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JEE MAINS & ADVANCED 2019

FULLY SOLVED

Nitin Jain

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ALGEBRA

By Nitin Jain

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Preface

I take great pleasure in presenting this book to students and teachers. This book is intended for students appearing for JEE (MAINS and ADVANCED) and other competitive exams of similar level.

Germination of this book comes from the thought of extending our knowledge to all students of 11th and 12th standard and who are aiming at JEE MAINS and JEE ADVANCED exams.

Theory related to each topic is thoroughly discussed from scratch. Then it quickly ventures into solved illustrations and Mc^2 (make concepts clear), a topical exercise to strengthen the concepts of a topic. All previous year JEE Main, Advanced and AIEEE questions are included at various places as per the requirement. In last, two level practice sessions are there for every chapter which includes all types of questions.

The purpose is not to stuff the student with more bare matter but to broaden the base. An attempt has therefore been made to avoid mystification by providing more questions to test basics.

The book is 100% solved and if you face any problem you can refer the solutions provided.

I am extremely indebted to colleagues, friends and students for pouring in suggestions in writing this book.

We shall welcome all suggestions for the improvement of the book.

Nitin Jain

Contents

ALGEBRA

Chapter 3

QUADRATIC EQUATIONS

QUADRATIC EQUATIONS

Quadratic equations are used in many areas of science and engineering. The path of a projectile is almost parabolic and we use a quadratic equation to find out where the projectile is going to hit. Parabolic antennas are other applications of quadratic equations.

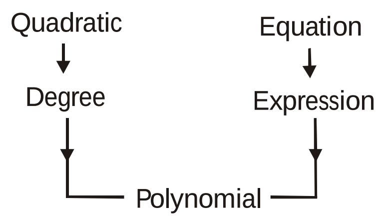
Applications of Quadratic Equations to other branches of Mathematics

1. Complex numbers arise out of the problem where we cannot find the square root of negative number.
2. Parabola is very closely related to Quadratic Equation.
3. It is used in Integration by Partial fraction.
4. It is used in solving second order linear homogeneous differential equation.
5. Used in solving piecewise function.
6. Sometimes used in curve sketching.

3.1 Solution and Formation of Quadratic Equation

Quadratic Equation

Let us start the discussion of quadratic equation with the help of following flowchart.



Here, before knowing what is quadratic equation, we have to understand the terms related to it i.e, expression, degree,polynomial etc.

Expression

It is defined as combinations of terms containing constant & variables.

Now to understand it, first we should know what is variables & constants.

a) Constants Constants are the elements(entities) whose values is always fixed i.e. which does not change in the whole system. Generally constants are denoted by a, b, c.

b) Variables Variables are those elements whose value changes under different cases/conditions, generally denoted by x, y, z.

So, expression can be written as $ax^2 + bx$,

$$2x + 3,\ 9x^3 + bx^2 + 3x - 1,\ 5\sin x + 2x.$$

Algebra

Equation

When any expression is equated to zero or one expression is equated to another expression, then the result would be an equation

Eg : $ax - bx^2 = 0$; $2x - 3 = 0$; $ax^2 + bx = cx^3 - d$

Roots of equations / zeroes of equation

The values of variable, which when substituted in equation satisfy the equation are said to be zeroes/roots of equation.

Eg : $x^2 - 9 = 0$

When we put $x = 3$ or -3 then LHS = RHS, so $x = 3$ & $x = -3$ are the two roots of equation $x^2 - 9 = 0$.

Polynomial

It is an algebraic expression containing any number of terms (remember, that number of terms in any of the polynomial must be finite)

Eg: $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ is a polynomial where $a_0, a_1, ...a_n$ are constants and x is variable.

Here, algebraic word has its own significance, which says that there is no trigonometric function, inverse trigonometric function, logarithmic function, exponential function & also there should be no term containing negative exponent of x.

Or in other words we can say that in polynomial there is no transcandent function.

Eg:

i) $ax^3 + bx^2 + cx$

ii) $ax^3 + bx^2 + c\sin 3$

iii) $ax^3 + bx^2 + c\sin x$

iv) $ax^3 + bx^2 + cx + \frac{d}{x}$

Here i) and ii) are polynomial while iii) and iv) are not polynomials, they are just expression.

Note

Now the question arises in your mind that if we write the expansion of sinx, cos x, ex etc., then the expression containing these functions are converted into polynomial, but it is not so. Because all of these expansion contains infinite number of terms, while polynomial must contain finite number of terms.

But we can use sin 3, e4, log 5 in polynomial (as in Eg (ii)) because these have constant values.

Types of polynomial

1) Real polynomial Polynomial in which both constants and variables are real quantities.

Eg : Let $a_o, a_1, a_2, ... a_n$ are real constants and x is a real variable then $f(x) = a_o + a_1x + a_2x^2 + ... a_nx^n$ is real polynomial.

Eg : $2x^2 + 3x$, $ax^3 + bx^2$

2) Complex polynomial Polynomial in which either constant or variable or both are complex (not purely real)

Or in other words we can say that any polynomial containing imaginary term are said to be complex polynomial.

Eg : $(2 + i)x^2 + x$, $2(ix)^3 + (ix)^2$.

Degree of a polynomial

The highest power of variable in a polynomial is termed as degree of polynomial.

Eg : $a_0x^n + a_1x^{n-1} + ... + a_n$ $(a_0 \neq 0)$ is polynomial of degree n, which is highest power of x. Here, remember that degree $\in$ N.

i) If degree of polynomial is 1, then polynomial is said to be linear polynomial.

Eg : $2x + 3$, $ax + b$; $a \neq 0$, is general form of linear polynomial.

ii) If degree of polynomial is 2, then polynomial is said to be binomial or quadratic polynomial.

Eg : $ax^2 + bx + c$, $a \neq 0$ is general form of quadratic polynomial.

iii) If degree of polynomial is 3, then polynomial is said to be trinomial or cubic polynomial.

Eg : $ax^3 + bx^2 + cx + d$; $a \neq 0$ is general form of cubic polynomial.

iv) If degree of polynomial is 4, then polynomial is said to be biquadratic or quartic polynomial.

Eg : $ax^4 + bx^3 + cx^2 + dx + e$; $a \neq 0$ is general form of biquadratic polynomial.

Polynomial Equation : If $y = f(x)$ is real polynomial of degree n, then $f(x) = 0$ is the corresponding real polynomial equation of degree n.

Note

Before determining degree of any polynomial first we have to make all power of variables as whole numbers.

Eg :

(i) $x^3 + \sqrt{x} = x$. Here the Degree $\neq 3$ as there is one irrational term i.e. $\sqrt{x}$

So, $x^3 - x = -\sqrt{x}$

On squaring

$x^6 - 4x^4 + x^2 - x = 0$

Hence degree = 6.

ii) $x + x^{\frac{1}{2}} + x^{\frac{1}{3}} = 0$. Here degree $\neq$ 1.

To eliminate all fractional powers put $x = t^6$ so equation becomes, $t^6 + t^3 + t^2 = 0$

Hence, degree $= 6$

iii) $x^{\frac{1}{2}} + x^{\frac{4}{3}} + x^{\frac{1}{4}} = 0$

To eliminate all fractional powers, put $x = t^{12}$ so equation becomes $t^6 + t^{16} + t^4 = 0$

Hence degree is 16.

Note

If $f(x) = x^{\frac{a_1}{b_1}} + x^{\frac{a_2}{b_2}} + x^{\frac{a_3}{b_3}}$ is any polynomial (Where $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ are fractions simplified in simplest form) then to determine its degree multiply numerator and denominator of $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$, such that denominator of all the terms are same, then highest value of numerator of these terms is degree of polynomial.

Eg : $x^{\frac{1}{2}} + x^{\frac{4}{3}} + x^{\frac{1}{4}} = 0$

fraction powers are $\frac{1}{2}, \frac{4}{3}, \frac{1}{4}$. Here to make denominator of all fractions to be same multiply numerator and denominator by 12, 8, 6 respectively. So they become $\frac{12}{24}, \frac{16}{24}, \frac{6}{24}$. So degree of polynomial is 16.

So, from above discussion we can say that in general Degree of equation = Numbers of roots of equation.

Eg :

1) If equation is $(x-2)(x-3)(x-4) = 0$

   then degree = 3 = Number of roots i.e. $x = 2, 3, 4$

2) If equation is $(x-2)^3 = 0$ then degree $= 3 \neq$ Number of roots(here only root is $x = 2$)

3) If equation is $(x-1)(x-2)^3 = 0$ then degree $= 4 \neq$ Number of roots(which are 2 only)

$\therefore$ We can say that, degree of equation and number of roots of equation are same if it does not have any repeated roots. If repeated roots are there in any equation then degree must be greater than number of distinct roots.

Quadratic Polynomial A polynomial of degree two which is of the form $ax^2 + bx + c$ $(a \neq 0)$ is said to be quadratic polynomial or quadratic expression in x.

Here, $a, b, c \in R, a \neq 0$

Quadratic equation

A quadratic polynomial (i.e. polynomial of degree 2) when equated to zero gives rise to quadratic equation.

Eg : $2x^2 - 3x - 4 = 0$, $3x^2 - 4x = 0$, $4x^2 = 0$

General form of quadratic equation is $ax^2 + bx + c = 0$

Here, 'a' is leading coefficient of quadratic (coefficient of x^2)

'c' is absolute term of quadratic.

'b' is coefficient of x

Solution of quadratic equation/ Roots of quadratic equation

The values of x which satisfy the quadratic equation is called its roots (or zeroes or solutions).

As degree of quadratic equation $ax^2 + bx + c = 0$ is 2, so it can have maximum of two roots. There are two methods to solve any quadratic equation.

1) Factorization Method

If α & β are roots of $ax^2 + bx + c = 0$ then it can be written as $ax^2 + bx + c = a(x-\alpha)(x-\beta) = 0$

Eg:

$$x^2 - 5x + 6 = 0$$

$$\Rightarrow x^2 - 3x - 2x + 6 = 0$$

$$\Rightarrow x(x-3) - 2(x-3) = 0$$

$$\Rightarrow (x-2)(x-3) = 0$$

$\therefore$ Roots of above quadratic equation are 2 and 3

2) Sridharacharya Method/Method to make perfect square

Let $ax^2 + bx + c = 0$ is any quadratic equation whose roots are α & β.

Then equation can be written as

$$a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = 0$$

$$x^2 + 2x \cdot \frac{b}{2a} + \frac{c}{a} = 0 \quad (\text{as } a \neq 0)$$

To make it perfect square add and subtract $\frac{b^2}{4a^2}$ in the above equation.

$$\Rightarrow x^2 + 2x \cdot \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

$$\Rightarrow \left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$$

$$\therefore \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So, the two values of x which satisfy the given equation i.e. two roots α & β are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \ \& \ \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Sum of roots (SOR)

$$\Rightarrow \alpha + \beta$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{a}$$

$$\therefore \alpha + \beta = \frac{-b}{a} = \frac{-(\text{coefficient of } x)}{(\text{coefficient of } x^2)}$$

Product of roots (POR)

$$\alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{c}{a}$$

$$\alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Formation of quadratic equation

If α & β are roots of $ax^2+bx+c=0$, then

$$\alpha + \beta = \frac{-b}{a} \qquad \& \qquad \alpha\beta = \frac{c}{a}$$

The equation $ax^2 + bx + c = 0$ can also be written as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \Rightarrow x^2 - \left(\frac{-b}{a}\right)x + \frac{c}{a} = 0$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$x^2 - (SOR)x + (POR) = 0$$

Relation between roots & coefficients of quadratic equations

If α & β are roots of $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -b/a \ \& \ \alpha\beta = c/a$$

1) $\alpha^2 + \beta^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta = \dfrac{b^2 - 2ac}{a^2}$

2) $\alpha - \beta = \sqrt{(\alpha - \beta)^2} = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta}$

$$= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \frac{\sqrt{b^2 - 4ac}}{|a|}$$

3) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$$= -\frac{b^3}{a^3} + \frac{3c}{a} \times \frac{b}{a} = \frac{3abc - b^3}{a^3}$$

4) $|\alpha^3 - \beta^3| = (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$

$$= (\alpha - \beta)\sqrt{\alpha^2 + \beta^2 + \alpha\beta}$$

$$= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}\,\left|(\alpha + \beta)^2 - \alpha\beta\right|$$

$$= \frac{\sqrt{b^2 - 4ac}}{|a|} \times \left|\frac{b^2 - 4ac}{a^2}\right|$$

5) $|\alpha^2 - \beta^2| = (\alpha - \beta)(\alpha + \beta)$

$$= |\alpha + \beta|\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$= \frac{\sqrt{b^2 - 4ac}}{|a|} \times \left|\frac{b}{a}\right|$$

Solved Examples

1. Solve the equation $\dfrac{3}{x - 5} + \dfrac{2x}{x - 3} = 5$

Sol. Given equation can be written as

$$3(x - 3) + 2x(x - 5) = 5(x - 5)(x - 3)$$

$$\therefore 3x^2 - 33x + 84 = 0$$

$$x = 4 \text{ or } x = 7$$

2. If the equation $(k - 2)x^2 - (k - 4)\,x - 2 = 0$ has difference of roots as 3 then find value of k

Sol. $|(\alpha - \beta)| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$

Now $\alpha + \beta = \dfrac{(k - 4)}{(k - 2)}$, $\alpha\beta = \dfrac{-2}{k - 2}$

$$\therefore |(\alpha - \beta)| = \sqrt{\left(\frac{k - 4}{k - 2}\right)^2 + \frac{8}{(k - 2)}}$$

$$3 = \frac{\sqrt{k^2 + 16 - 8k + 8(k - 2)}}{|(k - 2)|}$$

$$\Rightarrow 3 = \left|\frac{k}{k - 2}\right| \Rightarrow \frac{k}{k - 2} = 3 \text{ or } -3$$

k = 3 or k = 3/2

3. Let α and β be the roots of the quadratic equation $(x-2)(x-3)+(x-3)(x+1)+(x+1)(x-2)=0$.
Find the value of

$$\frac{1}{(\alpha+1)(\beta+1)}+\frac{1}{(\alpha-2)(\beta-2)}+\frac{1}{(\alpha-3)(\beta-3)}$$

Sol. On expanding the given quadratic equation the equation can be written as

$3x^2-8x+1=0$

$\alpha+\beta=\frac{8}{3}$ and $\alpha\beta=\frac{1}{3}$

Required expression can be written as

$$=\frac{1}{\alpha\beta+(\alpha+\beta)+1}+\frac{1}{\alpha\beta-2(\alpha+\beta)+4}+\frac{1}{\alpha\beta-3(\alpha+\beta)+9}$$

$$=\frac{1}{4}-1+\frac{3}{4}=-\frac{3}{4}+\frac{3}{4}=0$$

4. Let α and β be roots of the equation $2x^2-3x-7=0$. Without computing α and β, find a quadratic equation with $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots.

Sol. From given equation $\alpha+\beta=\frac{3}{2}$ & $\alpha\beta=\frac{-7}{2}$

For required equation sum of roots

$$=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{-37}{14}$$

Product of roots $=\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}=1$

Hence, required equation is

$x^2+\frac{37}{14}x+1=0$ or $14x^2+37x+14=0$

5. If α, β are the root of a quadratic equation $x^2-3x+5=0$ then the equation whose roots are $(\alpha^2-3\alpha+7)$ and $(\beta^2-3\beta+7)$ is

A) $x^2+4x+1=0$ B) $x^2-4x+4=0$
C) $x^2-4x-1=0$ D) $x^2+2x+3=0$

Sol. Since α, β are the roots of equation $x^2-3x+5=0$
So $\alpha^2-3\alpha+5=0$ & $\beta^2-3\beta+5=0$
$\therefore$ $\alpha^2-3\alpha=-5$ & $\beta^2-3\beta=-5$
So $(\alpha^2-3\alpha+7)$ and $(\beta^2-3\beta+7)$ becomes 2 & 2. (which are roots
$\therefore$ the required equation is $x^2-4x+4=0$

6. If $m(ax^2+2bx+c)+px^2+2qx+r$ can be expressed in the form $n(x+k)^2$, then show that $(ak-b)(qk-r)=(pk-q)(bk-c)$.

Sol. Given

$m(ax^2+2bx+c)+px^2+2qx+r=n(x+k)^2$

Equating the coefficients of similar power of x, we get

$ma+p=n$ ...(1)
$mb+q=nk$ ...(2)
$mc+r=nk^2$ ...(3)

Apply k(1)-(2)

$m(ak-b)+pk-q=0\Rightarrow m=-\frac{pk-q}{ak-b}$ ..(4)

Apply k(2)-(3)

$m(bk-c)+qk-r=0\Rightarrow m=-\frac{qk-r}{bk-c}$ ..(5)

From (4) and (5)

$-\frac{pk-q}{ak-b}=-\frac{qk-r}{bk-c}$

$\Rightarrow(ak-b)(qk-r)=(pk-q)(bk-c)$

7. Let p & q be the two roots of the equation. $mx^2+x(2-m)+3=0$. Let m_1, m_2 be the two values of m satisfying $\frac{p}{q}+\frac{q}{p}=\frac{2}{3}$. Determine the numerical value of $\frac{m_1}{m_2^2}+\frac{m_2}{m_1^2}$.

Sol. From given equation

$p+q=\frac{m-2}{m};pq=\frac{3}{m}$

$\therefore\frac{p}{q}+\frac{q}{p}=\frac{2}{3}\Rightarrow\frac{p^2+q^2}{pq}=\frac{2}{3}$

$\frac{(p+q)^2-2pq}{pq}=\frac{2}{3}$

On putting p + q & pq we get

$m^2-12m+4=0$

As m_1 & m_2 satisfy it so

$m_1+m_2=12$ and $m_1m_2=4$

Now $\frac{m_1}{m_2^2}+\frac{m_2}{m_1^2}=\frac{m_1^3+m_2^3}{(m_1m_2)^2}$

$$=\frac{(m_1+m_2)^3-3m_1m_2(m_1+m_2)}{(m_1m_2)^2}$$

$$=\frac{12^3-12.12}{16}=\frac{12^2.11}{16}=99$$

8. If $\exp\{(\sin^2x+\sin^4x+\sin^6x+.....\infty)\ln 2\}$ satisfies the quadratic equation $x^2-9x+8=0$, then find the value of $\frac{\cos x}{\cos x+\sin x}$, where $x\in\left(0,\frac{\pi}{2}\right)$.

Algebra

Sol. We have, $\sin^2 x + \sin^4 x + \sin^6 x + ...\infty$

$$= \frac{\sin^2 x}{1-\sin^2 x} = \tan^2 x$$

So, $\exp\left\{\left(\sin^2 x + \sin^4 x + \sin^{2n} x + ...\infty\right)\ln 2\right\}$

$$= \exp\left(\tan^2 x.\ln 2\right) = 2^{\tan^2 x}$$

Now the roots of the equation, $x^2 - 9x + 8 = 0$ are 1 and 8.
So, according to given condition

$2^{\tan^2 x} = 1$ and $2^{\tan^2 x} = 8$

$\therefore \tan^2 x = 0, 3$

As x lies in first quadrant so only possibility is $\tan x = \sqrt{3}$

$$\therefore \frac{\cos x}{\cos x + \sin x} = \frac{1}{1+\tan x} = \frac{\sqrt{3}-1}{2}$$

9. If α and β are the roots of the equation $ax^2 + bx + c = 0$ then find the equation whose roots are $\alpha^2 + \beta^2 . \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Sol. Here $\alpha + \beta = \frac{-b}{a}$ & $\alpha\beta = \frac{c}{a}$

Let S be the sum and P be the product of the roots of required equation

Now S = $(\alpha^2+\beta^2) + \frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$

$$= \left(\frac{b^2-2ac}{a^2}\right) + \left(\frac{b^2-2ac}{c^2}\right) = (b^2-2ac)\frac{(a^2+c^2)}{a^2c^2}$$

As the product

$$P = \frac{(\alpha^2+\beta^2)^2}{\alpha^2\beta^2} = \left(\frac{b^2-2ac}{a^2}\right)^2 \times \frac{1}{\frac{c^2}{a^2}}$$

Hence equation is
$(acx)^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0$

10. If α, β be roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$ then $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) =$

Sol. Here

$$\left.\begin{array}{l}\alpha+\beta=-p\ ;\ \alpha\beta=1\\ \gamma+\delta=-q\ ;\ \gamma\delta=1\end{array}\right\} \Rightarrow \alpha\beta = \gamma\delta$$

Now $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$
$= \{\alpha\beta - \gamma(\alpha + \beta) + \gamma^2\}\{\alpha\beta + \gamma(\alpha+\beta) + \delta^2\}$
$= \{1 + \gamma p + \gamma^2\}\{1 - p\delta + \delta^2\}$
$= [(\gamma^2 + 1) + rp][(\delta^2 + 1) - p\delta)]$
As γ & δ satisfy second equation so
$= (-q\gamma + \gamma p)(-q\delta - p\delta)$
$= \gamma\delta(q^2 - p^2) = 1(q^2 - p^2)$

11. If α, β are roots of the equation $ax^2 + bx + c = 0$ then find $\frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2}$

Sol. Since α, β are the root of the $ax^2 + bx + c = 0$ then $a\alpha^2 + b\alpha + c = 0 \Rightarrow \alpha(a\alpha + b) + c = 0$

$(a\alpha + b) = -\frac{c}{\alpha}$ & $(a\beta + b) = -\frac{c}{\beta}$

$$\therefore \frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2} = \frac{1}{\left(-\frac{c}{\alpha}\right)^2} + \frac{1}{\left(-\frac{c}{\beta}\right)^2}$$

$$\Rightarrow \frac{\alpha^2}{c^2} + \frac{\beta^2}{c^2} = \frac{\alpha^2+\beta^2}{c^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{c^2}$$

$$= \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{c^2} = \frac{b^2-2ac}{a^2c^2}$$

12. If $a(p+q)^2 + 2bpq + c = 0$ and $a(p+r)^2 + 2bpr + c = 0$ then show that $qr = p^2 + \frac{c}{a}$

Sol. The two given equations can be written as

$\Rightarrow aq^2 + 2(a+b)pq + (c + ap^2) = 0$ ....(1)

$\Rightarrow ar^2 + 2(a+b)pr + (c + ap^2) = 0$ ....(2)

From equation (1) and (2), we can say that q and r satisfy the quadratic equation,

$\Rightarrow ax^2 + 2(a+b)px + (c + ap^2) = 0$

Hence, Product of roots is given by

qr = $\frac{c+ap^2}{a}$

13. If α, β are the roots of $x^2 - p(x + 1) - c = 0$ then evaluate $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$

Sol. Here the equation is $x^2 - p(x + 1) - c = 0$

$\therefore \alpha+\beta = p,\ \alpha\beta = -(p+c)$

$\Rightarrow (\alpha+1)(\beta+1) = 1-c$

Now given expression

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$$

On, putting value of $1 - c = (\alpha + 1)(\beta + 1)$, we get

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha} = \frac{\alpha+1-\beta-1}{\alpha-\beta} = 1$$

14. If one root of the quadratic equation $ax^2 + bx + c = 0$ be n^{th} root of the other then

$(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} =$

Sol. Let one root be α and other β. such that $\beta^{1/n} = \alpha \Rightarrow \beta = \alpha^n$

Now $\alpha+\beta = -\frac{b}{a} \Rightarrow \alpha + \alpha^n = -\frac{b}{a}$ ..... (1)

and $\alpha\beta = \frac{c}{a} \Rightarrow \alpha.\alpha^n = \frac{c}{a}$

$\therefore \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$ ..... (2)

From (1) & (2)

$\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{n}{n+1}} = -\frac{b}{a}$

$\Rightarrow \left(a^n c\right)^{\frac{1}{n}+1} + \left(ac^n\right)^{\frac{1}{n}+1} = -b$

15. Let α, β be the roots of $ax^2+bx+c=0$, $\alpha_1, -\beta$ be the roots of $a_1x^2+b_1x+c_1=0$ then find quadratic equation whose roots are α, α_1.

Sol. Given $\alpha+\beta = -\frac{b}{a}$; $\alpha\beta = \frac{c}{a}$

also $\alpha_1 - \beta = -\frac{b_1}{a_1}$; $-\alpha_1\beta = \frac{c_1}{a_1}$

Now $(\alpha+\beta)+(\alpha_1-\beta) = -\frac{b}{a} - \frac{b_1}{a_1} = \alpha+\alpha_1$

$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-b/a}{c/a} = -\frac{b}{c}$

$\frac{1}{\alpha_1}-\frac{1}{\beta} = -\frac{b_1}{c_1}$

From above equations

$\frac{1}{\alpha}+\frac{1}{\alpha_1} = -\frac{b}{c}-\frac{b_1}{c_1} = \frac{\alpha+\alpha_1}{\alpha\alpha_1}$

Now the equation whose roots are α, α_1 is

$x^2 - (\alpha+\alpha_1)x + \alpha\alpha_1 = 0$

$\Rightarrow \frac{x^2}{\left(\frac{b}{a}+\frac{b_1}{a_1}\right)} + x + \frac{1}{\left(\frac{b}{c}+\frac{b_1}{c_1}\right)} = 0$

Make Concepts Clear 3.1

1. The roots of the equation $(x+2)^2 = 4(x+1) - 1$ are
A) ± 1 B) $\pm i$ C) 1, 2 D) – 1, – 2

2. A quadratic polynomial p(x) has $1+\sqrt{5}$ and $1-\sqrt{5}$ as roots and it satisfies p(1) = 2. Find the quadratic polynomial.

3. If r be the ratio of the roots of the equation $ax^2+bx+c=0$, show that $\frac{(r+1)^2}{r} = \frac{b^2}{ac}$

4. If the product of the roots of the quadratic equation $mx^2 - 2x + (2m-1) = 0$ is 3 then the value of m
A) 1 B) 2 C) – 1 D) 3

5. Solve the equation $\frac{x^2-3x}{x^2-1}+2+\frac{1}{x-1}=0$

6. If equation $\frac{x^2-bx}{ax-c} = \frac{k-1}{k+1}$ has equal and opposite roots then the value of k is
A) $\frac{a+b}{a-b}$ B) $\frac{a-b}{a+b}$ C) $\frac{a}{b}+1$ D) $\frac{a}{b}-1$

7. If α, β are roots of the equation $2x^2 - 35x + 2 = 0$, then the value of $(2\alpha-35)^3.(2\beta-35)^3$ is equal to
A) 1 B) 8 C) 64 D) None

8. The coefficient of x in the equation $x^2+px+q=0$ was wrongly written as 17 in place of 13 and the roots thus found to be –2 and –15.Find the roots of original equation

9. The quadratic equation $x^2+mx+n=0$ has roots which are twice those of $x^2+px+m=0$ and m, n and $p \neq 0$. Find the value of $\frac{n}{p}$.

10. If α and β are roots of the eqation $x^2 - 5x + 6 = 0$ then the value of $\alpha^3+\beta^3$ is
A) 35 B) 40 C) 45 D) None

11. If the roots of the equation $4x^2+ax+3=0$ are in the ratio 1 : 2 ; Show that the roots of the equation $ax^2+3x+a=2$ are imaginary.

12. If α, β be the roots $x^2+px-q=0$ and γ, δ be the roots of $x^2+px+r=0$ then $\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}=$
A) 1 B) q C) r D) q + r

13. If the difference of the roots of $x^2-px+q=0$ is unity, then prove that $p^2-4q=1$ and $p^2+4q^2=(1+2q)^2$.

14. If $\tan\alpha, \tan\beta$ are the roots of $x^2-px+q=0$ and $\cot\alpha, \cot\beta$ are the roots of $x^2-rx+s=0$ then find the value of rs in terms of p and q.

15. If α, β be the roots of $ax^2+bx+c=0$ and γ, δ those of $\ell x^2+mx+n=0$, then the equation whose roots are $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$ is
A) $a^2\ell^2x^2 - ab\ell mx + (b^2-4ac)n\ell + m^2ac = 0$
B) $a^2\ell^2x^2 + ab\ell mx - (b^2-4ac)n\ell = 0$
C) $a^2\ell^2x^2 - ab\ell mx + m^2ac = 0$
D) None of these

Answers

1. B	***2. –2/5 (x^2 – 2x – 4)***		***4. C***	***5. –(1/3)***	***6. B***	***7. C***
8. –3 and –10	***9. 8***	***10. A***	***12. A***	***14. rs = (p/q^2)***		***15. A***

Solutions Are On Page No. 3.73

3.2 Nature of Roots and Symmetric Function of Roots

Discriminant of Quadratic Equation

As we know that Roots of equation $ax^2 + bx + c = 0$ are given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \ \& \ \beta = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

then, quantity inside the square root is said to be discriminant of quadratic equation. Discriminant of equation is denoted by D or Δ.

$$D(\text{or } \Delta) = b^2 - 4ac$$

Nature of roots Nature of roots of quadratic equation i.e. whether the roots are real, rational, complex etc will depend on the value of discriminant

1) If $a, b, c \in R$ & $D > 0$ then roots are real & distinct.

If the give equation is $ax^2 + bx + c = 0$ then roots of this equation are given by

$$\frac{-b+\sqrt{D}}{2a} \ \& \ \frac{-b-\sqrt{D}}{2a}$$

As D >0, both the roots are real and distinct.

2) If $a, b, c \in R \ \& \ D = 0$ then roots are real & equal.

If the give equation is $ax^2 + bx + c = 0$ then roots of this equation are given by

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

As D = 0, so both roots are same i.e. $\frac{-b}{2a}$

3) If $a, b, c \in R \ \& \ D < 0$ then roots of equation are complex and roots are given by , $x = \frac{-b \pm i\sqrt{-D}}{2a}$

4) If $a, b, c \in Q \ \& \ D = 0$ then roots are rational and equal, and both roots are same i.e. $\frac{-b}{2a}$

5) If $a, b, c \in Q \ \& \ D > 0$ and D is perfect square then roots of equation are rational and unequal.

If $ax^2 + bx + c = 0$ is given equation whose discriminant is given by

$$D = b^2 - 4ac = D_1^2 \text{ (let)}$$

then roots of the equation are given by $\Rightarrow x = \frac{-b \pm D_1}{2a}$. As $D_1 \in Q \neq 0$, so roots are rational and unequal.

6) If $a, b, c \in Q \ \& \ D > 0$ and D is not a perfect square then roots of equation are irrational and distinct.

Roots of $ax^2 + bx + c = 0$ are given by,

$$x = \frac{-b+\sqrt{D}}{2a} \ \& \ \frac{-b-\sqrt{D}}{2a}$$

As D > 0 and D $\notin$ Q the roots are irrational and distinct.

7) If $a, b, c \in R \ \& \ D < 0$ then roots are complex conjugate of each other i.e. if one root is p + iq then other root must be p – iq.

Proof

Let $ax^2+bx+c=0$ be quadratic equation whose one root is $p+iq(q \neq 0)$ and let second root is β.

So, $\beta + p + iq = \frac{-b}{a}$ ....(1)

$\beta(p+iq) = \frac{c}{a}$ ....(2)

From (1) & (2) equation.

$\beta = \frac{-b}{a} - p - iq = \frac{c}{a(p+iq)}$ ....(3)

$$\frac{-b-a(p+iq)}{a} = \frac{c}{a(p+iq)}$$

$$\Rightarrow -b(p+iq) - a(p+iq)^2 = c$$

$$\Rightarrow \left[a(p^2-q^2)+bp+c\right] + i\left[2pqa+bq\right] = 0 + i0$$

Equating real & imaginary parts of above equation we get.

$a(p^2 - q^2) + bp + c = 0$ ....(4)

$q(2pa + b) = 0$ ....(5)

From (5) as $q \neq 0$ so, $2p = \frac{-b}{a}$

Put this in equation (3)

$$\beta = \frac{-b}{a} - p - iq = 2p - p - iq$$

$\therefore \beta = p - iq$, so second root is p - iq.

Note

If $a,b,c \notin R$ and one root of quadratic equation is $p + iq$ then it is not necessary that other root should be $p - iq$.

8) If $a, b, c \in Q$ & D is not a perfect square then roots of equation are exist in irrational conjugate pair i.e. if one root is $p+\sqrt{q}$ then other root must be $p-\sqrt{q}$. (q is not perfect square)

Proof

Let $ax^2 + bx + c = 0$ be quadratic equation whose one root is $p+\sqrt{q}\,(p,q \neq 0)$ & let β be other root.

So, $\beta + p + \sqrt{q} = \frac{-b}{a}$ ...(1)

$\beta\left(p+\sqrt{q}\right) = \frac{c}{a}$ ...(2)

From (1) & (2)

$\beta = \frac{-b}{a} - p - \sqrt{q} = \frac{c}{a\left(p+\sqrt{q}\right)}$ ...(3)

$$= \frac{-b - a\left(p+\sqrt{q}\right)}{a} = \frac{c}{\left(p+\sqrt{q}\right)a}$$

$$\Rightarrow -b\left(p+\sqrt{q}\right) - a\left(p+\sqrt{q}\right)^2 = c$$

$$\Rightarrow \left[a\left(p^2+q\right) + bp + c\right] + \sqrt{q}\,(2pa+b) = 0$$

Equating rational & irrational parts of above equation, we get

$a\left(p^2+q\right) + bp + c = 0$ ...(4)

$\sqrt{q}\,(2pa+b) = 0$ ...(5)

From (5) $2p = \frac{-b}{a}$ $(\text{as } q \neq 0)$

Put this value in (3)

$$\beta = 2p - p - \sqrt{q}$$

$$\beta = p - \sqrt{q}$$

So, second root of equation is $p+\sqrt{q}$

Note

If $a,b,c \notin Q$, and one root of quadratic equation is $p+\sqrt{q}$ then it is not necessary that other root is $p-\sqrt{q}$.

9) If $\frac{b}{a}$ & $\frac{c}{a} \in I$ & $\frac{\beta}{4a^2}$ is perfect square then both roots of quadratic equation are integers.

Let the quadratic equation be $ax^2 + bx + c = 0$ have two integral roots, say α & β.

Since, $\alpha, \beta \in I \Rightarrow \alpha + \beta = \frac{-b}{a} \in I$

$$\Rightarrow \alpha\beta = \frac{c}{a} \in I$$

Also, $\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\alpha, \beta = \frac{1}{2}\left(\frac{-b}{a} \pm \sqrt{\left(\frac{b}{a}\right)^2 - 4\frac{c}{a}}\right)$ ...(1)

Let us assume $\frac{b}{a} = p$ & $\frac{c}{a} = q$, $p, q \in I$

then $\alpha, \beta = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4q}\right)$ ...(2)

For, integral roots, $p^2 - 4q$ must be perfect square.

$\sqrt{p^2 - 4q}$ is even or odd depends on p being even or odd. In both situations α, β come out to be integers, since $-p \pm \sqrt{p^2 - 4q}$ is always even.

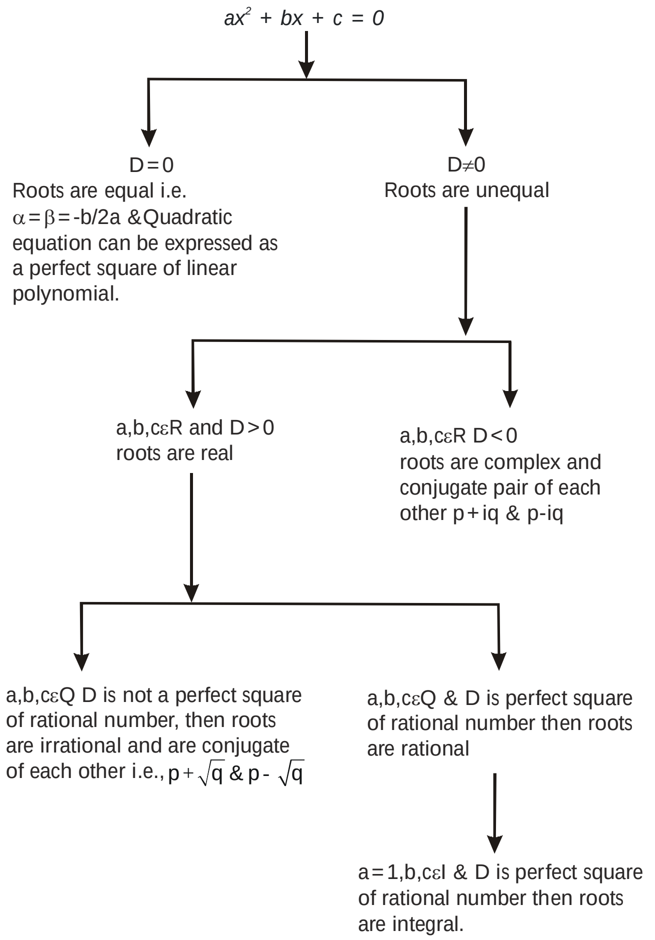
Hence, if $\frac{-b}{a} \in I$ and $\frac{c}{a} \in I$ & $\frac{b^2 - 4ac}{4a^2}$ is perfect square of an integer, then both roots are integers.

Note

In special case, if $a = 1$, $b,c \in I$ & $b^2 - 4ac$ is perfect square then both roots are integers.

Now the entire discussion on nature of roots we are showing diagramatically

Let α, β be roots of $ax^2 + bx + c = 0$ whose discriminant $D = b^2 - 4ac$



Symmetric functions of roots

If α,β are roots of $ax^2 + bx + c = 0$ & $f(\alpha,\beta)$ is any function of α & β. Then, $f(\alpha,\beta)$ is said to be symmetric function in α & β, if $f(\alpha,\beta) = f(\beta,\alpha)$. i.e. if α&β are interchanged in function $f(\alpha,\beta)$ and there is no change in function then function is said to be symmetric.

OR The function of roots $f(\alpha,\beta,\gamma,\delta...)$ which remains unaltered when any two of the roots are interchanged are called symmetric function of roots.

If f is symmetric then λf is also symmetric for $\lambda \in R$. So, the sum, difference, product, & division of two symmetric function is also symmetric.

