublets and one non-doublet)}$$

$$=\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}=3\left(\frac{1}{6^2}\times\frac{5}{6}\right)=\frac{15}{216}$$

and $P(X=3)=P\ \text{(three doublets)}\ =\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{216}.$

Thus, the required probability distribution is

X	0	1	2	3
$P(X)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Verification:

$$\text{Sum of the probabilities} = \sum_{i=1}^{n} p_i=\frac{125}{216}+\frac{75}{216}+\frac{15}{216}+\frac{1}{216}$$

$$=\frac{125+75+15+1}{216}=\frac{216}{216}=1$$

28.11 Binomial Distribution

In n independent trials of a random experiment, let X be the number of times an event A occurs. In each trial, event A has same probability as $P(A) = p$ referred to as success. Then in a trial non-occurrence of A is referred to as failure and given by $q = 1 - p$.

Here X can assume values from 0 to n. Now $X = r$ means A occurs in r trials and $(n - r)$ it does not occur this may look as

$$\underbrace{A\,A\,A.........A}_{r\text{ times}}\ \underbrace{\bar{A}\,\bar{A}.........\bar{A}}_{n-r\text{ times}}$$

here $(\bar{A})$ means complement of A. Using the assumption that trials are independent, that is, they do not influence each other, hence has the probability

$$\underbrace{p\,p\,........p}_{r\text{ times}}\ \underbrace{q\,q.........q}_{n-r\text{ times}}=p^r q^{n-r}$$

and it can be arranged in $\dfrac{n!}{r!(n-r)!}={}^nC_r$ ways.

Hence, the probability of getting r successes or occurrence of A in r trials out of n independent trials is

$$P(X=r)={}^nC_r p^r q^{n-r}$$

which is denoted as binomial distribution of random variable X.

- The probability of getting at least k successes in
$$P(X\ge k)=\sum_{r=k}^{n}{}^nC_r p^r q^{n-r}$$
- The probability of getting almost K successes is
$$P(X\le k)=\sum_{r=0}^{k}{}^nC_r p^r q^{n-r}$$
- $\displaystyle\sum_{r=0}^{n}{}^nC_r p^r q^{n-r}=(p+q)^n=1$

Let us consider a binomial experiment which has been repeated 'n' times. Let the probability of success and failure in any trial be p and q, respectively and we are interested in the probability of occurrence of exactly 'r' successes in these n trials. Now number of ways of choosing 'r' success in 'n' trials $= \boldsymbol{{}^nC_r}$. Probability of 'r' successes and $(n - r)$ failures is $p^r \times q^{n-r}$. Thus, probability of having exactly r successes $={}^nC_r\times p^r\times q^{n-r}$

Let 'X' be the random variable representing the number of successes. Then

$$P(X=r)={}^nC_r\times p^r\times q^{n-r}\quad (r=0, 1, 2, \ldots, n)$$

- Probability of utmost 'r' successes in n trials $=\displaystyle\sum_{\lambda=0}^{r}{}^nC_\lambda p^\lambda q^{n-\lambda}$
- Probability of at least 'r' successes in n trials $=\displaystyle\sum_{\lambda=r}^{n}{}^nC_\lambda p^\lambda\cdot q^{n-\lambda}$
- Probability of having 1st success at the r^{th} trial $=p\times q^{r-1}$

Illustration 28.18 Ten coins are tossed simultaneously. Find the probability of getting at least 7 heads.

Solution: In this case

$$n=10\,;\ p=1/2\,;\ q=1/2$$

$$P(X\ge 7)=P(X=7)+P(X=8)+P(X=9)+P(X=10)$$

$$=\left({}^{10}C_3+{}^{10}C_2+{}^{10}C_1+1\right)\frac{1}{2^{10}}=\frac{176}{2^{10}}$$

Illustration 28.19 Numbers are chosen at random, one at a time, from the two digit numbers 00, 01, 02, ⋯, 99 with replacement. An event E occurs if and only if the product of the two digits of a selected number is 18. If four numbers are selected, find the probability that E occurs at least 3 times.

Solution: The numbers, whose two digits product is 18, are 29, 36, 63, 92

$$P(E)=\frac{4}{100}=0.04$$

Since there is replacement, $P(E)$ remains the same for every selection. Four such selections are made.
Probability that E occurs, at least thrice

= Probability that E occurs thrice and fails to occur once + Probability that E occurs all the four times

$$= {}^4C_3\,(0.04)^3(0.96)^{4-3} + {}^4C_4\,(0.04)^4(0.96)^{4-4}$$
$$= {}^4C_3\,(0.04)^3\,(0.96) + (0.04)^4 = 0.00024832$$

Illustration 28.20 A set has n elements. A subset P of A is selected at random. All the elements of P are returned to A. The subset Q of A is formed. Find the probability that P and Q have no common element.

Solution: The subset P, formed first, may contain none, one, two,… or all elements of A. P and Q have no common elements. Any element x of A has one of the four possibilities.

1. $x \in P$; $x \in Q$
2. $x \in P$; $x \notin Q$
3. $x \notin P$; $x \in Q$
4. $x \notin P$; $x \notin Q$

Case 2, 3, 4 correspond to P and Q having no common element.
Hence, the probability that this element x is not in P or not in Q is 3/4. The same is true of the other elements.
Hence, the probability is

$$ {}^nC_n \left(\frac{3}{4}\right)^n \left(\frac{1}{4}\right)^0 $$

28.11.1 Recurrence Formula for Binomial Distribution

We know that

$$P(r) = {}^nC_r q^{n-r} p^r \quad [\textbf{Note: } P(r) \text{ means } P(X=r)]$$

$$P(r+1) = {}^nC_{r+1} q^{n-r-1} p^{r+1}$$

$$\frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}}{{}^nC_r}\cdot\frac{q^{n-r-1}}{q^{n-r}}\cdot\frac{p^{r+1}}{p^r} = \frac{n-r}{r+1}\cdot\frac{p}{q}$$

Hence,

$$P(r+1) = \frac{n-r}{r+1}\cdot\frac{p}{q}\,P(r)$$

which is the required recurrence formula for binomial distribution.

1. If $P(0)$ is known, then we find $P(1)$, $P(2)$, … with the help of recurrence formula.
2. If $p = \frac{1}{2}$, then $q = \frac{1}{2}$. Then the binomial distribution is called symmetrical binomial distribution.

28.11.2 Mean and Variance of Binomial Distribution

If $x_1, x_2, \ldots, x_n$ are the values of a random variable X and $p_1, p_2, \ldots, p_n$ are the corresponding probabilities, then mean (μ) and variance (σ^2) of the probability distribution are given by

$$\mu = \Sigma p_i x_i$$
$$= 0\cdot p(0) + 1\cdot p(1) + 2\cdot p(2) + \cdots + n\cdot p(n)$$
$$= 1\cdot {}^nC_1 pq^{n-1} + 2\cdot {}^nC_2 p^2 q^{n-2} + \cdots + n\,{}^nC_n p^n$$
$$= npq^{n-1} + 2\frac{n(n-1)}{2}\cdot p^2q^{n-2} + 3\cdot\frac{n(n-1)(n-2)}{1.2.3}\cdot p^3q^{n-3} + \cdots + n\cdot p^n$$
$$= np[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{1.2}p^2q^{n-3} + \cdots + p^{n-1}]$$
$$= np[\,{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 pq^{n-2} + {}^{n-1}C_2 p^2q^{n-3} + \cdots + {}^{n-1}C_{n-1}p^{n-1}]$$
$$= np(q+p)^{n-1} = np(1)^{n-1} \qquad [\because q+p=1]$$
$$= np$$

Hence, the mean of the distribution, that is, $\mu = np$.
Again variance σ^2 is given by

$$\sigma^2 = \Sigma p_i x_i^2 - \mu^2$$
$$= 0^2 p(0) + 1^2\cdot p(1) + 2^2\cdot p(2) + \cdots + n^2\cdot p(n) - \mu^2$$
$$= 1\cdot {}^nC_1 pq^{n-1} + 2^2\cdot {}^nC_2 p^2q^{n-2} + \cdots + n^2\cdot {}^nC_n p^n - \mu^2$$
$$= [npq^{n-1} + 4\frac{n(n-1)}{1\cdot 2}p^2q^{n-2} + 9\cdot\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}p^3q^{n-3} + \cdots + n^2\cdot p^n] - \mu^2$$
$$= np[q^{n-1} + 2\cdot(n-1)pq^{n-2} + 3\cdot\frac{(n-1)(n-2)p^2}{1.2}q^{n-3} + \cdots + np^{n-1}] - \mu^2$$
$$= np[\,{}^{n-1}C_0 q^{n-1} + 2\,{}^{(n-1)}C_1 p^1 q^{n-2} + 3\cdot {}^{(n-1)}C_2 p^2 q^{n-3} + \cdots + np^{n-1}] - \mu^2$$
$$= np[(\,{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 pq^{n-2} + {}^{n-1}C_2 p^2q^{n-3} + \cdots + {}^{n-1}C_{n-1}p^{n-1})] + (\,{}^{n-1}C_1 pq^{n-2} + 2\cdot {}^{n-1}C_2 p^2q^{n-3} + \cdots + (n-1\cdot {}^{n-1}C_{n-1}p^{n-1})] - \mu^2$$
$$= np[(q+p)^{n-1} + (n-1)p(\,{}^{n-2}C_0\cdot q^{n-2}) + {}^{n-2}C_1 pq^{n-3} + \cdots + {}^{n-2}C_{n-2}p^{n-2})] - \mu^2$$
$$\left[\text{As } r\cdot\frac{{}^{n-1}C_r}{n-r} = {}^{n-1}C_{r-1} \text{ for } r = 1, 2, 3, \ldots, n-1\right]$$
$$= np[(1)^{n-1} + (n-1)p\cdot(q+p)^{n-2}] - n^2p^2 \qquad [\text{As } \mu = np]$$
$$= np[1 + (n-1)p\cdot(1)^{n-2}] - n^2p^2$$
$$= np[1 + (n-1)p] - n^2p^2 = np[1 + np - p] - n^2p^2$$
$$= np[np + q] - n^2p^2 \qquad [\text{As } 1-p = q]$$
$$= n^2p^2 + npq - n^2p^2 = npq$$

Hence, variance of binomial distribution is given by $\sigma^2 = npq$.

- Standard Deviation (S.D.) for the binomial distribution is
$$\sigma = \sqrt{npq}$$
- Variance of binomial distribution is less than its mean. Since, variance $= npq \le np$
$$\text{Mean} = np$$

Hence, Variance $\le$ Mean of the Binomial Distribution.
That is, mean of the binomial distribution is always greater than the variance.

Illustration 28.21 Calculate $P(r)$ for $r = 1, 2, 3, 4$ and 5 by using the recurrence formula of the binomial distribution; use $p = \frac{1}{3}$ and $n = 5$. Hence, draw the histogram for the distribution.

Solution: We know that

$$P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} \cdot P(r) \quad (1)$$

[Recurrence formula for binomial distribution]

Putting $p = \frac{1}{3}, q = 1 - \frac{1}{3} = \frac{2}{3}$ and $n = 5$, we get

$$P(r+1) = \frac{5-r}{2(r+1)} P(r), r = 0, 1, 2, 3, 4$$

$$P(r) = {}^nC_r\, q^{n-r}\, p^r \,(n = 5; r = 0)$$

$$P(0) = {}^5C_0\, q^{5-0} p^0$$

$$P(0) = q^5 = \left(\frac{2}{3}\right)^5 = \frac{32}{243}\; P(0) = 0.13$$

Now, $r = 0, 1, 2, 3, 4$ in Eq. (1), we get

$$P(1) = \frac{5-0}{2(0+1)} P(0) = \frac{5}{2}(.13) = .33;$$

$$P(2) = \frac{5-1}{2(1+1)} P(1) = .33;$$

$$P(3) = \frac{5-2}{2(3)} P(2) = \frac{1}{2}(.33) = .16;$$

$$P(4) = \frac{5-3}{2(3+1)} P(3) = \frac{1}{4}(.16) = .04;$$

$$P(5) = \frac{5-4}{2(4+1)} P(4) = \frac{1}{10}(.04) = .004 = 0$$

Correct upto two decimal places.
The histogram is shown in Fig. 28.5.

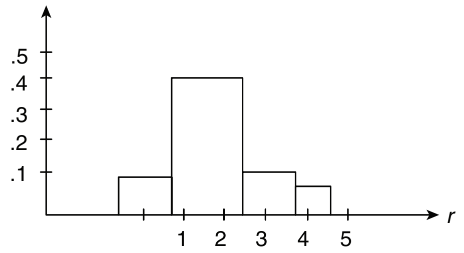


Figure 28.5

Illustration 28.22 Find the binomial distribution whose mean is 9 and whose standard deviation is $\frac{3}{2}$.

Solution: Let *p, q* be the probability of success and failure in any one trial and *n* be the number of trials. Then the binomial distribution is

$$(q+p)^n \quad (1)$$

For binomial distribution, we have

$$\text{Mean } = np = 9 \quad (2)$$

$$\text{S.D.} = \sqrt{npq} = \frac{3}{2} \quad (3)$$

From Eqs. (2) and (3),

$$\sqrt{9q} = \frac{3}{2} \text{ or } 9q = \frac{9}{4} \qquad \left[\text{since } q = \frac{1}{4}\right]$$

But

$$p + q = 1 \Rightarrow p = 1 - \frac{1}{4} = \frac{3}{4}$$

Since

$$np = 9 \Rightarrow n\left(\frac{3}{4}\right) = 9$$

Therefore, $$n = \frac{36}{3} = 12$$

Hence, the required binomial distribution is $\left(\frac{1}{4} + \frac{3}{4}\right)^{12}$.

28.12 Poisson Distribution

It is a limiting case of binomial distribution. If the number of events n is very large $(n \to \infty)$ and probability of success in each experiment is p ('p' being very small) and $np = \lambda$ (say) is finite, then

$$P(X = r) \text{ or } P(r) = \frac{e^{-\lambda} \lambda^r}{r!},$$

where $r = 0, 1, 2, \ldots$ and $\lambda = np$. Here λ is known or parameter of distribution.

$$P(r+1) = \frac{\lambda}{r+1} P(r) \text{ is known as recurrence formula.}$$

Note:

1. $\sum_{r=0}^{\infty} P(r) = 1$
2. If λ_1 and λ_2 are parameter of variables *X* and *Y*, then parameter of $X + Y$ will be $(\lambda_1 + \lambda_2)$.
3. In Poisson distribution, mean = variance = λ.

Illustration 28.23 If a random variable *X* has Poisson distribution such that $P(X = 1) = P(X = 2)$, then find $P(X = 4)$.

Solution:

$$\frac{e^{-\lambda} \lambda}{1!} = \frac{e^{-\lambda} \lambda^2}{2!} \Rightarrow \lambda = 2$$

$$P(X = 4) = \frac{e^{-\lambda} \lambda^4}{4!} = \frac{e^{-2}(16)}{24} = \frac{2}{3} e^{-2}$$

Illustration 28.24 If X is a Poisson variate such that $P(X = 2) = 9P(X = 4) + 90P(X = 6)$, then find Var (*X*).

Solution:

$$\frac{e^{-\lambda} \cdot \lambda^2}{2!} = 9\frac{e^{-\lambda} \cdot \lambda^4}{4!} + 90\frac{e^{-\lambda} \cdot \lambda^6}{6!}$$

$$\Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0 \Rightarrow (\lambda^2 + 4)(\lambda^2 - 1) = 0$$

$$\Rightarrow \lambda = 1 \Rightarrow \text{Variance } = \lambda = 1$$

28.13 Probability of Events in Experiments with Countable Infinite Sample Space

The sample space of some random experiments have infinite sample points, and hence the events also have infinite sample points in favour of their occurrence.

In some cases, the probability is calculated by relating the sample space or the sets representing the events, with lengths or areas of geometrical figures, etc. Following are the illustrations of such cases.

Illustration 28.25 If a be chosen at random in the interval [0, 5], show that the probability that the equation $4x^2 + 4ax + (a+2) = 0$, to have real roots is 3/5.

Solution: Equation $4x^2 + 4ax + (a + 2) = 0$ will have real roots, if discriminant,

$$D = b^2 - 4ac \geq 0$$

$$\Rightarrow 16a^2 - 16(a + 2) \geq 0$$

that is,

$$a^2 - a - 2 \geq 0 \text{ or } (a - 2)(a + 1) \geq 0$$

Hence, $a \geq 2$ or $a \leq -1$.

But in the interval [0, 5],

$$a \geq 2$$

$$\Rightarrow a \in [2, 5]$$

(See Fig. 28.6.)

Hence,

$$\text{Probability} = \frac{\text{Length of interval } [2, 5]}{\text{Length of interval } [0, 5]} = \frac{3}{5}$$

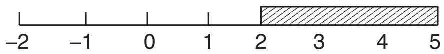


Figure 28.6

Illustration 28.26 A line is divided at random into three parts. What is the probability that they form the sides of a triangle?

Solution: Let $AP = x$, $BQ = y$ and $AB = a$. Then length of third side is $a - x - y$.

Since the sum of two sides of a triangle is greater than the third side, AP must be less than $\frac{a}{2}$, that is, $x < \frac{a}{2}$. Similarly, $y < \frac{a}{2}$ and

$$a - (x + y) < \frac{a}{2} \text{ or } x + y > \frac{a}{2} \quad (1)$$

For all possible cases of dividing the line,

$$0 < x < a, 0 < y < a \text{ and } x + y < a \quad (2)$$

Condition (2) corresponds to the triangular region OXY and condition (1) corresponds to the triangular region PQR (Fig. 28.7).

$$p = \frac{\text{Area of } \Delta PQR}{\text{Area of } \Delta OXY} = \frac{\frac{1}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)}{\frac{1}{2}(a)(a)} = \frac{1}{4}.$$

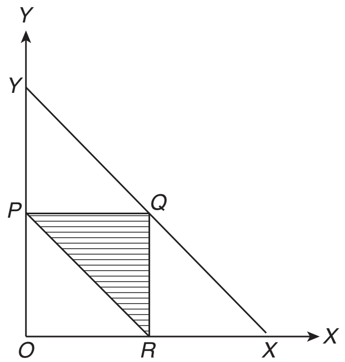


Figure 28.7

Illustration 28.27 On a straight line of length a, two points are taken at random. Find the chance that no part is greater than b.

Solution: Let

$$AB = a, AP = x \text{ and } BQ = y$$

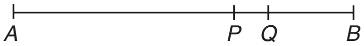
For favourable cases,

$$x < b, y < b \text{ and } a - x - y < b \quad (1)$$

For all possible cases,

$$x + y < a, 0 < x < a, 0 < y < a \quad (2)$$

The ΔOXY where $X = (a, 0)$, $Y = (0, a)$ represents all cases.



Let M be $(0, b)$, N be $(b, 0)$ and equation of KL be $x + y = a - b$.

Then the region $KLNQPM$ represents the favourable cases.

Two cases arise.

Case I: $b > \frac{a}{2}$

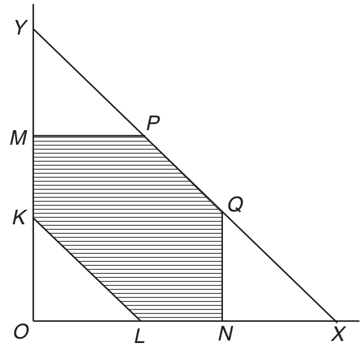


Figure 28.8

The conditions of (1) are represented by the region $KLNQPM$ in Fig. 28.8 and hence,

$$p = \frac{\frac{1}{2}\left\{a^2 - 3(a - b)^2\right\}}{\frac{1}{2}a^2}$$

Since $OL = a - b$, $NX = a - b$ and $MY = a - b$

$$= 1 - 3\left(\frac{a - b}{a}\right)^2$$

Case II: $b < \frac{a}{2}$

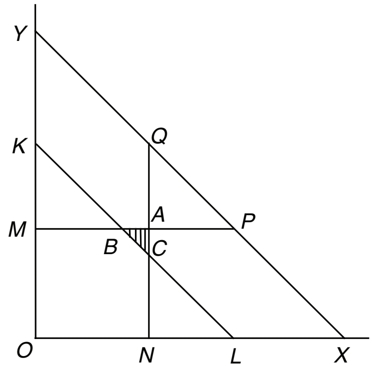


Figure 28.9

$$ON = b < \frac{a}{2}; OM = b < \frac{a}{2}$$

In this case, the favourable conditions correspond to the shaded region *ABC* (Fig. 28.9)

$$CA = NA - NC$$
$$= b - (a - 2b),$$

since (*ON*, *NC*) satisfies the equation $x + y = a - b = 3b - a$

Therefore,

$$\text{Probability} = \frac{\frac{1}{2} CA \times BA}{\frac{1}{2} OX \times OY} = \frac{\frac{1}{2}(3b - a)^2}{\frac{1}{2}a^2} = \frac{(3b - a)^2}{a^2}.$$

Your Turn 2

1. One bag contains 3 white and 2 black balls. A second bag contains 5 white and 3 black balls. If a bag is chosen at random and a ball is taken from it and it is found to be a white ball, what is the chance that it is taken from the first bag? **Ans.** $\frac{24}{49}$
2. 16 coins are tossed simultaneously. What is the chance of getting at least 14 heads? **Ans.** $\frac{137}{2^{16}}$
3. A dice is thrown 6 times; getting the face 6 up is considered a success. What is the probability that there are at least 3 successes? **Ans.** $\frac{2906}{6^6}$
4. If six throws are made with a pair of dice, then what is the chance of throwing doublets at least four times? **Ans.** $\frac{406}{6^6}$
5. Two real numbers x and y are selected such that $x^2 + y^2 \le 25$. Find the probability that the selected numbers satisfy $x^2 + y^2 \ge 4$. **Ans.** 21/25
6. Two numbers a and b are selected from the interval [0, 4]. Find the probability that $b \le a$. **Ans.** 1/2

28.14 Important Information

28.14.1 About Playing Cards

1. **(a)** A pack of 52 playing cards had 4 suits.
   (i) Clubs **(ii)** Hearts
   (iii) Diamonds **(iv)** Spades
   (b) Spades and clubs are **black**-faced cards.
   (c) Hearts and diamonds are **red**-faced cards.
   (d) Each suit consists of 13 cards.
   (e) The Aces, Kings, Queens, Jacks are called **face** cards or **honours** cards. These are four in number, one of each suit.
   (f) The Kings, Queens, Jacks are called **court** cards.
2. Game of Bridge: It is played by 4 players. Each player is given 13 cards.
3. Game of Whist: It is a game of cards played by two pairs of persons.

28.14.2 Divisibility Test

1. Divisibility by 2: A number is divisible by 2 if and only if its last digit is divisible by 2.
2. Divisibility by 3: A number is divisible by 3 if and only if the sum of its digit is divisible by 3.
3. Divisibility by 4: A number is divisible by 4 if its units' digit plus twice its ten's digit is divisible by 4.
4. Divisibility by 5: A number is divisible by 5 if and only if its unit's digit is divisible by 5 (i.e., if it ends in 0 or 5).
5. Divisibility by 6: A number is divisible by 6 if and only if its unit's digit is even and the sum of its digits divisible by 3.
6. Divisibility by 7: A number is divisible by 7 if and only if 3 × unit's digit + 2 × ten's digit –1 × hundred's digit –3 × thousand's digit –2 × ten thousand's digit + 1 × hundred thousand's digit is divisible by 7. If there are more digits present, the sequence of multipliers 3, 2, –1, –3, –2, 1 is repeated as often as necessary.

   Example: 3,201,828

   $$\begin{aligned} 3 \times 8 &= 24 \\ 2 \times 2 &= 4 \\ -1 \times 8 &= -8 \\ -3 \times 1 &= -3 \\ -2 \times 0 &= 0 \\ 1 \times 2 &= 2 \\ 3 \times 3 &= 9 \\ \hline \text{Total} &= 28 \text{ (7 divides 28)} \end{aligned}$$

   If the number has more than three digits, the following method is easier.

   An integer is divisible by 7 if and only if the difference of the number of its thousands and the remainder of its division by 1000 is divisible by 7 (i.e., the difference of the number formed by the last three digits from the number formed by the remaining digits is divisible by 7). For example:

   (i) The number 3,201,828 is divisible by 7, because the difference 3,201 – 828 = 2373, which is divisible by 7.

   (ii) The number 55246821 is divisible by 7,
   if 55246 – 821 = 54425 is divisible by 7,
   if 54 – 425 = –371 is divisible by 7 which is true.
7. Divisibility by 8: A number is divisible by 8 if and only if its unit's digit + 2 × ten's digit + 4 × hundred's is divisible by 8.
8. Divisibility by 9: A number is divisible by 9 if and only if the sum of its digits divisible by 9.
9. Divisibility by 10: A number is divisible by 10 if and only if the last digit is 0.
10. Divisibility by 11: A number is divisible by 11 if and only if its unit's digit –its ten's digit + its hundred's digit – its thousand's digit, divisible by 11.
11. Divisibility by 12: Test for divisibility by 4 and 3.
12. Divisibility by 13: A number is divisible by 13 if and only if 10 × unit's digit – 4 × ten's digit –1 × hundred's digit + 3 × thousand's digit + 4 × ten thousand's digit + 1 × hundred thousand's digit is divisible by 13. If there are more digits present, the sequence of multipliers 10, –4, –1, 3, 4, 1 is repeated as often as necessary. **Example:**

   3,046,583,852

   $$\begin{aligned} 10 \times 2 &= 20 \\ -4 \times 5 &= -20 \\ -1 \times 8 &= -8 \\ -3 \times 3 &= -9 \\ 4 \times 8 &= 32 \\ 1 \times 5 &= 5 \\ 10 \times 6 &= 60 \\ -4 \times 4 &= -16 \end{aligned}$$

$$\begin{aligned} -1\times 0 &= 0 \\ 3\times 3 &= 9 \\ \hline \text{Total} &= 91 \text{ (13 divides 91)} \end{aligned}$$

We illustrate the proof of these tests by demonstrating one case, that of three digit numbers and the divisor 7. Let N be a three digit number with digits a, b, and c. Then,

$$N = 100a + 10b + c$$

From the sum required by the test call it S; then,

$$2S = -2a + 4b + 6c$$

and $$N + 2S = 98a + 14b + 7c = 7(14a + 2b + c)$$

The sum $N + 2S$ is, therefore a multiple of 7, say $7M$. Now if N is multiple of 7, say $7P$, then $2S = 7M - 7P = 7(M - P)$, and it follows from this that S must also be divisible by 7. If conversely, S is a multiple of 7, say $7Q$, then

$$N = 7M - 14Q = 7(M - 2Q)$$

This tells us that N must be a multiple of 7.

Note: If a number has more than three digits then the criteria of divisibility by 7 can be applied for divisibility by 11 as well as 13 in exactly the same manner.

For example, the number 3046583852 is divisible by 13

1. If $3046583 - 852 = 3045731$ is divisible by 13.
2. If $3045 - 731 = 2314$ is divisible by 13.
3. If $2 - 314 = -312$ is divisible by 13 which is true.

Additional Solved Examples

1. A bag 'A' contains 3 white and 2 black balls. A bag 'B' contains 2 white and 4 black balls. First a bag is chosen and then a ball is drawn. What is the probability that it is white?

Solution: Probability of taking the first bag and then a white ball from it is

$$= \frac{1}{2}\cdot\frac{3}{5} = \frac{3}{10}$$

Probability of taking the 2nd bag and drawing a white ball from it is

$$= \frac{1}{2}\cdot\frac{2}{6} = \frac{1}{6}$$

The two events are exclusive.

Probability of one of them happening is

$$P(A \cup B) = P(A) + P(B) = \frac{3}{10} + \frac{1}{6} = \frac{7}{15}$$

Note: Since the events are mutually exclusive,

$$A \cap B = \phi \text{ and } P(A \cap B) = 0$$

2. A bag contains 2 black, 4 white and 3 red balls. One ball is drawn at random and kept aside. Then another ball is drawn and is also kept aside. This process is continued till all are drawn. Find the probability that the balls drawn are in the sequence 2 black, 4 white and 3 red.

Solution: The probability required is

= Product of the individual probabilities of the balls drawn in the given sequence

$$= \frac{2}{9}\cdot\frac{1}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}\times 1\times 1\times 1 = \frac{1}{1260}$$

3. There are two groups of subjects, one of which consists of 5 Science and 3 Engineering subjects; and the other consists of 3 Science and 5 Engineering subjects. An unbiased dice is cast. If number 3 or number 5 turns up; a subject is selected at random from the first group, otherwise a subject is selected from the 2nd group. Find the probability that an Engineering subject is selected ultimately.

Solution: If an Engineering subject selected is to come from the 1st group, for which the probability is $3/8$; first the group must be selected. For this we should get 3 or 5 in the dice. The probability of getting a 3 or 5 is

$$\frac{2}{6} = \frac{1}{3}$$

Therefore, probability that the Engineering subject finally selected is selected from the first group is

$$= \frac{1}{3}\cdot\frac{3}{8} = \frac{1}{8}$$

On a similar reasoning, the probability that the Engineering subject finally selected is selected from the 2nd group is

$$\frac{2}{3}\cdot\frac{5}{8} = \frac{5}{12}$$

$$\text{Required probability} = \frac{1}{8} + \frac{5}{12} = \frac{13}{24}$$

4. Each packet of certain items contains a coupon, which is equally likely to bear the letters A, N, S, H or U. If m packets are purchased, find the probability that the coupons cannot be used to spell ANSHU.

Solution: Let

E_1 be the event that A is not present,
E_2 be the event that N is not present,
E_3 be the event that S is not present,
E_4 be the event that H is not present,
E_5 be the event that U is not present,

Then

$$\text{Required probability} = \frac{\text{Number of favourable cases}}{\text{Total number of cases}}$$

$$= \frac{{}^5C_1\cdot 4^m - {}^5C_2\cdot 3^m + {}^5C_3\cdot 2^m - {}^5C_4\cdot 1^m + 0}{5^m}$$

5. A black dice and a red dice are rolled. Find the probability that

(i) The sum of their scores is divisible by 5.
(ii) The sum of their scores is 8; given that at least one dice shows a 3 or 4.

Solution: In any one case, the total number of ways of combining the six faces of one with the six faces of the other is 36. Now with regard to favourable cases.

(i) The sum of their scores is divisible by 5. Such ordered pairs are (1, 4); (2, 3); (3, 2); (4, 1); (5, 5); (4, 6); (6, 4).

Hence, required probability is $7/36$.

(ii) The sum of their scores is 8 given that at least one has a 3 or 4. The possible cases are

Red	Black	No. of cases
3	1, 2, 3, 4, 5, or 6	6
4	1, 2, 3, 4, 5, or 6	6
1, 2, 5, or 6	3	4
1, 2, 5, or 6	4	4

Thus, there are totally 20 cases of which there are three cases having sum 8 and atleast one die shows a 3 or 4, that is, (3, 5), (4, 4), (5, 3)
Hence,

$$\text{Required probability } = \frac{3}{20}$$

6. A set of 3 numbers are chosen from the set of numbers 1, 2, 3, $\cdots$, $(2n + 1)$. What is the probability that the numbers chosen are in A.P.?

Solution: Any three can be chosen in ${}^{(2n+1)}C_3$ ways. Regarding the favourable cases; if three numbers a, b, c chosen are to be in AP then $a + c = 2b$. Thus, the sum of the extremes being $2b$ should be always even. The two extreme numbers chosen are both odd there are $(n + 1)$ odd numbers or both even there are n even numbers.

Required probability is

$$\left({}^{(n+1)}C_2 + {}^{n}C_2\right)/\ {}^{(2n+1)}C_3 = \frac{\frac{(n+1)\,n}{2} + \frac{n(n-1)}{2}}{\frac{(2n+1)\,2n(2n-1)}{6}}$$

$$= \frac{n}{2}\cdot\frac{2n\,.\,6}{2n(4n^2-1)} = \frac{3n}{(4n^2-1)}$$

7. If the letters of the word REGULATIONS be arranged such that only R and E can change places and the rest have same order among themselves, what is the chance that there are exactly four letters between R and E?

Solution: There are 11 places corresponding to 11 letters and in these 11 places R and E can be arranged in ${}^{11}p_2$ ways.
Total possible cases where only 4 letters are between R and E

$RxxxxExxxxx$
$xRxxxxExxxx$
$xxRxxxxExxx$
$xxxRxxxxExx$
$xxxxRxxxxEx$
$xxxxxRxxxxE$

Reverse the order of R and E, so total favourable outcomes is 12.

$$\text{Probability } = \frac{2\times 6}{{}^{11}P_2} = \frac{2\times 6}{11\cdot 10} = \frac{6}{55}$$

Note: The working is based on the relative positions of R and E. Other letters do not play any part.

8. A and B cut a pack of cards alternately and every time the card is put back, the pack is well-shuffled after each cut. A start the game and the one who cuts a spade first win. What are the respective probabilities of winning?

Solution: Chance of cutting a spade is $\frac{1}{4}$ and the chance of not cutting a spade is $\frac{3}{4}$.

***A* starts**	**Probability**
A cuts a spade and wins	$1/4$
A loses in the first attempt, B also does. A wins in the second attempt	$\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}$
A loses in the first attempt and also in his second attempt and wins only in his third attempt	$\left(\frac{3}{4}\cdot\frac{3}{4}\right)\left(\frac{3}{4}\cdot\frac{3}{4}\right)\frac{1}{4}$

$\cdots$ this infinite sequence continues.

Hence,

$$\text{A's probability } = \frac{1}{4}\left\{1 + (3/4)^2 + (3/4)^4 + \ldots\infty\right\}$$

$$= \frac{1}{4}\cdot\frac{1}{1-\frac{9}{16}} = \frac{4}{7}$$

Therefore,

$$B\text{'s probability of win } = 1 - \frac{4}{7} = \frac{3}{7}$$

9. Cards are dealt one by one from a well-shuffled pack until an ace appears if cards are not replaced. Find the probability that exactly n cards are dealt before the first appears.

Solution: The question envisages that in the first n draws there is no ace at all. Hence, the probability of this happening $= \dfrac{{}^{48}C_n}{{}^{52}C_n}$.

The $(n + 1)^{\text{th}}$ draw is an ace, for which the probability is $\dfrac{4}{52-n}$.

Hence,

$$\text{required probability } = \frac{{}^{48}C_n}{{}^{52}C_n}\cdot\frac{4}{52-n}$$

$$= \frac{48!\,n!\,52-n!}{n!\,48-n!\,52!}\cdot\frac{4}{52-n} = \frac{4.(49-n)(50-n)(51-n)}{49\cdot 50\cdot 51\cdot 52}$$

10. p is the probability that a man aged x years will die in a year. Find the probability that out of n men $A_1, A_2, \cdots, A_n$ each aged x years, A_1 will die in a year and will be the first to die.

Solution: Let E_1 be the event that A_1 dies in a year.

$$P(E_1) = p$$

The probability that none of the n persons die in the year is $(1-p)^n$.
Probability that, at least, one of $A_1, A_2, \cdots, A_n$ dies in a year is

$$1-(1-p)^n$$

Probability that among them, A_1 is the one is $1/n$. The probability that A_1 is the one who dies first is $\frac{1}{n}[1-(1-p)^n]$.

11. The chance of one event happening is the square of the chance of another event happening. The odds against the first are the cubes of the odds against the second. Find the chance of each.

Solution: Let x be the chance of the first happening and y that of second.
Probability that the first does not happen is $1 - x$. Probability that the second does not happen is $1 - y$.
By the question,

(i) $x = y^2$

(ii) $\frac{1-x}{x} = \left(\frac{1-y}{y}\right)^3$

Hence,

$$\frac{1-y^2}{y^2} = \left(\frac{1-y}{y}\right)^3 \Rightarrow 1+y = \frac{(1-y)^2}{y} \Rightarrow y + y^2 = y^2 - 2y + 1$$

for $y \neq 1$

Therefore, $3y = 1$ or $y = \frac{1}{3}$ and hence $x = \frac{1}{9}$.

Hence, the chance of the first happening is $\frac{1}{9}$.

Chance of the second happening is $\frac{1}{3}$.

If we consider the case $y = 1$, then $x = 1$, which is true for sure event.

12. If A, B, C are events such that $P(A) = 0.3$; $P(B) = 0.4$; $P(C) = 0.8$; $P(AB) = 0.08$; $P(AC) = 0.28$; $P(ABC) = 0.09$ and if $P(A \cup B \cup C) \geq 0.75$, then show that $P(BC)$ lies in the interval $0.23 \leq x \leq 0.48$.

Solution: Let $P(BC) = x$. Then, $P(A \cup B \cup C) \geq 0.75$

$$\Rightarrow P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC) \geq 0.75$$

$$\Rightarrow 0.3 + 0.4 + 0.8 - 0.08 - 0.28 - x + 0.09 \geq 0.75$$

$$\Rightarrow x \leq 0.48$$

Also $P(A \cup B \cup C) \leq 1$,

$$\Rightarrow 0.3 + 0.4 + 0.8 - 0.08 - 0.28 - x + 0.09 \leq 1$$

$x \geq 1.5 - 1 + 0.09 - 0.36$ *i.e.*, $x \geq 0.23$

Therefore, $0.23 \leq x \leq 0.48$.

13. Bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find probability that it was drawn from bag B.

Solution: Let the event of drawing a red ball be denoted by L and the event of choosing bag A be denoted by M and the event of choosing bag B be denoted by N.

We have

$$P(M) = P(N) = \frac{1}{2}$$

Now

$P(L/M) =$ Probability of drawing a red ball from bag $A = \frac{3}{5}$ and $P(L/N) = \frac{5}{9}$

Required probability $= P(N/L)$

$$= \frac{P(N) \cdot P(L/N)}{P(N) \cdot P(L/N) + P(M) \cdot P(L/M)} \quad \text{(Bayes' formula)}$$

$$= \frac{\frac{1}{2} \cdot \frac{5}{9}}{\frac{1}{2} \cdot \frac{5}{9} + \frac{1}{2} \cdot \frac{3}{5}} = \frac{25}{52}$$

Note: The required probability p may be easily found without any elaborate procedure.

The probability of choosing bag B, and taking a red ball from it

$$P = \frac{\text{Taking a red ball from it}}{\text{The probability of choosing either bag}}$$

$$= \frac{\frac{1}{2} \cdot \frac{5}{9}}{\frac{1}{2} \cdot \frac{5}{9} + \frac{1}{2} \cdot \frac{3}{5}} = \frac{25}{52}$$

14. Cards are drawn one by one at random from a well-shuffled full pack of 52 cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, then show that the probability of obtaining 2 aces when N is equal to n is $\frac{(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}$, where $2 \leq n \leq 50$.

Solution: Suppose $N = 2$, that is, the number of cards to be drawn is 2.
In this case, both the cards have to be aces. Hence,

$$\text{Probability} = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{13} \cdot \frac{1}{17}$$

Suppose $N = 3$
In this case, 2 aces and 1 non–ace have to be drawn.
It could $(NA)\, A\, A$ or $A\, (NA)\, A$

$$\text{and the probability } = \frac{48}{52} \cdot \frac{4}{51} \cdot \frac{3}{50} + \frac{4}{52} \cdot \frac{48}{51} \cdot \frac{3}{50} = \frac{48}{13 \cdot 17 \cdot 25}$$

Suppose $N = 4$

$$\text{Probability } = 3 \cdot \frac{48}{52} \times \frac{47}{51} \times \frac{4}{50} \times \frac{3}{49}$$

(In this case, the first ace can be in the first or second or third draw)

Similarly, where $N = n$, (n^{th} draw is a definite ace and the first ace can be in any one of the first $(n - 1)$ draws.

$$\text{Probability } = (n-1) \cdot \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdots \frac{48-(n-3)}{52-(n-3)} \times \frac{4}{54-n} \times \frac{3}{53-n}$$

$$= \frac{(n-1)(52-n)(51-n)}{52 \cdot 51 \cdot 50 \cdot 49} \times 4 \times 3 = \frac{(n-1)(52-n)(51-n)}{50 \cdot 49 \cdot 13 \cdot 17},$$

where $2 \leq n \leq 50$

Alternate Solution:

Probability of making n draws until two aces appear is

$$\left\{\frac{{}^{48}C_{n-2} \times {}^{4}C_1}{{}^{52}C_{n-1}}\right\} \times \frac{3}{53-n} = \frac{(n-1)(52-n)(51-n)}{50 \cdot 49 \cdot 13 \cdot 17}$$

The first factor is the probability of drawing $(n-2)$ non-aces and one ace in the first $(n-1)$ draws while the second factor is the probability of drawing an ace in the n^{th} draw.

15. Suppose A and B are two equally strong table tennis players. Which of the following two events is more probable?
(a) A beats B in exactly 3 games out of 4 or
(b) A beats B in exactly 5 games out of 8.

Solution: Since A and B are equally strong players, the probability that A beats B is $\frac{1}{2}$ and the probability that A loses to B is also $\frac{1}{2}$.