Eg :

i) $\alpha+\beta,\ \alpha^2+\beta^2,\ \dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha},\ \left|\alpha^2-\beta^2\right|$,

$\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha+\beta^2\alpha+\gamma^2\beta+\alpha^2\gamma$

$\dfrac{\alpha^2+\beta^2}{\alpha^2\beta^2},(\alpha+\beta)\left(\alpha^2+\beta^2\right)$ are symmetric functions

(ii) $\alpha-\beta,\dfrac{\alpha}{\beta},\alpha^2\beta+\alpha\beta,\alpha\beta-\beta$ are non symmetric functions.

Rule to find number of terms in a symmetric function

If n is total number of root of the equation and 'r' is number of roots occuring in each term of symmetric function.

If K is number of roots having the same index (degree), then total number of terms in symmetric functions is $= \dfrac{n!}{(n-r)!\,k!}$

Solved Examples

16. Find nature of roots of the quadratic equation $x^2 - 2(a+b)x + 2(a^2+b^2) = 0$ (where $a \neq b$)

Sol. $A = 1,\ B = -2(a+b),\ C = 2(a^2+b^2)$
$D = B^2 - 4AC = 4(a+b)^2 - 4(1)\,2(a^2+b^2)$
$= 4a^2 + 4b^2 + 8ab - 8a^2 - 8b^2$
$= -4a^2 - 4b^2 + 8ab$
$= -4(a-b)^2 < 0$
So roots are imaginary and different

17. Find all the integral values of a for which the quadratic equation $(x-a)(x-10) + 1 = 0$ has integral roots.

Sol. $(x-a)(x-10) = -1$; where $x, a \in I$

$\therefore\ x-a=1\ \&\ x-10=-1$ or $x-a=-1\ \&\ x-10=1$

$\Rightarrow x=9\ \&\ 9-a=1$ or $x=11\ \&\ 11-a=-1$

$\therefore\ a = 8$ or 12

18. Form a quadratic equation with rational coefficients if one of its root is $\cot^2 18^0$.

Sol. $\cot^2 18^0 = \dfrac{1+\cos 36^0}{1-\cos 36^0} = \dfrac{1+\dfrac{\sqrt{5}+1}{4}}{1-\dfrac{\sqrt{5}+1}{4}} = 5+2\sqrt{5}$

Hence if $\alpha = 5+2\sqrt{5}$ then $\beta = 5-2\sqrt{5}$

$\therefore \alpha+\beta = 10; \alpha\beta = 25-20 = 5$

The quadratic equation is $x^2 - 10x + 5 = 0$

19. If $\alpha,\beta,\gamma,\delta$ be the roots of the equation $x^4+px^3+qx^2+rx+x=0$. Then $\left(1+\alpha^2\right)\left(1+\beta^2\right)\left(1+\gamma^2\right)\left(1+\delta^2\right)$ equals to

Sol. Since $\alpha,\beta,\gamma,\delta$ are the roots of the equation $x^4+px^3+qr^2+rx+x=0$

$\therefore x^4 + px^3 + qr^2 + rx + x \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$,

Substituting $x = i, -i$ in equation, we have

$(1-q+s) - i(p-r) = (i-\alpha)(i-\beta)(i-\gamma)(i-\delta)$

Take modules of both sides of above equation

$(1-q+s)^2 + (p-r)^2 = (1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)$

20. Let a & c be prime numbers and b an integer. Given that the quadratic equation $ax^2 + bx + c = 0$ has rational roots, show that one of the root is independent of the coefficients. Find the two roots.

Sol. $b^2 - 4ac = n^2 \Rightarrow (b-n)(b+n) = 4ac$

Case I : $b - n = 4a$ and $b + n = c$ not possible

Case II : $b - n = 4c$ and $b + n = a$ not possible

As $2b = 4a + c$ or $2b = 4c + a$ is odd

$\Rightarrow b$ is not an integer

Case III : $b - n = 2a$ and $b + n = 2c$

$\Rightarrow b = a + c$

Now $\alpha\beta = \frac{c}{a}$ & $\alpha + \beta = -\frac{b}{a}$

$= -\frac{a+c}{a} = -1 - \frac{c}{a}$

$\Rightarrow \alpha = -1$ and $\beta = -\frac{c}{a}$

21. If the equation $ax^2 + 2bx + c = 0$ has real roots $(a, b, c \in R)$ and $m, n \in R$ such that $m^2 > n > 0$ then prove that the equation $ax^2 + 2mbx + nc = 0$ has real roots.

Sol. Given equation is $ax^2 + 2bx + c = 0$ ....(1)

Since the roots of equation (1) are real

$\therefore 4b^2 - 4ac \geq 0 \Rightarrow 4b^2 \geq 4ac$ ....(2)

Now the discriminant of equation

$ax^2 + 2mbx + nc = 0$

$D = 4m^2b^2 - 4anc$ ....(3)

Since $m^2 > n > 0 \quad \therefore m^2 > n$ ....(4)

$\therefore$ From (2) and (4)

$m^2 . 4b^2 > 4ac.n$

$4m^2b^2 - 4acn > 0$

which is discriminant of 2nd equation

Hence roots of $ax^2 + 2mbx + nc = 0$ are real.

22. If the roots of the equation

$\left(1-q+\frac{p^2}{2}\right)x^2 + p(1+q)x + q(q-1) + \frac{p^2}{2} = 0$ are equal

then show that $p^2 = 4q$

Sol. As roots are equal $b^2 - 4ac = 0$

$\Rightarrow p^2(1+q)^2 = 4\left(1-q+\frac{p^2}{2}\right)\left\{q(q-1)+\frac{p^2}{2}\right\}$

$\Rightarrow p^2(1+q)^2 = \{4((1-q)+2p^2)\}\left\{q(q-1)+\frac{p^2}{2}\right\}$

$\Rightarrow p^2(1+q)^2 = [-4q(1-q)^2 + 2p^2q(q-1) + 2p^2(1-q) + p^4]$

$\Rightarrow p^2[(1+q)^2 - 2q^2 + 4q - 2 - p^2] = -4q(1-q)^2$

$\Rightarrow p^2[-q^2 + 6q - 1 - p^2] = -4q(1-q)^2$

$\Rightarrow -p^2(1-q)^2 + p^2(4q - p^2) = -4q(1-q)^2$

$\Rightarrow (1-q^2)(-p^2 + 4q) + p^2(4q - p^2) = 0$

$\Rightarrow (-p^2 + 4q)(p^2 + (1-q)^2) = 0$

$\Rightarrow$ As $p^2 + (1-q)^2 \neq 0$

$\therefore p^2 = 4q$

23. If both the roots of the quadratic equation $x^2 - (2n+18)x - n - 11 = 0$, $n \in I$, are rational, then find the value(s) of n.

Sol. Here the coefficients are integers, hence rational. Now the discriminant of given equation must be a perfect square of a rational number.

i.e. $4\left[(n+9)^2 + n + 11\right]$ must be perfect square

$\Rightarrow n^2 + 19n + 92$ must be perfect square of a whole number since $n \in I$

$\Rightarrow n^2 + 19n + 92 = m^2$ where $m \in W$

$\Rightarrow n = \frac{-19 \pm \sqrt{4m^2 - 7}}{2}$

$\Rightarrow 4m^2 - 7$ is perfect square of a whole number

$\Rightarrow 4m^2 - 7 = p^2$ where $p \in W$

$\Rightarrow 4m^2 - p^2 = 7$

$\Rightarrow (2m+p)(2m-p) = 7$

$\Rightarrow$ either $2m + p = \pm 1, 2m - p = \pm 7$

or $2m + p = \pm 7, 2m - p = \pm 1$

But as $m, p \in W$, so either $2m+p=1$ and $2m-p=7$ or $2m+p=7$ and $2m-p=1$,

$2p = -6$ (not acceptable as $p \in W$)

$2p = 6, 2m = 4$

$\Rightarrow m = 2$

$\Rightarrow n^2 + 19n + 92 = 4$

$\Rightarrow (n+8)(n+11) = 0 \Rightarrow n = -8$ or -11

Make Concepts Clear 3.2

1. If the roots of the equation $6x^2 - 7x + k = 0$ are rational then k is equal to
A) – 1 B) – 1, – 2 C) – 2 D) 1, 2
2. Show that the equation
$(a + b - 2c) x^2 + (b + c - 2a) x + (c + a - 2b) = 0$
have distinct rational roots.
3. If $a \neq 1, -2$ and $a \in Q$, show that the roots of the equation $(a^2 + a - 2) x^2 + (2a^2 + a - 3) x + a^2 - 1 = 0$ are rational
4. The roots of the equation
$(b + c)x^2 - (a + b + c) x + a = 0$
$(a, b, c \in Q, b + c \neq a)$ are
A) Irrational and different
B) Rational and different
C) Imaginary and different
D) Real and equal
5. If a, b, c, d and p are distinct non - zero real numbers such that $(a^2 + b^2 + c^2) p^2 - 2(ab + bc + cd) p + (b^2 + c^2 + d^2) \leq 0$, then show that a, b, c, d are in G.P.
6. Show that if the roots of the equation
$(a^2 + b^2) x^2 + 2x (ac + bd) + c^2 + d^2 = 0$ are real, they will be equal.
7. If $x^2 - ax + b = 0$ and $x^2 - px + q = 0$ have a root in common and the second equation has equal roots, show that $b+q=\frac{ap}{2}$.
8. If a, b, c are in G.P, then which of the following equations have equal roots –
A) $(b - c) x^2 + (c - a)x + (a - b) = 0$
B) $a(b - c) x^2 + b(c - a)x + c(a - b) = 0$
C) $(a^2 + b^2) x^2 - 2b(a + c)x + (b^2 + c^2) = 0$
D) None of these
9. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, find the value of
$(\beta+\gamma-\alpha)^3 + (\gamma+\alpha-\beta)^3 + (\alpha+\beta-\gamma)^3$
10. Show that if a, b, c, d are real numbersand $ad = 2(b+c)$, then atleast one root of equation $x^2+ax+b=0$, $x^2+dx+c=0$ has real roots.
11. If the roots of the equation $(x - a) (x - b) - k = 0$ be c and d, then prove that the roots of the equation $(x - c) (x - d) + k = 0$ are a and b.
12. If the root of $x^2 - ax + b = 0$ are real and differ by a quantity which is less than c (c > 0). Prove that b lies between $\left(\frac{a^2-c^2}{4}\right)$ and $\frac{a^2}{4}$
13. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, find the value of $\Sigma\frac{2\beta\gamma-\alpha^2}{\beta+\gamma-\alpha}$
14. Prove that the roots of equation $bx^2 + (b - c)x + b - c - a = 0$ are real if those of equation $ax^2 + 2bx + b = 0$ are imaginary and vice versa, where $a, b, c \in R$
15. Find all positive integers a, b such that each of the equations $x^2-ax+b=0$ and $x^2-bx+a=0$ has distinct positive integral roots.

Answers

1. D ***4. B*** ***8. C*** ***9.*** $24r - p^3$ ***13.*** $\frac{q^2}{r}$ ***15. a = 6, b = 5***

Solutions Are On Page No. 3.73

3.3 Identity, Common Roots, Higher Degree Equation

Roots Under Particular Conditions

Let the quadratic equations be $ax^2 + bx + c = 0$, $(a, b, c \in R)$ whose discriminant is $D = b^2 - 4ac > 0$ then

1) If c = 0 then equation becomes $ax^2 + bx = 0$.

So one root is zero, & other root is $\frac{-b}{a}$

2) If b = 0 then equation becomes $ax^2 + c = 0$

$\Rightarrow x^2 = \frac{-c}{a} \Rightarrow x = \pm\sqrt{\frac{-c}{a}}$

$\therefore$ Roots are equal in magnitude & opposite in sign.

Note

If a & c are of same sign then roots are complex whose imaginary part is not zero while if a and c are of opposite sign then roots are real.

3) If b = c = 0 then equation becomes $ax^2 = 0$

$\Rightarrow x^2 = 0$

$\therefore$ Both roots are zero.

4) If a = c then equation becomes $ax^2 + bx + a = 0$

So, POR = $\alpha\beta = 1$

So, roots of equation are reciprocal to each other i.e. two roots are $\alpha, \frac{1}{\alpha}$.

5) If one root of $ax^2+bx+c=0$ is infinite then $a=0$

Proof

Put $x=\frac{1}{t}$ in the equation we get,

$$\Rightarrow \frac{a}{t^2}+\frac{b}{t}+c=0$$

$$\Rightarrow ct^2+bt+a=0$$

as $x \to \infty$, so $t \to 0$

As one root of $ax^2 + bx + c = 0$ is infinite, so

one root of $ct^2 + bt + a = 0$ must be zero.

so, $a = 0$

So, we can say that if $a = 0$ then one root of

$ax^2 + bx + c = 0$ is infinite.

6) If both roots of $ax^2 + bx + c = 0$ are infinite then $a = b = 0$.

Put $x=\frac{1}{t}$ in the equation we get,

$$\Rightarrow \frac{a}{t^2}+\frac{b}{t}+c=0$$

$$\Rightarrow ct^2+bt+a=0$$

As both roots of $ax^2 + bx + c = 0$ are infinite so both roots of $ct^2 + bt + a = 0$ are zero then $b = a = 0$

$\therefore$ If $a = b = 0$ then both roots of $ax^2 + bx + c = 0$ are infinite

7) If $a.c < 0$ i.e. $(a > 0 \; \& \; c < 0)$ or $(a < 0 \; \& \; c > 0)$ then for the given quadratic equation

$$\alpha\beta \;=\; POR=\frac{c}{a}=\text{negative}$$

$\therefore$ Roots of quadratic equation are of opposite sign.

8) If a,b,c are of same sign i.e. all positive or all negative then for the given quadratic equation

$$\alpha+\beta \;=\; SOR=\frac{-b}{a}<0 \;\;\&$$

$$\alpha\beta \;=\; POR=\frac{c}{a}>0$$

$$\alpha+\beta=\text{negative} \;\&\; \alpha\beta=\text{positive}$$

$\therefore$ Both roots are negative.

9) If a and c are of same sign and b is of opposite sign i.e. $(a>0,b<0,c>0)$ or $(a<0,b>0,c<0)$ then for given quadratic equation

$$\alpha+\beta \;=\; SOR=\frac{-b}{a}>0 \;\;\&$$

$$\alpha\beta \;=\; POR=\frac{c}{a}>0$$

$\alpha + \beta$ = positive & $\alpha\beta$ = negative

$\therefore$ Both roots are positive.

10) Sign of a = sign of $b \neq$ sign of c i.e. $(a>0,b>0,c<0)$ or $(a<0,b<0,c>0)$ then for the given quadratic equation

$$\alpha+\beta \;=\; SOR=\frac{-b}{a}=\text{negative} \;\&$$

$$\alpha\beta = POR=\frac{c}{a}=\text{negative}$$

$\alpha + \beta$ = negative & $\alpha\beta$ = negative

So, roots are of opposite sign and negative root has greater magnitude.

11) If sign of b = sign of $c \neq$ sign of a i.e. $(a>0,b<0,c<0)$ or $(a<0,b>0,c>0)$ then for the given quadratic equation

$$\alpha+\beta \;=\; SOR=\frac{-b}{a}=\text{positive} \;\&$$

$$\alpha\beta \;=\; POR=\frac{c}{a}=\text{negative}$$

$\alpha + \beta$ = positive & $\alpha\beta$ = negative

So, roots are of opposite sign and positive root has greater magnitude.

12) If $a + b + c = 0$ then $b = -a - c$

Then given equation becomes $ax^2-(a+c)x + c = 0$

$a(x^2 - x) - c(x - 1) = 0$

$(x - 1)[a(x) - c) = 0$

So one root is 1 (unity) & other root is c/a.

13) If $a - b + c = 0$ then $b = a + c$

Equation becomes $ax^2 + (a + c)x + c = 0$

$a(x^2 + x) + c(x + 1) = 0$

$(x + 1)(ax + c) = 0$

So, one root is –1 & other root is –c/a.

Identity

The statement of equality between two expressions of x, that are equal for all values of x, for which expressions on the two sides are defined is called an identity.

Alternatively, f(x) = 0 is said to be identity in x if it is satisfied for all values of x in domain of f(x).

So, identity in x satisfied by all values of x for which expression is defined while equation in x is satisfied for some particular values of x.

Eg : i) $x^2 - 4x + 4 \equiv (x-2)^2$ is an identity in x as for each and every value of x, LHS = RHS

ii) $x^2 - 5x + 6 = 0$ is not an identity as it satisfy only for two values of x i.e. 2,3.

'$\equiv$' is used to denote identity whereas '=' for equation. So from above discussion we can say that every identity is an equation whereas its converse is not true.

Note

For a polynomial equation we can say that if number of roots are greater than degree than equation becomes identity & equation satisfy for all real values of x.

THEOREM

If a = b = c = 0, then equation $ax^2 + bx + c = 0$ is an identity.

$ax^2 + bx + c = 0$ is an identity in x than number of roots is greater than degree i.e. 2.

Proof

Let α, β, γ be roots of $ax^2+bx+c=0$ $(\alpha \neq \beta \neq \gamma)$

So, $a\alpha^2+b\alpha+c=0$ ...(1)

$a\beta^2+b\beta+c=0$ ...(2)

$a\gamma^2+b\gamma+c=0$ ...(3)

From (1) & (2) on applying (1)–(2)

$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0$

As $\alpha \neq \beta$

So, $a(\alpha+\beta)+b=0$ ...(4)

Similarly $a(\beta+\gamma)+b=0$ ...(5)

By subtracting (4) & (5)

$a(\alpha-\gamma)=0$ ...(6)

Now, as $\alpha \neq \gamma$ so $a=0$

Put $a=0$ in (4) we get $b=0$

Put $a=b=0$ in (1) we get $c=0$

So, $ax^2+bx+c=0$ becomes an identity if $a=b=c=0$

Concept Rockoner 1

If $(a^2-1)x^2+(a-1)x+(a+1)=0$ is an identity, then find possible values of a.

Explanation

For identity $a^2-1=a-1=a+1=0$

$\Rightarrow a^2-1=0 \Rightarrow a=\pm 1$ but neither a = 1 & a = –1 satisfies both a – 1 = 0 & a + 1 = 0. So for no value of a equation is identity.

Condition for common roots As in the quadratic equations, only two roots are possible so for two given equations either one root or both roots may be common.

1) Only one root is common

Let α is common root of two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$ then

$a_1\alpha^2+b_1\alpha+c_1=0$ ...(1)

$a_2\alpha^2+b_2\alpha+c_2=0$ ...(2)

By cramers rule or by cross multiplication

$$\frac{\alpha^2}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-\alpha}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or $$\frac{\alpha^2}{b_1c_2-b_2c_1} = \frac{\alpha}{a_2c_1-a_1c_2} = \frac{1}{a_1b_2-a_2b_1}$$

On eliminating α

$$\alpha = \frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1} = \frac{b_1c_2-b_2c_1}{a_2c_1-a_1c_2}$$

which is the required common root,

$(a_2c_1-a_1c_2)^2=(a_1b_2-a_2b_1)(b_1c_2-b_2c_1)$

is required condition for single common root.

Alternate method

Let α is common root of both equations, so

$a_1\alpha^2+b_1\alpha+c_1=0$ ...(1)

$a_2\alpha^2+b_2\alpha+c_2=0$ ...(2)

to make coefficient of α^2 in both equation equal.

Multiply (1) & (2) by a_2 & a_1 respectively.

$a_1a_2\alpha^2+b_1a_2\alpha+c_1a_2=0$ ...(3)

$a_1a_2\alpha^2+b_2a_1\alpha+c_2a_1=0$ ...(4)

On applying (3) – (4)

$\alpha(b_1a_2-b_2a_1)+c_1a_2-c_2a_1=0$ ...(5)

So,common root is $\alpha = \dfrac{c_2a_1-c_1a_2}{b_1a_2-b_2a_1}$

And condition of common root can be obtained by putting value of α in either of the equation (1) or (2)

$(a_2c_1-a_1c_2)^2=(a_1b_2-a_2b_1)(b_1c_2-b_2c_1)$ which is required condition for single common root of both equations.

Both roots are common

If both roots of two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$ are common, then both equations are identical. So, on comparing two equations,

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

Note

1. If two equations $a_1x^2+b_1x+c_1=0$ & $a_2x^2+b_2x+c_2=0$ ($a_1,a_2,b_1,b_2,c_1,c_2 \in Q$) have a common root and one equation has irrational or complex roots, then two equations has both roots common.

2. If $f(x) = 0$ & $g(x) = 0$ has a common root(s) then that common root is also root of $h(x)=af(x)+bg(x)=0$

Concept Rockoner 2

If one root of the equation $x^2 + 2x + 3k = 0$ and $2x^2 + 3x + 5k = 0$ is common then find value of k

Explanation

Since one root is common, let the root is α.

$$\frac{\alpha^2}{10k-9k}=\frac{\alpha}{6k-5k}=\frac{1}{3-4}$$

$$\alpha^2=-k \quad \text{...(1)}$$

$$\alpha=-k \quad \text{...(2)}$$

$$\therefore \alpha^2=k^2 z$$

$$\Rightarrow k^2 = -k \Rightarrow k^2 + k = 0$$

$$\Rightarrow k(k+1) = 0 \Rightarrow k = 0 \text{ and } k = -1$$

Relation between roots & coefficients of higher degree

If $\alpha_1,\alpha_2,\alpha_3.............\alpha_n$ are n roots of equation

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+......+a_n=0$$

where $a_0,a_1,....... a_n \in R$ & $a_0 \neq 0$ then

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+.......+a_n=$$

$$a_0(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n) =$$

$$f(x) = a_0[x^n-x^{n-1}\sum\alpha_1+x^{n-2}\sum\alpha_1\alpha_2.......+(-1)^n\Pi\alpha_1]$$

On comparing coefficients of LHS & RHS.

$$\sum\alpha_1=\frac{-a_1}{a_0}\ ;\ \sum\alpha_1\alpha_2=\frac{a_2}{a_0}$$

$$\sum\alpha_1\alpha_2\alpha_3=\frac{-a_3}{a_0}$$

$$\vdots$$

$$\Pi\alpha_1=(-1)^n\frac{a_n}{a_0}$$

So, with the help of this concept, we can say that

1) If α,β,γ are roots of $ax^3 + bx^2 + cx + d = 0$ then

$$\alpha+\beta+\gamma=\frac{-b}{a},\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a} \ \&$$

$$\alpha\beta\gamma=\frac{-d}{a}$$

2) $a_0,a_1,......a_n \in R$ & $p + iq$ is one root of $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...+ a_nx^0 = 0$ then $p - iq$ is also a root of the equation, i.e. complex roots exists in conjugate pairs.

3) Odd degree polynomial equation has atleast one real root. Because if complex roots are there then they always exist in pairs only.

4) If $a_0,a_1,....a_n \in Q$ and $p+\sqrt{q}$ is one of its root then $p-\sqrt{q}$ is also its root, i.e. irrational roots exist in conjugate pairs.

5) If $a_0,a_1,....a_n \in Q$ and $\sqrt{p}+\sqrt{q}$ is one of its root (where $\sqrt{p}$ & $\sqrt{q}$ are different irrationals) then $\sqrt{p}-\sqrt{q},-\sqrt{p}+\sqrt{q}$ & $-\sqrt{p}-\sqrt{q}$ are also its roots.

6) If $\sqrt{p}+\sqrt{q}+\sqrt{r}$ is one root of equation whose coefficients are rational then minimum degree of this polynomial equation is 8 (i.e. 2^3), & its roots are $\sqrt{p}+\sqrt{q}+\sqrt{r}$, $\sqrt{p}+\sqrt{q}-\sqrt{r}$, $\sqrt{p}-\sqrt{q}+\sqrt{r}$, $-\sqrt{p}+\sqrt{q}+\sqrt{r},\sqrt{p}-\sqrt{q}-\sqrt{r}$, $-\sqrt{p}+\sqrt{q}-\sqrt{r}$,

$$-\sqrt{p}-\sqrt{q}+\sqrt{r} \ \& -\sqrt{p}-\sqrt{q}-\sqrt{r}.$$

Equation in terms of roots of another equation

If α & β are roots of equation $ax^2 + bx + c = 0$ then the equation whose roots are

1) $-\alpha,-\beta$ is formed by replacing x by –x in the equation whose roots are α & β

$$a(-x)^2 + b(-x) + c = 0$$

Algebra

$\Rightarrow$ $ax^2 - bx + c = 0$ is required equation whose roots are $-\alpha$ & $-\beta$

2) $\frac{1}{\alpha},\frac{1}{\beta}$ is formed by replacing x by 1/x in

$ax^2 + bx + c = 0$

$$\Rightarrow a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c = 0$$

$cx^2 + bx + a = 0$ is the required equation whose roots are α^{-1} & β^{-1}

3) $\alpha + k, \beta + k$

Let $y = \alpha + k$ so, $\alpha = y - k$

As α is root of $ax^2 + bx + c = 0$ so it will satisfy this equation.

$a(y - k)^2 + b(y - k) + c = 0$

Replace y by x in above equation, we get

$a(x - k)^2 + b(x - k) + c = 0$ which is the required equation whose roots are $\alpha + k$ & $\beta + k$

So, to determine equation whose roots are $\alpha + k$ & $\beta + k$, replace x by (x - k) in the equation whose roots are α & β.

4) $\alpha - k, \beta - k$ Replace x by (x + k) in equation $ax^2 + bx + c = 0$ whose roots are α & β.

So, $a(x + k)^2 + b(x + k) + c = 0$ is the required equation whose roots are $\alpha - k, \beta - k$

5) $k\alpha, k\beta$

Let $y = k\alpha$ so $\alpha = \frac{y}{k}$

As α is root of $ax^2 + bx + c = 0$, so

$$a\left(\frac{y}{k}\right)^2 + b\left(\frac{y}{k}\right) + c = 0$$

Replace y by x in above equation,

$a\left(\frac{x}{k}\right)^2 + b\left(\frac{x}{k}\right) + c = 0$ is required equation whose roots are $k\alpha$ & $k\beta$.

So, to determine equation whose roots are $k\alpha$ & $k\beta$, replace x by $\frac{x}{k}$ in the equation whose roots are α&β.

6) $\frac{\alpha}{k},\frac{\beta}{k}$ is formed by replacing x by kx in the equation whose roots are α&β i.e.$ax^2 + bx + c = 0$

So, $a(kx)^2 + b(kx) + c = 0$ is required equation whose roots are $\frac{\alpha}{k},\frac{\beta}{k}$.

7) $\alpha^n, \beta^n \ (n \in N)$ is formed by replacing x by $x^{\frac{1}{n}}$ in the equation whose roots are α & β i.e.$ax^2 + bx + c = 0$

So, $ax^{\frac{2}{n}} + bx^{\frac{1}{n}} + c = 0$ is the required equation whose roots are α^n, β^n

8) $\alpha^{\frac{1}{n}}, \beta^{\frac{1}{n}} \ (n \in N)$ is formed by replacing x by x^n in the equation whose roots are α & β i.e.$ax^2 + bx + c = 0$

So, $ax^{2n} + bx^n + c = 0$ is the required equation whose roots are $\alpha^{1/n}, \beta^{1/n}$.

9) $p\alpha + q, p\beta + q$

Let $y = p\alpha + q$ i.e. $\alpha = \frac{y - q}{p}$, as α is root of $ax^2 + bx + c = 0$, so substitute this value of α in equation $ax^2 + bx + c = 0$, we will get

$$a\left(\frac{y-q}{p}\right)^2 + b\left(\frac{y-q}{p}\right) + c = 0$$

Replace y by x in above equation we get

$$a\left(\frac{x-q}{p}\right)^2 + b\left(\frac{x-q}{p}\right) + c = 0$$

which is required equation whose roots are $p\alpha + q$ & $p\beta + q$

So, to determine equation whose roots are $p\alpha + q$ & $p\beta + q$, replace x by $\frac{x - q}{p}$ in equation whose roots are α & β.

Note

1. This concept can also be used to polynomial equation of higher degree.

2. If Difference of roots of two quadratic equation are same then ratio of their discriminant is same as ratio of square of leading coefficient of two equation.

Let two equations are

$ax^2 + bx + c = 0$ whose roots are α, β

$px^2 + qx + r = 0$ whose roots are γ, δ

As, $|\alpha - \beta| = |\gamma - \delta|$

$$\Rightarrow (\alpha - \beta)^2 - 4\alpha\beta = (\gamma - \delta)^2 - 4\gamma\delta$$

$$\Rightarrow \frac{b^2 - 4ac}{a^2} = \frac{q^2 - 4pr}{p^2} \Rightarrow \frac{b^2 - 4ac}{q^2 - 4pr} = \frac{a^2}{p^2}$$

$$\Rightarrow \frac{D_1}{D_2} = \frac{a^2}{p^2}$$

Concept Rockoner 3

If α, β, γ are the roots of the equation $9x^3 - 7x + 6 = 0$ then the equation whose roots are $3\alpha + 2$, $3\beta + 2$, $3\gamma + 2$ is

Explanation

Let $y = 3\alpha + 2$, then $\alpha = (y - 2)/3$.
Since α is a root of $9x^3 - 7x + 6 = 0$
$9\alpha^3 - 7\alpha + 6 = 0$

$$\Rightarrow 9\left(\frac{y-2}{3}\right)^3 - 7\left(\frac{y-2}{3}\right) + 6 = 0$$

or $(y - 2)^3 - 7(y - 2) + 18 = 0$
or $y^3 - 6y^2 + 5y + 24 = 0$
Thus, the required equation is
$x^3 - 6x^2 + 5x + 24 = 0$

Solved Examples

24. Solve the equation

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2$$

Sol. The given equation is satisfied by $x = a$, by $x = b$, or by $x = c$. Since equation is of only of second degree in x, and it has 3 roots so it must be an identity

25. If one root of the equation $x^3 + 2x^2 + px + q = 0$ is $(\alpha - 1) - i\beta$ then prove that equation which has one root 2α is $x^3 - 2x^2 + px - q = 0.$ $(p, q \in R)$

Sol. Given equation is $x^3 + 2x^2 + px + q = 0$ ....(1)
One root of the equation is $(\alpha - 1) - i\beta$
$\therefore$ Second root will be $(\alpha - 1) + i\beta$.
Let the third root be γ then

$$\Rightarrow (\alpha - 1) - i\beta + (\alpha - 1) + i\beta + \gamma = -\frac{2}{1}$$

$\Rightarrow \gamma = -2\alpha$
We have to find an equation of which one root is $-\gamma = 2\alpha$, which can be obtained by replacing x by $-x$ in equation (1).
$-x^3 + 2x^2 - px + q = 0$
$\therefore$ $x^3 - 2x^2 + px - q = 0$ is required equation

26. If $3x^2 - 2\left(\frac{1}{d} - \frac{1}{a}\right)x + \left(\frac{1}{a^2} + \frac{2}{b^2} + \frac{2}{c^2} + \frac{1}{d^2}\right) - 2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd}\right) \le 0$ for real x and non zero real numbers a,b,c,d then show that a,b,c,d are in HP.

Sol. The given inequality reduces to

$$\left\{x - \left(\frac{1}{b} - \frac{1}{a}\right)\right\}^2 + \left\{x - \left(\frac{1}{c} - \frac{1}{b}\right)\right\}^2 + \left\{x - \left(\frac{1}{d} - \frac{1}{c}\right)\right\}^2 \le 0$$

As sum of squares can't be negative, so for existence all brackets must be zero

$$\therefore x = \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c}$$

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in AP,
So, a, b, c, d are in HP.

27. When $x = \frac{3 + 5\sqrt{-1}}{2}$, find the value of $2x^3 + 2x^2 - 7x + 70$

Sol. We know that complex roots exist in conjugate pairs, so the two roots are $\frac{3 \pm 5i}{2}$
Then quadratic equation formed by these two roots is
$2x^2 - 6x + 17 = 0$
Now $2x^3 + 2x^2 - 7x + 70$
$= x(2x^2 - 6x + 17) + 4(2x^2 - 6x + 17) + 2$
$= x \times 0 + 4 \times 0 + 2 = 2$

28. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of

i) $\Sigma\alpha^2$ ii) $\Sigma\frac{1}{\alpha}$

iii) $\Sigma\alpha^3$ iv) $\Sigma\beta^2\gamma^2$

v) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$

Sol. We have $\alpha + \beta + \gamma = -p$ ... (1)

$\beta\gamma + \gamma\alpha + \alpha\beta = q$ ... (2)

$\alpha\beta\gamma = -r$ ... (3)

i) Squaring (1), we have
$\Sigma\alpha^2 + 2\Sigma\alpha\beta = p^2$
Substituting the value from (2)

$$\Sigma\alpha^2 = p^2 - 2\Sigma\alpha\beta = p^2 - 2q$$

ii) $\Sigma\frac{1}{\alpha} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} = \frac{q}{-r} = \frac{q}{r}$

iii) We have $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$

$= (\alpha+\beta+\gamma)\left(\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha\right)$

$= -p\left\{(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)\right\}$

$= -p(p^2-3q)$

$\therefore\ \Sigma\alpha^3 = 3pq - p^3 + 3\alpha\beta\gamma$

$= 3pq - p^3 - 3r$

iv) Squaring (2), we get

$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 + 2\alpha\beta\gamma(\alpha+\beta+\gamma) = q^2$

$\therefore\ \Sigma\beta^2\gamma^2 = q^2 - 2(-r)(-p)$ from (1) and (3)

$= q^2 - 2pr$

v) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

$= 2\alpha\beta\gamma + \alpha\beta^2 + \alpha^2\beta + \beta\gamma^2 + \beta^2\gamma + \gamma\alpha^2 + \gamma^2\alpha$

But,

$\Sigma\alpha\beta^2 = (\alpha+\beta+\gamma)(\beta\gamma+\gamma\alpha+\alpha\beta) - 3\alpha\beta\gamma$

$= -pq - 3(-r) = 3r - pq$

Therefore,

$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = 2(-r) + (3r - pq)$

$= r - pq$

29. If $f(x) = ax^2 + bx + c$ and α, β are roots of $px^2 + qx + r = 0$ then show that

$f(\alpha)f(\beta) = \dfrac{(cp-ar)^2 - (bp-aq)(cq-br)}{p^2}$, hence or otherwise, show that if $ax^2+bx+c=0$ and $px^2+qx+r=0$ have common root then bp–aq, cp–ar and cq-br are in GP.

Sol. Here $\alpha+\beta = \dfrac{-q}{p}$ and $\alpha\beta = \dfrac{r}{p}$

Thus we have

$f(\alpha)f(\beta) = \left(a\alpha^2+b\alpha+c\right)\left(a\beta^2+b\beta+c\right)$

On putting the value of $\alpha+\beta$ & $\alpha\beta$ we get

$\dfrac{(cp-ar)^2-(bp-aq)(cq-br)}{p^2}$

If one root is common then either $f(\alpha)=0$ or $f(\beta)=0$,

Hence we have $f(\alpha)f(\beta)=0$

i.e. $(cp-ar)^2 = (bp-aq)(cq-br)$

Which shows that bp–aq, cp–ar and cq–br are in GP.

30. If $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have a common root and $\dfrac{a}{a_1}, \dfrac{b}{b_1}, \dfrac{c}{c_1}$ are in A.P., show that a_1, b_1, c_1 are in G.P.

Sol. If α be the common root, then

$a\alpha^2 + 2b\alpha + c = 0$

$a_1\alpha^2 + 2b_1\alpha + c_1 = 0$

Solving these together, we get

$\dfrac{\alpha^2}{bc_1-b_1c} = \dfrac{2\alpha}{a_1c-ac_1} = \dfrac{1}{ab_1-a_1b}$

On eliminating α, we get

$4(bc_1 - b_1c)(ab_1 - a_1b) = (a_1c - ac_1)^2$ .... (1)

Given $\dfrac{a}{a_1}, \dfrac{b}{b_1}, \dfrac{c}{c_1}$ are in A.P.

$\Rightarrow \left(\dfrac{b}{b_1}-\dfrac{a}{a_1}\right) = \left(\dfrac{c}{c_1}-\dfrac{b}{b_1}\right) = d$, (say)

and so $\left(\dfrac{c}{c_1}-\dfrac{a}{a_1}\right) = 2d$

$\Rightarrow (a_1b - ab_1) = da_1b_1, (b_1c - bc_1) = db_1c_1$

$(a_1c - ac_1) = 2da_1c_1$

Using above result in equation (1), we get

$4(-db_1c_1)(-da_1b_1) = 4d^2a_1^2c_1^2$

or $b_1^2 = a_1c_1$

$\Rightarrow$ a_1, b_1, c_1 are in G.P.

31. Two roots of a biquadratic $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ have their product equal to (–32). Find the value of k.

Sol. Let r_1, r_2, r_3, r_4 be 4 roots of equation

$r_1r_2r_3r_4 = -1984 \Rightarrow -32r_3r_4 = -1984$

$\Rightarrow r_3r_4 = \dfrac{1984}{32} = 62$

Hence, let the given equation

$x^4 - 18x^3 + kx^2 + 200x - 1984 = (x^2 + Ax - 32)(x^2 + Bx + 62)$

$= x^4 + (A+B)x^3 + (62 - 32 + AB)x^2 + (62A - 32B)x - 1984$

Equating coefficient, we get

$A + B = -18$

$AB + 30 = k$

$62A - 32B = 200$

On solving we get, $A = -4$ and $B = -14$

Hence $k = (-4)(-14) + 30 \Rightarrow k = 86$

32. If equations $x^3 + 3px^2 + 3qx + r = 0$ and $x^2 + 2px + q = 0$ have a common root, show that $4(p^2 - q)(q^2 - pr) = (pq - r)^2$

Sol. Let α be the common root, then

$\alpha^3 + 3p\alpha^2 + 3q\alpha + r = 0$ ...(1)

and $\alpha^2 + 2p\alpha + q = 0$ ...(2)

From (1) & (2), (1) – α (2), we get

$p\alpha^2 + 2q\alpha + r = 0$ ...(3)

From (2) and (3)

$\dfrac{\alpha^2}{2(pr-q^2)} = \dfrac{\alpha}{pq-r} = \dfrac{1}{2(q-p^2)}$

On eliminating α from above, we get

$4(q^2 - pr)(p^2 - q) = (pq - r)^2$

33. If α, β are the roots of $x^2 + px + q = 0$ and also of $x^{2n} + p^n x^n + q^n = 0$ and if $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are roots of $x^n + 1 + (x+1)^n = 0$ then n is odd integer (True/False)

Sol. Here $\alpha + \beta = -p$, $\alpha\beta = q$
in second equation
Let $t = x^n \Rightarrow$ When $x = \alpha$, $t = \alpha^n$
When $x = \beta$, $t = \beta^n$
$\therefore t^2 + p^n t + q^n = 0$
$\therefore \alpha^n + \beta^n = -p^n$, $\alpha^n\beta^n = q^n$

Again $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are roots of $x^n + 1 + (x+1)^n = 0$

$$\Rightarrow \left(\frac{\alpha}{\beta}\right)^n + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n = 0$$

$\Rightarrow \alpha^n + \beta^n + (\alpha + \beta)^n = 0$
$\Rightarrow -p^n + (-p)^n = 0$ which is true only if n is even integer.