(a) A beats B in exactly three games out of 4.
In this case, A has to lose one game and win three for which the probability is

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3=\frac{1}{16}$$

But this one game which he loses could be any one of the four. The four events LWWW, WLWW, WWLW and WWWL are mutually exclusive and the probability for each one is $\frac{1}{16}$.
Hence, probability of winning exactly 3 games out of 4 is

$$4\times\frac{1}{16}=\frac{1}{4}$$

(b) A beats B in exactly 5 games out of 8.
In this case, A has to lose 3 games out of 8. These events of the type LLLWWWWW can occur in $^8C_3$ ways and the corresponding probability is

$$^8C_3\times\left(\frac{1}{2}\right)^8=\frac{7}{32}$$

Now,

$$\frac{1}{4}=\frac{8}{32}>\frac{7}{32}$$

Hence, the first event is more probable than the second.

Previous Years' Solved JEE Main/AIEEE Questions

1. A pair of fair dice is thrown independently three times. The probability of getting a score of exactly 9 twice is
(A) 1/729 **(B)** 8/9 **(C)** 8/729 **(D)** 8/243
[AIEEE 2007]

Solution: Probability of getting score 9 in a single throw is,

$$\frac{4}{36}=\frac{1}{9}$$

Probability of not getting score 9 in a single throw is,

$$1-\frac{1}{9}=\frac{8}{9}$$

Therefore, probability of getting score 9 exactly twice is

$$^3C_2\times\left(\frac{1}{9}\right)^2\times\left(\frac{8}{9}\right)^{3-2}=\frac{8}{243}$$

Hence, the correct answer is option (D).

2. Two aeroplanes I and II bomb a target in succession. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only if the first misses the target. The probability that the target is hit by the second plane is
(A) 0.06 **(B)** 0.14 **(C)** 0.3 **(D)** 0.7
[AIEEE 2007]

Solution:

$$P_1(\text{Hit})=0.3;\overline{P_1}(\text{Hit})=0.7$$
$$P_2(\text{Hit})=0.2;\overline{P_2}(\text{Hit})=0.8$$

Probability that the target is hit by the second plane is obtained as follows:

$$\overline{P}_1P_2+\overline{P_1\overline{P}_2}\,\overline{P}_1P_2+\overline{P_1\overline{P}_2\overline{P}_1\overline{P}_2}\,\overline{P}_1P_2+\cdots$$
$$=0.7\times0.2+(0.7)(0.8)(0.7)(0.2)+(0.7)(0.8)(0.7)(0.8)(0.2)+\cdots$$
$$=0.14[1+(0.56)+(0.56)^2+\cdots]$$
$$=0.14\left[\frac{1}{1-0.56}\right]=\frac{0.14}{0.44}=\frac{7}{22}\approx0.3$$

Hence, the correct answer is option (C).

3. It is given that the events A and B are such that $P(A)=\frac{1}{4},P\left(\frac{A}{B}\right)=\frac{1}{2}$ and $P\left(\frac{B}{A}\right)=\frac{2}{3}$. Then $P(B)$ is
(A) $\frac{1}{6}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{1}{2}$
[AIEEE 2008]

Solution:

$$P\left(\frac{A}{B}\right)=\frac{P(A\cap B)}{P(B)}=\frac{1}{2};P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}=\frac{2}{3}$$

Hence,

$$\frac{P(A)}{P(B)}=\frac{3}{4}$$

However, $P(A) = 1/4$, which implies that, $P(B)=\frac{1}{3}$.

Hence, the correct answer is option (B).

4. In a binomial distribution $B\left(n,\ p=\frac{1}{4}\right)$, if the probability of at least one success is greater than or equal to $\frac{9}{10}$, then n is greater than
(A) $\frac{1}{\log_{10}4-\log_{10}3}$ **(B)** $\frac{1}{\log_{10}4+\log_{10}3}$
(C) $\frac{9}{\log_{10}4-\log_{10}3}$ **(D)** $\frac{4}{\log_{10}4-\log_{10}3}$
[AIEEE 2009]

Solution: Probability of at least one success is

$$=P(X\ge1)\ge\frac{9}{10}$$
$$1-P(\text{no success})\ge\frac{9}{10}\Rightarrow1-{}^nC_r\left(\tfrac{3}{4}\right)^{n-0}\left(\tfrac{1}{4}\right)^0\ge\frac{9}{10}$$
$$\left(\frac{3}{4}\right)^n\le1-\frac{9}{10}=\frac{1}{10}\Rightarrow\left(\frac{3}{4}\right)^n\le\frac{1}{10}\Rightarrow n\log_{10}\left(\frac{3}{4}\right)\le-1$$
$$\Rightarrow n(\log_{10}3-\log_{10}4)\le-1$$
$$\Rightarrow n\ge\frac{-1}{(\log_{10}3-\log_{10}4)}=\frac{1}{(\log_{10}4-\log_{10}3)}$$

Hence, the correct answer is option (A).

5. One ticket is selected at random from 50 tickets numbered 00, 01, 02, ⋯, 49. Then the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equals

(A) $\frac{1}{14}$ **(B)** $\frac{1}{7}$ **(C)** $\frac{5}{14}$ **(D)** $\frac{1}{50}$

[AIEEE 2009]

Solution: We have,

$$S = \{00, 01, 02, \ldots, 49\}$$

Let A be the event that sum of the digits on the selected ticket is 8. Then,

$$A = \{08, 17, 26, 35, 44\}$$

Let B be the event that the product of the digits is zero. Then

$$B = \{00, 01, 02, 03, \ldots, 09, 10, 20, 30, 40\}$$

Therefore,

$$A \cap B = \{8\}$$

Therefore, the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equals,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/50}{14/50} = \frac{1}{14}$$

Hence, the correct answer is option (A).

6. Four numbers are chosen at random (without replacement) from the set {1, 2, 3, ..., 20}.

Statement-1: The probability that the chosen numbers when arranged in some order will form an AP is $\frac{1}{85}$.

Statement-2: If the four chosen numbers from an AP, then the set of all possible values of common difference is {±1, ±2, ±3, ±4, ±5}.

(A) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation for Statement-1

(B) Statement-1 is true, Statement-2 is false

(C) Statement-1 is false, Statement-2 is true

(D) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation for Statement-1

[AIEEE 2010]

Solution:

$$N(S) = {}^{20}C_4$$

Statement-1: Common difference is 1;

Therefore, possible cases are $\{\{1,2,3,4\},\ldots,\{17,18,19,20\}\}$.

So, total number of cases = 17.

Common difference is 2;

Therefore, possible cases are $\{\{1,3,5,7\},\ldots,\{14,16,18,20\}\}$.

So, total number of cases = 14.

Similarly, if common difference is 3; total number of cases = 11

If common difference is 4; total number of cases = 8

If common difference is 5; total number of cases = 5

If common difference is 6; total number of cases = 2

$$\text{Probability} = \frac{17+14+11+8+5+2}{{}^{20}C_4} = \frac{1}{85}$$

Statement-2: Four numbers from {1, 2, ... 20} can be chosen with common difference ±6 also, that is, the numbers can be 1, 7,13, 19. Hence, Statement-2 is false.

Hence, the correct answer is option (B).

7. An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colour is

(A) $\frac{2}{7}$ **(B)** $\frac{1}{21}$ **(C)** $\frac{2}{23}$ **(D)** $\frac{1}{3}$

[AIEEE 2010]

Solution:

$$n(S) = {}^9C_3 \Rightarrow n(E) = {}^3C_1 \times {}^4C_1 \times {}^2C_1$$

The probability that the three balls have different colour is calculated as

$$\frac{{}^3C_1 \times {}^4C_1 \times {}^2C_1}{{}^9C_3} = \frac{3\times4\times2}{{}^9C_3} = \frac{24\times3!}{9!}\times6! = \frac{24\times6}{9\times8\times7} = \frac{2}{7}$$

Hence, the correct answer is option (A).

8. Consider 5 independent Bernoulli's trials each with probability of success p. If the probability of at least one failure is greater than or equal to $\frac{31}{32}$, then p lies in the interval

(A) $\left[\frac{3}{4},\frac{11}{12}\right]$ **(B)** $\left[0,\frac{1}{2}\right]$ **(C)** $\left[\frac{11}{12},1\right]$ **(D)** $\left[\frac{1}{2},\frac{3}{4}\right]$

[AIEEE 2011]

Solution:

$$n = 5; \text{Success} = p; \text{Failure} = q$$

We have,

$$P(\text{at least one failure}) \geq \frac{31}{32} \Rightarrow 1 - P(\text{no failure}) \geq \frac{31}{32}$$

$$\Rightarrow 1 - {}^5C_r q^{5-r} p^r \geq \frac{31}{32}$$

For all 5 successes, $r = 5$. So,

$$1 - {}^5C_5 q^{5-5} p^5 \geq \frac{31}{32} \Rightarrow 1 - p^5 \geq \frac{31}{32} \Rightarrow p^5 \leq \frac{1}{32}$$

$$\Rightarrow p^5 \leq \frac{1}{(2)^5} \Rightarrow p \leq \frac{1}{2}$$

Therefore,

$$p \in \left[0, \frac{1}{2}\right]$$

Hence, the correct answer is option (B).

9. If C and D are two events such that $C \subset D$ and $P(D) \neq 0$, then the correct statement among the following is

(A) $P(C|D) \geq P(C)$ **(B)** $P(C|D) < P(C)$

(C) $P(C|D) \geq \frac{P(D)}{P(C)}$ **(D)** $P(C|D) = P(C)$

[AIEEE 2011]

Solution: We have,

$$C \cap D = C \Rightarrow P(C \cap D) = P(C) \Rightarrow P\left(\frac{C}{D}\right) = \frac{P(C \cap D)}{P(D)} \geq P(C)$$

Hence, the correct answer is option (A).

10. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is

(A) $\frac{13}{3^5}$ **(B)** $\frac{11}{3^5}$ **(C)** $\frac{10}{3^5}$ **(D)** $\frac{17}{3^5}$

[JEE MAIN 2013]

Solution: We have,

p = Correct answer = 1/3; q = Incorrect answer = 2/3

Therefore, probability of either 4 or 5 correct answers is

$$^5C_4\left(\frac{1}{3}\right)^{5-1}\left(\frac{2}{3}\right)^1+{}^5C_5\left(\frac{1}{3}\right)^{5-0}={}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1+{}^5C_5\left(\frac{1}{3}\right)^5$$

$$=\frac{5\times 2}{(3)^5}+\frac{1}{(3)^5}=\frac{11}{3^5}$$

Hence, the correct answer is option (B).

11. Let A and B be two events such that $P(\overline{A\cup B})=\frac{1}{6}, P(A\cap B)=\frac{1}{4}$ and $P(\bar{A})=\frac{1}{4}$, where $\bar{A}$ stands for the complement of the event A. Then the events A and B are

(A) Independent but not equally likely
(B) Independent and equally likely
(C) Mutually exclusive and independent
(D) Equally likely but not independent

[JEE MAIN 2014 (OFFLINE)]

Solution:

$$P(\overline{A\cup B})=\frac{1}{6}\Rightarrow P(A\cup B)=1-\frac{1}{6}=\frac{5}{6}$$

$$P(A\cap B)=\frac{1}{4} \qquad (1)$$

$$P(\bar{A})=\frac{1}{4}\Rightarrow P(A)=1-\frac{1}{4}=\frac{3}{4}$$

Now,

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\frac{5}{6}=\frac{3}{4}+P(B)-\frac{1}{4} \text{ or } P(B)=\frac{5}{6}-\frac{3}{4}+\frac{1}{4}=\frac{10-9+3}{12}=\frac{4}{12}=\frac{1}{3}$$

$$P(A)\times P(B)=\frac{\cancel{3}}{4}\times\frac{1}{\cancel{3}}=\frac{1}{4} \qquad (2)$$

Therefore, from Eqs. (1) and (2), A and B are independent.

Since $P(A)\neq P(B)$, therefore not equally likely.

Hence, the correct answer is option (A).

12. If A and B are two events such that $P(A\cup B)=P(A\cap B)$, then the incorrect statement amongst the following statements is

(A) A and B are equally likely **(B)** $P(A\cap B')=0$
(C) $P(A'\cup B)=0$ **(D)** $P(A)+P(B)=1$

[JEE MAIN 2014 (ONLINE SET–1)]

Solution : Given

$$P(A\cup B)=P(A\cap B)$$
$$\Rightarrow P(A)+P(B)-P(A\cap B)=P(A\cap B)$$
$$\Rightarrow P(A)+P(B)=2P(A\cap B) \qquad (1)$$

Let us assume

$$P(A)=P(B)=P(A\cap B)$$

Now $P(A\cap B)\leq P(A)$ and $P(A\cap B)\leq P(A)$

If we assume $P(A\cap B)<P(A)$, then,

$$2P(A\cap B)=P(A\cap B)>P(A\cap B)<P(A)+P(B)$$

which contradicts (1).

Therefore, our assumption is wrong!

Thus

$$P(A)=P(B)=P(A\cap B)\Rightarrow P(A)=P(B)$$

Therefore, (A) is correct, (B) and (C) are also correct due to A and B being equally likely. Thus, (D) is the only option left.

Hence, the correct answer is option (D).

13. A set S contains 7 elements. A non-empty subset A of S and an element x of S are chosen at random. Then the probability that $x\in A$ is

(A) $\frac{1}{2}$ **(B)** $\frac{64}{127}$
(C) $\frac{63}{128}$ **(D)** $\frac{31}{128}$

[JEE MAIN 2014 (ONLINE SET–2)]

Solution: See Fig. 28.10.

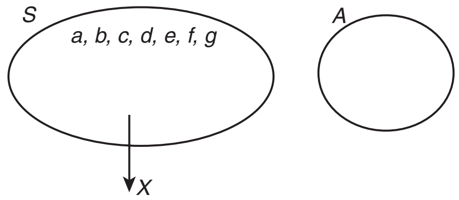


Figure 28.10

A can be selected in 2^7-1 ways.

Hence, number of non–empty subsets of S can be chosen in

$$^7C_1+{}^7C_2+\cdots+{}^7C_7=2^7-1 \text{ ways}$$

Now any x if not included in A can happen in 2^6-1 ways (Since sets made without x)

Therefore, number of sets which include x are

$$(2^7-1)-(2^6-1)=2^6(2-1)=2^6=64 \text{ ways}$$

Hence, probability that $x\in A$ is

$$\frac{64}{2^7-1}=\frac{64}{127}$$

Hence, the correct answer is option (B).

14. If X has a binomial distribution, $B(n, p)$ with parameters n and p such that $P(X=2)=P(X=3)$, then $E(X)$, the mean of variable X, is

(A) $2-p$ **(B)** $3-p$
(C) $\frac{p}{2}$ **(D)** $\frac{p}{3}$

[JEE MAIN 2014 (ONLINE SET–2)]

Solution: Since, $P(x=2)=P(X=3)$; and ${}^nC_r(1-p)^{n-r}p^r=P(X=r)$

Therefore,

$$^nC_2(1-p)^{n-2}p^2={}^nC_3(1-p)^{n-3}p^3$$

$$\Rightarrow(1-p)^{\cancel{n}-2-\cancel{n}+3}p^{-1}=\frac{^nC_3}{^nC_2}$$

$$=\frac{\cancel{n}(\cancel{n-1})(n-2)}{3\times\cancel{2}\times1}\times\frac{2\times1}{\cancel{n}(\cancel{n-1})}$$

$$\Rightarrow\frac{(1-p)}{p}=\frac{n-2}{3}\Rightarrow 3-3p=np-2p$$

$$\Rightarrow 3=np+p\Rightarrow np=3-p,\ \therefore\ E(X)=np=3-p$$

Hence, the correct answer is option (B).

15. A number x is chosen at random from the set {1, 2, 3, 4, ⋯, 100}. Define the event: A = the chosen number x satisfies $\frac{(x-10)(x-50)}{(x-30)}\geq 0$. Then $P(A)$ is

(A) 0.71 **(B)** 0.70 **(C)** 0.51 **(D)** 0.20

[JEE MAIN 2014 (ONLINE SET–3)]

Solution: See Fig. 28.11.

$$\frac{(x-10)(x-50)}{(x-30)}\geq 0$$

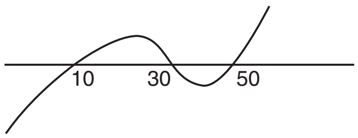


Figure 28.11

By wavy curve method, the solution is

$$\{10, 11, 12, \cdots 29\}\cup\{50, 51, \cdots 100\}$$

Since, $x\neq 30$, therefore, $n(A)=20+51=71$, therefore,

$$P(A)=\frac{n(A)}{n(\text{Given set})}=\frac{71}{100}=0.71$$

Hence, the correct answer is option (A).

16. Let A and E be any two events with positive probabilities

Statement-1: $P(E/A)\geq P(A/E)P(E)$

Statement-2: $P(A/E)\geq P(A\cap E)$.

(A) Both the statements are true

(B) Both the statements are false

(C) Statement-1 is true, Statement-2 is false

(D) Statement-1 is false, Statement-2 is true

[JEE MAIN 2014 (ONLINE SET–4)]

Solution: We know that

$$P(E/A)=\frac{P(E\cap A)}{P(A)} \quad (1)$$

$$P(A/E)=\frac{P(A\cap E)}{P(E)} \quad (2)$$

Dividing Eq. (1) by Eq. (2), we get

$$\frac{P(E/A)}{P(A/E)}=\frac{P(E)}{P(A)}$$

Now since $0<P(A)\leq 1$. Therefore,

$$\frac{P(E/A)}{P(A/E)}\geq P(E)\Rightarrow P(E/A)\geq P(A/E)\,P(E)$$

Therefore, Statement-1 is true.

Now since again $0<P(E)\leq 1$ thus, from (2), $P(A/E)\geq P(A\cap E)$ is true.

Hence, the correct answer is option (A).

17. If 12 identical balls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls is

(A) $55\left(\frac{2}{3}\right)^{10}$ **(B)** $220\left(\frac{1}{3}\right)^{12}$ **(C)** $22\left(\frac{1}{3}\right)^{11}$ **(D)** $\frac{55}{3}\left(\frac{2}{3}\right)^{11}$

[JEE MAIN 2015 (OFFLINE)]

Solution: Choose 3 balls out of 12 in ${}^{12}C_3$ ways and distribute the remaining 9 balls in two boxes in 2^9 ways. However,

$$\text{total number of possible ways}=(3)^{12}$$

Therefore,

$$\text{required probability }=\frac{{}^{12}C_3(2)^9}{(3)^{12}}$$

$$=\frac{12\times11\times10}{6}\times\frac{2^9}{3^{12}}=\frac{55}{3}\left(\frac{2}{3}\right)^{11}$$

Hence, the correct answer is option (D).

Note: This answer is correct only when the boxes are different and not identical.

18. Let X be a set containing 10 elements and $P(X)$ be its power set. If A and B are picked up at random from $P(X)$, with replacement, then the probability that A and B have equal number of elements, is

(A) $\frac{{}^{20}C_{10}}{2^{10}}$ **(B)** $\frac{(2^{10}-1)}{2^{20}}$

(C) $\frac{(2^{10}-1)}{2^{10}}$ **(D)** $\frac{{}^{20}C_{10}}{2^{20}}$

[JEE MAIN 2015 (ONLINE SET–1)]

Solution:

$$n(x)=10,\ n(P(x))=(2)^{10}=1024$$

$$A,B\in P(x)$$

$$n(\text{sample space})=2^{10}\times 2^{10}$$

$$n(E)=(1\times1+{}^{10}C_1\times{}^{10}C_1+{}^{10}C_2\times{}^{10}C_2+{}^{10}C_3\times{}^{10}C_3+\cdots+{}^{10}C_{10}\times{}^{10}C_{10})$$

$$=({}^{10}C_0)^2+({}^{10}C_1)^2+({}^{10}C_2)^2+\cdots+({}^{10}C_{10})^2={}^{20}C_{10}$$

Hence,

$$P(E)=\frac{{}^{20}C_{10}}{2^{10}\times2^{10}}=\frac{{}^{20}C_{10}}{2^{20}}$$

Hence, the correct answer is option (D).

19. If the lengths of the sides of a triangle are decided by the three throws of a single fair dice, then the probability that the triangle is of maximum area given that it is an isosceles triangle is

(A) $\frac{1}{26}$ **(B)** $\frac{1}{27}$ **(C)** $\frac{1}{21}$ **(D)** $\frac{1}{15}$

[JEE MAIN 2015 (ONLINE SET–2)]

Solution: Favourable case: all sides (6, 6, 6)

Total number of cases by $a+b>c$

{(1, 1, 1) (2, 2, 1), (2, 2, 2), (2, 2, 3)(3, 3, 1)… (3, 3, 5) (4, 4, 1) … (4, 4, 6) (5, 5, 1) … (5, 5, 6) (6, 6, 1)… (6, 6, 6)} = 27

Hence, probability $=\frac{1}{27}$

Hence, the correct answer is option (B).

20. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that dice A shows up four, E_2 is the event that dice B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is NOT true?

(A) E_1, E_2 and E_3 are independent.
(B) E_1 and E_2 are independent.
(C) E_2 and E_3 are independent.
(D) E_1 and E_3 are independent.

[JEE MAIN 2016 (OFFLINE)]

Solution: We have

$E_1 \to$ Dice A shows 4
$E_2 \to$ Dice B shows 2
$E_3 \to$ Sum odd (No. on A to be odd + No. on B to be even or No. on B to be odd + No. on A to be even)

Therefore,

$$P(E_1) = \frac{1}{6}$$
$$P(E_2) = \frac{1}{6}$$
$$P(E_3) = \frac{{}^2C_1 \cdot {}^3C_1 \cdot {}^3C_1}{6 \cdot 6} = \frac{1}{2}$$

Now

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$
$$P(E_1 \cap E_3) = P(E_1) \cdot P(E_3)$$
$$P(E_2 \cap E_3) = P(E_2) \cdot P(E_3)$$

Thus, the events are independent.

Hence, the correct answer is option (A).

21. If A and B are any two events such that $P(A) = 2/5$ and $P(A \cap B) = 3/20$, then the conditional probability, $P(A|(A' \cup B'))$, where A' denotes the complement of A, is equal to

(A) 11/20 **(B)** 5/17 **(C)** 8/17 **(D)** 1/4

[JEE MAIN 2016 (ONLINE SET–1)]

Solution: We have

$$P(A) = \frac{2}{5},\ P(A \cap B) = \frac{3}{20},\ P\left(\frac{A}{A' \cup B'}\right) = ?$$

Now,

$$P(A \cap B) + P(A \cap B)' = 1$$
$$P(A \cap B)' = 1 - \frac{3}{20} = \frac{17}{20}$$
$$P(A' \cup B') = \frac{17}{20}$$

Therefore,

$$P\left(\frac{A}{A' \cup B'}\right) = \frac{P(A \cap (A' \cup B'))}{P(A' \cup B')}$$
$$= \frac{P(A) - P(A \cap B)}{P(A' \cup B')}$$
$$= \frac{(2/5) - (3/20)}{17/20} = \frac{5}{17}$$

Hence, the correct answer is option (B).

22. An experiment succeeds twice as often as it fails. The probability of at least five successes in the six trials of this experiment is

(A) $\frac{496}{729}$ **(B)** $\frac{192}{729}$ **(C)** $\frac{240}{729}$ **(D)** $\frac{256}{729}$

[JEE MAIN 2016 (ONLINE SET–2)]

Solution: The probability of at least 5 successes be p. Therefore,

$$p = 2(1 - p)$$
$$p = 2 - 2p \Rightarrow p = \frac{2}{3}$$
$$p = \frac{2}{3}, q = \frac{1}{3}$$

Therefore, the probability of at least 5 successes is

$${}^6C_5 p^5 q + {}^6C_6 p^6 = 6\left(\frac{2^5}{3^6}\right) + \left(\frac{2^6}{3^6}\right) = 3\left(\frac{2^6}{3^6}\right) + \left(\frac{2^6}{3^6}\right) = 4\left(\frac{2^6}{3^6}\right) = \frac{256}{729}$$

Hence, the correct answer is option (D).

Previous Years' Solved JEE Advanced/ IIT-JEE Questions

1. One Indian and four American men and their wives are to be seated randomly around a circular table. Then the conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife is

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{1}{5}$

[IIT-JEE 2007]

Solution: Fixing four American couple and one Indian man in between any two couples, we have five different ways in which his wife can be seated, of which 2 cases are favourable. Therefore, the required probability is 2/5.

Hence, the correct answer is option (C).

2. Let $H_1, \cdots, H_n$ be mutually exclusive and exhaustive events with $P(H_i) > 0$, $i = 1, 2, \cdots, n$. Let E be any other event with $0 < P(E) < 1$.

Statement-1: $P(H_i | E) > P(E | H_i) \cdot P(H_i)$ for $i = 1, 2, \cdots, n$.
because

Statement-2: $\sum_{i=1}^{n} P(H_i) = 1$.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True

[IIT-JEE 2007]

Solution:

Statement-1: If $P(H_i \cap E) = 0$ for some i, then

$$P\left(\frac{H_i}{E}\right) = P\left(\frac{E}{H_i}\right) = 0$$

If $P(H_i \cap E) \neq 0\ \forall\ i = 1, 2, 3, \ldots, n,$ then

$$P\left(\frac{H_i}{E}\right)=\frac{P(H_i\cap E)}{P(H_i)}\times\frac{P(H_i)}{P(E)}$$
$$=\frac{P(E/H_i)\times P(H_i)}{P(E)}$$
$$>P(E/H_i)\times P(H_i)$$

Hence, Statement-1 may not always be true.

Statement-2: Clearly, we can write as

$$H_1\cup H_2\cup H_3\cup\cdots\cup H_n=S$$
$$\Rightarrow P(H_1)+P(H_2)+\cdots+P(H_n)=1$$

Hence, Statement-2 is true.

Hence, the correct answer is option (D).

3. Let E^c denote the complement of an event E. Let E, F, G be pair-wise independent events with $P(G)>0$ and $P(E\cap F\cap G)=0$. Then $P(E^c\cap F^c\,|\,G)$ equals

(A) $P(E^c)+P(F^c)$ **(B)** $P(E^c)-P(F^c)$

(C) $P(E^c)-P(F)$ **(D)** $P(E)-P(F^c)$

[IIT-JEE 2007]

Solution: We have

$$P(E^c\cap F^c|G)=\frac{P(E^c\cap F^c\cap G)}{P(G)}$$
$$=\frac{P(G)-P(G\cap E)-P(G\cap F)+P(E\cap F\cap G)}{P(G)}$$
$$=\frac{P(G)-P(G)P(E)-P(G)P(F)}{P(G)}$$
$$=1-P(E)-P(F)$$
$$=P(E^c)-P(F)$$

Hence, the correct answer is option (C).

4. Consider the system of equations $ax+by=0$, $cx+dy=0$, where $a, b, c, d\in\{0, 1\}$.

Statement-1: The probability that the system of equations has a unique solution is 3/8.

Statement-2: The probability that the system of equations has a solution is 1.

(A) Statement-1 is True, Statement -2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement -1 is True, Statement -2 is True; Statement-2 is NOT a correct explanation for Statement-1

(C) Statement -1 is True, Statement -2 is False

(D) Statement -1 is False, Statement -2 is True

[IIT-JEE 2008]

Solution: We have,

$$ax+by=0$$
$$cx+dy=0$$

Since, the system of homogenous equation is always consistent and has a solution.

Therefore, statement–2 is true.

Now,

$$\Delta=\begin{vmatrix}a & b\\ c & d\end{vmatrix}\quad\text{and}\quad a,b,c,d\in\{0,1\}$$
$$=ad-bc$$

No. of ways of selecting a, b, c, d from the set {0, 1} is

$$2\times2\times2\times2=16$$

If the system has unique solution, then $\Delta\neq0$

$\Rightarrow$ Either $ad=1$, $bc=0$ or $ad=0$, $bc=1$

$\Rightarrow$ Favourable cases $=6$

Therefore, probability that system of equation has unique solution is $\frac{6}{16}=\frac{3}{8}$.

Hence, the correct answer is option (B).

Paragraph for Questions 5–7: A fair dice is tossed repeatedly until a six is obtained. Let X denote the number of tosses required.

[IIT-JEE 2009]

5. The probability that $X=3$ equals

(A) $\frac{25}{216}$ **(B)** $\frac{25}{36}$ **(C)** $\frac{5}{36}$ **(D)** $\frac{125}{216}$

Solution:

$$P(X=3)=\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\frac{1}{6}=\frac{25}{216}$$

Hence, the correct answer is option (A).

6. The probability that $X\geq3$ equals

(A) $\frac{125}{216}$ **(B)** $\frac{25}{36}$ **(C)** $\frac{5}{36}$ **(D)** $\frac{25}{216}$

Solution:

$$P(X\leq2)=\frac{1}{6}+\frac{5}{6}\times\frac{1}{6}=\frac{11}{36}$$

$$\text{Required probability }=1-\frac{11}{36}=\frac{25}{36}$$

Hence, the correct answer is option (B).

7. The conditional probability that $X\geq6$ given $X>3$ equals

(A) $\frac{125}{216}$ **(B)** $\frac{25}{216}$ **(C)** $\frac{5}{36}$ **(D)** $\frac{25}{36}$

Solution: For $X\geq6$, the probability is

$$\frac{5^5}{6^6}+\frac{5^6}{6^7}+\cdots\infty=\frac{5^5}{6^6}\left(\frac{1}{1-5/6}\right)=\left(\frac{5}{6}\right)^5$$

For $X>3$

$$\frac{5^3}{6^4}+\frac{5^4}{6^5}+\frac{5^5}{6^6}+\cdots\infty=\frac{5^3}{6^4}\left[\frac{1}{1-\frac{5}{6}}\right]=\frac{5^3}{6^3}=\left(\frac{5}{6}\right)^3$$

Hence, the conditional probability is

$$\frac{(5/6)^5}{(5/6)^3}=\frac{25}{36}$$

Hence, the correct answer is option (D).

8. Let ω be a complex cube root of unity with $\omega\neq1$. A fair dice is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the dice, then the probability that $\omega^{r_1}+\omega^{r_2}+\omega^{r_3}=0$ is

(A) $\frac{1}{18}$ **(B)** $\frac{1}{9}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{1}{36}$

[IIT-JEE 2010]

Solution:

$$r_1, r_2, r_3 \in \{1,2,3,4,5,6\}$$

r_1, r_2, r_3 are of the form $3k$, $3k+1$, $3k+2$

$$\text{Required probability} = \frac{3!\times {}^2C_1\times {}^2C_1\times {}^2C_1}{6\times6\times6} = \frac{6\times8}{216} = \frac{2}{9}$$

Hence, the correct answer is option (C).

9. A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$, respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is

(A) $\frac{3}{5}$ **(B)** $\frac{6}{7}$

(C) $\frac{20}{23}$ **(D)** $\frac{9}{20}$

[IIT-JEE 2010]

Solution: Event G = original signal is green

E_1 = A receives the signal correct
E_2 = B receives the signal correct
E = signal received by B is green

P(signal received by B is green)

$$P(E) = P(GE_1E_2) + P(G\bar{E}_1\bar{E}_2) + P(\bar{G}E_1\bar{E}_2) + P(\bar{G}\bar{E}_1E_2)$$

$$= \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{3}{4} + \frac{4}{5}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4}\cdot\frac{3}{4}$$

$$P(E) = \frac{46}{5\times16}$$

$$P(G) = P(GE_1E_2) + P(G\bar{E}_1\bar{E}_2)$$

$$= \frac{4}{5}\cdot\frac{3}{4}\cdot\frac{3}{4} + \frac{4}{5}\cdot\frac{1}{5}\cdot\frac{1}{5}$$

$$= \frac{40}{5\times16}$$

Therefore,

$$P(G/E) = \frac{40/5\times16}{46/5\times16} = \frac{20}{23}$$

Hence, the correct answer is option (C).

Paragraph for Questions 10 and 11: Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.

[IIT-JEE 2011]

10. The probability of the drawn ball from U_2 being white is

(A) $\frac{13}{30}$ **(B)** $\frac{23}{30}$ **(C)** $\frac{19}{30}$ **(D)** $\frac{11}{30}$

Solution:

$H \to$ 1 ball from U_1 to U_2
$T \to$ 2 ball from U_1 to U_2
E: 1 ball drawn from U_2

$$P/W \text{ from } U_2 = \frac{1}{2}\times\left(\frac{3}{5}\times1\right) + \frac{1}{2}\times\left(\frac{2}{5}\times\frac{1}{2}\right) + \frac{1}{2}\times\left(\frac{{}^3C_2}{{}^5C_2}\times1\right)$$
$$+\frac{1}{2}\times\left(\frac{{}^2C_2}{{}^5C_2}\times\frac{1}{3}\right) + \frac{1}{2}\times\left(\frac{{}^3C_1\cdot{}^2C_1}{{}^5C_2}\times\frac{2}{3}\right) = \frac{23}{30}$$

Hence, the correct answer is option (B).

11. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is

(A) $\frac{17}{23}$ **(B)** $\frac{11}{23}$ **(C)** $\frac{15}{23}$ **(D)** $\frac{12}{23}$

Solution:

$$P\left(\frac{H}{W}\right) = \frac{P(W/H)\times P(H)}{P(W/T)\cdot P(T) + (W/H)\cdot P(H)} = \frac{\frac{1}{2}\left(\frac{3}{5}\times1 + \frac{2}{5}\times\frac{1}{2}\right)}{23/30} = \frac{12}{23}$$

Hence, the correct answer is option (D).

12. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denotes the probability of occurrence of the event T, then

(A) $P(E) = \frac{4}{5}, P(F) = \frac{3}{5}$ **(B)** $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$

(C) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$ **(D)** $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$

[IIT-JEE 2011]

Solution: Let,

$$P(E) = e \text{ and } P(F) = f$$

$$P(E\cup F) - P(E\cap F) = \frac{11}{25}$$

$$\Rightarrow e + f - 2ef = \frac{11}{25} \quad (1)$$

$$P(\bar{E}\cap\bar{F}) = \frac{2}{25}$$

$$\Rightarrow (1-e)(1-f) = \frac{2}{25}$$

$$\Rightarrow 1 - e - f + ef = \frac{2}{25} \quad (2)$$

From Eqs. (1) and (2), we have

$$ef = \frac{12}{25} \text{ and } e + f = \frac{7}{5}$$

Solving, we get

$$e = \frac{4}{5}, f = \frac{3}{5} \text{ or } e = \frac{3}{5}, f = \frac{4}{5}$$

Hence, the correct answers are options (A) and (D).

13. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event that the ship is operational and let X_1, X_2 and X_3 denote respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true?

(A) $P\left[X_1^c \middle| X\right]=\frac{3}{16}$

(B) P [Exactly two engines of the ship are functioning | X] $=\frac{7}{8}$

(C) $P\left[X \middle| X_2\right]=\frac{5}{16}$

(D) $P\left[X \middle| X_1\right]=\frac{7}{16}$

[IIT–JEE 2012]

Solution:

$$P(X_1)=\frac{1}{2}, P(X_2)=\frac{1}{4}, P(X_3)=\frac{1}{4}$$

$$P(X)=P(X_1\cap X_2\cap X_3^C)+P(X_1\cap X_2^C\cap X_3)$$
$$+P(X_1^C\cap X_2\cap X_3)+P(X_1\cap X_2\cap X_3)=\frac{1}{4}$$

(A) $P(X_1^C/X)=\frac{P(X\cap X_1^C)}{P(X)}=\frac{1/32}{1/4}=\frac{1}{8}$

(B) P [Exactly two engines of the ship are functioning | X] $=\frac{7/32}{1/4}=\frac{7}{8}$

(C) $P\left(\frac{X}{X_2}\right)=\frac{5/32}{1/4}=\frac{5}{8}$

(D) $P\left(\frac{X}{X_1}\right)=\frac{7/32}{1/2}=\frac{7}{16}$

Hence, the correct answers are options (B) and (D).

14. Four fair dice D_1, D_2, D_3 and D_4, each having six faces numbered 1, 2, 3, 4, 5, and 6, are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is

(A) $\frac{91}{216}$ **(B)** $\frac{108}{216}$ **(C)** $\frac{125}{216}$ **(D)** $\frac{127}{216}$

[IIT–JEE 2012]

Solution: Required probability is

$$1-\frac{6\cdot 5^3}{6^4}=1-\frac{125}{216}=\frac{91}{216}$$

Hence, the correct answer is option (A).

15. Let X and Y be two events such that $P\left(X \middle| Y\right)=\frac{1}{2}$, $P\left(Y \middle| X\right)=\frac{1}{3}$ and $P(X\cap Y)=\frac{1}{6}$. Which of the following is (are) correct?

(A) $P(X\cup Y)=\frac{2}{3}$

(B) X and Y are independent

(C) X and Y are not independent

(D) $P(X^C\cap Y)=\frac{1}{3}$

[IIT–JEE 2012]

Solution:

$$P\left(\frac{X}{Y}\right)=\frac{P(X\cap Y)}{P(Y)}=\frac{1}{2} \text{and} \frac{P(X\cap Y)}{P(X)}=\frac{1}{3}$$

$$P(X\cap Y)=\frac{1}{6}\Rightarrow P(Y)=\frac{1}{3} \text{and} P(X)=\frac{1}{2}$$

Clearly, X and Y are independent.

Also,

$$P(X\cup Y)=\frac{1}{2}+\frac{1}{3}-\frac{1}{6}=\frac{2}{3}$$

Hence, the correct answers are options (A) and (B).

16. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2},\frac{3}{4},\frac{1}{4},\frac{1}{8}$. Then the probability that the problem is solved correctly by at least one of them is

(A) $\frac{235}{256}$ **(B)** $\frac{21}{256}$ **(C)** $\frac{3}{256}$ **(D)** $\frac{253}{256}$

[JEE ADVANCED 2013]

Solution: Let us consider that

$$P(A)=\frac{1}{2}; \quad P(B)=\frac{3}{4};$$
$$P(C)=\frac{1}{4}; \quad P(D)=\frac{1}{8}.$$

Therefore,

$$\begin{aligned}P(A\cup B\cup C\cup D)&=1-\overline{P(A\cup B\cup C\cup D)}\\&=1-P(\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D})\\&=1-P(\bar{A})\,P(\bar{B})\,P(\bar{C})\,P(\bar{D})\\&=1-\frac{1}{2}.\frac{1}{4}.\frac{3}{4}.\frac{7}{8}\\&=1-\frac{21}{256}\\&=\frac{235}{256}\end{aligned}$$

Hence, the correct answer is option (A).

17. Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha-2\beta)p=\alpha\beta$ and $(\beta-3\gamma)p=2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1). Then $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3}=$ ______.

[JEE ADVANCED 2013]

Solution: We have

$$P(E_1)P(\bar{E}_2)P(\bar{E}_3)=\alpha \quad (1)$$

$$P(\bar{E}_1)P(E_2)P(\bar{E}_3)=\beta \quad (2)$$

$$P(\bar{E}_1)P(\bar{E}_2)P(E_3)=\gamma \quad (3)$$

$$P(\bar{E}_1)P(\bar{E}_2)P(\bar{E}_3)=\rho \quad (4)$$

Dividing Eq. (1) by Eq. (4), we get

$$\frac{P(E_1)}{P(\bar{E}_1)}=\frac{\alpha}{\rho}=\frac{\alpha}{\frac{\alpha\beta}{\alpha-2\beta}}=\frac{\alpha-2\beta}{\beta}$$

$$\Rightarrow \frac{P(\bar{E}_1)}{P(E_1)}=\frac{\beta}{\alpha-2\beta}$$

$$\Rightarrow \frac{1-P(E_1)}{P(E_1)}=\frac{\beta}{\alpha-2\beta}$$

$$\Rightarrow \frac{1}{P(E_1)}-1=\frac{\beta}{\alpha-2\beta}$$
$$\Rightarrow \frac{1}{P(E_1)}=\frac{\alpha-\beta}{\alpha-2\beta}$$
$$\Rightarrow P(E_1)=\frac{\alpha-2\beta}{\alpha-\beta} \quad (5)$$

Also

$$\frac{\alpha\beta}{\alpha-2\beta}=\frac{2\beta\gamma}{\beta-3\gamma}$$
$$\Rightarrow \frac{\alpha}{\alpha-2\beta}=\frac{2\gamma}{\beta-3\gamma}$$
$$\Rightarrow \alpha\beta-3\gamma\alpha=2\gamma\alpha-4\beta\gamma$$
$$\Rightarrow \alpha\beta=5\gamma\alpha-4\beta\gamma$$
$$\gamma=\frac{\alpha\beta}{5\alpha-4\beta} \quad (6)$$

Dividing Eq. (3) by Eq. (4), we get

$$\frac{P(E_3)}{P(\bar{E}_3)}=\frac{\gamma}{\rho}=\frac{\gamma}{\frac{\alpha\beta}{\alpha-2\beta}}=\frac{\gamma(\alpha-2\beta)}{\alpha\beta}$$
$$\frac{P(E_3)}{P(\bar{E}_3)}=\frac{\alpha-2\beta}{5\alpha-4\beta}$$
$$\frac{P(\bar{E}_3)}{P(E_3)}=\frac{5\alpha-4\beta}{\alpha-2\beta}$$
$$\frac{1-P(E_3)}{P(E_3)}=\frac{5\alpha-4\beta}{\alpha-2\beta}$$
$$\frac{1}{P(E_3)}=\frac{6\alpha-6\beta}{\alpha-2\beta}=\frac{6(\alpha-\beta)}{\alpha-2\beta}$$
$$P(E_3)=\frac{\alpha-2\beta}{6(\alpha-\beta)}$$
$$\frac{P(E_1)}{P(E_3)}=\frac{\frac{\alpha-2\beta}{\alpha-\beta}}{\frac{\alpha-2\beta}{6(\alpha-\beta)}}=6$$

Therefore,

$$\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3}=6$$

Hence, the correct answer is (5).

Paragraph for Questions 18 and 19: A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

[JEE ADVANCED 2013]

18. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3, the probability that all 3 drawn balls are of the same colour is

(A) $\frac{82}{648}$ **(B)** $\frac{90}{648}$ **(C)** $\frac{558}{648}$ **(D)** $\frac{566}{648}$

Solution: Following table shows the probability of drawing balls from the boxes:

B_1	B_2	B_3
1 W	2 W	3 W
3 R	3 R	4 R
2 B	4 B	5 B

$$P(WWW+RRR+BBB)=\left(\frac{1}{6}\times\frac{2}{9}\times\frac{3}{12}\right)+\left(\frac{3}{6}\times\frac{3}{9}\times\frac{4}{12}\right)+\left(\frac{2}{6}\times\frac{4}{9}\times\frac{5}{12}\right)$$
$$=\frac{6+36+40}{648}=\frac{82}{648}$$

Hence, the correct answer is option (A).

19. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is

(A) $\frac{116}{181}$ **(B)** $\frac{126}{181}$ **(C)** $\frac{65}{181}$ **(D)** $\frac{55}{181}$

Solution:

$$P\left(\frac{B_2}{WR}\right)=\frac{P\left(\frac{WR}{B_2}\right)\times P(B_2)}{P\left(\frac{WR}{B_1}\right)\cdot P(B_1)+P\left(\frac{WR}{B_2}\right)\cdot P(B_2)+P\left(\frac{WR}{B_3}\right)\cdot P(B_3)}$$
$$=\frac{\frac{{}^2C_1\times{}^3C_1}{{}^9C_2}\times\frac{1}{3}}{\left(\frac{{}^1C_1\times{}^3C_1}{{}^6C_2}\times\frac{1}{3}\right)+\left(\frac{{}^2C_1\times{}^3C_1}{{}^9C_2}\times\frac{1}{3}\right)+\left(\frac{{}^3C_1\times{}^4C_1}{{}^{12}C_2}\times\frac{1}{3}\right)}$$
$$=\frac{\frac{2\times3}{9\times4}}{\frac{3\times2}{6\times5}+\frac{2\times3}{9\times4}+\frac{3\times4\times2}{12\times11}}$$
$$=\frac{6\times5\times6\times11}{36\times181}$$
$$=\frac{55}{181}$$

Hence, the correct answer is option (D).

20. Three boys and two girls stand in a queue. The probability, that the number of boys ahead of every girl is at least one more than the number of girls ahead of her, is

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$

[JEE ADVANCED 2014]

Solution: According to question following possibilities are there

Case 1:

$G_1\,B_1\,G_2\,B_2\,B_3$
$B_1\,G_1\,B_2\,G_2\,B_3$
$G_1\,B_1\,B_2\,G_2\,B_3$

Girls separate

$\square B_1 \square B_2 \square B_3$

Out of 3 gaps, 2 are selected and girls are standing there in ${}^3C_2$ ways.

Next boys and girls permute separately in $3!\times2!$ ways.

Therefore,

$$\text{number of ways } = {}^3C_2 \times 3! \times 2! \\ = 3 \times 6 \times 2 = 36$$

Case 2:

$G_1\, G_2\, B_1 B_2\, B_3$

$B_1\, G_1\, G_2 B_2\, B_3$

$B_1\, B_2\, G_1 G_2 B_3$ Not possible

Girls together

Hence, places selected in ${}^2C_1 = 2$ ways (Gaps) and then permutation is $3! \times 2!$ ways. Therefore,

${}^2C_1 \times 3! \times 2! = 2 \times 6 \times 2 = 24$ ways

Therefore,

$$\text{Probability } = \frac{36+24}{5!} = \frac{\cancel{60}}{\cancel{120}\,2} = \frac{1}{2}$$

Hence, the correct answer is option (A).

Paragraph for Questions 21 and 22: Box 1 contains three cards bearing numbers 1, 2, 3, box 2 contains five cards bearing numbers 1, 2, 3, 4, 5; and box 3 contains seven cards bearing numbers 1, 2, 3, 4, 5, 6, 7. A card is drawn from each of the boxes. Let x_i be number of the card drawn from the i^{th} box, $i = 1,2,3$.

[JEE ADVANCED 2014]

21. The probability that $x_1 + x_2 + x_3$ is odd, is

(A) $\frac{29}{105}$ **(B)** $\frac{53}{105}$ **(C)** $\frac{57}{105}$ **(D)** $\frac{1}{2}$

Solution:

$$\underset{1}{|(1)(2)(3)|}\;\underset{2}{|(1)(2)(3)(4)(5)|}\;\underset{3}{|(1)(2)(3)(4)(5)(6)(7)|}$$

For $x_1 + x_2 + x_3$ to be odd, either all the numbers are odd.

$2 \times 3 \times 4 = 24$ ways or one odd and two even in

$$\underbrace{2\times 2\times 3}_{\text{odd from 1st}} + \underbrace{1\times 3\times 3}_{\text{odd from 2nd}} + \underbrace{1\times 2\times 4}_{\text{odd from 3rd}}$$

$$= 12 + 9 + 8 = 29 \text{ ways}$$

Therefore, total number of ways = 24 + 29 = 53 ways

$$\text{All possibilities} = 3 \times 5 \times 7 = 105$$

Hence, probability $= \frac{53}{105}$.

Hence, the correct answer is option (B).

22. The probability that x_1, x_2, x_3 are in an arithmetic progression, is

(A) $\frac{9}{105}$ **(B)** $\frac{10}{105}$ **(C)** $\frac{11}{105}$ **(D)** $\frac{7}{105}$

Solution: For x_1, x_2, x_3 to be in AP.

$$2x_2 = x_1 + x_3$$

Therefore, we require $(x_1 + x_3)$ to be even.

Hence, either both even in $1 \times 3 = 3$ ways

or both x_1, x_3 odd in $2 \times 4 = 8$ ways

$$\text{All possibilities} = 3 \times 5 \times 7 = 105$$

Therefore,

$$\text{Probability } = \frac{\text{Favorable ways}}{\text{All possibilities}} = \frac{8+3}{105} = \frac{11}{105}$$

Hence, the correct answer is option (C).

23. The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is ______.

[JEE ADVANCED 2015]

Solution: Let x = no. of times the coin is tossed.

Hence,

$$P\text{ (At least 2 heads)} \geq 0.96$$

$$\Rightarrow (1 - P(0 \text{ heads}) - P(1 \text{ head})) \geq 0.96$$

$$\Rightarrow 1 - \left(\frac{1}{2^n}\right) - \left(\underbrace{\frac{1}{2^{n-1}}\cdot\frac{1}{2} + \frac{1}{2^{n-1}}\cdot\frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\cdot\frac{1}{2}}_{n\text{-times}}\right) \geq 0.96$$

$$\Rightarrow 1 - \frac{1}{2^n} - \frac{n}{2^n} \geq 0.96$$

$$\Rightarrow \frac{n+1}{2^n} \leq 0.04 = \frac{1}{25}$$

$$\Rightarrow \frac{n+1}{2^n} \leq \frac{1}{25}$$

$$\Rightarrow 25n + 25 \leq 2^n$$

$$\Rightarrow n \geq 8$$

Therefore, least value of $n = 8$

Hence, the correct answer is (8).

Paragraph for Questions 24 and 25: Let n_1 and n_2 be the number of red and black balls, respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II.

[JEE ADVANCED 2015]

24. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is $\frac{1}{3}$, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

(A) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$

(B) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

(C) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$

(D) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

Solution: See Fig. 28.12.

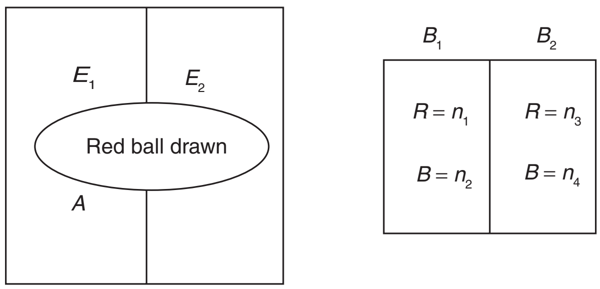


Figure 28.12

$$P\left(\frac{E_2}{A}\right)=\frac{1}{3}=\frac{P\left(\frac{A}{E_2}\right)\cdot P(E_2)}{P\left(\frac{A}{E_1}\right)\cdot P(E_1)+P\left(\frac{A}{E_2}\right)\cdot P(E_2)}$$

$$\Rightarrow \frac{1}{3}=\frac{\left(\frac{n_3}{n_3+n_4}\right)\cdot\frac{1}{2}}{\left(\frac{n_1}{n_1+n_2}\right)\cdot\frac{1}{2}+\left(\frac{n_3}{n_3+n_4}\right)\cdot\frac{1}{2}}$$

$$\Rightarrow \frac{1}{3}=\frac{n_3(n_1+n_2)}{n_1(n_3+n_4)+n_3(n_1+n_2)}$$

$$\Rightarrow \frac{1}{3}=\frac{1}{\frac{n_1(n_3+n_4)}{n_3(n_1+n_2)}+1} \Rightarrow \frac{n_1(n_3+n_4)}{n_3(n_1+n_2)}=2$$

which are satisfied by options (A) and (B).

Hence, the correct answers are options (A) and (B).

25. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the possible value of n_1 and n_2 is(are)

(A) $n_1=4$ and $n_2=6$
(B) $n_1=2$ and $n_2=3$
(C) $n_1=10$ and $n_2=20$
(D) $n_1=3$ and $n_2=6$

Solution:

$$P(\text{red from I after transfer}) = \frac{1}{3} \text{ (given)}$$

$$\Rightarrow P(\text{Red transfer from I})\cdot P\left(\frac{\text{Red from I}}{\text{Red transfer from I}}\right)$$

$$+P(\text{Black tranfer from I})\cdot P\left(\frac{\text{Red from I}}{\text{Black transfer from I}}\right)=\frac{1}{3}$$

$$\Rightarrow \left(\frac{n_1}{n_1+n_2}\right)\cdot\left(\frac{n_1-1}{n_1+n_2-1}\right)+\left(\frac{n_2}{n_1+n_2}\right)\cdot\left(\frac{n_1}{n_1+n_2-1}\right)=\frac{1}{3}$$

$$\Rightarrow 3(n_1(n_1-1)+n_2.n_1)=(n_1+n_2)(n_1+n_2-1)$$

$$2n_1^2-2n_1+n_1n_2-n_2^2+n_2=0$$

which are satisfied for $n_1=3, n_2=6$ or $n_1=10, n_2=20$

Hence, the correct answers are options (C) and (D).

26. A computer producing factory has only two plants T_1 and T_2. Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that *P* (computer turns out to be defective given that is produced in plant T_1) = 10*P*(computer turns out to be defective given that it is produced in plant T_2), where *P*(*E*) denotes the probability of an event *E*. A computer produces in the factory is randomly selected and it does not turn out to be defective. Then the probability that it is produced in plant T_2 is

(A) $\frac{36}{73}$ **(B)** $\frac{47}{79}$ **(C)** $\frac{78}{93}$ **(D)** $\frac{75}{83}$

[JEE ADVANCED 2016]

Solution: Let P_1 be the defective computers that are produced from plant T_1 and P_2 be that from plant T_2.

The total percentage of the defective computers produced is 7%.

Now, $P_1 = 10P$ and $P_2 = P$.

The computers produced that are defective:

$$\frac{20}{100}\times P_1+\frac{80}{100}\times P_2=\frac{7}{100}$$

$$20P_1+80P_2=7$$

$$200P+80P=7$$

$$P=\frac{7}{280}=\frac{1}{40}$$

P(computers that are not defective) $=1-\frac{1}{40}=\frac{39}{40}$

Now, the probability of the defective products is calculated as follows:

$$\frac{20}{100}P_1+\frac{80}{100}P_2=\frac{20}{100}\times\frac{1}{4}+\frac{80}{100}\times\frac{1}{40}=\frac{28}{400}=\frac{7}{100}$$

The probability of producing NOT defective computers is

$$1-\frac{7}{100}=\frac{93}{100}$$

The probability that plant T_2 produces NOT defective computers is calculated as follows:

$$\frac{(80/100)\times(39/40)}{(93/100)}=\frac{78}{93}$$

Hence, the correct answer is option (C).

Paragraph for Questions 27 and 28: Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2},\frac{1}{6}$ and $\frac{1}{3}$, respectively. Each team gets 3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let *X* and *Y* denote the total points scored by teams T_1 and T_2, respectively, after two games.

[JEE ADVANCED 2016]

27. $P(X > Y)$ is

(A) $\frac{1}{4}$ **(B)** $\frac{5}{12}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{7}{12}$

Common data for Questions 27 and 28:

- Probability of wining of T_1 against T_2 is = 1/2.
- Probability of drawing of T_1 against T_2 is = 1/6.
- Probability of losing of T_1 against T_2 is = 1/3.
- 3 points for win.
- 1 point for draw.
- 0 point for loss.

Solution: Now, $P(x > y) = P(T_1$ wins both game) + $P(T_1$ wins one game and one draw)

$$=\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times{}^2C_1\times\frac{1}{6}=\frac{1}{4}+\frac{1}{6}=\frac{3+2}{12}=\frac{5}{12}$$

Hence, the correct answer is option (B).

28. $P(X = Y)$ is

(A) $\frac{11}{36}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{13}{36}$ **(D)** $\frac{1}{2}$

Solution: Using the data given, we get

$P(x = y) = P$(both team win 1 game or both games draw)

$$= {}^2C_1 \times \frac{1}{2} \times \frac{1}{3} + \frac{1}{6} \times \frac{1}{6} = \frac{13}{36}$$

Hence, the correct answer is option (C).

Practice Exercise 1

1. Two fair dices are tossed. Let A be the event that the first dice shows an even number and B be the event that the second dice shows an odd number. The two event A and B are

(A) Mutually exclusive
(B) Independent and mutually exclusive
(C) Dependent
(D) None of these

2. Let A, B, C be three mutually independent events. Consider the two statements S_1 and S_2

$S_1 : A$ and $B \cup C$ are independent

$S_2 : A$ and $B \cap C$ are independent

Then

(A) Both S_1 and S_2 are true **(B)** Only S_1 is true
(C) Only S_2 is true **(D)** Neither S_1 nor S_2 is true

3. Two card are drawn successively with replacement from a pack of 52 cards. The probability of drawing two aces is

(A) $\frac{1}{169}$ **(B)** $\frac{1}{221}$ **(C)** $\frac{1}{2652}$ **(D)** $\frac{4}{663}$

4. A single letter is selected at random from the word 'PROBABILITY'. The probability that the selected letter is a vowel is

(A) $\frac{2}{11}$ **(B)** $\frac{3}{11}$ **(C)** $\frac{4}{11}$ **(D)** 0

5. The probability of happening an event A in one trial is 0.4. The probability that the event A happens at least once in three independent trials is

(A) 0.936 **(B)** 0.784 **(C)** 0.904 **(D)** 0.216

6. Two dice are thrown simultaneously. What is the probability of obtaining a multiple of 2 on one of them and a multiple of 3 on the other

(A) $\frac{5}{36}$ **(B)** $\frac{11}{36}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{3}$

7. A problem of mathematics is given to three students whose chances of solving the problem are 1/3, 1/4 and 1/5 respectively. The probability that the question will be solved is

(A) $\frac{2}{3}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{5}$ **(D)** $\frac{3}{5}$

8. Three persons work independently on a problem. If the respective probabilities that they will solve it are 1/3, 1/4 and 1/5, then the probability that none can solve it

(A) $\frac{2}{5}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{1}{3}$ **(D)** None of these

9. Two dices are thrown. The probability that the sum of the points on two dices will be 7, is

(A) $\frac{5}{36}$ **(B)** $\frac{6}{36}$ **(C)** $\frac{7}{36}$ **(D)** $\frac{8}{36}$

10. The probability that an event will fail to happen is 0.05. The probability that the event will take place on four consecutive occasions is

(A) 0.00000625 **(B)** 0.18543125
(C) 0.00001875 **(D)** 0.81450625

11. Three identical dice are rolled. The probability that same number will appear on each of them will be

(A) $\frac{1}{6}$ **(B)** $\frac{1}{36}$ **(C)** $\frac{1}{18}$ **(D)** $\frac{3}{28}$

12. The probability of hitting a target by three marksmen are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. The probability that one and only one of them will hit the target when they fire simultaneously, is

(A) $\frac{11}{24}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{8}$ **(D)** None of these

13. A determinant is chosen at random. The set of all determinants of order 2 with elements 0 or 1 only. The probability that value of the determinant chosen is positive, is

(A) 3/16 **(B)** 3/8 **(C)** 1/4 **(D)** None of these

14. For any two independent events E_1 and $E_2, P\{(E_1 \cup E_2) \cap (\bar{E}_1 \cap \bar{E}_2)\}$ is

(A) $< \frac{1}{4}$ **(B)** $> \frac{1}{4}$ **(C)** $\geq \frac{1}{2}$ **(D)** None of these

15. In order to get at least once a head with probability ≥ 0.9, the number of times a coin needs to be tossed is

(A) 3 **(B)** 4 **(C)** 5 **(D)** None of these

16. A fair coin is tossed repeatedly. If tail appears on first four tosses, then the probability of head appearing on fifth toss equals

(A) $\frac{1}{2}$ **(B)** $\frac{1}{32}$ **(C)** $\frac{31}{32}$ **(D)** $\frac{1}{5}$

17. The probability that in a year of the 22^{nd} century chosen at random there will be 53 Sundays is

(A) $\frac{3}{28}$ **(B)** $\frac{2}{28}$ **(C)** $\frac{7}{28}$ **(D)** $\frac{5}{28}$

18. Find the probability that the two digit number formed by digits 1, 2, 3, 4, 5 is divisible by 4 (while repetition of digit is allowed)

(A) $\frac{1}{30}$ **(B)** $\frac{1}{20}$ **(C)** $\frac{1}{40}$ **(D)** None of these

19. A bag x contains 3 white balls and 2 black balls and another bag y contains 2 white balls and 4 black balls. A bag and a ball out of it are picked at random. The probability that the ball is white, is

(A) 3/5 **(B)** 7/15 **(C)** 1/2 **(D)** None of these

20. In a throw of a dice the probability of getting one in even number of throw is

(A) $\frac{5}{36}$ **(B)** $\frac{5}{11}$ **(C)** $\frac{6}{11}$ **(D)** $\frac{1}{6}$

21. The letter of the word 'ASSASSIN' are written down at random in a row. The probability that no two *S* occur together is

(A) $\frac{1}{35}$ **(B)** $\frac{1}{14}$ **(C)** $\frac{1}{15}$ **(D)** None of these

22. A bag contains 6 red, 5 white and 4 black balls. Two balls are drawn. The probability that none of them is red, is

(A) $\frac{12}{35}$ **(B)** $\frac{6}{35}$ **(C)** $\frac{4}{35}$ **(D)** None of these

23. Out of 13 applicants for a job, there are 5 women and 8 men. It is desired to select 2 persons for the job. The probability that at least one of the selected persons will be a woman is

(A) $\frac{25}{39}$ **(B)** $\frac{14}{39}$ **(C)** $\frac{5}{13}$ **(D)** $\frac{10}{13}$

24. A committee consists of 9 experts taken from three institutions *A, B* and *C*, of which 2 are from *A*, 3 from *B* and 4 from *C*. If three experts resign, then the probability that they belong to different institutions is

(A) $\frac{1}{729}$ **(B)** $\frac{1}{24}$ **(C)** $\frac{1}{21}$ **(D)** $\frac{2}{7}$

25. Five digit numbers are formed using the digits 1, 2, 3, 4, 5, 6 and 8. What is the probability that they have even digits at both the ends?

(A) $\frac{2}{7}$ **(B)** $\frac{3}{7}$ **(C)** $\frac{4}{7}$ **(D)** None of these

26. A bag contains 3 red, 4 white and 5 black balls. Three balls are drawn at random. The probability of being their different colours is

(A) $\frac{3}{11}$ **(B)** $\frac{2}{11}$ **(C)** $\frac{8}{11}$ **(D)** None of these

27. Two numbers are selected randomly from the set $S=\{1,2,3,4,5,6\}$ without replacement one by one. The probability that minimum of the two numbers is less than 4 is

(A) $\frac{1}{15}$ **(B)** $\frac{14}{15}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{4}{5}$

28. The odds against a certain event is 5 : 2 and the odds in favour of another event is 6 : 5. If both the events are independent, then the probability that at least one of the events will happen is

(A) $\frac{50}{77}$ **(B)** $\frac{52}{77}$ **(C)** $\frac{25}{88}$ **(D)** $\frac{63}{88}$

29. A party of 23 persons take their seats at a round table. The odds against two persons sitting together are

(A) 10 : 1 **(B)** 1 : 11 **(C)** 9 : 10 **(D)** None of these

30. The probabilities of three mutually exclusive events are 2/3, 1/4 and 1/6. The statement is

(A) True **(B)** Wrong
(C) Could be either **(D)** Do not know

31. If *A* and *B* are two events such that $P(A)\neq 0$ and $P(B)\neq 1$, then $P\left(\frac{\overline{A}}{\overline{B}}\right)=$

(A) $1-P\left(\frac{A}{B}\right)$ **(B)** $1-P\left(\frac{\overline{A}}{B}\right)$ **(C)** $\frac{1-P(A\cup B)}{P(\overline{B})}$ **(D)** $\frac{P(\overline{A})}{P(\overline{B})}$

32. If two events *A* and *B* are such that $P(A^c)=0.3, P(B)=0.4$ and $P(AB^c)=0.5$, then $P[B/(A\cup B^c)]$ is equal to

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** None of these

33. Let $0<P(A)<1, 0<P(B)<1$, and $P(A\cup B)=P(A)+P(B)-P(A)P(B)$. Then

(A) $P(B/A)=P(B)-P(A)$ **(B)** $P(A^c\cup B^c)=P(A^c)+P(B^c)$
(C) $P(A\cup B)^c=P(A^c)P(B^c)$ **(D)** $P(A/B)=P(A)$

34. For a biased dice the probabilities for different faces to turn up are given below

Face :	1	2	3	4	5	6
Probability :	0.1	0.32	0.21	0.15	0.05	0.17

The dice is tossed and you are told that either face 1 or 2 has turned up. Then the probability that it is face 1, is

(A) $\frac{5}{21}$ **(B)** $\frac{5}{22}$ **(C)** $\frac{4}{21}$ **(D)** None of these

35. Two cards are drawn one by one from a pack of cards. The probability of getting first card an ace and second a coloured one is (before drawing second card first card is not placed again in the pack)

(A) $\frac{1}{26}$ **(B)** $\frac{5}{52}$ **(C)** $\frac{5}{221}$ **(D)** $\frac{4}{13}$

36. Eight coins are tossed simultaneously. The probability of getting at least 6 heads is

(A) $\frac{57}{64}$ **(B)** $\frac{229}{256}$ **(C)** $\frac{7}{64}$ **(D)** $\frac{37}{256}$

37. In a box containing 100 eggs, 10 eggs are rotten. The probability that out of a sample of 5 eggs none is rotten if the sampling is with replacement is

(A) $\left(\frac{1}{10}\right)^5$ **(B)** $\left(\frac{1}{5}\right)^5$ **(C)** $\left(\frac{9}{5}\right)^5$ **(D)** $\left(\frac{9}{10}\right)^5$

38. If *x* denotes the number of sixes in four consecutive throws of a dice, then $P(x=4)$ is

(A) $\frac{1}{1296}$ **(B)** $\frac{4}{6}$ **(C)** 1 **(D)** $\frac{1295}{1296}$

39. In a simultaneous toss of four coins, what is the probability of getting exactly three heads

(A) $\frac{1}{2}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** None of these

40. A coin is tossed 10 times. The probability of getting exactly six heads is

(A) $\frac{512}{513}$ **(B)** $\frac{105}{512}$ **(C)** $\frac{100}{153}$ **(D)** ${}^{10}C_6$

41. If a dice is thrown twice, then the probability of occurrence of 4 at least once is

(A) $\frac{11}{36}$ (B) $\frac{7}{12}$ (C) $\frac{35}{36}$ (D) None of these

42. If A and B are any two events, then the probability that exactly one of them occur is

(A) $P(A)+P(B)-P(A\cap B)$ (B) $P(A)+P(B)-2P(A\cap B)$

(C) $P(A)+P(B)-P(A\cup B)$ (D) $P(A)+P(B)-2P(A\cup B)$

43. The probability of happening at least one of the events A and B is 0.6. If the events A and B happens simultaneously with the probability 0.2, then $P(\overline{A})+P(\overline{B})=$

(A) 0.4 (B) 0.8 (C) 1.2 (D) 1.4

44. Let A and B be two events such that $P(A)=0.3$ and $P(A\cup B)=0.8$. If A and B are independent events, then $P(B)=$

(A) $\frac{5}{6}$ (B) $\frac{5}{7}$ (C) $\frac{3}{5}$ (D) $\frac{2}{5}$

45. For two given events A and B, $P(A\cap B)=$

(A) Not less than $P(A)+P(B)-1$

(B) Not greater than $P(A)+P(B)$

(C) Equal to $P(A)+P(B)-P(A\cup B)$

(D) All of the above

46. $P(A\cup B)=P(A\cap B)$ if and only if the relation between $P(A)$ and $P(B)$ is

(A) $P(A)=P(\overline{A})$ (B) $P(A\cap B)=P(A'\cap B')$

(C) $P(A)=P(B)$ (D) None of these

47. The two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both A and B occur simultaneously is 0.14. Then the probability that neither A nor B occurs is

(A) 0.39 (B) 0.25 (C) 0.904 (D) None of these

48. Given two mutually exclusive events A and B such that $P(A)=0.45$ and $P(B)=0.35$, then P (A or B) $=$

(A) 0.1 (B) 0.25 (C) 0.15 (D) 0.8

49. If $P(A)=P(B)=x$ and $P(A\cap B)=P(A'\cap B')=\frac{1}{3}$, then $x=$

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{6}$

50. In a certain population 10% of the people are rich, 5% are famous and 3% are rich and famous. The probability that a person picked at random from the population is either famous or rich but not both, is equal to

(A) 0. 07 (B) 0.08 (C) 0. 09 (D) 0. 12

51. If $P(A)=1/3$, $P(B)=1/2$ and $P(A\cup B)=5/6$, then events A and B are

(A) Mutually exclusive

(B) Independent as well as mutually exhaustive

(C) Independent

(D) Dependent only on A

52. Three dice are thrown simultaneously. What is the probability of obtaining a total of 17 or 18

(A) $\frac{1}{9}$ (B) $\frac{1}{72}$ (C) $\frac{1}{54}$ (D) None of these

53. The probability of getting head and tail alternately in three throws of a coin (or a throw of three coins), is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{3}{8}$

54. A bag contains 19 tickets numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. The probability that both the tickets will show even number, is

(A) $\frac{9}{19}$ (B) $\frac{8}{18}$ (C) $\frac{9}{18}$ (D) $\frac{4}{19}$

55. A box contains 3 white and 2 red balls. A ball is drawn and another ball is drawn without replacing first ball, then the probability of second ball to be red is

(A) $\frac{8}{25}$ (B) $\frac{2}{5}$ (C) $\frac{3}{5}$ (D) $\frac{21}{25}$

56. An unbiased dice is tossed until a number greater than 4 appears. The probability that an even number of tosses is needed is

(A) $\frac{1}{2}$ (B) $\frac{2}{5}$ (C) $\frac{1}{5}$ (D) $\frac{2}{3}$

57. Two cards are drawn without replacement from a well-shuffled pack. Find the probability that one of them is an ace of heart

(A) $\frac{1}{25}$ (B) $\frac{1}{26}$

(C) $\frac{1}{52}$ (D) None of these

58. Word 'UNIVERSITY' is arranged randomly. Then the probability that both 'I' does not come together, is

(A) $\frac{3}{5}$ (B) $\frac{2}{5}$ (C) $\frac{4}{5}$ (D) $\frac{1}{5}$

59. A bag contains 8 red and 7 black balls. Two balls are drawn at random. The probability that both the balls are of the same colour is

(A) $\frac{14}{15}$ (B) $\frac{11}{15}$ (C) $\frac{7}{15}$ (D) $\frac{4}{15}$

60. 5 boys and 5 girls are sitting in a row randomly. The probability that boys and girls sit alternatively is

(A) 5/126 (B) 1/126 (C) 4/126 (D) 6/125

61. If odds against solving a question by three students are 2 : 1, $5:2$ and $5:3$ respectively, then probability that the question is solved only by one student is

(A) $\frac{31}{56}$ (B) $\frac{24}{56}$ (C) $\frac{25}{56}$ (D) None of these

62. If A and B are two independent events such that $P(A\cap B')=\frac{3}{25}$ and $P(A'\cap B)=\frac{8}{25}$, then $P(A)=$

(A) $\frac{1}{5}$ (B) $\frac{3}{8}$ (C) $\frac{2}{5}$ (D) $\frac{4}{5}$

63. The probability of solving a question by three students are $\frac{1}{2},\frac{1}{4},\frac{1}{6}$, respectively. Probability of question is being solved will be

(A) $\frac{33}{48}$ (B) $\frac{35}{48}$ (C) $\frac{31}{48}$ (D) $\frac{37}{48}$

64. If E and F are independent events such that $0 < P(E) < 1$ and $0 < P(F) < 1$, then

(A) E and F^c (the complement of the event F) are independent
(B) E^c and F^c are independent
(C) $P\left(\frac{E}{F}\right) + P\left(\frac{E^c}{F^c}\right) = 1$
(D) All of the above

65. If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F, respectively and if $0 < P(F) < 1$, then

(A) $P(E/F) + P(\bar{E}/F) = 1$
(B) $P(E/F) + P(E/\bar{F}) = 1$
(C) $P(\bar{E}/F) + P(E/\bar{F}) = 1$
(D) $P(E/\bar{F}) + P(\bar{E}/\bar{F}) = 1$

66. A bag 'A' contains 2 white and 3 red balls and bag 'B' contains 4 white and 5 red balls. One ball is drawn at random from a randomly chosen bag and is found to be red. The probability that it was drawn from bag 'B' was

(A) $\frac{5}{14}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{5}{18}$ **(D)** $\frac{25}{52}$

67. A dice is tossed thrice. A success is getting 1 or 6 on a toss. The mean and the variance of number of successes

(A) $\mu = 1, \sigma^2 = 2/3$ **(B)** $\mu = 2/3, \sigma^2 = 1$
(C) $\mu = 2, \sigma^2 = 2/3$ **(D)** None of these

68. Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards, then the mean of the number of aces is

(A) 1/13 **(B)** 3/13 **(C)** 2/13 **(D)** None of these

69. A sample of 4 items is drawn at a random without replacement from a lot of 10 items. Containing 3 defective. If X denotes the number of defective items in the sample, then $P(0 < x < 3)$ is equal to

(A) $\frac{3}{10}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{1}{6}$

70. If E and F are events with $P(E) \le P(F)$ and $P(E \cap F) > 0$, then

(A) Occurrence of $E \Rightarrow$ Occurrence of F
(B) Occurrence of $F \Rightarrow$ Occurrence of E
(C) Non-occurrence of $E \Rightarrow$ Non–occurrence of F
(D) None of the above implications holds

71. An anti-aircraft gun take a maximum of four shots at an enemy plane moving away from it. The probability of hitting the plane at the first, second, third and fourth shot are 0.4, 0.3, 0.2 and 0.1, respectively. The probability that the gun hits the plane is

(A) 0.25 **(B)** 0.21 **(C)** 0.16 **(D)** 0.6976

72. If $(1+3p)/3, (1-p)/4$ and $(1-2p)/2$ are the probabilities of three mutually exclusive events, then the set of all values of p is

(A) $\frac{1}{3} \le p \le \frac{1}{2}$ **(B)** $\frac{1}{3} < p < \frac{1}{2}$
(C) $\frac{1}{2} \le p \le \frac{2}{3}$ **(D)** $\frac{1}{2} < p < \frac{2}{3}$

73. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of eleven cards numbered 2, 3, 4,…,12 is picked and the number on the card is noted. The probability that the noted number is either 7 or 8, is

(A) 0.24 **(B)** 0.244 **(C)** 0.024 **(D)** None of these

74. If $P(A) = 0.3, P(B) = 0.4, P(C) = 0.8, P(AB) = 0.08,$ $P(AC) = 0.28, P(ABC) = 0.09, P(A+B+C) \ge 0.75$ and $P(BC) = x$, then

(A) $0.23 \le x \le 0.48$
(B) $0.32 \le x \le 0.84$
(C) $0.25 \le x \le 0.73$
(D) None of these

75. If the integers m and n are chosen at random between 1 and 100, then the probability that a number of the form $7^m + 7^n$ is divisible by 5 equals

(A) $\frac{1}{4}$ **(B)** $\frac{1}{7}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{49}$

76. There are four machines and it is known that exactly two of them are faulty. They are tested, one by one, in a random order till both the faulty machines are identified. Then the probability that only two tests are needed is

(A) $\frac{1}{3}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{1}{4}$

77. Two persons A and B take turns in throwing a pair of dice. The first person to through 9 from both dice will be avoided the prize. If A throws first then the probability that B wins the game is

(A) $\frac{9}{17}$ **(B)** $\frac{8}{17}$ **(C)** $\frac{8}{9}$ **(D)** $\frac{1}{9}$

78. Six boys and six girls sit in a row. What is the probability that the boys and girls sit alternatively

(A) $\frac{1}{462}$ **(B)** $\frac{1}{924}$ **(C)** $\frac{1}{2}$ **(D)** None of these

79. Cards are drawn one by one at random from a well-shuffled full pack of 52 cards until two aces are obtained for the first time. If N is the number of cards required to be drawn, then $P_r\{N = n\}$, where $2 \le n \le 50$, is

(A) $\frac{(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}$

(B) $\frac{2(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}$

(C) $\frac{3(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}$

(D) $\frac{4(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}$

80. Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently equals

(A) – **(B)** $\dfrac{7}{15}$ **(C)** $\dfrac{2}{15}$ **(D)** $\dfrac{1}{3}$

81. If A and B are two events such that $P(A\cup B)=P(A\cap B)$, then the true relation is

(A) $P(A)+P(B)=0$

(B) $P(A)+P(B)=P(A)P\left(\dfrac{B}{A}\right)$

(C) $P(A)+P(B)=2P(A)P\left(\dfrac{B}{A}\right)$

(D) None of these

82. The probability of happening an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then the probability of happening neither A nor B is

(A) 0.6 **(B)** 0.2 **(C)** 0.21 **(D)** None of these

83. If A and B are two events, then the probability of the event that at most one of A, B occurs, is

(A) $P(A'\cap B)+P(A\cap B')+P(A'\cap B')$

(B) $1-P(A\cap B)$

(C) $P(A')+P(B')+P(A\cup B)-1$

(D) All of the these

84. For any two events A and B in a sample space

(A) $P\left(\dfrac{A}{B}\right)\geq\dfrac{P(A)+P(B)-1}{P(B)}$, $P(B)\neq 0$ is always true

(B) $P(A\cap\overline{B})=P(A)-P(A\cap B)$ does not hold

(C) $P(A\cup B)=1-P(\overline{A})P(\overline{B})$, if A and B are disjoint

(D) None of these

85. Urn A contains 6 red and 4 black balls and urn B contains 4 red and 6 black balls. One ball is drawn at random from urn A and placed in urn B. Then one ball is drawn at random from urn B and placed in urn A. If one ball is now drawn at random from urn A, the probability that it is found to be red, is

(A) $\dfrac{32}{55}$ **(B)** $\dfrac{21}{55}$ **(C)** $\dfrac{19}{55}$ **(D)** None of these

86. The probability that a leap year selected at random contains either 53 Sundays or 53 Mondays, is

(A) $\dfrac{2}{7}$ **(B)** $\dfrac{4}{7}$ **(C)** $\dfrac{3}{7}$ **(D)** $\dfrac{1}{7}$

87. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c, respectively. On these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Which of the following relations are true?

(A) $p+m+c=\dfrac{19}{20}$ **(B)** $p+m+c=\dfrac{27}{20}$

(C) $pmc=\dfrac{1}{10}$ **(D)** $pmc=\dfrac{1}{4}$

88. Three groups A, B, C are competing for positions on the Board of Directors of a company. The probabilities of their winning are 0.5, 0.3, 0.2, respectively. If the group A wins, the probability of introducing a new product is 0.7 and the corresponding probabilities for group B and C are 0.6 and 0.5, respectively. The probability that the new product will be introduced, is

(A) 0.18 **(B)** 0.35 **(C)** 0.10 **(D)** 0.63

89. A purse contains 4 copper coins and 3 silver coins, the second purse contains 6 copper coins and 2 silver coins. If a coin is drawn out of any purse, then the probability that it is a copper coin is

(A) 4/7 **(B)** 3/4 **(C)** 37/56 **(D)** None of these

90. An unbiased dice with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5, is

(A) 16/81 **(B)** 1/81 **(C)** 80/81 **(D)** 65/81

91. India plays two matches each with West Indies and Australia. In any match the probabilities of India getting point 0, 1 and 2 are 0.45, 0.05 and 0.50 respectively. Assuming that the outcomes are independents, the probability of India getting at least 7 points is

(A) 0.8750 **(B)** 0.0875 **(C)** 0.0625 **(D)** 0.0250

92. A box contains 24 identical balls, of which 12 are white and 12 are black. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is

(A) $\dfrac{5}{64}$ **(B)** $\dfrac{27}{32}$ **(C)** $\dfrac{5}{32}$ **(D)** $\dfrac{1}{2}$

Practice Exercise 2

Single/Multiple Correct Choice Type Questions

1. If E and F are two independent events; such that $P(E\cap F)=\dfrac{1}{6}$, $P\left(E^C\cap F^C\right)=\dfrac{1}{3}$ and $[P(E)-P(F)][1-P(F)]>0$, then

(A) $P(E)=\dfrac{1}{2}$ **(B)** $P(E)=\dfrac{1}{4}$

(C) $P(F)=\dfrac{1}{3}$ **(D)** $P(F)=\dfrac{2}{3}$

2. A drawer contains red and black balls. When two balls are drawn at random, the probability that they both are red is $\dfrac{1}{2}$. The number of balls in the drawer can be

(A) 21 **(B)** 11 **(C)** 4 **(D)** 3

3. A bag initially contains one red and two blue balls. An experiment consisting of selecting a ball at random, noting its colour and replacing it together with an additional ball of the same colour. If three such trials are made, then

(A) Probability that at least one blue ball is drawn is 0.9.

(B) Probability that exactly one blue ball is drawn is 0.2.

(C) Probability that all the drawn balls are red given that all the drawn balls are of same colour is 0.2.

(D) Probability that at least one red ball is drawn is 0.6.

4. Two persons A and B have $n + 1$ and n coins, respectively, which they toss simultaneously. Then the probability that A will have more heads than B is

(A) $\frac{1}{2}$ **(B)** $> \frac{1}{2}$ **(C)** $< \frac{1}{2}$ **(D)** $> \frac{1}{3}$

5. A sum of money is rounded off to the nearest rupee. The probability that the round-off error is at most 10 paise is

(A) 63/300 **(B)** 11/100 **(C)** 3/25 **(D)** 21/100

6. Consider the Cartesian plane R^2 and let X denote the subset of points for which both coordinates are integers. A coin of diameter $\frac{1}{2}$ is tossed randomly into the plane. The probability P that the coin covers a point of X satisfies

(A) $P = \frac{\pi}{16}$ **(B)** $P < \frac{\pi}{3}$ **(C)** $P > \frac{\pi}{30}$ **(D)** $P = \frac{1}{4}$

Comprehension Type Questions

Paragraph for Questions 7–9: There are four boxes A_1, A_2, A_3 and A_4. Box A_i has i cards and on each card a number is printed, the numbers are from 1 to i. A box is selected randomly, the probability of selection of box A_i is $\frac{i}{10}$ and then a card is drawn. Let E_i represents the event that a card with number 'i' is drawn.