34. If α, β, γ be the roots of the equation $x^3 + qx + r = 0$, hence prove that

$$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \frac{\alpha^3 + \beta^3 + \gamma^3}{3} \times \frac{\alpha^2 + \beta^2 + \gamma^2}{2}$$

Sol. Since α, β, γ are the roots of the equation

$$x^3 + qx + r = 0 \qquad \text{...(1)}$$

therefore
$$\left.\begin{array}{l} \alpha^3 + q\alpha + r = 0 \\ \beta^3 + q\beta + r = 0 \\ \gamma^3 + q\gamma + r = 0 \end{array}\right\} \qquad \text{...(2)}$$

Adding the corresponding sides of (2), we have

$$\sum \alpha^3 + q\sum \alpha + 3r = 0$$

Since $\sum \alpha = 0 \Rightarrow \sum \alpha^3 = -3r$ ...(3)

Also $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta = -2q$ ...(4)

Multiplying (1) throughout by x^2, we find that α, β, γ are three of the roots of the equation

$$x^5 + qx^3 + rx^2 = 0 \qquad \text{...(5)}$$

Substituting $x = \alpha, \beta, \gamma$ successively in (5) and adding, we get

$$\sum \alpha^5 + q\sum \alpha^3 + r\sum \alpha^2 = 0 \qquad \text{....(6)}$$

Substituting the values of $\sum \alpha^3$ and $\sum \alpha^2$ from (3) and (4) we get

$$\sum \alpha^5 = 5qr \qquad \text{.....(7)}$$

From (3), (4) and (7) we get

$$\frac{1}{5}\sum \alpha^5 = \left(\frac{1}{3}\sum \alpha^3\right)\left(\frac{1}{2}\sum \alpha^2\right)$$

Make Concepts Clear 3.3

1. If α, β, γ are roots of $x^3 + x^2 + 2x + 3 = 0$, then find the equation whose roots are $\alpha + \beta - \gamma$, $\beta + \gamma - \alpha$, $\gamma + \alpha - \beta$

2. The roots of the equation $4x^2 - (5a + 1)x + 5a = 0$ are α and β. If $\beta = 1 + \alpha$, Calculate the possible value of α and β.

3. Let $x^2 - 2ax + b^2 = 0$ and $x^2 - 2bx + a^2 = 0$ be two equations $(a > 0)$. Then the AM of the roots of the first equation is
A) AM of the roots of second
B) GM of the roots of second
C) Square root of the GM of the roots of second
D) None of these

4. For the roots of the equation $a - bx - x^2 = 0$ $(a > 0, b > 0)$ which statement is true
A) Positive and same sign
B) Negative and same sign
C) Greater root in magnitude negative and opposite in signs
D) Greater root in magnitude positive and opposite in signs

5. If $(p^2 - 1)x^2 + (p - 1)x + p^2 - 4p + 3 = 0$ be an identity in x, then find the value of p.

6. If p, q, r be in H.P. and p and r be different having same sign, then the root of the equation $px^2 + 2qx + r = 0$ will be
A) Real B) Equal
C) Imaginary D) None of these

7. Show that
$$\frac{(x+b)(x+c)}{(b-c)(c-a)} + \frac{(x+c)(x+a)}{(c-b)(a-b)} + \frac{(x+a)(x+b)}{(a-c)(b-c)} = 1$$
is an identity

8. If $(a^2 - 3a + 2)x^2 + (a^2 - 1)x + (a^2 - 6a + 4) = 0$ has three roots which are in AP. Then find the value of a.

9. If $a + b + c = 0; an^2 + bn + c = 0$ and $a + bn + cn^2 = 0$ where $n \neq 0, 1$, then prove that $a = b = c = 0$.

10. If the expression $x^2 - 11x + a$ and $x^2 - 14x + 2a$ must have a common factor and $a \neq 0$, then, the common factor is
A) $(x - 3)$ B) $(x - 6)$ C) $(x - 8)$ D) None

11. If the equation $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and the other root of the second equation is the reciprocal of the first, then prove that $(q - b)^2 - bq(p - a)^2 = 0$

12. If roots of $x^3 + bx^2 + cx - 1 = 0$ form an increasing GP

then find relation between b and c.

13. Form the equation whose roots are the squares of the sum and of the difference of the roots of $2x^2 + 2(m+n)x + m^2 + n^2 = 0$

Answers

1. $x^3 + x^2 + 7x - 17 = 0$ ***2. a = 3,*** α ***= (3/2),*** β ***= (5/2) ; a = – (1/5),*** α ***= – (1/2),*** β ***= (1/2)***

3. B ***4. C*** ***5. 1*** ***6. C*** ***8. No value of a*** ***10. C***

12. b + c = 0 ***13.*** $x^2 - 4mnx - (m^2 - n^2)^2 = 0$

Solutions Are On Page No. 3.75

3.4 Graph of Quadratic Expression & Range of Quadratic Expression

Let $y = f(x) = ax^2 + bx + c$ be any quadratic expression then it can be written as

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

$$\Rightarrow y = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

where D is discriminant of equation.

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{D}{4a}\right)$$

Put $x + \frac{b}{2a} = X$ & $y + \frac{D}{4a} = Y$

gives $X^2 = \frac{1}{a}Y$ which is equation of parabola. So, from here we can say that

1) Shape of quadratic expression is parabolic whose vertex is $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$.

2) If $a > 0$ then parabola open upwards while if $a < 0$ then parabola open downwards.

3) Axis of parabola is $X = 0 \Rightarrow x = -\frac{b}{2a}$ i.e axis of parabola is parallel to y axis.

4) Intersection with y-axis

$y = ax^2 + bx + c$, the point where it intersects with y axis have abscissa = 0. So, on putting $x = 0$ in the equation we get $y = c$.

Thus the required point of intersection with y axis is (0, c)

5) Intersection with x-axis

The point where the curve $y = ax^2 + bx + c$ intersects x axis, the ordinate will be zero i.e. $y = 0$.

So equation becomes $ax^2 + bx + c = 0$

From Sridharacharya method,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So from here the following conclusion can be made

a) If $D > 0$ then $f(x) = 0$ cut the x-axis at two real & distinct points (which are two roots of equation)

b) If $D = 0$ then $f(x) = 0$ touches x-axis (point of contact is single root of equation)

c) If $D < 0$ then $f(x) = 0$ neither touches, nor cuts x-axis (No real root of equation).

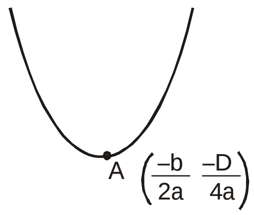
6) Greatest & least value of $f(x) = ax^2 + bx + c$ OR Range of f(x)

a) When domain of f(x) is R

$$y = ax^2 + bx + c$$

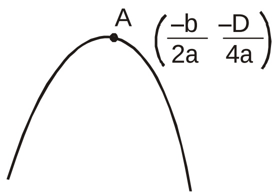
$\left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$ vertex of this parabola is $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

i) If $a > 0$ then parabola open upwards and whose least value is at vertex. So, least value of f(x) is $\frac{-D}{4a}$ at $x = \frac{-b}{2a}$. Also as there is no upper range so greatest value exist at $x = \pm\infty$ which is ∞. So, range is $\left[\frac{-D}{4a}, \infty\right)$



ii) If $a < 0$ then parabola open downwards and whose greatest value is at vertex. So, greatest value of f(x) is $\frac{-D}{4a}$ at $x = \frac{-b}{2a}$

Also as there is no lower range so least value exist at $x = \pm\infty$ which is $-\infty$



b) When domain is restricted to $x \in [x_1, x_2]$

i) If $-\frac{b}{2a} \in [x_1, x_2]$

Then min{f(x)} - min $\left\{f(x_1),\ f\left(-\frac{b}{a}\right),\ f(x_2)\right\}$

$$\min\{f(x)\} = \min\left\{f(x),\ -\frac{D}{4a},\ f(x_2)\right\}$$

And similarly

$$\max\{f(x)\} = \max\left\{f(x_1),\ f\left(-\frac{b}{2a}\right).\ f(x_2)\right\}$$

$$= \max\left\{f(x_1),\ \frac{-D}{4a}\ f(x_2)\right\}$$

Range of $f(x) \in [\min\{f(x)\}, \max\{f(x)\}]$

ii) If $-\frac{b}{2a} \notin [x_1, x_2]$

then min {f (x) } = min { $f(x_1), f(x_2)$}

max { f(x) = max { $f(x_1), f(x_2)$ }

$\therefore$ Range of f(x) $\in$ [min {f(x)}, max { f(x) }]

Concept Rockoner 4

If $\min\left(x^2 + (a-b)x + (1-a-b)\right) >$ $\max\left(-x^2 + (a+b)x - (1+a+b)\right)$ prove that $a^2 + b^2 < 4$ ·

Explanation

Let $f(x) = x^2 + (a-b)x + (1-a-b)$

$f(x)_{min} = -\frac{D}{4}$, where D_1 is the discriminant of f(x).

Let $g(x) = -x^2 + (a+b)x - (1+a+b)$

$g(x)_{max} = \frac{-D_2}{-4}$ where D_2 is the discriminant of g(x).

Thus

$$-\frac{(a-b)^2 - 4(1-a-b)}{4} > -\frac{(a+b)^2 - 4(1+a+b)}{-4}$$

$$\Rightarrow 4(1-a-b) - (a-b)^2 > (a+b)^2$$

$$\Rightarrow a^2 + b^2 < 4$$

Wavvy Curve Method (method of intervals)

It is the method to solve inequalities by the help of which we can determine that for what values of x function is positive & for what values of x function is negative.

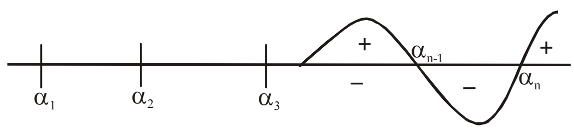
1) If f(x) = $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)....(x-\alpha_n)$

where $\alpha_1 < \alpha_2 < \alpha_3 < ..... < \alpha_n$

Procedure to solve

Mark $\alpha_1, \alpha_2, ...\alpha_n$ on real axis (number line) in increasing value, then put plus sign on right of the greteast number (i.e.α_n), then minus sign is put in interval which is next to the greatest interval on the left, then plus sign towards left & so on.

(In short, start with plus sign in extreme right and then put alternate negative & positive sign)



The function f(x) > 0 in the intervals where there is plus sign, while f(x) < 0 in the intervals where there is minus sign.

2) $f(x) = (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}.........(x-\alpha_n)^{k_n}$

where k_i's $\in N$ & α_i's $\in R$ & all α_i's are distinct such that $\alpha_1 < \alpha_2 < \alpha_3 < .... < \alpha_n$

Procedure to solve

1) The numbers $\alpha_1, \alpha_2, ...\alpha_n$ are marked on number line.

2) The plus sign is put in the interval to the right of the greatest of these numbers i.e. to the right of a_n.

3) The plus sign is put in the next interval to the left of α_n if k_n is an even number and minus sign if k_n is odd number.

4) The sign in the next interval on the left of α_{n-1} is put according to rule at α_{n-1}, f(x) changes sign if k_{n-1} is odd number while f(x) does not change if k_{n-1} is even number..
5) Now for rest of intervals we follow same procedure as in 4th step.
6) Now solution of inequality is union of intervals of all plus sign (for f(x) > 0) & union of intervals of all minus sign (for f(x) < 0).

Sign of Quadratic expression

Here we have to determine that for what value of x the quadratic expression, f(x) > 0 or f(x) < 0.

Let $f(x) = ax^2 + bx + c \qquad (a \neq 0)$

This can be determined by plotting the curve of $y = f(x)$. If curve lies above x-axis then f(x) > 0 and if curve lies below x-axis then f(x) < 0 .

$f(x) = ax^2 + bx + c = 0$

1) If a > 0 (parabola open upwards)

a) D > 0

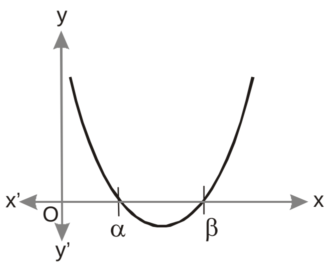
$$x = \frac{-b \pm \sqrt{D}}{2a}$$

as D > 0, so there will be two distinct values of x. where the curve cut x – axis and abscissae of these points are roots of above equation. Let α and β be roots of equation where $\alpha < \beta$.

From the adjoining figure we can say that the curve is below x axis only in interval $x \in (\alpha, \beta)$ else it is above x axis.

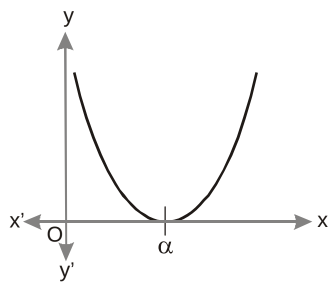
$f(x) > 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$

$f(x) < 0$ in $x \in (\alpha, \beta)$



b) If D = 0

As D = 0, so the only value of x is $-\dfrac{b}{2a}$ (which is the only root of f(x) = 0). So, curve touches x axis just at one point and abscissae of that point is root of f(x) = 0. From shown figure we can say that curve is totally above x axis except at root of f(x) = 0.

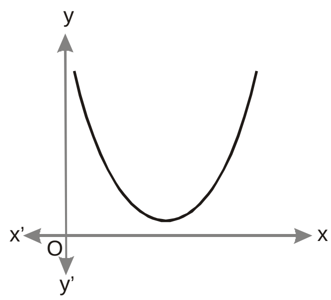


So, $f(x) > 0$ in $x \in (-\infty, \infty) - \{\alpha\}$

$f(x) < 0$ in $x \in \phi$

c) D < 0 :

The parabola neither touches nor intersect x-axis, so in this case there is no real root and total curve is above x axis.



In this case for all values of x, f(x) > 0.

Note

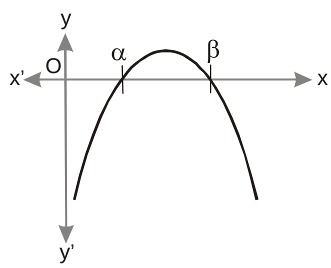
In any quadratic expression $f(x) = ax^2 + bx + c$, if $a > 0$ & $D < 0$ then for all values of x, $f(x) > 0$, so curve lies totally above x axis for all values of x.

2) If a < 0 & (parabola open downwards)

a) D > 0

D > 0 so there will be two different values of x where curve cut x axis, and abscissae of these points are roots of above equation. Let α and β the be two roots of equation where $\alpha < \beta$.

From the adjoining figure we can say that the curve is above x axis only in interval (α, β) else it is below x axis. then

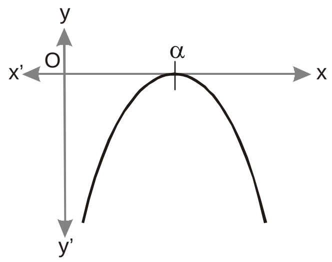


$f(x) > 0$ in $x \in (\alpha, \beta)$

$f(x) < 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$

b) D = 0:

parabola x-axis at one point, whose abscissae is root of f(x) = 0. Let α is root of f(x) = 0, then



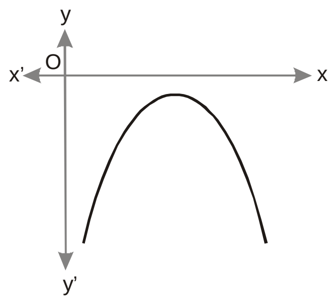
So, $f(x) > 0$ in $x \in \phi$

$f(x) < 0$ in $x \in (-\infty, \infty) - \{\alpha\}$

c) D < 0 :

Then parabola neither touches nor intersect x axis, under this case there is no real roots of f(x) = 0.

So, for all values of x, f(x) < 0



Note

In any quadratic expression $f(x)=ax^2+bx+c$, if $a < 0$ and $D < 0$ then for all values of x, $f(x) < 0$ so curve lies below x - axis for all values of x.

Greatest & least values of rational expression

Rational expression is an expression which is of the form of $\frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$.

Let $y = \frac{ax^2+bx+c}{px^2+qx+r}$ be any rational expression,

On cross multiplication

$\Rightarrow x^2(py-a) + x(qy-b) + ry - c = 0$ ....(1)

For real values of x, discriminant of (1) must be positive

$D = (qy-b)^2 - 4(py-a)(ry-c) \geq 0$

$y^2(q^2-4pr) + y(4ar+4pc-2qb) + b^2-4ac \geq 0$

Now, resulting inequality can be solved by the help of wavvy curve method, which is required range of rational expression

Concept Rockoner 5

Find the difference of maximum and minimum value of $\frac{x^2+4x+9}{x^2+9}$.

Explanation

Let $y = \frac{x^2+4x+9}{x^2+9}$

$(y-1)x^2 - 4x + 9(y-1) = 0$

For real value of x, $D \geq 0$

$16 - 36(y-1)^2 \geq 0$

$4 - 9(y-1)^2 \geq 0$

$\{2-3(y-1)\}\{2+3(y-1)\} \geq 0$

$(5-3y)(3y-1) \geq 0$

$\frac{1}{3} \leq y \leq \frac{5}{3}$

$\therefore$ Difference of maximum and minimum value is $\frac{5}{3} - \frac{1}{3} = \frac{4}{3}$

Solved Examples

35. Find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$. If two roots are equal.

Sol. Let roots be α, α and β.

$\therefore \alpha+\alpha+\beta = -\frac{20}{4}$

$\Rightarrow 2\alpha+\beta = -5$ ... (i)

$\therefore \alpha.\alpha + \alpha\beta + \alpha\beta = -\frac{23}{4}$

$\Rightarrow \alpha^2 + 2\alpha\beta = -\frac{23}{4}$ ... (ii)

& $\alpha^2\beta = -\frac{6}{4}$ ... (iii)

From equation (i) & (ii)

$\alpha^2 + 2\alpha(-5-2\alpha) = -\frac{23}{4}$

$\therefore \alpha = \frac{1}{2}, -\frac{23}{6}$

i) If $\alpha = \frac{1}{2}$, then from (i), we get $\beta = -6$

and these α & β satisfy equation (iii)

ii) If $\alpha = -\frac{23}{6}$, then from (i), we get $\beta = \frac{8}{3}$

but these α and β does not satisfy equation (iii)

$\therefore$ Roots are $\frac{1}{2}, \frac{1}{2}, -6$

Algebra

36. Obtain a polynomial of lowest degree with integral coefficient, whose one of the zeroes is $\sqrt{5}+\sqrt{2}$.

Sol. Other zeroes of polynomial are

$\sqrt{5}-\sqrt{2}, -\sqrt{5}+\sqrt{2}, -\sqrt{5}-\sqrt{2}$

Polynomial is given by

$\left(x-\sqrt{5}-\sqrt{2}\right)\left(x-\sqrt{5}+\sqrt{2}\right)(x+\sqrt{5}-\sqrt{2})(x+\sqrt{5}+\sqrt{2})$

$= x^4 - 14x^2 + 9$

37. For what value of a in the interval $\left[\frac{7\pi}{6}, \frac{7\pi}{4}\right]$ does the quadratic expression $x^2 \cot a + 2x\sqrt{\tan a} + \tan a$, assumes only positive values.

Sol. f(x) = $x^2 \cot a + 2x\sqrt{\tan a} + \tan a$,

For f(x) to be positive for all x, discriminant must be negative & coefficient of $x^2 > 0$

4tan a – 4 < 0 $\Rightarrow$ tan a < 1

i.e. $\left[\frac{0}{6}, \frac{\pi}{4}\right] \cup \left(\frac{\pi}{2}, \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2}, \frac{9\pi}{4}\right)$

And cot a >0, so a must lie in 1st or 3rd quadrant

i.e. $a \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$

Now intersection of above two inequalities, we get

$a \in \left[\frac{7\pi}{6}, \frac{5\pi}{4}\right)$

38. Show that the value of $\frac{\tan x}{\tan 3x}$ never lies between 1/3 and 3

Sol. Let $y = \frac{\tan x}{\tan 3x} = \tan x\left(\frac{1-3\tan^2 x}{3\tan x - \tan^3 x}\right)$

$= \frac{1-3\tan^2 x}{3-\tan^2 x}$ (where) $\tan x \ne 0, \pm\sqrt{3}$

i.e. $\tan x = \pm\sqrt{\frac{3y-1}{y-3}}$ as tan x is real and non zero, so

$\frac{3y-1}{y-3} > 0 \Rightarrow y < \frac{1}{3}$ or $y > 3$

Also $\tan x \ne \pm\sqrt{3} \Rightarrow \frac{3y-1}{y-3} \ne 3$ i.e. $-1 \ne 9$ which is always true for all y.

So, $y < \frac{1}{3}$ or $y > 3$ is required result.

39. If the expression $y = \frac{x^2+ax+3}{x^2+bx+4}$ takes real values for real values of x, then show that $4a^2 - 7ab + 49 > 0$

Sol. The given expression can be written as

$x^2(y-1) + x(by-a) + (4y-3) = 0$

$$x = \frac{(by-a) \pm \sqrt{(b^2-16)y^2 + (28-2ab)y + (a^2-12)}}{2(y-1)}$$

For x to be real, for every 'y' the term inside radical sign must be greater than or equal to zero, i.e.

$(b^2-16)y^2 + (28-2ab)y + (a^2-12) \ge 0$

So, leading coefficient must be positive and discriminant must be negative

$(ab-14)^2 - (a^2-12)(b^2-16) \le 0$

$3b^2 \le 7ab - 4a^2 - 1$

And $b^2 > 16$ (leading coefficient must positive)

On solving above inequalities, we get

$4a^2 - 7ab + 49 > 0$

40. If a_1, a_2, a_3 ...... a_n ($n \ge 2$) are real and $(n-1)\,a_1^2 - 2na_2 < 0$, then prove that at least two roots of the equation $x_n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$ are imaginary.

Sol. Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are the roots of the given equation.

then $\Sigma\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + ...+ \alpha_n = -a_1$

and $\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3\ .....+ \alpha_{n-1}\alpha_n = a_2$

Now $(n-1)\,a_1^2 - 2na_2$

$= (n-1)\,(\Sigma\alpha_1)^2 - 2n\Sigma\alpha_1\alpha_2 = \sum\sum_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2$

But given that $(n-1)\,a_1^2 - 2na_2 < 0$

$\therefore \sum\sum_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2 < 0$

Which is true only when at least two roots are imaginary.

41. Given that a,b,c are distinct real numbers such that the quadratic expressions ax^2+bx+c, bx^2+cx+a and cx^2+ax+b are always non negative, prove that values of the expression $\frac{a^2+b^2+c^2}{ab+bc+ca}$ does not lie between $(-\infty, 1] \cup (4, \infty)$

Sol. If $ax^2+bx+c \ge 0$ for every real x , then

$a > 0$ & $b^2 - 4ac \le 0$

Similarly for other two inequalities, we have

$b > 0$ & $c^2 - 4ab \le 0$ and

$c > 0$ & $a^2 - 4bc \le 0$

On adding all the above inequalities, we have

$a^2+b^2+c^2 - 4(ab+bc+ca) \le 0$

i.e $\frac{a^2+b^2+c^2}{ab+bc+ca} \le 4$ (as a,b,c >0)

Also we have

$(a^2+b^2+c^2) - (ab+bc+ca) =$

$\frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] > 0$

i.e. $\dfrac{a^2+b^2+c^2}{ab+bc+ca} > 1$

So, $1 < \dfrac{a^2+b^2+c^2}{ab+bc+ca} \le 4$

42. The roots of the equation $x^3 - x^2 + ax + b = 0$ are real and are in AP. Find interval in which a and b lie.

Sol. If x_1, x_2, x_3 are roots of given equation then

$x_1 + x_2 + x_3 = 1$ & $x_1 + x_3 = 2x_2$

On solving the above equations we get $x_2 = \dfrac{1}{3}$

So, on putting $x = 1/3$ in the given equation, we have

$$\frac{1}{27} - \frac{1}{9} + a\left(\frac{1}{3}\right) + b = 0$$

i.e. $a + 3b = \dfrac{2}{9}$ ...(1)

Now given equation can be written as

$$(x - 1/3)\left\{x^2 - \frac{2}{3}x + \left(a - \frac{2}{9}\right)\right\} = 0 \quad \text{...(2)}$$

As roots of above equation are real, so

$D = \dfrac{4}{9} - 4\left(a - \dfrac{2}{9}\right) \ge 0$, i.e. $a \le \dfrac{1}{3}$ ...(3)

From (1) and (3) we get $b \ge -\dfrac{1}{27}$

43. If $a, b, c \in R$ and $x^2 + bx + c = 0$ has no real roots. Prove that equation $x^2 + bx + c\ (x + a)\ (2x + b) = 0$ has real roots for every a.

Sol. As $b^2 - 4c < 0 \Rightarrow c > 0$

The second equation is

$(1 + 2c)x^2 + [b + c(b + 2a)]x + abc = 0$ whose discriminant

$$D = [b + c(b + 2a)]^2 - 4abc(1 + 2c)$$

Arranging D as a quadratic in a,

$$D = 4a^2c^2 - 4abc^2 + 2b^2(c + 1) \quad \text{...(1)}$$

Now roots of the second equation will be real for all a if the last expression is non–negative for all a. The necessary and sufficient condition for this is that the discriminant D* of the last expression is non – positive.

Indeed $D^* = 16b^2c^4 - 16b^2c^2(c + 1)^2$

$$= 16b^2c^2\left(c^2 - (c+1)^2\right) = 16b^2c^2(-2c - 1) < 0$$

(as $c > 0$)

$\Rightarrow$ The expression (1) is positive for all a.

44. If $a, b, c \in R, a \ne 0$ and $(b - 1)^2 - 4ac < 0$, show that the system of equations

$ax_1^2 + bx_1 + c = x_2$

$ax_2^2 + bx_2 + c = x_3$

.

.

.

$ax_n^2 + bx_n + c = x_1$

in n unknowns $x_1, x_2, x_3, \ldots\ldots., x_n$ has no solution.

Sol. The given equation can be written

$ax_1^2 + (b - 1)x_1 + c = x_2 - x_1$

$ax_2^2 + (b - 1)x_2 + c = x_3 - x_2$

.

.

.

$ax_n^2 + (b - 1)x_n + c = x_1 - x_n$

Since it is given that $(b - 1)^2 - 4ac < 0$, the expression $ax^2 + (b - 1)x + c$ never be zero. It can always be positive if 'a' is positive while negative if 'a' is negative.

Now if we add the above equations, we can see that sum of the expressions on the LHS cant be zero, whereas sum of RHS is zero.

So, no solution.

45. Consider the inequality, $9^x - a.3^x - a + 3 \le 0$, where a is a real parameter. Find the values of a so that.

A) The given inequality has atleast one negative solution.

B) The given inequality has atleast one positive solution.

C) Has atleast one solution in (1, 2)

D) Has atleast one real solution.

Sol. $9^x - a.3^x - a + 3 \le 0$

Let $t = 3^x \Rightarrow t^2 - at - a + 3 \le 0$

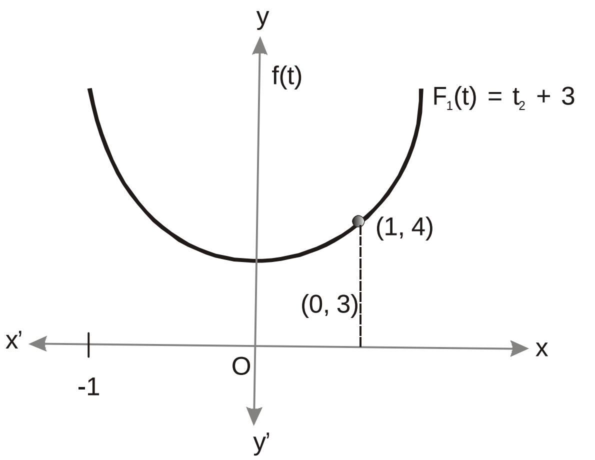
$t^2 + 3 \le a(t + 1)$ .... (i)

where $t \in R^+\ \forall x \in R$

a) For $x < 0$, $t \in (0,1)$. That means (i) should have atleast one solution in $t \in (0,1)$

From (i) it is obvious that $a \in R^+$.

Now $f_2(t) = a(t + 1)$ represents a straight line. It should meet the curve $f_1(t) = t^2 + 3$, atleast once in $t \in (0,1)$.



$f_1(0) = 3$, $f_1(1) = 4$, $f_2(0) = a$, $f_2(1) = 2a$
If $f_1(0) = f_2(0) \Rightarrow a = 3$; if $f_1(1) = f_2(1) \Rightarrow a = 2$
Hence, $a \in (2, \infty)$

b) For atleast one positive solution, $t \in (1, \infty)$. That means graphs of $f_1(t) = t^2 + 3$ and $f_2(t) = a(t+1)$ should meet at least once in $t \in (1, \infty)$.
If $a = 2$, both curve touch each other at (1, 4).
Hence $a \in (2, \infty)$

c) In this equation $t^2 - at - a + 3 = 0$ should not have both roots either in the interval $(-\infty, 3]$ or $[9, \infty)$. Discriminant of the given equation is $D = a^2 - 4(3 - a)$ which is greater than 0 in $(-\infty, -6] \cup [2, \infty)$. If both the roots are less than equal to 3, then $D \geq 0$, $9 - 3a - a + 3 \geq 0$ and $\frac{a}{2} \leq 3 \Rightarrow a \in (-\infty, -6] \cup [2, 3]$.
If both the roots are greater than equal to 9 then
$D \geq 0$, $81 - 9a - a + 3 \geq 0$ and $\frac{a}{2} \geq 9 \Rightarrow a \in \phi$
$\therefore$ Required solution will be $a \in (3, \infty)$.

d) In this case both graphs should meet atleast once in $t \in (0, \infty)$
For $a = 2$ both curves touch, hence $a \in [2, \infty)$.

Make Concepts Clear 3.4

1. The maximum value of $\ln\{(5-x)(x-3)\}$; $3 < x < 5$ is
   A) 0 B) $\ln 2$
   C) $\ln 4$ D) Not defined
2. The maximum value of $\left(\frac{1}{2}\right)^{x^2-3x+2}$ is
3. Find the range of $f(x) = 2x^2 - 3x + 2$ in [0, 2]
4. $\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3$ if x is such that
   A) $x < -4$ B) $-3 < x < 3/2$
   C) $x > 5/2$ D) All are correct
5. Solve $6x^3 - 11x^2 + 6x - 1 = 0$ if roots of the equation are in H.P.
6. Obtain the condition that $x^3 + 3px + q$ may have a factor of the form $(x-a)^2$.
7. Solve $\left|\frac{x^2-3x-1}{x^2+x+1}\right| < 3$
8. Let α, β, γ be the roots of $f(x) = x^3 + x^2 - 5x - 1 = 0$, then find the value of $\left|[\alpha] + [\beta] + [\gamma]\right|$, where [x] denotes GIF.
9. If α, β, γ are such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$ then find the value of $\alpha^4 + \beta^4 + \gamma^4$

Answers

1. A ***2.*** $\sqrt[4]{2}$ ***3.*** $\left[\frac{7}{8}, 4\right]$ ***4. D*** ***5.*** $1, \frac{1}{2}, \frac{1}{3}$ ***6.*** $4p^3 + q^2 = 0$

7. $x \in (-\infty, -2) \cup (-1, \infty)$ ***8. 3*** ***9. 18***

Solutions Are On Page No. 3.75

3.5 Location of Roots

In the previous sections we determine location (position) of roots wrt origin i.e., only whether the root is positive or negative. But now, here our interest also lies in location of roots in given interval. By the help of this concept we can able to answer following types of question.

a) Do both root occur in given interval.

b) Condition for exactly one root between given interval.

c) Condition for atleast one root between given interval.

Let $f(x) = ax^2 + bx + c$, is any quadratic expression and α, β are roots of $f(x) = 0$ $(\alpha \leq \beta)$

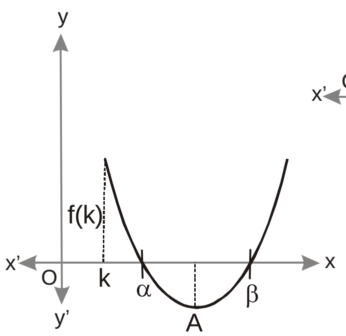
Let $k, k_1, k_2 \in R$, where $k_1 < k_2$.

So, here we can determine following cases and in each case we have to consider all cases i.e. 'a' positive or negative.

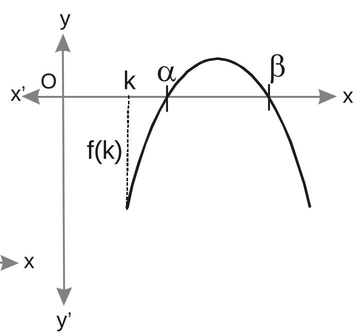
1) If both roots of f(x) = 0 are greater than k

$f(x) = ax^2 + bx + c$

a) If $a > 0$



b) If $a < 0$



$$\left.\begin{array}{c} D \geq 0 \\ k < \dfrac{-b}{2a} \\ f(k) > 0 \\ a > 0 \end{array}\right\}....(1) \qquad \left.\begin{array}{c} D \geq 0 \\ k < \dfrac{-b}{2a} \\ f(k) < 0 \\ a < 0 \end{array}\right\}....(2)$$

Here (1) and (2) satisfy condition when $\alpha, \beta > k$ when a > 0 and a < 0 respectively.

So we have to take intersection of (1) &(2).

$D \geq 0$ ... (A)

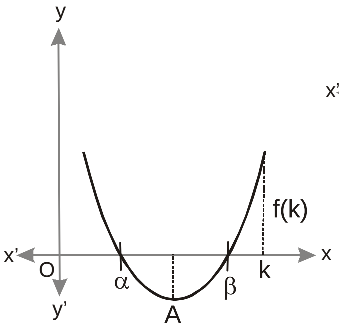
$k < \dfrac{-b}{2a}$ ... (B)

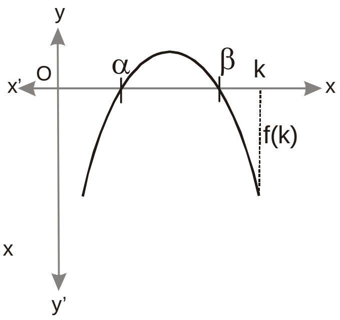
$af(k) > 0$ ... (C)

So intersection of (A), (B) and (C) will be required result when both roots are greater than k.

2) If both roots of f(x) = 0 are less than k

a) If a>0 b) If a<0





$$\left.\begin{array}{c} D \geq 0 \\ k > \dfrac{-b}{2a} \\ f(k) > 0 \\ a > 0 \end{array}\right\}....(1) \qquad \left.\begin{array}{c} D \geq 0 \\ k > \dfrac{-b}{2a} \\ f(k) < 0 \\ a < 0 \end{array}\right\}....(2)$$

Here (1) and (2) satisfy the condition $\alpha, \beta < k$ when a > 0 and a < 0 respectively.

So, intersection of (1) & (2) is

$D \geq 0$ ... (A)

$k > \dfrac{-b}{2a}$ ... (B)

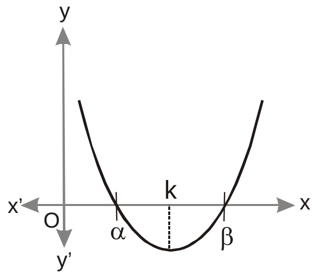
$af(k) > 0$ ... (C)

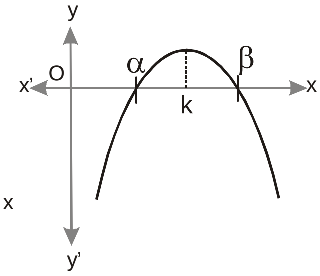
So intersection of (A), (B) and (C) will be required result when both roots are lesser than k.

3) If k lies between the roots of f(x) = 0

$f(x) = ax^2 + bx + c$

a) If a > 0 b) a < 0





$$\left.\begin{array}{c} D > 0 \\ f(k) < 0 \\ a > 0 \end{array}\right\}.....(1) \qquad \left.\begin{array}{c} D > 0 \\ f(k) > 0 \\ a < 0 \end{array}\right\}.....(2)$$

Note

Here $k > \dfrac{-b}{2a}$ or $k < \dfrac{-b}{2a}$ is not used as we don't know whether k is nearer to α or β.

Here (1) and (2) satisfy condition for $\alpha < k < \beta$ when a > 0 and a < 0 respectively. So intersection of (1) and (2) is

$D > 0$ ...(A)

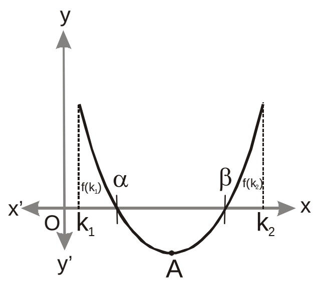
$af(k) < 0$ ...(B)

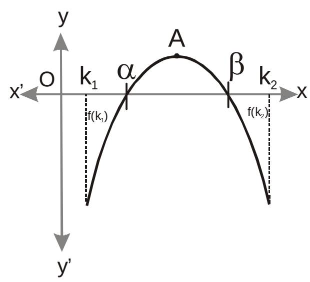
Intersection of A and B gives the required result.