7. $P(E_1)$ is equal to

(A) $\frac{1}{5}$ **(B)** $\frac{1}{10}$ **C)** $\frac{2}{5}$ **(D)** $\frac{1}{4}$

8. $P(A_3/E_2)$ is equal to

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$

9. Expectation of the number on the card is

(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.5

Paragraph for Questions 10–12: Sania Mirza is to play with Sharapova in a three set match. For a particular set, the probability of Sania winning the set is y and if she wins probability of her winning the next set becomes $\sqrt{y}$ else the probability that she wins the next one becomes y^2. There is no possibility that a set is to be abandoned. R is the probability that Sania wins the first set.

10. If $R = \frac{1}{2}$, then the probability that match will end in first two sets is nearly equal to

(A) 0.73 **(B)** 0.95 **(C)** 0.51 **(D)** 0.36

11. If $R = \frac{1}{2}$ and Sania wins the second set, then the probability that she has won first set as well is nearly equal to

(A) 0.74 **(B)** 0.46 **(C)** 0.26 **(D)** 0.54

12. If Sania loses the first set, then the values of R such that her probability of winning the match is still larger than that of her loosing are given by

(A) $R \in \left(\frac{1}{2}, 1\right)$ **(B)** $R \in \left[\left(\frac{1}{2}\right)^{\frac{1}{3}}, 1\right]$

(C) $R \in \left[\left(\frac{1}{2}\right)^{3/2}, 1\right]$ **(D)** No values of R

Paragraph for Questions 13–15: Let $n = 10k + r$ when $k, r \in N$, $0 \le r \le 9$. A number a is chosen at random from the set {1, 2, ..., n} and let p_n denote the probability that $a^2 - 1$ is divisible by 10.

13. If $r = 0$, then p_n equals

(A) $2k/n$ **(B)** $(k + 1)/n$ **(C)** $(2k + 1)/n$ **(D)** k/n

14. If $r = 9$, then p_n equals

(A) $2k/n$ **(B)** $2(k + 1)/n$ **(C)** $(2k + 1)/n$ **(D)** k/n

15. If $1 \le r \le 8$, then p_n equals

(A) $(2k - 1)/n$ **(B)** $(2k/n)$ **(C)** $(2k + 1)/n$ **(D)** k/n

Paragraph for Questions 16–18: A player 'A' plays a game against a machine. At each round he deposits one rupee in a slot and then flips a coin which has a probability p of showing a head. If the flipped coin show head, he gets back the rupee he deposited and one more rupee from the machine, else he loses his rupee. Let A starts with 10 rupee coins and $q = 1 - p$ (the probability of showing a tail), then

16. The probability that he will be drained out with all of his rupee coins exactly at the eleventh round is

(A) q^{11} **(B)** $1 - q^{11}$ **(C)** $pq^{10} + q^{11}$ **(D)** 0

17. The probability that all his money will be finished exactly at the twelfth round is

(A) q^{12} **(B)** $1 - q^{12}$ **(C)** ${}^{10}C_1\, pq^{11}$ **(D)** ${}^{12}C_2 p^2q^{10}$

18. The probability that he is left with no money by the 14th round or earlier is

(A) $q^{10}(1 + 10pq + 65p^2q^2)$

(B) $q^{14}(p^2q + 36pq + 7)$

(C) $q^{12} + 3pq^{13} + 3p^{13}q + p^{12}$

(D) $1 - {}^{10}C_1\, pq^{11} - {}^{10}C_2\, p^2q^{12}$

Matrix Match Type Questions

19. In a tournament, there are 12 players $S_1, S_2, \cdots, S_{12}$. They are divided into 6 pairs at random. From each game a winner is decided on the basis of a game played between the two players of the pair. Assuming all the pairs are of equal strength, then match the following:

List I	List II
(A) Probability that S_2 is among the losers is	**(p)** $\frac{5}{22}$
(B) Probability that exactly one of S_3 and S_4 is among the losers, is	**(q)** $\frac{5}{6}$
(C) Probability that both S_2 and S_4 are among the winners, is	**(r)** $\frac{1}{3}$
(D) If exactly one of S_3 and S_4 is among the six winner, then the probability that S_3 and S_4 are in different group is	**(s)** $\frac{1}{2}$
	(t) $\frac{6}{11}$

20. Match the following:

List I	List II
(A) One ball is drawn from a bag containing 4 balls and is found to be white. The events that the bag contains "1 white", "2 white", "3 white" and "4 white" are equally likely. If the probability that all the balls are white is $\frac{p}{15}$, then the value of p is	**(p)** 9
(B) From a set of 12 persons, if the number of different selection of a committee, its chairperson and its secretary (possibly same as chair person) is $13.2^{10}\,m$, then value of m is	**(q)** 3
(C) If $x, y, z > 0$ and $x + y + z = 1$, then the least value of $\frac{5x}{2-x} + \frac{5y}{2-y} + \frac{5z}{2-z}$ is	**(r)** 8
(D) If $\sum_{k=1}^{12} 12k \cdot {}^{12}C_k \cdot {}^{11}C_{k-1}$ is equal to $\frac{12\times 21\times 19\times 17\times\cdots\times 3}{11!}\times 2^{12}\times p$, then the value of p is	**(s)** 6
	(t) 12

21. A bag contains some white and some black balls, all combinations being equally likely. The total number of balls in the bag is 12. Four balls are drawn at random from the bag without replacement. Now match the entries from the following two lists:

List I	List II
(A) Probability that all the four balls are black is equal to	**(p)** $\frac{14}{33}$
(B) If the bag contains 10 black and 2 white balls, then the probability that all four balls are black is equal to	**(q)** $\frac{1}{5}$
(C) If all the four balls are black, then the probability that the selected bag contains 10 black balls, is equal to	**(r)** $\frac{70}{429}$
(D) Probability that two balls are black and two are white	**(s)** $\frac{13}{110}$
	(t) $\frac{13}{165}$

22. Let A and B be two independent events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, Now match the entries from the following two lists :

List I	List II
(A) $P\,(A\cup B)$ is equal to	**(p)** $\frac{1}{12}$
(B) $P\,(A/A\cup B)$ is equal to	**(q)** $\frac{1}{2}$
(C) $P\,(B/A'\cap B')$ is equal to	**(r)** $\frac{2}{3}$
(D) $P\,(A'/B')$ is equal to	**(s)** $\frac{1}{3}$
	(t) 0

Integer Type Questions

23. The probability of a man hitting a target in one fire is $1/5$. Find the minimum number of fire; he must follow in order to make his chances of hitting the target more than $\frac{3}{4}$.

24. There are 100 employee in sales office of a multinational company. Let $P(E_i)$ be the probability that exactly i out of 100 employee are infected with swine flu virus. If $P(E_i)$ is directly proportional to $i(i+2); 1\le i\le 100$. If the probability that a person selected at random is found to be positive for this virus is given by $\frac{108p}{575}$, then p is equal to ______.

25. If in an experiment of tossing of a fair coin 10 times, probability when no two heads are consecutive is p, then find the value of $64\,p$.

26. An urn contains 3 white balls, 5 black balls and 2 red balls. Two persons draw balls in turn, without replacement. The person who draws first a white ball wins the game. If a red ball is drawn, the game is a tie. Suppose $A_1 = \{$the player who begins the game is the winner$\}$, $A_2 = \{$the second participant is the winner$\}$ and $B = \{$the game is a tie$\}$. If $\lambda P(B) = 2$, then find the value of λ.

27. There are $N + 1$ identical boxes each containing N wall clocks. The r^{th} box contains $(r - 1)$ defective and $(N - r + 1)$ effective clocks for $1 \le r \le N + 1$. A box is selected at random and from this box a clock is chosen at random and is found to be effective. The probability that it is from kth box is $\frac{2N-2k+\lambda}{N^2+N}$, then find λ.

28. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of the subset P. A subset Q of A is again chosen at random. The probability such that $P\cap Q$ contains 2 elements is $\frac{{}^nC_a\cdot 3^{n-b}}{4^n}$, then find $a + b$.

Answer Key

Practice Exercise 1

1. (D)	**2.** (A)	**3.** (A)	**4.** (C)	**5.** (B)	**6.** (B)
7. (D)	**8.** (A)	**9.** (B)	**10.** (D)	**11.** (B)	**12.** (A)
13. (A)	**14.** (A)	**15.** (B)	**16.** (A)	**17.** (D)	**18.** (D)
19. (B)	**20.** (B)	**21.** (B)	**22.** (A)	**23.** (A)	**24.** (D)
25. (A)	**26.** (A)	**27.** (D)	**28.** (B)	**29.** (A)	**30.** (B)
31. (C)	**32.** (C)	**33.** (C), (D)	**34.** (A)	**35.** (C)	**36.** (D)
37. (D)	**38.** (A)	**39.** (C)	**40.** (B)	**41.** (A)	**42.** (B)
43. (C)	**44.** (B)	**45.** (D)	**46.** (C)	**47.** (A)	**48.** (D)
49. (A)	**50.** (C)	**51.** (A)	**52.** (C)	**53.** (B)	**54.** (D)
55. (B)	**56.** (B)	**57.** (B)	**58.** (C)	**59.** (C)	**60.** (B)
61. (C)	**62.** (A)	**63.** (A)	**64.** (D)	**65.** (A), (D)	**66.** (D)
67. (A)	**68.** (C)	**69.** (B)	**70.** (D)	**71.** (D)	**72.** (A)
73. (B)	**74.** (D)	**75.** (A)	**76.** (B)	**77.** (B)	**78.** (A)
79. (A)	**80.** (B)	**81.** (C)	**82.** (B)	**83.** (D)	**84.** (A)
85. (A)	**86.** (C)	**87.** (B), (C)	**88.** (D)	**89.** (C)	**90.** (A)
91. (B)	**92.** (C)				

Practice Exercise 2

1. (A), (C)	**2.** (A), (C)	**3.** (A), (B), (C), (D)	**4.** (A)	**5.** (D)	**6.** (A)
7. (C)	**8.** (B)	**9.** (A)	**10.** (A)	**11.** (A)	**12.** (B)
13. (A)	**14.** (B)	**15.** (C)	**16.** (D)	**17.** (C)	**18.** (A)

19. (A) $\to$ s, (B) $\to$ t, (C) $\to$ p, (D) $\to$ q
20. (A) $\to$ s, (B) $\to$ t, (C) $\to$ q, (D) $\to$ s
21. (A) $\to$ q, (B) $\to$ p, (C) $\to$ r, (D) $\to$ q
22. (A) $\to$ q, (B) $\to$ r, (C) $\to$ t, (D) $\to$ r
23. 7 **24.** 4 **25.** 9 **26.** 5
27. 2 **28.** 4

Solutions

Practice Exercise 1

1. They are mutually independent. Hence, the correct answer is option (D).
2. $B \cup C$ is independent to A, so S_1 is true.
   $B \cap C$ is also independent to A, so S_2 is true.
   Hence, the correct answer is option (A).
3. Required probability $= \left(\frac{4}{52}\right)^2 = \frac{1}{169}$
4. Since there are one A, two I and one O, hence, the required probability $= \frac{1+2+1}{11} = \frac{4}{11}$
5. Here $P(A) = 0.4$ and $P(\bar{A}) = 0.6$
   Probability that A does not happen at all $= (0.6)^3$
   Thus, required Probability $= 1-(0.6)^3 = 0.784$
6. Favourable cases for one are three, that is, 2, 4 and 6 and for other are two, that is, 3, 6. Hence,
   $$\text{required probability} = \left[\left(\frac{3\times 2}{36}\right)2 - \frac{1}{36}\right] = \frac{11}{36}$$
   {As same way happen when dice changes numbers among themselves}
7. The probability of students not solving the problem are
   $$1-\frac{1}{3} = \frac{2}{3},\ 1-\frac{1}{4} = \frac{3}{4} \text{ and } 1-\frac{1}{5} = \frac{4}{5}$$
   Therefore, the probability that the problem is not solved by any one of them $= \frac{2}{3}\times\frac{3}{4}\times\frac{4}{5} = \frac{2}{5}$
   Hence, the probability that problem is solved $= 1-\frac{2}{5} = \frac{3}{5}$
8. Required probability $= \left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right) = \frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5} = \frac{2}{5}$
9. Since favourable ways are 6. Total ways are 36.
   Hence,
   $$\text{required probability} = \frac{6}{36}$$
10. We have
   $$P(\bar{A}) = 0.05 \Rightarrow P(A) = 0.95$$
   Hence, the probability that the event will take place in 4 consecutive occasions is
   $$\{P(A)\}^4 = (0.95)^4 = 0.81450625$$
11. Same number can appear in 6 ways.

Hence,

$$\text{required probability } = \frac{6}{216} = \frac{1}{36}$$

12. Here,

$$P(A) = \frac{1}{2},\ P(B) = \frac{1}{3},\ P(C) = \frac{1}{4}$$

and $$P(\bar{A}) = \frac{1}{2}, P(\bar{B}) = \frac{2}{3}, P(\bar{C}) = \frac{3}{4}$$

Hence, required probability is

$$P(A)P(\bar{B})P(\bar{C}) + P(\bar{A})P(B)P(\bar{C}) + P(\bar{A})P(\bar{B})P(C)$$

$$\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\right) + \left(\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{4}\right) + \left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} = \frac{11}{24}$$

13. $n =$ total number of ways $= 2^4 = 16$

$m =$ Favourable number of ways $= 3$

Since the value of determinant is positive when it is

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

Hence,

$$\text{required probability } = \frac{3}{16}$$

14. Since $\overline{E_1} \cap \overline{E_2} = \overline{E_1 \cup E_2}$ and $(E_1 \cup E_2) \cap (\overline{E_1 \cup E_2}) = \phi$

$$P\{(E_1 \cup E_2) \cap (\overline{E_1} \cap \overline{E_2})\} = P(\phi) = 0 < \frac{1}{4}$$

15. Probability of getting at least one head in n tosses is

$$1 - \left(\frac{1}{2}\right)^n \ge 0.9 \Rightarrow \left(\frac{1}{2}\right)^n \le 0.1 \Rightarrow 2^n \ge 10 \Rightarrow n \ge 3$$

Hence, least value of $n = 4$.

16. Appearance of head on fifth toss does not depend on the outcomes of first four tosses. Hence,

$$P\text{ (head on } 5^{\text{th}} \text{ toss) } = \frac{1}{2}$$

17. We know a leap year is fallen within 4 years, so its probability is $\frac{25}{100} = \frac{1}{4}$

$$53^{\text{rd}} \text{ Sunday is a leap year } = \frac{1}{4} \times \frac{2}{7} = \frac{2}{28}$$

Similarly, probability of 53^{rd} Sunday in a non-leap year

$$= \frac{75}{100} \times \frac{1}{7} = \frac{3}{4} \times \frac{1}{7} = \frac{3}{28}$$

Therefore,

$$\text{required probability } = \frac{2}{28} + \frac{3}{28} = \frac{5}{28}$$

18. Total number of numbers $= (5)^2$

Favourable cases $= [12, 24, 32, 44, 52]$

Therefore,

$$\text{required probability } = \frac{5}{25} = \frac{1}{5}$$

19. Required probability $= \frac{1}{2}\left(\frac{3}{5} + \frac{2}{6}\right) = \frac{9+5}{30} = \frac{7}{15}$

20. Required probability $= \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^5\left(\frac{1}{6}\right) + \cdots$

$$= \frac{\frac{5}{6}\cdot\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} = \frac{5}{36 - 25} = \frac{5}{11}$$

21. Total ways of arrangements $= \frac{8!}{2! \cdot 4!}$

$$\bullet w \bullet x \bullet y \bullet z \bullet$$

Now 'S' can have places at dot's and in places of w, x, y, z we have to put $2A$'s, one I and one N. Therefore,

$$\text{favourable ways } = 5\left(\frac{4!}{2!}\right)$$

Hence,

$$\text{required probability } = \frac{5 \cdot 4!\, 2!\, 4!}{2!\, 8!} = \frac{1}{14}$$

22. Required probability $= \frac{{}^9C_2}{{}^{15}C_2} = \frac{9 \times 8}{15 \times 14} = \frac{12}{35}$

23. Required probability $= \frac{{}^5C_1 \times {}^8C_1}{{}^{13}C_2} + \frac{{}^5C_2}{{}^{13}C_2} = \frac{25}{39}$

24. Required probability $= \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1}{{}^9C_3} = \frac{2 \times 3 \times 4}{\left(\frac{9 \times 8 \times 7}{3 \times 2}\right)} = \frac{2}{7}$

25. By using digits 1, 2, 3, 4, 5, 6 and 8, total 5 digits numbers $= {}^7P_5$

And number of ways to form the numbers, they have even digit at both ends $= 4 \times 3 \times {}^5P_3$.

Hence,

$$\text{probability } = \frac{4 \times 3 \times {}^5P_3}{{}^7P_5} = \frac{2}{7}$$

26. Probability $= \frac{{}^3C_1 . {}^4C_1 . {}^5C_1}{{}^{12}C_3} = \frac{3}{11}$

27. Total ways $= 2!$ and ${}^6C_2 = 30$

Favourable cases $= 30 - 6 = 24$

Hence,

$$\text{required probability } = \frac{24}{30} = \frac{4}{5}$$

28. Let A and B be two given events. The odds against A are $5:2$, therefore

$$P(A) = \frac{2}{7}$$

The odds in favour of B are $6:5$, therefore

$$P(B) = \frac{6}{11}$$

The required probability $= 1 - P(\bar{A})P(\bar{B})$

$$= 1 - \left(1 - \frac{2}{7}\right)\left(1 - \frac{6}{11}\right) = \frac{52}{77}$$

29. Required probability $=\dfrac{(21)!2!}{(22)!}=\dfrac{1}{11}=\dfrac{1}{1+10}$

Therefore, odds against = 10 : 1.

30. Since here $P(A+B+C)=P(A)+P(B)+P(C)=\dfrac{2}{3}+\dfrac{1}{4}+\dfrac{1}{6}=\dfrac{13}{12}$, which is greater than 1.
Hence, the statement is wrong.

31. $P\left(\dfrac{\bar{A}}{\bar{B}}\right)=\dfrac{P(\bar{A}\cap\bar{B})}{P(\bar{B})}=\dfrac{P(\overline{A\cup B})}{P(\bar{B})}=\dfrac{1-P(A\cup B)}{P(\bar{B})}$

32. $P[B/(A\cup B^c)]=\dfrac{P(B\cap(A\cup B^c))}{P(A\cup B^c)}=\dfrac{P(A\cap B)}{P(A)+P(B^c)-P(A\cap B^c)}$

$$=\frac{P(A)-P(A\cap B^c)}{P(A)+P(B^c)-P(A\cap B^c)}=\frac{0.7-0.5}{0.8}=\frac{1}{4}$$

33. Since $P(A\cap B)=P(A)P(B)$.

It means A and B are independent events so A^c and B^c will also be independent. Hence,
$P(A\cup B)^c=P(A^c\cap B^c)=P(A^c)P(B^c)$ (Demorgan's law)

As A is independent of B, hence

$$P(A/B)=P(A),\ \{\because\ P(A\cap B)=P(B)P(A/B)\}$$

34. Required probability $=\dfrac{0.1}{0.1+0.32}=\dfrac{0.1}{0.42}=\dfrac{5}{21}$

35. $P(E_1)=\dfrac{4}{52}=\dfrac{1}{13}$, $P\left(\dfrac{E_2}{E_1}\right)=\dfrac{15}{51}=\dfrac{5}{17}$

$$P(E_1\cap E_2)=P(E_1).P\left(\frac{E_2}{E_1}\right)=\frac{1}{13}.\frac{5}{17}=\frac{5}{221}$$

36. The required probability is

$$^8C_6\left(\frac{1}{2}\right)^6.\left(\frac{1}{2}\right)^2+{}^8C_7\left(\frac{1}{2}\right)^7.\left(\frac{1}{2}\right)+{}^8C_8\left(\frac{1}{2}\right)^8=\frac{37}{256}$$

37. Let $P(\text{fresh egg})=\dfrac{90}{100}=\dfrac{9}{10}=p$

$P(\text{rotten egg})=\dfrac{10}{100}=\dfrac{1}{10}=q;\quad n=5,\ r=5$

So, the probability that none egg is rotten is

$$^5C_5\left(\frac{9}{10}\right)^5.\left(\frac{1}{10}\right)^0=\left(\frac{9}{10}\right)^5$$

38. Probability of coming 'six' in one throw is $\dfrac{1}{6}$.

Hence, required probability is given by

$$^4C_4\left(\frac{1}{6}\right)^4.\left(\frac{5}{6}\right)^0=\frac{1}{1296}$$

39. Required probability $={}^4C_3.\left(\dfrac{1}{2}\right)^3\left(\dfrac{1}{2}\right)=\dfrac{1}{4}$

40. $p=P$ (getting a head) $=\dfrac{1}{2},q=\dfrac{1}{2}$

Hence, required probability $=P$(six successes)

$$={}^{10}C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4=\frac{10!}{6!4!}.\frac{1}{2^{10}}=\frac{105}{512}$$

41. Probability of occurrence of '4' $=\dfrac{1}{6}$

Probability of inoccurrence of '4' $=\dfrac{5}{6}$

Hence, required probability is

$$^2C_1\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)+{}^2C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^0=\frac{11}{36}$$

42. Required probability $=A$ occurs and B does not occur or B occurs and A does not occur
$=P(A\cap\bar{B})+P(\bar{A}\cap B)=P(A)-P(A\cap B)+P(B)-P(A\cap B)$
$=P(A)+P(B)-2P(A\cap B)$

43. We are given that $P(A\cup B)=0.6$ and $P(A\cap B)=0.2$.

We know that if A and B are any two events, then

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$0.6=1-P(\bar{A})+1-P(\bar{B})-0.2$$
$$\Rightarrow P(\bar{A})+P(\bar{B})=2-0.8=1.2$$

44. $0.8=0.3+x-0.3x\Rightarrow x=5/7$

45. $P(A\cap B)=P(A)+P(B)-P(A\cup B)$
and

$$0\le P(A\cup B)\le 1$$

Therefore,

$$P(A\cap B)\ge P(A)+P(B)-1$$

and

$$P(A\cap B)\le P(A)+P(B)$$

46. $P(A)=P(B)$ As this gives

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

or

$$P(A)=2P(A)-P(A)$$
$$\Rightarrow P(A\cup B)=P(A\cap B)$$

47. $P(A^c\cap B^c)=1-P(A\cup B)=1-[0.25+0.5-0.14]=0.39$

48. $P(A\cup B)=P(A)+P(B)=0.45+0.35=0.8$

49. $P(A'\cap B')=1-P(A\cup B)\Rightarrow P(A\cup B)=\dfrac{2}{3}$

Now,

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\Rightarrow\frac{2}{3}=x+x-\frac{1}{3}\Rightarrow x=\frac{1}{2}$$

50. Here,

$$P(R)=\frac{10}{100}=0.1,\quad P(F)=\frac{5}{100}=0.05$$

(See Fig. 28.13)

Therefore,

$$P(F\cap R)=\frac{3}{100}=0.03$$

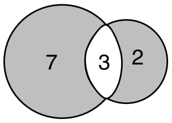


Figure 28.13

Hence,

$$\text{required probability} = P(R)+P(F)-2P(F\cap R)$$
$$= 0.1 + 0.05 - 2(0.03) = 0.09$$

51. $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\frac{5}{6}=\frac{1}{3}+\frac{1}{2}-P(A\cap B) \Rightarrow P(A\cap B)=0$$

Therefore, events A and B are mutually exclusive.

52. Three dices can be thrown in $6\times6\times6=216$ ways. A total 17 can be obtained as $(5,6,6)$, $(6,5,6)$, $(6,6,5)$. A total 18 can be obtained as $(6,6,6)$. Hence,

$$\text{the required probability} = \frac{4}{216}=\frac{1}{54}$$

53. Total probable ways $=8$

Favourable number of ways $=[HTH, THT]$

Hence, required probability $=\frac{2}{8}=\frac{1}{4}$.

54. The probability of getting an even number in first draw $=\frac{9}{19}$. The probability of getting an even number in second draw $=\frac{8}{18}$. Both are independent event and so required probability $=\frac{9}{19}\times\frac{8}{18}=\frac{4}{19}$.

55. The second ball can be red in two different ways

(i) First is white and second red

$$P(A)=\frac{3}{5}\times\frac{2}{4}=\frac{6}{20}$$

(ii) First is red and second is also red

$$P(B)=\frac{2}{5}\times\frac{1}{4}=\frac{2}{20}$$

Both are mutually exclusive events, hence, required probability is $\frac{6}{20}+\frac{2}{20}=\frac{8}{20}=\frac{2}{5}$.

56. Probability of success $=\frac{2}{6}=\frac{1}{3}=p$

Probability of failure $=1-\frac{1}{3}=\frac{2}{3}=q$

Probability that success occurs in even number of tosses

$$= P(FS)+P(FFFS)+P(FFFFFS)+\cdots = pq+q^3p+q^5p+\ldots$$
$$=\frac{pq}{1-q^2}=\frac{2}{5}$$

57. There are two conditions:

(i) When first is an ace of heart and second one is non-ace of heart $=\frac{1}{52}\times\frac{51}{51}=\frac{1}{52}$

(ii) When first is non-ace of heart and second one is an ace of heart $=\frac{51}{52}\times\frac{1}{51}=\frac{1}{52}$

Hence,

$$\text{required probability} = \frac{1}{52}+\frac{1}{52}=\frac{1}{26}$$

58. Total number of ways $=\frac{10!}{2!}$

Favourable number of ways for 'I' come together is $9!$

Thus, probability that 'I' come together $=\frac{9!\times2!}{10!}=\frac{2}{10}=\frac{1}{5}$

Hence,

$$\text{required probability} = 1-\frac{1}{5}=\frac{4}{5}$$

59. Required probability = Either the balls are red or the balls are black

$$=\frac{{}^8C_2}{{}^{15}C_2}+\frac{{}^7C_2}{{}^{15}C_2}=\frac{28+21}{105}=\frac{49}{105}=\frac{7}{15}$$

60. Let n = total no. of ways = 10!

m = favourable no. of ways = $2\times 5!5!$

Since the boys and girls can sit alternately in 5!5! ways if we begin with a boy and similarly, they can sit alternately in 5!5! ways if we begin with a girl. Hence,

$$\text{required probability} = \frac{m}{n}$$
$$=\frac{2\times5!5!}{10!}=\frac{2\times5!}{10\times9\times8\times7\times6}=\frac{1}{126}$$

61. The probability of solving the question by these three students are $\frac{1}{3},\frac{2}{7}$ and $\frac{3}{8}$, respectively.

$$P(A)=\frac{1}{3};\ P(B)=\frac{2}{7};\ P(C)=\frac{3}{8}$$

Then probability of question solved by only one student

$$=P(A\bar{B}\bar{C} \text{ or } \bar{A}B\bar{C} \text{ or } \bar{A}\bar{B}C)$$
$$=P(A)\,P(\bar{B})\,P(\bar{C})+P(\bar{A})\,P(B)P(\bar{C})+P(\bar{A})P(\bar{B})P(C)$$
$$=\frac{1}{3}\cdot\frac{5}{7}\cdot\frac{5}{8}+\frac{2}{3}\cdot\frac{2}{7}\cdot\frac{5}{8}+\frac{2}{3}\cdot\frac{5}{7}\cdot\frac{3}{8}=\frac{25+20+30}{168}=\frac{25}{56}$$

62. Since events are independent. So,

$$P(A\cap B')=P(A)\times P(B')=\frac{3}{25}$$
$$\Rightarrow P(A)\times\{1-P(B)\}=\frac{3}{25} \quad (1)$$

Similarly,

$$P(B)\times\{1-P(A)\}=\frac{8}{25} \quad (2)$$

On solving Eqs. (1) and (2), we get

$$P(A)=\frac{1}{5} \text{ and } \frac{3}{5}$$

63. (i) This question can also be solved by one student.

(ii) This question can be solved by two students simultaneously.

(iii) This question can be solved by three students all together.

$$P(A)=\frac{1}{2},\ P(B)=\frac{1}{4},P(C)=\frac{1}{6}$$
$$P(A\cup B\cup C)= P(A)+P(B)+P(C)$$
$$-[P(A)\cdot P(B)+P(B)\cdot P(C)+P(C)\cdot P(A)]+[P(A)\cdot P(B)\cdot P(C)]$$

$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}-\left[\frac{1}{2}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{6}+\frac{1}{6}\times\frac{1}{2}\right]+\left[\frac{1}{2}\times\frac{1}{4}\times\frac{1}{6}\right]=\frac{33}{48}$$

64. $P(E\cap F)=P(E)\cdot P(F)$

Now,

$$P(E\cap F^c)=P(E)-P(E\cap F)=P(E)[1-P(F)]=P(E)\cdot P(F^c)$$

and

$$P(E^c\cap F^c)=1-P(E\cup F)=1-[P(E)+P(F)-P(E\cap F)$$
$$=[1-P(E)][1-P(F)]=P(E^c)P(F^c)$$

Also,

$$P(E/F)=P(E) \text{ and } P(E^c/F^c)=P(E^c)$$
$$\Rightarrow P(E/F)+P(E^c/F^c)=1$$

65. $P(E/F)+P(\overline{E}/F)=\dfrac{P(E\cap F)+P(\overline{E}\cap F)}{P(F)}=\dfrac{P\{(E\cap F)\cup(\overline{E}\cap F)\}}{P(F)}$

[$\because E\cap F$ and $\overline{E}\cap F$ are disjoint]

$$=\frac{P\{(E\cup\overline{E})\cap F\}}{P(F)}=\frac{P(F)}{P(F)}=1$$

Similarly, we can show that (B) and (C) are not true while (D) is true.

$$P\left(\frac{E}{\overline{F}}\right)+P\left(\frac{\overline{E}}{\overline{F}}\right)=\frac{P(E\cap\overline{F})}{P(\overline{F})}+\frac{P(\overline{E}\cap\overline{F})}{P(\overline{F})}=\frac{P(\overline{F})}{P(\overline{F})}=1$$

66. Let E_1 be the event that the ball is drawn from bag A, E_2 the event that it is drawn from bag B and E that the ball is red. We have to find $P(E_2/E)$.

Since both the bags are equally likely to be selected, we have

$$P(E_1)=P(E_2)=\frac{1}{2}$$

Also $P(E/E_1)=3/5$ and $P(E/E_2)=5/9$.

Hence, by Bayes' theorem, we have

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}=\frac{\frac{1}{2}\cdot\frac{5}{9}}{\frac{1}{2}\cdot\frac{3}{5}+\frac{1}{2}\cdot\frac{5}{9}}=\frac{25}{52}$$

67. For binomial distribution, mean $=np$ and variance $=npq$

$$n=3,\ p=\frac{2}{6}=\frac{1}{3},\ q=1-p=1-\frac{1}{3}=\frac{2}{3}$$

So,

$$\text{mean } (\mu)=3\times\frac{1}{3}=1$$
$$\text{Variance } (\sigma^2)=3\times\frac{1}{3}\times\frac{2}{3}=\frac{2}{3}$$

68. Let X denote a random variable which is the number of aces. Clearly, X takes values, 1, 2.

Hence,

$$p=\frac{4}{52}=\frac{1}{13},\ q=1-\frac{1}{13}=\frac{12}{13}$$
$$P(X=1)=2\times\left(\frac{1}{13}\right)\times\left(\frac{12}{13}\right)=\frac{24}{169}$$
$$P(X=2)=2\cdot\left(\frac{1}{13}\right)^2\left(\frac{12}{13}\right)^0=\frac{2}{169}$$
$$\text{Mean}=\sum P_iX_i=\frac{24}{169}+\frac{2}{169}=\frac{26}{169}=\frac{2}{13}$$

69. Since the item are choosen without replacement. Hence,

$$P(X=x)=\frac{^3C_x+{}^7C_{4-x}}{^{10}C_4}$$

Putting $x=1,2$, we have

$$P(0<x<3)=\frac{^3C_1\times{}^7C_3}{210}+\frac{^3C_2\times{}^7C_2}{210}$$
$$=\frac{3\times35+3\times21}{210}=\frac{105+63}{210}=\frac{168}{210}=\frac{4}{5}$$

70. $P(E)\le P(F)\Rightarrow n(E)\le n(F)$

$$P(E\cap F)>0\Rightarrow E\cap F\ne\phi$$

These do not mean that E is a sub-set of F or F is a sub-set of E, that is, $E\subseteq F$ or $F\subseteq E$ or $\overline{E}\subseteq\overline{F}$.

71. Let $p_1=0.4,\ p_2=0.3,\ p_3=0.2$ and $p_4=0.1$.

P (the gun hits the plane) $=P$ (the plane is hit in once)

$=1-P$ (the plane is hit in none of the shots)

$=1-(1-p_1)(1-p_2)(1-p_3)(1-p_4)=0.6976$

72. Since $\dfrac{(1+3p)}{3}, \dfrac{(1-p)}{4}$ and $\left(\dfrac{1-2p}{2}\right)$ are the probabilities of the three mutually exclusive events, we must have

$$0\le\frac{1+3p}{3}\le1,\ 0\le\frac{1-p}{4}\le1 \text{ and } 0\le\frac{1-2p}{2}\le1$$
$$\Rightarrow -1\le3p\le2,\ -3\le p\le1 \text{ and } -1\le2p\le1$$
$$\Rightarrow -\frac{1}{3}\le p\le\frac{2}{3},\ -3\le p\le1 \text{ and } -\frac{1}{2}\le p\le\frac{1}{2}$$

Also as $\dfrac{1+3p}{3}, \dfrac{1-p}{4}$ and $\dfrac{1-2p}{2}$ are the probabilities of three mutually exclusive events.

$$0\le\frac{1+3p}{3}+\frac{1-p}{4}+\frac{1-2p}{2}\le1$$
$$\Rightarrow 0\le4+12p+3-3p+6-12p\le12 \Rightarrow \frac{1}{3}\le p\le\frac{13}{3}$$

Thus, the required value of p are such that

$$\text{Max.}\left\{-\frac{1}{3},-3,-\frac{1}{2},\frac{1}{3}\right\}\le p \text{ min.}\left\{\frac{2}{3},1,\frac{1}{2},\frac{13}{3}\right\}$$
$$\Rightarrow \frac{1}{3}\le p\le\frac{1}{2}$$

73. Required probability = probability that either the number is 7 or the number is 8.

That is, required probability $=P_7+P_8$

Now

$$P_7=\frac{1}{2}\cdot\frac{1}{11}+\frac{1}{2}\cdot\frac{6}{36}=\frac{1}{2}\left(\frac{1}{11}+\frac{1}{6}\right)$$
$$P_8=\frac{1}{2}\cdot\frac{1}{11}+\frac{1}{2}\cdot\frac{5}{36}=\frac{1}{2}\left(\frac{1}{11}+\frac{5}{36}\right)$$

Hence,

$$P=\frac{1}{2}\left(\frac{2}{11}+\frac{11}{36}\right)=0.244$$

74. There will be no x because $P(AB)$ can never be less than $P(ABC)$. Hence, the correct answer is option (D).

75. Since m and n are selected between 1 and 100, hence sample space $=100\times100$

Also $7^1=7,\ 7^2=49,\ 7^3=343,\ 7^4=2401,\ 7^5=16807$, etc. Hence, 1, 3, 7 and 9 will be the last digits in the powers of 7. Hence, for favourable cases

$n\ m\rightarrow$

$\downarrow$

1, 1 1, 2 1, 3… 1, 100

2, 1 2, 2 2, 3… 2, 100

....

100, 1 100, 2 100, 3…100, 100

For $m=1$; $n=3, 7, 11\ldots 97$

Therefore, favourable cases $=25$.

For $m=2$; $n=4, 8, 12\ldots100$

Therefore, favourable cases $=25$.

Similarly for every m, favourable n are 25.

Hence, total favourable cases $=100\times25$

and,

$$\text{required probability }=\frac{100\times25}{100\times100}=\frac{1}{4}$$

76. This is a problem of without replacement.

$$P=\frac{\text{one def. from 2 def.}}{\text{anyone from 4}}\times\frac{\text{1 def. from remaining 1 def.}}{\text{anyone from remaining 3}}$$

Hence,

$$\text{required probability }=\frac{2}{4}\times\frac{1}{3}=\frac{1}{6}$$

Alternate Solution: Number of ways in which two faulty machines may be detected (depending upon the test done to indentify the faulty machines) $={}^4C_2=6$

Number of favourable cases $=1$

[When faulty machines are identified in the first and the second test]

Hence,

$$\text{required probability }=\frac{1}{6}$$

77. The probability of throwing 9 with two dices $=\frac{4}{36}=\frac{1}{9}$

Hence, the probability of not throwing 9 with two dices $=\frac{8}{9}$

If A is to win he should throw 9 in 1st or 3rd or 5th attempt.
If B is to win, he should throw, 9 in 2nd, 4th attempt.

$$B\text{'s chances }=\left(\frac{8}{9}\right)\cdot\frac{1}{9}+\left(\frac{8}{9}\right)^3\cdot\frac{1}{9}+....=\frac{\frac{8}{9}\times\frac{1}{9}}{1-\left(\frac{8}{9}\right)^2}=\frac{8}{17}$$

78. Let $n=$ total number of ways $=12!$ and $m=$ favourable number of ways $=2\times6!\cdot6!$

Since the boys and girls can sit alternately in $6!\cdot6!$ ways if we begin with a boy and similarly they can sit alternately in $6!\cdot6!$ ways if we begin with a girl.

Hence,

$$\text{required probability }=\frac{m}{n}=\frac{2\times6!\cdot6!}{12!}=\frac{1}{462}$$

79. Here, the least number of draws to obtain 2 aces are 2 and the maximum number is 50 thus n can take value from 2 to 50. Since we have to make n draws for getting two aces, in $(n-1)$ draws, we get any one of the 4 aces and in the n^{th} draw we get one ace. Hence, the required probability is

$$\frac{{}^4C_1\times{}^{48}C_{n-2}}{{}^{52}C_{n-1}}\times\frac{3}{52-(n-1)}$$

$$=\frac{4\times(48)!}{(n-2)!(48-n+2)!}\times\frac{(n-1)!(52-n+1)!}{(52)!}\times\frac{3}{52-n+1}$$

$$=\frac{(n-1)(52-n)(51-n)}{50\times49\times17\times13}\quad\text{(on simplification)}$$

80. The number of ways to arrange 7 white and 3 black balls in a row $=\frac{10!}{7!\cdot3!}=\frac{10\cdot9\cdot8}{1\cdot2\cdot3}=120$

Number of blank places between 7 balls are 6. There is 1 place before first ball and 1 place after last ball. Hence, total number of places are 8.
Hence, 3 black balls are arranged on these 8 places so that no two black balls are together in number of ways.

$${}^8C_3=\frac{8\times7\times6}{1\times2\times3}=56$$

So, required probability $=\frac{56}{120}=\frac{7}{15}$

81. $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow P(A\cap B)=P(A)+P(B)-P(A\cap B)\ \{\because\ P(A\cap B)=P(A\cup B)\}$

$$\Rightarrow 2P(A\cap B)=P(A)+P(B)$$

$$\Rightarrow 2P(A)\cdot\frac{P(A\cap B)}{P(A)}=P(A)+P(B)$$

$$\Rightarrow 2P(A)\cdot P\left(\frac{B}{A}\right)=P(A)+P(B)$$

82. $P(\bar{A}\cap\bar{B})=P(\overline{A\cup B})=1-P(A\cup B)$

Since A and B are mutually exclusive, so

$$P(A\cup B)=P(A)+P(B)$$

Hence, required probability $=1-(0.5+0.3)=0.2$

83. Probability of occurrence of at most one of event $A, B=$ Probability of occurrence of event B not event A + Probability of occurrence of event A not event B + Probability of non–occurrence of event A and B

$=P(A'\cap B)+P(A\cap B')+P(A'\cap B')$

$=P(B)-P(A\cap B)+P(A)-P(A\cap B)+1-P(A\cup B)$

$=-P(A\cap B)+1$

$P(A')+P(B')+P(A\cup B)-1=1-P(A)+1-P(B)+P(A)$

$+P(B)-P(A\cap B)-1=1-P(A\cap B)$

84. We know that

$$P(A/B)=\frac{P(A\cap B)}{P(B)}$$

Also we know that

$$P(A \cup B) \le 1$$

$$\Rightarrow P(A)+P(B)-P(A \cap B) \le 1 \Rightarrow P(A \cap B) \ge P(A)+P(B)-1$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} \ge \frac{P(A)+P(B)-1}{P(B)} \Rightarrow P(A/B) \ge \frac{P(A)+P(B)-1}{P(B)}$$

85. Let the events are:

R_1 = A red ball is drawn from urn A and placed in B

B_1 = A black ball is drawn from urn A and placed in B

R_2 = A red ball is drawn from urn B and placed in A

B_2 = A black ball is drawn from urn B and placed in A

R = A red ball is drawn in the second attempt from A

Then, the required probability is

$$P(R_1R_2R)+(R_1B_2R)+P(B_1R_2R)+P(B_1B_2R)$$

$$= P(R_1)P(R_2)P(R)+P(R_1)P(B_2)P(R)+P(B_1)P(R_2)P(R)+P(B_1)P(B_2)P(R)$$

$$= \left(\frac{6}{10}\times\frac{5}{11}\times\frac{6}{10}\right)+\left(\frac{6}{10}\times\frac{6}{11}\times\frac{5}{10}\right)+\left(\frac{4}{10}\times\frac{4}{11}\times\frac{7}{10}\right)+\left(\frac{4}{10}\times\frac{7}{11}\times\frac{6}{10}\right)$$

$$= \frac{32}{55}$$

86. A leap year consists of 366 days comprising of 52 weeks and 2 days. There are 7 possibilities for these 2 extra days *viz.*

(i) Sunday, Monday, (ii) Monday, Tuesday,

(iii) Tuesday, Wednesday, (iv) Wednesday, Thursday,

(v) Thursday, Friday, (vi) Friday, Saturday and

(vii) Saturday, Sunday.