4) Both roots of f(x)=0 lies between k_1 & k_2

$f(x) = ax^2 + bx + c$

a) a > 0 b) a < 0





$$\left.\begin{array}{c} D \geq 0 \\ k_1 < \dfrac{-b}{2a} < k_2 \\ f(k_1) > 0 \\ f(k_2) > 0 \\ a > 0 \end{array}\right\}.........(1) \qquad \left.\begin{array}{c} D \geq 0 \\ k_1 < \dfrac{-b}{2a} < k_2 \\ f(k_1) < 0 \\ f(k_2) < 0 \\ a < 0 \end{array}\right\}.........(2)$$

Here (1) and (2) satisfy the condition when $k_1 \le \alpha, \beta \le k_2$ when a > 0 and a < 0 respectively. So, intersection of (1) and (2) is

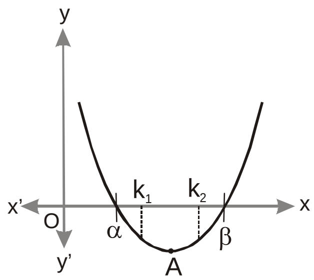
$$\begin{array}{ccc} D \ge 0 & & D \ge 0 \\ k_1 < \dfrac{-b}{2a} < k_2 & \text{or} & k_1 < \dfrac{-b}{2a} < k_2 \\ af(k_1) > 0 & & f(k_1)f(k_2) > 0 \\ af(k_2) > 0 & & \text{If a is known} \end{array}$$

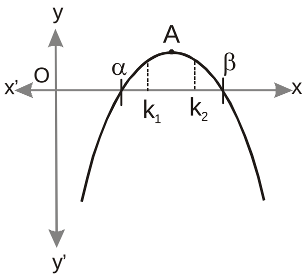
The intersection of inequalities in 1st or 2nd case would be the required result.

5) If k_1 & k_2 lies between the roots of f(x) =0

$f(x) = ax^2 + bx + c$

a) a>0 b) a<0





$$\left.\begin{array}{l} D > 0 \\ f(k_1) < 0 \\ f(k_2) < 0 \\ a > 0 \end{array}\right\} \ldots (1) \qquad \left.\begin{array}{l} D > 0 \\ f(k_1) > 0 \\ f(k_2) > 0 \\ a < 0 \end{array}\right\} \ldots (2)$$

Note

Here $k_1 < \dfrac{-b}{2a} < k_2$ is not used as it may be possible that both k_1 and k_2 lie towards α or towards β.

Here (1) & (2) satisfy condition for $\alpha < k_1 < k_2 < \beta$ when a > 0 and a < 0 respectively. So, intersection of (1) and (2) is

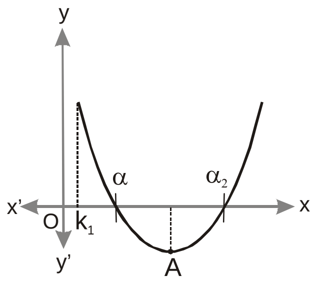
$$\begin{array}{ccc} D > 0 & & D > 0 \\ af(k_1) < 0 & \text{or} & f(k_1)f(k_2) > 0 \\ af(k_2) < 0 & & \text{If a is known} \end{array}$$

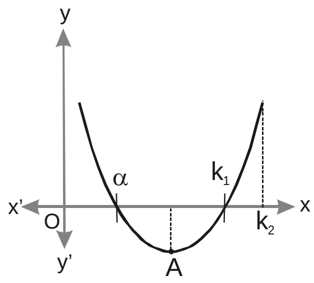
The intersection of inequalities in 1st or 2nd case would be the required result.

6) One root lies between k_1&k_2

Under this we have to consider four cases

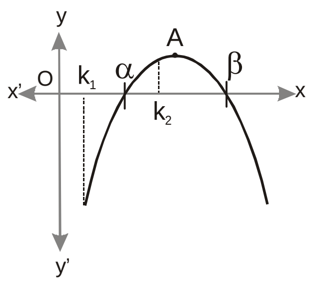
a) a>0 & $k_1<\alpha<k_2$ b) a>0 & $k_1<\beta<k_2$

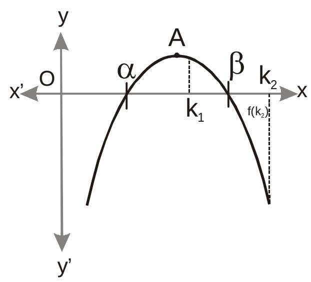




$$\left.\begin{array}{l} D > 0 \\ k_1 < \dfrac{-b}{2a} \\ f(k_1) > 0 \\ f(k_2) < 0 \end{array}\right\} \ldots (1) \qquad \left.\begin{array}{l} D > 0 \\ k_2 > \dfrac{-b}{2a} \\ f(k_1) < 0 \\ f(k_2) > 0 \end{array}\right\} \ldots (2)$$

c) a<0 & $k_1<\alpha< k_2$ d) a<0 & $k_1<\beta<k_2$





$$\left.\begin{array}{l} D > 0 \\ k_1 < \dfrac{-b}{2a} \\ f(k_1) < 0 \\ f(k_2) > 0 \end{array}\right\} \ldots (3) \qquad \left.\begin{array}{l} D > 0 \\ k_2 > \dfrac{-b}{2a} \\ f(k_1) > 0 \\ f(k_2) < 0 \end{array}\right\} \ldots (4)$$

Here as (1), (2), (3) & (4) satisfy required condition, so result is intersection of (1), (2), (3) & (4).

$D > 0$ ... (A)

$f(k_1)f(k_2) < 0$ ... (B)

So the intersection of (A) and (B) is required result.

THEOREM

If roots of $ax^2 + bx + c = 0$ are imaginary or complex then for $x \in R$, $ax^2 + bx + c$ & a have same sign.

Proof

Graphically if $ax^2+bx+c=0$ has complex roots then curve of expression neither touches nor intersects x-axis.

If a > 0 then curve is totally above x-axis so

$ax^2 + bx + c > 0.$

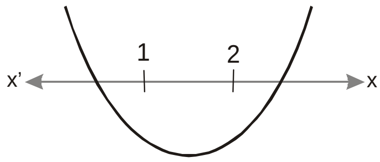
Similarly, if a < 0 then curve is totally below x-axis so $ax^2 + bx + c < 0$.

$\therefore$ a & $ax^2 + bx + c$ are of same sign.

Solved Examples

46. Find the values of a for which the inequality $x^2 + ax + a^2 + 6a < 0$ is satisfied for all $x \in [1, 2]$.

Sol. Let $f(x) = x^2 + ax + a^2 + 6a$. According to the required condition, the plot of $f(x)$ must look as shown . For the curve to look like this, the necessary and sufficient conditions are



$f(1) < 0$ and $f(2) < 0$

i.e. $1 + a + a^2 + 6a < 0$ ie. $4 + 2a + a^2 + 6a < 0$

i.e. $a^2 + 7a + 1 < 0$ i.e, $a^2 + 8a + 4 < 0$

i.e. $\frac{-7-3\sqrt{5}}{2} < a < \frac{-7+3\sqrt{5}}{2}$

i.e. $-4-2\sqrt{3} < a < -4+2\sqrt{3}$

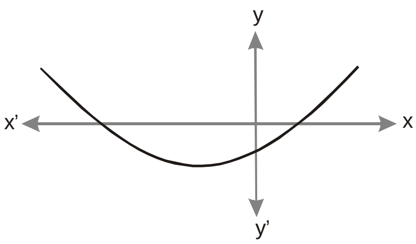
Now, taking intersection of above inequatities we get

$$a \in \left(\frac{-7-3\sqrt{5}}{2}, -4+2\sqrt{3}\right)$$

47. Prove that for any real value of a the inequality, $(a^2 + 3)x^2 + (a + 2)x - 5 < 0$ is true for at least one negative x.

Sol. $f(x) = (a^2 + 3)x^2 + (a + 2)x - 5$

when $f(0) < 0$ obviously there is atleast one negative x for which $f(x) < 0$.



$\therefore$ $f(0) = -5$ which is always true for any $a \in R$

48. Find the complete set of real values of a for which both roots of the quadratic equation.

$$(a^2-6a+5)x^2-\left(\sqrt{a^2+2a}\right)x + (6a-a^2-8) = 0$$

lie on either side of the origin.

Sol. Divide by given equation by $(a^2 - 6a + 5)$

$\therefore$ Equation is

$$x^2+\frac{\sqrt{a^2+2a}}{a^2-6a+5}\,x + \frac{6a-a^2-8}{a^2-6a+5} = 0$$

Coefficient of x must be real $\Rightarrow$ $a(a+2) \geq 0$

$\Rightarrow$ $a \in (-\infty, -2] \cup [0, \infty$ ... (1)

Since roots are on the either side of origin

$\Rightarrow$ Product $< 0 \Rightarrow \frac{6a-a^2-8}{a^2-6a+5} < 0)$

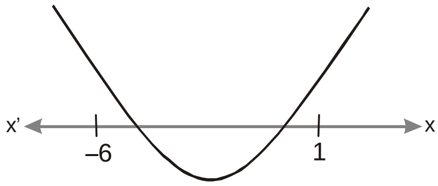
$a \in (-\infty, 1) \cup (2, 4) \cup (5, \infty)$ ...(2)

So, solution set is

$(-\infty, -2] \cup [0, 1) \cup (2, 4) \cup (5, \infty)$

49. Let $x_1, x_2 (x_1 \neq x_2)$ be the roots of the equation $x^2 + 2(m-3)x + 9 = 0$. For what values of m will the roots satisfy the inequality $-6 < x_1, x_2 < 1$.

Sol. Let $f(x) = x^2 + 2(m - 3)x + 9$.



For both roots to lie in the interval (–6, 1) the plot of f(x) must look as shown . For the curve to look like this, the necessary and sufficient conditions are

discriminant > 0

i.e., $(m - 3)^2 - 9 > 0 \Rightarrow m \in (-\infty, 0) \cup (6, \infty)$ ...(1)

$f(-6) > 0$

$\Rightarrow$ $m < \frac{27}{4}$ ... (2)

and $f(1) > 0$

$\Rightarrow$ $m > -2$ ... (3)

$-6 \frac{-2(m-3)}{2} < 1$

$\Rightarrow m < 9$ or $m > 2$ ... (4)

Intersection of above inequalities are,

$m \in \left(6, \frac{27}{4}\right)$

50. Find the range of values of a for which the equation $x^2-(a-5)x+\left(a-\frac{15}{4}\right)=0$ has at least one positive root.

Sol. $x^2-(a-5)x+\left(a-\frac{15}{4}\right)=0$

$\therefore D \geq 0 \Rightarrow (a-5)^2 - 4\left(a-\frac{15}{4}\right) \geq 0$

$\Rightarrow a^2 - 14a + 40 \geq 0$

$\Rightarrow a \in (-\infty, 4] \cup [10, \infty)$

Case I : When both roots are positive

$D \geq 0, a-5 > 0, a-\frac{15}{4} > 0$

$\Rightarrow D \ge 0, a > 5, a > \dfrac{15}{4}$

$\Rightarrow a \in [10, \infty)$

Case II : When exactly one root is positive

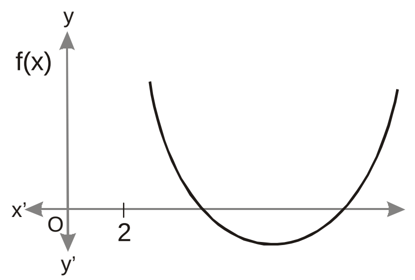
$\Rightarrow a - \dfrac{15}{4} \le 0,\ a \le \dfrac{15}{4}$

So, final result is union of case I & II

i.e., $a \in \left(-\infty, \dfrac{15}{4}\right] \cup [10, \infty)$

51. Let $x^2 - (m - 3)x + m = 0$ ($m \in R$) be a quadratic equation. Then find values of m for which.
a) Both roots are greater than 2
b) Roots are equal in magnitude and opposite in sign.
c) Both roots lie in the interval (1, 2)

Sol. a)



Condition – I : $D \ge 0$
$\Rightarrow (m - 3)^2 - 4m \ge 0 \Rightarrow m^2 - 10m + 9 \ge 0$
$\Rightarrow (m - 1)(m - 9) \ge 0$
$\Rightarrow m \in (-\infty, 1] \cup [9, \infty)$ ... (i)

Condition – II : f(2) > 0
$\Rightarrow 4 - (m - 3)2 + m > 0 \Rightarrow m < 10$ ... (ii)

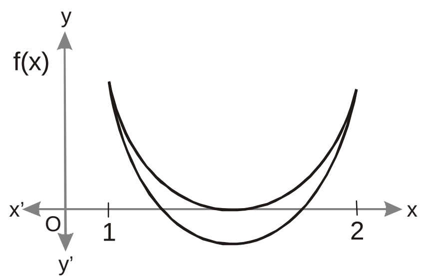
Condition – III : $-\dfrac{b}{2a} > 2$

$\Rightarrow \dfrac{m-3}{2} > 2 \Rightarrow m > 7$ ... (iii)

Intersection of (i), (ii) and (iii) gives $m \in [9, 10)$

b) Sum of roots = 0 $\Rightarrow$ m = 3
Product of roots < 0 $\Rightarrow$ m < 0
$\therefore m \in \phi$

c)



Condition – I : $D \ge 0 \Rightarrow m \in (-\infty, 1] \cup [9, \infty)$

Condition – II : $f(1) > 0 \Rightarrow 1 - (m - 3) + m > 0$
$\Rightarrow 4 > 0$ which is true $\forall m \in R$

Condition – III : $f(2) > 0 \Rightarrow m < 10$

Condition – IV : $1 < -\dfrac{b}{a} < 2 \Rightarrow 5 < m < 7$

intersection gives ϕ

52. For what values of 'a' exactly one root of the equation $2^a x^2 - 4^a x + 2^a - 1 = 0$, lies between 1 and 2.

Sol. Since exactly one root of the given equation lies between 1 and 2
we have f(1) f(2) < 0

Here $f(x) = 2^a x^2 - 4^a x + 2^a - 1$

$\therefore (2^a - 4^a + 2^a - 1)(4.2^a - 2.4^a + 2^a - 1) < 0$

$\Rightarrow (4^a - 2.2^a + 1)(4.4^a - 5.2^a + 1) < 0$

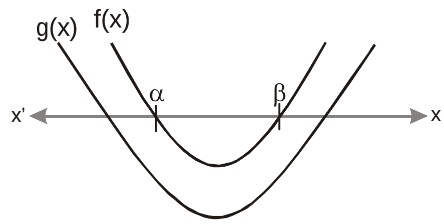
$\Rightarrow (2^a - 1)^2 (2.2^{2a} - 5.2^a + 1) < 0$

$\Rightarrow \therefore 2.2^{2a} - 5.2^a + 1 < 0 \Rightarrow \dfrac{1}{2} < 2^a < 1$

gives $a \in (-1, 0)$

53. For what real values of 'a' do the roots of the equation $x^2 - 2x - (a^2 - 1) = 0$ lie between the roots of the equation $x^2 - 2(a + 1)x + a(a - 1) = 0$

Sol. Let $f(x) = x^2 - 2x - (a^2 - 1)$
and $g(x) = x^2 - 2(a + 1)x + a(a - 1)$
According to the required condition, the plot of f(x) and g(x) must look as shown. The necessary and sufficient conditions are



$g(\alpha) < 0$ and $g(\beta) < 0$
where α, β are the roots of f(x) = 0, so that
$\alpha + \beta = 2,\ \alpha\beta = 1 - a^2$
and $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 2(a^2 + 1)$
Thus, we have
$g(\alpha) = \alpha^2 - 2(a + 1)\alpha + a(a - 1)$ ... (1)
and $g(\beta) = \beta^2 - 2(a + 1)\beta + a(a - 1)$ ... (2)
Now, $g(\alpha) < 0$ and $g(\beta) < 0$
$\Rightarrow g(\alpha)\, g(\beta) > 0$ and $g(\alpha) + g(\beta) < 0$
Multiplying equation (1) and (2) putting $\alpha\beta$ & $\alpha + \beta$ we get
$(a - 1)(3a + 1)(4a + 1) < 0$

i.e, $a \in \left(-\infty, -\dfrac{1}{3}\right) \cup \left(-\dfrac{1}{4}, 1\right)$ ... (3)

Adding equations (1) and (2), & putting values $\alpha + \beta$ & $\alpha\beta$ we get
$2a^2 - 3a - 1 < 0$

i.e. $\frac{3-\sqrt{17}}{4} < a < \frac{3+\sqrt{17}}{4}$ ... (4)

Taking intersection of (3) and (4), gives $a \in \left(-\frac{1}{4}, 1\right)$

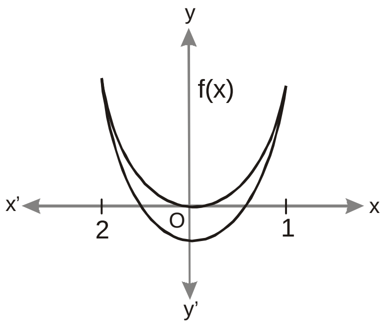
54. Find all the values of 'a' for which both the roots of the equation $(a-2)x^2 + 2ax + (a+3) = 0$ lies in the interval (–2, 1).

Sol. $(a-2)f(-2) > 0$

$\Rightarrow (a-2)(a-5) > 0$ .... (1)

$(a-2)f(1) > 0$

$\Rightarrow (a-2)(4a+1) > 0$ .... (2)



Intersection of (1) & (2) is $\Rightarrow a < -\frac{1}{4}, a > 5$

$D \ge 0 \Rightarrow 4a^2 - 4(a+3)(a-2) \ge 0$

$\Rightarrow a \le 6$ .... (4)

$-\frac{b}{2a} < 1 \Rightarrow \frac{2(a-1)}{a-2} > 0$

$\Rightarrow a \in (-\infty, 1) \cup (2, \infty)$ .... (5)

$-2 < -\frac{b}{2a} \Rightarrow \frac{-2a}{2(a-2)} > -2 \Rightarrow \frac{a-4}{a-2} > 0$

$\Rightarrow a \in (-\infty, 2) \cup (4, \infty)$ .... (6)

Intersection of 3, 4, 5 & 6

So, Complete solution is $a \in \left(-\infty, -\frac{1}{4}\right) \cup (5, 6]$

55. If the roots of $\sum_{k=1}^{n} (x+k-1)(x+k) = 10$ differ by 1, then find value of n.

Sol. We have $\sum_{k=1}^{n} (x+k-1)(x+k) = 10$

i.e., $\sum_{k=1}^{n} x^2 + x\sum_{k=1}^{n}(2k-1) + \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k = 10$

i.e., $nx^2 + [n(n+1) - n]x + \frac{n(n+1)(2n+1)}{6}$

$- \frac{n(n+1)}{2} - 10 = 0$

i.e, $x^2 + nx + \frac{n^2-1}{3} - \frac{10}{n} = 0$

Let the roots of the above equation be α, $\alpha + 1$. Then , we have

$2\alpha + 1 = -n$ and $\alpha^2 + \alpha = \frac{n^2-1}{3} - \frac{10}{n}$

Eliminating α, we have

$\left(\frac{n+1}{2}\right)^2 - \frac{n+1}{2} = \frac{(n-1)(n+1)}{3} - \frac{10}{n}$

$\Rightarrow (n-1)n(n+1) = 120 \Rightarrow n = 5.$

56. If $ax^2 - bx + c = 0$ have two distinct roots lying in the interval (0, 1) a, b, c $\in$ N , then prove that $\log_5 abc \ge 2$.

Sol. $\therefore \alpha + \beta = b/a$ and $\alpha\beta = c/a$ and $0 < \alpha < 1$, $0 < \beta < 1$

$\therefore 0 < 1 - \alpha < 1$ & $0 < 1 - \beta < 1$

Then $\frac{\alpha + (1-\alpha)}{2} \ge \sqrt{\alpha(1-\alpha)}$

$\Rightarrow \frac{1}{4} \ge \alpha(1-\alpha) > 0$

Similarly $\frac{1}{4} \ge \beta(1-\beta) > 0$

$\therefore \frac{1}{16} \ge \alpha\beta(1-\alpha)(1-\beta) > 0$

As α and β are distinct, so equality sign is absent

i.e. $0 < \alpha\beta(1 - (\alpha+\beta) + \alpha\beta) < \frac{1}{16}$

$\Rightarrow 0 < \frac{c}{a}\left(1 - \frac{b}{a} + \frac{c}{a}\right) < \frac{1}{16}$

$\Rightarrow 0 < c(a - b + c) < \frac{a^2}{16}$

$\because$ c (a – b + c) = Natrual number $(\because a, b, c \in N)$

Minimum values of c(a – b + c) = 1

$\therefore \frac{a^2}{16} > 1$, $\therefore a \ge 5$ $(\because a \in N)$

and condition for real roots, $b^2 - 4ac \ge 0$

$\Rightarrow b^2 \ge 4ac \Rightarrow b^2 \ge 20c$

$\therefore b \ge 5$ $(\because b \in N)$

and minimum value of c = 1. Hence $abc \ge 25$

$\Rightarrow \log_5(abc) \ge \log_5 5^2 \Rightarrow \log_5(abc) \ge 2$

Make Concepts Clear 3.5

1. If one root of the equation $x^2 - (p + 1)x + p^2 + p - 8 = 0$ is greater than 2 and the other root is smaller than 2, then p is such that

   A) $-\frac{11}{3} < p < 3$ B) $-2 < p < 3$

   C) $2 < p < 3$ D) None of these

2. If both the roots of the equation $x^2 - 9x + a = 0$ are positive and one is greater than 3 and other is less than 3. Find all possible values of a.

3. If both roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, then $a > 11/9$. Justify the statement

4. If the equation $x^2 + 2(k + 1)x + 9x - 5 = 0$ has only negative roots, then –

   A) $k \leq 0$ B) $k \geq 0$

   C) $k \geq 6$ D) $k \leq 6$

5. The coefficients of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$, satisfy $(a+b+c)(4a-2b+c) < 0$. Prove that this equation has 2 distinct real solutions

6. If α, β are the roots of the equation $4x^2 - 16x + \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$ then find the number of integral solution of λ.

7. For what values of p, the number 6 lies between the roots of the equation $x^2 + 2(p-3)x + 9 = 0$

8. If α, β are the roots of the equation, $x^2 - 2x - a^2 + 1 = 0$ and γ, δ are the roots of the equation. $x^2 - 2(a + 1)x + a(a - 1) = 0$ such that $\alpha, \beta \in (\gamma, \delta)$ then find the values of a.

9. If $ax^2 + bx + c = 0$ $(a > 0)$ has $\alpha \& \beta$ as its roots. If $\alpha < -2$ & $\beta > 2$ then prove that $4a + 2|b| + c < 0$

Answers

1. B ***2. $0 < a < 18$*** ***3. True*** ***4. C*** ***6. 3*** ***7. $p \in \left(-\infty, -\frac{3}{4}\right)$***

8. $a \in \left(-\frac{1}{4}, 1\right)$

Solutions Are On Page No. 3.76

3.6 Equations Reducible to Quadratic and other Special Equations

1. If equation is of type $ax^4 + bx^3 + cx^2 + bx + a = 0$

To convert this equation into quadratic divide the given equation by x^2, then resulting equation is

$$ax^2 + bx + c + \frac{b}{x} + \frac{a}{x^2} = 0$$

$$\Rightarrow a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$$

Put $x + \frac{1}{x} = t \Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2$

$$\Rightarrow a(t^2 - 2) + bt + c = 0$$
$$\Rightarrow at^2 + bt - 2a + c = 0$$

Which becomes quadratic in t, from where we got two roots of above equation i.e. two values of t.

Let the roots be t_1 & t_2, then

$$x + \frac{1}{x} = t_1 \;\;\&\;\; x + \frac{1}{x} = t_2$$

From where we got 4 value of x. Which are 4 roots of given equation.

2. If equation is of type $(x-a)(x-b)(x-c)(x-d) = Ax^2$ where $ab = cd$

The given equation can be written as

$$[(x-a)(x-b)][(x-c)(x-d)] = Ax^2 \quad ...(1)$$

$$[x^2 - x(a+b) + ab][x^2 - x(c+d) + cd] = Ax^2$$

$$[x^2 - x(a+b) + ab][x^2 - x(c+d) + ab] = Ax^2 \quad ...(2)$$

(As $ab = cd$)

Divide LHS & RHS of equation (2) by x^2

$$\frac{[x^2 - x(a+b) + ab]}{x}\frac{[x^2 - x(c+d) + ab]}{x} = A$$

$$\left(x + \frac{ab}{x} - (a+b)\right)\left(x + \frac{ab}{x} - (c+d)\right) = A \quad ...(3)$$

Put $x+\frac{ab}{x}=t$

So equation (3) becomes

$(t-a-b)(t-c-d)=A$ ...(4)

Which becomes quadratic in t, from where we got two values of t, which are roots of equation (4).

Let roots of (4) be t_1 & t_2 then

$$x+\frac{ab}{x}=t_1 \;\&\; x+\frac{ab}{x}=t_2$$

On solving above two equations we got 4 values of x which are 4 roots of given equation.

3. If equation is of type $(x-a)(x-b)(x-c)(x-d)=A$ where $a+b=c+d$:

The above equation can be written as,

$$[(x-a)(x-b)][(x-c)(x-d)]=A \text{ ....(1)}$$

$$[x^2-x(a+b)+ab][x^2-x(c+d)+cd]=A$$

$$[x^2-x(a+b)+ab][x^2-x(a+b)+cd]=A \text{ ...(2)}$$

(As $a+b=c+d$)

Now put $x^2-x(a+b)=t$ in equation (2), so it becomes

$\therefore\ (t+ab)(t+cd)=A$ ...(3)

which becomes quadratic in t, from where we get two values of t, which are roots of equation. (3)

Let roots of equation (3) be t_1 & t_2, then

$x^2-x(a+b)=t_1$ & $x^2-x(a+b)=t_2$

On solving above two equations we get 4 values of x which are 4 roots of given equation.

Concept Rockoner 6

Solve : $(x+9)(x-3)(x-7)(x+5)=385$

Explanation

$(x+9)(x-7)(x-3)(x+5)=385$

$(x^2+2x-63)(x^2+2x-15)=385$

Put $x^2+2x=y$ .... (i)

$\therefore\ (y-63)(y-15)=385$

$\therefore\ y^2-78y+560=0$

$\therefore\ y=70$ or 8

When $y=70$, (i) gives $x^2+2x-70=0$

$\therefore x=-1\pm\sqrt{71}$

when $y=8$, (i) gives $x^2+2x-8=0$

$\Rightarrow\ (x+4)(x-2)=0$

$\therefore$ Solutions are $-4, 2$ & $-1\pm\sqrt{71}$

4. If equation is of type $ax^{2n}+bx^n+c=0$ where $a\neq 0$, $n\geq 2$ & $n\in N$

To solve such equations put $x^n=t$, then given equation reduces to $at^2+bt+c=0$ which is quadratic in t.

For this equation we will get two value of t, let t_1 & t_2 then $x^n=t_1$ & $x^n=t_2$. On solving these two equations for x we will get the required roots of the given equation.

5. If equation is of type $(x-a)^{2n}+(x-b)^{2n}=A$

This type of equation can be solved by change of variables i.e by making a substitution. Here variable x is changed to variable y, such that

$$y=\frac{x-a+x-b}{2}\Rightarrow x=y+\frac{a+b}{2}$$

So on this substitution given equation becomes

$$\left(y-\left(\frac{a}{2}-\frac{b}{2}\right)\right)^{2n}+\left(y+\left(\frac{a}{2}-\frac{b}{2}\right)\right)^{2n}=A$$

LHS of this equation can be simplified bythe help of binomial expansions .

6. If equation is of type

$$\frac{Ax}{ax^2+b_1x+c}\pm\frac{Bx}{ax^2+b_2x+c}=C \quad \textbf{or}$$

$$\frac{ax^2+b_1x+c}{ax^2+b_2x+c}\pm\frac{ax^2+b_3x+c}{ax^2+b_4x+c}=A \;\textbf{;(where } a\neq 0\textbf{)}$$

To solve such equation, divide numerator & denominator by x then it becomes

$$\left.\begin{aligned}&\frac{A}{ax+b_1+c/x}\pm\frac{B}{ax+b_2+c/x}=C \quad \text{or}\\&\frac{ax+b_1+c/x}{ax+b_2+c/x}\pm\frac{ax+b_3+c/x}{ax+b_4+c/x}=A\end{aligned}\right\} \text{ ....(1)}$$

Now, put $ax+c/x=t$, then equation (1) becomes

$$\left.\begin{aligned}&\frac{A}{t+b_1}\pm\frac{B}{t+b_2}=C\\&\frac{t+b_1}{t+b_2}\pm\frac{t+b_3}{t+b_4}=D\end{aligned}\right\} \text{ ...(2)}$$

which becomes quadratic in t, from where we get two values of t which are roots of equation (2).

Let roots of equation (2) be t_1 & t_2, then

$$ax+\frac{c}{x}=t_1 \;\&\; ax+\frac{c}{x}=t_2$$

On solving above two equations we get 4 values of x, which are roots of given equation.

Equations involving absolute functions

1) If equation is of type |f(x)| = g(x)

This example is similar to collection of solutions of

$$\begin{cases} f(x) = g(x); \text{if } f(x) > 0 \\ -f(x) = g(x); \text{if } f(x) < 0 \end{cases} \quad ...(1)$$

$$g(x) \geq 0 \quad ...(2)$$

So,if $|f(x)| = a \Rightarrow f(x) = \pm a$

2) If equation is of type f(|x|) = g(x)

It is similar to collection of solutions of

$$\begin{cases} f(x) = g(x); x \geq 0 \\ f(-x) = g(x); x \leq 0 \end{cases}$$

3) If equation is of type |f(x)| = f(x)

As LHS of above equation is always positive, so above equation will have no solution if RHS is negative. So, it's solution is $f(x) \geq 0$

4) If equation is of type |f(x) + g(x)| = |f(x)| + |g(x)|

On squaring both sides of given equation.

$f^2(x) + g^2(x) + 2f(x).g(x) = f^2(x) + g^2(x) + 2|f(x)g(x)|$

$\Rightarrow f(x)\,g(x) = |f(x)\,g(x)|$

Which gives that $f(x)\,g(x) \geq 0$

So, solution of given equation can be obtained by solving inequality $f(x)\,g(x) \geq 0$, which can be solved by wavy curve method.

5) If equation is of form $|f_1(x)| + |f_2(x)| + ... + |f_n(x)| = g(x)$ (where $f_i(x)$ are function of x & g(x) is constant or function of x)

To solve such equation first of all determine critical point of all the modulus functions i.e. value of x where $f_i(x)$ is zero, then plot these points on number line in increasing order.

On plotting these points on number line, the number line is divided in n + 1 sub intervals. Then for each interval determine sign of $f_i(x)$ and so for each interval make the separate equation which should be free from modulus function, and then solve equations for each interval separately.

Collection of solutions in all these intervals is solution of given equation.

Equation containing exponential function

1) If equation is of type $a^{f(x)} = 1$ ($a > 0, a \neq 1$)

Then it can be written as $a^{f(x)} = a^0$

$\therefore f(x) = 0$ is the required solution of this equation.

2) If equation is of type $f(a^x)=A$

To solve such equation put $a^x = t$

So, given equation becomes $f(t) = A$

Now solve resulting equation for t and put its each root to be equal to a^x to find values of x which satisfy original equation.

3) If equation is of type $\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} = 0$ (where $\alpha, \beta, \gamma \in R_0$) where $b^2 = ac$

Divide the given equation by $b^{f(x)}$, then it becomes

$$\Rightarrow \alpha\left(\frac{a}{b}\right)^{f(x)} + \beta + \gamma\left(\frac{c}{b}\right)^{f(x)} = 0$$

As $b^2 = ac$ so $\frac{c}{b} = \left(\frac{a}{b}\right)^{-1}$

$$\Rightarrow \alpha\left(\frac{a}{b}\right)^{f(x)} + \beta + \gamma.\left(\frac{a}{b}\right)^{-f(x)} = 0$$

Put $\left(\frac{a}{b}\right)^{f(x)} = t$

So, above equation becomes

$$\Rightarrow \alpha t + \beta + \frac{\gamma}{t} = 0$$

$$\Rightarrow \alpha t^2 + \beta t + \gamma = 0$$

Now, solve this equation for t, let roots, be t_1 & t_2,

Now put

$$\left(\frac{a}{b}\right)^{f(x)} = t_1 \;\; \& \;\; \left(\frac{a}{b}\right)^{f(x)} = t_2$$

Now, from these two equation we can determine values of x, which is required solution of original equation.

Concept Rockoner 7

Solve the equation $64.\,9^x - 84.\,12^x + 27.16^x = 0$

Explanation

Here $9 \times 16 = (12)^2$

then we divide its both sides by 12^x and obtain

$$\Rightarrow 64.\left(\frac{3}{4}\right)^x - 84 + 27.\left(\frac{4}{3}\right)^x = 0 \quad .... (1)$$

Let $\left(\frac{3}{4}\right)^x = t$, then equation (1) reduce in the form

$64t^2 - 84t + 27 = 0$

Hence $t_1 = \frac{3}{4}$ and $t_2 = \frac{9}{16}$

then $\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^t$ and $\left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^2$

$\therefore$ Roots are 1 & 2

Equation containing logarithm function

1) If equation is of form $\log_a f(x) = b$ (where $a>0, a \neq 1$)

$\log_a f(x) = b$

gives $f(x) = a^b$; $f(x) > 0$

and No solution for $f(x) < 0$

2) If equation is of type $\log_{f(x)} a = b$ $(a>0)$

If $a \neq 1 \Rightarrow f(x) = a^{1/b}$ $(b \neq 0)$

If $a = 1$ & $b = 0$ then $f(x)$ becomes identity provided that $f(x) > 0$ and $f(x) \neq 1$.

3) If equation is of type

$$\log_a f(x) = \log_a g(x) \log_{f(x)} (a > 0, a \neq 1)$$

For the existence of two logarithm $f(x) > 0$ and $g(x) > 0$.

As base of LHS and RHS is same, so

$\log_a f(x) = \log_a g(x)$ equivalent to $f(x) = g(x)$

So, required solution is $f(x) = g(x) > 0$

4) If equation is of type $a = \log_{g(x)} a$ $(a > 0)$

For the existence of this equation $f(x) > 0$, $f(x) \neq 1$ and $g(x) > 0$ $g(x) \neq 1$.

On taking reciprocal of given equation it becomes $\log_a f(x) = \log_a g(x)$

As base of two logarithms are same so solution of given equation is

$$\left.\begin{array}{l} f(x) = g(x) > 0 \\ f(x) = g(x) \neq 1 \end{array}\right\}$$

5) If equation is of type

$$\log_{f(x)} h(x) = \log_{f(x)} g(x)$$

For such type of equation, before determining solution we have to determine the domain of all the functions used in the equation.

Here, all $f(x)$, $g(x)$ & $h(x)$ should be positive and $f(x) \neq 1$

So, $\log_{f(x)} h(x) = \log_{f(x)} g(x)$ is equivalent to

$h(x) = g(x)$ if $f(x) > 0$ and $f(x) \neq 1$

So, solution set is obtained from intersection of following inequalities

$f(x) > 0$ & $f(x) \neq 1$ & $h(x) = g(x) > 0$

6) If equation is of type $\log_{f(x)} g(x) + \log_{f(x)} h(x) = \log_{f(x)} I(x)$

Before determining solution of this equation we have to determine domain of all functions used in the equation.

Here, for the existing of all logarithms $f(x)$, $g(x)$, $h(x)$, $I(x)$ all should be positive and $f(x) \neq 1$

So, given equation is equivalent to

$g(x)\, h(x) = I(x)$ if $f(x) > 0$ & $f(x) \neq 1$

So, solution set is obtained from intersection of following inequalities.

$f(x) > 0$, $g(x) > 0$, $h(x) > 0$, $I(x) > 0$

$f(x) \neq 1$

$g(x)\, h(x) = I(x)$

Irrational equations

It is the equation in which variable quantities are under radical sign.

One method of solving irrational equation is to raise both sides of equation to a power which is LCM of exponent of all radical sign in given equation.

While raising the power, if it is even number, then domain of the irrational function may changed, so roots may increase on raising the power.

So, after raising the power of the given equation we have to verify the resulting roots with original equation. The values which satisfy the original equation are roots of equation and values which are not satisfied original equation are known as extraneous roots.

Note

Before solving any irrational equation or logarithmic equation find domain of all functions which contain in equation, and at last check that roots of equation lies in the domain or not. If they not lie in domain of any of functions used in equation then that value is not the root of equation. (or that root is extraneous root).

Solved Examples

57. Solve $|x-4|^{\left(\frac{x^2-10x+24}{x-3}\right)}=1$

Sol. Taking log on both sides of the given equation, we have

$$\frac{\left(x^2-10x+24\right)}{x-3}\log|x-4|=0$$

i.e., $\dfrac{(x-6)\,(x-4)}{(x-3)}\log|x-4|=0$ .... (1)

The value of x satisfying equation (1) are
$x = 4, 6$
and $|x-4| = 1$, i.e., $x = 3, 5$.
However, $x = 3, 4$ are not permissible, since the left hand expression of equation (1) is not defined for these values of x. Therefore, the solution is $x\in\{5, 6\}$.

58. Solve the equation $3\sqrt{(x+3)}-\sqrt{(x-2)}=7$

Sol. We have $3\sqrt{(x+3)}-\sqrt{(x-2)}=7$

$\Rightarrow\ 3\sqrt{(x+3)}=7+\sqrt{(x-2)}$

Squaring both sides of the equation, we obtain

$9x+27=49+x-2+14\sqrt{(x-2)}$

$\Rightarrow (4x-10)=7\sqrt{(x-2)}$

Again squaring both sides, we obtain
$\Rightarrow\ 16x^2-129x+198=0$

$\Rightarrow\ x_1=6$, and $x_2=\dfrac{33}{16}$

$x = 6$ satisfies the original equation but

$x=\dfrac{33}{16}$ does not satisfies the original equation.

Hence $x = 6$ is only solution.

59. Solve the equation $\dfrac{1-2(\log x^2)^2}{\log x-2(\log x)^2}=1$

Sol. The given equation can be rewritten in the form

$$\frac{1-2(2\log x)^2}{\log x-2(\log x)^2}=1\Rightarrow\frac{1-8(\log x)^2}{\log x-2(\log x)^2}-1=0$$

Let $\log x = t$ then $\dfrac{1-8t^2}{t-2t^2}-1=0$

$$\Rightarrow\ \frac{1-8t^2-t+2t^2}{t-2t^2}=0$$

$$\Rightarrow 6t^2+t-1=0\left(t\pm 0,\frac{1}{2}\right)$$

$$\Rightarrow\begin{cases}t=-\dfrac{1}{2}\\[2mm] t=\dfrac{1}{3}\end{cases}\Rightarrow\begin{cases}\log x=-\dfrac{1}{2}\\[2mm] \log x=\dfrac{1}{3}\end{cases}\Rightarrow\begin{cases}x_1=10^{-1/2}\\ x_2=10^{1/3}\end{cases}$$

Hence $x_1=\dfrac{1}{\sqrt{10}}$ and $x_2=\sqrt[3]{10}$ are the roots of the original equation.

60. Solve the equation

$(x+2)(x+3)(x+8)(x+12)=4x^2$

Sol. Since $(-2)(-12)=(-3)(-8)$ we can write the equaiton as

$(x+2)(x+3)(x+8)(x+12)=4x^2$ ...(i)

$\Rightarrow\left(x^2+14x+24\right)\left(x^2+11x+24\right)=4x^2$ ...(ii)

Dividing by x^2 on both sides of (ii) we get

$\Rightarrow\left(x+\dfrac{24}{x}+14\right)\left(x+\dfrac{24}{x}+11\right)=4$ ...(iii)

Put $x+\dfrac{24}{x}=y$ then equation (iii) can be reduced to $(y+14)(y+11)=4$ or

$\Rightarrow y=-10,-15$

Thus the original equation is equivalent to the collection of equations :

$$\left[\begin{array}{l}x+\dfrac{24}{x}=-15\\[2mm] x+\dfrac{24}{x}=-10\end{array}\right.\quad\text{i.e.}\quad\left[\begin{array}{l}x^2+15x+24=0\\ x^2+10x+24=0\end{array}\right.$$

Solving this collection, we get

$$x_1=\frac{-15-\sqrt{129}}{2},\ x_2=\frac{-15+\sqrt{129}}{2},x_3=-6,\ x_4=-4.$$

61. Prove that $(x+1)(x+2)(x+3)(x+4)+1$ is the square of a trinomial

Sol. As given expression is the square of a trinomial, then

$(x+1)(x+2)(x+3)(x+4)+1=\left(x^2+ax+b\right)^2$,

On comparing the coefficients of x^3 and x^2 in both sides. We get

$$\begin{cases}2a=10\\ a^2+2b=35,\end{cases}$$

Hence we find $a = 5$ and $b = 5$. At last we have to check that these values of a & b also satisfy constant & coefficient of x. (So the given expression is perfect square of $\left(x^2+5x+5\right)^2$.

62. Find the values of a for which the equation
$(x^2+x+2)^2-(a-3)(x^2+x+2)(x^2+x+1)+(a-4)(x^2+x+1)^2=0$
has at least one real root.

Sol. The given equation can be written as

$(y+1)^2-(a-3)\,y\,(y+1)+(a-4)y^2=0$
[putting $x^2+x+1=y$]
$\Rightarrow\ (5-a)y+1=0$

$$\Rightarrow\ x^2+x+1-\frac{1}{a-5}=0$$

i.e., $x^2+x+\dfrac{a-6}{a-5}=0$

whose roots will be real if discriminant ≥ 0

$$\Rightarrow\ 1-\frac{4(a-6)}{a-5}\geq 0$$

$$\Rightarrow\ 5<a\leq\frac{19}{3}$$

63. Show that the equation

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+.....+\frac{L^2}{x-l}=x-m$$

where a, b, c,.....,*l* are real numbers, all different, cannot have any imaginary root.

Sol. Let $\alpha+i\beta,\ \beta\neq 0$, be an imaginary root of the equation, then $\alpha-i\beta$ is also a root.

$$\therefore\ \frac{A^2}{\alpha+i\beta-a}+\frac{B^2}{\alpha+i\beta-b}+...+\frac{L^2}{\alpha+i\beta-1}=\alpha+i\beta-m \qquad ...\ (1)$$

and

$$\frac{A^2}{\alpha-i\beta-a}+\frac{B^2}{\alpha-i\beta-b}+...+\frac{L^2}{\alpha-i\beta-1}=\alpha-i\beta-m \qquad ...\ (2)$$

Subtracting (1) from (2) gives

$$\left[\frac{A^2}{(\alpha-a)^2+\beta^2}+\frac{B^2}{(\alpha-b)^2+\beta^2}+...+\frac{L^2}{(\alpha-1)^2+\beta^2}+1\right]=0$$

This is possible only when $\beta=0$. Hence the equation cannot have any imaginary root.

64. If a > 1, show that the solution of the equation

$$\left(a+\sqrt{a^2-1}\right)^{x^2-2x}+\left(a-\sqrt{a^2-1}\right)^{x^2-2x}=2a$$

is indepdendent of a.

Sol. Let us put $x^2-2x=\alpha$, and

$$y=a+\sqrt{a^2-1}=\frac{1}{a-\sqrt{a^2-1}}$$

The given equation can now be written as

$$y^\alpha+\frac{1}{y^\alpha}=2a\ \Rightarrow (y^\alpha)^2-2ay^\alpha+1=0$$

$$y^\alpha=a\pm\sqrt{a^2-1}=y^{\pm 1}\ \Rightarrow\ \alpha=\pm 1$$

i.e. $x^2-2x\pm 1=0$
whose roots are independent of a.

65. Solve the equation $ax^4+bx^3+cx^2+dx+e=0$ if $a+b=b+c+d=d+e$.

Sol. The equation contains five coefficients : a, b, c, d and e and there exist two relationships among them. Thus, three coefficients remain arbitrary. Let us express all the coefficients in terms of any three. We have $a=c+d,\ e=b+c$.
The equation takes the form
$(c+d)x^4+bx^3+cx^2+dx+(b+c)=0$
$\Rightarrow c(x^4+x^2+1)+dx\,(x^3+1)+b(x^3+1)=0$
But $x^3+1=(x+1)\,(x^2-x+1)$
$x^4+x^2+1=(x^2+x+1)\,(x^2-x+1)$
The equation is now rewritten as
$(x^2-x+1)\{\,c(x^2+x+1)+dx\,(x+1)+b(x+1)\}=0$

Equating the first factor to zero, we find $x=\dfrac{1}{2}\pm i\dfrac{\sqrt{3}}{2}$.

The remaining two roots are found by solving the second quadratic equation.

66. Solve the following equation for x

$$\left(15+4\sqrt{14}\right)^t+\left(15-4\sqrt{14}\right)^t=30$$

where $t=x^2-2\,|x|$

Sol. Let $y=\left(15+4\sqrt{14}\right)^t$, then $\left(15-4\sqrt{14}\right)^t=\dfrac{1}{y}$ Now given equation reduces to $y+\dfrac{1}{y}=30$ or

$y^2-30y+1=0$

$\Rightarrow y=15\pm 4\sqrt{14}$

Now $y=15+4\sqrt{14}$

$\Rightarrow\ \left(15+4\sqrt{14}\right)^t=\left(15+4\sqrt{14}\right)$

$\Rightarrow\ t=1$

$\therefore\ x^2-2\,|x|=1$

$\Rightarrow x=\pm\left(1+\sqrt{2}\right)$

Again $y=15-4\sqrt{14}$

$\Rightarrow\ \left(15+4\sqrt{14}\right)^t=\left(15+4\sqrt{14}\right)^{-1}$

$\therefore\ t=-1$

$\therefore\ x^2-2\,|x|=-1\ \Rightarrow x=\pm 1$

Hence $x=\pm 1,\ \pm\left(1+\sqrt{2}\right)$

67. If the product of two roots of the equation $4x^4-24x^3+31x^2+6x-8=0$ is 1, find the roots.

Sol. Suppose the roots are $\alpha,\beta,\gamma,\delta$ and $\alpha\beta=1$

As $\alpha\beta\gamma\delta=-2\ \Rightarrow\gamma\delta=-2$

So the given equation can be written as

$$x^4-6x^3+\frac{31}{4}x^2+\frac{3}{2}x-2=(x^2(\alpha+\beta)x+1)(x^2(\gamma+\delta)x-2)$$

Algebra

On comparing $\alpha+\beta=\frac{5}{2}$ & $\gamma+\delta=\frac{5}{2}$

On solving $\alpha\beta=1$ & $\alpha+\beta=\frac{5}{2}$ we get $\alpha=\frac{1}{2}$ & $\beta=2$

Similarly, $\gamma=-\frac{1}{2}$ & $\delta=4$.

68. Solve the equation
$(12x - 1)(6x - 1)(4x - 1)(3x - 1) = 5$

Sol. The given equation can be written as

$$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right)=\frac{5}{12.6.4.3} \quad ...(i)$$

Since $\frac{1}{12}<\frac{1}{6}<\frac{1}{4}<\frac{1}{3}$ and $\frac{1}{6}-\frac{1}{12}=\frac{1}{3}-\frac{1}{4}$

We can introduce a new variable

$$y=\frac{1}{4}\left[\left(x-\frac{1}{12}\right)+\left(x-\frac{1}{6}\right)+\left(x-\frac{1}{4}\right)+\left(x-\frac{1}{3}\right)\right]$$

$$y=x-\frac{5}{24}$$

Substituting $x=y+\frac{5}{24}$ in (i) we obtain

$$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right)=\frac{5}{12.6.4.3}$$

$$\Rightarrow\left[y^2-\left(\frac{1}{24}\right)^2\right]\left[y^2-\left(\frac{3}{24}\right)^2\right]=\frac{5}{12.6.4.3}$$

Hence we find that $y^2=\frac{49}{24^2}$

i.e., $y_1=\frac{7}{24}$ and $y_2=-\frac{7}{24}$

$\therefore$ The corresponding roots of the original equation are

$-\frac{1}{12}$ and $\frac{1}{2}$.

69. Solve the equation $(x + b + c)(x + a + c)(x + a + b)(a + b + c) - abcx = 0$

Sol. Put $a + b + c = p$ and make the substitution $x + p = y$.
We have $(y - a)(y - b)(y - c)p - abc(y - p) = 0$
Hence,

$$p\left\{y^3-(a+b+c)y^2+(ab+ac+bc)y\right\}-abcy=0$$

or $$y\begin{Bmatrix}(a+b+c)y^2-(a+b+c)^2y \\ +(ab+ac+bc)(a+b+c)-abc\end{Bmatrix}=0$$

And so, we find three values for y : one of them is zero, the other two are obtained as the roots of a quadratic equation.
Then it is easy to find the corresponding values of x.

70. For any real x, prove that the value of the expression

$2(k-x)\left(x+\sqrt{x^2+k^2}\right)$ cannot exceed $2k^2$

Sol. Let $y=2(k-x)\left(x+\sqrt{x^2+k^2}\right)$

i.e., $x+\sqrt{x^2+k^2}=\frac{y}{2(k-x)}$ .... (1)

i.e., $\frac{k^2}{\sqrt{x^2+k^2}-x^2}=\frac{y}{2(k-x)}$

i.e., $\sqrt{x^2+k^2}-x^2=\frac{2k^2(k-x)}{y}$ ....(2)

Subtracting (2) from (1), we get

$$2x=\frac{y}{2(k-x)}-\frac{2k^2(k-x)}{y}$$

$$\Rightarrow 4\left(k^2-y\right)x^2-4k\left(2k^2-y\right)x+\left(4k^4-y^2\right)=0$$

Since x is real , so we have Disc ≥ 0

$$16k^2\left(2k^2-y\right)^2\geq 16\left(k^2-y\right)\left(4k^4-y^2\right)$$

i.e. $2k^2y^2-y^3\geq 0$

$y\leq 2k^2$

Make Concepts Clear 3.6

1. Solve $x^{x^2+5x+6}=1$
2. Solve the equation $5^{2x} - 24.5^x - 25 = 0$
3. Solve : $\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}$
4. If $0\leq x\leq\pi$. then the solution of the equation $16^{\sin^2 x}+16^{\cos^2 x}=10$ is given by x equal to

   A) $\frac{\pi}{6},\frac{\pi}{3}$ B) $\frac{\pi}{3},\frac{\pi}{2}$

   C) $\frac{\pi}{6},\frac{\pi}{2}$ D) None of these
5. Solve for x
   $(x-1)^3+(x-2)^3+(x-3)^3+(x-4)^3+(x-5)^3=0$
6. Prove that if the equation x^2+9y^2-4x+3 is satisfied for all real values of x and y, then x must lie between 1 and 3 and y must lie between -1/3 and 1/3.
7. Solve for x, $(x^2-4x)^2-(x-2)^2-16=0$
8. Show that all the roots of the equation $ax^3+x^2+x+1=0$ cannot be real, where $a\in R$.
9. Solve $6x^5+11x^4-33x^3-33x^2+11x+6=0$
10. Let {x} and [x] denote the fractional and integral part respectively of real number x, then solve $4\{x\}=x+[x]$
11. Solve the equation :
    $\frac{a}{x+a}+\frac{b}{x+b}+\frac{c}{x+c}=3$

Answers

1. No Solution 2. 2 3. 9 4. A 5. $x = 3, 3 \pm \sqrt{6}i$ ***7. 2, 5, – 1***

9. $-1, 2, \frac{1}{2}, -3, -\frac{1}{3}$ ***10. 0 and 5/3 11.*** $0, -\frac{1}{3}\left\{(a+b+c) \pm \sqrt{(a^2+b^2+c^2-bc-ca-ab)}\right\}$

Solutions Are On Page No. 3.76

3.7 Inequalities Involving Rational, Exponential, Logarithm Function etc.

Inequalities involving rational algebraic function

Rational algebraic function is of form of $\frac{f(x)}{g(x)}$ where $g(x) \neq 0$

Rational algebraic inequalities are of form $\frac{f(x)}{g(x)} \geq 0$ or $\frac{f(x)}{g(x)} \leq 0$,these inequalitites can be solved directly by the help of wavy curve (after factorising f(x) & g(x)).

1) $\frac{f(x)}{g(x)} \geq 0 \Rightarrow \begin{matrix} f(x).g(x) \geq 0 \\ g(x) \neq 0 \end{matrix} \Rightarrow \begin{matrix} f(x) \geq 0 \ \& \ g(x) > 0 \text{ or} \\ f(x) \leq 0 \ \& \ g(x) < 0 \end{matrix}$

So, solution is

$x \in \{x : f(x) \geq 0, g(x) > 0\} \cup \{x : f(x) \leq 0 , g(x) < 0\}$

2) $\frac{f(x)}{g(x)} \leq 0 \Rightarrow \begin{matrix} f(x)g(x) \leq 0 \\ g(x) \neq 0 \end{matrix} \Rightarrow \begin{matrix} f(x) \geq 0 \ \& \ g(x) < 0 \text{ or} \\ f(x) \leq 0 \ \& \ g(x) > 0 \end{matrix}$

So, solution of given equation is

$x \in \{x : f(x) \geq 0, g(x) < 0\} \cup \{x : f(x) \leq 0 , g(x) > 0\}$

Equalities involving modulus function

1) If $|x| \leq a$

$\Rightarrow x^2 \leq a^2 \Rightarrow x^2 - a^2 \leq 0$

$(x-a)(x+a) \leq 0$

So, required result is $-a \leq x \leq a$

2) If $|x| \geq a$

$\Rightarrow x^2 \geq a^2 \Rightarrow x^2 - a^2 \geq 0$

$\Rightarrow (x-a)(x+a) \geq 0$

$\Rightarrow x \geq a$ or $x \leq -a$

3) If inequation is of type $f(|x|) > g(x)$

Solution of this inequality is collection (or union) of following solutions.

$$\begin{cases} f(x) > g(x) \ ; x \geq 0 \\ f(-x) > g(x) \ ; x \leq 0 \end{cases}$$

4) If inequation is of type $|f(x)|>g(x)$

It is equivalent to collection of following solutions.

$$\begin{cases} f(x) > g(x) \text{ or } f(x) < -g(x) \ ; \ g(x) > 0 \\ \text{always true} \ ; \quad g(x) < 0 \end{cases}$$

5) If inequation is of type $|f(x)| \geq |g(x)|$

To solve such inequation, square both sides.

$f^2(x) \geq g^2(x)$ & then solve by wavvy curve method.

Note

If in any inequality or equality there are multi modulus function i.e. modulus inside modulus then try to remove modulus sign from inner modulus first and then eliminate outer modulus.

Inequalities involving exponential function

1) If inequation is of type $a^x>b$ (where $a>0$ & $a \neq 1$)

Solution of $a^x > b$ can be obtained by taking log of both sides and then make cases for a & b

a) $x \in (\log_a b, \infty)$ for $a>1, b>0$

b) $x \in (-\infty, \log_a b)$ for $0<a<1, b>0$

c) $x \in R$ for $a > 0$, $b < 0$.

Result is union of all above 3 inequalities.

2) If inequation is of type $a^{f(x)}>b$ ($a>0$ & $a \neq 1$)

Solution of $a^{f(x)} > b$ can be obtained by taking log of both sides and then make cases for a and b.

a) $x \in$ domain of f(x) if $b \leq 0$

b) $f(x) > \log_a b$ if $a>1$ & $b>0$

c) $f(x) < \log_a b$ if $0<a<1$ & $b>0$

Result is union of all the above inequalities.

3) If Inequation is of type f(ax) > 0

To solve such inequality, put $a^x = t$

So, inequation becomes $f(t) \geq 0$.

Now, solve $f(t) \geq 0$ by the help of wavvy curve method.

4) If Inequation is of type

$$\alpha a^{f(x)} + \beta b^{f(x)} + \gamma c^{f(x)} \geq 0$$

(where $\alpha, \beta, \gamma \in R_0$) such that $b^2 = ac$.

To solve such inequality we have to divide the inequation by $b^{f(x)}$

$$\Rightarrow \alpha\left(\frac{a}{b}\right)^{f(x)} + \beta + \gamma\left(\frac{c}{b}\right)^{f(x)} \geq 0$$

As $b^2 = ac$ so $\frac{c}{b} = \left(\frac{a}{b}\right)^{-1}$

Put $\left(\frac{a}{b}\right)^{f(x)} = t$

$$\Rightarrow \alpha t + \beta + \frac{\gamma}{t} \geq 0 \Rightarrow \frac{\alpha t^2 + \beta t + \gamma}{t} \geq 0$$

Now, first Solve this inequality for t and then by using $\left(\frac{a}{b}\right)^{f(x)} = t$ solve it for x.

Inequations using logarithmic function

1) If inequation is of type $\log_a f(x) \geq 0$

Then we have to take following cases

If $a > 1$ then $f(x) \geq 1$

If $0 < a < 1$ then $0 < f(x) \leq 1$

2) If inequation is of type $\log_a f(x) \geq b$

Here we have following two cases wrt base a.

If $a > 1$ then $f(x) \geq a^b$

If $0 < a < 1$ then $0 < f(x) \leq a^b$

3) If inequation is of type $\log_{g(x)} f(x) \geq 0$

First of all before determining solution we have to find domain of all the functions used in the inequality, and then make cases wrt base.

If $g(x) > 1$ then $f(x) \geq 1$

If $0 < g(x) < 1$ then $0 < f(x) \leq 1$

then we have to take union of above two cases.

4) If inequation is of type $\log_{g(x)} f(x) \geq b$

First of all before determining solution we have to find domain of all the functions used in the inequality, and then make cases wrt base.

If $g(x) > 1$ then $f(x) \geq (g(x))^b$ & $f(x) > 0$

If $0 < g(x) < 1$ then $0 < f(x) \leq (g(x))^b$ & $f(x) > 0$

In last, first we have to take intersection of all the inequalities in each case, then finally we have do take union of above two cases i.e.

$$x \in \left\{x : \left\{f(x) > g(x)^b \cap f(x) > 0\right\} \cup \left\{0 < f(x) < g(x)^b \cap f(x) > 0\right\}\right\}$$

5) If inequation is of type $\log_{g(x)} f_1(x) \geq \log_{g(x)} f_2(x)$

Before solving this inequality, we have to determine domain of all the function used in the inequality. Final result of inequality always lies in common domain.

To solve this inequality we have to make cases wrt base of logarithm.

If $g(x) > 1$ then $f_1(x) \geq f_2(x) > 0$

If $0 < g(x) < 1$ then $0 < f_1(x) \leq f_2(x)$

Now, find solution, which is union of above two results.

6) If inequation is of type $\log_{f_1(x)} g(x) \geq \log_{f_2(x)} g(x)$

Again to solve such inequation, we have to determine domain of all functions .

To solve the inequality make following cases wrt base of logarithm.

i) If $g(x) > 1$ then $0 < f_1(x) \leq f_2(x)$; $f(x)$ & $f_2(x) \neq 1$

ii) If $0 < g(x) < 1$ then $0 < f_2(x) < f_1(x)$; $f_1(x)$ & $f_2(x) \neq 1$

Now final solution is union of above two cases.

7) If inequation is of type $\log_a f_1(x) \geq \log_a f_2(x)$

If $a > 1$ then $f_1(x) \geq f_2(x) > 0$

If $0 < a < 1$ then $0 < f_1(x) \leq f_2(x)$

Final result is union of above two inequalities.

Inequation involving irrational functions

While solving irrational inequalities, we have to check the domain, and also if n^{th} root is there then we directly take n^{th} power of both sides if n is odd, while if n is even then we cant do n^{th} power directly, i.e., before doing so we have do check the domain.

1) If inequation is of type $\sqrt[2n]{f(x)} < g(x)\ (n \in N)$

As 2n is even, so following inequalities we have to solve

$f(x) \ge 0$, $g(x) > 0$, $f(x) < (g(x))^{2n}$

Final result is intersection of above inequalities

2) If inequation is of type $\sqrt[2n+1]{f(x)} < g(x)$

As 2n + 1 is odd, so given equation is equivalent to $f(x) < (g(x))^{2n+1}$

Which can be solved by wavvy curve method.

3) If inequation is of type $\sqrt[2n]{f(x)} > g(x)$

As 2n is even, so we have to make following cases.

If $g(x) \ge 0$ then $f(x) > (g(x))^{2n}$

If $g(x) < 0$ then $f(x) \ge 0$

Final result is union of above two cases.

4) If inequation is of type $\sqrt[2n+1]{f(x)} > g(x)$

As 2n + 1 is odd, so given equation is equivalent to $f(x) > (g(x))^{2n+1}$

Solved Examples

71. Solve the inequation $\sqrt{(-x^2+4x-3)} > 6-2x$

Sol. First for the existence of inequality

We have $-x^2+4x-3 \ge 0$

$\Rightarrow x \in [1,3]$ ...(1)

Now for $x \in [1,3] - 6 - 2x > 0$

So, above inequality can be written as $-x^2+4x-3 > (6-2x)$

$\Rightarrow x \in \left(\frac{13}{5}, 3\right)$ ...(2)

Hence $x \in \left(\frac{13}{5}, 3\right)$ is required solutions

72. Show that $(x-1)(x-3)(x-4)(x-6)+10$ is positive for all real values of x.

Sol. Taking the first and last factors together, and also the other two the given expression becomes

$\left(x^2-7x+6\right)\left(x^2-7x+12\right)+10$

$=\left(x^2-7x\right)^2+18\left(x^2-7x\right)+82$

Assuming this as a quadratic expression in $\left(x^2-7x\right)$,

$D = 18^2 - 4 \times 82 = 4(81-82) = -4 < 0$

Which is clearly always positive for real values of c.

73. Solve the inequation $3^{x+2} > \left(\frac{1}{9}\right)^{\frac{1}{x}}$

Sol. We have $3^{x+2} > \left(3^{-2}\right)^{\frac{1}{x}} \Rightarrow 3^{x+2} > 3^{-\frac{2}{x}}$

$\Rightarrow x+2 > -\frac{2}{x}$ (as base > 1)

$\Rightarrow \frac{x^2+2x+2}{x} > 0 \Rightarrow \frac{(x+1)^2+1}{x} > 0$

$\Rightarrow \frac{1}{x} > 0 \Rightarrow x \in (0,\infty)$

74. Solve the inequation $\log_{\left(\frac{25-x^2}{16}\right)}\left(\frac{24-2x-x^2}{14}\right) > 1$

Sol. The given inequation is valid for

$\frac{24-2x-x^2}{14} > 0$ and $\frac{25-x^2}{16} > 0$

$\therefore x^2+2x-24 < 0$ and $x^2 - 25 < 0$

$\Rightarrow -6 < x < 4$ and $-5 < x < 5$

Combining both inequality

$-5 < x < 4$ ... (1)

Now consider the following cases :

Case – I : If $0 < \frac{25-x^2}{16} < 1 \Rightarrow 9 < x^2 < 25$

$\therefore x \in (-5, -3) \cup (3, 5)$ ... (2)

$\therefore$ The given inequation convert in the form

$\frac{24-2x-x^2}{14} < \frac{25-x^2}{16}$

$\Rightarrow x^2 + 16x - 17 > 0$

$\therefore x \in (-\infty, -17) \cup (1, \infty)$ ... (3)

Combining (1), (2) and (3), we get

$x \in (3, 4)$ ... (4)

Case – II : If $\frac{25-x^2}{16} < 1 \Rightarrow x \in (-3, 3)$ ... (5)

$\therefore$ The given inequation convert in the form

$\frac{24-2x-x^2}{14} > \frac{25-x^2}{16}$

$\Rightarrow x^2 + 16x - 17 < 0$

$\therefore x \in (-17, 1)$ ... (6)

Combining (1), (5) and (6), we get $x \in (-3, 1)$

So final solution is $x \in (-3, 1) \cup (3, 4)$

75. Let S denote the set of all real values of the parameter a for which every solution of the inequality $\log_{1/2}x^2 \ge \log_{1/2}(x+2)$ is a solution of the inequality $49x^2 - 4a^4 \le 0$. Find S.

Sol. We have

$\log_{1/2}x^2 \geq \log_{1/2}(x + 2)$

i.e., $x^2 \leq x + 2 \Rightarrow -1 \leq x \leq 2$ ... (1)

and $49x^2 - 4a^4 \leq 0$

i.e., $\frac{-2a^2}{7} \leq x \leq \frac{2a^2}{7}$ ... (2)

According to the given condition, set (1) is a subset of (2). Hence, we have

$\frac{-2a^2}{7} \leq -1$ and $2 \leq \frac{2a^2}{7}$

i.e, $a^2 \geq \frac{7}{2}$ and $a^2 \geq 7 \Rightarrow a^2 \geq 7$

i.e., $a \in (-\infty, -\sqrt{7}] \cup [\sqrt{7}, \infty)$

76. Solve the inequation $(x^2 + x + 1)^x < 1$

Sol. Taking logarithm both sides on base 10.

then $x \log(x^2 + x + 1) < 0$

which is equivalent to the collection of systems

$$\left[\begin{cases} x>0, \\ \log(x^2+x+1)<0 \end{cases} \& \begin{cases} x<0, \\ \log(x^2+x+1)>0 \end{cases}\right.$$

$$\Leftrightarrow \left[\begin{cases} x>0, \\ x^2+x+1<1, \end{cases} \& \begin{cases} x>0, \\ x^2+x+1>1, \end{cases}\right.$$

$$\Leftrightarrow \left[\begin{cases} x>0, \\ -1<x<0 \end{cases} \& \begin{cases} x<0, \\ x>0 \text{ and } x<-1 \end{cases}\right.$$

$$\Leftrightarrow \begin{cases} x \in \phi \\ x<-1 \end{cases}$$

Consequently the interval $x \in (-\infty, -1)$ is the set of all solutions of the original inequation.

77. Find the values of a for which the inequality $x^2 + |x - a| - 3 < 0$ is satisfied by at least one negative x.

Sol. We have

$x^2 + |x - a| - 3 = x^2 + x - (a + 3)$; if $x \geq a$

$= x^2 - x + (a - 3)$; if $x < a$

Let $f(x) = x^2 + x - (a + 3)$

and $g(x) = x^2 - x + (a - 3)$

The coefficient of x^2 is positive in both the expressions and hence, the plot of f(x) as well as g(x) represents a parabola open upwards.

Now, $f(x) < 0$ for some negative x.

$\Rightarrow$ a portion of the parabola must lie in the third quadrant

Case – I : (one root negative, one root positive)

product of roots < 0, i.e., $a > -3$.

Case – II : (one root negative one root 0)

$f(0) = 0$ gives $a = -3$ and other root equal to -1 which is negative.

Hence $a = -3$ is acceptable.

Case – III : (both roots negative and distinct)

discriminant > 0

i.e., $1 + 4(a + 3) > 0$, i.e., $a > -\frac{13}{4}$ ...(1)

and product of roots > 0, i.e., $a < -3$ ... (2)

and sum of roots < 0, i.e., $-1 < 0$ which is true $\forall a \in R$. ... (3)

Intersection of inequalities (1), (2) and (3) gives $-\frac{13}{4} < a < -3$.

Now, taking union of the inequalities obtained in the three case gives $a > -\frac{13}{4}$ ... (4)

Similarly, for $g(x) < 0$ for some negative x, we have

Case – I : (one root negative, one root positive)

product of roots < 0, i.e., $a < 3$.

Case – II : (one root negative, one root 0)

$f(0) = 0$ gives $a = 3$ and other root equal to 1 which is positive.

Hence, $a = 3$ is not acceptable.

Case – III : (both roots negative and distinct)

discriminant > 0

i.e., $1 - 4(a + 3) > 0$, i.e., $a < -\frac{11}{4}$ ... (5)

and product of roots > 0, i.e., $a > 3$ ... (6)

and sum of roots < 0, i.e. $1 < 0$ which is not true for any real a ... (7)

Intersection of inequalities (5), (6) and (7) gives $a \in \phi$

Now, taking union of the inequalities obtained in the three cases gives $a < 3$. ...(8)

The required value of a is obtained by taking intersection of (4) and (8), which gives $a \in \left(-\frac{13}{4}, 3\right)$

Make Concepts Clear 3.7

1. The set of all real numbers x for which $x^2 - |x+2| + x > 0$ is

A) $(-\infty, -2) \cup (2, \infty)$ B) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

C) $(-\infty, -1) \cup (1, \infty)$ D) $(\sqrt{2}, \infty)$

2. Find the integral solutions of the follwing systems of inequalities

i) $5x - 1 < (x+1)^2 < 7x - 3$

ii) $\frac{x}{2x+1} \geq \frac{1}{4}, \frac{6x}{4x-1} < \frac{1}{2}$

3. Find the set of all x for which $\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$

4. For all ‘x’, $x^2+2ax+(10-3a)>0$, then the interval in which ‘a’ lies is
A) $a<-5$ B) $-5<a<2$
C) $a>5$ D) $2<a<5$

Answers

1. B ***2. a) 3*** ***b) No integral solution exists*** ***3.*** $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

4. B

Solutions Are On Page No. 3.77

3.8 Theory of Equations

Resolving of 2nd degree equation in product of two rational factors

Here, we have to determine under what condition the second degree general equation will be expressed as a product of two rational factors. (linear factors)

$f(x,y) = ax^2+2hxy+by^2+2gx+2fy+c$

Now, write this expression in descending power of x and equate it to zero i.e.

$ax^2+2x(hy+g)+by^2+2fy+c=0$

On solving this equation we get,

$$x = \frac{-2(hy+g) \pm \sqrt{4(hy+g)^2 - 4a(by^2+2fy+c)}}{2a}$$

Now given equation will be product of two linear factors, if discriminant of above equation is perfect square.

$\therefore y^2(h^2-ab)+2y(hg-af)+g^2-ac = \text{perfect square}$

which is quadratic in y.

As, any quadratic equation is perfect square, if its discriminant. = 0

$\therefore D=4(hg-af)^2-4(h^2-ab)(g^2-ac)=0$

Gives

$$D=abc+2fgh-af^2-bg^2-ch^2=0=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Is required condition under which the 2nd degree general equation has two rational factors.

Remainder theorem

Let P(x) be any polynomial of degree greater than or equal to one & ‘a’ is any real number. If P(x) is divided by (x–a) then remainder is equal to P(a).

As (x–a) is of 1st degree, so remainder is constant and degree of quotient is one less then degree of P(x), If R is remainder and Q is quotient, then

$P(x)=(x-a)Q(x)+R$

On putting $x=a$

Remainder, $R=P(a)$

Factor theorem

Let P(x) be any polynomial of degree greater than or equal to one & ‘a’ is any real number. If $P(a)=0$ then (x–a) is factor of P(x) or if (x–a) is factor of P(x) then $P(a)=0$.

THEOREM

If f(x) has integral coefficients & ‘a’ is integral root of f(x) and m is any integer different from ‘a’ then a-m divides f(m).

Proof

On dividing f(x) by (x–m), let Q(x) is quotient & f(m) is remainder, then $f(x)=(x-m)Q(x)+f(m)$

Put $x=a$

$f(a)=(a-m)Q(a)+f(m)$

As $f(a)=0$, so $f(m)=(m-a)Q(a)$

Hence, a–m divides f(m).

THEOREM

If f(x) is polynomial which vanishes when x has values $\alpha_1,\alpha_2,....\alpha_n$, no two of which are equal, then

$(x-\alpha_1)(x-\alpha_2).......(x-\alpha_n)$ is factor of f(x).

Proof

As α_1 is root of f(x), so by the help of remainder theorem

$f(x)=(x-\alpha_1)f_1(x)......(1)$ ($f_1(x)$ is another polynomial) As f(x) also vanish when $x=\alpha_2$ so (1) becomes

$f(\alpha_2)=(\alpha_2-\alpha_1)f_1(\alpha_2)$

So $f_1(\alpha_2)=0$ and consequently $x-\alpha_2$ is factor of $f_1(x)$

$f_1(x)=(x-\alpha_2)f_2(x)$

$f(x)=(x-\alpha_1)(x-\alpha_2)f_2(x)$

In the similar process, we get

$f(x)=(x-\alpha_1)(x-\alpha_2).......(x-\alpha_n)\ f_n(x)$

So, $(x-\alpha_1)(x-\alpha_2).......(x-\alpha_n)$ is factor of f(x).

Newton's Theorem

If α, β are roots of $ax^2 + bx + c = 0$ and $s_n = \alpha^n + \beta^n$ then for $n > 2$, $n \in N$ then $as_n + bs_{n-1} + cs_{n-2} = 0$

Proof

$as_n + bs_{n-1} + cs_{n-2}$

$a(\alpha^n + \beta^n) + b(\alpha^{n-1} + \beta^{n-1}) + c(\alpha^{n-2} + \beta^{n-2})$

$\Rightarrow a\alpha^n + b\alpha^{n-1} + c\alpha^{n-2} + a\beta^n + b\beta^{n-1} + c\beta^{n-2}$

$\Rightarrow \alpha^{n-2}(a\alpha^2 + b\alpha + c) + \beta^{n-2}(a\beta^2 + b\beta + c)$

$= 0$ (as α, β are roots of $ax^2 + bx + c = 0$)

$\therefore as_n + bs_{n-1} + cs_{n-2} = 0$ (proved)

Descartes rule of sign

By the help of this rule we can determine maximum number of real roots in any polynomial equation.

Let f(x) is a polynomial function arranged in descending power of x.

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n \quad ...(1)$$

where $a_0, a_1, a_2, ..., a_n \in R$. & $a_0 \neq 0$

$$f(-x) = a_0(-x)^n + a_1(-x)^{n-1} + a_2(-x)^{n-2} + ... + a_n \quad ...(2)$$

1) Maximum number of positive real roots is equal to number of sign changes in coefficients of f(x). Let number of sign changes is p.
2) Maximum number of negative real roots is equal to number of sign changes in coefficients of f(–x). Let number of sign changes is q.
3) So from (1) & (2) Maximum number of real roots in f(x) is p + q
4) So Minimum number of imaginary (non real) roots in f(x) is n–(p + q).

Note

1. In any polynomial equation if all odd degree terms are absent and all other terms have same sign, then number of real roots of that equation is zero.

2. If in an equation, all the coefficients are of same sign then the equation has no positive root.

3. If in an equation, all the coefficients of even power are positive (or all negative) and coefficients of odd powers of x are of opposite sign (to that of even powers coefficient) then equation has no negative roots.

Division of polynomial by x – a

1. Synthetic division

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...a_n$ is a polynomial function of degree n and $q(x) = b_0x^{n-1} + b_1x^{n-2} + ... + b_{n-1}$ is a polynomial function of degree n–1.

Let q(x) is quotient when f(x) is divided by x-a. And r(x) is its remainder.

$\therefore f(x) = (x-a)q(x) + r(x)$

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

$= (x-a)(b_0x^{n-1} + b_1x^{n-2} + ... + b_{n-1}) + r$

$= b_0x^n + x^{n-1}(b_1 - ab_0) + x^{n-2}(b_2 - ab_1) + ...ab_{n-1} + r$

On comparing LHS & RHS

$a_0 = b_0 \quad \Rightarrow \quad b_0 = a_0$

$a_1 = b_1 - ab_0 \quad \Rightarrow \quad b_1 = a_1 + ab_0 = a_1 + aa_0$

$a_2 = b_2 - ab_1 \quad \Rightarrow \quad b_2 = a_2 + ab_1 = a_2 + a(a_1 + aa_0)$

: :

: :

$a_n = r - ab_{n-1} \quad r = a_n + ab_{n-1}$

So, $q(x) = a_0x^{n-1} + (a_1 + aa_0)x^{n-2} + ... + b_{n-1}$

So remainder r(x) when f(x) is divided by (x – a) is

$r(x) = r = a_n + ab_{n-1}$

2. Horner's Method of synthetic division

Let f(x) is any polynomial function of degree x which has to be divided by (x – a)

$f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$

	a_0	a_1	a_2	a_3 ....a_{n-1}		a_n
a	-	ab_0	ab_1	ab_2	ab_{n-2}	ab_{n-1}
	a_0	b_1	b_2	b_3	b_{n-1}	r

(where $a_0 = b_0$)

In the first row we write the coefficient of given polynomial f(x). We divide the polynomial f(x) by x–a, by writing a in left corner out of box.