Let us consider two events :

A: the leap year contains 53 Sundays

B: the leap year contains 53 Mondays.

Then we have

$$P(A)=\frac{2}{7},\ P(B)=\frac{2}{7},\ P(A \cap B)=\frac{1}{7}$$

Hence, required probability $= P(A \cup B)$

$$= P(A)+P(B)-P(A \cap B)=\frac{2}{7}+\frac{2}{7}-\frac{1}{7}=\frac{3}{7}$$

87. Let M, P and C be the events of passing in mathematics, physics and chemistry, respectively.

$$P(M \cup P \cup C)=\frac{75}{100}=\frac{3}{4}$$

$$P(M \cap P)+P(P \cap C)+P(M \cap C)-2P(M \cap P \cap C)=\frac{50}{100}=\frac{1}{2}$$

$$P(M \cap P)+P(P \cap C)+P(M \cap C)-2P(M \cap P \cap C)=\frac{40}{100}=\frac{2}{5}$$

Therefore,

$$m(1-p)(1-c)+p(1-m)(1-c)+c(1-m)(1-p)+mp(1-c)+mc$$

$$(1-p)+pc(1-m)+mpc=\frac{3}{4}$$

$$\Rightarrow m+p+c-mc-mp-pc+mpc=\frac{3}{4} \qquad (1)$$

Similarly,

$$mp(1-c)+pc(1-m)+mc(1-p)+mpc=\frac{1}{2}$$

$$\Rightarrow mp+pc+mc-2mpc=\frac{1}{2} \qquad (2)$$

$$mp(1-c)+pc(1-m)+mc(1-p)=\frac{2}{5}$$

$$\Rightarrow mp+pc+mc-3mpc=\frac{2}{5} \qquad (3)$$

From Eqs. (2) and (3),

$$mpc=\frac{1}{2}-\frac{2}{5}=\frac{1}{10}$$

From Eqs. (1) and (2),

$$m+p+c-mpc=\frac{3}{4}+\frac{1}{2}$$

Hence,

$$m+p+c=\frac{3}{4}+\frac{1}{2}+\frac{1}{10}=\frac{15+10+2}{20}=\frac{27}{20}$$

88. Let E be the event that a new product is introduced.

Then $P(A)=0.5$, $P(B)=0.3$, $P(C)=0.2$ and

$P(E/A)=0.7$, $P(E/B)=0.6$, $P(E/C)=0.5$.

Since, A, B and C are mutually exclusive and exhaustive events.

$$P(E)=P(A)\cdot P(E/A)+P(B)\cdot P(E/B)+P(C)\cdot P(E/C)$$

$$=0.5\times0.7+0.3\times0.6+0.2\times0.5=0.35+0.18+0.10=0.63$$

89. Required probability $=\frac{1}{2}\cdot\frac{4}{7}+\frac{1}{2}\cdot\frac{6}{8}=\frac{37}{56}$

90. P(minimum face value not less than 2 and maximum face value is not greater than 5) is

$$P(2 \text{ or } 3 \text{ or } 4 \text{ or } 5)=\frac{4}{6}=\frac{2}{3}$$

Hence,

$$\text{required probability } = {}^4C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^0=\frac{16}{81}$$

91. Matches played by India are four. Maximum points in any match are 2.

Therefore, maximum points in four matches can be 8 only.

Therefore, probability $(P)=p(7)+p(8)$

$$p(7)={}^4C_1(0.05)(0.5)^3=0.0250$$

$$p(8)=(0.5)^4=0.0625$$

$$\Rightarrow P=0.0875$$

92. To get 3 white balls in first 6 draw and then a white again in 7^{th} draws.

$$P={}^6C_3\times\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)\Rightarrow P=\frac{5}{32}$$

Practice Exercise 2

1.
$$P(E \cap F)=P(E)\cdot P(F)=\frac{1}{6} \qquad (1)$$

$$P(E^C \cap F^C)=[1-P(E)][1-P(F)]=\frac{1}{3}$$

$$P(E)+P(F)=\frac{5}{6} \quad (2)$$

$$\Rightarrow |P(E)-P(F)|=\frac{1}{6}$$

As

$$[P(E)-P(F)][1-P(F)]>0$$

$$P(E)>P(F) \quad P(E)-P(F)=\frac{1}{6} \quad (3)$$

Solving Eqs. (2) and (3), we get

$$P(E)=\frac{1}{2},\ P(F)=\frac{1}{3}$$

2. Let the drawer contains p balls of which m are red.
Probability of drawing two red balls at random is

$$\frac{^mC_2}{^pC_2}=\frac{1}{2}$$

$$\Rightarrow 2m(m-1)=p(p-1)$$

$$\Rightarrow 2m^2-2m-p^2+p=0$$

$$\Rightarrow m=\frac{2\pm\sqrt{4-8(p-p^2)}}{4}=\frac{1\pm\sqrt{1-2p+2p^2}}{2}$$

$$\Rightarrow 1-2p+2p^2 \text{ should be an odd perfect square.}$$

That is, $p=21, 4$ but $p\neq 3$, when 3 balls out of 4 are red. Therefore, 15 balls out of 21 are red.

3. (i) $P(E_1)=1-P(R\,R\,R)=1-\left[\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\right]=0.9$

(ii) $P(E_2)=3\,P(B\,R\,R)=3.\frac{2}{3}\cdot\frac{1}{4}\cdot\frac{2}{5}=0.2$

(iii) $P(E_3)=P(R\,R\,R/R\,R\,R\cup B\,B\,B)=\dfrac{P(RRR)}{P(RRR)+P(BBB)}$

But $P(B\,B\,B)=\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}=\frac{8}{20}\Rightarrow P(E_3)=\frac{0.1}{0.1+0.4}=0.2$

(iv) $P(E_4)=1-P(B\,B\,B)=1-\frac{2}{5}=0.6$

4. Let α and β be the number of heads and tails thrown by A respectively, then

$$\alpha+\beta=n+1 \quad (1)$$

Let γ and δ be the number of heads and tails thrown by B respectively, then

$$\gamma+\delta=n \quad (2)$$

Required probability is $P(\alpha>\gamma)=p$ (say).
Due to symmetry, P (A will have more heads than B) $= P$ (A will have more tails than B), that is, (3)

$$P(\alpha>\gamma)=P(\beta>\delta)=p$$

Now

$$P(\alpha>\gamma)=1-P(\alpha\le\gamma)$$

Also

$$\alpha\le\gamma\Leftrightarrow n+1-\beta\le n-\delta\Leftrightarrow 1+\delta\le\beta$$
$$\Leftrightarrow \delta<\beta \text{ [From Eqs. (1) and (2)]}$$

Hence,

$$P(\alpha\le\gamma)=P(\delta<\beta)$$

$$\Rightarrow P=1-P(\delta<\beta)=1-P$$

Therefore,

$$P=\frac{1}{2}$$

5. Sample space is { –0.50, –0.49, ..., –0.01, 0, 0.01, ..., 0.49} and the event is {–0.10, –0.09, ..., 0.10}.
Hence, the probability is 21/100.

6. See Fig. 28.14. Let S denote the set of points inside a square with corners (x, y), $(x, y+1)$, $(x+1, y)$, $(x+1, y+1)$, x and y are integers.
Clearly, each of the four points belong to the set X.
Let P denote the set of points in S with distance less than $\frac{1}{4}$ from any corner point. P consists of four quarter circles each of radius $\frac{1}{4}$.

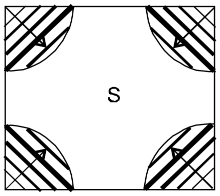


Figure 28.14

A coin, whose centre falls in S, will cover a point of X if and only if its centre falls in P.

Hence,

$$\text{required probability, } p=\frac{\text{Area of } P}{\text{Area of } S}=\frac{\pi\left(\frac{1}{4}\right)^2}{1\times1}=\frac{\pi}{16}$$

7. $P(E_1)=\frac{1}{10}\times1+\frac{2}{10}\times\frac{1}{2}+\frac{3}{10}\times\frac{1}{3}+\frac{4}{10}\times\frac{1}{4}=\frac{2}{5}$

8. $P(A_3/E_2)=\dfrac{\frac{3}{10}\times\frac{1}{3}}{\frac{2}{10}\times\frac{1}{2}+\frac{3}{10}\times\frac{1}{3}+\frac{4}{10}\times\frac{1}{4}}=\frac{1}{3}$

9. Expectation $=\frac{4}{10}\times1+\frac{3}{10}\times2+\frac{2}{10}\times3+\frac{1}{10}\times4=2$

10. Required probability $=\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\left(1-\frac{1}{4}\right)=0.728$

11. $P(\text{I}/\text{II})=\dfrac{P(\text{I}).P(\text{II}/\text{I})}{P(\text{II})}=\dfrac{\frac{1}{2}\cdot\frac{1}{\sqrt{2}}}{\frac{1}{2}\cdot\frac{1}{\sqrt{2}}+\frac{1}{8}}=0.74$

12. Probability that Sania wins the match $=(1-R)\,R^2\,R=R^3\,(1-R)$
Probability Sania loses the match $=(1-R)\,R^2\,(1-R)+(1-R)\,(1-R^2)$

Hence,

$$R^3\,(1-R)>(1-R)\,R^2\,(1-R)+(1-R)\,(1-R^2)$$

$$\Rightarrow R^3>R^2\,(1-R)+(1-R^2)=1-R^3$$

$\Rightarrow 2R^3 > 1$

$\Rightarrow R \in \left[\left(\frac{1}{2}\right)^{\frac{1}{3}}, 1\right]$

Common Solution for Questions 13–15:

$n = 10k + r, k, r \in N, 0 \le r \le 9$

Unit place of a^2 will contain 0, 1, 4, 5, 6, 9 only.

Hence, $a^2 - 1$ is divisible by 10 only if unit place of a^2 contain 1.

If unit place of a^2 is 1, then unit place of a will be 1 or 9.

13. $n = 10k + r$

$$r = 0$$

$n = 10k$, no. of a whose unit place is 1 or 9

$\Rightarrow k = 1, n = 10$, no. of a whose unit place is 2

$\Rightarrow k = 2, n = 20$, no. of a whose unit place is 4

$\Rightarrow k = k, n = 10k$, no. of a whose unit place is $2k$

Therefore

$$p_n = \frac{2k}{n}$$

14. $n = 10k + 9$

Number of a whose unit place is 1 or $9 = 2(k + 1)$

Therefore,

$$p_n = \frac{2(k+1)}{n}$$

15. Number of a whose unit place is 1 or $9 = 2k + 1$.

Therefore,

$$p_n = \frac{2k+1}{n}$$

16. A can be drawn out only at even numbered round. Therefore, A will not be drained out at the 11^{th} round.

17. To finish at the 12^{th} round he must have exactly 1 head in the first 10 rounds, and a tail at the 11^{th} and the 12^{th} round. The probability of this is ${}^{10}C_1\, pq^{11}$.

18. To drain out at the 14^{th} round, two cases arise

(i) He gets exactly 2 heads in the first 10 rounds.
Therefore, probability in this case is

$${}^{10}C_2\, p^2 q^8 . q^4 = 45p^2q^{12}$$

(ii) He gets exactly 1 head in the first 10 rounds and then exactly one head at the next two rounds.
Therefore, probability in this case is

$${}^{10}C_1\, pq^9 . {}^2C_1\, pq . q^2$$
$$= 20p^2q^{12}$$

To drain out earlier than 14^{th} round, two cases arise

(i) He gets no head in the first 10 rounds.
Therefore, probability in this case is q^{10}.

(ii) He gets exactly one head in first 10 rounds and then no heads.
Therefore, probability in this case is

$${}^{10}C_1\, q^9 . p . q^2 = 10pq^{11}$$

Therefore,

required probability $= 65\, p^2q^{12} + q^{10} + 10pq^{11}$

19. (A) $\frac{{}^{11}C_5}{{}^{12}C_6} = \frac{1}{2}$

(B) Let E_1 be the event that S_3 and S_4 are in same group and E_2 be the event that S_3 and S_4 are in different group. Then

$$P(E_1) = \frac{1}{11}, P(E_2) = \frac{10}{11}$$

Let E be the event that exactly one of S_3 and S_4 is among the losers. Then

$$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) = \frac{1}{11} \cdot 1 + \frac{10}{11} \cdot \frac{1}{2} = \frac{6}{11}$$

(C) S_2 and S_4 should be in different groups for both being winner,

$$\text{Required probability} = \frac{10}{11}\left(\frac{1}{2} \cdot \frac{1}{2}\right) = \frac{5}{22}$$

(D) E_1 be the event that S_3 and S_4 are in same group E_2 be the event that S_3 and S_4 are in different group

$$P(E_1) = \frac{1}{11}, P(E_2) = \frac{10}{11}$$

then,

$$\text{required probability} = \frac{\frac{10}{11} \times \frac{1}{2}}{\frac{10}{11} \times \frac{1}{2} + \frac{1}{11} \times 1} = \frac{5}{6}$$

20. (A) Let E_1, E_2, E_3, E_4 be the events that the bag contains 1 white, 2 white, 3 white, 4 white ball, respectively.

Let $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$

Let W be the event that the ball drawn is white. Then

$$P(W) = \sum P(E_i) P(W/E_i) = \frac{1}{4}\left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4}\right) = \frac{5}{8}$$

Now,

$$P(E_4/W) = \frac{P(E_4)P(W/E_4)}{P(W)} = \frac{1/4}{5/8} = \frac{2}{5}$$

Therefore,

$$\frac{2}{5} = \frac{p}{15} \Rightarrow p = 6$$

(B) ${}^{12}C_1 + {}^{12}C_2\, ({}^2C_1 + 2.{}^2C_2) + {}^{12}C_3\, ({}^3C_1 + 2.{}^3C_2) + \cdots + {}^{12}C_{12}\, ({}^{12}C_1 + 2.{}^{12}C_2)$

$= ({}^{12}C_1 + 2.{}^{12}C_2 + 3.{}^{12}C_3 + \cdots + 12.{}^{12}C_2) + 2\, ({}^{12}C_2.{}^2C_2 + {}^{12}C_3.{}^3C_2 + \cdots + {}^{12}C_{12}.{}^{12}C_2)$

$= \sum_{r=1}^{12} r\ {}^{12}C_r + 12 \times 11 \times \sum_{r=2}^{12} {}^{10}C_{r-2} = 12 \times 2^{11} + 12 \times 11 \times 2^{10}$

$= 12 \times 2^{10}\,(2 + 11) = 13 \times 2^{10} \times 12$

Hence,

$$13 \times 2^{10} \times 12 = 13 \times 2^{10} \times m$$

Therefore, $m = 12$.

(C) $\frac{5x}{2-x} + \frac{5y}{2-y} + \frac{5z}{2-z} = 5\left[\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z}\right]$

$$= 5\left[\frac{x-2+2}{2-x} + \frac{y-2+2}{2-y} + \frac{z-2+2}{2-z}\right] = 5$$

$$\left[-3+2\left[\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}\right]\right]$$

Now

$$2-x+2-y+2-z=5$$

Therefore,

$$\frac{5}{3}\geq\frac{3}{\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}}$$

That is,

$$\frac{1}{2-x}+\frac{1}{2-y}+\frac{1}{2-z}\geq\frac{9}{5}$$

Hence,

$$\frac{5x}{2-x}+\frac{5y}{2-y}+\frac{5z}{2-z}\geq 5\left[-3+2\cdot\frac{9}{5}\right]=3$$

Therefore, least value is 3.

(D) $\sum_{k=1}^{12} 12\cdot K\cdot {}^{12}c_k\cdot {}^{11}c_{k-1}=12^2\sum_{k=1}^{12}({}^{11}c_{k-1})^2=12^2\frac{22!}{11!\,11!}$

$$=12\cdot\frac{21\cdot 19\cdots 3}{11!}\cdot 2^{12}\cdot 6$$

Therefore, $p=6$.

21. Let E_i denotes the event that the bag contains i black and $(12-i)$ white balls $(i=0, 1, 2, \cdots, 12)$ and A denotes the event that the four balls drawn are all black. Then

$P(E_i)=\frac{1}{13}\ (i=0,1,2,\cdots,12)\,;\ P\left(\frac{A}{E_i}\right)=0$ for $i=0,1,2,3$; $P\left(\frac{A}{E_i}\right)$

$=\frac{{}^iC_4}{{}^{12}C_4}$ for $i\geq 4$

(A) $P(A)=\sum_{i=0}^{12}P(E_i)\,P\left(\frac{A}{E_i}\right)=\frac{1}{13}\times\frac{1}{{}^{12}C_4}\left[{}^4C_4+{}^5C_4+\cdots+{}^{12}C_4\right]$

$$=\frac{{}^{13}C_5}{13\times{}^{12}C_4}=\frac{1}{5}$$

(B) Clearly,

$$P\left(\frac{A}{E_{10}}\right)=\frac{{}^{10}C_4}{{}^{12}C_4}=\frac{14}{33}$$

(C) By Bayes' theorem,

$$P\left(\frac{E_{10}}{A}\right)=\frac{P(E_{10})P\left(\frac{A}{E_{10}}\right)}{P(A)}=\frac{\frac{1}{13}\times\frac{14}{33}}{\frac{1}{5}}=\frac{70}{429}$$

(D) Let B denote the probability of drawing 2 white and 2 black balls then

$$P\left(\frac{B}{E_i}\right)=0 \text{ if } i=0,\,1 \text{ or } 11,\,12$$

$$P\left(\frac{B}{E_i}\right)=\frac{{}^iC_2\times{}^{12-i}C_2}{{}^{12}C_4} \text{ for } i=2,3,\ldots,10$$

Therefore,

$$P(B)=\sum_{i=0}^{12}P(E_i)\,P\left(\frac{B}{E_i}\right)=\frac{1}{13}\times\frac{1}{{}^{12}C_4}\,[{}^2C_2\times{}^{10}C_2+{}^3C_2\times{}^9C_2$$

$$+{}^4C_2\times{}^8C_2+\ldots+{}^{10}C_2\times{}^2C_2]$$

$$=\frac{1}{13}\times\frac{1}{{}^{12}C_4}\,[2\{{}^2C_2\times{}^{10}C_2+{}^3C_2\times{}^9C_2+{}^4C_2\times{}^8C_2+$$

$${}^5C_2\times{}^7C_2\}+{}^6C_2\times{}^6C_2]$$

$$=\frac{1}{13}\times\frac{1}{495}\,(1287)=\frac{1}{5}$$

22. We have,

$$P(A\cap B)=P(A)\cdot P(B)=\frac{1}{12}$$

(A) $P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{3}+\frac{1}{4}-\frac{1}{12}=\frac{1}{2}$

(B) $P\left(\frac{A}{A\cup B}\right)=\frac{P(A)}{P(A\cup B)}=\frac{2}{3}$

(C) $P\left(\frac{B}{A'\cap B'}\right)=\frac{P[B\cap(A'\cap B')]}{P(A'\cap B')}=\frac{P(\phi)}{P(A'\cap B')}=0$

(D) $P\left(\frac{A'}{B'}\right)=P(A')=\frac{2}{3}$

23. Probability of hitting the target in one fire $p=\frac{1}{5}$.

Then, the probability of hitting the target at least once in n fires is

1– (Probability of not hitting the target)

$$=1-\left(\frac{4}{5}\right)^n>\frac{3}{4}$$

$$\left(\frac{4}{5}\right)^n<\frac{1}{4} \text{ as } \left(\frac{4}{5}\right)^6>\frac{1}{4} \text{ and } \left(\frac{4}{5}\right)^7<\frac{1}{4}$$

The least value of $n=7$.

24. $P(E_i)=ki(i+2)\ \ (\because\ P(E_1)+P(E_2)+\cdots+P(E_{100})=1)$

Therefore,

$$k\sum_{i=1}^{100}i(i+2)=1$$

$$\Rightarrow k=\frac{6}{100.101.207}$$

Now,

$$P(E)=\sum_{i=1}^{100}P(E_i)\cdot P\left(\frac{E}{E_i}\right)=\sum_{i=1}^{100}ki(i+2)\frac{i}{100}=\frac{432}{575}$$

25. Let a_n denotes the number of outcomes in which no two consecutive head occurs in n tosses clearly $\mathbf{a}_1=2$ and $\mathbf{a}_2=3$. Let us consider last outcome is fixed as tail then we cannot have two consecutive heads in first $(n-1)$ tosses in $\mathbf{a}_{n-1}$ ways and if last outcome is head we must have tail at $(n-1)^{\text{th}}$ toss and cannot have two consecutive heads in first $(n-2)$ tosses in $\mathbf{a}_{n-2}$ ways. So,

$$\mathbf{a}_n = \mathbf{a}_{n-1} + \mathbf{a}_{n-2} \text{ for } n \geq 3$$
$$\Rightarrow \mathbf{a}_{10} = 144$$

Hence,

$$p = \frac{144}{2^{10}} = \frac{9}{64} \Rightarrow 64p = 9$$

26. We know,

$$P(A_1) = P(\omega) + P(BB\omega) + P(BBBB\omega)$$
$$= \frac{3}{10} + \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8} + \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{3}{6}$$
$$= \frac{3}{10} + \frac{1}{12} + \frac{1}{12\times 7} = \frac{332}{840} = \frac{83}{210}$$

Also,

$$P(A_2) = (B\omega) + (BBB\omega) + P(BBBBB\omega)$$
$$= \frac{5}{10}\cdot\frac{3}{9} + \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{3}{7} + \frac{5}{10}\cdot\frac{4}{9}\cdot\frac{3}{8}\cdot\frac{2}{7}\cdot\frac{1}{6}\cdot\frac{3}{5}$$
$$= \frac{1}{6} + \frac{1}{28} + \frac{1}{420} = \frac{86}{420} = \frac{43}{210}$$
$$P(B) = 1 - P(A_1) - P(A_2) = \frac{2}{5}$$

Therefore, $\lambda = 5$.

27. Probability of selection of any box is $\frac{1}{N+1}$.

Let E be the event that the wall clock selected is effective. Then

$$P(E) = P(B_1)\,P(E/B_1) + P(B_2)\,P(E/B_2) + \cdots + P(B_{N+1})\,P(E/B_{N+1})$$

$$= \frac{1}{(N+1)}\left[1 + \frac{N-1}{N} + \frac{N-2}{N} + \cdots + \frac{1}{N} + 0\right] = \frac{1+2+\cdots+N}{N(N+1)} = \frac{1}{2}$$

$$P(B_k/E) = \frac{P(B_k)\cdot P(E/B_k)}{P(E)} = \frac{\frac{1}{N+1}\cdot\frac{(N-K+1)}{N}}{1/2} = \frac{2N-2K+2}{N^2+N}$$

Therefore, $\lambda = 2$.

28. Let $A = \{a_1, a_2, \cdots, a_n\}$. For each $a_i \in A (1 \leq i \leq n)$, we have the following four cases ;

(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \notin P$ and $a_i \in Q$

(iii) $a_i \in P$ and $a_i \notin Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

Thus, the total number of ways of choosing P and Q is 4^n

$P \cap Q$ contains exactly two element in $({}^nC_2)\,(3^{n-2})$.

Hence, the probability of $P \cap Q$ contains two elements is

$$\frac{{}^nC_2 \cdot 3^{n-2}}{4^n}$$

Here,

$$a = 2, b = 2 \Rightarrow a + b = 4$$

Solved JEE 2017 Questions

JEE Main 2017

1. For three events A, B and C
P(Exactly one of A or B occurs) = P(Exactly one of B or C occurs)
P(Exactly one of C or A occurs) $= \frac{1}{4}$ and
P(All the three events occur simultaneously) $= \frac{1}{16}$.

Then the probability that at least one of the events occurs, is:

(A) $\frac{7}{16}$ **(B)** $\frac{7}{64}$

(C) $\frac{3}{16}$ **(D)** $\frac{7}{32}$

(OFFLINE)

Solution: We have

$$P(\text{Exactly one of A or B occurs}) = P(\text{A}) + P(\text{B}) - 2P(\text{A} \cap \text{B}) = \frac{1}{4}$$

$$P(\text{Exactly one of } B \text{ or } C \text{ occurs}) = P(\text{B}) + P(\text{C}) - 2P(\text{B} \cap \text{C}) = \frac{1}{4}$$

$$P(\text{Exactly one of C or A occurs}) = P(\text{C}) + P(\text{A}) - 2P(\text{C} \cap \text{A}) = \frac{1}{4}$$

Adding all, we get

$$2\sum P(\text{A}) - 2\sum P(\text{A} \cap \text{B}) = \frac{3}{4}$$

Therefore,

$$\sum P(\text{A}) - \sum P(\text{A} \cap \text{B}) = \frac{3}{8}$$

Now, it is given that all the three events occur simultaneously, which is given by

$$P(\text{A} \cap \text{B} \cap \text{C}) = \frac{1}{16}$$

Therefore, the probability that at least one of the events occurs, is

$$P(\text{A} \cup \text{B} \cup \text{C}) = \sum P(\text{A}) - \sum P(\text{A} \cap \text{B}) + P(\text{A} \cap \text{B} \cap \text{C})$$
$$= \frac{3}{8} + \frac{1}{16} = \frac{7}{16}$$

Hence, the correct answer is option (A).

2. If two different numbers are taken from the set {0, 1, 2, 3, …, 10}; then, the probability that their sum as well as absolute difference are both multiple of 4, is:

(A) $\frac{12}{55}$ **(B)** $\frac{14}{45}$

(C) $\frac{7}{55}$ **(D)** $\frac{6}{55}$

(OFFLINE)

Solution: We have

$$n(s) = {}^{11}C_2 = 55$$

Now, the favourable events are as follows:

$$\begin{bmatrix} (0,4) \\ (0,8) \\ (2,6),(2,10) \\ (4,8),(6,10) \end{bmatrix}$$

So, the probability that the sum as well as absolute difference, which are both multiple of 4 is

$$\frac{\text{Favarouble events}}{\text{Total events}} = \frac{6}{55}$$

Hence, the correct answer is option (D).

3. Three persons P, Q and R independently try to hit a target. If the probabilities of their hitting the target are $\frac{3}{4}, \frac{1}{2}$ and $\frac{5}{8}$, respectively, then the probability that the target is hit by P or Q but not by R is

(A) $\frac{39}{64}$ **(B)** $\frac{21}{64}$

(C) $\frac{15}{64}$ **(D)** $\frac{9}{64}$

(ONLINE)

Solution: We have the following probabilities:

- The probability that the target is hit by the person P is $\frac{3}{4}$.
- The probability that the target is not hit by the person P is $1 - \frac{3}{4} = \frac{1}{4}$.
- The probability that the target is hit by the person Q is $\frac{1}{2}$.
- The probability that the target is not hit by the person Q is $1 - \frac{1}{2} = \frac{1}{2}$.
- The probability that the target is hit by the person R is $\frac{5}{8}$.
- The probability that the target is not hit by the person R is $1 - \frac{5}{8} = \frac{3}{8}$.

Here, we have used the fact that if the probability of occurrence of an event is p, then the probability of non-occurrence of an event is $q = 1 - p$.
Therefore, the probability that the target is hit by P or Q and not by R is

(Probability that the target is hit by P and not by Q and R)
+ (Probability that the target is hit by Q and not by P and R)
+ (Probability that the target is hit by both P and Q and not by R)

$$= \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{3}{8}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{3}{8}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{3}{8}\right)$$
$$= \frac{9}{64} + \frac{3}{64} + \frac{9}{64}$$
$$= \frac{9+3+9}{64} = \frac{21}{64}$$

Hence, the correct answer is option (B).

4. An unbiased coin is tossed eight times. The probability of obtaining at least one head and at least one tail is

(A) $\frac{63}{64}$ (B) $\frac{255}{256}$

(C) $\frac{127}{128}$ (D) $\frac{1}{2}$

(ONLINE)

Solution: It is given that an unbiased coin is tossed eight times; thus, $n = 8$.

The probability of obtaining head is

$$p = \frac{1}{2}$$

The probability of obtaining tail is

$$q = \frac{1}{2}$$

If x is the number of heads, then the probability that at least one head and at least one tail is obtained is expressed as

1 – (Probability of all head + Probability of all tail) = 1 – [$p(x = 8)$ + $p(x = 0)$]

Using identity $p(x = r) = {}^nC_r p^r q^{n-r}$, we can write as

1 – (Probability of all head + Probability of all tail)

$$= 1 - \left({}^8C_8 p^8 q^{8-8} + {}^8C_0 p^0 q^{8-0}\right)$$

$$= 1 - \left[\frac{8!}{8!(8-8)!}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^0 + \frac{8!}{0!(8-0)!}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^8\right]$$

$$\left[\text{Here, we have used } {}^nC_4 = \frac{n!}{r!(n-r)!}\right]$$

$$= 1 - \left[\left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8\right]$$

$$= 1 - 2 \times \left(\frac{1}{2}\right)^8 = 1 - 2 \times \frac{1}{2^8}$$

$$= 1 - \frac{1}{2^7} = 1 - \frac{1}{128} = \frac{128-1}{128} = \frac{127}{128}$$

Hence, the correct answer is option (C).

5. Let E and F be two independent events. The probability that both E and F happen is $\frac{1}{12}$ and the probability that neither E nor F happens is $\frac{1}{2}$, then a value of $\frac{P(E)}{P(F)}$ is

(A) $\frac{1}{3}$ (B) $\frac{5}{12}$

(C) $\frac{3}{2}$ (D) $\frac{4}{3}$

(ONLINE)

Solution: Let $P(E) = x$ and $P(F) = y$. Now,

$$P(E \cap F) = \frac{1}{12}$$

$$\Rightarrow P(E)P(F) = \frac{1}{12} \Rightarrow xy = \frac{1}{12}$$

Also,

$$P(E' \cap F') = \frac{1}{2}$$

$$\Rightarrow (1 - P(E))(1 - P(F)) = \frac{1}{2}$$

$$\Rightarrow (1-x)(1-y) = \frac{1}{2} \Rightarrow 1 - x - y + xy = \frac{1}{2}$$

$$\Rightarrow x + y = 1 + xy - \frac{1}{2}$$

$$\Rightarrow x + y = 1 + \frac{1}{12} - \frac{1}{2}$$

$$\Rightarrow x + y = \frac{1}{2} + \frac{1}{12} = \frac{7}{12} \qquad (1)$$

Now,

$$(x - y)^2 = (x + y)^2 - 4xy$$

$$\Rightarrow (x-y)^2 = \frac{49}{144} - \frac{1}{3} \Rightarrow \frac{49-98}{144} \Rightarrow \frac{1}{144}$$

$$\Rightarrow (x - y) = \frac{1}{12} \Rightarrow x - y = \frac{1}{2} \qquad (2)$$

From Eqs. (1) and (2), we get

$$(x+y)(x-y) = \frac{7}{12} + \frac{1}{12}$$

$$\Rightarrow x = \frac{4}{12}; y = \frac{3}{12}$$

$$\Rightarrow \frac{x}{y} = \frac{4}{3} = \frac{P(E)}{P(F)}$$

Hence, the correct answer is option (D).

6. From a group of 10 men and 5 women, four member committees are to be formed each of which must contain at least one woman. Then the probability for these committees to have more women than men is

(A) $\frac{1}{11}$ (B) $\frac{21}{220}$

(C) $\frac{2}{23}$ (D) $\frac{3}{11}$

(ONLINE)

Solution: We have

$${}^5C_1 \times {}^{10}C_3 + {}^5C_2 \times {}^{10}C_2 + {}^5C_3 \times {}^{10}C_1 + {}^5C_4$$

$$\Rightarrow \frac{5!}{4!} \times \frac{10!}{7!3!} + \frac{5!}{2!3!} \times \frac{10!}{8!2!} + \frac{5!}{3!2!} \times \frac{10!}{9!1!} + \frac{5!}{4!}$$

$$= 5 \times \frac{10 \times 9 \times 8}{3 \times 2} + \frac{5 \times 4}{2 \times 1} \times \frac{10 \times 9}{2} + \frac{5 \times 4}{2 \times 1} \times 10 + 5$$

$$= 600 + 450 + 100 + 5 = 1155 \rightarrow \text{Total number of ways}$$

The number of ways to form a committee having more women than men is

$${}^5C_2 \times {}^{10}C_2 + {}^5C_4 = \frac{5!}{6!2!} \times \frac{10!}{9!} + \frac{5!}{4!} = 10 \times 10 + 5 = 105$$

Therefore, the probability for thee committees to have more women than men is

$$\frac{105}{1155} = \frac{1}{11}$$

Hence, the correct answer is option (A).

JEE Advanced 2017

1. Let X and Y be two events such that $P(X) = \frac{1}{3}$, $P(X|Y) = \frac{1}{2}$ and $P(Y|X) = \frac{2}{5}$. Then

(A) $P(Y) = \frac{4}{15}$ (B) $P(X'|Y) = \frac{1}{2}$

(C) $P(X \cap Y) = \frac{1}{5}$ (D) $P(X \cup Y) = \frac{2}{5}$

Solution: It is given that

$$P(X) = \frac{1}{3} \quad (1)$$

$$P(X|Y) = \frac{1}{2}$$

and $$P(Y|X) = \frac{2}{5}$$

Now,

$$P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{1}{2} \quad (2)$$

where $\cap$ denotes intersection and

$$P(Y|X) = \frac{P(Y \cap X)}{P(X)} = \frac{2}{5} \quad (3)$$

From Eq. (2), we get

$$P(X \cap Y) = \frac{P(Y)}{2}$$

From Eq. (3), we get

$$P(Y \cap X) = \frac{2}{5}P(X) = \frac{2}{5} \times \frac{1}{3} = \frac{2}{15}$$

also

$$P(X \cap Y) = P(Y \cap X) = \frac{2}{15}$$

$$\Rightarrow \frac{P(Y)}{2} = \frac{2}{15} \quad \Rightarrow P(Y) = \frac{4}{15}$$

Now

$$P(X'|Y) = \frac{P(X' \cap Y)}{P(Y)} = \frac{P(Y) - P(X \cap Y)}{P(Y)}$$

Substituting all values, we get

$$P(X'|Y) = \frac{\left(\frac{4}{15} - \frac{2}{15}\right)}{4/15} = \frac{\left(\frac{4-2}{15}\right)}{4/15} = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow P(X'|Y) = \frac{1}{2}$$

We know that

$$P(X \cap Y) = \frac{2}{15}$$

Now

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

Substituting all values, we get

$$P(X \cup Y) = \frac{1}{3} + \frac{4}{15} - \frac{2}{15} = \frac{1}{3} + \frac{2}{15} = \frac{5+2}{15} = \frac{7}{15}$$

$$\Rightarrow P(X \cup Y) = \frac{7}{15}$$

Hence, the correct answers are options (A) and (B).

2. Three randomly chosen non-negative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is

(A) $\frac{36}{55}$ (B) $\frac{6}{11}$

(C) $\frac{1}{2}$ (D) $\frac{5}{11}$

Solution: It is given that the integers x, y and z satisfy

$$x + y + z = 10$$

The total number of non-negative integers satisfying this equation is

$$^{10+3-1}C_{3-1} = {}^{12}C_2 = \frac{12!}{10! \times 2!} = \frac{12 \times 11}{2} = 66$$

Suppose z is even, let $z = 2k$. Therefore,

$$x + y + 2k = 10$$
$$\Rightarrow x + y = 10 - 2k$$

Then, the total number of non-negative solutions is

$$11 + 9 + 7 + 5 + 3 + 1 = 36$$

$$\left(\begin{array}{l}\text{since for } k = 0, 1, 2, 3, 4, 5, \text{ we have} \\ ^{10-2k+2-1}C_{2-1} = {}^{11-2k}C_1 \text{ solutions}\end{array}\right)$$

Therefore, the probability that the integer z should be even is

$$\frac{36}{66} = \frac{6}{11}$$

Hence, the correct answer is option (B).

Appendix

Chapterwise Solved JEE 2018 Questions

Chapter 17: Inverse Trigonometric Functions

JEE Advanced 2018

1. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right)$$
$$= \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$$

lying in the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is ___________.

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $[0, \pi]$, respectively.)

Solution

(2) Given:

$$\sin^{-1}\left(\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right)$$
$$= \frac{\pi}{2} - \cos^{-1}\left(\sum_{i=1}^{\infty}\left(\frac{-x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i\right)$$

$$\sum_{i=1}^{\infty} x^{i+1} - x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i = \sum_{i=1}^{\infty}\left(\frac{-x}{2}\right)^i - \sum_{i=1}^{\infty}(-x)^i$$

$$\Rightarrow \frac{x^2}{1-x} - x\left(\frac{x}{2-x}\right) = \frac{-x/2}{1+x/2} - \frac{(-x)}{1+x}$$

$$\Rightarrow x = 0$$

or $$\frac{-x}{x-1} + \frac{x}{x-2} = \frac{-1}{x+2} + \frac{1}{x+1}$$

$$\Rightarrow x(x+2)(x+1) = (x-1)(x-2)$$

$$\Rightarrow x^3 + 2x^2 + 5x - 2 = 0 \qquad \text{[say, } f(x)\text{]}$$

Differentiating, we get

$$3x^2 + 4x + 5 = 0$$

$$x = \frac{-4 \pm \sqrt{16-60}}{2\times 3}$$

Thus, $f(0) < 0$ and $f\left(\frac{1}{2}\right) > 0$.

Hence, there are two roots in interval $\left(\frac{-1}{2}, \frac{1}{2}\right)$.

Chapter 18: Matrices and Determinants

JEE Main 2018

2. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$, then the ordered pair (A, B) is equal to

(A) $(-4, 3)$ **(B)** $(-4, 5)$
(C) $(4, 5)$ **(D)** $(-4, -5)$ **(Offline)**

Solution

(B) Given:

$$\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

$C_1 \to C_1 + C_2 + C_3$:

$$\begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

$$\Rightarrow (5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

$C_2 \to C_2 - C_1$:

$$(5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x-4 & 0 \\ 1 & 0 & -x-4 \end{vmatrix} = (A \times Bx)(x-A)^2$$

$$(5x-4)(x+4)^2 = (A+Bx)(x-A)^2$$

That is, $A = -4$ and $B = 5$.

3. Suppose A is any 3×3 non-singular matrix and $(A - 3I)(A - 5I) = O$, where $I = I_3$ and $O = O_3$. If $\alpha A + \beta A^{-1} = 4I$, then $\alpha + \beta$ is equal to

(A) 8 **(B)** 7
(C) 13 **(D)** 12

(Online)

Solution

(A) Given: $(A - 3I)(A - 5I) = 0$

$$\Rightarrow A^2 - 8A + 15I = 0$$
$$\Rightarrow A - 8I + 15A^{-1} = 0$$
$$\Rightarrow \frac{1}{2}A + \frac{15}{2}A^{-1} = 4I$$

Given: $\alpha A + \beta A^{-1} = 4I$

So, $\alpha = \frac{1}{2}, \beta = \frac{15}{2}$

Therefore, $\alpha + \beta = \frac{16}{2} = 8$

4. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = A^{20}$. Then the sum of the elements of the first column of B is

(A) 210 **(B)** 211
(C) 231 **(D)** 251 **(Online)**

Solution

(3) Given:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, B = A^{20}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 10 & 4 & 1 \end{bmatrix}$$

$$A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 20 & 1 & 0 \\ 210 & 210 & 1 \end{bmatrix}$$

Therefore, the sum of the elements of the first column is

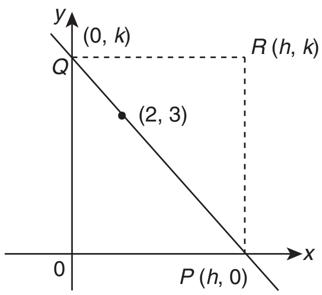
$$1 + 20 + 210 = 231$$

5. A straight line through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle $OPRQ$ is completed, then the locus of R is

(A) $2x + 3y = xy$ **(B)** $3x + 2y = xy$
(C) $3x + 2y = 6xy$ **(D)** $3x + 2y = 6$ **(Offline)**

Solution

(B) From the given data, we can draw as



From above figure, we can say

$$\begin{bmatrix} 0 & k & 1 \\ 2 & 3 & 1 \\ h & 0 & 1 \end{bmatrix} = 0$$

$$-(2 - h) + (1(-3h)) = 0$$
$$-2y + xy - 3x = 0$$
$$3x + 2y = xy$$

6. If the system of linear equations $x + ky + 3z = 0$; $3x + ky - 2z = 0$; $2x + 4y - 3z = 0$ has a non-zero solution (x, y, z), then $\frac{xz}{y^2}$ is equal to

(A) 10 **(B)** – 30
(C) 30 **(D)** –10 **(Offline)**

Solution

(A) Given:

$$x + ky + 3z = 0, 3x + ky - 2z = 0, 2x + 4y - 3z = 0$$

For non-zero solutions, we know that

$$\begin{bmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{bmatrix} = 0$$

$$-3k + 8 - k(-9 + 4) + 3(12 - 2k) = 0$$
$$-3k + 8 + 5k + 36 - 6k = 0$$
$$-4k = -44$$
$$k = 11$$

Therefore,
$$x + 11y + 3z = 0$$
$$3x + 11y + 2z = 0$$
$$2x + 4y - 3z = 0$$

Now, $z = t$; therefore,

$$x + 11y = -3t$$
$$3x + 11y = 2t$$

$$2x = 5t$$
$$\Rightarrow x = \frac{5t}{2}$$

Therefore, $$y = \frac{-3z - x}{11} = \frac{-3t - \frac{5t}{2}}{11}$$
$$= \frac{-11t}{2 \times 11} = -\frac{t}{2}$$

Then, $\frac{xz}{y^2}$ is obtained as follows:

$$\frac{xz}{y^2} = \frac{\left(\frac{5t \times t}{2}\right)}{\left(\frac{t^2}{4}\right)} \Rightarrow \frac{5}{2} \times 4 \Rightarrow 10$$

7. Let A be a matrix such that $A \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is a scalar matrix and $|3A| = 108$. Then A^2 equals

(A) $\begin{bmatrix} 4 & -32 \\ 0 & 36 \end{bmatrix}$ **(B)** $\begin{bmatrix} 36 & 0 \\ -32 & 4 \end{bmatrix}$

(C) $\begin{bmatrix} 4 & 0 \\ -32 & 36 \end{bmatrix}$ **(D)** $\begin{bmatrix} 36 & -32 \\ 0 & 4 \end{bmatrix}$ **(Online)**

Solution

(D) Let us consider

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } A \cdot \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Therefore, $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & 2a + 3b \\ c & 2c + 3d \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

That is, $a = \lambda$, $2a + 3b = 0$, $c = 0$, $2c + 3d = \lambda$ and

$$b = \frac{-2a}{3}$$
$$3d = \lambda \Rightarrow d = \lambda / 3$$
$$A = \begin{bmatrix} \lambda & \frac{-2a}{3} \\ 0 & \frac{\lambda}{3} \end{bmatrix} \Rightarrow 3A = \begin{bmatrix} 3\lambda & -2\lambda \\ 0 & \lambda \end{bmatrix}$$

and $$|3A| = 3\lambda^2 = 108$$
$$\Rightarrow \lambda^2 = \frac{108}{3} = 36$$

Therefore, $$A^2 = \begin{bmatrix} \lambda & \frac{-2\lambda}{3} \\ 0 & \frac{\lambda}{3} \end{bmatrix}\begin{bmatrix} \lambda & \frac{-2\lambda}{3} \\ 0 & \frac{\lambda}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^2 & \frac{-2\lambda^2}{3} - \frac{2\lambda^2}{9} \\ 0 & \frac{\lambda^2}{9} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 36 & -32 \\ 0 & 4 \end{bmatrix}$$

8. Let S be the set of all real values of k for which the system of linear equations $x + y + z = 2$; $2x + y - z = 3$; $3x + 2y + kz = 4$ has a unique solution. Then, S is

(A) an empty set **(B)** equal to $\{0\}$

(C) equal to R **(D)** equal to $R - \{0\}$ **(Online)**

Solution

(D) Given system of linear equations is

$$x + y + z = 2$$
$$2x + y - z = 3$$
$$3x + 2y + kz = 4$$

For unique solution, $\Delta \neq 0$. Now,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} = 1(k + 2) - 1(2k + 3) + 1(4 - 3)$$

$$= k + 2 - 2k - 3 + 1 = -k$$

Thus, $\Delta = -k \neq 0$

Therefore, $k \in R - \{0\}$

9. If the system of linear equations $x + ay + z = 3$; $x + 2y + 2z = 6$; $x + 5y + 3z = b$ has no solution, then

(A) $a = -1, b = 9$ **(B)** $a = -1, b \neq 9$

(C) $a \neq -1, b = 9$ **(D)** $a = 1, b \neq 9$ **(Online)**

Solution

(B) The given system of linear equations is

$$x + ay + z = 3$$
$$x + 2y + 2z = 6$$
$$x + 5y + 3z = b$$

Now, $$\Delta = \begin{vmatrix} 1 & a & 1 \\ 1 & 2 & 2 \\ 1 & 5 & 3 \end{vmatrix} = 1(6 - 10) - a(3 - 2) + 1(5 - 2)$$

$$= -4 - a + z = -(a + 1)$$

Therefore,

$$\Delta_1 = \begin{vmatrix} 3 & a & 1 \\ 6 & 2 & 2 \\ b & 5 & 3 \end{vmatrix} = 3(6 - 10) - a(18 - 2b) + (30 - 2b)$$

$$= -12 - 18a + 2ab + 30 - 2b$$
$$= 18 - 2b - 18a + 2ab$$
$$= 18(1 - a) - 2b(1 - a)$$
$$= 2(g - b)(1 - a)$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 6 & 2 \\ 1 & b & 3 \end{vmatrix} = 1(18 - 2b) - 3(3 - 2) + 1(b - 6)$$

$$= 18 - 2b - 3 + b - 6 = (g - b)$$

and

$$\Delta_3 = \begin{vmatrix} 1 & a & 3 \\ 1 & 2 & b \\ 1 & 5 & b \end{vmatrix} = 1(2b - 30) - a(b - 6) + 3(5 - 2)$$

$$= 2b - 30 - ab + 6a + 9$$
$$= b(2 - a) - 3(7 - 2a)$$

For no solution, $\Delta = 0$ and at least one of the values of Δ_1, Δ_2 and Δ_3 is non-zero.

$$\Delta = 0 \Rightarrow a = -1$$
$$\Delta_1 = 0 \Rightarrow a = 1, b = 9$$
$$\Delta_2 = 0 \Rightarrow b = 9$$

Therefore, if the given system of equations has no solution, then $a = -1$ and $b \neq 9$.

JEE Advanced 2018

10. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

(A) $x + 2y + 3z = b_1, 4y + 5z = b_2$ and $x + 2y + 6z = b_3$
(B) $x + y + 3z = b_1, 5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
(C) $-x + 2y - 5z = b_1, 2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$
(D) $x + 2y + 5z = b_1, 2x + 3z = b_2$ and $x + 4y - 5z = b_3$

(Paper-2)

Solution

(A), (D) The given system of equations is

$$-x + 2y + 5z = b_1$$
$$2x - 4y + 3z = b_2$$
$$x - 2y + 2z = b_3$$

$$\Rightarrow A = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -4 & 3 \\ 1 & -2 & 2 \end{vmatrix}$$
$$= -1(-8 + 6) - 2(4 - 3) + 5(-4 + 4)$$
$$= 0 \Rightarrow \text{atleast one soluion is possible.}$$

Therefore, we have

$$D_1 = D_2 = D_3 = 0$$
$$\Rightarrow b_1 + 7b_2 - 13b_3 = 0 \qquad (1)$$

- **Option (A):** The given set of equations are

$$x + 2y + 3z = b_1,\ 4y + 5z = b_2 \text{ and } x + 2y + 6z = b_3$$

$$\Rightarrow A = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 4 & 3 \\ 1 & 2 & 6 \end{vmatrix}$$
$$= 1(24 - 10) + 2(-0 + 5) + 3(0 - 4)$$
$$= 14 + 10 - 12 = 12$$
$$\Rightarrow A \neq 0$$

That is, in this case, there is a unique solution.
Hence, option (A) is correct.

- **Option (B):** The given set of equations are

$$x + y + 3z = b_1,\ 5x + 2y + 6z = b_2 \text{ and } -2x - y - 3z = b_3$$

$$\Rightarrow A = \begin{vmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{vmatrix}$$
$$= 1(-6 + 6) - 1(-15 + 12) + 3(-5 + 4)$$
$$\Rightarrow A = 0 + 3 - 3 = 0$$

However, $D_1 = 0$ if $D_2 = 0$. Therefore,

$$b_1 + b_2 - 3b_3 = 0$$

Thus, $b_1 + b_2 = 0$ and $b_3 = 0$.

However, this does not satisfy Eq. (1). Therefore, this system of equations has no solution.
Hence, option (B) is incorrect.

- **Option (C):** The given set of equations are

$$-x + 2y - 5z = b_1,\ 2x - 4y + 10z = b_2 \text{ and } x - 2y + 5z = b_3$$

$$\Rightarrow A = \begin{vmatrix} -1 & 2 & -5 \\ 2 & -4 & 10 \\ 1 & -2 & 5 \end{vmatrix}$$
$$= -1(-20 + 20) - 2(10 - 10) - 5(-4 + 4)$$
$$\Rightarrow A = 0$$

That is, this has infinitely many solutions.
Hence, option (C) is incorrect.

- **Option (D):** The given set of equations are

$$x + 2y + 5z = b_1,\ 2x + 3z = b_2 \text{ and } x + 4y - 5z = b_3$$

$$\Rightarrow A = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & -5 \end{vmatrix}$$
$$= 1(0 - 12) - 2(-10 - 3) + 5(8)$$
$$\Rightarrow A = -12 + 26 + 40 = 54$$
$$\Rightarrow A \neq 0$$

That is, in this case, there is a unique solution.
Hence, option (D) is correct.

11. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is ________.

(Paper-2)

Solution

(4) Given: Here, P is a 3×3 matrix.

Let us consider

$$P=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Therefore,

$$|P|=a_{11}\times\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\times\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\times\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The maximum possibility of $|P|$ can be 6 if we have the following:

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=\pm2$$

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=\pm2$$

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=\pm2$$

However, if $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=\pm2$ and $\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=\pm2$, then

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=0$$

Therefore, $|P|\neq 6$. Thus, the next possibility of $|P|$ is 4.

Therefore, $P=\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ can be one possibility.

Chapter 19: Limit, Continuity and Differentiability

JEE Main 2018

12. For each $t\in\mathbb{R}$, let $[t]$ be the greatest integer less than or equal to t. Then $\lim\limits_{x\to0+} x\left[\left(\frac{1}{x}\right)+\left(\frac{2}{x}\right)+\cdots+\left(\frac{15}{x}\right)\right]$

(A) is equal to 15. **(B)** is equal to 120.

(C) does not exist (in $\mathbb{R}$). **(D)** is equal to 0. **(Offline)**

Solution

(B) The given limit is

$$\lim_{x\to0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\cdots+\left[\frac{15}{x}\right]\right)$$

$$\lim_{x\to0^+} \left(x\left[\frac{1}{x}\right]+x\left[\frac{2}{x}\right]+\cdots+x\left[\frac{15}{x}\right]\right)=1+2+3+\cdots+15$$

$$=\frac{15}{2}(15+1)=120$$

13. $\lim\limits_{x\to0}\dfrac{x\tan2x-2x\tan x}{(1-\cos2x)^2}$ equals

(A) $\frac{1}{4}$ **(B)** 1

(C) $\frac{1}{2}$ **(D)** $-\frac{1}{2}$ **(Online)**

Solution

(C) Given:

$$\lim_{x\to0}\frac{x\tan2x-2x\tan x}{(1-\cos 2x^2)}$$

$$\lim_{x\to0}\frac{\frac{2x\tan x}{1-\tan^2 x}-2x\tan x}{(1-1+2\sin^2 x)^2}$$

$$\Rightarrow \lim_{x\to0}\frac{2x\tan x}{1-\tan^2 x}\left(\frac{1-1+\tan^2 x}{4\sin^4 x}\right)$$

$$\Rightarrow \lim_{x\to0}\frac{1}{2}\left(\frac{x}{\sin x}\right)\left(\frac{\tan^3 x}{x^3}\right)\left(\frac{x^3}{\sin^3 x}\right)=\frac{1}{2}$$

14. $\lim\limits_{x\to0}\dfrac{(27+x)^{1/3}-3}{9-(27+x)^{2/3}}$ equals

(A) $\frac{1}{3}$ **(B)** $-\frac{1}{3}$

(C) $-\frac{1}{6}$ **(D)** $\frac{1}{6}$ **(Online)**

Solution

(C) Given:

$$\lim_{x\to0}\frac{(27+x)^{\frac{1}{3}}-3}{9-(27+x)^{\frac{2}{3}}}$$

$$\lim_{x\to0}\frac{3\left[\left(1+\frac{x}{27}\right)^{\frac{1}{3}}-1\right]}{9\left[1-\left(1+\frac{x}{27}\right)^{\frac{2}{3}}\right]}$$

$$\lim_{x\to0}\frac{3\left[1+\frac{1}{3}\left(\frac{x}{27}\right)-1\right]}{9\left[1-1-\frac{2}{3}\left(\frac{x}{27}\right)\right]}$$

$$\lim_{x\to0}\frac{1}{3}\frac{\frac{x}{81}}{\frac{-2}{3}\cdot\frac{x}{27}}=\frac{-1}{6}$$

15. Let $S = \{t \in \mathbb{R} : f(x) = |x - \pi| \cdot (e^{|x|} - 1)\sin|x|$ is not differentiable at $t\}$. Then, the set S is equal to

(A) $\{0\}$ **(B)** $\{\pi\}$

(C) $\{0, \pi\}$ **(D)** ϕ (an empty set)

(Offline)

Solution

(D) Given: $$f(x) = |x - \pi| \cdot (e^{|x|} - 1)\sin|x|$$

According to the given options, we have to check only at $x = 0$ and at $x = \pi$.

- **At $x = 0$:**

$$\text{LHD} = \lim_{h\to 0^+} \frac{f(0-h) - f(0)}{-h}$$
$$= \lim_{h\to 0^+} \frac{(\pi + h) \times (e^h - 1)\sin h}{-h}$$
$$\text{RHD} = \lim_{h\to 0^+} \frac{(\pi - h)(e^h - 1)\sin h}{h} = 0$$

Therefore, it is differentiable at $x = 0$.

- **At $x = \pi$:**

$$f(\pi) = 0$$
$$\text{LHD} = \lim_{h\to 0^+} \frac{f(\pi - h) - f(\pi)}{-h}$$
$$= \lim_{h\to 0^+} \frac{h(e^{\pi-h} - 1)\sin h}{h} = 0$$
$$\text{RHD} = \lim_{h\to 0} \frac{f(\pi + h) - f(\pi)}{h}$$
$$= \lim_{h\to 0} \frac{-h(e^{\pi+h} - 1)\sin h}{h} = 0$$

Therefore, it is differentiable at $x = \pi$. Hence, the set S is equal to ϕ (an empty set).

16. Let $S = \{(\lambda, \mu) \in \mathbb{R}\times\mathbb{R} : f(t) = (|\lambda|e^{|t|} - \mu)\sin(2|t|), t \in \mathbb{R},$ is a differentiable function$\}$. Then S is a subset of

(A) $\mathbb{R}\times[0,\infty)$ **(B)** $[0,\infty)\times\mathbb{R}$

(C) $\mathbb{R}\times(-\infty,0)$ **(D)** $(-\infty,0)\times\mathbb{R}$

(Online)

Solution

(A) Given: $$f(t) = (|\lambda|e^{|t|} - \mu)\sin(2|t|)$$

Checking the differentiability at $t = 0$:

$$\text{LHL} = \lim_{t\to 0} f(t) = \lim_{t\to 0^-} (|\lambda|e^{|t|} - \mu)\sin(2|t|)$$
$$\lim_{t\to 0^-} (|\lambda|e^h - \mu)\sin 2h = 0$$
$$\text{RHL} = \lim_{t\to 0} f(t) = \lim_{h\to 0} (|\lambda|e^h - \mu)\sin 2h = 0$$
$$\text{LHL} = \text{RHL} = f(0)$$

Therefore, $f(t)$ is continuous for all t.

$$\text{LHD} = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{2(|\lambda|e^h - \mu)\sin 2h}{2h}$$
$$= -2(|\lambda|e^0 - \mu) = -2(|\lambda| - \mu)$$
$$\text{RHD} = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = 2(|\lambda| - \mu)$$
$$\text{LHD} = \text{RHD} \Rightarrow 4(|\lambda| - \mu) = 0 \Rightarrow \mu = \lambda$$
$$\Rightarrow \lambda \in \mathbb{R},\ \mu \ge 0$$

Therefore, S is a subset of $\mathbb{R}\times[0,\infty)$.

17. Let $f(x) = \begin{cases} (x-1)^{\frac{1}{2-x}}, & x > 1, x \ne 2 \\ k & , x = 2 \end{cases}$

The value of k for which f is continuous at $x = 2$ is

(A) 1 **(B)** e

(C) e^{-1} **(D)** e^{-2} **(Online)**

Solution

(C) Given:

$$f(x) = \begin{cases} (x-1)^{\frac{1}{2-x}} & , x > 1, x \ne 2 \\ k & , x = 2 \end{cases}$$
$$f(2) = \lim_{x\to 2} f(x)$$
$$k = e^{\lim\limits_{x\to 2}\left(-(x-1-1)\frac{1}{x-2}\right)} = e^{-1} = \left(\frac{1}{e}\right)$$

18. If the function f defined as $f(x) = \frac{1}{x} - \frac{k-1}{e^{2x}-1}$, $x \ne 0$, is continuous at $x = 0$, then the ordered pair $(k, f(0))$ is equal to

(A) (3, 2) **(B)** (3, 1)

(C) (2, 1) **(D)** $\left(\frac{1}{3}, 2\right)$ **(Online)**

Solution

(B) Given: $f(x) = \frac{1}{x} - \frac{k-1}{e^{2x}-1}; x \ne 0$

Here, $f(x)$ is continuous at $x = 0$. Therefore,

$$f(0) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{k-1}{e^{2x}-1}\right)$$
$$= \lim_{x\to 0} \frac{\left(1 + (2x) + \frac{1}{2!}(2x)^2 + \cdots (-1 - x(k-1))\right)}{2x^2\left(\frac{e^{2x}-1}{2x}\right)}$$

Therefore, clearly, $k = 3$ for $f(0) = 1$

JEE Advanced 2018

19. Let f: $\mathbb{R}\to\mathbb{R}$ and g: $\mathbb{R}\to\mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x)-g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is(are) TRUE?

(A) $f(2) < 1 - \log_e 2$ **(B)** $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$ **(D)** $g(1) < 1 - \log_e 2$

(Paper-1)

Solution

(B), (C) Given: $f'(x)=(e^{f(x)-g(x)})g'(x)$ $\forall$ for all $x \in \mathbb{R}$

$$f(1)=g(2)=1$$

Now,
$$f'(x)=e^{f(x)-g(x)}g'(x)$$
$$f'(x)=e^{f(x)}e^{-g(x)}g'(x)$$
$$\Rightarrow f'(x)=\frac{e^{f(x)}}{e^{g(x)}}g'(x) \Rightarrow \frac{f'(x)}{e^{f(x)}}=\frac{g'(x)}{e^{g(x)}}$$
$$\Rightarrow e^{-f(x)}f'(x)=e^{-g(x)}g'(x)$$
$$\Rightarrow \frac{d}{dx}(e^{-f(x)})=\frac{d}{dx}(e^{-g(x)})$$

Integrating, we get
$$e^{-f(x)}=e^{-g(x)}+C$$

- **For $x = 1$:** $e^{-f(1)} = e^{-g(1)} + c \Rightarrow e^{-1} + e^{-g(1)} + c$
- **For $x = 2$:** $e^{-f(2)} = e^{-g(2)} + c \Rightarrow e^{-f(2)} = e^{-1} + c$

Subtracting the above two equations, we get

$$e^{-1}-e^{-f(2)}=e^{-g(1)}-e^{-1}$$
$$\Rightarrow \frac{1}{e}-\frac{1}{e^{f(2)}}=\frac{1}{e^{g(1)}}-\frac{1}{e} \qquad (1)$$
$$\Rightarrow \frac{1}{e^{g(1)}}=\frac{2}{e}-\frac{1}{e^{f(2)}} \Rightarrow \frac{1}{e^{g(1)}}=\frac{2e^{f(2)}-e}{e\cdot e^{f(2)}}$$
$$\Rightarrow \frac{1}{e^{g(1)}}=\frac{2e^{f(2)}-e}{e^{f(2)+1}} \Rightarrow e^{g(1)}=\frac{e^{f(2)+1}}{2e^{f(2)}-e}>0$$

Therefore, $e^{f(2)+1}>2e^{f(2)}-e$

or $e^{f(2)}>\dfrac{e}{2}$

or $f(2) > 1 - \ln 2$ [option (B)]

Also, using Eq. (1), we get

$$\frac{1}{e^{f(2)}}=\frac{2}{e}-\frac{1}{e^{g(1)}}=\frac{2e^{g(1)}-e}{e^{g(1)+1}}$$
$$e^{f(2)}=\frac{e^{g(1)+1}}{2e^{g(1)}-e}>0$$

Therefore, $g(1) > 1 - \ln 2$ [option (C)]

20. Let f: $[0, \infty) \to \mathbb{R}$ be a continuous function such that $f(x)=1-2x+\int_0^x e^{x-t}f(t)dt$ for all $x\in [0,\infty)$. Then, which of the following statement(s) is(are) TRUE?

(A) The curve $y = f(x)$ passes through the point (1, 2).
(B) The curve $y = f(x)$ passes through the point (2, –1).
(C) The area of the region $\{(x, y) \in[0,1]\times\mathbb{R}: f(x)\le y\le\sqrt{1-x^2}\}$ is $\dfrac{\pi-2}{4}$.
(D) The area of the region $\{(x, y) \in[0,1]\times\mathbb{R}: f(x)\le y\le\sqrt{1-x^2}\}$ is $\dfrac{\pi-1}{4}$. **(Paper-1)**

Solution

(B), (C) We have
$$f(x)=1-2x+\int_0^x e^{x-t}f(t)dt$$
$$f(x)=1-2x+e^x\int_0^x e^{-t}f(t)dt$$

Multiplying both sides by e^{-x}, we get
$$e^{-x}f(x)=e^{-x}(1-2x)+\int_0^x e^{-t}f(t)dt$$

Differentiating, we get
$$-e^{-x}f(x)+e^{-x}f'(x)=-e^{-x}(1-2x)+e^{-x}(-2)+e^{-t}f(t)\Big|_0^x$$
$$\Rightarrow -e^{-x}f(x)+e^{-x}f'(x)=-e^{-x}(1-2x)-2e^{-x}+e^{-x}f(x)$$
$$\Rightarrow -f(x)+f'(x)=-(1-2x)-2+f(x)$$
$$\Rightarrow f'(x)-2f(x)=2x-3$$

Here, the integrating factor is e^{-2x}. Therefore,
$$e^{-2x}f'(x)-2e^{-2x}f(x)=e^{-2x}(2x-3)$$
$$\Rightarrow \frac{d}{dx}(e^{-2x}f(x))=e^{-2x}(2x-3)$$

Integrating it, we get
$$e^{-2x}f(x)=\int e^{-2x}(2x-3)dx$$

Using integration by parts, we get
$$e^{-2x}f(x)=(2x-3)\int e^{-2x}dx-\int\left(2\int e^{-2x}dx\right)dx$$
$$\Rightarrow e^{-2x}f(x)=\frac{(2x-3)e^{-2x}}{-2}-\frac{e^{-2x}}{2}+c$$
$$\Rightarrow f(x)=\frac{(2x-3)}{-2}-\frac{1}{2}+ce^{2x}$$

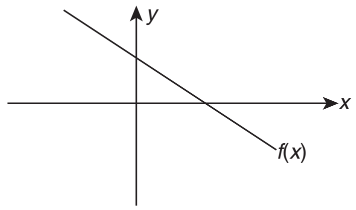
Therefore, $f(0)=\dfrac{3}{2}-\dfrac{1}{2}+c \Rightarrow f(0) = 1 + c$

That is, $c = 0$. Therefore,
$$f(x)=\frac{2x-3}{-2}\frac{-1}{2}=-x+\frac{3}{2}-\frac{1}{2}$$

or $f(x) = 1 - x$

These lines pass through (2, –1).

Hence, option (B) is correct.



The area of the region $(x,y)\in[0,1]\times R: f(x)\le y\le\sqrt{1-x^2}$ is

$$\text{Area}\Big|_{\text{region}}=\frac{\pi}{4}-\frac{1}{2}=\frac{\pi-2}{4}$$ [option (C)]

21. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$ is _____.

Solution

(8) Given: $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$.

That is,

$$\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}} = (\log_2 9)^{2/\log_2(\log_2 9)} \times 7^{\frac{1}{2\log_4 7}}$$
$$= (\log_2 9)^{2\log_{\log_2 9} 9^2} \times 7^{\log_7 4/2}$$
$$= 4 \times 2 = 8$$

22. For any positive integer n, define $f_n: (0, \infty) \to \mathbb{R}$ as $f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$ for all $x \in (0, \infty)$.

[Here, the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.]

Then, which of the following statement(s) is(are) TRUE?

(A) $\sum_{j=1}^{5} \tan^2(f_j(0)) = 55$

(B) $\sum_{j=1}^{10} (1 + f_j'(0))\sec^2(f_j(0)) = 10$

(C) For any fixed positive integer n, $\lim_{x\to\infty} \tan(f_n(x)) = \frac{1}{n}$

(D) For any fixed positive integer n, $\lim_{x\to\infty} \sec^2(f_n(x)) = 1$

(Paper-2)

Solution

(D) Let us check all options as follows:

- **Options (A) and (B):** It is given that

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)$$

Which can be rewritten as follows:

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right)$$
$$= \sum_{j=1}^{n} \tan^{-1}(x+j) - \tan^{-1}(x+j-1)$$
$$= \tan^{-1}\left[\frac{n}{1+x(n+x)}\right]$$

Therefore,

$$f_n'(x) = \left[\frac{1}{1+x(n+x)}\right]$$
$$= \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

We know that $f_n(0) = \tan^{-1} n$. Therefore,

$$\tan^2(\tan^{-1} n) = n^2$$

As in this case, $x = 0$ is not in the given domain. That is, $x \in (0, \infty)$.

Hence, options (A) and (B) are incorrect.

- **Option (C):** For any fixed positive integer n, we have

$$\lim_{x\to\infty} \tan(f_n(x)) = \lim_{x\to\infty}\left[\frac{n}{1+x(n+x)}\right] = 0$$

Hence, option (C) is incorrect.

- **Option (D):**

For any fixed positive integer n, we have

$$\lim_{x\to\infty} \sec^2(f_n(x)) = \lim_{x\to\infty} 1+\tan^2(f_n(x)) = 0$$
$$= 1 + \lim_{x\to\infty} \tan^2(f_n(x)) = 1$$

Hence, only option (D) is correct.

23. Let $f: (0, \pi) \to \mathbb{R}$ be a twice differentiable function such that $\lim_{t\to x} \frac{f(x)\sin t - f(t)\sin x}{t-x} = \sin^2 x$ for all $x \in (0, \pi)$. If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is(are) TRUE?

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

Solution

(B), (C), (D) Given: $f: (0, \pi) \to \mathbb{R}$ is twice differentiable function such that

$$\lim_{t\to x} \frac{f(x)\sin t - f(t)\sin x}{t-x} = \sin^2 x \quad \forall\, x \in (0, \pi)$$

Also, it is given that

$$f\left(\frac{\pi}{6}\right) = \frac{-\pi}{12}$$

Using L'Hospitals rule, we get

$$\lim_{t\to x} \frac{f(x)\cos t - f'(t)\sin x}{1} = \sin^2 x$$
$$\Rightarrow f(x)\cos x - f'(x)\sin x = \sin^2 x$$

Rearranging this, we get

$$-\left(\frac{f'(x)\sin x - f(x)\cos x}{\sin^2 x}\right) = 1$$

This equation can be written in differential form:

$$-\frac{d}{dx}\left(\frac{f(x)}{\sin x}\right) = 1$$

Integrating this, we get

$$-\int \frac{d}{dx}\left(\frac{f(x)}{\sin x}\right)dx = \int dx$$
$$\Rightarrow \frac{-f(x)}{\sin x} = x + c$$

For $x = \frac{\pi}{6}$:

$$\frac{-f\left(\frac{\pi}{6}\right)}{\left(\sin\frac{\pi}{6}\right)} = \frac{\pi}{6} + c \Rightarrow \frac{+(\pi/12)}{1/2} = \frac{\pi}{6} + c \ \Rightarrow c = 0$$

Therefore, $\frac{-f(x)}{\sin x} = x \Rightarrow f(x) = -x\sin x$

Therefore, $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\sin\frac{\pi}{4} = \frac{-\pi}{4}\times\frac{1}{2}$ [option (A) is false]

Now, $f(x) = -x\sin x$

Since $\sin x > x - \frac{x^3}{6}$, we get

$$-x\sin x < -x^2 + \frac{x^4}{6}$$

Therefore, $f(x) < -x^2 + \frac{x^4}{6}\ \forall x \in (0,\pi)$ [option (B) is true]

Now, $f(x) = -x\sin x$

That is, $f'(x) = -\sin x - x\cos x$

Now,

$$f'(x) = 0$$
$$\Rightarrow -\sin x - x\cos x = 0$$
$$\Rightarrow \sin x = -x\cos x$$
$$\Rightarrow \tan x = -x$$

Thus, there exists $\alpha \in (0,\pi)$ for which $f'(x) = 0$.
As *f*(*x*) is continuous in $[0, \pi]$ and differentiable in $(0,\pi)$.
Hence, option (C) is true.

Now, $f(x) = -x\sin x$

Therefore,

$$f'(x) = -\sin x - x\cos x$$
$$f''(x) = x\sin x - \cos x - \cos x = x\sin x - 2\cos x$$

Therefore,

$$f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\frac{\pi}{2} - 2\cos\frac{\pi}{2} + \left(-\frac{\pi}{2}\sin\frac{\pi}{2}\right)$$
$$= -2\cos\frac{\pi}{2} = 0$$

Therefore, $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

Hence, option (D) is true.

24. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with *f*(0) = 0. If *y* = *f*(*x*) satisfies the differential equation $\frac{dy}{dx}$ = (2 + 5*y*) (5*y* − 2), then the value of $\lim\limits_{x\to\infty} f(x)$ is ____________.

Solution

(0.4) We have the differential equation

$$\frac{dy}{dx} = (2+5y)(5y-2)$$

Rearranging this, we get

$$\frac{dy}{dx} = 25y^2 - 4 \Rightarrow \frac{dy}{25y^2-4} = dx$$

Integrating it, we get

$$\int \frac{1}{25y^2-4}dy = \int dx = \int \frac{1}{(5y+2)(5y-2)}dy = \int dx$$

That is,

$$\frac{1}{4}\int\left(\frac{1}{5y-2} - \frac{1}{5y+2}\right)dy = \int dx$$
$$\Rightarrow \frac{1}{4}\left(\frac{\ln(5y-2)}{5} - \frac{\ln(5y+2)}{5}\right) = x + c$$
$$\Rightarrow \frac{1}{20}\ln\left|\frac{5y-2}{5y+2}\right| = x + c$$

Given: $f(0) = 0 \Rightarrow c = 0 \ \Rightarrow \ln\left|\frac{5y-2}{5y+2}\right| = 20x$

Now, $x \to -\infty \Rightarrow \ln\left|\frac{5y-2}{5y+2}\right| \to -\infty$

$$\Rightarrow \left|\frac{5y-2}{5y+2}\right| \to 0 \Rightarrow y \to \frac{2}{5}$$

Therefore, $\lim\limits_{x\to-\infty} f(x) = \frac{2}{5} = 0.4$

25. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function with *f*(0) = 1 and satisfying the equation *f*(*x* + *y*) = *f*(*x*)*f* ′(*y*) + *f* ′(*x*)*f*(*y*) for all $x, y \in \mathbb{R}$. Then, the value of $\log_e(f(4))$ is ____.

Solution

(2) We have

$$f(x+y) = f(x)f'(y) + f'(x)f(y)$$

and $f(0) = 1$

- For $x = 0,\ y = 0$:

$$f(0) = f(0)f'(0) + f'(0)f(0)$$
$$\Rightarrow f(0) = 2f(0)f'(0)$$
$$\Rightarrow f'(0) = \frac{1}{2}$$

- For $y = 0$:

$$f(x) = f(x)f'(0) + f'(x)f(0)$$

Substituting $f(0)=1$ and $f'(0)=\frac{1}{2}$, we get

$$f(x)=f(x)\times\frac{1}{2}+f'(x)\times 1$$
$$\Rightarrow f(x)-\frac{f(x)}{2}=f'(x)$$
$$\Rightarrow \frac{1}{2}f(x)=f'(x)$$
$$\Rightarrow \frac{f'(x)}{f(x)}=\frac{1}{2}$$

Integrating this, we get

$$\ln(f(x))=\frac{x}{2}+c$$

Using $f(0)=1$, we get $c=0$.

Therefore, $\ln(f(x))=\frac{x}{2}$

That is, $\ln(f(4))=\frac{4}{2}=2$

Hence, the value of $\log_e(f(4))=2$.

26. Let $f_1:\mathbb{R}\to\mathbb{R}, f_2:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$,

$f_3:\left(-1, e^{\frac{\pi}{2}}-2\right)\to\mathbb{R}$ and

$f_4:\mathbb{R}\to\mathbb{R}$ be functions defined by

(i) $f_1(x)=\sin\left(\sqrt{1-e^{-x^2}}\right)$,

(ii) $f_2(x)=\begin{cases}\dfrac{|\sin x|}{\tan^{-1}x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

(iii) $f_3(x)=[\sin(\log_e(x+2))]$, where, for $t\in\mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x)=\begin{cases}x^2\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$.

LIST–I	LIST–II
P. The function f_4 is	**1. NOT** continuous at $x=0$
Q. The function f_2 is	**2.** continuous at $x=0$ and **NOT** differentiable at $x=0$
R. The function f_3 is	**3.** differentiable at $x=0$ and its derivative is **NOT** continuous at $x=0$
S. The function f_4 is	**4.** differentiable at $x=0$ and its derivative is continuous at $x=0$

The correct option is:

(A) P→2; Q→3; R→1; S→4
(B) P→4; Q→1; R→2; S→3
(C) P→4; Q→2; R→1; S→3
(D) P→2; Q→1; R→4; S→3

Solution

(D) Let us check for each given item:

(i) It is given that

$$f_1(x)=\sin\sqrt{1-e^{-x^2}}$$

That is, $f_1'(x)=\cos\sqrt{1-e^{-x^2}}\times\dfrac{1}{2\sqrt{1-e^{-x^2}}}(0-e^{-x^2}\cdot(-2x))$

At $x=0$: $f_1'(x)$ does not exist. Therefore, $f_1(x)$ is continuous at $x=0$ and it is not differentiable at $x=0$.

(ii) It is given that

$$f_2(x)=\begin{cases}\dfrac{|\sin x|}{\tan^{-1}x}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$$

Now, $\lim\limits_{x\to 0^+}\left(\dfrac{\sin x}{x}\dfrac{x}{\tan^{-1}x}\right)=1$

Therefore, $f_2(x)$ is not continuous at $x=0$.

(iii) It is given that

$$f_3(x)=[\sin(\log_e(x+2))]$$

Now, $0<(x+2)<e^{\pi/2}$

Taking $\log_e$ on both sides, we get

$$0<\log_e(x+2)<\frac{\pi}{2}$$

Taking sin on both sides, we get

$$\sin 0<\sin(\log_e(x+2))<\sin\frac{\pi}{2}$$
$$\Rightarrow 0<\sin(\log_e(x+2))<1\Rightarrow f_3(x)=0$$

Therefore, $f_3(x)$ is differentiable at $x=0$ and its derivative is continuous at $x=0$.

$$f_4(x)=\begin{cases}x^2\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

$$f_4'(x)=2\sin\left(\frac{1}{x}\right)-x^2\cdot\frac{1}{x^2}\cos\left(\frac{1}{x}\right)=2\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)$$

This is oscillating. Thus, $f_4(x)$ is differentiable at $x=0$ and its derivative is not continuous at $x=0$.

Therefore, the correct mapping is P→2; Q→1; R→4; S→3.

Hence, option (B) is correct.

Chapter 20: Differentiation*

JEE Main 2018

Chapter 21: Applications of Derivatives

JEE Main 2018

27. If $f(x)=\begin{vmatrix}\cos x & x & 1\\ 2\sin x & x^2 & 2x\\ \tan x & x & 1\end{vmatrix}$, then $\lim\limits_{x\to 0}\dfrac{f'(x)}{x}$

(A) does not exist.
(B) exists and is equal to 2.
(C) exists and is equal to 0.
(D) exists and is equal to –2. **(Online)**

Solution

(D) Given: $$f(x)=\begin{vmatrix}\cos x & x & 1\\ 2\sin x & x^2 & 2x\\ \tan x & x & 1\end{vmatrix}$$

So,

$$f'(x)=\begin{vmatrix}-\sin x & 1 & 0\\ 2\sin x & x^2 & 2x\\ \tan x & x & 1\end{vmatrix}+\begin{vmatrix}\cos x & x & 1\\ 2\cos x & 2x & 2\\ \tan x & x & 1\end{vmatrix}+\begin{vmatrix}\cos x & x & 1\\ 2\sin x & x^2 & 2x\\ \sec^2 x & 1 & 0\end{vmatrix}$$

$$\Rightarrow f'(x)=-\sin x(x^2-2x^2)-(2\sin x-2x\tan x)+\cos x(0-2x)$$
$$-x(2\sin x\cdot 0-2x\sec^2 x)+(2\sin x-x^2\sec^2 x)$$
$$=+x^2\sin x-2\sin x+2x\tan x-2x\cos x+2x^2\sec^2 x$$
$$-x^2\sec^2 x+2\sin x$$
$$\Rightarrow f'(x)=x^2\sin x+2x\tan x-2x\cos x+x^2\sec^2 x$$

Then, $$\frac{f'(x)}{x}=x\sin x+2\tan x-2\cos x+x\sec^2$$

Therefore,

$$\lim_{x\to 0}\frac{f'(x)}{x}=\lim_{x\to 0}(x\sin x+2\tan x-2\cos x-x\sec^2)=-2$$

28. If $x^2+y^2+\sin y=4$, then the value of $\dfrac{d^2y}{dx^2}$ at the point (–2, 0) is

(A) –34 **(B)** –32
(C) 4 **(D)** –2 **(Online)**

Solution

(A) Given: $$x^2+y^2+\sin y=4$$

Now, $$2x+2y\frac{dy}{dx}+\cos y\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{-2x}{(2y+\cos y)}$$

$$\left.\frac{dy}{dx}\right|_{(-2,\,0)}=\frac{4}{0+1}=4$$

Now, $$-\frac{d^2y}{dx^2}=\frac{2(2y+\cos y)-2x\left(2\dfrac{dy}{dx}-\sin y\dfrac{dy}{dx}\right)}{(2y+\cos y)^2}$$

$$-\left.\frac{d^2y}{dx^2}\right|_{(-2,\,0)}=\frac{2(0+1)+4(2(4)-0)}{(0+1)}=34$$

$$\Rightarrow-\left.\frac{d^2y}{dx^2}\right|_{(-2,\,0)}=-34$$

29. If $f(x)=\sin^{-1}\left(\dfrac{2\times 3^x}{1+9^x}\right)$, then $f'\left(-\dfrac{1}{2}\right)$ equals

(A) $-\sqrt{3}\log_e\sqrt{3}$ **(B)** $\sqrt{3}\log_e\sqrt{3}$
(C) $-\sqrt{3}\log_e 3$ **(D)** $\sqrt{3}\log_e 3$ **(Online)**

Solution

(B) Given: $$f(x)=\sin^{-1}\left(\frac{2.3^x}{1+(3^x)^2}\right)$$

Let $3^x=\tan\theta$, then

$$f(\theta)=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$
$$\Rightarrow f(\theta)=\sin^{-1}(\sin 2\theta)=2\theta$$
$$\Rightarrow f(x)=2\tan^{-1}(3^x)$$

So, $$f'(x)=2\frac{1}{(1+9^x)}\cdot 3^x\ \ln 3$$

$$f'\left(\frac{-1}{2}\right)=\frac{2}{1+9^{-1/2}}3^{-1/2}\ \ln 3=\sqrt{3}\ \ln\sqrt{3}$$

30. If $x=\sqrt{2^{\operatorname{cosec}^{-1}t}}$ and $y=\sqrt{2^{\sec^{-1}t}}$ ($|t|\geq 1$), then $\dfrac{dy}{dx}$ is equal to

(A) $\dfrac{y}{x}$ **(B)** $\dfrac{x}{y}$
(C) $-\dfrac{y}{x}$ **(D)** $-\dfrac{x}{y}$ **(Online)**

Solution

(C) Given: $x=\sqrt{2^{\operatorname{cosec}^{-1}t}}$ and $y=\sqrt{2^{\sec^{-1}t}}$ ($|t|\geq 1$)

Now, $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

*No question from this chapter appeared in 2018 JEE papers.

$$= \frac{\frac{1}{2\sqrt{2^{\sec^{-1}t}}}\, 2^{\sec^{-1}t}\ln 2\left(\frac{1}{t\sqrt{t^2-1}}\right)}{\frac{1}{2\sqrt{2^{\operatorname{cosec}^{-1}t}}}\, 2^{\operatorname{cosec}^{-1}t}\ln 2\left(\frac{1}{t\sqrt{t^2-1}}\right)}$$

$$= \frac{\sqrt{2^{\sec^{-1}t}}}{\sqrt{2^{\operatorname{cosec}^{-1}t}}} = \frac{-y}{x}$$

31. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in \mathbb{R} - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is

(A) -3 **(B)** $-2\sqrt{2}$
(C) $2\sqrt{2}$ **(D)** 3 **(Offline)**

Solution

(C) Given:

$$f(x) = x^2 + \frac{1}{x^2} \text{ and } g(x) = x - \frac{1}{x}$$

Therefore,

$$h(x) = \frac{f(x)}{g(x)} \Rightarrow \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} \Rightarrow \frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)}$$

For $x - \frac{1}{x} < 0$, $\quad \frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)} \le -2\sqrt{2}$

Therefore, $-2\sqrt{2}$ is the local maximum value.