Write a_0 in left corner in 3rd row, then multiply a by a_0(or b_0) and write below a_1. Now add a_1 & ab_0 and write below ab_0 in third row and write sum as b_1.

Now multiply a by b_1 and write below a_2 in 2^n row.

Then add a_2 and ab_1 to write this b_2 in third row.

And continue this process till end and the last term in right corner of 3rd row is remainder.

Note

If any term is absent then write the coefficient of that term to be zero.

Division of polynomial by $x^2 - ax - b$

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$ be a polynomial of degree n. Let q(x) & R(x) be quotient & remainder when f(x) is divided by $x^2 - ax - b$.

Here, degree of q(x) is n–2 & degree of R(x) is 1.

Let $q(x) = b_0x^{n-2} + b_1x^{n-3} + b_2x^{n-4} + ... + b_{n-2}$ &

$R(x) = mx + n$

	a_0	a_1	a_2	a_3	a_{n-2}	a_{n-1}	a_n
a	0	ab_0	ab_1	ab_2	ab_{n-3}	ab_{n-2}	0
b	0	0	bb_0	bb_1	bb_{n-4}	bb_{n-3}	bb_{n-2}
	a_0 $\parallel$ b_0	b_1	b_2	b_3	b_{n-2}	m	n

In the first row write the coefficients of f(x). On left side of vertical line write a and b in 2nd and 3rd row. Below a_0 write 0 in 2nd and 3rd row and add a_0, 0 & 0 and write in 4th row (i.e. below horizontal line) and let $a_0 = b_0$. Now multiply a by b_0 and write below a_1 and add a_1, ab_0 & 0 (i.e b_1) & write in 4th row. Now multiply a by b_1 & b by b_0 and write below a_2, and add a_2, ab_1 & bb_0 (i.e. b_2) & write in 4th row. This process continueous till end.

Last two terms in 4th row are coefficients of R(x), & rest are coefficients of q(x).

So, from here we get remainder as well as quotient when f(x) is divided by (x - a).

Taylor's theorem

If f(x) is a polynomial of degree n then,

$$f(x+h) = f(h) + \frac{x}{1!}f^1(h) + \frac{x^2}{2!}f^{II}(h) + ... + \frac{x^n}{n!}f^n(h)$$

where $f^r(x)$ is rth derivative of f(x)

THEOREM

The equation whose roots diminished by h then those of f(x) = 0 is f(x + h) = 0.

Proof

Let α is root of f(x) = 0

$\Rightarrow f(\alpha) = 0$

$\Rightarrow f(\alpha + h - h) = 0$

Now as we have to determine the equation whose root is $\alpha - h$, So put $\alpha - h = x$ gives f(x + h) = 0 whose root is $\alpha - h$.

Method to determine f(x+h)

1. Horners Method : f(x+h) can be easily calculated by Horners method.

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

h	a_0	a_1	a_2	a_3	a_{n-1}	a_n	
	0	ha_0	hb_1	hb_2	hb_{n-2}	hb_{n-1}	
h	$b_0 = a_0$	b_1	b_2	b_3	b_{n-1}	k_n	
	0	hc_0	hc_1	hc_2...........	hc_{n-2}		
h	c_0	c_1	c_2	c_3	k_{n-1}		
	0	hd_0	hd_1	hd_2			
h	d_0	d_1	d_2	d_3	k_{n-2}		
					k_{n-3}		
	k_0						

then $k_0, k_1, k_2....k_n$ are coefficient of x^n, x^{n-1}, ...x^0 of f(x + h).

Note

f(x + h) can also be calculated by Taylors theorem.

THEOREM

If f(x) = 0 is an equation of degree n, then to eliminate rth term, f(x) = 0 can be transformed to f(x+h)=0 where h is constant such that $f^{(n-r+1)}(h) = 0$

Proof

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$ be polynomial equation of degree n.

Then by Taylor's theorem

$$f(x+h) = f(h) + \frac{x}{1!}f^I(h) + \frac{x^2}{2!}f^{II}(h) + ...$$

$$+ \frac{x^{n-(r-1)}}{[n-(r-1)]!}f^{n-r+1}(h) + ... + \frac{x^n}{n!}f^n(h)$$

As we have to eliminate rth term, so

$$\frac{x^{n-r+1}}{(n-r+1)!}f^{n-r+1}(h) = 0$$

As $x \neq 0$ so $f^{n-r+1}(h) = 0$

Note

To determine equation after removing rth term can also be determined by the help of Horner's method.

Reciprocal Equation

A polynomial equation f(x) = 0 is said to be reciprocal equation, if α & $\frac{1}{\alpha}$ both are roots of f(x) = 0 of same multiplicity.

THEOREM

An equation $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ is said to be reciprocal equation if either $a_i = a_{n-i}$ or $a_i = -a_{n-i}$ for every i.

Proof

Let $\alpha_1, \alpha_2, ...\alpha_n$ be roots of $f(x) = 0$

then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n}$ be roots of $f\left(\frac{1}{x}\right) = 0$

$$f\left(\frac{1}{x}\right) = g(x) = a_0\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + ... + a_n$$

$$g(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 = 0$$

As f(x) is reciprocal equation, so f(x) = k g(x) for $k \in R_0$

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n =$

$k(a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0)$

On comparing coefficients of respective powers of LHS and RHS.

$a_0 = ka_n$, $a_1 = ka_{n-1}, ..., a_n = ka_0$

As $a_0 = ka_n$ & $a_n = ka_0$

$\therefore a_0 = k^2a_0 \Rightarrow k = \underline{+}1$

So, we can say that $a_i = \pm a_{n-i}$ for $i \in [1, n]$ then f(x) = 0 & g(x) = 0 has same roots, and is also a reciprocal equation.

Types of reciprocal equation

A reciprocal equation

$f(x) = a_0x^n + a_1x^{n-1} + ... + a_n = 0$ is divided in two categories.

1) If $a_i = a_{n-i}$ then reciprocal equation is said to be reciprocal equation of **class one**.

2) If $a_i = a_{n-i}$ then reciprocal equation is said to be reciprocal equation of **class two**.

Now following points we have to remember while dealing with reciprocal equation of class one and class two.

i) For an odd degree reciprocal equation of class one –1 is its root.

ii) For an odd degree reciprocal equation of class two 1 is its root.

iii) For an even degree reciprocal equation of class two 1 & –1 are its roots.

THEOREM

If f(x) is reciprocal equation of degree n then $f(x) = \pm x^n f(1/x)$

Proof

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...+ a_n$ be reciprocal equation of degree n.

a) If equation is of class one then $a_i = a_{n-i}$, then

$$f(x) = x^n\left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_n}{x^n}\right) \quad ...(1)$$

By replacing a_i by a_{n-i} in above equation (1)we get

$$\Rightarrow f(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + ... + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \quad ...(2)$$

$$\text{As } f\left(\frac{1}{x}\right) = \frac{a_o}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + ... + a_n \quad ...(3)$$

So, from (2) and (3) equations we get, $f(x) = x^n f(1/x)$

b) If Equation is of class two then $a_i = -a_{n-i}$, then

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

$$f(x) = -x^n\left(-a_0 - \frac{a_1}{x} - \frac{a_2}{x^2}........\frac{a_n}{x^n}\right) \quad ...(1)$$

By replacing a_i by - a_{n-i} in equation (1) we get

$$f(x) = -x^n\left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + ... + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)$$

$$\text{As } f\left(\frac{1}{x}\right) = \frac{a_o}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + ... + a_n$$

So,from above two equation $f(x) = -x^n f(1/x)$

So, from (a) & (b), if f(x) is reciprocal equation of either class one or of class two then $f(x) = \pm x^n f(1/x)$

Solving reciprocal equation

1) If reciprocal equation of even degree (i.e. 2n) then divide equation by x^n and then

a) If equation is of class one then put $x+1/x=t$

b) If equation is of class two then put $x–1/x=t$

2) If reciprocal equation of odd degree (i.e. $2n+1$), then

a) If equation is of class one then divide the equation by $x+1$, then quotient of equation will be of degree $2n$, which can be solved by previous method (i.e (a) or (b)).

b) If equation is of class two then divide the equation by $x–1$ then quotient of equation is of degree $2n$, which can be solved by previous methods.

Rational root theorem

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$ is any polynomial equation, (where $a_0, a_1, ...a_n \in$ Int) having $\frac{p}{q}$ as its rational root (p&q are coprime) then p is factor of a_n & q is factor of a_0.

Proof

As $\frac{p}{q}$ is factor of $f(x)=0$

So, $a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + a_2\left(\frac{p}{q}\right)^{n-2} + ... + a_n = 0$

$$a_0p^n + a_1p^{n-1}q + a_2p^{n-2}q^2 + ... + a_nq^n = 0 \qquad ...(1)$$

Equation (1) can be written as

$$a_0p^n = -q\left(a_1p^{n-1} + a_2p^{n-2}q + ... + a_nq^{n-1}\right) \qquad ...(2)$$

As both factors of LHS & RHS are integers, so q is factor of a_0p^n. But also p and q are coprime, so **q is factor of a_0**.

Equation (1) can be written as

$$a_nq^n = -p\left(a_0p^{n-1} + a_1p^{n-2}q + a_2p^{n-3}q^2 + ... + a_{n-1}q^{n-1}\right) ...(3)$$

As both factors of LHS & RHS are integers, so p is factor of a_nq^n. But as p & q are coprime, so **p is factor of a_n.**

Hence Proved

Note

1. Every rational root of equation

$x^n + a_1x^{n-1} + a_2x^{n-2} + ...a_n = 0$ (where a_i's are integers) must be integer. And every rational root must be divisor of a_n.

2. In $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...a_n = 0$. *If a_0 & a_n are large values then there can be many possibilities of rational roots, which is not an easy task to check all of them whether they are roots or not. So we can reduce these number of possible rational roots by the help of following methods.*

1) **Graphical method :** Plot the curve of given polynomial equation, from there we can determine approximately that value where curve cut x-axis.

Then by rational root theorem we can check only those value which is nearly to the values of x where curve cut x axis.

2) By descarte's rule we can eliminate some of the rational rules which can't be roots of equation.

3) **Lower & Upper bound theorem :** By this theorem we can minimise possible number of rational roots.

To determine lower bound select that value from possible rational roots (let a) such that all roots lesser than that value (a) gives same sign for f(x).

To determine upper bound select that value from possible rational roots of f(x) (say b) such that all roots greater than that value (b) gives same sign for f(x).

So, from possible rational roots we don't have to check that values which are lesser than 'a' or greater than 'b'.

General strategy of solving polynomial equation

Following steps has to follow to find rational roots of a polynomial equation.

1) By using rational root theorem, determine all the possible rational roots.

2) By using descarte's rule determine how many positive and negative real roots of equation may have. (In some cases possible positive or negative rational roots may be eliminated).

3) By using synthetic root division check the possible positive & negative rational roots.

4) By graphing utility we can narrow down the number of possible choices.

5) By upper & lower bound method narrow down the number of possible choices of number of rational roots.

6) If a is root of equation, then find quotient by dividing the equation by (x – a). Second factor can be determined from quotient by synthetic division.

7) Similarly we can determine other rational roots.

Common roots in polynomial equation

If $\phi(x)$ is highest common factor of f(x) and g(x), then roots of $\phi(x) = 0$ are common roots of f(x) = 0 and g(x) = 0.

Method of subtraction to determine common roots

If α is common root of f(x) = 0 & g(x) = 0 then α is also root of f(x) – g(x) = 0, but all roots of f(x) – g(x) = 0 are not common roots of f(x) = 0 & g(x) = 0

Proof

Let α is common root of $f(x)=0$ & $g(x)=0$ & β is any real number such that f(x) and g(x) has same value at $x = \beta$.

$f(\alpha) = 0$ & $g(\alpha) = 0$

$f(\beta) = g(\beta) = k$(let)

So, $f(\alpha) - g(\alpha) = 0$ & $f(\beta) - g(\beta) = 0$

So, α&β are roots of $f(x) - g(x) = 0$, but out of these only α is common root of $f(x) = 0$ and $g(x) = 0$.

Note

Before subtracting two equations i.e before determining $f(x) - g(x) = 0$, make coefficient of highest power of x in both $f(x) = 0$ & $g(x) = 0$ to be same.

If degree of $f(x) = 0$ and $g(x) = 0$ are not same, then before finding $f(x) - g(x) = 0$ make degree of f(x) & g(x) to be same, by multiplying respective power of x.

Multiple roots in polynomial equation

If f(x) contains a factor $(x-\alpha)^r$, then $f^I(x)$ contains a factor $(x-\alpha)^{r-1}$. So, if f(x) and $f'(x)$ has common root then f(x) has multiple root.

If α is r-multiple root of f(x). Whose degree is n, then

$$f(x) = (x-\alpha)^r \phi(x) \qquad ...(1)$$

Where $\phi(x)$ is polynomial of degree n-r, which does not have α as its root.

$$f'(x) = (x-\alpha)^r \phi'(x) + r(x-\alpha)^{r-1}\phi(x)$$

$$= (x-\alpha)^{r-1}[r\phi(x) + (x-\alpha)\phi^I(x)]$$

$$f'(x) = (x-\alpha)^{r-1} g(x) \qquad ...(2)$$

where g(x) is polynomial of degree (n–r) which does not have α as its root.

So, α is (r–1) multiple root of $f^I(x)$.

$\therefore$ If α is r multiple root of $f(x)=0$, then α is

(r–1) multiple root of $f^I(x)=0$, and (r–2) multiple root of $f^{II}(x)=0$ and so on.

Intermediate value theorem for polynomials

If f(x) is a polynomial such that $f(a) \neq f(b)$ and f(x) takes every value of x lie between a and b.

1) If f(a) & f(b) are of opposite sign then there will be odd number of roots of $f(x)=0$ between [a,b] (or atleast one root lie between [a,b]).
2) If f(a) and f(b) are of same sign then there will be even number of roots of $f(x)=0$ between [a,b] (minimum zero roots between [a,b]).
3) Every equation of an odd degree has atleast one real root whose sign is opposite to that of its last term, provided that the leading coefficient is positive.

   $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

   Put $x = \infty, 0, -\infty$ successively then

   $F(\infty) = \infty$, $f(0) = a_n$, $f(-\infty) = -\infty$

   If a_n is positive then $f(x) = 0$ has a root lying between 0 and $-\infty$ and if a_n is negative then $f(x)=0$ has a root lying between 0 & ∞.

Note

For sufficiently large values of x, f(x) has same sign as a_0.

4) Every equation which is of even degree whose absolute term is negative and leading coefficient is positive has atleast two real roots of opposite sign.

   Put $x=0, \infty, -\infty$ in $f(x)=0$ successively then

   $f(\infty) = \infty$, $f(0) = a_n$, $f(-\infty) = \infty$

   As a_n is negative , so one root of $f(x)=0$ lying between $(-\infty, 0)$ & other root lying between $(0, \infty)$.
5) If equation contain only even powers of x, and all coefficients are of same sign then the equation does not have any real root(or all roots are complex).
6) If equation contain only odd powers of x, and all coefficients are of same sign, then the equation has only one real root i.e. $x = 0$.

Rolle's theorem for polynomial

If f(x) is a continuous function at $[\alpha,\beta]$and differentiable polynomial function between (α,β), where α,β are roots of $f(x) = 0$.

Then f(x) satisfies all the conditions of rolle's theorem, then there is atleast one number $\gamma \in (\alpha,\beta)$ such that $f^I(\gamma) = 0$ or in other words $x = \gamma$ is root of $f^I(x) = 0$

Note

Between any two roots of polynomial equation $f(x) = 0$, there is always a root of $f'(x) = 0$

1) If all the roots of $f(x) = 0$ are real and distinct then all the roots of $f^I(x) = 0$ are also real and its roots are different from roots of $f(x) = 0$.
2) Maximum one root of $f(x)=0$ lies between two consecutive roots of $f^I(x)=0$.
3) If $f'(x) = 0$ has r real roots, then $f(x)=0$, cannot have more than $(r+1)$ real roots.
4) If $f^r(x) = 0$ is equation obtained by r^{th} derivative of f(x), and this equation has p imaginary roots then $f(x) = 0$ also has atleast p imaginary roots.
5) If all the roots $\beta_1, \beta_2, ...$of $f^I(x)=0$ are known, we can find the number of real roots of $f(x)=0$ by considering

sign of $f(\beta_1)$, $f(\beta_2)$....... A single root of $f(x)=0$ or no root of $f(x)=0$ lies between β_1,&β_2 according as $f(\beta_1)$ & $f(\beta_2)$ are of opposite sign or same sign respectively.

Nature of roots of cubic polynomial

Let $f(x) = ax^3 + bx^2 + cx + d$ (where $a,b,c \in R$) and $a>0$.

$f'(x) = 3ax^2 + 2bx + c$

Its discriminant is, $D = 4(b^2-3ac)$

Nature of roots of $f(x) = 0$ will depend upon discriminant of $f'(x) = 0$

1) If $D < 0$

As coefficient of x^2 in $f^{|}(x) = 0$ is positive and $D<0$, so $f^{|}(x)$ is always positive.

$f^{|}(x) > 0$

So, f(x) is strictly increasing, so f(x) cut x-axis at only one point and so only one real root of $f(x)=0$ exist.

2) If $D > 0$

So, $f^{|}(x) = 0$ has two real roots.

Let roots of $f^{|}(x) = 0$ be α & β

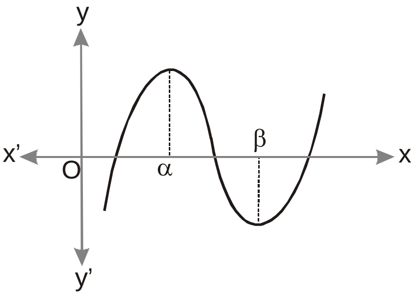
$\therefore f^{|}(x) = 3a(x-\alpha)(x-\beta)$

So $f^{|}(x)<0$ if $x \in (\alpha,\beta)$ and so in this interval f(x) is decreasing,while $f'(x) > 0$ if $x \in (-\infty,\alpha) \cup (\beta,\infty)$ and so in this interval f(x) is increasing.

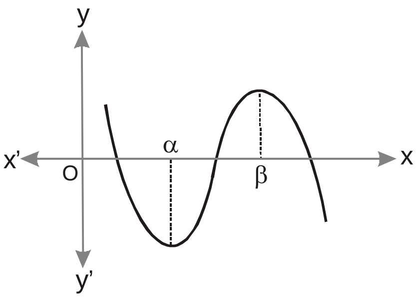
a) If $f(\alpha)f(\beta)<0$

In this case $f(x)=0$ has 3 real and distinct roots.

For $a>0$



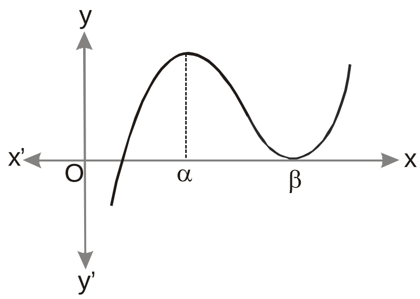
For $a<0$



b) If $f(\alpha)f(\beta)=0$

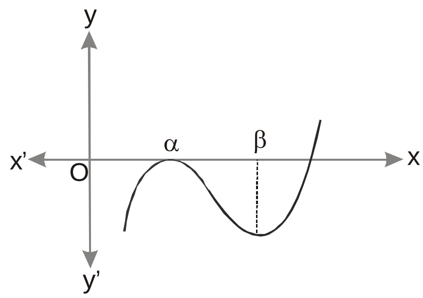
In this case $f(x)=0$ has 3 real roots out of which two are equal roots.

For $a>0$



2) If $D > 0$

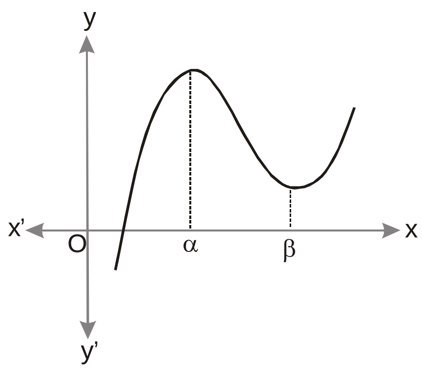
For $a<0$



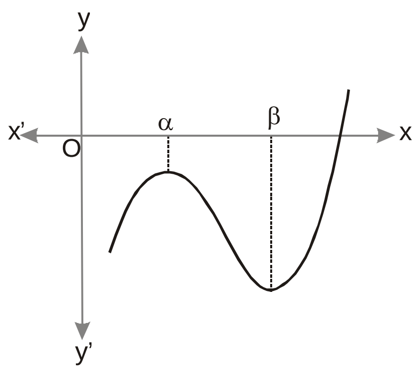
c) If $f(\alpha)f(\beta)>0$

In this case $f(x)=0$ has only one real root or $f(x)=0$ has two complex roots.

For $a>0$



For $a<0$



Algebra

3) If D = 0

It says that $f'(x)$ will become perfect square.

$$f'(x) = 3(x-\alpha)^2 \quad (\alpha = \beta)$$

$$f(x) = (x-\alpha)^3 + \lambda$$

a) If $\lambda = 0$ then f(x) = 0 has 3 equal real roots.

b) If $\lambda \neq 0$ then f(x) = 0 has atleast one real root.

Solved Examples

78. Find the character of the roots of $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 1 = 0$

Sol. $f'(x) = 12x^3 - 24x^2 - 12x + 24 = 12(x^2-1)(x-2)$

The roots of $f'(x) = 0$ are –1, 1, 2

When $\begin{array}{lcccc} x = -\infty & -1 & 1 & 2 & \infty \\ f(x) = \infty & - & + & + & \infty \end{array}$

Therefore f(x) = 0 has two real roots, one lying between $-\infty$ and –1 and the other between –1 and1.
The other two roots are imaginary

79. Show that the equation

$x^2 - xy + y^2 = 4(x+y-4)$ will not be satisfied for any real value of x and y except x = 4 and y = 4

Sol. Rewriting the given equation as a quadratic in y

$$y^2 - y(x+4) + (x^2+4x+16) = 0$$

Since y is real, discriminant ≥ 0

$$(x+4)^2 - 4(x^2-4x+16) \geq 0$$

$$\Rightarrow -3x^2 + 24x - 48 \geq 0$$

$$\Rightarrow x^2 - 8x + 16 \leq 0 \ \Rightarrow (x-4)^2 \leq 0$$

Which is possible only if x = 4
On putting x = 4 in given equation we get y = 4

80. Without actual division prove that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

Sol. Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2$ be the given polynomials. Then

$$g(x) = x^2 - 3x + 2 = (x-1)(x-2)$$

In order to prove that f(x) is exactly divisible by & g(x), it is sufficient to prove that x–1 and x –2 are factors of f(x). For this it is sufficient to prove that f(1) = 0 and f(2) = 0

Now $f(1) = 2\times1^4 - 6\times1^3 + 3\times1^2 + 3\times1 - 2 = 0$

and $f(2) = 2\times2^2 - 6\times2^3\times2^2 + 3\times2 - 2 = 0$

$\Rightarrow (x-1)$ and $(x-2)$ are factors of f(x).

Hence f(x) is exactly divisible by g(x)

81. Solve in R the equation

$$2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + \dots + 2x + 3 = 0$$

Sol. $2x^{99} + 3x^{98} + 2x^{97} + 3x^{96} + \dots + 2x + 3$

$$= (2x+3)(x^{98} + x^{96} + x^{94} + \dots + 1)$$

$$= (2x+3)\frac{(x^{100}-1)}{x^2-1}$$

The equation $x^{100} - 1 = 0$ has only two real roots, namely ±1 which are not acceptable. Therefore the given equation has only one real root, namely –3/2.

82. If $\alpha \,\&\, \beta$ are the roots of the equation $x^2 - ax + b = 0$ and $v_n = \alpha^n + \beta^n$. Show that $v_{n+1} = av_n - bv_{n-1}$ and hence obtain the value of $\alpha^5 + \beta^5$.

Sol. $\alpha + \beta = a;\ \alpha\beta = b;\ v_n = \alpha^n + \beta^n$

$$v_{n+1} = \alpha^{n+1} + \beta^{n+1}$$

$$= (\alpha+\beta)(\alpha^n + \beta^n) - \alpha\beta^n - \beta\alpha^n$$

$$= av_n - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$$

Now

$$\alpha^5 + \beta^5 = v_5 = av_4 - bv_3 = a(av_3 - bv_2) - bv_3$$

$$= (a^2-b)v_3 - abv_2$$

$$= (a^2-b)[av_2 - bv_1] - abv_2$$

$$= (a(a^2-b) - ab)v_2 - b(a^2-b)v_1$$

$$= (a^3-2ab)(a^2-2b) - ab(a^2-b)$$

$$= a^5 - 2a^3b - 2a^3b + 4ab^2 - a^3b + ab^2$$

$$= a^5 - 5a^3b + 5ab^2$$

83. Let f(x) be a polynomial leaving the remainder A when divided by x – a and the remainder B when divided by x – b ($a \neq b$). Find the remainder left by this polynomial when divided by (x – a) (x – b).

Sol. Since the product (x – a) (x – b) is a second degree trinomial when divided by it, the polynomial f(x) will necessarily leave a remainder which is a first degree polynomial in x, let $\alpha x + \beta$. Thus, there exists the following identity f(x) = (x – a) (x – b) Q(x)$+\alpha x+\beta$.
It only remains to determine α and β. Putting in this identity first x = a and then x = b, we get
f(a) = $\alpha a + \beta$, f(b) = $\alpha b + \beta$.

But according to question $a\alpha + \beta = A \,\&\, b\alpha + \beta = B$

$$\therefore \alpha = \frac{1}{a-b}(A-B).\ \beta = \frac{aB - bA}{a-b}$$

84. Let [a] denote the greatet integer less than or equal to a. Given that the quadratic equation $x^2 + [a^2 - 5a + b + 4]\, x + b = 0$ has roots – 5 and 1, find 'a'

Sol. Since – 5 and 1 are the roots of the equation.

$x^2 + [a^2 - 5a + b + 4]\, x + b = 0$

$\Rightarrow [a^2 - 5a + b + 4] = 4$ and $b = -5$

$\Rightarrow [a^2 - 5a - 1] = 4 \quad \Rightarrow \quad 4 \le a^2 - 5a - 1 < 5$

$\Rightarrow a^2 - 5a - 5 \ge 0$ and $a^2 - 5a - 6 < 0$

$$\Rightarrow \left(a \le \frac{5-3\sqrt{5}}{2} \text{ or } a \ge \frac{5+3\sqrt{5}}{2}\right)$$

and $-1 < a < 6$

$$\therefore\ a \in \left(-1, \frac{5-3\sqrt{5}}{2}\right] \cup \left[\frac{5+3\sqrt{5}}{2}, 6\right)$$

85. Solve for "x".

$1! + 2! + 3! + \ldots.. + (x-1)! + x! = k^2$ and $k \in I$

Sol. For $x < 4$, the given equation has the only solutions $x = 1, k = \pm 1$ and $x = 3, k = \pm 3$. Now let us prove that there are no solutions for $x \ge 4$. The expressions.

$$\left.\begin{array}{ll} 1!+2!+3!+4! & =33 \\ 1!+2!+3!+4!+5! & =153 \\ 1!+2!+3!+4!+5!+6! & =873 \\ 1!+2!+3!+4!+5!+6!+7! & =5913 \end{array}\right\} \text{ ends with the digit 3}$$

Now for $x \ge 4$ the last digit of the sum $1! + 2! + \ldots\ldots + x!$ is equal to 3 and therefore this sum can not be equal to a square of a whole number k (because a square of a whole number can not end with 3).

86. Let $a_0, a_1, \ldots\ldots, a_{n-1}$ be real numbers where $n \ge 1$ and let $f(x) = x^n + a_{n-1}x^n + \ldots\ldots + a_0$ be such that $|f(0)| = f(1)$ and each root of $f(x) = 0$ is real and between 0 and 1. Prove that the product of the roots doesn't exceed $\frac{1}{2^n}$.

Sol. Let $f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$

where $\alpha_1, \alpha_2, \ldots.. \alpha_n$ are the roots of $f(x) = 0$

Since $|f(0)| = f(1)$

$$\Rightarrow \alpha_1 . \alpha_2 . \alpha_3 \ldots.. \alpha_n = (1-\alpha_2)(1-\alpha_2)\ldots..(1-\alpha_n)$$

$$\Rightarrow (\alpha_1.\alpha_2.\alpha_3\ldots\alpha_n)^2 = \alpha_1(1-\alpha_1)\,\alpha_2(1-\alpha_2)\ldots\alpha_n(1-\alpha_n)$$

$$\Rightarrow (\Pi\alpha_i)^2 = \Pi\alpha_i\,(1-\alpha_i)\,;\ i = 1,2,\ldots.n$$

Now,

$$(\Pi\alpha_i)^2 = \Pi\alpha_i\,(1-\alpha_i) \le \Pi\left(\frac{\alpha_i + (1-\alpha_i)}{2}\right)^2 = \frac{1}{2^{2n}}$$

(since G.M. $\le$ A.M.)

87. Prove that the equation $x^6 + 2x^3 + 5 + ax^3 + a = 0$ has atmost two real roots for all values of $a \in R - \{-5\}$.

Sol. The given expression

$$(x^3+1)^2 + a(x^3+1) + 4 = 0$$

If discriminant of the above equation is less than zero i.e. $D < 0$

Then we have six complex roots and no real roots.

If $D \ge 0$, $x^2 + 1 = t$, then the equation reduces to

$$f(t) = t^2 + at + 4 = 0$$

we will get two real roots and other roots will be complex except when $t = 1$ is one of the roots

$$\Rightarrow f(1) = 0 \Rightarrow a = -5$$

88. Prove that if a polynomial $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ with integral coefficients assumes the value 7 for four integral values of x then it cannot take the value 14 for any integral value of x.

Sol. Let the polynomial P(x) take on the value 7 at the points x = a, x = b, x = c and x = d. Then a, b, c and d are four integral roots of the equation $P(x) - 7 = 0$. This means that the polynomial $P(x) - 7$ is divisible by x–a, x–b, x–c and x–d, that is

$$P(x) - 7 = (x-a)(x-b)(x-c)(x-d)q(x)$$

where $q(x)$ may be equal to 1.

Now let us suppose that the polynomial P(x) assumes the value 14 for an integral value x = A. On substituting x = A into the last equaility we obtain

$$7 = (A-a)(A-b)(A-c)(A-d)q(A)$$

which is impossible because the integral numbers A–a, A–b, A–c and A–d are all distinct and the number 7 cannot be factored into five integers among which at least four are different.

89. If $x^4 - 14x^2 + 24x - k = 0$ has four real and unequal roots, prove that k must lie between 8 and 11.

Sol. Let $f(x) = x^4 - 14x^2 + 24x - k$ Then

$$f'(x) = 4x^3 - 28x + 24 = 4(x-1)(x-2)(x+3)$$

$\therefore f'(x) = 0$ has the roots –3, 1 , 2

As $f'(x)$ has 3 distinct roots so f(x) has four distinct roots if f(–3), f(1), f(2) are of alternately.

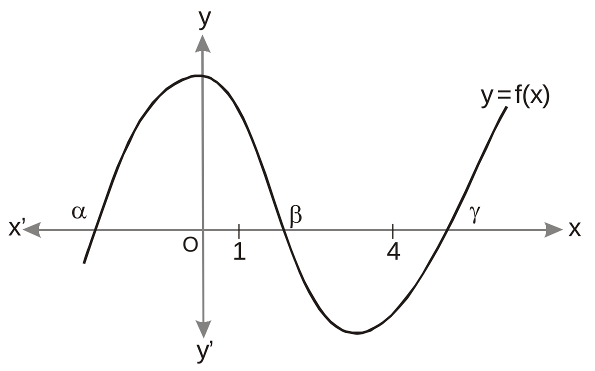
i.e. $-117-k<0$, $11-k > 0$ and $8-k<0$, or $k>-117$, $k<11$, $k>8$. Therefore $8<k<11$.

90. Given the cubic equation $x^3 - 2kx^2 - 4kx + k^2 = 0$. If one of the root is less than 1, other root is in the interval (1, 4) and the third root is greater than 4, then the value of k lies in the internal $\left(a + \sqrt{b}, b\left(a + \sqrt{b}\right)\right)$ where $a, b \in N$. Find the value of a and b.

Sol. $f(x) = x^3 - 2kx^2 - 4kx + k^2 = 0$

Note that $f(0) = k^2 > 0$

$f(1) > 0 \Rightarrow 1 - 2k - 4k + k^2 > 0$



$\left[k-\left(3+2\sqrt{2}\right)\right]\left[k-\left(3-2\sqrt{2}\right)\right]>0$ .... (1)

Also f(4) < 0

$\Rightarrow 64-32k-16k+k^2<0$

$(k-24)^2<512$

$\left[k-8\left(3-2\sqrt{2}\right)\right]\left[k-8\left(3+2\sqrt{2}\right)\right]<0$ ....(2)

(1) $\cap$ (2) $\Rightarrow 3+2\sqrt{2}<k<8\left(3+2\sqrt{2}\right)$

$3+\sqrt{8}<k<8\left(3+\sqrt{8}\right)$

$\therefore a=3; b=8$

91. Let a, b, c be positive integers and consider all the quadratic equations of the form $ax^2 - bx + c = 0$ which have two distinct real roots in the open interval $0 < x < 1$. Find the least positive integer a for which such a quadratic equation exists.

Sol. Let α, β be two distinct real roots of $f(x) = ax^2 - bx + c = 0$ lying in (0, 1).
Then $f(x) = a(x-\alpha)(x-\beta)$
Now $f(0)f(1) = a^2\alpha(1-\alpha)\beta(1-\beta)$
As $0<\alpha<1$, $0<\alpha(1-\alpha)\le 1/4$, with equality holding for $\alpha = 1/2$. Since $0<\alpha, \beta<1$ and $\alpha\neq\beta$.

$$0<\alpha(1-\alpha)\beta(1-\beta)<\frac{1}{16}$$

$$\Rightarrow 0<f(0)f(1)<\frac{a^2}{16} \quad \text{... (1)}$$

As a, b, c are positive integers and $c(a-b+c)>0$
$f(0)f(1) = c(a-b+c)\ge 1$ ... (2)
$[\because f(0)f(1) > 0]$

(1) and (2) imply $\frac{a^2}{16}>1$, i.e. $a\ge 5$. Since the roots of $f(x) = 0$ are real and distinct, its discriminant $= b^2 - 4ac > 0$.

$\Rightarrow b^2 > 4ac \ge 20$ $[\because c\ge 1]$

Hence the minimum possible value of b is 5. Let us try the least values of a, b and c, that is, a = 5, b = 5 and c = 1. It is easy to check that $5x^2 - 5x + 1 = 0$ has two distinct real roots lying between 0 and 1. Thus, the least positive integral value of a is 5.

92. Solve completely the equation

$$x^4-5x^3+11x^2-13x+6=0$$

using the fact that two of its roots α and β are connected by the relation $3\alpha+2\beta=7$

Sol. From the given relation $\beta=\frac{7-3\alpha}{2}$

Replacing x by $\frac{7-3x}{2}$ in

$f(x)=x^4-5x^3+11x^2-13x+6$

we get

$$\left(\frac{7-3x}{2}\right)^4-5\left(\frac{7-3x}{2}\right)^3+11\left(\frac{7-3x}{2}\right)^2-13\frac{7-3x}{2}+6$$

Let it reduce to

$\phi(x)=81x^4-486x^3+1152x^2-1242x+495$

The H.C.D. of f(x) and $\phi(x)$ is x–1.

$\therefore \alpha=1$ and $\beta=\frac{7-3}{2}=2$

Let the other roots of f(x) = 0 by γ and δ.
By the relation between roots and coefficients,
$1+2+\gamma+\delta=5$ or $\gamma+\delta=2$
and $1.2.\gamma.\delta=6$ or $\gamma\delta=3$

Solving a quadratic equation we get $\gamma=1+i\sqrt{2}$ and $\delta=1-i\sqrt{2}$

Hence the roots are $1, 2, 1\pm i\sqrt{2}$

93. For what real values of the parameter a, does the equation $x^4 + 2ax^3 + x^2 + 2ax + 1 = 0$ have at least two distinct negative roots.

Sol. We can see that zero cannot be a root, therefore the given equation can be written as

$$x^2+\frac{1}{x^2}+2a\left(x+\frac{1}{x}\right)+1=0$$

i.e., $\left(x+\frac{1}{x}\right)^2+2a\left(x+\frac{1}{x}\right)-1=0$

gives $x+\frac{1}{x}=\frac{-2a\pm\sqrt{4a^2+4}}{2}=-a\pm\sqrt{a^2+1}$

i.e, $x^2-\lambda x+1=0$ where $\lambda=-a\pm\sqrt{a^2+1}$
Let $f(x) = x^2-\lambda x+1$
According to the given condition, both the roots must be negative. Hence we have discriminant > 0

i.e., $\lambda^2>4$

i.e., $\lambda<-2$ or $\lambda>2$ ... (1)

and product of roots > 0
i.e. 1 > 0 ... (2)
which is true for every $a\in R$

and sum of roots < 0

i.e., $\lambda < 0$ ... (3)

Intersection of inequalities (1), (2) and (3), gives $\lambda < -2$

Since $-a + \sqrt{a^2+1}$ is positive for every a, hence, the possible values of a are given by the inequality.

$-a-\sqrt{a^2+1} < -2$

i.e., $a + \sqrt{a^2+1} > 2$

i.e., $a^2 + 1 > (2 - a)^2$

gives $a > \frac{3}{4}$

94. If $\alpha \in N$ and α satisfies the equation. If $x^2+ax+b+1=0$, where $a, b \neq -1$ are integers. Then prove that a^2+b^2 is a composite.