For $x - \frac{1}{x} > 0$, $\quad \frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)} \ge 2\sqrt{2}$

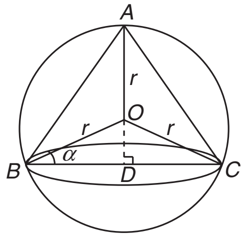
Therefore, $2\sqrt{2}$ is the local minimum value.

32. If a right circular cone, having maximum volume, is inscribed in a sphere of radius 3 cm, then the curved surface area (in cm^2) of this cone is

(A) $6\sqrt{2}\pi$ **(B)** $6\sqrt{3}\pi$
(C) $8\sqrt{2}\pi$ **(D)** $8\sqrt{3}\pi$ **(Online)**

Solution

(D) The given situation is depicted in the following figure:



Let $\angle OBD = \alpha$. Therefore, from ΔOBD, we get the following:

$$BD = r\cos\alpha = r_1$$
$$OD = r\sin\alpha$$
$$AD = r + OD = r + r\sin\alpha = h$$

The volume of cone is given by

$$V = \frac{1}{3}\pi r_1^2 h$$

Therefore,

$$V = \frac{1}{3}\pi r^2\cos^2\alpha(r + r\sin\alpha) = \frac{\pi r^3}{3}\cos^2\alpha(1 + \sin\alpha)$$

$$\frac{dV}{d\alpha} = \frac{\pi r^3}{3}\cos^2\alpha(\cos\alpha) - \frac{\pi r^3}{3}2\cos\alpha\sin\alpha(1 + \sin\alpha)$$

$$= \frac{\pi r^3}{3}\cos\alpha(\cos^2\alpha - 2\sin\alpha - 2\sin^2\alpha) = 0$$

Since $\cos\alpha \neq 0$, we get

$$\cos^2\alpha - 2\sin\alpha - 2\sin^2\alpha = 0$$
$$\Rightarrow 1 - 3\sin^2\alpha - 2\sin\alpha = 0$$
$$\Rightarrow 3\sin^2\alpha + 2\sin\alpha - 1 = 0$$

That is, $\quad (3\sin\alpha - 1)(\sin\alpha + 1) = 0$

$$\sin\alpha + 1 \neq 0,\ \sin\alpha = \frac{1}{3}$$

So, the maximum value of volume occurs at

$$\sin\alpha = \frac{1}{3} \Rightarrow \alpha = \sin^{-1}\frac{1}{3}$$

$$\left.\frac{d^2V}{d\alpha^2}\right|_{\alpha = \sin^{-1}1/3} < 0$$

The surface area of curved surface is

$$\pi r_1 l \qquad (l = AC)$$
$$= \pi r\cos\alpha\sqrt{AD^2 + CD^2}$$
$$= \pi r\cos\alpha\sqrt{h^2 + r_1^2}$$
$$= \pi r\cos\alpha\sqrt{(r + r\sin\alpha)^2 + (r\cos\alpha)^2}$$
$$= \pi r^2\cos\alpha\sqrt{2 + 2\sin\alpha}$$

For, $\alpha = \sin^{-1}\frac{1}{3}$:

$$= 9\pi\sqrt{1 - \frac{1}{9}}\sqrt{2 + 2/3} = 9\pi\sqrt{\frac{8}{9}}\cdot\sqrt{\frac{8}{3}} = 8\sqrt{3}\pi$$

33. Let $f(x)$ be a polynomial of degree 4 having extreme values at $x = 1$ and $x = 2$. If $\lim_{x\to 0}\left(\frac{f(x)}{x^2} + 1\right) = 3$, then $f(-1)$ is equal to

(A) $\frac{9}{2}$ **(B)** $\frac{5}{2}$
(C) $\frac{3}{2}$ **(D)** $\frac{1}{2}$ **(Online)**

Solution

(A) Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$$\lim_{x\to 0}\left(\frac{f(x)}{x^2}+1\right) = 3 \Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2} = 2$$

$$\Rightarrow \lim_{x\to 0}\frac{ax^4 + bx^3 + cx^2 + dx + e}{x^2} = 2$$

For finite limit, e and d should equal to 0. Therefore,

$$\lim_{x\to 0}\frac{ax^4 + bx^3 + cx^2}{x^2} = 2$$

$$\Rightarrow \lim_{x\to 0}(ax^2 + bx + c) = 2 \Rightarrow c = 2$$

So, $f(x) = ax^4 + bx^3 + 2x^2$

Then $f'(x) = 4ax^3 + 3bx^2 + 4x$

Since $f'(1) = 0$ and $f'(2) = 0$, we get

$$4a + 3b + 4 = 0 \text{ and } 32a + 12b + 8 = 0$$

$$\begin{array}{l} 8a + 3b + 2 = 0 \\ 4a + 3b + 4 = 0 \\ -\quad -\quad - \\ \hline 4a - 2 = 0 \end{array}$$

$$\Rightarrow a = \frac{1}{2}$$

$$2 + 3b + 4 = 0 \Rightarrow b = -2$$

$$f(x) = \frac{1}{2}x^4 - 2x^3 + 2x^2$$

$$f(-1) = \frac{1}{2} + 2 + 2 = \frac{9}{2}$$

34. Let M and m be respectively the absolute maximum and the absolute minimum values of the function, $f(x) = 2x^3 - 9x^2 + 12x + 5$ in the interval [0, 3]. Then $M - m$ is equal to

(A) 5 **(B)** 9

(C) 4 **(D)** 1 **(Online)**

Solution

(B) Given: $f(x) = 2x^3 - 9x^2 + 12x + 5$ in the interval [0, 3].

$$f(0) = 2(0) - 9(0) + 12(0) + 5 = 5$$
$$f(1) = 2(1)^3 - 9(1)^2 + 12(1) + 5 = 1 - 9 + 12 + 5 = 9$$
$$f(2) = 2(2)^3 - 9(2)^2 + 12(2) + 5 = 16 - 36 + 24 + 5 = 9$$
$$f(3) = 2(3)^3 - 9(3)^2 + 12(3) + 5 = 54 - 81 + 36 + 5 = 14$$
$$f(3) = M = 14$$
$$f(0) = m = 5$$

Therefore, $M - m \Rightarrow 14 - 5 = 9$

JEE Advanced 2018

35. For each positive integer n, let $y_n = \frac{1}{n}((n+1)(n+2)\ldots(n+n))^{1/n}$. For $x \in \mathbb{R}$, let $[x]$ be the greatest integer less than or equal to x. If $\lim_{x\to\infty} y_n = L$, then the value of $[L]$ is ___. **(Paper-1)**

Solution

(1) It is given that

$$y_n = \frac{1}{n}((n+1)(n+2)\cdots(n+n))^{1/n}$$

or
$$y_n = \prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{1/n}$$

Taking log on both sides of this equation, we get

$$\log y_n \log \prod_{k=1}^{n}\left(1+\frac{k}{n}\right)^{\frac{1}{n}} = \frac{1}{n}\sum_{k=1}^{n}\log\left(1+\frac{k}{n}\right)$$

Taking limit $n\to\infty$:

$$\lim_{n\to\infty}\log y_n = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\left(1+\frac{k}{n}\right)$$

Therefore,
$$\log L = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\left(1+\frac{k}{n}\right)$$
$$= \int_0^1 \log(1+x)dx$$

Substituting $1 + x = t$, we get

$$dx = dt$$

Therefore,
$$\log L = \int_1^2 \log t\, dt = t\log t - t\Big|_1^2$$

Therefore,
$$\log L = 2(\log 2) - 2 - 1(\log 1) + 1$$
$$\log L = \log 2^2 - 1$$
$$\log L = \log 4 - \log e$$

Therefore,
$$\log L = \log\frac{4}{e} \Rightarrow [L] = 1$$

Chapter 22: Indefinite Integration

JEE Main 2018

36. The integral

$$\int \frac{\sin^2 x\cos^2 x}{(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x)^2}dx$$

is equal to

(A) $-\frac{1}{3(1+\tan^3 x)}+C$ **(B)** $\frac{1}{1+\cot^3 x}+C$

(C) $-\frac{1}{1+\cot^3 x}+C$ **(D)** $-\frac{1}{3(1+\tan^3 x)}+C$

(where C is a constant of integration) **(Offline)**

Solution

(A) Let us consider

$$I=\int\frac{\sin^2 x\cos^2 x}{[(\sin^2 x+\cos^2 x)(\sin^3 x+\cos^3 x)]^2}dx=\int\frac{\sin^2 x\cos^2 x}{(\sin^3 x+\cos^3 x)^2}dx$$

$$=\int\frac{\cos^4 x\left(\frac{\sin^2 x}{\cos^2 x}\right)}{\cos^6 x(\sin^3 x+\cos^3 x)^2}dx=\int\frac{\cos^4 x\left(\frac{\sin^2 x}{\cos^2 x}\right)}{\cos^6 x(\sin^3 x+\cos^3 x)^2}dx$$

$$=\frac{\tan^2 x\sec^2 x}{(1+\tan^3 x)}dx$$

Let us substitute as $1+\tan^3 x=t$, we get

$$3\tan^2 x\sec^2 x dx=dt$$

$$\tan^2 x\sec^2 x dx=\frac{1}{3}dt$$

Thus,

$$I=\frac{1}{3}\int\frac{1}{t^2}dt=\frac{1}{3}\left(-\frac{1}{t}\right)+c$$

$$=-\frac{1}{3(1+\tan^3 x)}+c$$

37. If $f\left(\frac{x-4}{x+2}\right)=2x+1,\ (x\in R-\{1,-2\})$, then $\int f(x)dx$ is equal to (where C is a constant of integration)

(A) $12\log_e|1-x|+3x+C$ **(B)** $-12\log_e|1-x|-3x+C$

(C) $12\log_e|1-x|-3x+C$ **(D)** $-12\log_e|1-x|+3x+C$

(Online)

Solution

(B) Given: $f\left(\frac{x-4}{x+2}\right)=2x+1$

Let $\frac{x-4}{x+2}=y\Rightarrow x-4=(x+2)y$

$$\Rightarrow x(1-y)=2y+4$$

$$\Rightarrow x=\frac{2y+4}{1-y}$$

Therefore,

$$f(y)=\frac{2(2y+4)}{(1-y)}+1=\frac{4y+8+1-y}{1-y}=\frac{3y+9}{1-y}=3\left(\frac{y+3}{1-y}\right)$$

Therefore,

$$\int f(y)dy=-3\int\frac{y+3-1+1}{y-1}dy$$

$$=-3\int\left(1+\frac{4}{y-1}\right)dy=-3y-12\ln|1-y|+c$$

$$=-12\ln|1-x|-3x+c$$

38. If $\int\frac{2x+5}{\sqrt{7-6x-x^2}}dx=A\sqrt{7-6x-x^2}+B\sin^{-1}\left(\frac{x+3}{4}\right)+C$ (where C is a constant of integration), then the ordered pair (A, B) is equal to

(A) (2, 1) **(B)** (−2, −1)

(C) (−2, 1) **(D)** (2, −1) **(Online)**

Solution

(B) Given:

$$\int\frac{2x+5}{\sqrt{7-6x-x^2}}dx=\int\frac{a\frac{d}{dx}(7-6x-x^2)+b}{\sqrt{7-6x-x^2}}dx$$

$$2x+5=a(-6-2x)+b$$

Now, comparing x coefficient and constant coefficient, we get

$$5=-6a+b \text{ and } -2a=2\Rightarrow a=-1$$

$$5=6+b\Rightarrow b=-1$$

$$-\int\frac{-(-6-2x)}{\sqrt{7-6x-x^2}}dx-\int\frac{dx}{\sqrt{16-(x+3)^2}}$$

$$=-2\sqrt{7-6x-x^2}-\sin^{-1}\left(\frac{x+3}{4}\right)+c$$

Therefore, $A=-2$, $B=-1$. That is, the ordered pair (A, B) is equal to $(-2, -1)$.

39. If $\int\frac{\tan x}{1+\tan x+\tan^2 x}dx=x-\frac{K}{\sqrt{A}}\tan^{-1}\left(\frac{K\tan x+1}{\sqrt{A}}\right)+C$ (C is a constant of integration), then the ordered pair (K, A) is equal to

(A) (2, 1) **(B)** (−2, 3)

(C) (2, 3) **(D)** (−2, 1) **(Online)**

Solution

(C) Given:

$$I=\int\frac{\tan}{1+\tan x+\tan^2 x}dx$$

$$=\int\left(1-\frac{\sec^2 x}{1+\tan x+\tan^2 x}\right)dx$$

Let $\tan x=t$, that is, $\sec^2 x dx=dt$.

So,

$$I=x-\int\frac{dt}{1+t+t^2}$$

$$=x-\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=x-\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\sqrt{3}/2}\right)+C$$

$$=x-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x+1}{\sqrt{3}}\right)+C$$

Therefore, $K=2$ and $A=3$.

Chapter 23: Definite Integration

JEE Main 2018

40. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ is

(A) $\frac{\pi}{2}$ **(B)** 4π

(C) $\frac{\pi}{4}$ **(D)** $\frac{\pi}{8}$ **(Offline)**

Solution

(C) Given:

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} = \int_0^{\pi/2}\left(\frac{\sin^2 x}{1+2^x} + \frac{\sin^2 x}{1+2^{-x}}\right) dx$$

$$\left[\text{Using property} \int_{-\theta}^{+\theta} f(x)dx = \int_0^{\theta}(f(x) + (-x))dx\right]$$

$$I = \int_0^{\pi/2} \sin^2 x\, dx = \int_0^{\pi/2} \frac{(1-\cos(2x))}{2} dx$$

$$= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right)_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}$$

41. The value of the integral $\int_{-\pi/2}^{\pi/2} \sin^4 x\left(1+\log\left(\frac{2+\sin x}{2-\sin x}\right)\right)dx$ is

(A) 0 **(B)** $\frac{3}{4}$

(C) $\frac{3}{8}\pi$ **(D)** $\frac{3}{16}\pi$ **(Online)**

Solution

(C) Let $I = \int_{-\pi/2}^{\pi/2} \sin^4 x\, dx\left(1+\log\left(\frac{2+\sin x}{2-\sin x}\right)\right) dx \quad (1)$

Substituting $x = -x$:

$$I = \int_{-\pi/2}^{\pi/2} \sin^4 x\, dx\left(1+\log\left(\frac{2-\sin x}{2+\sin x}\right)\right) dx \quad (2)$$

From Eqs. (1) and (2), we get

$$2I = 2\int_{-\pi/2}^{\pi/2} \sin^4 x\, dx$$

$$I = 2\int_0^{\pi/2} \sin^4 x\, dx \text{ and } I = 2\int_0^{\pi/2} \cos^4 x\, dx$$

So, $$2I = 2\int_0^{\pi/2} (\sin^4 x + \cos^4 x)dx$$

$$I = \int_0^{\pi/2} (1 - 2\sin^2 x\cos^2 x)dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi/2} \sin^2 2x\, dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\int_0^{\pi/2}\left(\frac{1-\cos 4x}{2}\right) dx$$

$$= \frac{\pi}{2} - \frac{1}{2}\left(\frac{x}{2} - \frac{1}{8}\sin 4x\right)\Bigg|_0^{\pi/2}$$

$$I = \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}$$

42. The value of integral $\int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx$ is

(A) $\pi\sqrt{2}$ **(B)** $\pi(\sqrt{2}-1)$

(C) $\frac{\pi}{2}(\sqrt{2}+1)$ **(D)** $2\pi(\sqrt{2}-1)$ **(Online)**

Solution

(B) Given: $\int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx$

Let $I = \int_{\pi/4}^{3\pi/4} \frac{x}{1+\sin x} dx$. That is,

$$I = \int_{\pi/4}^{3\pi/4} \frac{x(1-\sin x)}{1-\sin^2 x} dx$$

$$I = \int_{\pi/4}^{3\pi/4} x(\sec^2 x - \tan x \sec x)\, dx$$

Now,

$$\int (x\sec^2 x - x\tan x\sec x)\, dx$$

$$= \int x\sec^2 dx - \int x\tan x\sec x\, dx$$

$$= x\tan x - \int \tan x\, dx - \left(x\sec x - \int \sec x\, dx\right)$$

$$= x(\tan x - \sec x) - \ln|\sec x| + \ln|\sec x + \tan x| + c$$

So, $I = x(\tan x - \sec x) - \ln|\sec x| + \ln|\sec x + \tan x|\Big\|_{\pi/4}^{3\pi/4}$

$$= \frac{3\pi}{4}(-1+\sqrt{2}) - \ln\sqrt{2} + \ln(\sqrt{2}+1) - \frac{\pi}{4}(1-\sqrt{2})$$

$$+ \ln\sqrt{2} - \ln(\sqrt{2}+1)$$

$$= \pi(\sqrt{2}-1)$$

43. If

$$I_1 = \int_0^1 e^{-x}\cos^2 x\,dx,\ I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx \text{ and } I_3 = \int_0^1 e^{-x^3}dx \text{ then:}$$

(A) $I_2 > I_3 > I_1$ **(B)** $I_2 > I_1 > I_3$
(C) $I_3 > I_2 > I_1$ **(D)** $I_3 > I_1 > I_2$ **(Online)**

Solution

(C) For $x \in (0, 0)$:

$$e^{-x^3} > e^{-x^2} > e^{-x}$$

So, $$I_3 = \int_0^1 e^{-x^3}dx,\ I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx,\ I_1 = \int_0^1 e^{-x}\cos^2 x\,dx$$

$$I_2 > I_1$$

$$\int_0^1 e^{-x^3}dx > \int_0^1 e^{-x^2}\cos^2 x\,dx$$

Therefore, $$I_3 > I_2 > I_1$$

44. If $f(x) = \int_0^x t(\sin x - \sin t)dt$ then

(A) $f'''(x) + f''(x) = \sin x$

(B) $f'''(x) + f''(x) - f'(x) = \cos x$

(C) $f'''(x) + f'(x) = \cos x - 2x\sin x$

(D) $f'''(x) - f''(x) = \cos x - 2x\sin x$ **(Online)**

Solution

(D) Given:

$$f(x) = \int_0^x t(\sin x - \sin t)dt$$

$$f(x) = \sin x\int_0^x t dt - \int_0^x t\sin t\,dt$$

$$f(x) = (\sin x)x + \cos x\int_0^x t dt - x\sin x$$

$$f(x) = \cos x\int_0^x t dt = x\cos x$$

Therefore, $$f'(x) = \cos x - x\sin x$$

and $$f''(x) = -\sin x - \sin x - x\cos x = -2\sin x - x\cos x$$
$$f'''(x) = -2\cos x - \cos x + x\sin x$$

Therefore, $$f'''(x) - f'(x) = \cos x - 2x\sin x$$

JEE Advanced 2018

45. The value of the integral $\int_0^{1/2} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{1/4}}dx$ is________.

(Paper-2)

Solution

(2) The given integral is

$$I = \int_0^{1/2} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{1/4}}dx$$

$$I = \int_0^{1/2} \frac{1+\sqrt{3}}{\left[(x+1)^2\frac{(1-x)^6}{(1+x)^6}\times(1+x)^6\right]^{1/4}}\cdot dx$$

$$I = \int_0^{1/2} \frac{1+\sqrt{3}}{\left[(1+x)^8\left(\frac{1-x}{1+x}\right)^6\right]^{1/4}}dx = \int_0^{1/2}\frac{1+\sqrt{3}}{(1+x)^2\left(\frac{1-x}{1+x}\right)^{3/2}}\cdot dx$$

Substituting $\frac{1-x}{1+x} = t$, we get

$$\frac{(1+x)(-dx)-(1-x)dx}{(1+x)^2} = dt$$

$$\Rightarrow \frac{-dx - xdx - dx + xdx}{(1+x)^2} = dt$$

$$\Rightarrow \frac{-2dx}{(1+x)^2} = dt$$

or $$\frac{dx}{(1+x)^2} = \frac{-dt}{2}$$

and $$x = 0 \Rightarrow t = 1$$

$$x = \frac{1}{2} \Rightarrow t = \frac{1}{3}$$

Therefore, the given integral becomes as follows:

$$I = \int_1^{1/3}\frac{(1+\sqrt{3})}{t^{3/2}}\frac{-dt}{2} = \frac{-(1+\sqrt{3})}{2}\int_1^{1/3}\frac{dt}{t^{3/2}}$$

$$I = \frac{-(1+\sqrt{3})}{2}\left(\left.\frac{t^{-1/2}}{(-1/2)}\right|_1^{1/3}\right) = \frac{-(1+\sqrt{3})}{2}(-2)\left[\left(\frac{1}{3}\right)^{-1/2} - 1^{-1/2}\right]$$

Therefore,

$$I = -\left(\frac{1+\sqrt{3}}{2}\right)(-2)(\sqrt{3}-1)$$

$$= (\sqrt{3}+1)(\sqrt{3}-1)$$

$$= \sqrt{3}^2 - 1^2 = 2$$

Chapter 24: Area Under the Curve

JEE Main 2018

46. Let $g(x) = \cos x^2, f(x) = \sqrt{x}$ and $\alpha, \beta (\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then, the area (in sq. units) bounded by the curve $y = (g \circ f)(x)$ and the lines $x = \alpha, x = \beta$ and $y = 0$ is

(A) $\frac{1}{2}(\sqrt{3}+1)$ **(B)** $\frac{1}{2}(\sqrt{3}+\sqrt{2})$

(C) $\frac{1}{2}(\sqrt{2}-1)$ **(D)** $\frac{1}{2}(\sqrt{3}-1)$ **(Offline)**

Solution

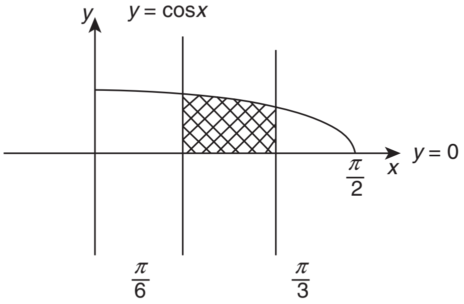
(D) Given:

$$g(x) = \cos x^2 \text{ and } f(x) = \sqrt{x}$$
$$18x^2 - 9\pi x + \pi^2 = 0$$
$$18x^2 - 6\pi x - 3\pi x + \pi^2 = 0$$
$$(6x - \pi)(3x + \pi) = 0$$

So, $\alpha = (6x - \pi), \beta = (3x + \pi)$

Now, $6x - \pi = 0, 3x + \pi = 0$

$$x = \frac{\pi}{6}, \frac{\pi}{2}$$
$$y = (g \circ f)(x) = \cos x$$



Now, the area (in sq. units) bounded by the given curve is

$$A = \int_{\pi/6}^{\pi/3} \cos x\,dx = (\sin x)\Big|_{\pi/6}^{\pi/3}$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{1}{2}(\sqrt{3} - 1)$$

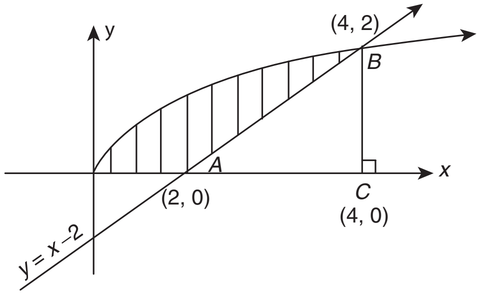
47. The area (in sq. units) of the region $\{x \in \mathbb{R} : x \geq 0, y \geq 0, y \geq x - 2 \text{ and } y \leq \sqrt{x}\}$, is

(A) $\frac{13}{3}$ **(B)** $\frac{8}{3}$

(C) $\frac{10}{3}$ **(D)** $\frac{5}{3}$ **(Online)**

Solution

(C) Area of region $\left\{x \in R;\ x \geq 0,\ y \geq 0,\ y \geq x - 2\ \&\ y \leq \sqrt{x}\right\}$



$$y = \sqrt{x}$$
$$\sqrt{x} = x - 2$$
$$(\sqrt{x})^2 - (\sqrt{x}) - 2 = 0$$
$$(\sqrt{x} - 2)(\sqrt{x} + 1) = 0$$
$$\sqrt{x} = 2 \Rightarrow x = 4$$

Therefore, the area of the region is

$$\int_0^4 \sqrt{x}\,dx - \text{Area of } \Delta ABC = \frac{2}{3}x^{3/2}\Big|_0^4 - \frac{1}{2} \times 2 \times 2$$
$$= \frac{2}{3} \times 4^{3/2} - 2$$
$$= \frac{16}{3} - 2 = \frac{10}{3} \text{ sq. unit}$$

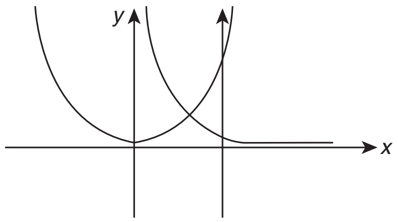
48. If the area of the region bounded by the curves, $y = x^2$, $y = \frac{1}{x}$ and the lines $y = 0$ and $x = t\ (t > 1)$ is 1 sq. unit, then t is equal to

(A) $e^{3/2}$ **(B)** $\frac{4}{3}$

(C) $\frac{3}{2}$ **(D)** $e^{2/3}$ **(Online)**

Solution

(D) Given: The region bounded by the curves, $y = x^2$, $y = \frac{1}{x}$ and the lines $y = 0$ and $x = t\ (t > 1)$



So, area bounded by these curves is given by

$$\int_x^1 x^2\,dx + \int \frac{1}{x}\,dx = 1$$
$$\frac{1}{3} + \ln t = 1$$
$$t = e^{2/3}$$

Chapter 25: Differential Equations

JEE Main 2018

49. Let $y = y(x)$ be the solution of the differential equation $\sin x\dfrac{dy}{dx} + y\cos x = 4x,\ x \in (0, \pi)$. If $y\left(\dfrac{\pi}{2}\right) = 0$, then $y\left(\dfrac{\pi}{6}\right)$ is equal to

(A) $-\dfrac{8}{9\sqrt{3}}\pi^2$ **(B)** $-\dfrac{8}{9}\pi^2$

(C) $-\dfrac{4}{9}\pi^2$ **(D)** $\dfrac{8}{9\sqrt{3}}\pi^2$ **(Offline)**

Solution

(B) The given differential equation is

$$\sin x\frac{dy}{dx} + y\cos x = 4x,\ x \in (0, \pi)$$

That is, $$\frac{dy}{dx} + y\cos x = 4x\operatorname{cosec} x$$

$$\text{I.F} = e^{\int \cot x dx} = \sin x$$

So, $$y\sin x = \int 4x\operatorname{cosec} x \cdot \sin x\, dx + c$$

$$y\sin x = 2x^2 + c$$

Since $y\left(\dfrac{\pi}{2}\right) = 0$, we get

$$0 = 2\left(\frac{\pi}{2}\right)^2 + c$$

$$c = \frac{-\pi^2}{2}$$

So, $$y\sin x = 2x^2 - \frac{\pi^2}{2}$$

Therefore, for $y\left(\dfrac{\pi}{6}\right)$, we have

$$y\sin\left(\frac{\pi}{6}\right) = 2\left(\frac{\pi}{6}\right)^2 - \frac{\pi^2}{2} \Rightarrow \frac{y}{2} = \pi^2\left(\frac{1}{18} - \frac{1}{2}\right) \Rightarrow y = -\frac{8\pi^2}{9}$$

50. Let $y = y(x)$ be the solution of the differential equation $\dfrac{dy}{dx} + 2y = f(x)$, where

$$f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

If $y(0) = 0$, then $y\left(\dfrac{3}{2}\right)$ is

(A) $\dfrac{e^2+1}{2e^4}$ **(B)** $\dfrac{1}{2e}$

(C) $\dfrac{e^2-1}{e^3}$ **(D)** $\dfrac{e^2-1}{2e^3}$ **(Online)**

Solution

(D) Given: $$\frac{dy}{dx} + 2y = f(x)$$

Let $$\frac{d}{dx}(ye^{2x}) = f(x)e^{2x}$$

$$ye^{2x} = \int f(x)e^{2x}\, dx + c$$

For $x \in [0, 1]$: $$y \cdot e^{2x} = \int e^{2x}\, dx + c = \frac{e^{2x}}{2} + c$$

At $x = 0, y = 0$: $$0 = \frac{1}{2} + c \Rightarrow c = -\frac{1}{2}$$

So, $$y = \frac{1}{2} - \frac{e^{-2x}}{2}$$

For $x \notin [0, 1]$: $$y \cdot e^{2x} = c \Rightarrow y = ce^{-2x}$$

Therefore, $$y(3/2) = \frac{c}{e^3} = \frac{e^2-1}{2e^3}\left(c = \frac{e^{2-}}{2}\right)$$

51. The curve satisfying the differential equation, $(x^2 - y^2)dx + 2xydy = 0$ and passing through the point (1, 1) is

(A) a circle of radius one. **(B)** a hyperbola.

(C) an ellipse. **(D)** a circle of radius two.

(Online)

Solution

(A) Given: $$(x^2 - y^2)dx + 2xy\, dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(x^2 - y^2)}{2xy}$$

Let $y = vx, \dfrac{dy}{dx} = v + \dfrac{dv}{dx}$.

$$v + x\frac{dv}{dx} = \frac{-(1-v^2)}{2v} = \frac{v^2 - 1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v} - v = -\left(\frac{1+v^2}{2v}\right)$$

$$\int \frac{2v}{1+v^2}dv = -\int \frac{dx}{x} + \ln c$$

$$\ln(1 + v^2) = -\ln x + \ln c = \ln\frac{c}{x}$$

$$1 + v^2 = \frac{c}{x}$$

$$x^2 + y^2 = \frac{c}{x}$$

which passes through (1, 1). Therefore,

$$c = 2$$

So, $$x^2 + y^2 - 2x = 0$$

Therefore, the curve is a circle.

52. The differential equation representing the family of ellipses having foci either on the x-axis or on the y-axis, centre at the origin and passing through the point (0, 3) is

(A) $xyy'' + x(y')^2 - yy' = 0$ **(B)** $x + yy'' = 0$

(C) $xyy' + y^2 - 9 = 0$ **(D)** $xyy' - y^2 + 9 = 0$

(Online)

Solution

(D) We know that general equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and passes through the point (0, 3). Therefore,

$$\frac{x^2}{a^2} + \frac{y^2}{9} = 1 \quad (1)$$

Differentiating the Eq. (1) with respect to x, we get

$$\frac{2x}{a^2} + \frac{2y}{9}y' = 0$$

$$\frac{x}{a^2} = \frac{-y}{9}y'$$

$$\frac{1}{a^2} = \frac{-y}{9x}y' \quad (2)$$

From Eqs. (1) and (2), the differential equation is

$$\frac{-xy}{9}y' + \frac{y^2}{9} = 1$$

$$xyy' - y^2 + 9 = 0$$

Chapter 26: Vector Algebra

JEE Main 2018

53. Let $\vec{u}$ be a vector coplanar with the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. If $\vec{u}$ is perpendicular to $\vec{a}$ and $\vec{u}\cdot\vec{b} = 24$, then $|\vec{u}|^2$ is equal to

(A) 315 **(B)** 256

(C) 84 **(D)** 336

(Offline)

Solution

(D) Let $\vec{u} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\vec{u}\cdot\vec{a} = 0$$

Therefore, $2x + 3y - z = 0$ (1)

Given: $\vec{u}\cdot\vec{b} = 24$

$$y + z = 24 \quad (2)$$

Since all three given vectors are coplanar, we get

$$|\vec{u}\ \vec{a}\ \vec{b}| = 0$$

That is,
$$\begin{vmatrix} x & y & z \\ 2 & 3 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4x - 2y + 2z = 0$$

That is, $2x - y + z = 0$ (3)

Adding Eqs. (2) and (3), we have

$$2x + 2z = 24$$

$$x + z = 12 \quad (4)$$

Now, substituting the value of y from Eq. (2) and value of x from Eq. (4) in Eq. (1), we get

$$24 - 2z + 3(24 - z) - z = 0$$

That is, $96 = 6z$

$$\Rightarrow z = 16$$

and hence $x = -4$ and $y = 8$. Therefore,

$$\vec{u} = -4\hat{i} + 8\hat{j} + 16\hat{k}$$

$$|\vec{u}|^2 = 16 + 64 + 256 \Rightarrow 336$$

54. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + 2\vec{b} + 2\vec{c} = \vec{0}$, then $|\vec{a}\times\vec{c}|$ is equal to

(A) $\frac{\sqrt{15}}{4}$ **(B)** $\frac{1}{4}$

(C) $\frac{15}{16}$ **(D)** $\frac{\sqrt{15}}{16}$ **(Online)**

Solution

(A) Given: $\vec{a} + 2\vec{b} + 2\vec{c} = 0$ and $|\vec{a}|^2 = |\vec{b}|^2 = |\vec{c}|^2 = 1$

$$\Rightarrow \vec{a} = 2\vec{c} = -2\vec{b}$$

$$\Rightarrow |\vec{a} + 2\vec{c}|^2 = |2\vec{b}|^2$$

$$\Rightarrow |\vec{a}|^2 + 4|\vec{c}|^2 + 4\vec{a}\cdot\vec{c} = 4|\vec{b}|^2$$

$$\Rightarrow 5 + 4\vec{a}\cdot\vec{c} = 4 \Rightarrow \vec{a}\cdot\vec{c} = \frac{-1}{4}$$

We know that $|\vec{a}\times\vec{c}|^2 + (\vec{a}\cdot\vec{c})^2 = |\vec{a}|^2|\vec{c}|^2$

$$|\vec{a}\cdot\vec{c}|^2 + \frac{1}{16} = 1 \Rightarrow |\vec{a}\times\vec{c}|^2 = \frac{15}{16}$$

$$\Rightarrow |\vec{a}\times\vec{c}|^2 = \frac{\sqrt{15}}{4}$$

55. If the position vectors of the vertices A, B and C of a ΔABC are, respectively, $4\hat{i} + 7\hat{j} + 8\hat{k}$, $2\hat{i} + 3\hat{j} + 4\hat{k}$ and $2\hat{i} + 5\hat{j} + 7\hat{k}$, then the position vector of the point, where the bisector of $\angle A$ meets BC is

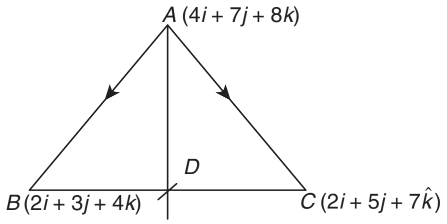
(A) $\frac{1}{2}(4\hat{i} + 8\hat{j} + 11\hat{k})$ **(B)** $\frac{1}{3}(6\hat{i} + 11\hat{j} + 15\hat{k})$

(C) $\frac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$ **(D)** $\frac{1}{4}(8\hat{i} + 14\hat{j} + 19\hat{k})$

(Online)

Solution

(C) From the given data, we plot the graph as shown in the following figure:



D is the point of intersection of internal angle bisector of a with BC line

Now, $\overrightarrow{AB} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - (4\hat{i} + 7\hat{j} + 8\hat{k}) = -2\hat{i} - 4\hat{j} - 4\hat{k}$

and $\overrightarrow{AC} = (2\hat{i} + 5\hat{j} + 7\hat{k}) - (4\hat{i} + 7\hat{j} + 8\hat{k}) = -2\hat{i} - 2\hat{j} - \hat{k}$

So, $|\overrightarrow{AB}| = 6, |\overrightarrow{AC}| = 3$

$$\frac{BD}{CD} = \frac{AB}{AC} = \frac{6}{3} = \frac{2}{1}$$

Therefore, the position vector of point D is

$$\frac{\left(2\left(2\hat{i} + 5\hat{j} + 7\hat{k}\right) + 1\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)\right)}{3}$$

$$= \frac{1}{3}\left(6\hat{i} + 13\hat{j} + 18\hat{k}\right)$$

56. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{j} - \hat{k}$ and a vector $\vec{b}$ be such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a}.\vec{b} = 3$. Then $|\vec{b}|$ equals

(A) $\frac{11}{3}$ **(B)** $\frac{11}{\sqrt{3}}$

(C) $\sqrt{\frac{11}{3}}$ **(D)** $\frac{\sqrt{11}}{3}$ **(Online)**

Solution

(C) Given:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{c} = \hat{j} - \hat{k}, \vec{a} \cdot \vec{b} = 3$$

$$\vec{a} \times \vec{b} = \vec{c}$$

$$\Rightarrow \vec{a} \times (\vec{a} \times \vec{b}) = \vec{a} \times \vec{c}$$

$$\Rightarrow (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b} = \vec{a} \times \vec{c}$$

$$\Rightarrow 3\vec{a} - 3\vec{b} = -2\hat{i} + \hat{j} + \hat{k}$$

$$\Rightarrow 3\hat{i} + 3\hat{j} + 3\hat{k} - 3\vec{b} = -2\hat{i} + \hat{j} + \hat{k}$$

$$\Rightarrow \vec{b} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$$

Therefore, $$|\vec{b}| = \frac{\sqrt{25 + 4 + 4}}{3}$$

$$\Rightarrow |\vec{b}| = \frac{\sqrt{11}}{3}$$

JEE Advanced 2018

57. Let $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a} \cdot \vec{b} = 0$. For some $x, y \in \mathbb{R}$, let $\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})$. If $|\vec{c}| = 2$ and the vector $\vec{c}$ is inclined at the same angle α to both $\vec{a}$ and $\vec{b}$, then the value of $8\cos^2\alpha$ is ___. **(Paper-1)**

Solution

(3) It is given that

$$\vec{c} = x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b}) \qquad (1)$$

Dot multiplying Eq. (1) by $\vec{a}$, we get

$$\vec{c} \cdot \vec{a} = (x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})) \cdot \vec{a}$$

$$\vec{c} \cdot \vec{a} = x\vec{a} \cdot \vec{a} + y\vec{b} \cdot \vec{a} + (\vec{a} \times \vec{b}) \cdot \vec{a}$$

$$\Rightarrow \vec{c} \cdot \vec{a} = x|a|^2 + (\vec{a} \times \vec{b}) \cdot \vec{a}$$

Since $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a} \cdot \vec{b} = 0$. Then, $(\vec{a} \times \vec{b}) = 1$

$$|\vec{c}|(1)\cos\alpha = x(1)^2 + (1)(1)$$

$$\Rightarrow x = 2\cos\alpha \qquad (2)$$

Now, dot multiplying Eq. (1) by $\vec{b}$, we get

$$\vec{c} \cdot \vec{b} = (x\vec{a} + y\vec{b} + (\vec{a} \times \vec{b})) \cdot \vec{b}$$

$$\vec{c} \cdot \vec{b} = x\vec{a} \cdot \vec{b} + y\vec{b} \cdot \vec{b} + (\vec{a} \times \vec{b}) \cdot \vec{b}$$

$$\Rightarrow \vec{c} \cdot \vec{b} = y|\vec{b}|^2 + (\vec{a} \times \vec{b}) \cdot \vec{b}$$

Since $\vec{a}$ and $\vec{b}$ be two unit vectors such that $\vec{a} \cdot \vec{b} = 0$. Then, $(\vec{a} \times \vec{b}) = 1$

$$|\vec{c}|(1)\cos\alpha = y(1)^2 + (1)(1)$$

$$\Rightarrow y = 2\cos\alpha \qquad (3)$$

Squaring Eq. (1) on both sides, we get

$$|\vec{c}|^2 = x^2 + y^2 + |\vec{a} \times \vec{b}|^2$$

Given: $|\vec{c}| = 2$. Therefore,

$$4 = x^2 + y^2 + (\vec{a} \times \vec{b})^2$$

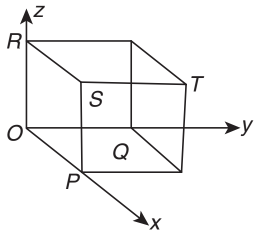
or $$4 = 8\cos^2\alpha + 1$$

$$\Rightarrow 8\cos^2\alpha = 3$$

58. Consider the cube in the first octant with sides OP, OQ and OR of length 1, along the x-axis, y-axis and z-axis, respectively, where $O(0, 0, 0)$ is the origin. Let $S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT. If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, then the value of $|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})|$ is _______. **(Paper-2)**

Solution

(0.5) Let us depict the given geometrical situation in a diagram as shown in the following figure:



The coordinates are:

$$Q \equiv (0,1,0)$$
$$S \equiv \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$
$$T \equiv (1,1,1)$$
$$P \equiv (1,0,0)$$
$$R \equiv (0,0,1)$$

As it is given that If $\vec{p} = \overrightarrow{SP}$, $\vec{q} = \overrightarrow{SQ}$, $\vec{r} = \overrightarrow{SR}$ and $\vec{t} = \overrightarrow{ST}$, we have the following:

$$\vec{p} = \overrightarrow{SP} = \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k} = \frac{1}{2}(\hat{i} - \hat{j} - \hat{k})$$
$$\vec{q} = \overrightarrow{SQ} = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k} = \frac{1}{2}(-\hat{i} + \hat{j} - \hat{k})$$
$$\vec{r} = \overrightarrow{SR} = -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k} = \frac{1}{2}(-\hat{i} - \hat{j} + \hat{k})$$
$$\vec{t} = \overrightarrow{ST} = \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k} = \frac{1}{2}(\hat{i} + \hat{j} + \hat{k})$$

Therefore,

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{4}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix}$$

$$\vec{p} \times \vec{q} = \frac{1}{4}(\hat{i}(1+1) - \hat{j}(-1-1) + \hat{k}(1-1))$$

Therefore,

$$\vec{p} \times \vec{q} = \frac{1}{4}(2\hat{i} + 2\hat{j}) = \frac{1}{2}(\hat{i} + \hat{j})$$

and
$$\vec{r} \times \vec{t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\vec{r} \times \vec{t} = \frac{1}{4}(\hat{i}(-1-1) - \hat{j}(-1-1) + \hat{k}(-1+1))$$

Therefore, $\vec{r} \times \vec{t} = \frac{1}{4}(-2\hat{i} + 2\hat{j}) = \frac{1}{2}(-\hat{i} + \hat{j})$

Therefore,
$$(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{vmatrix}$$
$$= \frac{1}{4}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \frac{1}{4}(\hat{k}(1+1))$$
$$= \frac{1}{2}\hat{k}$$

Therefore, the value of $|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})|$ is obtained as follows:

$$\left|(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})\right| = \frac{1}{2} = 0.5$$

Chapter 27: Three-Dimensional Geometry

JEE Main 2018

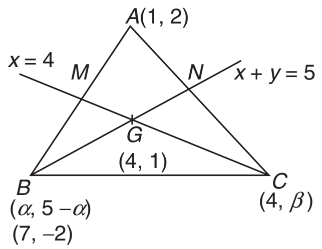
59. In a triangle ABC, coordinates of A are (1, 2) and the equations of the medians through B and C are respectively, $x + y = 5$ and $x = 4$. Then area of ΔABC (in sq. units) is

(A) 12 **(B)** 4

(C) 5 **(D)** 9 **(Online)**

Solution

(D) We draw the triangle ABC as shown in the following figure:



The centroid G is (4, 1).