Sol. Let α and β be the two roots of the equation where $\alpha \in N$. Then

$\alpha+\beta=-a$ ...(1)

$\alpha.\beta = b+1$ ...(2)

$\therefore \beta=-a-\alpha$ is an integer. Also, since $b+1\neq 0, \beta\neq 0$.

From Eq (1) and Eq (2) We get

$a^2+b^2=(\alpha+\beta)^2+(\alpha\beta-1)^2$

$=\alpha^2+\beta^2+\alpha^2\beta^2+1=(1+\alpha^2)(1+\beta^2)$

Now, as $\alpha \in N$ and β is a non – zero integer, $1+\alpha^2>1$ and $1+\beta^2>1$.

Hence a^2+b^2 is composite number.

Make Concepts Clear 3.8

1. If the expression $3x^2+2pxy+2y^2+2ax-4y+1$ can be resolved into linear factors, then prove that p must be one of the root of $p^2+4ap+2a^2+6=0$
2. Show that $a^2x^4+bx^3+cx^2+dx+f^2$ will be perfect square if $ad=\pm bf$ & $4a^2c-b^2=\pm 8a^3f$
3. Show that $(x-1)^2$ is a factor of $x^n-nx+n-1$
4. Find the integral roots of the equation $x^4-x^3-19x^2+49x-30=0$
5. If p(x) is a polynomial with integer coefficients and a,b,c are three distinct integers, then show that it is impossible to have p(a) = b, p(b) =c and p(c) = a.
6. Let u, v be two real numbers such that u, v and uv are roots of a cubic polynomial with rational coefficients. Prove that uv is rational.
7. If a, b, c, d are non - negative real number such that a + b + c + d = 1 then prove that $ab + bc + cd \leq \frac{1}{4}$
8. If $ax^2+by^2+cz^2+2ayz+2bxz+2cxy$ is resolvable into linear factors, then show that a + b + c = 0 or a = b = c.
9. Show that the polynomial $x(x^{n-1}-na^{n-1})+a^n(n-1)$ is divisible by $(x-a)^2$
10. Prove that there does not exist a natural number n whose product of digits is $n^3 - 25n^2 + 151n$.

Answers

4. 1,2,3,-5

Solutions Are On Page No. 3.78

Practice Session-1 for JEE MAIN

Single Choice

1. If b > a, then the equation (x – a) (x – b) – 1 = 0 has
A) Both roots in (a, b)
B) Both roots in $(-\infty, a)$
C) Both roots in $(b, +\infty)$
D) One root in $(-\infty, a)$ and the other in $(b, +\infty)$

2. If $0 \le x \le 1000$ and $\left[\frac{x}{2}\right]+\left[\frac{x}{3}\right]+\left[\frac{x}{5}\right]=\frac{31}{30}x$, where [x] denotes greatest integer function, then possible number of values of x is
A) 31 B) 32 C) 33 D) 34

3. The minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+x^3+\frac{1}{x^3}}$ for x > 0 is
A) 2 B) 4 C) 6 D) 10

4. In a triangle P Q R. $\angle R = \frac{\pi}{4}$ If $\tan\left(\frac{p}{3}\right)$ and $\tan\left(\frac{Q}{3}\right)$ are the roots of the equation $ax^2 + bx + c = 0$, then
A) a + b = c B) b + c = 0
C) a + c = b D) b = c

5. The set of exhaustive values of a for which the equation $|ax - 2| = 2x^2 + ax + 4$ has at least one positive roots is
A) $(-\infty, 0]$ B) $(-\infty, -2]$
C) $(-\infty, -2] \cup [2, \infty)$ D) $[2, \infty)$

6. If roots of equation $x^2 - 5x + 16 = 0$ are α, β and roots of equation $x^2 + px + q = 0$ are $\alpha^2 + \beta^2$ and $\frac{\alpha\beta}{2}$, then
A) p = 1 and q = – 56 B) p = – 1 and q = – 56
C) p = 1 and q = 56 D) p = – 1 and q = 56

7. The product of the real roots of the equation
$x^2 + 18x + 30 = 2\sqrt{x^2+18x+45}$ is
A) 10 B) 20
C) 10 D) None of these

8. If one root of equation $x^2 - 4ax + a + f(a) = 0$ is three times of the other then minimum value of f(a) is
A) $-\frac{1}{6}$ B) $-\frac{1}{10}$ C) $-\frac{1}{5}$ D) $-\frac{1}{12}$

9. The value of $\sqrt{1+\sqrt{7+\sqrt{1+\sqrt{7+.........\infty}}}}$ is
A) 2 B) 1 C) 0 D) $\frac{3}{2}$

10. The equation $a_8x^8 + a_7x^7 + a_6x^6 + ... + a_0 = 0$ has all its roots positive and real (where a_8 = 1, a_7 = – 4, $a_0 = \frac{1}{2^8}$), then
A) $a_1 = \frac{1}{2^8}$ B) $a_1 = \frac{1}{2^4}$
C) $a_2 = \frac{7}{2^4}$ D) $a_2 = \frac{7}{2^4}$

11. Suppose a cubic polynomial $f(x) = x^3 + px^2 + qx + 72$ is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ (where a, b, p, q are constants and ($a \ne b$). Find the sum of the squares of the roots of the cubic polynomial.
A) 146 B) 144 C) 140 D) 136

12. The value of '|x|' satisfying the equation
$x^4 - 2\left(x\sin\left(\frac{\pi}{2}x\right)\right)^2 + 1 = 0$
A) 1 B) 2
C) 0 D) No value of 'x'

13. If x is real, then the expression $\frac{x^2-bc}{2x-b-c}$ has no value lying between
A) 0 and b B) b and c
C) b and – c D) – b and – c

14. If the inequality $\frac{mx^2+3x+4}{x^2+2x+2} < 5$ is satisfied $\forall x \in R$, then
A) $m < \frac{1}{2}$ B) $m < \frac{91}{3}$ C) $m < \frac{71}{24}$ D) $m > \frac{91}{3}$

15. If $x = 2 + 2^{2/3} + 2^{1/3}$ then the value of $x^3 - 6x^2 + 6x$ is
A) 3 B) 2 C) 1 D) – 2

16. If $\alpha, \beta, \gamma, \delta$ are roots of $x^4 + x^2 + 1 = 0$, then the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ & δ^2 is
A) $x^4 - x^2 + 1 = 0$ B) $x^4 + x^2 - 1 = 0$
C) $(x^2 + x + 1)^2 = 0$ D) $(x^2 - x + 1) = 0$

17. If $0 < a < b < c$, and the roots α, β of the equation $ax^2 + bx + c = 0$ are imaginary, then

A) $|\alpha| \neq |\beta|$ B) $|\alpha| > 1$
C) $|\beta| < 1$ D) $a = |\beta|$

18. Let $P(x) = x^2 + bx + c$, where b and c are integer. If $P(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, find the value of P(1).
A) 0 B) 2 C) 3 D) 4

19. Find the value of x satisfying the equation $\left|||x^2 - x + 4| - 2| - 3\right| = x^2 + x - 12.$

A) 11 B) 22 C) $\frac{11}{2}$ D) $\frac{11}{4}$

20. P(x) is a polynomial with integral coefficients such that for four distinct integers a, b, c, d, P(a) = P(b) = P(c) = P(d) = 3. If P(e) = 5 then e =
A) 1 B) 3
C) 4 D) No value of e

21. The value(s) of 'p' for which the equation $ax^2 - px + ab = 0$ and $x^2 - ax - bx + ab = 0$ may have a common root, given a, b are non zero real numbers, is
A) $a + b^2$ B) $a(1 + b)$
C) $ab(1 + b)$ D) $b(1 + a)$

22. If minimum $\{x^2 + (a - b)x + (1 - a - b)\}$ > maximum $(-x^2 + (a + b)x - (1 + a + b))$ Then range of $a^2 + b^2$ is

A) $[4, \infty)$ B) $(4, \infty)$ C) $[2, \infty)$ D) $[0, \infty)$

23. The value of m for which will $2x^2 + mxy + 3y^2 - 5y - 2$ can be split into two factors is
A) ± 7 B) ± 2 C) 0 D) ± 5

24. The equation $\left(\frac{10}{9}\right)^x = -3x^2 + 2x - \frac{9}{11}$ has
A) No solution B) Exactly one solution
C) Exactly two solution D) None of these

25. The number of real roots of quadratic equation $\sum_{k=1}^{n} (x - k)^2 = 0 \ (n > 1)$ is
A) 1 B) 2 C) n D) 0

26. If $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_4$ are the roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x\cos\beta - \sin\beta = 0$ then $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4)$
A) $\cot\beta$ B) $\sin\beta$ C) $\tan\beta$ D) $\cos\beta$

27. If $a + b + c > \frac{9c}{4}$ and equation $ax^2 + 2bx - 5c = 0$ has non–real complex roots, then
A) $a > 0, c > 0$ B) $a > 0, c < 0$
C) $a < 0, c < 0$ D) $a < 0, c > 0$

28. If $f(x) = x^3 + ax^2 + bx + c = 0$ has roots α, β, γ and a, b, c are real and if the roots of $x^3 + a_1x^2 + b_1x + c_1 = 0$ are $(\alpha - \beta)^2$, $(\beta - \gamma)^2$ and $(\gamma - \alpha)^2$ then $c_1 = 0$. Roots of $f(x) = 0$ are
A) Real and distinct
B) Such that at least two of them are equal
C) Such that two of them are non real
D) Real and equal

29. If $x^2 + 5 = 2x - 4\cos(a + bx)$, where $a, b \in (0, 5)$, is satisfied for atleast one real x, then the maximum value of a + b is equal to
A) 3π B) 2π
C) π D) None of these

30. The number of real roots of the equation $(x^2 + 2)^2 + 8x^2 = 6x(x^2 + 2)$ is
A) 0 B) 2 C) 4 D) 6

31. $5^x + \left(2\sqrt{3}\right)^{2x} - 169 \leq 0$ true in the interval

A) $(-\infty, 2)$ B) (2, 4)
C) $(2, \infty)$ D) (0, 4).

32. The equation $(a + 2)x^2 + (a - 3)x = 2a - 1$, $a \neq -2$ has rational roots for
A) All rational values of a except a = – 2
B) All real values of a except a = – 2
C) Rational values of $a > \frac{1}{2}$
D) None of these

33. f(x) and g(x) are quadratic polynomials and $|f(x)| \geq |g(x)|$, $\forall x \in R$. Also f (x) = 0 have real roots. Then number of distinct roots of equation $h(x)h''(x) + h'(x)^2 = 0$ are (where h (x) = f (x) g (x))
A) 0 B) 2 C) 3 D) 4

34. If $\frac{x^2 + ax + 3}{x^2 + x + a}$, takes all real values for possible real values of x, then
A) $4a^3 + 39 \geq 0$ B) $4a^3 + 39 < 0$
C) $a < \frac{1}{4}$ D) $a \geq \frac{1}{4}$

35. Sum of the rational roots of $x^5 = \frac{133x - 78}{133 - 78x}$ is

A) $\frac{2}{9}$ B) $\frac{9}{2}$ C) $\frac{13}{6}$ D) $\frac{6}{13}$

36. If r_1, r_2, r_3 are the radii of the escribed circles of a triangle ABC and r is the radius of its incircle, then the root (s) of the equation $x^2 - r(r_1r_2 + r_2r_3 + r_3r_1)x + r_1r_2r_3 - 1 = 0$
A) 1 B) $r_1 = r_2 + r_3$
C) r D) $r_1r_2r_3 - 1$

37. If $ax^2 + 2bx - 3c = 0$ has no real roots and $\frac{3c}{4} < a + b$, then the range of c is
A) (– 1, 1) B) (0, 1) C) $(0, \infty)$ D) $(-\infty, 0)$

38. Let x be a positive real number. Find the maximum possible value of the expression $y = \frac{x^2 + 2 - \sqrt{x^4 + 4}}{x}$.

A) $\left(\sqrt{2} + 1\right)$ B) $2\left(\sqrt{2} + 1\right)$
C) $2\left(\sqrt{2} - 1\right)$ D) $\left(\sqrt{2} - 1\right)$

39. Let f(x) be a function such that f(x) = x – [x], where [x] is the greatest integer less than or equal to x. Then the number of solutions of the equation $f(x)+f\left(\frac{1}{x}=1\right)$ is (are)

A) 0 B) 1 C) 2 D) Infinite

40. If both the roots of $(2a-4)\,9^x-(2a-3)\,3^x+1=0$ are non-negative, then

A) $0<a<2$ B) $2<a<\frac{5}{2}$

C) $a<\frac{5}{4}$ D) $a>3$

41. If $x_1>x_2>x_3$ and x_1, x_2, x_3 are roots of

$\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}$; (a, b, > 0) and

$x_1-x_2-x_3=c$, then a, c, b are in.

A) A.P. B) G.P. C) H.P. D) None

42. If $p(x)=ax^2+bx$ and $q(x)=lx^2+mx+n$ with p (1) = q (1) ; p (2) – q (2) = 1 and p (3) – q (3) = 4, then p (4) – q (4) is

A) 3 B)6 C) 9 D) 10

43. The number of irrational roots of the equation $(x^2-3x+1)(x^2+3x+2)(x^2-9x+20)=-30$ is

A) 0 B) 2 C) 4 D) 6

44. If $ax^2-bx+5=0$ does not have two distinct real roots, the minimum value of 5a + b

A) – 1 B) 0 C) – 2 D) None

45. The equation $(x\in R),\ \sqrt{x^2+1}-\frac{1}{\sqrt{x^2-\frac{5}{3}}}=x$

A) Has no root B) Exactly one root

C) Two roots D) Four roots

46. The number of real solutions of the system of equations

$x=\frac{2z^2}{1+z^2},\ y=\frac{2x^2}{1+x^2},\ z=\frac{2y^2}{1+y^2}$ is

A) 1 B) 2 C) 3 D) 4

47. If the graph of $y=ax^3+bx^2+cx+d$ is symmetric about the line x = k then the value of a + k is

A) c B) c^2-bd C) $-\frac{c}{2b}$ D) $-\frac{b}{2c}$

48. Let a , b be two distinct roots of $x^4+x^3-1=0$ and $p(x)=x^6+x^4+x^3-x^2-1$

A) ab is a root of p (x) = 0

B) a + b is a root of p (x) = 0

C) Both a + b and ab are roots of p (x) = 0

D) None of ab, a + b is a root of p (x) = 0

Comprehension Linked Passages

Passage 1

α, β are roots of $ax^2+bx+c=0$ and γ, δ are roots of $px^2+qx+r=0$ then answer the following questions.

49. If $\alpha, \beta, \gamma, \delta$ are in AP, then its common difference is

A) $\frac{1}{2}\left(\frac{b}{a}-\frac{q}{p}\right)$ B) $\frac{1}{4}\left(\frac{b}{a}-\frac{q}{p}\right)$

C) $\frac{1}{3}\left(\frac{c}{a}-\frac{q}{p}\right)$ D) $\frac{1}{2}\left(\frac{c}{a}-\frac{q}{p}\right)$

50. If $\alpha, \beta, \frac{1}{\gamma}, \frac{1}{\delta}$ are in AP, then $\frac{b^2-4ac}{q^2-4pr}$ is

A) $\frac{a^2}{p^2}$ B) $\frac{c^2}{p^2}$ C) $\frac{b^2}{p^2}$ D) $\frac{a^2}{r^2}$

51. If $\alpha, \beta, \gamma, \delta$ are in GP, then its common ratio is

A) $\left(\frac{ar}{cp}\right)^{\frac{1}{4}}$ B) $\left(\frac{cr}{ap}\right)^{\frac{1}{4}}$ C) $\left(\frac{ar}{cp}\right)^{\frac{1}{2}}$ D) $\left(\frac{cr}{ap}\right)^{\frac{1}{2}}$

Passage 2

If α, β, γ be the roots of the equation $ax^3+bx^2+cx+d=0$. To obtain the equation whose roots are $f(\alpha), f(\beta), f(\gamma)$, where f is a function, we put $y=f(\alpha)$ and obtain $\alpha=f^{-1}(y)$.

For example, if α, β, γ are the roots of $ax^3+bx^2+cx+d=0$. To find equation whose roots are $\alpha^2, \beta^2, \gamma^2$, we put $y=\alpha^2 \Rightarrow \alpha=\sqrt{y}$. As α is a root of $ax^3+bx^2+cx+d=0$.

We get $ay^{3/2}+by+c\sqrt{y}+d=0$ or $\sqrt{y}\,(ay+c)=-(by+d)$

On squaring both sides, we get $y(a^2y^2+2acy+c^2)=b^2y^2+2bdy+d^2$

or $a^2y^3+(2ac-b^2)\,y^2+(c^2-2bdy-d^2=0$.

This is desired equation

On the basis of above information, answer the following questions :

52. If α, β are the roots of the equation $ax^2+bx+c=0$, then the roots of the equation $a(x+1)^2-b(x+1)(x-2)+c(x-2)^2=0$ are

A) $\frac{2\alpha+1}{\alpha-1}, \frac{2\beta+1}{\beta-1}$ B) $\frac{2\alpha-1}{\alpha+1}, \frac{2\beta-1}{\beta+1}$

C) $\frac{\alpha+1}{\alpha-2}, \frac{\beta+1}{\beta-2}$ D) $\frac{2\alpha+3}{\alpha-1}, \frac{2\beta+3}{\beta-1}$

53. If α, β are the roots of the equation $2x^2+4x-5=0$, the equation whose roots are the reciprocals of $2\alpha-3$ and $2\beta-3$ is

A) $x^2+10x-11=0$ B) $11x^2+10x+1=0$

C) $x^2+10x+11=0$ D) $11x^2-10x+1=0$

54. If α, β are the roots of the equation $px^2-qx+r=0$, then the equation whose roots are $\alpha^2+\frac{r}{p}$ and $\beta^2+\frac{r}{p}$ is

A) $p^3x^2+pq^2x+r=0$

B) $px^2-qx+r=0$

C) $p^3x^2-pq^2x+q^2r=0$

D) $px^2+qx-r=0$

Passage 3

Consider the equation $x^4+(1-2k)\,x^2+k^2-1=0$ where k is real. If x^2 is imaginary, or $x^2<0$, the equation has no real roots. If $x^2>0$ the equation has real roots

55. The equation has no real roots if $k\in$

A) $(-\infty,-1)\cup\left(\frac{5}{4},\infty\right)$ B) (–1, 1)

C) $\left(1,\frac{5}{4}\right)$ D) None of these

56. The equation has only two real roots if $k \in$

A) $(-\infty, -1)$ B) (0, 1) C) (1, 2) D) (– 1, 1)

57. The equation has only one or three distinct real roots if $x \in$

A) {0, 1} B) {– 1, 0, 1} C) {– 1, 1} D) {0}

Passage 4

Let a_m (m = 1, 2, ...,p) be the possible integral values of *a* for which the graphs of $f(x) = ax^2 + 2bx + b$ and $g(x) = 5x^2 - 3bx - a$ meets at same point for all real values of b.

Let $t_r = \prod_{m=1}^{p}(r - a_m)$ and $S_n = \sum_{r=1}^{n} t_r$, $n \in N$.

58. The minimum possible value of *a* is

A) $\frac{1}{5}$ B) $\frac{5}{26}$ C) $\frac{3}{38}$ D) $\frac{2}{43}$

59. The sum of values of n for which S_n vanishes is

A) 8 B) 9 C) 10 D) 15

60. The value of $\sum_{r=5}^{\infty} \frac{1}{t_r}$ is equal to

A) $\frac{1}{3}$ B) $\frac{1}{6}$ C) $\frac{1}{15}$ D) $\frac{1}{18}$

Practice Session-2 for JEE ADVANCED

Single Choice

1. The number of integral roots of the equation $x^4 + \sqrt{x^4 + 20} = 22$ is

A) 0 B) 2 C) 4 D) 8

2. The number of triplets (x, y, z) that satisfy the following system of equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1;\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$

$\frac{-x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is equal to

A) 8 B) 6 C) 1 D)Infinite

3. The sum of the roots of the equation $2^{33x-2}+2^{11x+2}=2^{22x+1}+1$ is

A) $\frac{1}{11}$ B) $\frac{2}{11}$ C) $\frac{3}{11}$ D) $\frac{4}{11}$

4. Let a, b, p, q $\in$ Q and suppose that $f(x) = x^2 + ax + b = 0$, $g(x) = x^3 + px + q = 0$ have a common irrational root, then

A) f (x) divides g (x) B) g (x) $\equiv$ x f(x)
C) g (x) $\equiv$ (x – b – q) f (x) D) $g(x) = x^2 f(x) + 1$

5. If $\alpha, \beta, \gamma, \delta$ are roots of the equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$ then the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)(1+\delta^2)$ is

A) 9 B) 11 C) 13 D) 5

6. If each pair of the equation $x^2 + ax + b = 0$, $x^2 + bx + c = 0$ and $x^2 + cx + a = 0$ has common root, then product of all common root is

A) $\sqrt{abc}$ B) $2\sqrt{abc}$

C) $\sqrt{ab+bc+ca}$ D) $2\sqrt{ab+bc+ca}$

7. The number of values of triplets (a, b, c) for which $a \cos 2x + b \sin^2 x + c = 0$ is an identity

A) 0 B) 1 C) 2 D) Infinite

8. If a, b, c are in H.P., then the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$

A) has real and distinct roots
B) has equal roots and each is equal to one
C) has no real root
D) has 0 as a root

9. If α, β are roots of $x^2 + x + 1 = 0$, then the value of $\lim \sum_{r=1}^{n}\left(\alpha^r+\beta^r\right)$ is

A) 0 B) 1 C) 2 D) 3

10. If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are the roots $x^n + ax + b = 0$, then $(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)\ldots(\alpha_1-\alpha_n) =$

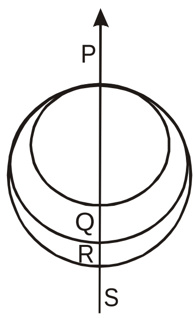
A) $n\alpha_1^{n-1}$ B) a

C) $n\alpha_1^{n-1}+a$ D) $n\alpha_1^{n-1}-a$

11. In a square matrix A of order 3, $a_{i,i}$'s are the sum of the roots of the equation $x^2 - (a+b)x + ab = 0$; $a_{i,i+1}$'s are the product of the roots, $a_{i,i-1}$'s are all unity and the rest of the elements are all zero. The value of the det. (A) is equal to

A) 0 B) $(a + b)^3$
C) $a^3 - b^3$ D) $(a^2 + b^2)(a + b)$

12. PQRS is a common diameter of three circles. The area of the middle circle is the average of the other two. If PQ = 2 and RS = 1, then the length of QR is



A) $\sqrt{6}+1$ B) $\sqrt{6}-1$ C) 5 D) 4

13. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5,100]$. Total number of different values of 'n' so that the given equation has integral roots is

A) 8 B) 3 C) 6 D) 4

14. If both the roots of $k(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k(2x^2 + 1) + px + 4x^2 - 2 = 0$ are common, then the value of 2r – p is

A) –1 B) 0 C) 1 D) 2

15. Let f be a continuous function defined on [– 2009, 2009] such that f(x) is irrational for each x $\in$ [– 2009, 2009] and $f(0) = 2+\sqrt{3}+\sqrt{5}$. The equation $f(2009)x^2 + 2f(0)x + f(2009) = 0$ has

A) Only rational roots
B) Only irrational roots
C) One rational and one irrational root

D) Imaginary roots

16. If both the roots of the equation $ax^2 + x + c - a = 0$ are imaginary and $c > -1$, then

A) $3a > 2 + 4c$ B) $3a < 2 + 4c$
C) $c < a$ D) $a > 0$

17. If α, β are the roots of the equation $6x^2 - 6x + 1 = 0$ then

$\frac{1}{2}\left(a+b\alpha+c\alpha^2+d\alpha^3\right)+\frac{1}{2}\left(a+b\beta+c\beta^2+d\beta^3\right) =$

A) $a + b + c + d$ B) $\frac{a}{1}+\frac{b}{2}+\frac{c}{3}+\frac{d}{4}$
C) $a + b + c - d$ D) $a + b - c - d$

18. Let $(a_1, a_2, a_3, a_4, a_5)$ denote a rearrangement of $(3, -5, 7, 4, -9)$, then the equation $a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0$ has

A) At least two real roots
B) All four real roots
C) Only imaginary roots
D) Two real and two imaginary roots

19. Find x if $4^x + 6^x = 9^x$

A) $\frac{\ln\left(\sqrt{5}-1\right)+\log 2}{\ln 2-\ln 3}$ B) $\frac{\ln\left(\sqrt{5}-1\right)+\ln 2}{\ln 2-\ln 3}$

C) $\frac{\ln\left(\sqrt{5}-1\right)-\ln 2}{\ln 2-\ln 3}$ D) $\frac{\ln(\sqrt{5}-1)+\ln 2}{\ln 3-\ln 2}$

20. The values of α and β such that equation $x^2 + 2x + 2 + e^{\alpha} - \sin\beta = 0$ have a real solution is

A) $\alpha,\beta\in R$ B) $\alpha\in(0,1)$, $\beta\in\left(\frac{\pi}{2},2\pi\right)$

C) $\alpha\in(0,\infty)$ and $\beta\in\left(\frac{\pi}{2},2\pi\right)$ D) None of these

21. The minimum integral value of α for which the quadratic equation $(\cos^{-1}\alpha)x^2-(\tan^{-1}\alpha)^{3/2}x+2(\cot^{-1}\alpha)^2=0$ has both positive roots

A) 1 B) 4 C) 2 D) 3

22. Let α, β are roots of the equation $\sin^2 x + a\sin x + b = 0$ and also of $\cos^2 x + c\cos x + d = 0$. If $\cos(\alpha+\beta)$

$= \frac{a_0a^2+a_1b^2+a_2c^2+a_3d^2}{b_0a^2+b_1b^2+b_2c^2+b_3d^2}$, then $\frac{-a_0+a_1+a_2+a_3}{b_0+b_1+b_2+b_3}$ equals

A) 0 B) -1

C) 1 D) $\frac{-a_0a_1a_2a_3}{b_0b_1b_2b_3}$

23. Let $A = \{x \mid x^2 + (m-1)x - 2(m+1) = 0, x \in R\}$
$B = \{x \mid (m-1)x^2 + mx + 1 = 0, x \in R\}$
Number of values of *m* such that $A \cup B$ has exactly 3 distinct elements, is

A) 4 B) 5 C) 6 D) 7

24. If $f(x)=-3x+\prod_{i=1}^{3}\left(x-a_i\right)+\sum_{i=1}^{3}a_i$ where $a_i<a_{i+1}$ then $f(x) = 0$ has

A) Only one real root
B) Three real roots of which two of them are equal
C) Three distinct real roots
D) Three equal roots

25. Let a, b, c, d be four integers such that ad is odd and bc is even, then $ax^3 + bx^2 + cx + d = 0$ has

A) At least one irrational root
B) All three rational roots
C) All three integral roots
D) None of these

26. Given that β_1, β_3 be roots of the equation $Ax^2 - 4x + 1 = 0$ and β_2, β_3 the roots of the equation $Bx^2 - 6x + 1 = 0$. If β_1, β_2, β_3, β_4 are in HP; then the integral values of A and B respectively are

A) -3, 8 B) -3, 16 C) 3, 8 D) 3, 16

27. Let $m, n \in N$ and $p(x) = 1 + x + ... + x^m$, $p(x)$ will divide $p(x^n)$ if

A) HCF (m, n) = 1 B) HCF (m + 1, n) = 1
C) HCF (m, n + 1) = 1 D) HCF(m + 1, n + 1) = 1

28. If $(b^2 - 4ac)^2 (1 + 4a^2) < 64a^2$, $a < 0$, then maximum value of quadratic expression $ax^2 + bx + c$ is always less than

A) 0 B) 2 C) -1 D) -2

29. If $a \in R$ and the equation $(a-2)(x-[x])^2 + 2(x-[x]) + a^2 = 0$ (which [x] denotes G.IF) has no integral solution and has exactly one solution in (2, 3) then a lies in the interval

A) (0, 1) B) (1, 2) C) $(-1, 3)$ D) $(-1, 0)$

30. Let $f(x) = x^3 + 3x^2 + 6x + 2009$ and

$g(x)=\frac{1}{x-f(1)}+\frac{2}{x-f(2)}+\frac{3}{x-f(x)}$.

The number of real solutions of $g(x) = 0$ is

A) 0 B) 1 C) 2 D) inifinite

31. A quadratic equation with integral coefficients has two prime numbers as its roots. If the sum of the coefficients of the equation is prime, then the sum of the roots is

A) 2 B) 5 C) 7 D) 11

32. If the equations $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ have two common roots, then

A) $a=b\neq c$ B) $a = -b = c$
C) $a = b = c = 2$ D) $a + b + c = 3$

33. The set S of all real "x" for which $(x^2 - x + 1)^{x-1} < 1$ contains

A) $(-5, -1)$ B) $(-1, 1)$ C) $(-1, 0]$ D) $(-3, 1)$

34. Let P(x) be a polynomial such that $P(x^2 + 2) = x^{17} - 3x^5 + x^3 - 3$, then

A) P(x) = 0 has at least 34 roots
B) P(x) = 0 has exactly 17 roots
C) Sum of the roots of P(x) = 0 is -193.
D) None of these

35. If the polynomial $f(x) = 1 - x + x^2 - x^3 ... - x^{19} + x^{20}$ is expressed as $g(y) = a_0 + a_1y + a_2y^2 + ... + a_{20}y^{20}$ where $y = x - 4$, then the value of $a_0 + a_1 + a_2 + ... + a_{20}$ is

A) $\frac{5^{21}-1}{6}$ B) $\frac{5^{20}}{6}$ C) $\frac{1+5^{20}}{6}$ D) $\frac{1+5^{21}}{6}$

36. Let $f(x) = x^2 - bx + c$, b is a odd positive integer, $f(x) = 0$ have two prime numbers as roots and $b + c = 35$. Then the global minimum value of f (x) is

A) $-\frac{183}{4}$ B) $\frac{173}{16}$

C) $-\frac{81}{4}$ D) Data not sufficient

37. Let α and β be the roots of the quadratic equation $x^2+px+p^3=0\ (p\neq 0)$. If (α, β) is a point on the parabola $y^2 = x$, then roots of the quadratic equation are

A) 4 and – 2 B) – 4 and – 2
C) 4 and 2 D) – 4 and 2

38. Let a, b, c be the three roots of the equation $x^3 + x^2 - 333x - 1002 = 0$ then find the value of $a^3 + b^3 + c^3$.
A) 2002 B) 2006 C) 2008 D) 2010

39. If the roots of equation $ax^2 + bx + 10 = 0$ are not real and distinct, where $a, b \in R$ and m and n are values of a and b respectively for which $5a + b$ is minimum, then the family of lines $(4x + 2y + 3) + n(x - y - 1) = 0$ are concurrent at
A) (1, – 1) B) $\left(-\frac{1}{6}, -\frac{7}{6}\right)$
C) (1, 1) D) none of these

40. The values of 'a' for which the quadratic expression $ax^2 + |2a - 3|x - 6$ is positive for exactly two integral values of x is
A) $\left(-\frac{3}{4}, -\frac{3}{5}\right)$ B) $\left(-\frac{3}{4}, -\frac{3}{5}\right]$
C) $\left[-\frac{3}{4}, -\frac{3}{5}\right]$ D) None of these

41. Let a, b, c be reals, all different from –1 and 1, such that $a + b + c = abc$. Then, The value of K such that
$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} = \frac{K(abc)}{(1-a^2)(1-b^2)(1-c^2)}$$ is
A) 2 B) 4 C) 8 D) 1

42. Given $x, y \in R$, $x^2 + y^2 > 0$. If the maximum and minimum value of the expression $E = \frac{x^2 + y^2}{x^2 + xy + 4y^2}$ are M and m, and A denotes the average value of M and m, compute (2007) A.
A) 1335 B) 1336 C) 1337 D) 1338

43. If α, β, γ are roots of $7x^3 - x - 2 = 0$ then the value of $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$ is
A) 1 B) 2 C) – 3 D) – 4

44. Let f (x) be a quadratic expression with positive integral coefficients such that for every $\alpha, \beta, \in R, \beta > \alpha, \int_\alpha^\beta f(x)dx > 0$
Let $g(t) = f''(t) f(t)$ and $g(0) = 12$, then
A) 16 such quadratics are possible
B) Minimum value of f(1) is 6
C) Maximum value of f(1) is 11
D) All the above

45. If $\sum_{i=0}^{n} (2i+1)^2 a_i b_n = 10; \sum_{i=0}^{n} (2i+3)^2 a_i b_n = 18$, $\sum_{i=0}^{n} (2i+5)^2 a_i b_n = 32$, then $\sum_{i=0}^{n} (2i+7)^2 a_i b_n$ is equal to
A) 50 B) 52 C) 60 D) 72

46. Corresponding to the equation
$$\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = a(a-1)$$ mark the wrong option
A) Four real roots, if $a > 2$
B) Two real roots if $1 < a < 2$
C) No real root if $a < -1$
D) Four real roots if $a < -1$

Multiple Choice

47. If α, β, γ are the roots of the equation $x^3 + bx^2 + 3x - 1 = 0$ $(\alpha \le \beta \le \gamma, \alpha, \beta, \gamma$ are in H.P.) then
A) One of the roots must be 1
B) One root is smaller than 1, other is greater than 1
C) $b \in [-3, \infty)$
D) All the roots must be equal

48. If $x^3 + 3x^2 - 9x + c$ is of the form of $(x-\alpha)^2(x-\beta)$ then c is equal to
A) 27 B) – 27 C) 5 D) – 5

49. If *S* is the set of all real *x* such that $\frac{(2x-1)}{(2x^3+3x^2+x)}$ is positive, then *S* contains
A) $\left(-\infty, -\frac{3}{2}\right)$ B) $\left(-\frac{3}{2}, -\frac{1}{4}\right)$
C) $\left(-\frac{1}{4}, \frac{1}{2}\right)$ D) $\left(\frac{1}{2}, 3\right)$

50. If $\left(\frac{1}{3}\right)^{\log_{\frac{1}{3}}\left(x^2 - \frac{10x}{3} + 1\right)} \le 1$, then the interval in which x belongs to
A) $\left[0, \frac{1}{3}\right)$ B) $\left[0, \frac{1}{3}\right]$
C) $\left[3, \frac{10}{3}\right)$ D) $\left(3, \frac{10}{3}\right]$

51. Let Δ^2 be the discriminant and α, β be the roots of the equation $ax^2 + bx + c = 0$.Then, $2a\alpha + \Delta$ and $2a\beta + \Delta$ can be the roots of the equation
A) $x^2 + 2bx + b^2 = 0$
B) $x^2 - 2bx + b^2 = 0$
C) $x^2 + 2bx - 3b^2 + 16ac = 0$
D) $x^2 - 2bx - 3b^2 + 16ac = 0$

52. If $b^2 \ge 4ac$ for the equation $ax^4 + bx^2 + c = 0$ then all the roots of the equation will be real if
A) $b > 0, a < 0, c > 0$ B) $b < 0, a > 0, c > 0$
C) $b > 0, a > 0, c > 0$ D) $b > 0, a < 0, c < 0$

53. In a Δ ABC, tan A and tan B satisfy the inequation $\sqrt{3x^2 - 4x} + \sqrt{3} < 0$. Then
A) $a^2 + b^2 - ab < c^2$ B) $a^2 + b^2 > c^2$
C) $a^2 + b^2 + ab > c^2$ D) All of the above

54. Let $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$, then $f(x) = 0$ has

A) Exactly one real root in (2, 3)
B) Exactly one real root in (3, 4)
C) At least one real root in (2, 3)
D) None of these

55. Let $|a| < |b|$ and a, b are the roots of the equation $x^2 - |\alpha|\, x - |\beta| = 0$. If $|\alpha| < b - 1$, then the equation $\log_{|a|}\left(\frac{x}{b}\right)^2 - 1 = 0$ has at least one

A) Root lying between $(-\infty, a)$
B) Roots lying between (b, ∞)
C) Negative root
D) Positive root

56. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with the given cubic equation, then
A) a, b are roots of $x^2 + x + 2 = 0$
B) $a = b = 0$
C) $a = b = 3$
D) $a = 0, b = 3$

57. If a, b are the real roots of $x^2 + px + 1 = 0$ and c, d are the real roots of $x^2 + qx + 1 = 0$ then $(a - c)(b - c)(a + d)(b + d)$ is divisible by
A) $a - b - c - d$ B) $a + b + c - d$
C) $a + b + c + d$ D) $a + b - c - d$

58. If α, β, γ be the non zero real roots of the equation $x^3 + px^2 + qx + r = 0$ satisfying the relation $\alpha\beta + 1 = 0$ then
A) $\gamma = r$
B) $r^2 + pr + q = -1$
C) $\frac{1}{\alpha} + \frac{1}{\beta} = p + r$
D) Harmonic mean of $\alpha, \beta, \gamma = \frac{-3r}{q}$

59. Let α, β, γ be roots of $x^3 + px + q = 0$, then
A) An equation whose roots are $\alpha^3, \beta^3, \gamma^3$ is $x^3 + 3qx^2 + (3q^2 + p^3)y + q^3 = 0$
B) $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$
C) An equation whose roots are $\beta^3 + \gamma^3, \gamma^3 + \alpha^3, \alpha^3 + \beta^3$ is $x^3 + 6qx^2 + (p^3 + 12q^2)x + 3p^3q + 8q^3 = 0$
D) $\alpha^4 + \beta^4 + \gamma^4 = 2p^2$

60. $\frac{\pi^e}{x-e} + \frac{e^\pi}{x-\pi} + \frac{\pi^\pi + e^e}{x-\pi-e} = 0$ has
A) One real root in (e, π) and other in $(\pi - e, e)$
B) One real root in (e, π) and other in $(\pi, \pi + e)$
C) Two real roots in $(\pi - e, \pi + e)$
D) No real roots

61. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$. Suppose $|f(x)| \le 1\ \forall x \in [0, 1]$, then
A) $|a| \le 8$ B) $|b| \le 8$
C) $|c| \le 1$ D) $|a| + |b| + |c| \le 17$

Comprehension Linked Passages

Passage 1

Let us define a new function antimodulus as

$$[x] = \begin{cases} -x\ ; x > 0 \\ x\ ;\ x \le 0 \end{cases}$$

62. A solution of the equation $[x - 1] = 2x + 3$ is α, then a possible value of α is
A) 0 B) – 4
C) 5 D) None of these

63. If α satisfies equation $x^2 + kx + 5 = 0$, then possible values of k are
A) $\frac{21}{4}$ B) $\frac{21}{4}, 5$
C) 5, 6, – 2 D) None of these

64. If α found as above satisfies the equation $x^2 + kx + 5 = 0$, then another root of quadratic equation $x^2 + kx + 5 = 0$ is .
A) $\frac{5}{4}$ B) $\frac{3}{2}$ C) $-\frac{3}{2}$ D) $-\frac{5}{4}$

Passage 2

Let $(a+\sqrt{b})^{Q(x)} + (a-\sqrt{b})^{Q(x)-2\lambda} = A$, where $\lambda \in N$, $A \in R$ and $a^2 - b = 1$

$\therefore (a+\sqrt{b})(a-\sqrt{b}) = 1 \Rightarrow (a+\sqrt{b}) = (a-\sqrt{b})^{-1}$ and

$(a-\sqrt{b}) = (a+\sqrt{b})^{-1}$ i.e,

$(a\pm\sqrt{b}) = (a+\sqrt{b})^{\pm 1}$ or $(a-\sqrt{b})^{\pm 1}$

65. If $(4+\sqrt{15})^{[x]} + (4-\sqrt{15})^{[x]} = 62$, where [.] denotes the greatest integer function, then
A) $x \in [-3,-2) \cup [1,2)$ B) $x \in [-3,2) \cup [-2,1)$
C) $x \in [-2,-1) \cup [2,3)$ D) $x \in [-2,3) \cup [-1,2)$

66. Solutions of $(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \frac{4}{2-\sqrt{3}}$ are
A) $1 \pm \sqrt{3}, 1$ B) $1 \pm \sqrt{2}, 1$
C) $1 \pm \sqrt{3}, 2$ D) $1 \pm \sqrt{2}, 2$

67. The number of real solutions of the equation $\left(15+4\sqrt{14}\right)^t + \left(15-4\sqrt{14}\right)^t = 30$ are, (where $t = x^2 - 2|x|$)
A) 0 B) 2 C) 4 D) 6

Passage 3

Let $f(x) = ax^2 + bx + c$ be a quadratic expression, where a, b, c are real and a is nonzero. If $f(x) = 0$ has imaginary roots, then for all x, f(x) has same sign as that of a. Also if a differentiable function f(x) is such that $f(a) = f(b)$ then equation $f'(x) = 0$ has atleast one real root between a and b.

68. If the equation $f(x) = 0$ has no real root then $(a + b + c)c$ is
A) 0 B) > 0 C) < 0 D) Not real

69. If $2a + 3b + 6c = 0$ then the equation $f(x) = 0$ has
A) Imaginary roots
B) No root between 0 and 1
C) Atleast one root between 0 and 2
D) None of these

70. If $f(x) = x$ has non real roots, then the equation $f(f(x)) = x$
A) Has all real and distinct roots
B) Has some real and some non real roots
C) Has all real and equal roots
D) Has all non real roots

Passage 4

If f(x) is a differentiable function wherever it is continuous and $f'(c_1) = f'(c_2) = 0$, $f''(c_1) \cdot f''(c_2) < 0$, $f(c_1) = 5$, $f(c_2) = 0$ and $(c_1 < c_2)$.

71. If f(x) is continuous in $[c_1, c_2]$ and $f''(c_1) - f''(c_2) > 0$, then minimum number of roots of $f'(x) = 0$ in $[c_1 - 1, c_2 + 1]$ is
A) 2 B) 3 C) 4 D) 5

72. If f(x) is continuous in $[c_1, c_2]$ and $f''(c_1) - f''(c_2) < 0$, then minimum number of roots of $f'(x) = 0$ in $[c_1 - 1, c_2 + 1]$ is
A) 1 B) 2 C) 3 D) 4

73. If f(x) is continuous in $[c_1, c_2]$ and $f''(c_1) - f''(c_2) > 0$, then minimum number of roots of $f(x) = 0$ in $[c_1 - 1, c_2 + 1]$ is
A) 2 B) 3 C) 4 D) 5

Passage 5

Let x_1, x_2, x_3, x_4 be the roots (real or complex) of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$. If $x_1 + x_2 + x_3 + x_4$ and a, b, c, d $\in R$, then

74. If a = 2, then the value of b – c is
A) –1 B) 1 C) – 2 D) 2

75. If b < 0 then how many different values of 'a' we may have
A) 3 B) 2 C) 1 D) 0

76. If b + c = 1 and $a \neq -2$, then for real values of 'a' then $c \in$
A) $\left(-\infty, \frac{1}{4}\right)$ B) $(-\infty, 3)$ C) $(-\infty, 1)$ D) $(-\infty, 4)$

Passage 6

Given $|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$ $\forall x \in R$ and $d = b^2 - 4ac > 0$ and $D = B^2 - 4AC > 0$.

77. Which of the following must be true ?
A) $|a| \geq |A|$ B) $|a| \leq |A|$
C) $|a| = |A|$ D) All of these

78. Which of the following must be true ?
A) $|d| \leq |D|$ B) $|d| \geq |D|$
C) $|d| = |D|$ D) None of these

79. If D and d are not necessarily positive, then which of the following must be true ?
A) The roots of $ax^2 + bx + c = 0$ and $Ax^2 + Bx + C = 0$ are equal
B) The roots of $ax^2 + bx + c = 0$ and $Ax^2 + Bx + C = 0$ are equal only when they are real.
C) The roots of $ax^2 + bx + c = 0$ and $Ax^2 + Bx + C = 0$ are equal only when they are imaginary.
D) The roots of $ax^2 + bx + c = 0$ and $Ax^2 + Bx + C = 0$ may not be equal.

Passage 7

A polynomial p (x) of degree two or less which takes values y_0, y_1, y_2 at three distinct values x_0, x_1, x_2 respectively, is given by

$$p(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

80. A polynomial of degree 2 which takes values y_0, y_0, y_1 at points $x_0, x_0 + t, x_1$ $(t \neq 0)$ is given by

A) $\frac{(x-x_1)(2x_0-x_1-x)}{(x_0-x_1)(x_0-x_1+t)}y_0 + \frac{(x-x_0)(x-x_0-t)}{(x_1-x_0)(x_1-x_0-t)}y_1$

B) $\frac{(x-x_1)(2x_0-x_1-x)}{(x_0-x_1)(x_0-x_1-t)}\left(y_0 + \frac{(x_1-x_0)\,y_1}{x_1-x_0-t}\right)$

C) $\frac{(x-x_0)(2x_0-x_1-x)}{(x_0-x_1)(x_0-x_1+t)}\left(y_0 + \frac{(x_0-x_1)\,y_1}{x_1+t-x_0}\right)$

D) None of these

81. Let p be a polynomial of degree 2 and $a \neq 0, 1$, then
A) p (x) = x [(x – 1) p (0) + x p (a)] + x (x – 1) p(a)
B) p (x) = x (x – a) p (0) + a p(a) [x (x – a) + x^2]
C) $p(x) = \left[1 - \frac{x(x-a)}{1-a}\right]p(0) + \frac{x(1-x)}{1-a}\left(\frac{p(a)-p(0)}{a}\right) + \frac{x(x-a)}{1-a}p(1)$
D) None of these

82. A polynomial of degree 2 which takes values – 1, 0 and 1 at –1, 0 and 1 respectively is
A) $x^2 - x + 1$ B) $x^2 - x + 3$
C) $x^2 + x - 1$ D) None of these

Integer Type

83. Let $-1 < p < 1$, show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$

84. Show that the following equation can have atmost one real root $3x^5 - 5x^3 + 21x + 3 \sin x + 4 \cos x + 5 = 0$

85. If $x^2 - x \cos(A + B) + 1$ is a factor of the expression, $2x^4 + 4x^3 \sin A \sin B - x^2(\cos 2A + \cos 2B) + 4x \cos A \cos B - 2$. Then find the other factor.

86. If x, y, z are distinct positive numbers such that $x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}$ then the value of x y z is

87. If a, b, c are positive numbers each different from 1 such that $(\log_b a . \log_c a - \log_a a) + (\log_a b . \log_c b - \log_b b) + (\log_a c . \log_b c - \log_c c) = 0$ then find abc, where $a \neq b \neq c$.

88. The number of solutions of the equation $8[x^2 - x] + 4[x] = 13 + 12[\sin x]$ is, where [.] denotes G. I. F

89. Find the range of values of a, such that $f(x) = \frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32}$ is always negative.

90. Find a quadratic equation whose sum and product of the roots are the values of the expressions (cosec $10° - \sqrt{3}$ sec10°) and (0.5 cosec10° – 2 sin70°) respectively. Also express the roots of this quadratic in terms of tangent of an angle lying in $\left(0, \frac{\pi}{2}\right)$.

91. If $|\log_3 x| - \log_3 x - 3 < 0$, the value of x lies in the interval $\left(\frac{1}{\sqrt{k}}, \infty\right)$, then find value of $(k)^{1/3}$.

92. If f(x) is continuous function and attains only rational values if f(0) = 3 then number of different solutions of the equation $f(1)\,x^2 + 2f(2)\,x + f(3) = 0$ are

93. If $f(x) = 27x^3 + \frac{1}{x^3}$ and α, β are the roots of $3x + \frac{1}{x} = 2$ then $-\frac{f(\alpha)}{10}$ is equal to

94. Let $P(x) = 4x^2 + 6x + 4$ and $Q(y) = 4y^2 - 12y + 25$. Find the unique pair of real numbers (x, y) that satisfy $P(x).Q(y) = 28$.

95. Find the number of polynomials p(x) with integral coefficients satisfying the conditions. p(1) = 2, p(3) = 1

96. A quadratic polynomial $f(x) = x^2 + ax + b$ is formed with one of its zeros being $\frac{4+3\sqrt{3}}{2+\sqrt{3}}$ where *a* and *b* are integers. Also $g(x) = x^4 + 2x^3 - 10x^2 + 4x - 10$ is a biquadratic polynomial such that $g\left(\frac{4+3\sqrt{3}}{2+\sqrt{3}}\right) = c\sqrt{3} + d$ where *c* and *d* are also integers. Find the values of a, b, c and d.

97. All the value of parameter m, for which $x^2 + 2(m-1)x + m + 5$ is positive for x > 1, lie in the interval $(-k, \infty)$ then find 'k'

98. Let there be a quotient of two natural numbers in which the denominator is one less than the square of the numerator. If we add 2 to both numerator & denominator, the quotient will exceed $\frac{1}{3}$ & if we subtract 3 from numerator & denominator, the quotient will lie between 0 & $\frac{1}{10}$. Determine the quotient.

99. A quadratic polynomial y = f (x) satisfies $f(x) = \left[\frac{f(x+1) - f(x-1)}{2}\right]^2$ for all real x. Find the leading coefficient of the quadratic polynomial and hence find the value of $[f(0) - f(-1)] + [f(0) - f(1)]$.

100. If p, q, r be the roots of $x^3 - ax^2 + bx - c = 0$, show that the area of the triangle whose sides are p, q and r is $\frac{\sqrt{a(4ab - a^3 - 8x)}}{4}$.

101. The equation $2(\log_3 x)^2 - |\log_3 x| + a = 0$ has exactly four real solutions if $a \in \left(0, \frac{1}{K}\right)$, then the value of K is

102. Number of positive integer n for which $n^2 + 96$ is a perfect square is

103. The number of roots of the equation $2^x + 2^{x-1} + 2^{x-2} = 7^x + 7^{x-1} + 7^{x-2}$ is

104. The set of real parameter 'a' for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all real solutions, is given by $\left[\frac{m}{n}, \infty\right)$ where *m* and *n* are relatively prime positive integers, then the value of (m + n) is

105. If the range of the function $f(x) = \frac{x^2 + ax + b}{x^2 + 2x + 3}$ is [– 5, 4], a, b $\in$ N, then find the value of $(a^2 + b^2)$.

Matrix Type Match The Following

106. Match the following.

Column I	Column II
A) If $x^2 + x - a = 0$ has integral roots and $a \in N$, then a can be equal to	P) 2
B) If the equation $ax^2 + 2bx + 4c = 16$ has no real roots and $a + c > b + 4$, then integral value of c can be equal to	Q) 12
C) If the equation $x^2 + 2bx + 9b - 14 = 0$ has only negative roots, then integral values of b can be	R) 3
D) If N be the number of solutions of the equation $\|x - \|4 - x\|\| - 2x = 4$, then the value of N is	S) 20

107. Match the following :

Column I	Column II
A) The roots of the equation $x^2 + 2(a-3)x + 9 = 0$ lie between – 6 and 1 and it is also given that 2, h_1, h_2, ... h_{10}, [a] are in HP, where [a] denotes the integral part of a, then $h_2 \times h_9$ is equal to	P) 1
B) If cosec $(\theta - \alpha)$, cosec θ, cosec $(\theta + \alpha)$ are in AP $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the number of values of α is	Q) 3
C) If the equation $ax^2 + b\|x\| + c = 0$ has two distinct real $(b^2 - 4ac \neq 0)$ roots, then the value of $\frac{c}{a}$ could be	R) 12

D) The number of pairs of positive integers (x, y) where x and y are prime numbers and $x^2 - 2y^2 = 1$, is — S) – 3

T) 0

108. For the following questions, match the items in column-I to one or more items in column-II

Column I	Column II
A) If ${}^8C_{k+2} + 2.{}^8C_{k+3} + {}^8C_{k+4} > {}^{10}C_4$, then the quadratic equations whose roots are α, β and α^k, β^k have *m* common roots, then *m* =	P) 1
B) If the number of solutions of the equation $\left\|2x^2 - 5x + 3\right\| + (x-1) = 0$ is (are) *n*, then *n* =	Q) 2
C) If the constant term of the quadratic expression $\sum_{k+1}^{n}\left(x - \frac{1}{k+1}\right)\left(x - \frac{1}{k}\right)$ as $n \to \infty$ is *p*, then *p* =	R) 0
D) The equation $x^2 + 4a^2 = 1 - 4ax$ and $x^2 + 4b^2 = 1 - 4bx$ have only one root in common, then the value of $\|a - b\|$ is	S) –1
	T) –2

109.

Column I	Column II
A) If $a + b + 2c = 0$ $(c \neq 0)$ then the equation $ax^2 + bx + c = 0$ has	P) At least one root in (– 2, 0)
B) Let a, b, c, $\in$ R such that $2a - 3b + 6c = 0$, then equation $ax^2 + bx + c = 0$ has	Q) At least one root in (– 1, 0)
C) Let a, b, c be non-zero real numbers such that	R) At least one root in (– 1, 1)
$\int_0^1 \left(1 + \cos_x^8\right)\left(ax^2 + bx + c\right).dx =$	S) At least one root in (0, 1)
$\int_0^2 (1 + \cos_x^8)(ax^2 + bx + c).dx,$	T) At least one root in (0, 2)
Then the equation $ax^2 + bx = c = 0$ has	

110. Let α, β γ be three numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{2}, \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{9}{4}$ and $\alpha + \beta + \gamma = 2$

Column I	Column II
A) $\alpha \beta \gamma$	P) 6
B) $\beta\gamma + \gamma\alpha + \alpha\beta$	Q) 8
C) $\alpha^2 + \beta^2 + \gamma^2$	R) – 2
D) $\alpha^3 + \beta^3 + \gamma^3$	S) – 1

Practice Session-3
Challengers

Single Choice

1. Consider two quadratic expressions f(x) = $ax^2 + bx + c$ and g(x) = $ax^2 + px + q$, (a, b, c, p, q $\in$ R, b $\neq$ p) such that their discriminants are equal. If f(x) = g(x) has a root x = α, then

A) α will be A.M. of the roots of f(x) = 0 and g(x) = 0

B) α will be A.M. of the roots of f(x) = 0

C) α will be A.M. of the roots of f(x) = 0 or g(x) = 0

D) α will be A.M. of the roots of g(x) = 0

2 If α, β, γ are the roots of the equation $x^3 - 7x + 7 = 0$, then the vaule of $\alpha^{-4} + \beta^{-4} + \gamma^{-4}$ is

A) $\frac{11}{7}$ B) $-\frac{11}{7}$ C) $\frac{3}{7}$ D) $-\frac{3}{7}$

3. The equation $x^5 - 5ax + 4b = 0$ has
A) Three real roots if $a^5 > b^4$
B) Only one real root if $a^5 > b^4$
C) Five real roots
D) None of these

4. The value of the expression $x^4 - 8x^3 + 18x^2 - 8x + 2$ when $x = 2 + \sqrt{3}$ is

A) 0 B) 1 C) 2 D) 3

5. If f(x) = 0 is a R. E. of second type and fifth degree then a root of f(x) = 0 is
A) 0 B) 1 C) – 1 D) 2

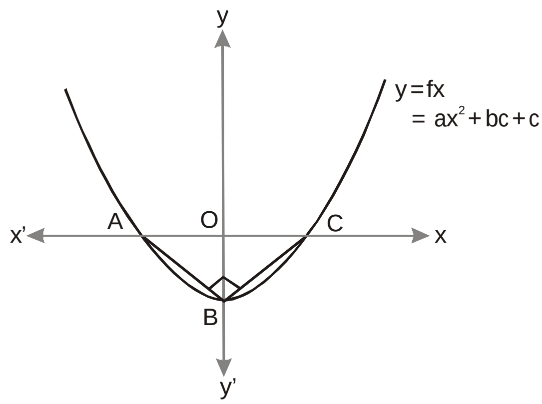
6. If x + y + z = 5 and xy + yz + zx = 3, then least and largest value of x are

A) $\frac{10}{3}, 5$ B) $-1, \frac{13}{3}$

C) $\frac{17}{3}, 7$ D) None of these

7. If a, b $\in$ R and $ax^2 + bx + 6 = 0$, a $\neq$ 0 does not have two distinct real roots, then
A) Minimum possible value of 3a + b is – 2
B) Minimum possible value of 3a + b is 2
C) Minimum possible value of 6a + b is – 2
D) Minimum possible value of 6a + b is 1

8. In the given figure vertices of ΔABC lie on y = f(x) = $ax^2 + bx + c$. The Δ ABC is right angled isosceles triangles whose hypotenuse $AC = 4\sqrt{2}$ units, then y = f(x) is given by



A) $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$ B) $y = \frac{x^2}{2} - 2$

C) $y = x^2 - 8$ D) $y = x^2 - 2\sqrt{2}$

9. Let f (x) be a real valued function satisfying a f(x) + b f (– x) = $px^2 + qx + r$ $\forall x \in R$ where p, q, r $\in$ R – {0} and a, b $\in$ R such that $|a| \neq |b|$. Then the condition that f(x) = 0 will have real roots is

A) $\left(\frac{a+b}{a-b}\right)^2 \leq \frac{q^2}{4pr}$ B) $\left(\frac{a+b}{a-b}\right)^2 \leq \frac{4pr}{q^2}$

C) $\left(\frac{a+b}{a-b}\right)^2 \geq \frac{q^2}{4pr}$ D) $\left(\frac{a+b}{a-b}\right)^2 \geq \frac{4pr}{q^2}$

Integer Type

10. If $(x^2 - 7x + 12).f(x) = (x^2 + 7x + 12).g(x)$ then prove that h(x) = f(x). g(x) + $x^4 - 25x^2 + 144$ has four real roots. Also find them

11. If α, β are roots of the equation $x^2 - 34x + 1 = 0$, evaluate $\sqrt[4]{\alpha} - \sqrt[4]{\beta}$, where $\sqrt[4]{.}$ denotes the principal value.

12. Find the absolute value of the difference of the real roots of the equation.

$$x^2 - 2^{2010}x + \left|x - 2^{2009}\right| + 2\left(2^{4017} - 1\right) = 0$$

13. Prove that $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) \geq 0$ for all $x \in R$. If equality holds then find the ratio of the roots of the equation $ax^2 + 2bx + c = 0$.

14. If the equation $x^4 + px^3 + qx^2 + rx + 5 = 0$ has four positive roots, then find the minimum value of pr.

Matrix Type Match The Following

15. Match the following, consider $f(x) = x^4 + 32x + k$ where $k \in R$

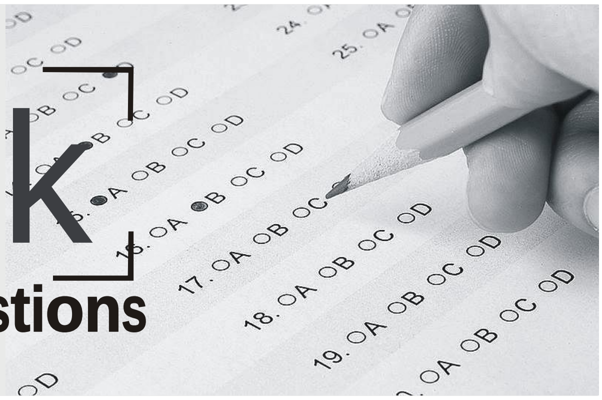
Column I	Column II
A) For $k > 50$, $f(x) = 0$ has	P) No real roots
B) For $-1 < k < 0$, $f(x) = 0$ has	Q) Two real roots with one zero root and one negative root
C) $10 < k < 20$, $f(x) = 0$ has	R) Two real roots with one positive and one negative root
D) For $k = 0$, $f(x) = 0$ has	S) Two real roots with both being negative

16. For $a \neq 0$ the equation $ax^2 + b|x| + c = 0$ has exactly k real solutions and p real roots.

Column I	Column II
A) If $k = 1$, $p = 1$, then there must be	P) $ab < 0$
B) If $k = 2$, $p = 2$, then there must be	Q) $ab = 0$
C) If $k = 3$, then there must be	R) $ac < 0$
D) If $k = 4$, then there must be	S) $ab > 0$
	T) $ac > 0$

Flashback

Previous Year Questions



Single Choice

1. Both the roots of the given equation
$(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are always **[IIT 1980]**

A) Positive B) Negative
C) Real D) Imaginary

2. The number of real solutions of the equation **[IIT 1982]**

$|x|^2 - 3|x| + 2 = 0$ is

A) 4 B) 1 C) 3 D) 2

3. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is **[IIT 1982]**

A) $-4 < x \le 0$ B) $0 < x < 1$
C) $-100 < x < 100$ D) $-\infty < x < \infty$

4. If $x_1, x_2, \ldots, x_n$ are any real numbers and n is any positive integer, then **[IIT 1982]**

A) $n\sum_{i=1}^{n} x_i^2 < \left(\sum_{i=1}^{n} x_i\right)^2$ B) $\sum_{i=1}^{n} x_i^2 \ge \left(\sum_{i=1}^{n} x_i\right)^2$

B) $\sum_{i=1}^{n} x_i^2 \ge n\left(\sum_{i=1}^{n} x_i\right)^2$ D) None of these

5. If $a+b+c = 0$, then the quadratic equation $3ax^2+2bx+c = 0$ has **[IIT 1983]**

A) At least one root in (0, 1)
B) One root in (2, 3) and the other in (–2, –1)
C) Imaginary roots
D) None of these

6. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has **[IIT 1984]**

A) No root B) One root
C) Two equal roots D) Infinitely many roots

7. If a, b and c are distinct positive numbers, then the expression $(b + c - a)(c + a - b)(a + b - c) - abc$ is **[IIT 1986]**

A) Positive B) Negative
C) Non–positive D) Non–negative

8. Let f(x) be a quadratic expression which is positive for all real x. If $g(x) = f(x) - f'(x) + f''(x)$, then for any real x **[IIT 1990]**

A) $g(x) > 0$ B) $g(x) \ge 0$
C) $g(x) \le 0$ D) $g(x) < 0$

9. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has **[IIT 1997]**

A) No solution B) One solution
C) Two solutions
D) More than two solutions

10. If the roots of the equation $x^2-2ax+a^2+a-3=0$ are real and less than 3, then **[IIT 1999]**

A) $a < 2$ B) $2 \le a \le 3$
C) $3 < a \le 4$ D) $a > 4$

11. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in **[AIEEE 2003]**

A) Arithmetic progression
B) Geometric progression
C) Harmonic progression
D) Arithmetic-geometric progression

12. The value of 'a' for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, is **[AIEEE 2003]**

A) $\frac{2}{3}$ B) $-\frac{2}{3}$ C) $\frac{1}{3}$ D) $-\frac{1}{3}$

13. If one root is square of the other root of the equation $x^2+px+q=0$, then the relation between p and q is **[IIT 2004]**

A) $p^3 - (3p-1)q + q^2 = 0$
B) $p^3 - q(3p+1) + q^2 = 0$
C) $p^3 + q(3p-1) + q^2 = 0$
D) $p^3 + q(3p+1) + q^2 = 0$

14. If $(1-p)$ is a root of quadratic equation $x^2 + px + (1-p) = 0$, then its roots are **[AIEEE 2004]**

A) 0, 1 B) – 1, 1 C) 0, – 1 D) – 1, 2

15. The value of a for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - a - 1 = 0$ assume the least value is **[AIEEE 2005]**

A) 2 B) 3 C) 0 D) 1

16. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval **[AIEEE 2005]**

A) [4, 5] B) $(-\infty, 4)$ C) $(6, \infty)$ D) (5,6]

17. If the equation $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x = 0,\ a_1 \neq 0,\ n \geq 2,$ has positive root x = α, then the equation $na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1 = 0$ has a positive root, which is **[AIEEE 2005]**

A) Equal to α
B) Greater than or equal to α
C) Smaller than α
D) Greater than α

18. If a, b, c are the sides of a triangle ABC such that $x^2 - 2(a+b+c)x + 3\lambda(ab+bc+ca) = 0$ then **[IIT 2006]**

A) $\lambda < \frac{4}{3}$ B) $\lambda < \frac{5}{3}$

C) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$ D) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$

19. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30^0 and tan 15^0 respectively, then the value of 2 + q – p is **[AIEEE 2006]**

A) 3 B) 0 C)1 D)2

20. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is **[AIEEE - 2007]**

A) (–3, 3) B) $(-3, \infty)$ C) $(3, \infty)$ D) $(-\infty, -3)$

21. Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then, the value of r is **[IIT 2007]**

A) $\frac{2}{9}(p-q)(2q-p)$ B) $\frac{2}{9}(q-p)(2p-q)$

C) $\frac{2}{9}(q-2p)(2q-p)$ D) $\frac{2}{9}(2p-q)(2q-p)$

22. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3 then, the common root is **[AIEEE 2008]**

A) 2 B) 1 C) 4 D) 3

23. If the roots of the equation $bx^2 + cx + a = 0$ is imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is **[AIEEE 2009]**

A) Greater than 4ab B) Less than 4ab
C) Greater than – 4ab D) Less than – 4ab

24. Let p and q be real numbers such that $p \neq 0, p^3 \neq q$ and $p^3 \neq -q$. If α and β are non–zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is **[IIT 2010]**

A) $(p^3+q)x^2 - (p^3+2q)x + (p^3+q) = 0$

B) $(p^3+q)x^2 - (p^3-2q)x + (p^3+q) = 0$

C) $(p^3-q)x^2 - (5p^3-2q)x + (p^3-q) = 0$

D) $(p^3-q)x^2 - (5p^3+2q)x + (p^3-q) = 0$

25. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are **[AIEEE 2011]**

A) – 4, – 3 B) 6, 1 C) 4, 3 D) – 6, – 1

26. A value of b for which the equations $x^2 + bx - 1 = 0$, $x^2 + x + b = 0$ have one root in common is **[IIT 2011]**

A) $-\sqrt{2}$ B) $-i\sqrt{3}$ C) $i\sqrt{5}$ D) $\sqrt{2}$

27. Let α and β be the root & of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[IIT 2011]**

A) 1 B) 2 C) 3 D) 4

28. The number of real roots of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ is **[AIEEE - 2012]**

A) 2 B) 1 C) Infinite D) None

29. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1] **[JEE Main 2013]**

A) Lies between 1 and 2 B) Lies between 2 and 3
C) Lies between –1and 0 D) Does not exist.

30. If $a \in R$ and the equation $-3(x-[x])^2 + 2(x-[x]) + a^2 = 0$ (where, [x] denotes the greatest integer $\leq$ x)has no integral solution, then all possible values of a lie in the interval **[JEE Main 2014]**

A) $(-1, 0) \cup (0, 1)$ B) (1, 2)

C) (–2, –1) D) $(-\infty, -2) \cup (2, \infty)$

31. Let α and β be the roots of equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q and r are in AP and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is **[JEE Main 2014]**

A) $\frac{\sqrt{61}}{9}$ B) $\frac{2\sqrt{17}}{9}$ C) $\frac{\sqrt{34}}{9}$ D) $\frac{2\sqrt{13}}{9}$

32. The quadratic equation p (x) = 0 with real coefficients has purely imaginary roots. Then, the equation p [p (x)] = 0 has **[JEE Main 2014]**

A) Only purely imaginary roots
B) All real roots
C) Two real and two purely imaginary roots
D) Neither real nor purely imaginary roots

Assertion Reason

33. Let a, b, c, p, q be the real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, \frac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$.

Statement I: $(p^2 - q)(b^2 - ac) \geq 0$ and

Statement II: $b \notin pa$ or $c \notin pa$. **[IIT 2008]**

34. Let $f: R \to R$ be a continuous function defined by $f(x) = \frac{1}{e^x + 2e^{-x}}$.

Statement I $f(c) = \frac{1}{3}$, for some $c \in R$.

Statement II $0 < f(x) \leq \frac{1}{2\sqrt{2}}, \forall x \in R$. **[AIEEE 2010]**

Multiple Choice

35. Let $a \in R$ and $f: R \to R$ be given by $f(x) = x^5 - 5x + a$. Then, **[IIT 2002]**

A) f(x) has three real roots, if a > 4

B) f(x) has only one real root, if a > 4

C) f(x) has three real roots, if a < – 4

D) f(x) has three real roots, if – 4 < a < 4

Comprehension Linked Passages

Passage 1

If a continuous f defined on the real line R, assumes positive and negative values in R, then the equation $f(x)=0$ has a root in R. For example, if it is known that a continuous function f on R is positive at some point and its minimum values is negative, then the equation $f(x)=0$ has a root in R. Consider $f(x) = ke^{x} - x$ for all real x where k is real constant. **[IIT 2007]**

36. The line y = x meets $y = ke^x$ for $k \leq 0$ at

A) No point B) One point

C) Two points D) More than two points

37. The positive value of k for which $ke^x - x = 0$ has only one root is

A) $\frac{1}{e}$ B) 1 C) e D $\log_e 2$

38. For k > 0, the set of all values of k for which $ke^x - x = 0$ has two distinct roots, is

A) $\left(0, \frac{1}{e}\right)$ B) $\left(\frac{1}{e}, 1\right)$

C) $\left(\frac{1}{e}, \infty\right)$ D) (0, 1)

Passage 2

Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Lets be the sum of all distinct real roots of f (x) and let $t = |s|$ **[IIT 2010]**

39. The real numbers s lies in the interval

A) $\left(-\frac{1}{4}, 0\right)$ B) $\left(-11, -\frac{3}{4}\right)$

C) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ D) $\left(0, \frac{1}{4}\right)$

40. The area bounded by the curve y = f(x) and the lines x = 0, y = 0 and x = t, lies in the interval

A) $\left(\frac{3}{4}, 3\right)$ B) $\left(\frac{24}{64}, \frac{11}{16}\right)$

C) (9, 10) D) $\left(0, \frac{21}{64}\right)$

41. The function f'(x) is

A) Increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

B) Decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

C) Increasing in (– t, t)

D) Decreasing in (– t, t)

Integer Type

42. Show that the square of $\frac{\sqrt{26 - 15\sqrt{3}}}{5\sqrt{2} - \sqrt{(38 + 5\sqrt{3})}}$ is a rational number. **[IIT 1978]**

43. Show that for any triangle with sides a, b, c ;

$3(ab + bc + ca) \leq (a + b + c)^2 \leq 4(ab + bc + ca)$ **[IIT 1979]**

44. For what values of m, does the system of equations $3x + my = m$ and $2x - 5y = 20$ has solution satisfying the conditions x > 0, y > 0 ? **[IIT 1980]**

45. Solve for $x\,(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$ **[IIT 1985]**

46. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$ where $ac \neq 0$, show that the equation $P(x).Q(x)=0$ has atleast two real roots. **[IIT 1985]**

47. For $a \leq 0$, determine all real roots of the equation

$x^2 - 2a|x - a| - 3a^2 = 0$ **[IIT 1986]**

48. Solve the equation $\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$ **[IIT 1987]**

49. If α, β are roots of $ax^2 + bx + c = 0$ and α^4, β^4 are roots of $lx^2 + mx + n = 0$, then prove that the roots of the equation $a^2lx^2 - 4aclx + 2c^2l + a^2m = 0$ are always real and opposite in sign. **[IIT 1989]**

50. If α is root of $ax^2 + bx + c = 0$, β is root of $-ax^2 + bx + c = 0$ and γ is root of $ax^2 + 2bx + 2c = 0$, then show that γ lies between α and β. **[IIT 1989]**

51. The equation $ax^2 + bx + c = 0$ has two real roots α and β such that $\alpha < -1, \beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$. **[IIT 1995]**

52. Find the set of all solutions the equation $2^{|y|} - \left|2^{y-1} - 1\right| = 2^{y-1} + 1$ **[IIT 1997]**

53. Let $f(x) = Ax^2 + Bx + C$ where, A, B, C are real numbers. Prove that if f(x) is an integer whenever x is an integer, then the numbers 2A, A + B and C are all integers. Conversely prove that if the numbers 2A, A + B and C are all integers, then f(x) is an integer whenever x is an integer. **[IIT 1998]**

54. If α, β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$ and $\alpha + \delta$, $\beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$, $(A \neq 0)$ for some constant δ, then prove that $\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$ **[IIT 2000]**

55. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^3x^2 + abcx + c^3 = 0$ in terms of α, β. **[IIT 2001]**

56. Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations $3x - y - z = 0$, $-3x + z = 0$, $-3x + 2y + z = 0$. Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is **[IIT 2009]**

57. The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is **[IIT 2009]**

ANSWER KEY TO PS - 1

1.D	2.D	3.C	4.A	5.B	6.B	7.B	8.D	9.A	10.B	11.A
12.A	13.B	14.C	15.B	16.C	17.B	18.D	19.C	20.D	21.B	22.B
23.A	24.A	25.D	26.A	27.B	28.B	29.A	30.B	31.A	32.A	33.D
34. B	35.C	36.D	37.D	38.C	39.D	40.B	41.C	42.C	43.D	44.A
45.B	46.A	47.C	48.A	49.B	50.D	51.A	52.A	53.B	54.C	55.A
56.D	57.B	58.B	59.C	60.D						

ANSWER KEY TO PS - 2

1.B	2.A	3.B	4.A	5.C	6.A	7.D	8.B
9.B	10.C	11.D	12.B	13.A	14.B	15.A	16.B
17.B	18.A	19.C	20.D	21.C	22.C	23.D	24.C
25.A	26.C	27.B	28.B	29.D	30.C	31.B	32.D
33.A	34.D	35.D	36.C	37.A	38.B	39.B	40.A
41.B	42.D	43.C	44.D	45.B	46.C	47.A,D	48. B,C
49.A,D	50. A,D	51.A,C	52. B,D	53.A,C	54.A,B	55. A,B	56.A,B,C
57. C,D	58. A,B,C,D	59.A,B,C,D	60.B,C	61.A,B,C,D	62.B	63.A	64.D
65.C	66.B	67.C	68.B	69.C	70.D	71.C	72.B
73.A	74.B	75.C	76.A	77.B	78.A	79.D	80.D
81.C	82.D	85. $2x^2+2x\cos(A-B)-2$		86.1	87.1	88.No solution	
89. $\left(-\infty,-\frac{1}{2}\right)$		90. $x^2-4x+1=0$; $a=\tan\left(\frac{\pi}{12}\right)$; $b=\tan\left(\frac{5\pi}{12}\right)$				91.3	
92.1	93.1	94. $\left(-\frac{3}{4},\frac{3}{2}\right)$		95.0	96. $a=2$, $b=-11$, $c=4$, $d=-1$		
97.1	98. $\frac{4}{15}$	99. $a=\frac{1}{4}$ and $[f(0)-f(-1)]+[f(0)-f(1)]=-\frac{1}{2}$				101.8	
102.4	103.1	104.7	105. 277	106. A–(P,Q,S), B–(Q,S), C–(P,Q,S), D–(R)			
107. A–(R), B–(P), C–(S), D–(P)				108. A–(Q), B–(P), C–(P), D–(P)			
109. A–(R,S,T), B–(P,Q,R), C–(R,S,T)				110. A– R, B – S, C – P, D – Q			

ANSWER KEY TO PS - 3

1.A	2.C	3.A	4.B	5.B	6.B	7.A	8.A
9.D	10. $x=\pm 3, \pm 4$		11. ± 2	12.2	14.80		
15.A – P; B – R; C – S; D – Q				16.A – Q ; B – R ; C – P ; D – PT			

ANSWER KEY TO FLASH BACK

1.C	2. A	3. D	4.D	5. A	6. A	7.B	8.A
9. A	10. A	11. C	12. A	13. A	14.C	15.D	16. B
17.C	18.A	19. A	20. A	21.D	22. A	23.C	24. B
25. B	26. B	27.C	28. D	29. D	30.A	31.D	32.D
33.B	34.A	35.B, D	36.B	37.A	38.A	39.C	40.A
41.B	44. $m \in \left(-\infty, -\frac{15}{2}\right) \cup (30, \infty)$			45. $x = \pm 2, \pm\sqrt{2}$			
47. $x = \left\{a\left(1-\sqrt{2}\right), a\left(\sqrt{6}-1\right)\right\}$			48. –1/4	52. $\{-1\} \cup [1, \infty)$		55. $x = \alpha^2\beta, \alpha\beta^2$	
56.7	57. 2						

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