Let coordinate of $B(\alpha, 5-\alpha)$ and $C(4, \beta)$.

The midpoint of AB is the median of $\angle C$ and it has value $M\left(\frac{\alpha+1}{2}, \frac{1-\alpha}{2}\right)$.

So, $$\frac{\alpha+1}{2} = 4 \Rightarrow \alpha = 7$$

Similarly, midpoint of AC is the median of $\angle B$ and it has the value $N\left(\frac{5}{2}, \frac{\beta+2}{2}\right)$.

So, $$\frac{5}{2} + \frac{\beta+2}{2} = 5 \Rightarrow \beta = 3$$

Therefore, area of ΔABC is

$$\left|\frac{1}{2}1(-2-3) + 7(3+2) + 4(2+2)\right| = \frac{1}{2}\left|-5+7+16\right| = 9$$

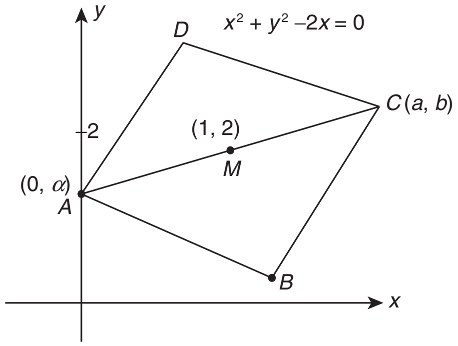
60. The sides of a rhombus $ABCD$ are parallel to the lines, $x - y + 2 = 0$ and $7x - y + 3 = 0$. If the diagonals of the rhombus intersect at $P(1, 2)$ and the vertex A (different from the origin) is on the y-axis, then the ordinate of A is

(A) $\frac{5}{2}$ **(B)** $\frac{7}{4}$

(C) 2 **(D)** $\frac{7}{2}$ **(Online)**

Solution

(A) From the given data, we plot the graph as shown in the following figure:



Now, the midpoint of AC is

$$\frac{a+0}{2} = 1 \Rightarrow a = 2$$

$$\frac{b+\alpha}{2} = 2 \Rightarrow b + \alpha = 4$$

The sides of a rhombus $ABCD$ are parallel to the lines $x - y + 2 = 0$ and $7x - y + 3 = 0$.

Equation of parallel diagonals is

$$\frac{x-y+2}{\sqrt{2}} \pm \frac{7x-y+3}{5\sqrt{2}}$$

$$\Rightarrow 2x + 4y - 7 = 0 \text{ and } 12x - 6y + 13 = 0$$

Therefore,

$$\text{slope} = \frac{-1}{2}2 = \text{slope of } AC$$

Therefore, $$\frac{2-\alpha}{1-0} = \frac{-1}{2} \text{ or } 2$$

$$\Rightarrow 2 - \alpha = \frac{1}{2} \text{ or } 2$$

$$\Rightarrow \alpha = \frac{5}{2} \text{ or } 0$$

Therefore, the ordinate of A is $\frac{5}{2}$.

61. The foot of the perpendicular drawn from the origin, on the line, $3x + y = \lambda (\lambda \neq 0)$ is P. If the line meets x-axis at A and y-axis at B, then the ratio $BP : PA$ is

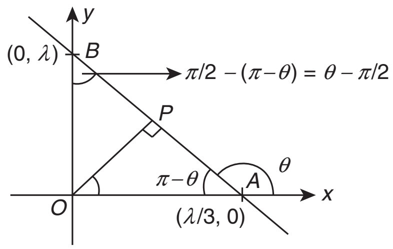
(A) 1 : 3 **(B)** 3 : 1

(C) 1 : 9 **(D)** 9 : 1 **(Online)**

Solution

(D) Given: $3x + y = \lambda$

From the given data, we plot the graph as shown in the following figure:



Slope of line $= -3 = \tan\theta$

Now, from ΔOAB: $\cos(\pi - \theta) = \frac{AP}{OA}$

$$\cos\theta = \frac{-3AP}{\lambda} \Rightarrow AP = \frac{-\lambda}{3}\cos\theta$$

Now, from ΔOBP:

$$\cos(\theta - \pi/2) = \frac{BP}{OB}$$

$$\sin\theta = \frac{BP}{\lambda} \Rightarrow BP = \lambda\sin\theta$$

$$\frac{BP}{AP} = \frac{-\lambda\sin\theta}{\left(\frac{\lambda}{3}\cos\theta\right)} = -3\frac{\sin\theta}{\cos\theta} = -3\tan\theta = \frac{9}{1}$$

62. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane, $x + y + z = 7$ is

(A) $\frac{2}{3}$ **(B)** $\frac{1}{3}$

(C) $\sqrt{\frac{2}{3}}$ **(D)** $\frac{2}{\sqrt{3}}$ **(Offline)**

Solution

(C) We have the points $A(5, -1, 4)$ and $B(4, -1, 3)$ which join a line segment. Now,

$$AB = \sqrt{(4-5)^2 + (-1-(-1))^2 + (3-4)^2} = \sqrt{1+1} = \sqrt{2}$$

The direction ratio of AB is <1, 0, 1>.

Let the angle between line AB and the plane be θ.

$$\sin\theta = \frac{2}{\sqrt{6}}$$

$$\cos\theta = \frac{1}{\sqrt{3}}$$

The projection of line AB on the plane is

$$AB\cos\theta = \sqrt{\frac{2}{3}}$$

63. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$, then the distance of the origin from the plane, containing the lines L_1 and L_2 is

(A) $\frac{1}{3\sqrt{2}}$ **(B)** $\frac{1}{2\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}}$ **(D)** $\frac{1}{4\sqrt{2}}$ **(Offline)**

Solution

(A) Given:

First two planes: $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$

Equation of plane that passes through the line of intersection of the first two planes is

$$2x - 2y + 3z - 2 + \lambda(x - y + z + 1) = 0$$

$$x(\lambda + 2) - y(2 + \lambda) + z(\lambda + 3) + (\lambda - 2) = 0 \qquad (1)$$

Equation (1) should have infinite number of solutions with the last two given planes, that is,

$$x + 2y - z - 3 = 0$$
$$3x - y + 2z - 1 = 0$$

Therefore,

$$\begin{vmatrix} (\lambda+2) & -(\lambda+2) & (\lambda+3) \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 0$$

On solving, we get $\lambda = 5$.

Substituting this value of λ in Eq. (1), we get

$$7x - 7y + 8z + 3 = 0$$

Therefore, the perpendicular distance from the origin (0, 0, 0) of the plane is

$$\frac{3}{\sqrt{167}} = \frac{1}{3\sqrt{2}}$$

64. A variable plane passes through a fixed point (3, 2, 1) and meets x, y and z axes at A, B and C respectively. A plane is drawn parallel to yz-plane through A, a second plane is drawn parallel zx-plane through B and a third plane is drawn parallel to xy-plane through C. Then, the locus of the point of intersection of these three planes, is

(A) $\frac{x}{3}+\frac{y}{2}+\frac{z}{1}=1$ **(B)** $x+y+z=6$

(C) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{11}{6}$ **(D)** $\frac{3}{x}+\frac{2}{y}+\frac{1}{z}=1$

(Online)

Solution

(B) Equation of plane is given as

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Therefore, the equation of plane passing through (3, 2, 1) is

$$\frac{3}{a}+\frac{2}{b}+\frac{1}{c}=1$$

Points on axes are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Therefore, the locus of intersection point of plane through point A, B and C. Parallel to yz-, zx- and xy- plane, respectively, is

$$\frac{3}{x}+\frac{2}{y}+\frac{1}{c}=1$$

65. An angle between the plane, $x + y + z = 5$ and the line of intersection of the planes, $3x + 4y + z - 1 = 0$ and $5x + 8y + 2z + 14 = 0$, is

(A) $\sin^{-1}(\sqrt{3/17})$ **(B)** $\cos^{-1}(\sqrt{3/17})$

(C) $\cos^{-1}(3/\sqrt{17})$ **(D)** $\sin^{-1}(3/\sqrt{17})$

(Online)

Solution

(A) Given:

$$3x + 4y + z - 1 = 0 \text{ and } 5x + 8y + 2z + 14 = 0$$

The direction of line of intersection of plans is given by

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 1 \\ 5 & 8 & 2 \end{vmatrix}$$

$$= \hat{i}(8-8) - \hat{j}\,(6-5) + \hat{k}(24-20) = -\hat{j} + 4\hat{k}$$

Direction ratios of line of intersection is (0, −1, 9)

Therefore, angle between plane $x + y + z = 5$ and line or intersection is θ.

$$\sin\theta = \left|\frac{(\hat{i}+\hat{j}+\hat{k})\cdot(-\hat{j}+4\hat{k})}{\sqrt{1+1+1}\,\sqrt{1+16}}\right| = \frac{3}{\sqrt{3}\,\sqrt{7}} = \sqrt{\frac{3}{17}}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{17}\right)$$

66. An angle between the lines whose direction cosines are given by the equations, $l + 3m + 5n = 0$ and $5lm - 2mn + 6nl = 0$, is

(A) $\cos^{-1}\left(\frac{1}{3}\right)$ **(B)** $\cos^{-1}\left(\frac{1}{4}\right)$

(C) $\cos^{-1}\left(\frac{1}{6}\right)$ **(D)** $\cos^{-1}\left(\frac{1}{8}\right)$ **(Online)**

Solution

(C) Given: $l + 3m + 5n = 0$

$$\Rightarrow l = -(3m + 5n) \quad (1)$$

and $5lm - 2mn + 6nl = 0$

$$\Rightarrow l = \frac{2mn}{5m + 6n} \quad (2)$$

From Eqs. (1) and (2), we get

$$-(3m + 5n) = \frac{2mn}{(5m + 6n)}$$

$$(3m + 5n)(5m + 6n) + 2mn = 0$$

$$15m^2 + 25mn + 18mn + 30n^2 + 2mn = 0$$

$$15m^2 + 30n^2 + 45mn = 0$$

$$m^2 + 2n^2 + 3mn = 0$$

$$(m + 2n)(m + n) = 0$$

$$m = -2n \text{ or } m - n$$

For, $m = -n$:

$$l = -(-3n + 5n) = -2n \Rightarrow (-2n, -n, n)$$

Direction ratio is (2, 1, −1).

For, $m = -2n$:

$$l = -(-6n + 5n) = n$$

Direction ratio $(n_1, -2n, n) = (1, -2, 1)$.

$$m = -2n, l = -(-6n + 5n) = n \Rightarrow \text{direction ratio}$$

$$(n, -2n, n) = (1, -2, 1)$$

Therefore, the angle between the lines is given as

$$\cos\theta = \left|\frac{2.1 + 1(-2) + (-1)(1)}{\sqrt{4+1+1}\sqrt{4+1+1}}\right|$$

$$\Rightarrow \cos\theta = \frac{1}{6} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{6}\right)$$

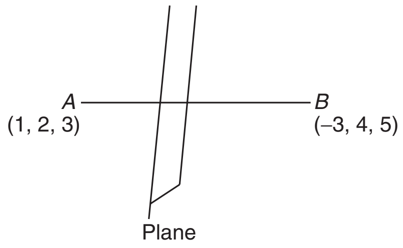
67. A plane bisects the line segment joining the points (1, 2, 3) and (−3, 4, 5) at right angles. Then this plane also passes through the point:

(A) (−3, 2, 1) **(B)** (3, 2, 1)

(C) (−1, 2, 3) **(D)** (1, 2, −3) **(Online)**

Solution

(A) From the given data, we can plot the graph as shown in the following figure:



Plane bisects the line AB perpendicularly. Therefore, the mid-point of AB is

$$\left(\frac{-3+1}{2}, \frac{4+2}{2}, \frac{3+5}{2}\right) = (-1, 3, 4)$$

Now, the direction ratio of AB is (−4, 2, 2).

Therefore, equation of plane is given by

$$-4(x + 1) + 2(y - 3) + 2(z - 4) = 0$$

$$\Rightarrow -4x - 4 + 2y - 6 + 2z - 8 = 0$$

$$\Rightarrow -4x + 2y + 2z - 18 = 0$$

$$\Rightarrow -2x + y + z = 9$$

Only (−3, 2, 1) satisfies the plane.

68. The sum of the intercepts on the coordinate axes of the plane passing through the point (−2, −2, 2) and containing the line joining the points (1, −1, 2) and (1, 1, 1), is

(A) 4 **(B)** −4

(C) −8 **(D)** 12 **(Online)**

Solution

(B) From the given data, we can say that this case refers to an equation of a plane:

$$\begin{vmatrix} x+2 & y+2 & z-2 \\ -2-1 & -2-(-1) & 2-2 \\ -2-1 & -2-1 & 2-1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+2 & y+2 & z-2 \\ -3 & -1 & 0 \\ -3 & -3 & 1 \end{vmatrix} = 0$$

$$-(x + 2) + 3(y + 2) + 6(z - 2) = 0$$

$$x - 3y - 6z + 8 = 0$$

Therefore, the sum of intercepts is $= -8 + \frac{8}{3} + \frac{8}{6} = -4$

69. If the angle between the lines, $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{5-x}{-2} = \frac{7y-14}{p} = \frac{z-2}{4}$ is $\cos^{-1}\left(\frac{2}{3}\right)$, then p is equal to

(A) $\frac{7}{2}$ **(B)** $\frac{2}{7}$

(C) $-\frac{7}{4}$ **(D)** $-\frac{4}{7}$ **(Online)**

Solution

(A) Given:

$$\frac{x}{2}=\frac{y}{2}=\frac{z}{1},\ \frac{x-5}{2}=\frac{y-2}{p/7}=\frac{z-3}{4}$$

The angle between both lines is given by

$$\cos^{-1}\left(\frac{2}{3}\right)=\cos^{-1}\left(\frac{4+\frac{2p}{7}+4}{3\sqrt{4+\frac{p^2}{49}+16}}\right)$$

$$\Rightarrow \frac{2}{3}=\frac{56+2p}{3\sqrt{p^2+980}}$$

$$\Rightarrow \sqrt{p^2+980}=p+28$$

$$\Rightarrow p^2+980=p^2+56p+784$$

$$\Rightarrow 56p=196 \Rightarrow p=\frac{7}{2}$$

JEE Advanced 2018

70. In a triangle PQR, let $\angle PQR=30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is(are) TRUE?

(A) $\angle QPP=45°$.

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP=120°$.

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3}-15$.

(D) The area of the circumcircle of the triangle PQR is 100π.

(Paper-1)

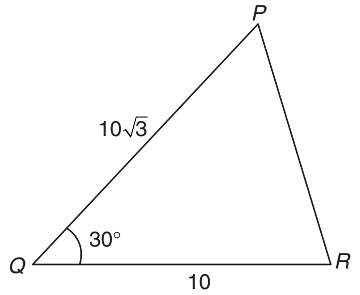
Solution

(B), (C), (D)

Let us check all four options as follows:

- **Option (A):** It is given that

$$\angle PQR=30°$$
$$PQ=10\sqrt{3}$$
$$QR=10$$



From this figure, we know that

$$PR=\sqrt{(PQ)^2+(RQ)^2-2(PQ)(RQ)\cos 30°}$$

Therefore, $\cos 30°=\dfrac{(10\sqrt{3})^2+(10)^2-(PR)^2}{2(10\sqrt{3})(10)}$

$$\Rightarrow \frac{\sqrt{3}}{2}=\frac{100\times 3+100-PR^2}{2\times 100\times\sqrt{3}}$$

$$\Rightarrow \frac{2\times 100\times 3}{2}=300+100-PR^2$$

$$\Rightarrow 300=400-PR^2$$

$$\Rightarrow PR^2=400-300=100$$

$$\Rightarrow PR=10$$

Since $QR=PR=10$, we get

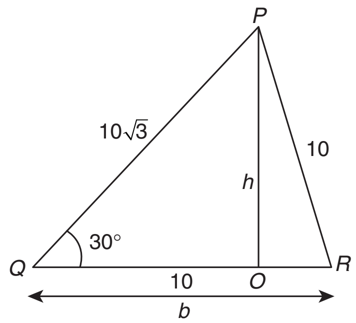
$$\angle PQR=\angle QPR$$

Therefore, $\angle QPR=30°$.

Hence, option (A) is false.

- **Option (B):** Now, the area of ΔPQR is

$$\frac{1}{2}bh=\frac{1}{2}\times 10\sqrt{3}\times 10\times\sin 30°$$
$$=\frac{1}{2}\times 10\times 10\sqrt{3}\times\frac{1}{2}$$
$$=25\sqrt{3}$$



Also,

$$\angle QRP=180°-(\angle PQR+\angle QPR)$$
$$=180°-(30°+30°)$$
$$=120°$$

Hence, option (B) is true.

- **Option (C):** Now, the radius of incircle is

$$r=\frac{2(\text{Area of }\Delta PQR)}{S}$$

where $S=PQ+PR+RQ$. Therefore,

$$r=\frac{2\times 25\sqrt{3}}{10+10+10\sqrt{3}}$$
$$=\frac{25\sqrt{3}}{10+5\sqrt{3}}=\frac{25\sqrt{3}}{5(2+\sqrt{3})}=\frac{5\sqrt{3}}{2+\sqrt{3}}$$
$$=\frac{5\sqrt{3}}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$$
$$=\frac{5\sqrt{3}(2-\sqrt{3})}{(2^2-\sqrt{3}^2)}=\frac{5\sqrt{3}(2-\sqrt{3})}{4-3}$$

Therefore, the radius of incircle is

$$5\sqrt{3}(2-\sqrt{3}) = 10\sqrt{3}-15$$

Hence, option (C) is true.

- **Option (D):** Now, the area of the circumcircle is πR^2, where

$$R = \frac{\text{Product of three sides of } \Delta PQR}{4(\text{Area of } \Delta PQR)} = \frac{10\times10\times10\sqrt{3}}{4\times25\sqrt{3}} = 10$$

$$R = \frac{S}{4(\text{Area of } \Delta PQR)} = \frac{10\times10\times10\sqrt{3}}{4\times25\sqrt{3}} = 10$$

Therefore, the area of circumcircle is

$$\pi(10)^2 = 100\pi$$

Hence, option (D) is correct.

71. Let $P_1: 2x + y - z = 3$ and $P_2: x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) TRUE?

(A) The line of intersection of P_1 and P_2 has direction ratios 1, 2, – 1.

(B) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2.

(C) The acute angle between P_1 and P_2 is 60°.

(D) If P_3 is the plane passing through the point (4, 2, –2) and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$.

(Paper-1)

Solution

(C), (D) Let us check all four options as follows:

- **Option (A):** It is given that

$$P_1 \Rightarrow 2x + y - z = 3$$
$$P_2 \Rightarrow x + 2y + z = 2$$

Let the direction ratios of the line of intersection of plane be a, b and c. Then, the equations of the normal of the plan are

$$2a + b - c = 0$$

and $$a + 2b + c = 0$$

That is,

$$\frac{a}{(1\times1)-(-1\times2)} = \frac{b}{(-1\times1)-(2\times1)} = \frac{c}{(2\times2)-(1\times1)}$$

$$\Rightarrow \frac{a}{1+2} = \frac{b}{-1-2} = \frac{c}{4-1}$$

$$\Rightarrow -3a = 3b$$

$$\Rightarrow a = -b$$

and $3b = -3c \Rightarrow c = -b$

Therefore, $a = 1$, $b = -1$ and $c = 1$.

The direction ratios of the line of intersection are 1, –1, 1.

Hence, option (A) is false.

- **Option (B):** It is given that

$$\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{2}{3}$$

$$\frac{\left(x-\frac{4}{3}\right)}{3} = \frac{\left(\frac{1}{3}-y\right)}{+3} = \frac{z}{3}$$

or $$\frac{\left(x-\frac{4}{3}\right)}{3} = \frac{\left(y-\frac{1}{3}\right)}{-3} = \frac{z}{3}$$

Therefore, the given line is parallel to the line of intersection of the two planes P_1 and P_2. Hence, option (B) is false.

Now, acute angle between the two planes P_1 and P_2 is

$$\cos\theta = \frac{2\times1+1\times2-1\times1}{\sqrt{2^2+1^2+1^2}\times\sqrt{1^2+2^2+1^2}} = \frac{2+2-1}{6}$$

$$\cos\theta = \frac{3}{6} = \frac{1}{2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) \Rightarrow \theta = 60°$$

Therefore, the acute angle between the two planes P_1 and P_2 is 60°.

Hence, option (C) is correct.

- **Option (D):** It is given that plane P_3 passes through the point (4, 2, –2). Therefore, the equation of plane P_3 is

$$(x-4) - (y-2) + (z+2) = 0$$

Thus, $$P_3 \Rightarrow x - y + z = 0$$

Now, the distance of plane P_3 from the point (2, 1, 1) is obtained as

$$\frac{2-1+1}{\sqrt{1+1+1}} = \frac{2}{\sqrt{3}}$$

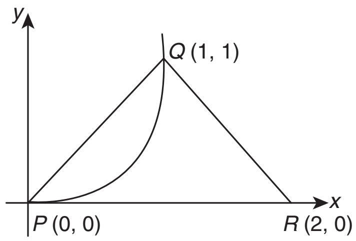
Thus, option **(D)** is true.

72. A farmer F_1 has a land in the shape of a triangle with vertices at $P(0, 0)$, $Q(1, 1)$ and $R(2, 0)$. From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n$ $(n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is ___________. **(Paper-1)**

Solution

(4) We know that the remaining area of land of the farmer F_1 is given by:

$$\text{Area} = \int_0^1 (x - x^n)dx$$



Given: The area taken by farmer is 30% of the area of ΔPQR.
The area of ΔPQR is given as

$$\frac{1}{2}\times 2\times 1=1$$

Thus, 30% of ΔPQR is found to be

$$1\times\frac{30}{100}=0.3$$

Therefore, $$0.3=\int_0^1 (x-x^n)dx=\left[\frac{x^2}{2}-\frac{x^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow 0.3=\frac{1}{2}-\frac{1}{n+1}\Rightarrow 0.3=0.5-\frac{1}{n+1}$$

or $$\frac{1}{n+1}=0.5-0.3=0.2$$

or $$n+1=\frac{1}{0.2}=5\Rightarrow n=4$$

Therefore, the value of n is 4.

73. Let P be a point in the first octant, whose image Q in the plane $x + y = 3$ (that is, the line segment PQ is perpendicular to the plane $x + y = 3$ and the mid-point of PQ lies in the plane $x + y = 3$) lies on the z-axis. Let the distance of P from the x-axis be 5. If R is the image of P in the xy-plane, then the length of PR is ________________. **(Paper-2)**

Solution

(8) Given: P is a point in first octant Q is image of P in plane $x+y=3$ lies on z-axis midpoint of PQ lies in $x+y=3$ plane. Therefore,

$$P\equiv(\alpha,\beta,\gamma)$$
$$Q\equiv(0,0,k)$$

The midpoint of PQ is $\left(\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma+k}{2}\right)$.

Since the midpoint satisfies $x+y=3$, we get

$$\alpha+\beta=6 \qquad (1)$$

Now, $$(\alpha-0,\beta-0,\gamma-k)=(p,p,0)$$
$$\Rightarrow \alpha=\beta \text{ and } \gamma=k$$

From Eq. (1), we get

$$\alpha=\beta=3$$

Now, the distance of point P from x-axis is 5. Therefore,

$$\Rightarrow \beta^2+\gamma^2=5^2\Rightarrow 3^2+\gamma^2=5^2$$
$$\Rightarrow 9+\gamma^2=25\ \Rightarrow \gamma^2=16\Rightarrow \gamma=4$$

Therefore, $\gamma=k=4$. Hence, the length of PR is

$$2k=2\times 4=8$$

Chapter 28: Probability

JEE Main 2018

74. A box A contains 2 white, 3 red and 2 black balls. Another box B contains 4 white, 2 red and 3 black balls. If two balls are drawn at random, without replacement, from a randomly selected box and one ball turns out to be white while the other ball turns out to be red, then the probability that both balls are drawn from box B is

(A) $\frac{9}{16}$ **(B)** $\frac{7}{16}$

(C) $\frac{9}{32}$ **(D)** $\frac{7}{8}$

(Online)

Solution

(B) Given:

Box *A*	**Box *B***
2 White ball	4 White ball
3 Red ball	2 Red ball
2 Black ball	3 Black ball

Probability of taking A and B box is

$$P(A)=P(B)=\frac{1}{2}$$

Probability of drawing one red and one white ball is given by

$$P(r,w)=P(A).P\left(\frac{r,w}{A}\right)+P(B)\,P\left(\frac{r,w}{B}\right)$$
$$=\frac{1}{2}\cdot\frac{2}{7}\cdot\frac{3}{6}+\frac{1}{2}\cdot\frac{4}{9}\cdot\frac{2}{8}$$

Therefore, the required probability is

$$P\left(\frac{B}{r,w}\right)=\frac{\frac{1}{2}\cdot\frac{4}{9}\cdot\frac{2}{8}}{\frac{1}{2}\cdot\frac{2}{7}\cdot\frac{3}{6}+\frac{1}{2}\cdot\frac{4}{9}\cdot\frac{2}{8}}$$
$$=\frac{\frac{1}{8}}{\frac{1}{14}+\frac{1}{18}}=\frac{1}{18}\cdot\frac{14.18}{14+18}=\frac{14}{14+18}=\frac{7}{7+9}=\frac{7}{16}$$

75. A player X has a biased coin whose probability of showing heads is p and a player Y has a fair coin. They start playing a game with their own coins and play alternately. The player who throws a head first is a winner. If X starts the game, and the probability of winning the game by both the players is equal, then the value of p is

(A) $\frac{1}{5}$ **(B)** $\frac{1}{3}$

(C) $\frac{2}{5}$ **(D)** $\frac{1}{4}$ **(Online)**

Solution

(B) Probability that X wins the game is

$$P(H) + P(T\,TH) + P(TTTTH) + P(TTTTTTH) + \dots + \infty$$

$$= p + (1-p)\frac{1}{2}p + (1-p)^2\frac{1}{2^2}.p + \dots\infty$$

$$= \frac{p}{1-\dfrac{1-p}{2}} = \frac{2p}{1+p} \quad (1)$$

Probability that Y wins the game is

$$P(TH) + P(TTTH) + P(TTTTTH) + \dots$$

$$= \frac{(1-p)}{2} + \frac{(1-p)^2}{4} + \frac{(1-p)^3}{8} + \dots$$

$$= \frac{1-p/2}{1-\dfrac{(1-p)}{2}} = \frac{1-p}{1+p} \quad (2)$$

We know that

Probability of X wins = Probability of Y wins

$$\frac{2p}{1+p} = \frac{1-p}{1+p} \Rightarrow 2p = 1-p$$

$$\Rightarrow p = \frac{1}{3}$$

76. Let A, B, and C be three events, which are pair-wise independent and $\bar{E}$ denotes the complement of an event E. If $P(A \cap B \cap C) = 0$ and $P(C) > 0$, then $P[(\bar{A} \cap \bar{B})|C]$ is equal to

(A) $P(\bar{A}) - P(B)$ **(B)** $P(A) - P(\bar{B})$

(C) $P(\bar{A}) - P(\bar{B})$ **(D)** $P(\bar{A}) + P(\bar{B})$ **(Online)**

Solution

(A) We have

$$P\left[\left(\bar{A}\cap\bar{B}\right)|C\right] = \frac{P\left[\left(\bar{A}\cup\bar{B}\right)\cap C\right]}{P(C)}$$

$$= \frac{P(C) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)}{P(C)}$$

$$= \frac{P(C) - P(A) - P(C) + P(B)P(C)}{P(C)}$$

$$= 1 - P(A) - P(B)$$

$$= P(\bar{A}) - P(B) \text{ or } P(\bar{B}) - P(A)$$

77. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is

(A) $\frac{2}{5}$ **(B)** $\frac{1}{5}$

(C) $\frac{3}{4}$ **(D)** $\frac{3}{10}$ **(Offline)**

Solution

(A) Let R = Red balls, B = Black balls. Therefore,

$$4R + 6B = 10$$

That is, $$P = \left(\frac{4}{10}\times\frac{6}{12}\right) + \left(\frac{6}{10}\times\frac{4}{12}\right)$$

$$= \frac{24}{120} + \frac{24}{120} = \frac{48}{120} = \frac{2}{5}$$

78. Two different families A and B are blessed with equal number of children. There are 3 tickets to be distributed amongst the children of these families so that no child gets more than one ticket. If the probability that all the tickets go to the children of the family B is $\frac{1}{12}$, then the number of children in each family is

(A) 3 **(B)** 4

(C) 5 **(D)** 6 **(Online)**

Topic: Conditional Probability

(C) Let n number of children in each family.

Number of ways in which all tickets goes to only B family is

$$^{n}C_3 \cdot 3!$$

Total number of ways to distribute the tickets is

$$^{2n}C_3 \cdot 3!$$

So, the probability that all the tickets go to the children of the family B is

$$\frac{1}{12} = \frac{^{n}C_3 . 3!}{^{2n}C_3 . 3!}$$

$$\Rightarrow \frac{^{n}C_3}{^{2n}C_3} = \frac{1}{12} \Rightarrow n = 5$$

JEE Advanced 2018

Paragraph A for Questions 71 and 72: There are five students S_1, S_2, S_3, S_4 and S_5 in a music class and for them there are five seats R_1, R_2, R_3, R_4 and R_5 arranged in a row, where initially the seat R_i is allotted to the student S_i, i = 1, 2, 3, 4, 5. But, on the examination day, the five students are randomly allotted the five seats.

79. The probability that, on the examination day, the student S_1 gets the previously allotted seat R_1, and NONE of the remaining students gets the seat previously allotted to him/her is

(A) $\frac{3}{40}$ **(B)** $\frac{1}{8}$

(C) $\frac{7}{40}$ **(D)** $\frac{1}{5}$ **(Paper-1)**

Solution

(A) The probability that S_1 gets previously allotted S_2, S_3, S_4, and the probability S_5 does not get previously allotted seat is

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$$

Hence, the required probability is

$$\frac{4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)}{5!}=\frac{\left(1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}\right)}{5}=\frac{\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}\right)}{5}$$

$$=\frac{\left(\frac{12-4+1}{24}\right)}{5}=\frac{9}{24\times 5}=\frac{3}{8\times 5}=\frac{3}{40}$$

80. For i = 1, 2, 3, 4, let T_i denote the event that the students S_i and S_{i+1} do **NOT** sit adjacent to each other on the day of the examination. Then, the probability of the event $T_1 \cap T_2 \cap T_3 \cap T_4$ is

(A) $\frac{1}{15}$ **(B)** $\frac{1}{10}$

(C) $\frac{7}{60}$ **(D)** $\frac{1}{5}$

Solution

(C) The probability of the event $T_1 \cap T_2 \cap T_3 \cap T_4$ is

$$\frac{\text{Total arrangements} - n(T_1 \cup T_2 \cup T_3 \cup T_4)}{\text{Total arrangements}}$$

Now, the probability that at least one pair of students sit adjacent to each other is

$$^4C_1 \times 4! \times 2! = \frac{4!}{1! \times 3!} \times 4! \times 2! = 4 \times 4 \times 3 \times 2 \times 2 = 192$$

The probability that at least two pairs of students sit adjacent to each other is

$$^3C_1 \times 3! \times 2! + {}^3C_1 + 3! \times 2! \times 2! = \frac{3!}{2!} \times 3! \times 2! + \frac{3!}{2!} \times 3! \times 2! \times 2!$$
$$= 36 + 72 = 108$$

The probability that at least three pairs of students sit adjacent to each other is

$$^2C_1 \times 2! \times 2! + {}^2C_1 \times 2 \times 2! + {}^2C_1 \times 2 \times 2!$$
$$= 2! \times 2! \times 2! + 2! \times 2 \times 2! + 2! \times 2 \times 2!$$
$$= 8 + 8 + 8 = 2!$$

The total arrangements is

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Therefore, the required probability is

$$\frac{120-192+108-24+2}{120}=\frac{14}{120}=\frac{7}{60}$$

Other Topics

JEE Main 2018

Principles of Mathematical Induction

81. The Boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to

(A) p **(B)** q

(C) $\sim q$ **(D)** $\sim p$ **(Offline)**

Solution

(D) We have

$$\begin{aligned}\sim(p \vee q) \vee (\sim p \wedge q) &= (\sim p \vee \sim q) \vee (\sim p \wedge q)\\ &= \sim p \wedge (\sim q \vee q)\\ &= \sim p \wedge (t)\\ &= \sim p\end{aligned}$$

82. If $(p \wedge -q) \wedge (p \wedge r) \rightarrow \sim p \vee q$ is false, then the truth values of p, q and r are, respectively,

(A) F, T, F **(B)** T, F, T

(C) T, T, T **(D)** F, F, F **(Online)**

Solution

(2) Given $(p \wedge \sim q) \wedge (p \wedge r) \rightarrow \sim p \vee q$ is false.

Now, the truth table is

p	q	r	$\sim p \wedge q$
T	T	T	T
T	F	T	F
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T
F	T	F	T
F	F	F	T

Only possible solution of (p, q, r) is (T, F, T) or (T, F, F).

83. Consider the following two statements:

Statement p: The value of sin 120° can be derived by taking $\theta = 240°$ in the equation $2\sin\frac{\theta}{2}=\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}$.

Statement q: The angles A, B, C and D of any quadrilateral $ABCD$ satisfy the equation $\cos\left(\frac{1}{2}(A+C)\right)+\cos\left(\frac{1}{2}(B+D)\right)=0$.

Then the truth values of p and q are, respectively,

(A) F, T **(B)** T, F

(C) T, T **(D)** F, F **(Online)**

Solution

(A) For statement p: $\theta = 240°$

$$2\sin\left(\frac{240°}{2}\right)=\sqrt{1+\sin 240°}-\sqrt{1-\sin 240°}$$

$$2\sin 120° = \sqrt{1-\frac{\sqrt{3}}{2}}-\sqrt{1+\frac{\sqrt{3}}{2}}$$

$$2\cdot\frac{\sqrt{3}}{2}=\sqrt{\frac{4-2\sqrt{3}}{4}}-\sqrt{\frac{4+2\sqrt{3}}{4}}$$

Linear Inequalities

84. The number of values of k for which the system of linear equations,

$$(k+2)x+10y=k$$
$$kx+(k+3)y=k-1$$

has **no solution**, is

(A) 1 **(B)** 2
(C) 3 **(D)** infinitely many

(Online)

Solution

(A) Given:

$$(k+2)x+10y=k$$
$$kx+(k+3)y=k-1$$

For no solution:

$$\frac{k+2}{k}=\frac{10}{k+3}\neq\frac{k}{k-1}$$
$$\Rightarrow (k+2)(k+3)=10k$$
$$\Rightarrow k^2-5k+6=0$$
$$\Rightarrow k=2, 3$$

However, for $k \neq 2$ for $k = 2$, both lines are identical and hence we have only one value of k, that is, $k = 3$.

Therefore, the number of values of k is 1.

Mathematical Reasoning

85. If $p \rightarrow (\sim p \vee \sim q)$ is false, then the truth values of p and q are, respectively,

(A) F, F **(B)** T, F
(C) F, T **(D)** T, T **(Online)**

Solution

(D) Given: $p \rightarrow (\sim p \vee \sim q)$

Now, let us make truth table of the given statement:

p	Q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$p \rightarrow (\sim p \vee \sim q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

About the Book

Wiley Mathematics Problem Book covers the complete mathematics course for JEE. It is focused on the development of problem-solving skills in JEE aspirants. The chapter flow of the book is closely aligned with the JEE (Main) syllabus and its coverage in the classroom. However, the topics required for JEE (Advanced) are also covered. The problems presented systematically cover all important concepts pertaining to the topic and the possible questions that can be framed on them.

Key Features

- Focused mainly on the preparation of JEE (Main) but also suitable for JEE (Advanced)
- **All concepts and formulae** as per JEE (Main) and JEE (Advanced) syllabus.
- **Topic-wise Illustrations** and **Additional Solved Problems** for grasping the concepts.
- Topic-wise questions provided in **Your Turn** sections for practicing the concepts.
- Solved last **10-year JEE-Main/AIEEE** questions and **10-year JEE-Advanced/IIT-JEE** questions provided (2007-2016).
- Practice Questions at two levels, i.e. **Practice Exercise 1** (JEE Main) and **Practice Exercise 2** (JEE Advanced).
- All **question types** as per JEE (Main and Advanced) covered:
  - Single Correct Choice Type
  - Multiple Correct Choice Type
  - Passage Type
  - Matrix-Match Type
  - Integer Type
- **Answer Key and Solutions** for all Practice Exercises provided.
- Covers Solved JEE 2017 (Main and Advanced) Questions Chapterwise.
- **Appendix:** Chapterwise Solved JEE 2018 (Main and Advanced) Questions.

About Maestro Series

- **Idea:** Best-suited content designed as per the needs of JEE aspirants.
- **Process:** Developed in consultation with experienced teachers after in-depth study of syllabus and relative weightage of topics.
- **Two-fold advantage:**
  - **Conceptual strength** provided by authoritative yet precise content as per syllabus requirement.
  - **Assessment as per JEE** through thought provoking end-of-chapter exercises as per JEE question pattern.

Wiley India Pvt. Ltd.
Customer Care +91 120 6291100
csupport@wiley.com
www.wileyindia.com
www.wiley.com

WILEY

ISBN 978-81-265-7630-2

9 788126 576302